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Notation

• \( \mathbb{R} = (-\infty, \infty) \) stands for the real line, \( \mathbb{R}_+ \) for \([0, \infty)\), and \( \mathbb{R}_+^* \) for \((0, \infty)\).

• \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \ldots\} \) and \( \mathbb{Z}^* = \mathbb{Z} \setminus \{0\} = \{\pm1, \pm2, \ldots\} \).

• For any \( \tau \in \mathbb{R} \) we denote by \( \lceil \tau \rceil \) and \( \{\tau\} \) the integer and the fractional part of \( \tau \), respectively, defined as
  \[
  \lceil \tau \rceil = \max\{m \in \mathbb{Z} : m \leq \tau\} \quad \{\tau\} = \tau - \lceil \tau \rceil.
  \]

• For any \( \lambda \in \mathbb{C} \), \( \Re \lambda \) and \( \Im \lambda \) denote the real and imaginary parts of \( \lambda \), respectively.

• \( |\cdot| \) stands for the norm of a Banach space \( X \), as well as for the absolute value of a real number or the modulus of a complex number.

• Generic elements of \( X \) will be denoted by \( u, v, w \ldots \)

• \( L(X) \) is the Banach space of all bounded linear operators \( \Lambda : X \to X \) equipped with the uniform norm \( \|\Lambda\| = \sup_{|u| \leq 1} |\Lambda u| \).

• For any metric space \((X, d)\), \( C_b(X) \) denotes the Banach space of all bounded uniformly continuous functions \( f : X \to \mathbb{R} \) with norm \( \|f\|_{\infty, X} = \sup_{u \in X} |f(u)| \).

For any \( f \in C_b(X) \) and \( \delta > 0 \) we call

\[
\text{osc} f(\delta) = \sup \{|f(x) - f(y)| : x, y \in X, d(x, y) \leq \delta\}
\]

the oscillation of \( f \) over sets of diameter \( \delta \).

• Given a Banach space \((X, |\cdot|)\) and a closed interval \( I \subseteq \mathbb{R} \) (bounded or unbounded), we denote by \( C_b(I; X) \) the Banach space of all bounded uniformly continuous functions \( f : I \to X \) with norm

\[
\|f\|_{\infty, I} = \sup_{s \in I} |f(s)|.
\]

We denote by \( C_b^1(I; X) \) the subspace of \( C_b(I; X) \) consisting of all functions \( f \) such that the derivative

\[
f'(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t}
\]

exists for all \( t \in I \) and belongs to \( C_b(I; X) \).

• \( D(A) \) denotes the domain of a linear operator \( A : D(A) \subset X \to X \).

• \( \Pi_\omega = \{\lambda \in \mathbb{C} : \Re \lambda > \omega\} \) for any \( \omega \in \mathbb{R} \).
1 Semigroups of bounded linear operators

Preliminaries

Let \((X, |·|)\) be a (real or complex) Banach space. We denote by \(\mathcal{L}(X)\) the Banach space of all bounded linear operators \(\Lambda : X \rightarrow X\) with norm

\[\|\Lambda\| = \sup_{|u| \leq 1} |\Lambda u| .\]

We recall that, for any given \(A, B \in \mathcal{L}(X)\), the product \(AB\) remains in \(\mathcal{L}(X)\) and we have that

\[\|AB\| \leq \|A\| \|B\| .\]  \hspace{1cm} (1.0.1)

So, \(\mathcal{L}(X)\) ia a Banach algebra.

**Proposition 1** Let \(A \in \mathcal{L}(X)\) be such that \(\|A\| < 1\). Then \((I - A)^{-1} \in \mathcal{L}(X)\) and

\[(I - A)^{-1} = \sum_{n=0}^{\infty} A^n .\]  \hspace{1cm} (1.0.2)

**Proof.** We observe that the series on the right-hand side of (1.0.2) is totally convergent in \(\mathcal{L}(X)\). So,

\[\Lambda := \sum_{n=0}^{\infty} A^n \in \mathcal{L}(X) .\]

Moreover,

\[(I - A)\Lambda = \sum_{n=0}^{\infty} (I - A)A^n = I = \sum_{n=0}^{\infty} A^n (I - A) = \Lambda (I - A) .\]  \hspace{1cm} \(\Box\)

1.1 Uniformly continuous semigroups

**Definition 1** A semigroup of bounded linear operators on \(X\) is a map

\[S : [0, \infty) \rightarrow \mathcal{L}(X)\]

with the following properties:

(a) \(S(0) = I\),

(b) \(S(t + s) = S(t)S(s)\) for all \(t, s \geq 0\).

We will use the equivalent notation \(\{S(t)\}_{t \geq 0}\) and the abbreviated form \(S(t)\).
Definition 2 The infinitesimal generator of a semigroup of bounded linear operators \( S(t) \) is the map \( A : D(A) \subset X \to X \) defined by

\[
\begin{align*}
D(A) &= \{ u \in X : \exists \lim_{t \downarrow 0} \frac{S(t)u - u}{t} \}, \\
Au &= \lim_{t \downarrow 0} \frac{S(t)u - u}{t} \quad \forall u \in D(A)
\end{align*}
\] (1.1.1)

Exercise 1 Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of a semigroup of bounded linear operators \( S(t) \). Prove that

(a) \( D(A) \) is a subspace of \( X \),

(b) \( A \) is a linear operator.

Definition 3 A semigroup \( S(t) \) of bounded linear operators on \( X \) is uniformly continuous if

\[
\lim_{t \downarrow 0} \|S(t) - I\| = 0.
\]

Proposition 2 Let \( S(t) \) be a uniformly continuous semigroup of bounded linear operators. Then there exists \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that

\[
\|S(t)\| \leq Me^{\omega t} \quad \forall t \geq 0.
\]

Proof. Let \( \tau \geq 0 \) be such that \( \|S(t) - I\| \leq 1/2 \) for all \( t \in [0, \tau] \). Then

\[
\|S(t)\| \leq \|I\| + \|S(t) - I\| \leq \frac{3}{2} \quad \forall t \in [0, \tau].
\]

Since every \( t \geq 0 \) can be represented as \( t = \lfloor t/\tau \rfloor \tau + \{t/\tau\} \tau \), we have that

\[
\|S(t)\| \leq \|S(\lfloor t/\tau \rfloor \tau)\|^{\lfloor t/\tau \rfloor} \|S(\{t/\tau\} \tau)\| \leq \left( \frac{3}{2} \right)^{\lfloor t/\tau \rfloor + 1} \leq \left( \frac{3}{2} \right)^{\frac{t}{\tau} + 1} = Me^{\omega t}
\]

with \( M = 3/2 \) and \( \omega = \log(3/2)/\tau \).

Corollary 1 A semigroup \( S(t) \) is uniformly continuous if and only if

\[
\lim_{s \to t} \|S(s) - S(t)\| = 0 \quad \forall t \geq 0.
\]

Example 1 let \( A \in \mathcal{L}(X) \). Then

\[
e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n
\]

is a uniformly continuous semigroup of bounded linear operators on \( X \). More precisely, the following properties hold.
\( (a) \) \( \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \) converges for all \( t \geq 0 \) and \( e^{tA} \in \mathcal{L}(X) \).

**Proof.** Indeed, the series is totally convergent in \( \mathcal{L}(X) \) because

\[
\sum_{n=0}^{\infty} \left\| \frac{t^n}{n!} A^n \right\| \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \|A\|^n < \infty.
\]

□

\( (b) \) \( e^{(t+s)A} = e^{tA} e^{sA} \) for all \( s, t \geq 0 \).

**Proof.** We have that

\[
e^{(t+s)A} = \sum_{n=0}^{\infty} \frac{(t+s)^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \binom{n}{k} t^k s^{(n-k)}
\]

where the last term coincides with the Cauchy product of the two series giving \( e^{tA} \) and \( e^{sA} \).

□

\( (c) \) \( Ae^{tA} = e^{tA} A \) for all \( t \geq 0 \).

\( (d) \) \( \|e^{tA} - I\| = \| \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n \| \leq t \|A\| e^{t \|A\|} \) for all \( t \geq 0 \).

\( (e) \) \( \left\| \frac{e^{tA} - I}{t} - A \right\| = \| \sum_{n=2}^{\infty} \frac{t^{n-1}}{n!} A^n \| \leq t \|A\|^2 e^{t \|A\|} \) for all \( t \geq 0 \).

Notice that property \((e)\) shows that \( A \) is the infinitesimal generator of \( e^{tA} \).

**Theorem 1** For any linear operator \( A : D(A) \subset X \rightarrow X \) the following properties are equivalent:

\( (a) \) \( A \) is the infinitesimal generator of a uniformly continuous semigroup,

\( (b) \) \( D(A) = X \) and \( A \in \mathcal{L}(X) \).

**Proof.** Example 1 shows that \((b) \Rightarrow (a)\). Let us prove that \((a) \Rightarrow (b)\). Let \( \tau > 0 \) be fixed such that

\[
\left\| I - \frac{1}{\tau} \int_{0}^{\tau} S(t)dt \right\| < 1.
\]

Then the bounded linear operator \( \int_{0}^{\tau} S(t)dt \) is invertible. For all \( h > 0 \) we have that

\[
\frac{S(h) - I}{h} \int_{0}^{\tau} S(t)dt = \frac{1}{h} \left( \int_{0}^{\tau} S(t+h)dt - \int_{0}^{\tau} S(t)dt \right)
\]

\[= \frac{1}{h} \left( \int_{h}^{\tau+h} S(t)dt - \int_{0}^{\tau} S(t)dt \right) = \frac{1}{h} \left( \int_{\tau}^{\tau+h} S(t)dt - \int_{0}^{h} S(t)dt \right).
\]
Hence
\[
\frac{S(h)-I}{h} = \frac{1}{h} \left( \int_{\tau}^{\tau+h} S(t)dt - \int_{0}^{h} S(t)dt \right) \left( \int_{0}^{\tau} S(t)dt \right)^{-1} \downarrow h \downarrow 0
\]
\[
A = (S(\tau) - I) \left( \int_{0}^{\tau} S(t)dt \right)^{-1}.
\]
This shows that \( A \in \mathcal{L}(X) \). \( \square \)

Let \( A \in \mathcal{L}(X) \). For any \( u_0 \in X \), a solution of the Cauchy problem
\[
\begin{cases}
  u'(t) = Au(t) & t > 0 \\
  u(0) = u_0
\end{cases}
\]  
(1.1.2)
is a function \( u \in C^1([0, \infty[: X) \) which satisfies (1.1.2) pointwise.

