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DYNAMICS CONTROL BY A TIME-VARYING FEEDBACK

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Abstract. We consider a smooth bracket-generating control-affine system in $\mathbb{R}^d$ and show that any orientation-preserving diffeomorphism of $\mathbb{R}^d$ can be approximated, in a very strong sense, by a diffeomorphism included in the flow generated by a time-varying feedback control which is polynomial with respect to the state variables and trigonometric-polynomial with respect to the time variable.

1. Introduction

We consider a control-affine system

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q), \quad q \in \mathbb{R}^d, \tag{1}$$

with $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, where f_i are smooth (i.e., C^∞) vector fields on $\mathbb{R}^d$. Moreover, we assume that $\{f_1, \dots, f_m\}$ is a bracket-generating family of vector fields, i.e., $\mathrm{Lie}_q\{f_1, \dots, f_m\} = \mathbb{R}^d$, for any $q \in \mathbb{R}^d$, where $\mathrm{Lie}_q\{f_1, \dots, f_m\}$ is the linear hull of all iterated Lie brackets of the fields $f_1, \dots, f_m$ evaluated at q.

Feedback control (or time-invariant feedback control) is a mapping

$$v = (v_1, \dots, v_m) : \mathbb{R}^d \to \mathbb{R}^m.$$

We can set $u_i = v_i(q)$ and obtain a *closed loop* system

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} v_i(q) f_i(q), \quad q \in \mathbb{R}^d. \tag{2}$$

It is very interesting to know what kind of dynamics we can realize by an appropriate choice of the feedback control. Of course, a smooth or at least Lipschitz feedback is preferable if we want system (2) to correctly define a dynamical system. Unfortunately, we cannot expect too much. In particular, if $f_0 = 0$, then system (2) with a continuous feedback control

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cannot have locally asymptotically stable equilibria as it was observed by R. Brockett [2].

J.-M. Coron suggested to use time-varying periodic with respect to time feedback controls

$$v : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \quad v(t+1, q) = v(t, q), \quad q \in \mathbb{R}^d,\ t \in \mathbb{R},$$

for system (1) and proved that asymptotic stability can be successfully achieved by a C^∞ time-varying feedback (see [4, 5] or [6, Sec. 11.2]).

In this paper, we focus on the transformation $q(0) \mapsto q(1)$ in virtue of the system

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} v_i(t, q) f_i(q), \quad q \in \mathbb{R}^d, \tag{3}$$

associated to the time-varying feedback control and demonstrate that practically any type of discrete-time dynamics can be realized in this way.

More precisely, let $\Phi_v : q(0) \mapsto q(1)$ be the transformation of $\mathbb{R}^d$ which sends the initial value of any solution of system (3) to its value at $t = 1$. We denote by $\mathrm{Diff}_0(\mathbb{R}^d)$ the group of orientation-preserving diffeomorphisms of $\mathbb{R}^d$. Let $P \in \mathrm{Diff}_0(\mathbb{R}^d)$, $\mathcal{O}_P$ be a C^∞-neighborhood of P and N be a positive integer. We prove (see Theorem 8) that there exists a polynomial with respect to q and trigonometric polynomial with respect to t time-varying feedback control v such that $\Phi_v \in \mathcal{O}_P$ and the N-jets of Φ_v and P at the origin coincide. Moreover, construction of the time-varying feedback v is surprisingly simple.

Let us fix notation. We denote by $\mathrm{Diff}(\mathbb{R}^d)$ the group of diffeomorphisms of $\mathbb{R}^d$ and by $\mathrm{Vec}\,\mathbb{R}^d$ the space of vector fields on $\mathbb{R}^d$. We assume that $\mathrm{Diff}(\mathbb{R}^d)$, $\mathrm{Diff}_0(\mathbb{R}^d)$, $\mathrm{Vec}\,\mathbb{R}^d$, and $C^\infty(\mathbb{R}^d)$ are endowed with the standard topology of the uniform convergence of the partial derivatives of any order on any compact of $\mathbb{R}^d$. Given a set $\mathcal{F}$ of vector fields on $\mathbb{R}^d$, we denote by

$$\mathrm{Gr}\mathcal{F} = \{e^{t_1 f_1} \circ \cdots \circ e^{t_k f_k} \mid t_i \in \mathbb{R},\ f_i \in \mathcal{F},\ k \in \mathbb{N}\}$$

the subgroup of $\mathrm{Diff}(\mathbb{R}^d)$ generated by flows of vector fields in $\mathcal{F}$ and by

$$\mathrm{Gr}_S\mathcal{F} = \{e^{a_1 f_1} \circ \cdots \circ e^{a_k f_k} \mid a_i \in C^\infty(\mathbb{R}^d),\ f_i \in \mathcal{F},\ k \in \mathbb{N}\}$$

the subgroup of $\mathrm{Diff}(\mathbb{R}^d)$ generated by flows of vector fields in $\mathcal{F}$ rescaled by smooth functions on $\mathbb{R}^d$. We consider time-varying vector fields $V_t(q)$ on $\mathbb{R}^d$ that are smooth with respect to $q \in \mathbb{R}^d$ and locally integrable with respect to $t \in \mathbb{R}$. All vector fields under consideration are supposed to satisfy the growth condition $V_t(q) \leq \varphi(t)(1 + |q|)$, where φ is a locally integrable function. This condition guarantees completeness of the vector field.

