Controllability on the Group of Diffeomorphisms

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Exponentials

Let M be a smooth connected manifold. Let V, V_t be complete.

Autonomous v.f. $V \in \mathrm{Vec}M$	Nonautonomous v.f. $V_t \in \mathrm{Vec}M$
$\begin{cases} \dot q(t) = V(q(t)) \\ q(0) = q_0 \,. \end{cases}$	$\begin{cases} \dot q(t) = V_t(q(t)) \\ q(t_0) = q_0 \,, \end{cases}$

For every fixed t

$e^{tV} (= \exp(tV))$	$\overrightarrow{\exp} \int_{t_0}^{t} V_\tau \, d\tau$

is a diffeomorphism of M which maps any $q_0 \in M$ to the value of the solution at time t of the system.

Our goal

Let $\mathcal{F} \subset \mathrm{Vec}M$ be a family of vector fields we set

$$\mathrm{Gr}\mathcal{F} = \{e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k} \ : \ t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\} .$$

Our purpose is to study the relation between $\mathrm{Gr}\mathcal{F}$ and $\mathrm{Diff}_0(M)$.

Thurston 1971

If M is compact then the group $\mathrm{Diff}_0(M)$ is simple.

If $\mathcal{F} = \mathrm{Vec}M$ then $\mathrm{Gr}\mathcal{F}$ is a *normal* subgroup of $\mathrm{Diff}_0(M)$. Therefore

$$\mathrm{Gr}(\mathrm{Vec}M) = \mathrm{Diff}_0(M)$$

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$$\mathrm{Gr}(\mathrm{Vec}M) = \mathrm{Diff}_0(M)$$

The main result

Theorem

If

- M is compact
- $\mathrm{Gr}\mathcal{F}$ acts transitively on M,

then

$$\mathrm{Gr}\left\{af : a \in C^{\infty}(M),\ f \in \mathcal{F}\right\} = \mathrm{Diff}_0 M.$$

Control Systems

By *Control System* we mean a system of the form:

$$\dot{q} = f_u(q), \quad q \in M, u \in U,$$

where

- $q \in M$ is called *state*;
- $u \in U$ is called *control*;
- $U \subset \mathbb{R}^m$ is called *set of control parameters*.

We represent the control system by a family of vector fields

$$\mathcal{F} = \{f_u \, : \, u \in U\} \subset \mathrm{Vec}M\,.$$

Control systems $\iff$ Families of vector fields

Controllability

Reachable set

$$\mathcal{R}_q = \{q \circ e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k} \ : f_i \in \mathcal{F}, k \in \mathbb{N}, t_i \geq 0\}\,.$$

Orbit

$$\begin{aligned}\mathcal{O}_q &= \{q \circ e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k} \ : f_i \in \mathcal{F}, k \in \mathbb{N}, t_i \in \mathbb{R}\} \\ &= \{q \circ P \ : \ P \in \mathrm{Gr}\mathcal{F}\}\,.\end{aligned}$$

If a family $\mathcal{F}$ is symmetric, namely if $\mathcal{F} = -\mathcal{F}$, then $\mathcal{R}_q = \mathcal{O}_q$.

Definition: Controllability

A system $\mathcal{F}$ is *controllable* $\iff$ $\mathcal{R}_q = M$, for every $q \in M$.

Remark

$\mathrm{Gr}\mathcal{F}$ acts transitively on M $\iff$ $\mathcal{O}_q = M$, for every $q \in M$.

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If M is compact and $\mathrm{Gr}\mathcal{F}$ acts transitively on M, then

$$\mathrm{Gr}\left\{af \,:\, a \in C^{\infty}(M),\ f \in \mathcal{F}\right\} = \mathrm{Diff}_0 M.$$

If $\mathcal{F}$ is symmetric then

Controllability on M

$$\Downarrow$$

Controllability “on” $\mathrm{Diff}_0(M)$

The main result

Main Theorem

Let M be a compact connected manifold and $\mathcal{F} \subset \mathrm{Vec} M$.
If $\mathrm{Gr}\mathcal{F}$ acts transitively on M, then there exist

- a neighborhood $\mathcal{O}$ of the identity in $\mathrm{Diff}_0(M)$;
- a positive integer μ

such that every $P \in \mathcal{O}$ can be presented in the form

$$P = e^{a_1 f_1} \circ \cdots \circ e^{a_\mu f_\mu},$$

for some $f_1, \ldots, f_\mu \in \mathcal{F}$ and $a_1, \ldots, a_\mu \in C^\infty(M)$.