**Proposition 3** Problem (1.1.2) has a unique solution given by \( u(t) = e^{tA}u_0 \).

**Proof.** The fact that \( u(t) = e^{tA}u_0 \) solves (1.1.2) follows from Example 1. Let \( v \in C^1([0, \infty[: X) \) be another solution of (1.1.2). Fix any \( t > 0 \) and set \( U(s) = e^{(t-s)A}v(s) \) for all \( s \in [0, t] \). Then
\[
U'(s) = -Ae^{(t-s)A}v(s) + e^{(t-s)A}Av(s) = 0 \quad \forall s \in [0, t].
\]
Therefore, \( U \) is constant on \([0, t]\) by Corollary 6 of Appendix A. So, \( v(t) = U(t) = U(0) = e^{tA}u_0 \). \( \square \)

**Example 2** Consider the integral equation
\[
\begin{cases}
  \frac{\partial u}{\partial t}(t, x) = \int_{0}^{1} k(x, y)u(t, y)dy & t > 0 \\
  u(0, x) = u_0(x)
\end{cases}
\]  
(1.1.3)
where \( k \in L^2([0, 1] \times [0, 1]) \) and \( u_0 \in L^2(0, 1) \). Problem (1.1.3) can be recast in the abstract form (1.1.2) taking \( X = L^2(0, 1) \) and
\[
Au(x) = \int_{0}^{1} k(x, y)u(t, y)dy \quad \forall x \in X.
\]
Then Proposition 3 insures that (1.1.3) has a unique solution \( u \in C^1([0, \infty[: X) \) given by \( u(t) = e^{tA}u_0 \).
1.2 Strongly continuous semigroups

Example 3 (Translations on \( \mathbb{R} \)) Let \( C_b(\mathbb{R}) \) be the Banach space of all bounded uniformly continuous functions \( f : \mathbb{R} \to \mathbb{R} \) with the uniform norm

\[
|f|_{\infty, \mathbb{R}} = \sup_{x \in \mathbb{R}} |f(x)|.
\]

For any \( t \in \mathbb{R}_+ \) define

\[
(S(t)f)(x) = f(x + t) \quad \forall f \in C_b(\mathbb{R}).
\]

The following holds true.

1. \( S(t) \) is a semigroup of bounded linear operators on \( C_b(\mathbb{R}) \).

2. \( S(t) \) fails to be uniformly continuous.

   \[\text{Proof.} \quad \text{For any } n \in \mathbb{N} \text{ the function} \quad f_n(x) = e^{-nx^2} \quad (x \in \mathbb{R}) \]
   belongs to \( C_b(\mathbb{R}) \) and has norm equal to 1. Therefore, for any \( t > 0 \)
   \[
   \|S(t) - I\| \geq |S(t)f_n - f_n|_{\infty, \mathbb{R}} = \sup_{x \in \mathbb{R}} |e^{-n(x+t)^2} - e^{-nx^2}| \geq 1 - e^{-nt^2}.
   \]
   Since this is true for any \( n \), we have that \( \|S(t) - I\| \geq 1 \). \( \square \)

3. For all \( f \in C_b(\mathbb{R}) \) we have that \( |S(t)f - f|_{\infty, \mathbb{R}} \to 0 \) as \( t \downarrow 0 \).

   \[\text{Proof.} \quad \text{Indeed,} \quad |S(t)f - f|_{\infty, \mathbb{R}} = \sup_{x \in \mathbb{R}} |f(x + t) - f(x)| \leq \text{osc}_f(t) \xrightarrow{t \downarrow 0} 0 \] \( \square \)

**Definition 4** A semigroup \( S(t) \) of bounded linear operators on \( X \) is called strongly continuous (or of class \( C_0 \), or even a \( C_0 \)-semigroup) if

\[
\lim_{t \downarrow 0} S(t)u = u \quad \forall u \in X. \tag{1.2.1}
\]

**Theorem 2** Let \( S(t) \) be a \( C_0 \)-semigroup of bounded linear operators on \( X \). Then there exist \( \omega \geq 0 \) and \( M \geq 1 \) such that

\[
\|S(t)\| \leq Me^{\omega t} \quad \forall t \geq 0. \tag{1.2.2}
\]
Proof. We first prove the following:

\[ \exists \tau > 0 \text{ and } M \geq 1 \text{ such that } \|S(t)\| \leq M \quad \forall t \in [0, \tau]. \]  

We argue by contradiction assuming there exists a sequence \( t_n \downarrow 0 \) such that \( \|S(t_n)\| \geq n \) for all \( n \geq 1 \). Then, the principle of uniform boundedness implies that, for some \( u \in X \), \( \|S(t_n)u\| \to \infty \) as \( n \to \infty \), in contrast with (1.2.1).

Now, given \( t \in \mathbb{R}_+ \), let \( n \in \mathbb{N} \) and \( \delta \in [0, \tau] \) be such that \( t = n\tau + \delta \).

Then, in view of (1.2.3),

\[ \|S(t)\| = \|S(\delta)S(\tau)^n\| \leq M \cdot M^n = M \cdot (M^{1/\tau})^{n\tau} \leq M \cdot (M^{1/\tau})^t \]

which yields (1.2.2) with \( \omega = \frac{\log M}{\tau} \).

\[ \square \]

Corollary 2 Let \( S(t) \) be a \( C_0 \)-semigroup of bounded linear operators on \( X \). Then for every \( u \in X \) the map \( t \mapsto S(t)u \) is continuous from \( \mathbb{R}_+ \) into \( X \).

Definition 5 A \( C_0 \)-semigroup of bounded linear operators on \( X \) is called uniformly bounded if \( S(t) \) satisfies (1.2.2) with \( \omega = 0 \). If, in addition, \( M = 1 \), we say that \( S(t) \) is a contraction semigroup.

Exercise 2 Prove that the translation semigroup of Example 3 satisfies

\[ \|S(t)\| = 1 \quad \forall t \geq 0. \]

So, \( S(t) \) is a contraction semigroup.

Exercise 3 For any fixed \( p \geq 1 \), let \( X = L^p(\mathbb{R}) \) and define, \( \forall f \in X \),

\[ (S(t)f)(x) = f(x + t) \quad \forall x \in \mathbb{R}, \forall t \geq 0. \]

(1.2.4)

Prove that \( S \) is \( C_0 \)-semigroup which fails to be uniformly continuous.

Solution. Suppose \( S \) is uniformly continuous and let \( \tau > 0 \) be such that \( \|S(t) - I\| < 1/2 \) for all \( t \in [0, \tau] \). Then by taking \( f_n(x) = n^{1/p} \chi_{[0,1/n]}(x) \) for \( p < \infty \) and \( n > 1/\tau \) we have that \( |f_n| = 1 \) and

\[ |S(\tau)f_n - f_n| = \left( \int_{\mathbb{R}} n|\chi_{[0,1/n]}(x + \tau) - \chi_{[0,1/n]}(x)|^p dx \right)^{1/p} = 2^{1/p}. \]

\[ \square \]

Exercise 4 Given a uniformly bounded \( C_0 \)-semigroup, \( \|S(t)\| \leq M \), define

\[ |u|_S = \sup_{t \geq 0} |S(t)u|, \quad \forall u \in X. \]

(1.2.5)

Show that:

(a) \( \cdot \mid_S \) is a norm on \( X \),

(b) \( |u| \leq |u|_S \leq M|u| \) for all \( u \in X \), and

(c) \( S \) is a contraction semigroup with respect to \( \cdot \mid_S \).
1.3 The infinitesimal generator of a $C_0$-semigroup

**Theorem 3** Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $C_0$-semigroup of bounded linear operators on $X$, denoted by $S(t)$. Then the following properties hold true.

(a) For all $t \geq 0$

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(s)u \, ds = S(t)u \quad \forall u \in X.$$ 

(b) For all $t \geq 0$ and $u \in X$

$$\int_0^t S(s)u \, ds \in D(A) \quad \text{and} \quad A \int_0^t S(s)u \, ds = S(t)u - u.$$ 

(c) $D(A)$ is dense in $X$.

(d) For all $u \in D(A)$ and $t \geq 0$ we have that $S(t)u \in D(A)$, $t \mapsto S(t)u$ is continuously differentiable, and

$$\frac{d}{dt} S(t)u = AS(t)u = S(t)Au.$$ 

(e) For all $u \in D(A)$ and all $0 \leq s \leq t$ we have that

$$S(t)u - S(s)u = \int_s^t S(\tau)Au \, d\tau = \int_s^t AS(\tau)u \, d\tau.$$ 

**Proof.** We remind the reader that all integrals are to be understood in the Cauchy sense.

(a) This point is an immediate consequence of the strong continuity of $S$.

(b) For any $t \geq h > 0$ we have that

$$\frac{S(h) - I}{h} \left( \int_0^t S(s)u \, ds \right) = \frac{1}{h} \int_0^t (S(h+s) - S(s))u \, ds$$

$$= \frac{1}{h} \left( \int_h^{t+h} S(s)u \, ds - \int_0^t S(s)u \, ds \right)$$

$$= \frac{1}{h} \left( \int_t^{t+h} S(s)u \, ds - \int_0^h S(s)u \, ds \right).$$

Therefore, by (a),

$$\lim_{h \downarrow 0} \frac{S(h) - I}{h} \left( \int_0^t S(s)x \, ds \right) = S(t)x - x$$

which proves (b).
(c) This point follows from (a) and (b).