Given a time-varying vector field $V_t(q)$ on $\mathbb{R}^d$, let

$$P_t : \mathbb{R}^d \to \mathbb{R}^d, \quad t \in \mathbb{R},$$

be the (nonstationary) flow generated by the differential equation $\dot{q} = V_t(q)$. In other words,

$$\frac{\partial P_t}{\partial t}(q) = V_t(P_t(q)), \quad P_0(q) \equiv q.$$

In the sequel, we will use the "chronological" notation $P_t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau$ to denote such a flow.

Recall that if $\mathcal{F}$ is a bracket-generating family of vector fields, then by the Rashevski–Chow theorem (see [3,8]), for every q_0, the orbit $\mathcal{O}_{q_0}$ of the family is the whole space $\mathbb{R}^d$; moreover, according to the orbit theorem of Sussmann ([9] or [1, Chap. 5]), any smooth vector field can be presented as a linear combination of vector fields from $\mathcal{F}$ transformed by diffeomorphisms from $\mathrm{Gr}\mathcal{F}$. In particular, it is possible to take $X_1, \ldots, X_d$ linearly independent at a point $q \in \mathbb{R}^d$ and such that $X_i = P^i{}_* f_i$, $i = 1, \ldots, d$, with $P^i \in \mathrm{Gr}\mathcal{F}$ and $f_i \in \mathcal{F}$.

The main result proved in this paper is as follows.

Theorem. *Let $\{f_1, f_2, \ldots, f_m\}$ be a bracket-generating family of vector fields on $\mathbb{R}^d$. Consider the control system*

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^d, \tag{4}$$

with controls u_i such that:

(i) *u_i is polynomial with respect to $q \in \mathbb{R}^d$,*

(ii) *u_i is a trigonometric polynomial with respect to $t \in [0, 1]$*

for every $i = 1, \ldots, m$.

Fix positive integers N and k, $\varepsilon > 0$, and a ball B in $\mathbb{R}^d$. For any $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$, there exist controls $u_1(t,q), \ldots, u_m(t,q)$ such that, if P is the flow at time 1 of the system, then

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

Proof is divided into four parts. In Sec. 2, we consider a bracket-generating family of vector fields closed under multiplication by smooth functions on $\mathbb{R}^d$, say $\mathcal{F}$, and then we prove that the group of diffeomorphisms generated by flows of vector fields in this family is dense in the connected component of the identity of the group of diffeomorphisms. In Sec. 3, we use the classical implicit-function theorem to prove that the Nth jet of a diffeomorphism in $\mathrm{Diff}_0(\mathbb{R}^d)$ sufficiently close to the identity can be represented as the Nth jet of an element in $\mathrm{Gr}\mathcal{F}$. Then, using Proposition 2, we can extend this result to every diffeomorphism in $\mathrm{Diff}_0(\mathbb{R}^d)$. The results of Secs. 2 and 3 are combined together in Sec. 4 to prove that it is possible to find an element in the group $\mathrm{Gr}\mathcal{F}$ with the same Nth jet of a given diffeomorphism and also close to it in the C^∞-topology. This result, as showed in Sec. 5, implies the main result in the driftless case, namely

$f_0 \equiv 0$, and with controls $u_i(t, \cdot)$ that are piecewise constant with respect to t. Therefore, we use the Brouwer fixed-point theorem to prove that it is possible to perturb the map

$$(u_1, \ldots, u_m) \mapsto J_0^N \left(\overrightarrow{\exp} \int_0^1 \sum_{i=1}^m u_i(t, \cdot) f_i(\cdot)\, dt \right),$$

without losing surjectivity. This argument leads to the proof of the theorem.

2. An approximation result

We start with a simple modification of a standard relaxation result (see [1, Chap. 8] or [7]). Its proof is done in the appendix for convenience of the reader.

Proposition 1. *Let $X_1, \ldots, X_k$ be smooth vector fields on $\mathbb{R}^d$ and $\mathcal{A}$ be a closed subspace of $C^\infty(\mathbb{R}^d)$. Then, for any time-varying vector field of the form*

$$V_t = \sum_{i=1}^k a_i(t, \cdot) X_i,$$

where $a_i(t, \cdot) \in \mathcal{A}$ and $0 \le a_i(t, q) \le \varphi(t)$ for some locally integrable φ, $i = 1, \ldots, k$, there exists a sequence of time-varying, piecewise constant with respect to t, vector fields Z_t^n such that

$$Z_t^n \in \{aX_i \mid a \in \mathcal{A}, \imath = 1, \ldots, k\} \quad \text{for any } t \in [0, 1]$$

and

$$\overrightarrow{\exp} \int_0^t Z_\tau^n\, d\tau \longrightarrow \overrightarrow{\exp} \int_0^t V_\tau\, d\tau \quad \text{as } n \to \infty$$

in the standard topology and uniformly with respect to $t \in [0, 1]$.

