Controllability on the Group of Diffeomorphisms

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Ph.D thesis
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*= with Andrei A. Agrachev

Exponentials

Let M be a smooth connected manifold. Let V, V_t be complete.

Autonomous v.f. $V \in \mathrm{Vec}M$	Nonautonomous v.f. $V_t \in \mathrm{Vec}M$
$\begin{cases} \dot{q}(t) = V(q(t)) \\ q(0) = q_0 \,. \end{cases}$	$\begin{cases} \dot{q}(t) = V_t(q(t)) \\ q(t_0) = q_0 \,, \end{cases}$

For every fixed t

e^{tV}	$\overrightarrow{\exp} \int_{t_0}^{t} V_\tau \, d\tau$

is a diffeomorphism of M which maps any $q_0 \in M$ to the value of the solution at time t of the system.

Our goal

Let $\mathcal{F} \subset \mathrm{Vec}M$ be a family of vector fields we set

$$\mathrm{Gr}\mathcal{F} = \{e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k} \ : \ t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\}.$$

Our purpose is to study the relation between $\mathrm{Gr}\mathcal{F}$ and $\mathrm{Diff}_0(M)$.

Thurston 1971

If M is compact then the group $\mathrm{Diff}_0(M)$ is simple.

If $\mathcal{F} = \mathrm{Vec}M$ then $\mathrm{Gr}\mathcal{F}$ is a *normal* subgroup of $\mathrm{Diff}_0(M)$. Therefore

$$\mathrm{Gr}(\mathrm{Vec}M) = \mathrm{Diff}_0(M)$$

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$$\mathrm{Gr}(\mathrm{Vec} M) = \mathrm{Diff}_0(M)$$

The main result

Theorem

If M is compact and $\mathrm{Gr}\mathcal{F}$ acts transitively on M, then

$$\mathrm{Gr}\left\{af : a \in C^\infty(M),\ f \in \mathcal{F}\right\} = \mathrm{Diff}_0 M.$$

Remark (Lobry)

The set of pairs (f_1, f_2) such that $\mathrm{Gr}\{f_1, f_2\}$ acts transitively on M is dense in $\mathrm{Vec}M \times \mathrm{Vec}M$.

The main result

Main Theorem

Let M be a compact connected manifold and $\mathcal{F} \subset \mathrm{Vec}M$.
If $\mathrm{Gr}\mathcal{F}$ acts transitively on M, then there exist

- a neighborhood $\mathcal{O}$ of the identity in $\mathrm{Diff}_0(M)$;
- a positive integer μ

such that every $P \in \mathcal{O}$ can be presented in the form

$$P = e^{a_1 f_1} \circ \cdots \circ e^{a_\mu f_\mu},$$

for some $f_1, \ldots, f_\mu \in \mathcal{F}$ and $a_1, \ldots, a_\mu \in C^\infty(M)$.

Remark

The number of exponentials μ does not depend on P.

Open Problem

Estimate μ.

The main result

Main Theorem

Let M be a compact connected manifold and $\mathcal{F} \subset \mathrm{Vec}M$.
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Control Systems

By *Control System* we mean a system of the form:

$$\dot{q} = f_u(q), \quad q \in M, u \in U,$$

where

- $q \in M$ is called *state*;
- $u \in U$ is called *control*;
- $U \subset \mathbb{R}^m$ is called *set of control parameters*.

We represent the control system by a family of vector fields

$$\mathcal{F} = \{f_u \,:\, u \in U\} \subset \mathrm{Vec}M \,.$$

Control systems $\iff$ Families of vector fields

Attainable sets

The set of points reachable is called *attainable set*.

Attainable set

$$\mathcal{A}_q = \{q \circ e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k} \ : \ t_i \geq 0, f_i \in \mathcal{F}, k \in \mathbb{N}\}.$$

We consider a larger set: the *Orbit*

Orbit

$$\begin{aligned} \mathcal{O}_q &= \{q \circ e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k} \ : \ t_i \in \mathbb{R}, f_i \in \mathcal{F}, k \in \mathbb{N}\} \\ &= \{q \circ P \ : \ P \in \mathrm{Gr}\mathcal{F}\}. \end{aligned}$$

If a family $\mathcal{F}$ is symmetric, namely if $\mathcal{F} = -\mathcal{F}$, then the attainable sets coincide with the orbits, i.e. $\mathcal{A}_q = \mathcal{O}_q$.