(d) For all \( u \in D(A) \), \( t \geq 0 \), and \( h > 0 \) we have that

\[
\frac{S(h) - I}{h} S(t)u = S(t) \frac{S(h) - I}{h} u \rightarrow S(t)Au \quad \text{as} \quad h \downarrow 0.
\]

Therefore, \( S(t)u \in D(A) \) and \( AS(t)u = S(t)Au = \frac{d^2}{dt} S(t)u \). In order to prove the existence of the left derivative, observe that for all \( 0 < h < t \)

\[
\frac{S(t-h)u - S(t)u}{-h} = S(t-h) \frac{S(h) - I}{h} u.
\]

Moreover, by (1.2.2),

\[
\left| S(t-h) \frac{S(h) - I}{h} u - S(t)Au \right| \\
\leq \left| S(t-h) \cdot \frac{S(h) - I}{h} u - S(h)Au \right| \\
\leq Me^{\omega t} \left| \frac{S(h) - I}{h} u - S(h)Au \right| \xrightarrow{h \downarrow 0} 0.
\]

Therefore

\[
\frac{S(t-h)u - S(t)u}{-h} \rightarrow S(t)Au = AS(t)u \quad \text{as} \quad h \downarrow 0,
\]

showing that the left and right derivatives coincide.

(e) This point follows from (d).

The proof is complete. \( \square \)

**Exercise 5** Show that the infinitesimal generator of the \( C_0 \)-semigroup of left translations on \( \mathbb{R} \) we introduced in Example 3 is given by

\[
\left\{ \begin{array}{ll}
D(A) = C^1_b(\mathbb{R}) \\
Af = f' & \forall f \in D(A).
\end{array} \right.
\]

**Solution.** For any \( f \in C^1_b(\mathbb{R}) \) we have that

\[
\left| \frac{S(t)f - f'}{t} \right|_{\infty, \mathbb{R}} = \sup_{x \in \mathbb{R}} \left| \frac{f(x + t) - f(x)}{t} - f'(x) \right| \leq \text{osc}_{f'}(t) \xrightarrow{t \downarrow 0} 0
\]

Therefore, \( C^1_b(\mathbb{R}) \subset D(A) \) and \( Af = f' \) for all \( f \in C^1_b(\mathbb{R}) \). Conversely, let \( f \in D(A) \). Then, \( Af \in C^1_b(\mathbb{R}) \) and

\[
\sup_{x \in \mathbb{R}} \left| \frac{f(x + t) - f(x)}{t} - Af(x) \right| \xrightarrow{t \downarrow 0} 0.
\]

So, \( f'(x) \) exists for all \( x \in \mathbb{R} \) and equals \( Af(x) \). Thus, \( D(A) \subset C^1_b(\mathbb{R}) \). \( \square \)
**Exercise 6** Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a uniformly bounded semigroup $\|S(t)\| \leq M$. Prove the Laundau-Kolmogorov inequality:

$$|Au|^2 \leq 4M^2 |u| |A^2u| \quad \forall u \in D(A^2), \quad (1.3.1)$$

where

$$\begin{align*}
D(A^2) &= \{ u \in D(A) : Au \in D(A) \} \\
A^2u &= A(Au), \quad \forall u \in D(A^2). \quad (1.3.2)
\end{align*}$$

**Solution.** Assume $M = 1$. For any $u \in D(A^2)$ and all $t \geq 0$ we have

$$\int_0^t (t-s)S(s)A^2u \, ds = [(t-s)S(s)Au]_{s=0}^{s=t} + \int_0^t S(s)Au \, ds$$

$$= -tAu + [S(s)u]_{s=0}^{s=t} = -tAu + S(t)u - u.$$

Therefore, for all $t > 0$,

$$|Au| \leq \frac{1}{t} |S(t)u - u| + \frac{1}{t} \int_0^t (t-s)|S(s)A^2u| \, ds$$

$$\leq \frac{2}{t} |u| + \frac{t}{2} |A^2u|. \quad (1.3.3)$$

If $A^2u = 0$, then the above inequality yields $Au = 0$ by letting $t \to \infty$. So, (1.3.1) is true in this case. On the other hand, for $A^2u \neq 0$ the function of $t$ on the right-hand side of (1.3.3) attains its minimum at

$$t_0 = \frac{2|u|^{1/2}}{|A^2u|^{1/2}}.$$

By taking $t = t_0$ in (1.3.3) we obtain (1.3.1) once again.

(Question: how to treat the case of $M \neq 1$? Hint: remember Exercise 4.) □

**Exercise 7** Use the Landau-Kolmogorov inequality to deduce the interpolation inequality

$$|f'|_{\infty, \mathbb{R}} \leq 4 |f|_{\infty, \mathbb{R}} |f''|_{\infty, \mathbb{R}} \quad \forall f \in C^2_0(\mathbb{R}).$$

### 1.4 The Cauchy problem with a closed operator

We recall that $X \times X$ is a Banach space with norm

$$\|(u, v)\| = |u| + |v| \quad \forall (u, v) \in X \times X.$$

**Definition 6** An operator $A : D(A) \subset X \to X$ is said to be closed if its graph

$$\text{graph}(A) := \{ (u, v) \in X \times X : u \in D(A), v = Au \}$$

is a closed subset of $X \times X$. 

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The following characterisation of closed operators is straightforward.

**Proposition 4** The linear operator $A : D(A) \subset X \to X$ is closed if and only if, for any sequence $\{x_n\} \subset D(A)$, the following holds:

$$\begin{cases} u_n \to u \\ Au_n \to v \end{cases} \implies u \in D(A) \text{ and } Au = v. \quad (1.4.1)$$

**Example 4** In the Banach space $X = C_b(\mathbb{R})$, the linear operator

$$\begin{cases} D(A) = C^1_b(\mathbb{R}) \\ Af = f' \quad \forall f \in D(A). \end{cases}$$

is closed. Indeed, for any sequence $\{f_n\} \subset C^1_b(\mathbb{R})$ such that

$$\begin{cases} f_n \to f \quad \text{in } C_b(\mathbb{R}) \\ f'_n \to g \quad \text{in } C_b(\mathbb{R}), \end{cases}$$

we have that $f \in C^1_b(\mathbb{R})$ and $f' = g$.

**Example 5** In the Banach space $X = C([0,1])$ with the uniform norm, the linear operator

$$\begin{cases} D(A) = C^1([0,1]) \\ (Af)(x) = f'(0) \quad \forall x \in [0,1] \end{cases}$$

fails to be closed. Indeed, for any $n \geq 1$ let

$$f_n(x) = \frac{\sin(nx)}{n} \quad x \in [0,1].$$

Then

$$\begin{cases} D(A) \ni f_n \to 0 \quad \text{in } C_b(\mathbb{R}) \\ Af_n = 1 \quad \forall n \geq 1, \end{cases}$$

in contrast with (1.4.1).

**Exercise 8** Prove that if $A : D(A) \subset X \to X$ is a closed operator and $B \in \mathcal{L}(X)$, then $A + B : D(A) \subset X \to X$ is also closed. What about $BA$?

**Exercise 9** Prove that, if $A : D(A) \subset X \to X$ is a closed operator and $f \in C([a,b]; D(A))$, then

$$A \int_a^b f(t)dt = \int_a^b Af(t)dt. \quad (1.4.2)$$
Solution. Let $\pi_n = \{t^n_i\}_{i=0}^{\infty} \in \Pi(a,b)$ be such that $\text{diam}(\pi_n) \to 0$ and let $\sigma_n = \{s^n_i\}_{i=1}^{\infty} \in \Sigma(\pi_n)$. Then $\int_a^b f(t)dt \in D(A)$ and

$$
\begin{cases}
D(A) \ni S^n_{\pi_n}(f) = \sum_{i=1}^{\infty} f(s^n_i)(t^n_i - t^n_{i-1}) \to \int_a^b f(t)dt \\
AS^n_{\pi_n}(f) = \sum_{i=1}^{\infty} Af(s^n_i)(t^n_i - t^n_{i-1}) \to \int_a^b Af(t)dt
\end{cases}
(n \to \infty)
$$

Therefore, by Proposition 4, $\int_a^b f(t)dt \in D(A)$ and (1.4.2) holds true. \qed

Proposition 5 The infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(t)$ is a closed operator.

Proof. Let $A : D(A) \subset X \to X$ be the infinitesimal generator of $S(t)$ and let $\{u_n\} \subset D(A)$ be as in (1.4.1). By Theorem 3–(d) we have that, for all $t \geq 0$,

$$
S(t)u_n - u_n = \int_0^t S(s)Au_n dx.
$$

Hence, taking the limit as $n \to \infty$ and dividing by $t$, we obtain

$$
\frac{S(t)u - u}{t} = \frac{1}{t} \int_0^t S(s)vdu.
$$

Passing to the limit as $t \downarrow 0$, we conclude that $Au = v$. \qed

Remark 1 From Proposition 5 it follows that the domain $D(A)$ of the infinitesimal generator of a $\mathcal{C}_0$-semigroup is a Banach space with the graph norm

$$
|u|_{D(A)} = |u| + |Au| \quad \forall u \in D(A).
$$

Exercise 10 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^2$. Define

$$
\begin{cases}
D(A) = H^2(\Omega) \cap H^1_0(\Omega) \\
Au = \Delta u \quad \forall u \in D(A).
\end{cases}
$$

Prove that $A$ is a closed operator on the Hilbert space $X = L^2(\Omega)$.

Solution. Let $u_i \in H^2(\Omega) \cap H^1_0(\Omega)$ be such that

$$
\begin{cases}
u_i \to u \\
\Delta u_i \to v
\end{cases}
\quad \text{in } L^2(\Omega).
$$

By elliptic regularity, we have that

$$
\|u_i - u_j\|_{2,\Omega} \leq C\|\Delta u_i - \Delta u_j\|_{0,\Omega}
$$

for some constant $C > 0$. Hence, $\{u_i\}$ is a Cauchy sequence in the Hilbert space $H^2(\Omega) \cap H^1_0(\Omega)$. So, $u \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\Delta u = v$. \qed
Given a closed operator \( A : D(A) \subset X \to X \), let us consider the Cauchy problem with initial datum \( u_0 \in X \)

\[
\begin{cases}
  u'(t) = Au(t), & t > 0 \\
  u(0) = u_0.
\end{cases}
\]  

(1.4.3)

**Definition 7** A classical solution of problem (1.4.3) is a function

\[ u \in C(\mathbb{R}_+; X) \cap C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(A)) \]

such that \( u(0) = u_0 \) and \( u'(t) = Au(t) \) for all \( t > 0 \).