Proposition 2 (approximation)**.** *Let $\mathcal{F} \subseteq \operatorname{Vec} \mathbb{R}^d$ be a bracket-generating family of vector fields on $\mathbb{R}^d$ such that*

$$af \in \mathcal{F} \ \text{ for any } \ a \in C^\infty(\mathbb{R}^d),\ f \in \mathcal{F}. \tag{5}$$

Then, for any orientation-preserving diffeomorphism P of $\mathbb{R}^d$, there exists a sequence $\{P_n\}_n \subset \mathrm{Gr}\mathcal{F}$ such that

$$P_n \longrightarrow P \quad \text{as } n \to \infty$$

in the standard topology.

Proof. First, note that any orientation-preserving diffeomorphism of $\mathbb{R}^d$ is isotopic to the identity. Indeed, let P be an orientation-preserving diffeomorphisms of $\mathbb{R}^d$. Without loss of generality, we can assume that P fixes the origin just taking the isotopy $H^1(t,\cdot) = P - (1-t)P(0)$. Now, rename for simplicity $P := H^1(0,\cdot)$ and consider another isotopy

$$H^2(t,q) = P(tq)/t, \quad t \in (0,1], \qquad \text{and} \qquad H^2(0,q) = \lim_{t\to 0} P(tq)/t.$$

Since P preserves the orientation, $H^2(0,\cdot)$ belongs to the connected component of the identity of the group of linear invertible operators on $\mathbb{R}^d$, $GL^+(d,\mathbb{R})$.

Let $P^t \subset \mathrm{Diff}_0(\mathbb{R}^d)$ be a path such that $P^0 = \mathrm{Id}$ and $P^1 = P$. Consider the time-varying vector field

$$V_t = \left(P^t\right)^{-1} \circ \frac{d}{dt} P^t.$$

We have

$$\overrightarrow{\exp} \int_0^t V_\tau \, d\tau = P^t.$$

Recall that, since $\mathcal{F}$ is bracket-generating family, it is possible to take $X_1, \dots, X_d$ such that $X_i = P^i_* f_i$ with $P^i \in \mathrm{Gr}\mathcal{F}$, $f_i \in \mathcal{F}$, $i = 1, \dots, d$, and

$$V_t = \sum_{i=1}^d a_i(t,\cdot) X_i,$$

where $a_i(t,\cdot) \in C^\infty(\mathbb{R}^d)$ for any $t \in [0,1]$.

By Proposition 1, there exists a sequence $Z^n_t \in \{\alpha X_i \mid \alpha \in C^\infty(\mathbb{R}^d), i = 1, \dots, d\}$ such that

$$\overrightarrow{\exp} \int_0^t Z^n_\tau \, d\tau \to P^t \quad \text{as } n \to \infty$$

and the convergence is uniform with respect to $t \in [0,1]$.

Let $P_n := \overrightarrow{\exp} \int_0^1 Z^n_t \, dt$; then

$$P_n \to P \quad \text{as } n \to \infty.$$

It remains to prove that $P_n \in \mathrm{Gr}\mathcal{F}$ for every n. Since Z^n_t is piecewise constant in t, so, for any fixed $n \in \mathbb{N}$, there exist disjoint segments $I_1, \dots, I_{h_n}$ covering $[0,1]$ and functions $\alpha_1, \dots, \alpha_{h_n} \in C^\infty(\mathbb{R}^d)$ such that

$$Z^n_t = \alpha_k X_{i_k} \quad \forall t \in I_k, \quad k = 1, \dots, h_n.$$

Hence

$$P_n = \overrightarrow{\exp} \int_0^1 Z^n_t \, dt = e^{|I_1|\alpha_1 X_{i_1}} \circ \dots \circ e^{|I_{h_n}|\alpha_{h_n} X_{i_{h_n}}}$$

$$= e^{|I_1|\alpha_1 P^{i_1}_* f_{i_1}} \circ \dots \circ e^{|I_h|\alpha_h P^{i_{h_n}}_* f_{i_{h_n}}} = (P^{i_1})^{-1} \circ e^{|I_1|(\alpha_1 \circ P^{i_1}) f_{i_1}} \circ P^{i_1} \circ \cdots$$

$$\circ\, (P^{i_{h_n}})^{-1} \circ e^{|I_h|(\alpha_{h_n} \circ P^{i_{h_n}}) f_{i_{h_n}}} \circ P^{i_{h_n}}. \quad (6)$$

Now let $\beta_k = |I_k|(\alpha_k \circ P^{i_k})$; then

$$P_n = (P^{i_1})^{-1} \circ e^{\beta_1 f_{i_1}} \circ P^{i_1} \circ \cdots \circ (P^{i_{h_n}})^{-1} \circ e^{\beta_{h_n} f_{i_{h_n}}} \circ P^{i_{h_n}},$$

and $P_n \in \mathrm{Gr}\mathcal{F}$ by assumption (5). □

In other words, we have proved that if $\mathcal{F}$ is a bracket-generating family of vector fields, then $\mathrm{Gr}_S\mathcal{F}$ is dense in the connected component of the identity of $\mathrm{Diff}(\mathbb{R}^d)$ endowed with the standard C^∞-topology.