Controllability

Definition: Controllability

A system $\mathcal{F}$ is *controllable* $\iff$ $\mathcal{A}_q = M,$ for every $q \in M$.

Remark

$\mathrm{Gr}\mathcal{F}$ acts transitively on M $\iff$ $\mathcal{O}_q = M,$ for every $q \in M$.

If $\mathcal{F}$ is symmetric then

Controllability on M

$\Downarrow$

“Controllability” on $\mathrm{Diff}_0(M)$

Bracket Generating families

Definition

- $\mathrm{Lie}(\mathcal{F}) = \mathrm{span}\{[f_1, [\ldots [f_{k-1}, f_k] \ldots]] \ : f_1, \ldots, f_k \in \mathcal{F}, k \in \mathbb{N}\}$
- $\mathrm{Lie}_q \mathcal{F} = \{f(q) \ : f \in \mathrm{Lie}(\mathcal{F})\}$.

Definition

We say that the family $\mathcal{F}$ is *bracket generating* if

$$\mathrm{Lie}_q \mathcal{F} = T_q M \quad \text{for every } q \in M\,.$$

Theorem (Chow–Rashevsky)

Let $\mathcal{F}$ be a *bracket generating* family of vector fields. Then

$$\mathcal{O}_q = M, \quad \text{for any } q \in M\,.$$

Application to control systems

Corollary

Let $\{f_1, \dots, f_m\}$ be bracket generating. Consider the system

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i\,, \quad q \in M\,, \tag{1}$$

with controls that are

- *piecewise constant in t,*
- *smooth in q.*

For every $P \in \mathrm{Diff}_0(M)$ there exist controls $u_i(t,q)$ such that

$$P = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{m} u_i(t,\cdot) f_i \, dt.$$

Outline of the proof

- Localization of the problem;
- Use the controllability assumption to consider a full-dimensional case;
- Restriction to a 1-dimensional problem with parameters;
- Linearize the diffeomorphism.

Localization

Lemma (Palis–Smale)

Let $\bigcup_j U_j = M$ *be an open covering of* M *and* $\mathcal{O}$ *be a neighborhood of identity in* $\mathrm{Diff}_0 M$.
Then the group $\mathrm{Diff}_0 M$ *is generated by the subset*

$$\{P \in \mathcal{O} : \exists\, j \text{ such that } \operatorname{supp} P \subset U_j\}.$$

Where $\operatorname{supp} P = \overline{\{x \in M : P(x) \neq x\}}$.

Orbit Theorem

Theorem (Orbit Theorem of Sussmann)

$\mathcal{O}_q$ is a connected submanifold of M. Moreover,

$$T_p\mathcal{O}_q = \operatorname{span}\{q \circ \operatorname{Ad} Pf \ : \ P \in \operatorname{Gr}\mathcal{F}, f \in \mathcal{F}\}, \quad p \in \mathcal{O}_q\,.$$

Recall that transitivity of the action of $\operatorname{Gr}\mathcal{F}$ on $M \implies \mathcal{O}_q = M$.
If $X_1(q), \ldots, X_n(q)$ form a basis of T_qM then

$$e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n} \in \operatorname{Gr}\{af : a \in C^\infty(M),\ f \in \mathcal{F}\}$$

for every $a_1, \ldots, a_n \in C^\infty(M)$.
Indeed $X_i = \operatorname{Ad} P_i f_i$ for $i = 1, \ldots, n$ with $P_i \in \operatorname{Gr}\mathcal{F}$, $f_i \in \mathcal{F}$

The problem reduces to

Given $X_1, \dots, X_n$ such that

$$\mathrm{span}\{X_1(0), \dots, X_n(0)\} = \mathbb{R}^n .$$

We have to prove that there exist

- an open neighborhood $U \subset \mathbb{R}^n$;
- a open subset of $\mathcal{O} \subset \mathrm{Diff}_0(U)$;

such that every $P \in \mathcal{O}$ can be written as

$$P = e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n} . \tag{2}$$

In the following we study analytical properties of map

$$(a_1, \dots, a_n) \mapsto e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n}\big|_U .$$

Outline of the proof

- Localization of the problem;
- Use the controllability assumption to consider a full-dimensional case;
- Restriction to a 1-dimensional problem with parameters;
- Linearize the diffeomorphism.