Our next result ensures the existence and uniqueness of a classical solution to (1.4.3) for initial data in \( D(A) \), provided \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup of bounded linear operators on \( X \).

**Proposition 6** Let \( A : D(A) \subset X \to X \) be the infinitesimal generator of a \( C_0 \)-semigroup of bounded linear operators on \( X \), \( S(t) \).

Then, for every \( u_0 \in D(A) \), problem (1.4.3) has a unique classical solution \( u \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(A)) \) given by \( u(t) = S(t)u_0 \) for all \( t \geq 0 \).

**Proof.** The fact that \( u(t) = S(t)u_0 \) satisfies (1.4.3) is point (d) of Theorem 3.

To show that \( u \) is the unique solution of the problem let \( v \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(A)) \) be any solution of (1.4.3), fix \( t > 0 \), and set

\[
U(s) = S(t-s)v(s), \quad \forall s \in [0,t].
\]

Then, for all \( s \in ]0,t[ \) we have that

\[
\frac{U(s+h) - U(s)}{h} = S(t-s-h)v'(s) + AS(t-s)v(s)
\]

\[
= S(t-s-h)\frac{v(s+h) - v(s)}{h} - S(t-s)v'(s)
\]

\[
+ \left( \frac{S(t-s-h) - S(t-s)}{h} + AS(t-s) \right) v(s).
\]

Now, point (d) of Theorem 3 immediately yields

\[
\lim_{h \to 0} \frac{S(t-s-h) - S(t-s)}{h} v(s) = -AS(t-s)v(s).
\]

Moreover,

\[
S(t-s-h)\frac{v(s+h) - v(s)}{h} - S(t-s)v'(s)
\]

\[
= S(t-s-h)\left( \frac{v(s+h) - v(s)}{h} - v'(s) \right)
\]

\[
+ \left( S(t-s-h) - S(t-s) \right) v'(s).
\]

\[1\] Here \( D(A) \) is regarded as a Banach space with the graph norm.
where

\[
(S(t-s-h) - S(t-s))v'(s) \xrightarrow{h \to 0} 0
\]

by the strong continuity of \(S(t)\), while

\[
\left| S(t-s-h) \left( \frac{v(s+h) - v(s)}{h} - v'(s) \right) \right| \leq Me^{\omega(t-s-h)} \left| \frac{v(s+h) - v(s)}{h} - v'(s) \right| \xrightarrow{h \to 0} 0
\]

in view of (1.2.2). Therefore,

\[
U'(s) = -AS(t-s)v(s) + S(t-s)Av(s) = 0, \quad \forall s \in ]0, T[.
\]

So, \(U\) is constant and \(u(t) = U(t) = U(0) = v(t)\). \(\square\)

**Exercise 11** Let \(S(t)\) and \(T(t)\) be \(C_0\)-semigroups with infinitesimal generators \(A : D(A) \subset X \to X\) and \(B : D(B) \subset X \to X\), respectively. Show that

\[
A = B \implies S(t) = T(t) \quad \forall t \geq 0.
\]

**Example 6 (Transport equation in \(C_b(\mathbb{R})\))** Returning to the left-translation semigroup on \(C_b(\mathbb{R})\) of Example 3, by Proposition 6 and Exercise 5 we conclude that for each \(f \in C^1_b(\mathbb{R})\) the unique solution of the problem

\[
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \frac{\partial u}{\partial x}(t,x) & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\
u(0,x) &= f(x)
\end{aligned}
\]

is given by \(u(t,x) = f(x+t)\).

### 1.5 Resolvent and spectrum of a closed operator

Let \(A : D(A) \subset X \to X\) be a closed operator on a complex Banach space \(X\).

**Definition 8** The **resolvent set** of \(A\), \(\rho(A)\), is the set of all \(\lambda \in \mathbb{C}\) such that \(\lambda I - A : D(A) \to X\) is bijective. The set \(\sigma(A) = \mathbb{C} \setminus \rho(A)\) is called the **spectrum** of \(A\). For any \(\lambda \in \rho(A)\) the linear operator

\[
R(\lambda, A) := (\lambda I - A)^{-1} : X \to X
\]

is called the **resolvent** of \(A\).

**Example 7** On \(X = C([0,1])\) with the uniform norm consider the linear operator \(A : D(A) \subset X \to X\) defined by

\[
\begin{aligned}
D(A) &= C^1([0,1]) \\
Af &= f', \quad \forall f \in D(A)
\end{aligned}
\]
is closed (compare to Example 4). Then \( \sigma(A) = \mathbb{C} \) because for any \( \lambda \in \mathbb{C} \) the function \( f_\lambda(x) = e^{\lambda x} \) satisfies

\[
\lambda f_\lambda(x) - f_\lambda'(x) = 0 \quad \forall x \in [0, 1].
\]

On the other hand, for the closed operator \( A_0 \) defined by

\[
\begin{align*}
D(A_0) &= \{ f \in C^1([0, 1]) : f(0) = 0 \} \\
A_0 f &= f', \quad \forall f \in D(A_0),
\end{align*}
\]

we have that \( \sigma(A_0) = \emptyset \). Indeed, for any \( g \in X \) the problem

\[
\begin{align*}
\lambda f(x) - f'(x) &= g(x) \quad x \in [0, 1] \\
f(0) &= 0
\end{align*}
\]

admits the unique solution

\[
f(x) = -\int_0^x e^{\lambda(x-s)}g(s) \, dx \quad (x \in [0, 1])
\]

which belongs to \( D(A_0) \).

**Proposition 7 (properties of \( R(\lambda, A) \))** Let \( A : D(A) \subset X \to X \) be a closed operator on a complex Banach space \( X \). Then the following holds true.

(a) \( R(\lambda, A) \in \mathcal{L}(X) \) for any \( \lambda \in \rho(A) \).

(b) For any \( \lambda \in \rho(A) \)

\[
AR(\lambda, A) = \lambda R(\lambda, A) - I. \tag{1.5.1}
\]

(c) The resolvent identity holds:

\[
R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \quad \forall \lambda, \mu \in \rho(A). \tag{1.5.2}
\]

(d) For any \( \lambda, \mu \in \rho(A) \)

\[
R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A). \tag{1.5.3}
\]

**Proof.** Let \( \lambda, \mu \in \rho(A) \).

(a) Since \( A \) is closed, so is \( \lambda I - A \) and aslo \( R(\lambda, A) = (\lambda I - A)^{-1} \). So, \( R(\lambda, A) \in \mathcal{L}(X) \) by the closed graph theorem.

(b) This point follows from the definition of \( R(\lambda, A) \).
(c) By (1.5.1) we have that
\[ [\lambda R(\lambda, A) - AR(\lambda, A)]R(\mu, A) = R(\mu, A) \]
and
\[ R(\lambda, A)[\mu R(\mu, A) - AR(\mu, A)] = R(\lambda, A). \]
Since \( AR(\lambda, A) = R(\lambda, A)A \) on \( D(A) \), (1.5.2) follows.

(d) Apply (1.5.2) to compute
\[ R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \]
\[ R(\mu, A) - R(\lambda, A) = (\lambda - \mu)R(\mu, A)R(\lambda, A). \]

Adding the above identities side by side yields the conclusion.

The proof is complete. \( \square \)

**Theorem 4 (analyticity of \( R(\lambda, A) \))** Let \( A : D(A) \subset X \to X \) be a closed operator on a complex Banach space \( X \). Then the resolvent set \( \rho(A) \) is open in \( \mathbb{C} \) and for any \( \lambda_0 \in \rho(A) \) we have that
\[ |\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|} \implies \lambda \in \rho(A) \] (1.5.4)
and the resolvent \( R(\lambda, A) \) is given by the (Neumann) series
\[ R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}. \] (1.5.5)

Consequently, \( \lambda \mapsto R(\lambda, A) \) is analytic on \( \rho(A) \) and
\[ \frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1} \quad \forall n \in \mathbb{N}. \] (1.5.6)

**Proof.** For all \( \lambda \in \mathbb{C} \) and \( \lambda_0 \in \rho(A) \) we have that
\[ \lambda I - A = \lambda_0 I - A + (\lambda - \lambda_0)I = [I - (\lambda_0 - \lambda)R(\lambda_0, A)](\lambda_0 I - A). \]
This operator is bijective if and only if \([I - (\lambda_0 - \lambda)R(\lambda_0, A)]\) is invertible, which is the case if \( \lambda \) satisfies (1.5.4). Then
\[ R(\lambda, A) = R(\lambda_0, A)[I - (\lambda_0 - \lambda)R(\lambda_0, A)]^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}. \]
The analyticity of \( R(\lambda, A) \) and (1.5.6) follows from (1.5.5). \( \square \)
Theorem 5 (integral representation of $R(\lambda, A)$) Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $C_0$-semigroup of bounded linear operators on $X$, $S(t)$, and let $M \geq 1$ and $\omega \in \mathbb{R}$ be such that
\begin{equation}
\|S(t)\| \leq Me^{\omega t} \quad \forall t \geq 0.
\end{equation}
Then $\rho(A)$ contains the half-plane
\begin{equation}
\Pi_\omega = \{ \lambda \in \mathbb{C} : \Re \lambda > \omega \}
\end{equation}
and
\begin{equation}
R(\lambda, A)u = \int_0^\infty e^{-\lambda t}S(t)u \, dt \quad \forall u \in X, \ \forall \lambda \in \Pi_\omega.
\end{equation}