3. Get the jet

In this section, given a bracket-generating family of vector fields $\mathcal{F}$, we find a diffeomorphism in the group $\mathrm{Gr}_S\mathcal{F}$ whose Nth jet is exactly the Nth jet of a given diffeomorphism on $\mathbb{R}^d$. The main tool used is the classical implicit-function theorem.

Proposition 3. *Let $\mathcal{F}$ be a bracket-generating family of vector fields on $\mathbb{R}^d$ and $N > 0$ a positive integer. For any diffeomorphism $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ sufficiently close to the identity, there exists $P \in \mathrm{Gr}_S\mathcal{F}$ such that*

$$J_0^N(P) = J_0^N(\Phi).$$

Proof. Consider a frame $X_1, \dots, X_d$ of linearly independent in $0 \in \mathbb{R}^d$ vector fields. Let $\mathbf{X}$ be the space of polynomials of degree less or equal than N in d variables and let $\mathbf{Y}$ be the jet-group of N-order jets at 0 of smooth orientation-preserving diffeomorphisms, i.e., $\mathbf{Y} = J_0^N(\mathrm{Diff}_0(\mathbb{R}^d))$. Note that $\dim \mathbf{X} < \infty$ and $\dim \mathbf{Y} < \infty$.

Consider the map

$$F : \mathbf{X}^d \to \mathbf{Y}, \quad (u_1, \dots, u_d) \mapsto J_0^N(e^{u_1 X_1} \circ \cdots \circ e^{u_d X_d}). \quad (7)$$

We want to prove that the implicit-function theorem can be applied. Let us calculate the differential of F at $0 \in \mathbf{X}^d$

$$D_0F(a_1, \dots, a_d) = \frac{\partial F}{\partial u_1}\Big|_{u_1 = \ldots = u_d = 0} a_1 + \cdots + \frac{\partial F}{\partial u_d}\Big|_{u_1 = \ldots = u_d = 0} a_d$$
$$= a_1 J_0^N(X_1) + \cdots + a_d J_0^N(X_d).$$

We claim that $D_0F : \mathbf{X}^d \to T_{\mathrm{Id}}\mathbf{Y}$ is invertible. Indeed,

$$T_{\mathrm{Id}}\mathbf{Y} = T_{\mathrm{Id}} J_0^N(\mathrm{Diff}_0(\mathbb{R}^d)) = J_0^N(T_{\mathrm{Id}}\,\mathrm{Diff}_0(\mathbb{R}^d)) = J_0^N(\mathrm{Vec}(\mathbb{R}^d)),$$

so for every $V \in J_0^N(\mathrm{Vec}(\mathbb{R}^d))$, there exist $b_1, \dots, b_d$ such that

$$V = J_0^N(b_1 X_1 + \cdots + b_d X_d) = J_0^N(b_1) J_0^N(X_1) + \ldots + J_0^N(b_d) J_0^N(X_d).$$

Every element $V \in T_{\mathrm{Id}}\mathbf{Y}$ is the image of d polynomials of degree less or equal than N, $a_i = J_0^N(b_i)$. Therefore, there exists a neighborhood $\mathcal{O}$ of Id in $\mathbf{Y}$ such that F is locally surjective on $\mathcal{O}$. Namely, for every $\psi \in \mathcal{O}$, there exist $(u_1, \dots, u_d) \in \mathbf{X}^d$ such that $F(u_1, \dots, u_d) = \psi$.

If Φ is sufficiently close to the identity, then $J_0^N(\Phi) \in \mathcal{O}$. Therefore, there exist polynomials $v_1, \dots, v_d \in \mathbf{X}^d$ such that

$$J_0^N(e^{v_1 X_1} \circ \cdots \circ e^{v_d X_d}) = J_0^N(\Phi).$$

It remains to prove that $P = e^{v_1 X_1} \circ \cdots \circ e^{v_d X_d} \in \mathrm{Gr}_S \mathcal{F}$, but according to the orbit theorem, for $i = 1, \dots, d$, we have that $X_i = P^i_* f_i$, where $f_i \in \mathcal{F}$ and $P^i \in \mathrm{Gr}\mathcal{F}$. Let

$$P^i = e^{t^i_1 f^i_1} \circ e^{t^i_2 f^i_2} \circ \cdots \circ e^{t^i_{s_i} f^i_{s_i}}$$