Restriction to a single direction

Let $X_1, \ldots, X_n \in \mathrm{Vec}\mathbb{R}^n$ such that

$$\mathrm{span}\{X_1(0), \ldots, X_n(0)\} = \mathbb{R}^n \,.$$

For

- U neighborhood of the origin in $\mathbb{R}^n$;
- $\mathcal{U}$ neighborhood of the identity in $\mathrm{Diff}_0(U)$;

small enough every $P \in \mathcal{U}$ splits into the composition

$$P = \varphi_1 \circ \cdots \circ \varphi_n \big|_U \,,$$

where $\varphi_i \in \mathrm{Diff}(U)$ and preserves the 1-foliation generated by the trajectories of the equation $\dot{q} = X_i(q)$, for every $i = 1, \ldots, n$. Namely of the form

$$\varphi_i = \overrightarrow{\exp} \int_0^1 a(t, \cdot) X_i \, dt \,.$$

Idea

The linear diffeomorphism of $U \subset \mathbb{R}$, say

$$x \mapsto \alpha x\big|_U , \quad \alpha \neq 1, \quad (\alpha > 0)\,,$$

is the exponential of the linear vector field $\log(\alpha)x\frac{\partial}{\partial x}$

- It is possible to take a nonempty open subset of $\mathcal{U}$ such that the linearization of every φ_k is not trivial.
- The change of coordinates that linearizes can be recovered from the solution of the PDE:

$$a(t,x,y)\frac{\partial u}{\partial x}(t,x,y) + \frac{\partial u}{\partial t}(t,x,y) + b(t,x,y)u(t,x,y) = 0, \tag{3}$$

with $t, x \in \mathbb{R}, y \in \mathbb{R}^{n-1}$ □

Improvements

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i \,, \quad q \in M \,,$$

with $\{f_1, \ldots, f_m\}$ bracket generating .
For every $P \in \mathrm{Diff}_0(M)$ there exist controls $u_i(t,q)$ that are

(i) piecewise constant w.r.t. t,

(ii) smooth w.r.t. q.

such that P is the flow at time 1 of the system.

- Is it possible to assume controls more regular?
- Is it possible to add a drift to the system?

The Second Result

Let $\{f_1, f_2, \ldots, f_m\}$ bracket generating. Consider the system

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^n,$$

with controls u_i such that, for every $i = 1, \ldots, m$:

(i) u_i is polynomial with respect to $q \in \mathbb{R}^n$;

(ii) u_i is a trigonometric polynomial with respect to $t \in [0,1]$.

Let k be a positive integer and consider

$$J_0^k(P)(z) = P(0) + (DP(0)) \cdot z + \frac{1}{2}(D^2P(0)) \cdot z^{\otimes 2} + \cdots + \frac{D^k P(0)}{k!} \cdot z^{\otimes k}.$$

Let r be a positive integer, $\varepsilon > 0$, and B ball in $\mathbb{R}^n$. For any $P \in \mathrm{Diff}_0(\mathbb{R}^n)$, there exist controls $u_1(t,q), \ldots, u_m(t,q)$ such that, if Φ is the flow at time 1 of the system then

$$J_0^k(\Phi) = J_0^k(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.$$

The Second Result

Let $\{f_1, f_2, \ldots, f_m\}$ bracket generating. Consider the system

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^n,$$

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$$J_0^k(\Phi) = J_0^k(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.$$

The core of the method

We have to study analytical properties of map

$$(a_1, \ldots, a_n) \mapsto e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n}\big|_U . \qquad (4)$$