Proof. We have to prove that, given any $\lambda \in \Pi_\omega$ and $u \in X$, the equation
\begin{equation}
\lambda v - Av = u
\end{equation}
is satisfied. Existence: observe that $v := \int_0^\infty e^{-\lambda t}S(t)u \, dt \in X$ because $\Re \lambda > \omega$. Moreover, for all $h > 0$,
\[
\frac{S(h)v - v}{h} = \frac{1}{h} \left\{ \int_0^\infty e^{-\lambda (t+h)}u \, dt - \int_0^\infty e^{-\lambda t}S(t)u \, dt \right\}
\]
\[
= \frac{1}{h} \left\{ e^{\lambda h} \int_h^\infty e^{-\lambda t}S(t)u \, dt - \int_0^\infty e^{-\lambda t}S(t)u \, dt \right\}
\]
\[
= \frac{e^{\lambda h} - 1}{h} v - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda t}S(t)u \, dt.
\]
So
\[
\lim_{h \downarrow 0} \frac{S(h)v - v}{h} = \lambda v - u
\]
which in turn yields that $v \in D(A)$ and (1.5.10) holds true. Uniqueness: let $v \in D(A)$ be a solution of (1.5.10). Then
\[
\int_0^\infty e^{-\lambda t}S(t)u \, dt = \int_0^\infty e^{-\lambda t}S(t)(\lambda v - Av) \, dt
\]
\[
= \lambda \int_0^\infty e^{-\lambda t}S(t)v \, dt - \int_0^\infty e^{-\lambda t} \frac{d}{dt}S(t)v \, dt = v
\]
which implies that $v$ is given by (1.5.9).

Proposition 8 Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed linear operators in $X$ and suppose $B \subset A$, that is,
\[
D(B) \subset D(A) \quad \text{and} \quad Au = Bu \quad \forall x \in D(B).
\]
If $\rho(A) \cap \rho(B) \neq \emptyset$, then $A = B$. 

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Proof. It suffices to show that $D(A) \subset D(B)$. Let $u \in D(A)$, $\lambda \in \rho(A) \cap \rho(B)$, and set

$$v = \lambda u - Au \quad \text{and} \quad w = R(\lambda, B)v.$$ 

Then $w \in D(B)$ and $\lambda w - Bw = \lambda u - Au$. Since $B \subset A$, $\lambda w - Bw = \lambda w - Aw$. Thus, $(\lambda I - A)(u - w) = 0$. So, $u = w \in D(B)$. □

Example 8 (Right-translation semigroup on $\mathbb{R}_+$) On the real Banach space

$$X = \{ f \in \mathcal{C}_b(\mathbb{R}_+) : f(0) = 0 \}$$

with the uniform norm, consider the right-translation semigroup

$$(S(t)f)(x) = \begin{cases} f(x-t) & x > t \\ 0 & x \in [0,t] \end{cases} \quad \forall x, t \geq 0.$$ 

It is easy to check that $S$ is a $C_0$-semigroup on $X$ with $\|S(t)\| = 1$ for all $t \geq 0$. In order to characterize its infinitesimal generator $A$, let us consider the operator $B : D(B) \subset X \to X$ defined by

$$D(B) = \{ f \in X : f' \in X \}$$

$$Bf = -f', \quad \forall f \in D(B).$$

We claim that:

(i) $B \subset A$

**Proof.** Let $f \in D(B)$. Then, for all $x, t \geq 0$ we have

$$\frac{(S(t)f)(x) - f(x)}{t} = \begin{cases} -\frac{f(x)}{t} = -f'(x_t), & 0 \leq x \leq t \\ \frac{f(x-t) - f(x)}{t} = -f'(x_t) & x \geq t \end{cases}$$

with $0 \leq x - x_t \leq t$. Therefore

$$\sup_{x \geq 0} \left| \frac{(S(t)f)(x) - f(x)}{t} + f'(x) \right| \leq \sup_{|x-y| \leq t} |f'(x) - f'(y)| \to 0 \quad \text{as} \quad t \downarrow 0$$

because $f'$ is uniformly continuous. □

(ii) $1 \in \rho(B)$

**Proof.** For any $g \in X$ the unique solution $f$ of the problem

$$\begin{cases} f \in D(B) \\ f(x) + f'(x) = g(x) \quad \forall x \geq 0 \end{cases}$$

is given by

$$f(x) = \int_0^x e^{s-x} g(s) \, ds \quad (x \geq 0).$$

Since $1 \in \rho(A)$ by Proposition 5, Proposition 8 yields that $A = B$. 19
1.6 The Hille-Yosida generation theorem

**Theorem 6** Let $M \geq 1$ and $\omega \in \mathbb{R}$. For a linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:

(a) $A$ is closed, $D(A)$ is dense in $X$, and

$$\rho(A) \supseteq \Pi_\omega = \{ \lambda \in \mathbb{C} : \Re \lambda > \omega \} \quad (1.6.1)$$

$$\|R(\lambda, A)^k\| \leq \frac{M}{(\Re \lambda - \omega)^k} \quad \forall k \geq 1, \forall \lambda \in \Pi_\omega \quad (1.6.2)$$

(b) $A$ is the infinitesimal generator of a $C_0$-semigroup, $S(t)$, such that

$$\|S(t)\| \leq Me^{\omega t} \quad \forall t \geq 0. \quad (1.6.3)$$

**Proof of (b) $\Rightarrow$ (a)** The fact that $A$ is closed, $D(A)$ is dense in $X$, and (1.6.1) holds true has already been proved, see Theorem 3-(c), Proposition 5, and Theorem 5. In order to prove (1.6.2) observe that, by using (1.5.9) to compute the $k$-th derivative of the resolvent of $A$, we obtain

$$\frac{d^k}{d\lambda^k} R(\lambda, A)u = (-1)^k \int_0^\infty t^k e^{-\lambda t} S(t) u \, dt \quad \forall u \in X, \forall \lambda \in \Pi_\omega.$$ 

Therefore,

$$\left\| \frac{d^k}{d\lambda^k} R(\lambda, A) \right\| \leq M \int_0^\infty t^k e^{-(\Re \lambda - \omega) t} \, dt = \frac{M k!}{(\Re \lambda - \omega)^{k+1}}$$

where the integral is easily computed by induction. The conclusion follows recalling (1.5.6). \(\square\)

**Lemma 1** Let $A : D(A) \subset X \to X$ be as in (a) of Theorem 6. Then:

(i) For all $u \in X$

$$\lim_{n \to \infty} nR(n, A)u = u. \quad (1.6.4)$$

(ii) The Yosida Approximation $A_n$ of $A$, defined as

$$A_n = nAR(n, A) \quad (n \geq 1) \quad (1.6.5)$$

is a sequence of bounded operator on $X$ which satisfies

$$A_n A_m = A_m A_n \quad \forall n, m \geq 1 \quad (1.6.6)$$

and

$$\lim_{n \to \infty} A_n u = Au \quad \forall u \in D(A). \quad (1.6.7)$$
(iii) For all $m, n > 2\omega$, $u \in D(A)$, $t \geq 0$ we have that

\[ \|e^{tA_n}\| \leq Me^{\frac{\omega t}{n}} \leq Me^{2\omega t} \]  

(1.6.8)

\[ |e^{tA_n}u - e^{tA_m}u| \leq M'te^{2\omega t}|A_n u - A_m u|. \]  

(1.6.9)

Consequently, for all $u \in D(A)$ the sequence $u_n(t) := e^{tA_n}u$ is Cauchy in $C([0,T]; X)$ for any $T > 0$.

Proof of (i): owing to (1.5.1), for any $u \in D(A)$ we have that

\[ |nR(n,A)u - u| = |AR(n, A)u| = |R(n, A)Au| \leq \frac{M|Au|}{n - \omega} \xrightarrow{(n \to \infty)} 0, \]

where we have used (1.6.2) with $k = 1$. Moreover, again by (1.6.2),

\[ \|nR(n, A)\| \leq \frac{Mn}{n - \omega} \leq 2M \quad \forall n > 2\omega. \]

We claim that the last two inequalities yield the conclusion because $D(A)$ is dense in $X$. Indeed, let $u \in X$ and fix any $\varepsilon > 0$. Let $u_\varepsilon \in D(A)$ be such that $|u_\varepsilon - u| < \varepsilon$. Then

\[ |nR(n, A)u - u| \leq |nR(n, A)(u - u_\varepsilon)| + |nR(n, A)u_\varepsilon - u_\varepsilon| + |u_\varepsilon - u| \]

\[ < (2M + 1)\varepsilon + \frac{M|Au_\varepsilon|}{n - \omega} \xrightarrow{(n \to \infty)} (2M + 1)\varepsilon. \]

Since $\varepsilon$ is arbitrary, (1.6.4) follows.

Proof of (ii): observe that $A_n \in \mathcal{L}(X)$ because

\[ A_n = n^2 R(n, A) - nI \quad \forall n \geq 1. \]  

(1.6.10)

Moreover, in view of (1.5.3) we have that

\[ A_nA_m = [n^2 R(n, A) - nI][m^2 R(m, A) - mI] \]

\[ = [m^2 R(m, A) - mI][n^2 R(n, A) - nI] = A_mA_n. \]

Finally, owing to (1.6.4), for all $u \in D(A)$ we have that

\[ A_n u = nAR(n, A)u = nR(n, A)Au \xrightarrow{(n \to \infty)} Au. \]

Proof of (iii): recalling (1.6.10) we have that

\[ e^{tA_n} = e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k}t^k}{k!}R(n, A)^k, \quad \forall t \geq 0. \]

Therefore, in view of (1.6.2),

\[ \|e^{tA_n}\| \leq Me^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k}t^k}{k!(n - \omega)^k} = Me^{\frac{\omega t}{n - \omega}} \leq Me^{2\omega t}. \]
for all \( t \geq 0 \) and \( n > 2 \omega \). This proves (1.6.8).