with $f^i_j \in \mathcal{F}$. Therefore,

$$\begin{aligned}
P &= e^{v_1 P^1_* f_1} \circ \cdots \circ e^{v_d P^d_* f_d} \\
&= P^1 \circ e^{(P^1)^{-1}(v_1) f_1} \circ \left(P^1\right)^{-1} \circ \cdots \circ P^d \circ e^{\left(P^d\right)^{-1}(v_d) f_d} \circ \left(P^d\right)^{-1} \\
&= \underbrace{e^{t^1_1 f^1_1} \circ \cdots \circ e^{t^1_{s_1} f^1_{s_1}}}_{P^1} \circ e^{(P^1)^{-1}(v_1) f_1} \circ \underbrace{e^{-t^1_{s_1} f^1_{s_1}} \circ \cdots \circ e^{-t^1_1 f^1_1}}_{(P^1)^{-1}} \circ \cdots \\
&\quad \circ \underbrace{e^{t^d_1 f^d_1} \circ \cdots \circ e^{t^d_{s_d} f^d_{s_d}}}_{P^d} \circ e^{\left(P^d\right)^{-1}(v_d) f_d} \circ \underbrace{e^{-t^d_{s_d} f^d_{s_d}} \circ \cdots \circ e^{-t^d_1 f^d_1}}_{(P^d)^{-1}} \\
&= e^{w_1 g_1} \circ \cdots \circ e^{w_\ell g_\ell} \qquad (8)
\end{aligned}$$

with $g_1, \dots, g_\ell \in \mathcal{F}$ and $\ell = d + 2(s_1 + \cdots + s_d)$. Therefore, $P \in \mathrm{Gr}_S \mathcal{F}$ and the proposition follows. □

Now we consider an arbitrary diffeomorphism $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$. By Proposition 2, there exists a sequence $\{P_n\}_n \subset \mathrm{Gr}_S \mathcal{F}$ that tends to Φ. Thus, for sufficiently large n, the last proposition can be applied to $P_n^{-1} \circ \Phi$ and we have the following result.

Corollary 4. *Let $\mathcal{F} \subseteq \mathrm{Vec}\,\mathbb{R}^d$ be a bracket-generating family of vector fields and $N > 0$ a positive integer. For every $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$, there exists $P \in \mathrm{Gr}_S \mathcal{F}$ such that*

$$J_0^N(P) = J_0^N(\Phi).$$

4. Geometric statement of the main result

The purpose of this section is to link the results of the last two sections in order to find an element in the group $\mathrm{Gr}_S \mathcal{F}$ with the same Nth jet of a given diffeomorphism and also close to it in the C^∞-topology.

Proposition 5. *Let $\mathcal{F} \subseteq \mathrm{Vec}\mathbb{R}^d$ be a bracket-generating family of vector fields. Let N and k be positive integers, $\varepsilon > 0$, and B be a ball in $\mathbb{R}^d$. For any $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$, there exists $P \in \mathrm{Gr}_S \mathcal{F}$ such that*

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

Proof. We can assume that $J_0^N(\Phi) = \mathrm{Id}$. Indeed, by Corollary 4, there exists $Q \in \mathrm{Gr}_S\mathcal{F}$ such that $J_0^N(Q) = J_0^N(\Phi)$. Then we consider, instead of Φ, the diffeomorphism $\Psi = \Phi \circ Q^{-1}$ which has trivial jet.

The idea of the proof is the same as in Proposition 2. Since $J_0^N(\Phi) = \mathrm{Id}$, Φ can be written as

$$\Phi(x) = x + g(x),$$

where $J_0^N(g) = 0$. Consider the one-parameter family of diffeomorphisms with trivial jet

$$\Phi_t(x) = x + tg(x).$$

This is a path in $\mathrm{Diff}(\mathbb{R}^d)$ from $\Phi_0 = \mathrm{Id}$ to $\Phi_1 = \Phi$. Let V_t be a nonautonomous vector field such that

$$\Phi_t = \overrightarrow{\exp} \int_0^t V_\tau \, d\tau.$$

Let $X_1, \dots, X_d$ be a frame of vector fields linearly independent at 0 such that $X_i = \mathrm{Ad}\, P^i f_i$, $P^i \in \mathrm{Gr}\mathcal{F}$, and $f_i \in \mathcal{F}$. Therefore,

$$V_t = \sum_{i=1}^d a_i(t, \cdot) X_i,$$

where $a_i(t, \cdot) \in C^\infty(\mathbb{R}^d)$ for any $t \in [0, 1]$. Note that, since $J_0^N(\Phi_t) = \mathrm{Id}$ and the vector fields X_i are linearly independent, $J_0^N(a_i(t, \cdot)) = 0$ for any $t \in [0, 1]$.

Now let $\mathcal{A}$ be the closed subspace of $C^\infty(\mathbb{R}^d)$ of smooth functions α such that $J_0^N(\alpha) = 0$. By Proposition 1, there exists a sequence $Z_t^n \in \{\alpha X_i \mid \alpha \in \mathcal{A},\ i = 1 \dots, d\}$ that is piecewise constant in t and

$$\overrightarrow{\exp} \int_0^t Z_\tau^n \, d\tau \to \Phi_t \quad \text{as } n \to \infty$$

in the C^∞-topology and uniformly with respect to $t \in [0, 1]$.