Consider

- the space $\mathbf{X}$ of polynomials of degree $\leq k$, in n variables;
- the jet–group $\mathbf{Y} = J_0^k(\mathrm{Diff}_0(\mathbb{R}^n))$;

and consider the map:

$$\begin{array}{rccc} F: & \mathbf{X}^n & \longrightarrow & \mathbf{Y} \\ & (a_1, \ldots, a_n) & \longmapsto & J_0^k(e^{a_1 X_1} \circ \ldots \circ e^{a_n X_n}) \end{array}$$

$\dim \mathbf{X} < \infty$ and $\dim \mathbf{Y} < \infty$

Implicit Function Theorem applied to F

$$\begin{array}{rccc} F: & \mathbf{X}^n & \longrightarrow & \mathbf{Y} \\ & (a_1, \ldots, a_n) & \longmapsto & J_0^k(e^{a_1 X_1} \circ \ldots \circ e^{a_n X_n}) \end{array}$$

- $F(0, \ldots, 0) = \mathrm{Id}$;
- $T_{\mathrm{Id}}\mathbf{Y} = J_0^k(\mathrm{Vec}(\mathbb{R}^n))$;
- $D_0F(a_1, \ldots, a_n) = a_1 J_0^k(X_1) + \ldots + a_n J_0^k(X_n)$.

D_0F is surjective

Thus F is locally surjective.
Moreover:

- F is continuous;
- F has a right inverse;
- the right inverse of F is continuous.

Relaxation

Theorem

If $\mathrm{Gr}\mathcal{F}$ acts transitively on $\mathbb{R}^n$. For any $P \in \mathrm{Diff}_0(\mathbb{R}^n)$, there exists a sequence

$$\{P_j\}_j \subset \mathrm{Gr}\{af \ : \ a \in C^\infty(\mathbb{R}^n), f \in \mathcal{F}\}$$

such that

$$P_j \to P, \quad \text{as } j \to \infty$$

in the C^∞–topology.

Proposition

If $V_t = \sum_{i=1}^{m} a_i(t,\cdot)X_i, \Rightarrow \exists\, Z_t^n$ sequence of ***piecewise constant*** *w.r.t. t vector fields s.t. $Z_t^n \in \{aX_i \mid a \in C^\infty, i = 1\ldots,m\}, \quad \forall t, n$, and*

$$\overrightarrow{\exp}\int_0^t Z_\tau^n\, d\tau \longrightarrow \overrightarrow{\exp}\int_0^t V_\tau\, d\tau, \quad \text{as } n \to \infty$$

in the C^∞–topology and uniformly w.r.t. $t \in [0,1]$.

Relaxation

Theorem

If $\mathrm{Gr}\mathcal{F}$ *acts transitively on* $\mathbb{R}^n$. *For any* $P \in \mathrm{Diff}_0(\mathbb{R}^n)$, *there exists a sequence*

$$\{P_j\}_j \subset \mathrm{Gr}\{af \ : \ a \in C^\infty(\mathbb{R}^n), f \in \mathcal{F}\}$$