Remark

The number of exponentials μ does not depend on P.

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Bracket Generating families

- $\mathrm{Lie}(\mathcal{F}) = \mathrm{span}\{[f_1,[\dots[f_{k-1},f_k]\dots]] \,:\, f_1,\dots,f_k \in \mathcal{F}, k \in \mathbb{N}\}$
- $\mathrm{Lie}_q\,\mathcal{F} = \{f(q) \,:\, f \in \mathrm{Lie}(\mathcal{F})\}$.

Definition

We say that the family $\mathcal{F}$ is *bracket generating* if

$$\mathrm{Lie}_q\,\mathcal{F} = T_qM \quad \text{for every } q \in M\,.$$

Theorem (Chow–Rashevsky)

Let $\mathcal{F}$ be a *bracket generating* family of vector fields. Then

$$\mathcal{O}_q = M, \quad \text{for any } q \in M\,.$$

Application to control systems

Corollary

Let $\{f_1, \ldots, f_m\}$ *be bracket generating. Consider the system*

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i\,, \quad q \in M\,, \tag{1}$$

with controls that are

- *piecewise constant in* t,
- *smooth in* q.

For every $P \in \mathrm{Diff}_0(M)$ *there exist controls* $u_i(t,q)$ *such that*

$$P = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{m} u_i(t,\cdot) f_i \, dt.$$

Remark

M non-compact $\implies$ controls measurable in t.

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Outline of the proof

Step 1 Localization of the problem;

- compactness of M,
- cut-off functions and a result by Palis & Smale

Step 2 Considering a full-dimensional case;

- Controllability assumption,
- Orbit Theorem of Sussmann

Step 3 Restriction to a 1-dimensional problem with parameters;

- Implicit Function Theorem

Step 4 Linearization of diffeomorphisms

Proof Idea

Consider a linear ODE on $U \subset \mathbb{R}$:

$$\begin{cases} \dot{x} = \beta x \\ x(0) = x_0 \end{cases}$$

the solution $\varphi(t, x_0) = e^{t\beta} x_0$ at time $t = 1$ is the linear diffeomorphisms, namely a rescaling of a factor e^{β}.

The linear diffeomorphism of $U \subset \mathbb{R}$,

$$x \mapsto \alpha x\big|_U, \quad \alpha \neq 1, \quad (\alpha > 0),$$

is the exponential of the linear vector field $\log(\alpha) x \frac{\partial}{\partial x}$

The change of coordinates that linearizes the diffeomorphism can be recovered from the solution of a first order linear PDE.

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Improvements

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i\,, \quad q \in M\,,$$

with $\{f_1, \ldots, f_m\}$ bracket generating .
For every $P \in \mathrm{Diff}_0(M)$ there exist controls $u_i(t,q)$ that are

(i) piecewise constant w.r.t. t,

(ii) smooth w.r.t. q.

such that P is the flow at time 1 of the system.

- Is it possible to assume controls more regular?
- Is it possible to add a drift to the system?

Get the Jet

Let $\{f_1, f_2, \dots, f_m\}$ bracket generating. Consider the system

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^n,$$

with controls u_i such that, for every $i = 1, \dots, m$:

(i) u_i is polynomial with respect to $q \in \mathbb{R}^n$;

(ii) u_i is a trigonometric polynomial with respect to $t \in [0, 1]$.

Let r and k be positive integers, $\varepsilon > 0$, and B ball in $\mathbb{R}^n$. For any $P \in \mathrm{Diff}_0(\mathbb{R}^n)$, there exist controls $u_1(t,q), \dots, u_m(t,q)$ such that, if Φ is the flow at time 1 of the system then

$$J_0^k(\Phi) = J_0^k(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.$$

Where

$$J_0^k(P)(z) = P(0) + (DP(0)) \cdot z + \frac{1}{2}(D^2 P(0)) \cdot z^{\otimes 2} + \cdots + \frac{D^k P(0)}{k!} \cdot z^{\otimes k}.$$

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Conclusion and Open Problems

- Is it possible to realize *exactly* a diffeomorphism as exponential of a control affine system with drift?
- Is it possible to assume the control to be more regular?

- What about the group of volume preserving diffeomorphisms?
- Is it possible to realize a volume preserving diffeomorphism as composition of exponential of divergence free vector fields?

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Thank you for your attention

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¡Gracias!