Thus, if $P_n = \overrightarrow{\exp} \int_0^1 Z_\tau^n \, d\tau$, then

$$P_n \to \Phi \quad \text{as } n \to \infty$$

in the standard topology. Now, for any n, we have that $P_n \in \mathrm{Gr}_S\mathcal{F}$ for the chain of Eqs. (6). Moreover P_n has trivial jet. Indeed, since the sequence Z_t^n is piecewise constant, there exist intervals $I_1, \dots, I_h$ such that

$$Z_t^n = \alpha_i X_{j_i} \quad \text{for any } t \in I_i,$$

with $j_i \in \{1, \dots, d\}$. Hence

$$J_0^N(P_n) = J_0^N\left(\overrightarrow{\exp} \int_0^1 Z_t^n \, dt\right) = J_0^N\left(e^{|I_1|\alpha_1 X_{j_1}}\right) \circ \dots \circ J_0^N\left(e^{|I_h|\alpha_h X_{j_h}}\right)$$

$$= e^{|I_1| J_0^N(\alpha_1) J_0^N(X_{j_1})} \circ \cdots \circ e^{|I_h| J_0^N(\alpha_h) J_0^N(X_{j_h})} = \mathrm{Id},$$

and the result is proved. □

5. Main result

In this last section, we prove the main result using Proposition 5 and a fixed-point argument. We start giving an equivalent formulation of Proposition 5 in terms of flows of the system:

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^d. \tag{9}$$

Assume that $\mathcal{F} = \{f_1, \ldots, f_m\}$ is a bracket-generating family of vector field on $\mathbb{R}^d$. By Proposition 3, there exist smooth functions $a_1, \ldots, a_k$ such that

$$J_0^N(\Phi) = J_0^N\left(e^{a_1 f_{i_1}} \circ \cdots \circ e^{a_k f_{i_k}}\right), \tag{10}$$

where $i_j \in \{1, \ldots, m\}$. Now there exist m functions $u_1(t,q), \ldots, u_m(t,q)$ piecewise constant in t such that

$$J_0^N(\Phi) = J_0^N\left(\overrightarrow{\exp} \int_0^1 \sum_{i=1}^{m} u_i(t,\cdot) f_i \, dt\right). \tag{11}$$

Lemma 6. *Let $\{f_1, f_2, \ldots, f_m\}$ be a bracket-generating family of vector fields on $\mathbb{R}^d$. Consider the control system*

$$\dot{q} = \sum_{i=1}^{m} u_i(t,q) f_i(q), \quad q \in \mathbb{R}^d, \tag{12}$$

where the controls u_i are piecewise constant with respect to $t \in [0,1]$ and smooth with respect to $q \in \mathbb{R}^d$ for every $i = 1, \ldots, m$. Let N and k be positive integers, $\varepsilon > 0$, and B be a ball in $\mathbb{R}^d$. For any $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$, there exist controls $u_1(t,q), \ldots, u_m(t,q)$ such that, if P is the flow at time 1 of system (12)*, then*

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

It remains to prove the last result adding a drift f_0 to system (12). Moreover, we want to have a certain regularity for the controls. Both these results can be proved with a fixed point argument. Indeed, let $\mathbf{U}$ the space of m-tuples of controls $u(t,q)$ piecewise constant in t and smooth with respect to q. Consider the map

$$\begin{aligned} \tilde{F} : \mathbf{U} &\longrightarrow J_0^N(\mathrm{Diff}_0(\mathbb{R}^d)), \\ (u_1, \ldots, u_m) &\longmapsto J_0^N\left(\overrightarrow{\exp} \int_0^1 \sum_{i=1}^{m} u_i(t,\cdot) X_i \, dt\right). \end{aligned} \tag{13}$$

This map is continuous and, by the last lemma, is also surjective. Moreover, $\tilde{F}$ has a continuous right inverse. Indeed, there is a smooth correspondence between the time-varying feedback controls $u_1, \dots, u_m$ and the functions $a_1, \dots, a_k$ in (10). By the implicit-function theorem applied to the map F in (7), we have that the right inverse of F is continuous and so is the right inverse of $\tilde{F}$.

In the next lemma we prove, using a fixed-point argument, that every small perturbation of a continuous surjective map with continuous right inverse and with finite-dimensional target space is also surjective.

Lemma 7. *Let X be a topological space, $\varepsilon > 0$, and $F : X \to \mathbb{R}^n$ be a continuous and surjective with continuous right inverse. If $G : X \to \mathbb{R}^n$ is continuous and is such that $\sup_{x\in K} |F(x) - G(x)| < \varepsilon$ for any $K \subseteq X$ compact, then G is surjective.*

Proof. Let F^{-1} be the right inverse of F. We define, for every $\bar{y}$ in $\mathbb{R}^n$, the map $\chi_{\bar{y}}(y) = y - G \circ F^{-1}(y) + \bar{y}$. Let $\delta = \varepsilon + \|\bar{y}\|$; then for every $y \in B_\delta = B_\delta(0)$ we have

$$\begin{aligned}\|\chi_{\bar{y}}(y)\| \le \|y - G \circ F^{-1}(y)\| + \|\bar{y}\| &\le \sup_{y\in B_\delta} \|y - G \circ F^{-1}(y)\| + \|\bar{y}\| \\ &\le \sup_{x\in F^{-1}(B_\delta)} \|F(x) - G(x)\| + \|\bar{y}\| < \varepsilon + \|\bar{y}\| = \delta.\end{aligned}$$