Next, observe that, for any \( u \in D(A) \), \( u_n(t) := e^{tA_n}u \) satisfies
\[
\begin{cases}
(u_n - u_m)'(t) = A_n(u_n - u_m)(t) + (A_n - A_m)u_m(t) & \forall t \geq 0 \\
(u_n - u_m)(0) = 0.
\end{cases}
\]

Therefore, for all \( t \geq 0 \) we have that
\[
e^{tA_n}u - e^{tA_m}u = \int_0^t e^{(t-s)A_n}(A_n - A_m)e^{sA_m}u \, ds
\]

because \( A_n \) and \( e^{sA_m}u \) commute in view of (1.6.6). Thus, by combining (1.6.11) and (1.6.8) we obtain
\[
|e^{tA_n}u - e^{tA_m}u| \leq M^2 \int_0^t e^{2\omega(t-s)}e^{2\omega s}|A_nu - A_mu|, ds
\]

In view of (1.6.7), the last inequality shows that \( e^{tA_n}u \) is a Cauchy sequence in \( C([0,T];X) \) for any \( T > 0 \), thus completing the proof. \( \square \)

**Exercise 12** Use a density argument to prove that \( e^{tA_n}u \) is a Cauchy sequence on all compact subsets of \( \mathbb{R}_+ \) for all \( u \in X \).

**Solution.** Let \( u \in X \) and fix any \( \varepsilon > 0 \). Let \( u_\varepsilon \in D(A) \) be such that \( |u_\varepsilon - u| < \varepsilon \). Then for all \( m, n > 2 \omega \) we have that
\[
|e^{tA_n}u - e^{tA_m}u| \leq |e^{tA_n}(u - u_\varepsilon)| + |(e^{tA_n} - e^{tA_m})u_\varepsilon| + |e^{tA_m}(u_\varepsilon - u)|
\]

Since \( \varepsilon \) is arbitrary, recalling point (iii) above the conclusion follows. \( \square \)

**Proof of (a) \( \Rightarrow \) (b)** On account of Lemma 1 and Exercise 12, we have that \( e^{tA_n}u \) is a Cauchy sequence on all compact subsets of \( \mathbb{R}_+ \) for all \( u \in X \). Consequently, the limit (uniform on all \( [0,T] \subset \mathbb{R}_+ \))
\[
S(t)u = \lim_{n \to \infty} e^{tA_n}u, \quad \forall u \in X,
\]

defines a \( C_0 \)-semigroup of bounded linear operators on \( X \). Moreover, passing to the limit as \( n \to \infty \) in (1.6.8), we conclude that \( \|S(t)\| \leq Me^{\omega t}, \forall t \geq 0 \).
Let us identify the infinitesimal generator of $S(t)$. By (1.6.8), for $u \in D(A)$ we have that
\[
\left| \frac{d}{dt} e^{tA_n}u - S(t)Au \right| \leq |e^{tA_n}Au - e^{tA_n}u| + |e^{tA_n}Au - S(t)Au|.
\]
uniformly on all compact subsets of $\mathbb{R}_+$ by (1.6.12). Therefore, for all $T > 0$ and $u \in D(A)$ we have that
\[
\begin{align*}
\left\{ e^{tA_n}u \xrightarrow{n \to \infty} S(t)u \quad \text{uniformly on } [0, T]. \right. 
\end{align*}
\]
This implies that
\[
S'(t)u = S(t)Au, \quad \forall u \in D(A), \forall t \geq 0. \tag{1.6.13}
\]
Now, let $B : D(B) \subset X \to X$ be the infinitesimal generator of $S(t)$. Then $A \subset B$ in view of (1.6.13). Moreover, $\Pi_\omega \subset \rho(A)$ by assumption (a) and $\Pi_\omega \subset \rho(B)$ by Proposition 5. So, on account of Proposition 8, $A = B$. \(\square\)

**Remark 2** The above proof shows that condition (a) in Theorem 6 can be relaxed as follows:

(a') $A$ is closed, $D(A)$ is dense in $X$, and
\[
\rho(A) \supseteq \omega, \infty \tag{1.6.14}
\]
\[
\|R(n,A)^k\| \leq \frac{M}{(n-\omega)^k} \quad \forall k \geq 1, \forall n > \omega. \tag{1.6.15}
\]

**Remark 3** When $M = 1$, the countably many bounds in condition (a) follow from (1.6.2) for $k = 1$, that is,
\[
\|R(\lambda,A)\| \leq \frac{1}{|\Re \lambda - \omega|} \quad \forall k \geq 1, \forall \lambda \in \Pi_\omega.
\]

**Example 9 (parabolic equations in $L^2(\Omega)$)** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^2$. Define
\[
\begin{align*}
D(A) &= H^2(\Omega) \cap H_0^1(\Omega) \\
Au &= \sum_{i,j=1}^n D_j(a_{ij} D_j)u + \sum_{i=1}^n b_i D_i u + cu \quad \forall u \in D(A).
\end{align*}
\]
where

(H1) $a_{ij} \in C^1(\Omega)$ satisfies $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$ and
\[
\sum_{i,j=1}^n a_{ij}(x)\xi_j \xi_i \geq \theta|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, x \in \Omega
\]
\( b_i \in L^\infty(\Omega) \) for all \( i = 1, \ldots, n \) and \( c \in L^\infty(\Omega) \).

In order to apply the Hille-Yosida theorem to show that \( A \) is the infinitesimal generator of a \( \mathcal{C}_0 \)-semigroup \( S(t) \) on \( L^2(\Omega) \), one can check that the following assumptions are satisfied.

1. \( D(A) \) is dense in \( L^2(\Omega) \).

   [This is a known property of Sobolev spaces (see, for instance, [3]).]

2. \( A \) is a closed operator.

   Proof. Let \( u_k \in D(A) \) be such that
   \[
   u_k \xrightarrow{k \to \infty} u \quad \text{and} \quad Au_k \xrightarrow{k \to \infty} f.
   \]
   Then, for all \( h, k \geq 1 \) we have that \( v_{hk} := u_h - u_k \) satisfies
   \[
   \begin{cases}
   \sum_{i,j=1}^n D_j(a_{ij}D_j)v_{hk} + \sum_{i=1}^n b_iD_i v_{hk} + cv_{hk} = f_{hk} & \text{in } \Omega \\
   v_{hk} = 0 & \text{on } \partial\Omega.
   \end{cases}
   \]
   So, elliptic regularity insures that
   \[
   \|v_{hk}\|_{2,\Omega} \leq C(\|f_{hk}\|_{0,\Omega} + \|v_{hk}\|_{0,\Omega})
   \]
   for some constant \( C > 0 \). The above inequality implies that \( \{u_k\} \) is a Cauchy sequence in \( D(A) \) and this yields \( f = Au \).

3. \( \exists \omega \in \mathbb{R} \) such that \( \rho(A) \supset [\omega, \infty[ \).

   [This follows from elliptic theory (see, for instance, [3]).]

4. \( \|R(\lambda, A)\| \leq \frac{1}{\lambda - \omega} \) for all \( k \geq 1 \) and \( \lambda > \omega \).

   [This follows from elliptic theory (see, for instance, [3]).]

Then, for any \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), the function \( u(t, x) = (S(t)u_0)(x) \) is the unique solution of the initial-boundary value problem
\[
\begin{cases}
\frac{\partial u}{\partial t} = \sum_{i,j=1}^n D_j(a_{ij}D_j)u + \sum_{i=1}^n b_iD_iu + cu & \text{in } ]0, \infty[ \times \Omega \\
u = 0 & \text{on } ]0, \infty[ \times \partial\Omega \\
u(0, x) = u_0(x) & x \in \Omega.
\end{cases}
\]

in the class
\[
C^1([0, \infty); L^2(\Omega)) \cap C([0, \infty); H^2(\Omega) \cap H^1_0(\Omega)).
\]
1.7 Asymptotic behaviour of $C_0$-semigroups

Let $S(t)$ be a $C_0$-semigroup of bounded linear operators on $X$.

**Definition 9** The number

$$\omega_0(S) = \inf_{t > 0} \frac{\log \|S(t)\|}{t}$$  \hspace{1cm} (1.7.1)

is called the type or growth bound of $S(t)$.

**Proposition 9** The growth bound of $S$ satisfies

$$\omega_0(S) = \lim_{t \to \infty} \frac{\log \|S(t)\|}{t} < \infty.$$  \hspace{1cm} (1.7.2)

Moreover, for any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\|S(t)\| \leq M_\varepsilon e^{(\omega_0(S)+\varepsilon)t} \quad \forall t \geq 0.$$  \hspace{1cm} (1.7.3)

**Proof.** The fact that $\omega_0(S) < \infty$ is a direct consequence of (1.7.1). In order to prove (1.7.2) it suffices to show that

$$\limsup_{t \to \infty} \frac{\log \|S(t)\|}{t} \leq \omega_0(S).$$  \hspace{1cm} (1.7.4)

For any $\varepsilon > 0$ let $t_\varepsilon > 0$ be such that

$$\frac{\log \|S(t_\varepsilon)\|}{t_\varepsilon} < \omega_0(S) + \varepsilon.$$  \hspace{1cm} (1.7.5)

Let us write any $t \geq t_\varepsilon$ as $t = nt_\varepsilon + \delta$ with $n = n(\varepsilon) \in \mathbb{N}$ and $\delta = \delta(\varepsilon) \in [0, t_\varepsilon]$. Then, by (1.2.2) and (1.7.5),

$$\|S(t)\| \leq \|S(\delta)\| \|S(t_\varepsilon)\|^n \leq Me^{\omega\delta} e^{n(\omega_0(S)+\varepsilon)} = Me^{(\omega-\omega_0(S)-\varepsilon)\delta} e^{(\omega_0(S)+\varepsilon)t}$$

which proves (1.7.3) with $M_\varepsilon = Me^{(\omega-\omega_0(S)-\varepsilon)\delta}$. Moreover, taking the logarithm of both sides of the above inequality we get

$$\frac{\log \|S(t)\|}{t} \leq \omega_0(S) + \varepsilon + \frac{\log M + (\omega - \omega_0(S) - \varepsilon)\delta}{t}$$

and (1.7.4) follows as $t \to \infty$. \hfill \Box

**Definition 10** For any operator $A : D(A) \subset X \to X$ we define the spectral bound of $A$ as

$$s(A) = \sup \{ \Re \lambda : \lambda \in \sigma(A) \}.$$  \hspace{1cm} (1.7.6)