such that

$$P_j \to P, \quad \text{as } j \to \infty$$

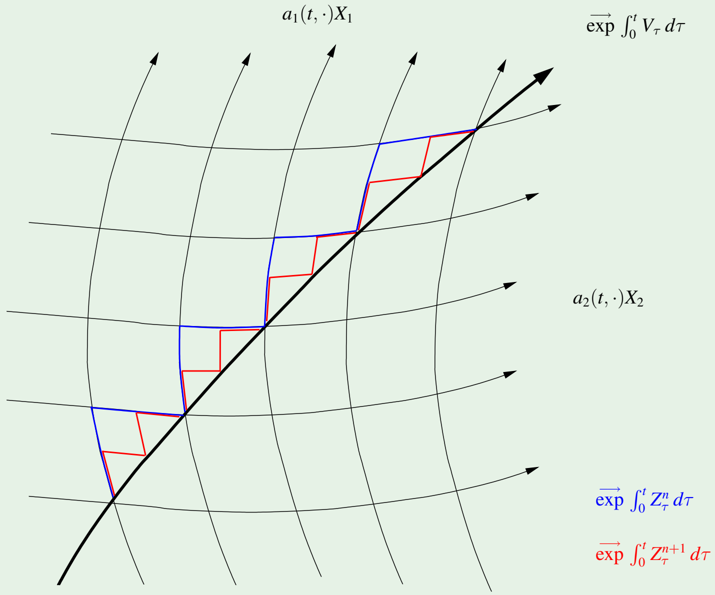
in the C^∞*–topology.*

Proposition

If $V_t = \sum_{i=1}^m a_i(t,\cdot)X_i, \Rightarrow \exists\ Z_t^n$ *sequence of* ***piecewise constant*** *w.r.t.* t *vector fields s.t.* $Z_t^n \in \{aX_i \mid a \in \mathcal{C}^\infty, i = 1\ldots, m\}, \quad \forall t, n,$ *and*

$$\overrightarrow{\exp}\int_0^t Z_\tau^n\, d\tau \longrightarrow \overrightarrow{\exp}\int_0^t V_\tau\, d\tau, \quad \text{as } n \to \infty$$

in the C^∞*–topology and uniformly w.r.t.* $t \in [0,1]$.


$a_1(t,\cdot)X_1$
$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau$
$a_2(t,\cdot)X_2$
$\overrightarrow{\exp} \int_0^t Z^n_\tau \, d\tau$
$\overrightarrow{\exp} \int_0^t Z^{n+1}_\tau \, d\tau$

Back to Control Systems

Let $\{f_1, \ldots, f_m\}$ be a bracket–generating family and consider the control–affine system

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^n .$$

for every $P \in \mathrm{Diff}_0(\mathbb{R}^d)$:

- there exist $u_i(t,\cdot)$ piecewise constant in t

such that

$$J_0^k(P) = J_0^k(e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n}) = J_0^k\left(\overrightarrow{\exp} \int_0^1 \sum_{i=1}^{m} u_i(t,\cdot) f_i \, dt \right) .$$

and

$\overrightarrow{\exp} \int_0^1 \sum_{i=1}^{m} u_i(t,\cdot) f_i \, dt$ is arbitrary close to P.

Lemma

Consider the control system

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i(q)\,, \qquad q \in \mathbb{R}^n\,,$$

with

- $\{f_1, f_2, \ldots, f_m\}$ *bracket generating;*
- u_i *piecewise constant with respect to* $t \in [0,1]$*;*
- u_i *smooth with respect to* q*.*

Let N *and* r *be positive integers,* $\varepsilon > 0$*, and* B *ball in* $\mathbb{R}^n$*. For any* $P \in \mathrm{Diff}_0(\mathbb{R}^n)$*, there exist controls* $u_1(t,q), \ldots, u_m(t,q)$ *such that, if*

$$\Phi = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{m} u_i(t,\cdot) f_i \, dt\,.$$

then

$$J_0^k(\Phi) = J_0^k(P) \quad \textit{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.$$

If $\mathbf{U}$ is the space of controls $u(t,q)$:

- smooth w.r.t. q;
- piecewise constant w.r.t. t.

By Implicit Function Theorem the map:

$$\begin{array}{rccc} \tilde{F}: & \mathbf{U}^m & \longrightarrow & \mathbf{Y} \\ & (u_1,\dots,u_m) & \longmapsto & J_0^k(\overrightarrow{\exp}\int_0^1 \sum_{i=1}^m u_i(t,\cdot) f_i\,dt) \end{array}$$

is continuous, surjective and with continuous right inverse.

Remark

Let $\varepsilon > 0$. If $G : \mathbf{U}^m \to \mathbf{Y}$ is s.t. $\sup_{x\in K} |\tilde{F}(x) - G(x)| < \varepsilon$ for any K compact,
then G is surjective too.

Small perturbations of map $\tilde{F}$ remain surjective.

Theorem

Let $\{f_1, f_2, \dots, f_m\}$ be a bracket generating family of vector fields on $\mathbb{R}^n$. Consider the control system

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^n,$$

with controls u_i such that, for every $i = 1, \dots, m$:

(i) u_i *is smooth w.r.t.* $q \in \mathbb{R}^n$*;*

(ii) u_i *is piecewise constant w.r.t.* $t \in [0,1]$.