Thus, $\chi_{\bar{y}}(B_\delta) \subseteq B_\delta$ and, since the map $\chi_{\bar{y}}$ is continuous, by the Brouwer fixed-point theorem, there exists $\tilde{y} \in B_\delta$ such that

$$\chi_{\bar{y}}(\tilde{y}) = \tilde{y},$$

namely,

$$G \circ F^{-1}(\tilde{y}) = \bar{y}.$$

We have proved that, for every $y \in \mathbb{R}^n$, there exists $x \in X$ such that $y = G(x)$. □

Now we can prove the main result.

Theorem 8. *Let $\{f_1, f_2, \dots, f_m\}$ be a bracket-generating family of vector fields on $\mathbb{R}^d$. Consider the control system*

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i(t, q) f_i(q), \quad q \in \mathbb{R}^d, \tag{14}$$

with controls u_i such that:

(i) *u_i is a polynomial with respect to $q \in \mathbb{R}^d$,*
(ii) *u_i is a trigonometric polynomial with respect to $t \in [0, 1]$*

for every $i = 1, \ldots, m$. Fix positive integers N and k, $\varepsilon > 0$, and B ball of $\mathbb{R}^d$. For any $\Phi \in \mathrm{Diff}_0(\mathbb{R}^d)$, there exist controls $u_1(t,q), \ldots, u_m(t,q)$ such that, if P is the flow at time 1 of system (14), then

$$J_0^N(P) = J_0^N(\Phi) \quad \text{and} \quad \|P - \Phi\|_{C^k(B)} < \varepsilon.$$

Proof. The proof splits into three steps. First, we prove that it suffices to consider controls that are polynomials with respect to $q \in \mathbb{R}^d$, then we add the drift to the system, and finally we find controls that are trigonometric polynomials with respect to t by smoothing the time dependence of the piecewise constant controls.

Let us start with the first step and note that, as a consequence of the density of polynomials in the space of smooth functions on a bounded set and by Lemma 7, we can assume that $u_i(t,q)$ is a polynomial in q for every $t \in [0,1]$ and for every $i = 1, \ldots, m$.

Now set $\mathbf{Y} = J_0^N(\mathrm{Diff}_0(\mathbb{R}^d))$ and consider the family of continuous maps

$$F_\varrho : \mathbf{U} \longrightarrow \mathbf{Y},$$

$$(u_1, \ldots, u_m) \longmapsto J_0^N \left(\overrightarrow{\exp} \int_0^\varrho \varrho f_0 + \sum_{i=1}^m u_i(t,\cdot) X_i \, dt \right).$$

We claim that, if there exists $\varrho > 0$ such that F_ϱ is surjective, then so is F_ϱ for $\varrho = 1$. Indeed,

$$F_\varrho(u_1(t,\cdot), \ldots, u_m(t,\cdot)) = F_1 \left(\frac{u_1(t/\varrho,\cdot)}{\varrho}, \ldots, \frac{u_m(t/\varrho,\cdot)}{\varrho} \right).$$

Similarly, the map

$$\tilde{F}_\varrho(u_1, \ldots, u_m) = J_0^N \left(\overrightarrow{\exp} \int_0^\varrho \sum_{i=1}^m u_i(t,\cdot) X_i \, dt \right)$$

is surjective for every $\varrho > 0$ since it is equal to the map $\tilde{F}$ defined in (13) up to rescalings of the time dependence of the controls u_i. For small $\varrho > 0$, we see that F_ϱ is a small perturbation of $\tilde{F}_\varrho$. Thus, Lemma 7 can be applied and F_1 is surjective.

Finally, for any control $u(t,q)$ that is piecewise constant in t and polynomial in q, we can write

$$u(t,q) = \sum_{|\alpha|=0}^{N} a_\alpha(t) q^\alpha,$$

where α is a multi-index and $a_\alpha(t)$ is piecewise constant. For every α, the function a_α admits a Fourier expansion of the form

$$a_\alpha(t) = \sum_{j=0}^{\infty} \eta_\alpha^j \cos(2\pi j t) + \xi_\alpha^j \sin(2\pi j t).$$

Consider the trigonometric polynomial

$$a_\alpha^n(t) = \sum_{j=0}^{n} \eta_\alpha^j \cos(2\pi j t) + \xi_\alpha^j \sin(2\pi j t);$$

then $a_\alpha^n(t) \to a_\alpha(t)$ as $n \to \infty$ in $L^1[0,1]$. So let

$$u^n(t,q) = \sum_{|\alpha|=0}^{N} a_\alpha^n(t) q^\alpha;$$

then

$$u^n(t,q) \to u(t,q) \quad \text{as } n \to \infty, \tag{15}$$

and the convergence is uniform with all derivatives on compact sets of $\mathbb{R}^d$ and in $L^1[0,1]$ with respect to t.