**Corollary 3** Let $S(t)$ be a $C_0$-semigroup on $X$ with infinitesimal generator $A$. Then

$$-\infty \leq s(A) \leq \omega_0(S) < +\infty.$$  \hspace{1cm} (1.7.7)
Proof. By combining Theorem 5 and (1.7.3) we conclude that

$$\Pi_{\omega_0(S)+\varepsilon} \subset \rho(A) \quad \forall \varepsilon > 0.$$ 

Therefore, $$s(A) \leq \omega_0(S) + \varepsilon$$ for all $$\varepsilon > 0$$. The conclusion follows. \(\Box\)

**Example 10** For fixed $$T > 0$$ and $$p \geq 1$$ let $$X = L^p(0, T)$$ and

$$(S(t)f)(x) = \begin{cases} f(x-t) & x \in [t, T] \\ 0 & x \in [0, t) \end{cases} \quad \forall x \in [0, T], \forall t \geq 0.$$

Then $$S$$ is a $$C_0$$-semigroup of bounded linear operators on $$X$$ which satisfies $$\|S(t)\| \leq 1$$ for all $$t \geq 0$$. Moreover, observe that $$S$$ is nilpotent, that is, we have $$S(t) \equiv 0$$, $$\forall t \geq T$$. Deduce that $$\omega_0(S) = -\infty$$. So, the spectral bound of the infinitesimal generator of $$S(t)$$ also equals $$-\infty$$.

**Example 11** ($$-\infty < s(A) = \omega_0(S)$$) In the Banach space

$$X = C_b(\mathbb{R}^+; \mathbb{C})$$,

with the uniform norm, the left-translation semigroup

$$(S(t)f)(x) = f(x+t) \quad \forall x, t \geq 0$$

is a $$C_0$$-semigroup of contractions on $$X$$ which satisfies $$\|S(t)\| = 1$$ (Exercise). Therefore

$$\omega_0(S) = 0.$$

The infinitesimal generator of $$S(t)$$ is given by

$$\begin{cases} D(A) = C^1_b(\mathbb{R}^+; \mathbb{C}) \\ Af = f' \quad \forall f \in D(A). \end{cases}$$

By Theorem 5 we have that

$$\rho(A) \supset \{ \lambda \in \mathbb{C} : \Re \lambda > 0 \}.$$

We claim that

$$\sigma(A) \supset \{ \lambda \in \mathbb{C} : \Re \lambda \leq 0 \}.$$ 

Indeed, for any $$\lambda \in \mathbb{C}$$ the function $$f_\lambda(x) := e^{\lambda x}$$ satisfies $$\lambda f - f' = 0$$. Moreover, $$f_\lambda \in D(A)$$ for $$\Re \lambda \leq 0$$. Therefore

$$s(A) = 0.$$
Example 12 \((s(A) < \omega_0(S))\) Let us denote by \(C_0(\mathbb{R}_+; \mathbb{C})\) the Banach space of all continuous functions \(f : \mathbb{R}_+ \to \mathbb{C}\) such that
\[
\lim_{x \to \infty} f(x) = 0
\]
with the uniform norm. Let \(X\) be the space of all functions \(f \in C_0(\mathbb{R}_+; \mathbb{C})\) such that
\[
\|f\| := \sup_{x \in \mathbb{R}_+} |f(x)| + \int_0^\infty |f(x)|e^x dx < \infty.
\]

Exercise 13 Prove that \((X, \|\cdot\|)\) is a Banach space.

Once again, the left-translation semigroup
\[
(S(t)f)(x) = f(x + t) \quad \forall x, t \geq 0
\]
is a \(C_0\)-semigroup of contractions on \(X\). Indeed, for all \(t \geq 0\)
\[
\|S(t)f\| = \sup_{x \in \mathbb{R}_+} |f(x + t)| + \int_0^\infty |f(x + t)|e^x dx
\leq \sup_{x \in \mathbb{R}_+} |f(x)| + e^{-t} \int_0^\infty |f(x)|e^x dx.
\]

Exercise 14 Prove that \(\|S(t)\| = 1\) for all \(t \geq 0\)

Therefore
\[
\omega_0(S) = 0.
\]
The infinitesimal generator of \(S(t)\) is given by
\[
\begin{cases}
D(A) = \{ f \in X : f' \in X \} \\
Af = f' & \forall f \in D(A).
\end{cases}
\]

For any \(\lambda \in \mathbb{C}\) the function \(f_\lambda(x) := e^{\lambda x}\) satisfies \(\lambda f - f' = 0\) and \(f_\lambda \in D(A)\) for \(\Re \lambda < -1\). So,
\[
s(A) \geq -1. \quad (1.7.6)
\]

We claim that
\[
\rho(A) \supset \{ \lambda \in \mathbb{C} : \Re \lambda > -1 \}. \quad (1.7.7)
\]
Indeed, a change of variables shows that, for any \(g \in X\), the function
\[
f(x) = \int_0^\infty e^{-\lambda t} (S(t)g)(x)dt = \int_0^\infty e^{-\lambda t} g(x + t)dt \quad (x \geq 0)
\]
satisfies $\lambda f - f' = g$. Consequently, if we show that $f \in X$, then $f \in D(A)$ follows and so $\lambda \in \rho(A)$. To check that $f \in X$ observe that, for all $x \geq 0$, 

$$
|f(x)| \leq \int_0^\infty |e^{-\lambda t}g(x + t)| dt \\
= \int_0^\infty e^{-t\Re \lambda} |g(x + t)| e^{x + t} e^{-x - t} dt \\
= e^{-x} \int_0^\infty e^{-t(\Re \lambda)} e^{x + t} |g(x + t)| dt \\
\leq e^{-x} \int_x^\infty e^{s} |g(s)| ds
$$

which insures that $f \in C_0(\mathbb{R}_+; \mathbb{C})$. Furthermore, by (1.7.8) we compute 

$$
\int_0^\infty |f(x)| e^x dx \leq \int_0^\infty dx \int_0^\infty e^{-t(\Re \lambda)} e^{x + t} |g(x + t)| dt \\
= \int_0^\infty e^{-t(\Re \lambda)} dt \int_0^\infty e^{x + t} |g(x + t)| dx \\
\leq \int_0^\infty e^{-t(\Re \lambda)} dt \int_0^\infty e^{\tau} |g(\tau)| d\tau < \infty.
$$

From (1.7.6) and (1.7.7) it follows that $s(A) = -1 < 0 = \omega_0(S)$.

**Exercise 15** Let $S(t)$ be a $C_0$-semigroup of bounded linear operators on $X$. Prove that $\omega_0(S) < 0$ if and only if 

$$
\lim_{t \to +\infty} \|S(t)\| = 0. \quad (1.7.9)
$$

**Solution.** One only needs to show that (1.7.9) implies that $\omega_0(S) < 0$. Let $t_0 > 0$ be such that $\|S(t_0)\| < 1/e$. For any $t > 0$ let $n \in \mathbb{N}$ be the unique integer such that 

$$
nt_0 \leq t < (n + 1)t_0. \quad (1.7.10)
$$

Then 

$$
\|S(t)\| = \|S(nt_0)S(t - nt_0)\| \leq \frac{Me^{\omega(t - nt_0)}}{e^n} \leq \frac{Me^{\omega t_0}}{e^n}.
$$

Therefore, on account of (1.7.9), we conclude that 

$$
\frac{\log \|S(t)\|}{t} \leq \frac{\log (Me^{\omega t_0})}{t} - \frac{n}{t} \\
\leq \frac{\log (Me^{\omega t_0})}{t} - \left( \frac{1}{t_0} - \frac{1}{t} \right) \quad \forall t > 0.
$$

Taking the limit as $t \to +\infty$ we conclude that $\omega_0(S) < 0$. \qed
Exercise 16 Let $S(t)$ be the $C_0$-semigroup on $L^2(\Omega)$ associated with the initial-boundary value problem

$$
\begin{align}
\frac{\partial u}{\partial t} &= \Delta u & \text{in } [0, \infty) \times \Omega \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } [0, \infty) \times \partial \Omega \\
u(0, x) &= u_0(x) & x \in \Omega
\end{align}
$$

(1.7.11)

Show that $\omega_0(S) < 0$.

Solution. We know from Example 9 that the infinitesimal generator of $S(t)$ is the operator $A$ defined by

$$
D(A) = H^2(\Omega) \cap H^1_0(\Omega) \\
Au = \Delta u & \forall u \in D(A).
$$

For $u_0 \in D(A)$, let $u(t, x) = (S(t)u_0)(x)$. Then $u$ satisfies (1.7.11). So

$$
\frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} |u(t, x)|^2 dx \right) = -\frac{1}{2} \int_{\Omega} |Du(t, x)|^2 dx & \forall t > 0.
$$

Moreover, by Poincaré’s inequality we have that

$$
\int_{\Omega} |u(t, x)|^2 dx \leq c(\Omega) \int_{\Omega} |Du(t, x)|^2 dx.
$$

Therefore,

$$
\frac{d}{dt} |u(t)|^2 \leq -\frac{2}{c(\Omega)} |u(t)|^2
$$

which ensures, by Gronwall’s lemma, that

$$
|u(t)| \leq e^{-t/c(\Omega)} |u_0| & \forall t > 0.
$$

By a density argument, one concludes that the above inequality holds true for any $u_0 \in L^2(\Omega)$, so that $\omega_0(S) \leq -1/c(\Omega)$.