Let N and r be positive integers, $\varepsilon > 0$, and B ball in $\mathbb{R}^n$. For any $P \in \mathrm{Diff}_0(\mathbb{R}^n)$, there exist controls $u_1(t,q), \dots, u_m(t,q)$ such that, if

$$\Phi = \overrightarrow{\exp} \int_0^1 f_0 + \sum_{i=1}^{m} u_i(t,\cdot) f_i \, dt .$$

then

$$J_0^k(\Phi) = J_0^k(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.$$

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with controls u_i such that, for every $i = 1, \ldots, m$:

- (i) u_i *is smooth w.r.t.* $q \in \mathbb{R}^n$*;*
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with controls u_i such that, for every $i = 1, \ldots, m$:

- (i) u_i is polynomial w.r.t. $q \in \mathbb{R}^n$;
- (ii) u_i is piecewise constant w.r.t. $t \in [0,1]$.

Let N and r be positive integers, $\varepsilon > 0$, and B ball in $\mathbb{R}^n$. For any $P \in \mathrm{Diff}_0(\mathbb{R}^n)$, there exist controls $u_1(t,q), \ldots, u_m(t,q)$ such that, if

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then

$$J_0^k(\Phi) = J_0^k(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.$$

Theorem

Let $\{f_1, f_2, \dots, f_m\}$ *be a bracket generating family of vector fields on* $\mathbb{R}^n$*. Consider the control system*

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^n,$$

with controls u_i *such that, for every* $i = 1, \dots, m$*:*

- (i) u_i *is polynomial w.r.t.* $q \in \mathbb{R}^n$*;*
- (ii) u_i *is trigonometric polynomial w.r.t.* $t \in [0,1]$*.*

Let N *and* r *be positive integers,* $\varepsilon > 0$*, and* B *ball in* $\mathbb{R}^n$*. For any* $P \in \mathrm{Diff}_0(\mathbb{R}^n)$*, there exist controls* $u_1(t,q), \dots, u_m(t,q)$ *such that, if*

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then

$$J_0^k(\Phi) = J_0^k(P) \quad \text{and} \quad \|\Phi - P\|_{C^r(B)} < \varepsilon.$$

An application of Nash–Moser

$$
\begin{array}{rccc}
F: & C^\infty(M)^n & \longrightarrow & \mathrm{Diff}_0(M) \\
 & (a_1,\dots,a_n) & \longmapsto & e^{a_1X_1}\circ\cdots\circ e^{a_nX_n}
\end{array}
\qquad (5)
$$

The problem is to prove:

1. F is locally onto;
2. small perturbations of the map F are locally surjective too.

Remark

Recall that point 1 implies the Main Theorem.

An alternative proof of Main Theorem

Proposition

Let $X_i \in \mathrm{Vec}\mathbb{R}^n$, $i = 1, \dots, n$, *such that*

$$\mathrm{span}\{X_1(0), \dots, X_n(0)\} = \mathbb{R}^n.$$

Then, there exist $\varrho > 0$ *and an open subset* $\mathcal{U} \subset C_0^\infty(B_\varrho)^n$, *such that the mapping*

$$\begin{array}{rcl} F : \mathcal{U} & \to & \mathrm{Diff}_0(B_\varrho), \\ (a_1, \dots, a_n) & \mapsto & \left(e^{a_1 X_1} \circ \cdots \circ e^{a_n X_n}\right)\Big|_{B_\varrho}, \end{array} \tag{6}$$

is an *open map* *from* $\mathcal{U}$ *into* $\mathrm{Diff}_0(B_\varrho)$, *where*

$$B_\varrho = \left\{e^{s_1 X_1} \circ \cdots \circ e^{s_n X_n}(0) : |s_i| < \varrho,\ i = 1, \dots, d\right\}.$$

Classical Implicit Function Theorem does not apply

It is possible to prove that

1. F maps C^k functions into C^k diffeomorphisms;
2. $D_{\mathbf{a}}F$ maps C^k functions into C^k vector fields;
3. $D_{\mathbf{a}}F^{-1}$ maps C^k vector fields into C^{k-1} functions.