Let G_n be the family of continuous maps

$$G_n : \mathbf{U} \longrightarrow \mathbf{Y},$$

$$(u_1, \ldots, u_m) \longmapsto J_0^N \left(\overrightarrow{\exp} \int_0^1 f_0 + \sum_{i=1}^{m} u_i^n(t, \cdot) X_i \, dt \right).$$

By the convergence in (15), $G_n \to F_1$ as $n \to \infty$ for every $(u_1, \ldots, u_m) \in \mathbf{U}$; then there exists n_0 integer for which Lemma 7 applies. Therefore, the map G_{n_0} is surjective and the theorem is proved. □

Remark 1. Clearly, the statement of Theorem 8 also holds if we consider the jet at a point $q \in \mathbb{R}^d$. Moreover, it is possible to fix a finite number of points in $\mathbb{R}^d$, say $q_1, \ldots, q_\ell$, and find an admissible diffeomorphism arbitrarily close to a given one that realize its Nth jet at all the points $q_1, \ldots, q_\ell$ at the same time.

6. Appendix

Here we prove Proposition 1. The proof is based on the following well-known fact (see, e.g., [1, Lemma 8.2]).

Lemma 9. *Let Z_t and Z_t^n, where $t \in [0,1]$ and $n = 1, 2, \ldots$, be nonautonomous vector fields on M. If*

$$\int_0^t Z_\tau^n \, d\tau \to \int_0^t Z_\tau \, d\tau \quad \text{as } n \to \infty$$

in the standard C^∞-topology and uniformly with respect to $t \in [0,1]$, then

$$\overrightarrow{\exp} \int_0^t Z_\tau^n \, d\tau \to \overrightarrow{\exp} \int_0^t Z_\tau \, d\tau \quad \text{as } n \to \infty$$

in the same topology.

Proof of Proposition 1. First, note that we can assume, without loss of generality, that $a_i(t,\cdot)$ is piecewise constant in t for every $i = 1,\dots,k$. Indeed, for any $i = 1,\dots,k$, the sequence

$$a_i^n(t,q) = n\sum_{j=1}^{n}\int_{\frac{j-1}{n}}^{\frac{j}{n}} a_i(\tau,q)\,d\tau\,\chi_j^n(t), \tag{16}$$

where $\chi_j^n(t)$ is the characteristic function of the interval $\left[\dfrac{j-1}{n},\dfrac{j}{n}\right]$, is such that

$$\int_0^t \sum_{i=1}^{k} a_i^n(\tau,\cdot)X_i\,d\tau \to \int_0^t V_\tau\,d\tau \quad \text{as } n\to\infty$$

uniformly with respect to t and in the C^∞-topology. Therefore, Lemma 9 allows us to suppose that $a_i(t,\cdot)$ is piecewise constant in t for every i.

Let ℓ be a positive integer such that V_t is constant on $\left[\dfrac{j-1}{\ell},\dfrac{j}{\ell}\right]$ for every $j = 1,\dots,\ell$. We can write

$$a_i(t,q) = \sum_{j=1}^{\ell} a_i^j(q)\chi_j^\ell(t), \tag{17}$$

where $a_i^j(q) \geq 0$ for every $q \in \mathbb{R}^d$. Let

$$\alpha^j = \sum_{i=1}^{k} a_i^j \tag{18}$$

and let $\{\varepsilon_n\}$ be a sequence of nonnegative smooth functions of $\mathbb{R}^d$ such that $\varepsilon_n(0) = 0$ for every n and $\varepsilon_n \to 0$ as $n\to\infty$ in the C^∞-topology. Then $\alpha_n^j = \alpha^j + \varepsilon_n$ is strictly positive on $\mathbb{R}^d \setminus \{0\}$ for every j and n.

Now, for every positive integer n and $j = 1,\dots,\ell$, let $b_n^{j,i} = a_i^j/\alpha_n^j$. Consider the following family of intervals:

$$A_n^{j,i} = \bigcup_{m=0}^{n-1}\left[\frac{j-1}{\ell}+\frac{m}{n\ell}+\frac{b_n^{j,1}+\cdots+b_n^{j,i-1}}{n\ell},\right.$$
$$\left.\frac{j-1}{\ell}+\frac{m}{n\ell}+\frac{b_n^{j,1}+\cdots+b_n^{j,i}}{n\ell}\right)$$

for $i = 2,\dots,k$, and

$$A_n^{j,1} = \bigcup_{m=0}^{n-1}\left[\frac{j-1}{\ell}+\frac{m}{n\ell},\ \frac{j-1}{\ell}+\frac{m}{n\ell}+\frac{b_n^{j,1}}{n\ell}\right).$$

The sequence of vector fields

$$Z_t^n = \alpha_n^j X_i, \quad \text{if } t \in A_n^{j,i}, \tag{19}$$

is such that

$$\int_0^t Z_\tau^n \, d\tau \to \int_0^t V_\tau \, d\tau \quad \text{as } n \to \infty$$

in the standard topology and uniformly with respect to $t \in [0, 1]$. The statement then follows from Lemma 9. □

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