Therefore

- $D_{\mathbf{a}}F^{-1}$ "loses derivatives" $\iff$ The inverse of $D_{\mathbf{a}}F$ is unbounded.
- We have to look to map F as a map between Fréchet spaces.
- We need to apply the Nash–Moser Implicit Function Theorem.

Tame Spaces

Stated in terms of Tame Spaces and Tame Maps (Sergeraert 1970)

Definition (Graded Fréchet space)

A Fréchet space F with a family of seminorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ s.t.

$$\|f\|_0 \leq \|f\|_1 \leq \|f\|_2 \leq \dots .$$

- The space $C^\infty(B)$ is a graded Fréchet with the family

$$\|f\|_n = \sup_{1\leq k\leq n} \sup_{x\in B} |f^{(k)}(x)|.$$

Spaces of smooth functions are something more:

- $C^\infty(B)$ and $\mathrm{Vec}M$ are Tame Spaces;
- $\mathrm{Diff}_0(M)$ is a Tame Manifold.

Tame space means "scale of Banach spaces".

Tame Maps

Definition (Tame Estimates)

Let $\mathbf{X}$ and $\mathbf{Y}$ tame spaces and $F : U \subset \mathbf{X} \to \mathbf{Y}$. F satisfies tame estimates of degree r and base b if there exists $C = C(n)$ such that

$$\|F(a)\|_n \leq C(\|a\|_{n+r} + 1),$$

for every $n \geq b$, $a \in U$.

Definition (Tame Map)

A map $F : U \subset \mathbf{X} \to \mathbf{Y}$ is a smooth tame map if it is differentiable and together with its differential satisfies tame estimates in a neighorhood of each point.

Example

The map $\mathrm{Exp} : \mathrm{Vec}M \to \mathrm{Diff}(M)$ that sends $f \mapsto e^f$ is a tame map.

Hamilton's version of Nash–Moser Theorem

Theorem (Nash–Moser)

Let $\mathbf{X}$ and $\mathbf{Y}$ be tame spaces and

$$F : U \subset \mathbf{X} \to \mathbf{Y}$$

a smooth tame map. If

- $D_a F(\xi) = \eta$ *has a solution for every* $a \in U$ *and for every* η;
- $DF^{-1} : \mathcal{O} \times \mathbf{Y} \to \mathbf{X}$ *is a smooth tame map.*

Then F is locally surjective. Moreover in a neighborhood of any point F has a smooth tame right inverse.

The method is:

- prove that F is tame;
- prove that $D_a F(\xi)$ is tame both in $a \in \mathcal{O}$ and $\xi \in \mathbf{X}$;
- invert DF not only in one point, but in all the neighborhood U;
- prove that $(D_a F)^{-1}$ is tame;

Open Problems

We have that for $u_1(t), \ldots, u_\nu(t)$ piecewise constant

$$F(a) = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{\nu} u_i(t) a_i f_{j_i} \, dt \, ,$$

is locally surjective. Consider the truncated fourier series of $u_i(t)$, say $u_i^k(t)$. Is the map

$$F_k(a) = \overrightarrow{\exp} \int_0^1 \sum_{i=1}^{\nu} u_i^k(t) a_i f_{j_i} \, dt \, ,$$

locally surjective too?

- No fixed point argument applies;
- Nash–Moser method (Newton iteration scheme) is the right tool.

"Of course the problem is hard! But this is SISSA... not a small mediocre university!"

"Certo che il problema è difficile! Ma questa è la SISSA... mica una piccola università mediocra!"

Main Theorem

Let M be a compact connected manifold and $\mathcal{F} \subset \mathrm{Vec}M$.
If $\mathrm{Gr}\mathcal{F}$ acts transitively on M, then there exist

- a neighborhood $\mathcal{O}$ of the identity in $\mathrm{Diff}_0(M)$;
- a positive integer μ

such that every $P \in \mathcal{O}$ can be presented in the form

$$P = e^{a_1 f_1} \circ \cdots \circ e^{a_\mu f_\mu} \,,$$

for some $f_1, \ldots, f_\mu \in \mathcal{F}$ and $a_1, \ldots, a_\mu \in C^\infty(M)$.