\hfill \Box

1.8 Strongly continuous groups

Definition 11 A strongly continuous group, or a $C_0$-group, of bounded linear operators on $X$ is a map $G: \mathbb{R} \to \mathcal{L}(X)$ with the following properties:

(a) $G(0) = I$ and $G(t + s) = G(t)G(s)$ for all $t, s \in \mathbb{R}$,

(b) for all $u \in X$

$$
\lim_{t \to 0} G(t)u = u.
$$

(1.8.1)
Definition 12 The infinitesimal generator of a $C_0$-group of bounded linear operators on $X$, $G(t)$, is the map $A : D(A) \subset X \to X$ defined by
\[
\begin{cases}
D(A) = \{ u \in X : \exists \lim_{t \to 0} \frac{S(t)u - u}{t} \} \\
Au = \lim_{t \to 0} \frac{S(t)u - u}{t} \quad \forall u \in D(A)
\end{cases}
\]

Theorem 7 Let $M \geq 1$ and $\omega \geq 0$. For a linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:

(a) $A$ is the infinitesimal generator of a $C_0$-group, $G(t)$, such that
\[
\|G(t)\| \leq Me^{\omega |t|} \quad \forall t \in \mathbb{R}.
\]

(b) $A$ and $-A$ are the infinitesimal generators of $C_0$-semigroups, $S_+(t)$ and $S_-(t)$ respectively, satisfying
\[
\|S_\pm(t)\| \leq Me^{\omega t} \quad \forall t \geq 0.
\]

(c) $A$ is closed, $D(A)$ is dense in $X$, and
\[
\rho(A) \supseteq \{ \lambda \in \mathbb{C} : |\Re \lambda| > \omega \}
\]
\[
\|R(\lambda, A)^k\| \leq \frac{M}{(|\Re \lambda| - \omega)^k} \quad \forall k \geq 1, \forall |\Re \lambda| > \omega
\]

Remark 4 Let $A$ and $S_\pm(t)$ be as in point (b) above. We claim that

(i) $S_+(t)S_-(s) = S_-(s)S_+(t)$ for all $s, t \geq 0$,

(ii) $S_+(t)^{-1} = S_-(t)$ for all $t \geq 0$.

Indeed, recall that
\[
S_+(t) = \lim_{n \to \infty} e^{tA_n}, \quad S_-(t) = \lim_{n \to \infty} e^{tB_n}
\]
where
\[
A_n = nAR(n, A), \quad B_n = -nAR(n, -A) = nAR(-n, A)
\]
are the Yosida approximations of $A$ and $-A$, respectively. Since $A_n$ and $B_m$ commute in view of (1.5.3), so do $e^{tA_n}$ and $e^{tB_n}$ and (i) holds true. Consequently,
\[
S(t) := S_+(t)S_-(t) \quad (t \geq 0)
\]
is also a $C_0$-semigroup and, for all $u \in D(A) = D(-A)$, we have that
\[
\frac{S(t)u - u}{t} = S_+(t) \frac{S_-(t)u - u}{t} + \frac{S_+(t)u - u}{t} \xrightarrow{t \to 0} -Au + Au = 0.
\]
So, $\frac{d}{dt} S(t)u = 0$ for all $t \geq 0$. Hence, $S(t)u = u$ for all $t \geq 0$ and $u \in D(A)$. By density, $S(t)u = u$ for all $x \in X$, which yields $S_+(t)^{-1} = S_-(t)$. □
Define, for all \( t \geq 0 \),
\[
S_+(t) = G(t) \quad \text{and} \quad S_-(t) = G(-t).
\]
Then it can be checked that \( S_{\pm}(t) \) is \( C_0 \)-semigroup satisfying (1.8.3). Moreover, observing that
\[
S_-(t)u - u |_{t} = G(-t)u - u \quad \text{and} \quad -G(t)G(t)u - u |_{t},
\]
it is easy to show that \( \pm A \) is the infinitesimal generator of \( S_{\pm}(t) \).

**Proof of (b) \( \Rightarrow \) (c)** By the Hille-Yosida theorem we conclude that \( A \) is closed, \( D(A) \) is dense in \( X \), and
\[
\rho(A) \supseteq \Pi_\omega = \{ \lambda \in \mathbb{C} : \Re \lambda > \omega \}
\]
\[
\| R(\lambda, A)^k \| \leq \frac{M}{(\Re \lambda - \omega)^k} \quad \forall k \geq 1, \forall \lambda \in \Pi_\omega.
\]
Since
\[
(\lambda I + A)^{-1} = -(-\lambda I - A)^{-1}, \quad (1.8.6)
\]
we have that \( -\rho(A) = \rho(-A) \supseteq \Pi_\omega \), or
\[
\rho(A) \supseteq -\Pi_\omega = \{ \lambda \in \mathbb{C} : \Re \lambda < -\omega \},
\]
and
\[
\| R(\lambda, A)^k \| = \| R(-\lambda, -A)^k \| \leq \frac{M}{(-\Re \lambda - \omega)^k} \quad \forall k \geq 1, \forall \lambda \in -\Pi_\omega. \quad \square
\]

**Proof of (c) \( \Rightarrow \) (a)** Recalling (1.8.6), by the Hille-Yosida theorem it follows that \( \pm A \) is the infinitesimal generator of a \( C_0 \)-semigroup, \( S_{\pm}(t) \), satisfying (1.8.3). For all \( u \in X \) define
\[
G(t)u = \begin{cases} 
S_+(t)u & (t \geq 0) \\
S_-(t)u & (t < 0).
\end{cases}
\]
Then, it follows that (1.8.1) and (1.8.2) hold true, and \( A \) is the infinitesimal generator of \( G(t) \). Let us check that \( G(t + s) = G(t)G(s) \) for all \( t \geq 0 \) and all \( s \leq 0 \) such that \( t + s \geq 0 \). Recalling point (ii) of Remark 4, we have that
\[
G(t)G(s) = S_+(t)S_-(s) = S_+(t + s)S_+(s)S_+(s)^{-1} = G(t + s). \quad \square
\]
1.9 Additional exercises

Exercise 17 Let $S$ be $C_0$-semigroup of bounded linear operators on $X$ and let $K \subset X$ be compact. Prove that for every $t_0 \geqslant 0$

$$\lim_{t \to t_0} \sup_{u \in K} |S(t)u - S(t_0)u| = 0. \quad (1.9.1)$$

Solution. We may assume $S \in G(M, 0)$ for some $M > 0$ without loss of generality. Let $t_0 > 0$ and fix any $\varepsilon > 0$. Since $K$ is totally bounded, there exist $u_1, \ldots, u_{N_\varepsilon} \in X$ such that

$$K \subset \bigcup_{n=1}^{N_\varepsilon} B\left(u_n, \frac{\varepsilon}{M}\right).$$

Moreover, there exists $\tau > 0$ such that

$$|t - t_0| < \tau \implies |S(t)u_n - S(t_0)u_n| < \varepsilon \quad \forall n = 1, \ldots, N_\varepsilon.$$

Thus, for all $|t - t_0| < \tau$ we have that, if $u \in K$ is such that $u \in B\left(u_n, \frac{\varepsilon}{M}\right)$, then

$$|S(t)u - S(t_0)u| \leqslant |S(t)u - S(t)u_n| + |S(t_0)u_n - S(t_0)u|$$

$$\leqslant 2M|u - u_n| + \varepsilon < 3\varepsilon.$$

So, the limit of $|S(t)u - S(t_0)u|$ as $t \to t_0$ is uniform on $K$. \hfill \Box

Exercise 18 Let $A : D(A) \subset X \to X$ be a closed operator satisfying (1.6.2) but suppose $D(A)$ fails to be dense in $X$. In the Banach space $Y := D(A)$, define the operator $B$, called the part of $A$ in $Y$, by

$$\begin{cases} D(B) = \{u \in D(A) : Au \in Y\} \\ Bu = Au \quad \forall u \in D(B). \end{cases}$$

Prove that $B$ is the infinitesimal generator of a $C_0$-semigroup on $Y$.

Solution. $R(\lambda, A)(Y) \subset D(B)$ for all $\lambda \in \mathbb{C}$ such that $\Re \lambda > \omega$. Indeed, owing to (1.5.1) for all $u \in D(A)$ we have that

$$\lim_{n \to \infty} nR(n, A)u = \lim_{n \to \infty} \{R(n, A)Au + u\} = u. \quad (1.9.2)$$

Since $\|nR(n, A)\|$ is bounded, (1.9.2) holds true for all $u \in Y$. Hence, $D(B)$ is dense in $Y$. So, $B$ satisfies in $Y$ all the assumptions of Theorem 6. \hfill \Box
Exercise 19 Let $X$ be a Banach space and let $A : D(A) \subset X \to X$ be the infinitesimal generator of a uniformly bounded semigroup. Define, for $n \geq 1,$

$$D(A^n) := \{ u \in D(A^{n-1}) : A^{n-1}u \in D(A) \}.$$ 

(i) Prove the following extension of the Landau-Kolmogorov inequality (1.3.1):

$$|A^k u| \leq (2M)^{k(n-k)} |A^n u|^\frac{k}{n} |u|^{\frac{n-k}{n}} \quad \forall u \in D(A^n), \forall 0 \leq k \leq n \quad (1.9.3)$$

**Solution:** proceed by induction. The conclusion is trivial for $n = 1.$ Assume (1.9.3) holds true for $n$ and let $u \in D(A^{n+1}).$ Then, in view of (1.3.1), we have that

$$|A^n u| \leq 2M |A^{n+1} u|^\frac{1}{2} |A^{n-1} u|^\frac{1}{2} \leq 2M |A^{n+1} u|^\frac{1}{2} \left( (2M)^n |A^n u|^\frac{2-n}{n} |u|^\frac{1}{n} \right)^\frac{1}{2} = (2M)^{\frac{n+1}{2}} |A^{n+1} u|^\frac{1}{2} |A^n u|^\frac{n-1}{2n} |u|^\frac{1}{2n}.$$

Therefore,

$$|A^n u| \leq (2M)^n |A^{n+1} u|^\frac{1}{n+1} |u|^{\frac{1}{n+1}}, \quad (1.9.4)$$

which is (1.9.3) for $n + 1$ with $k = n.$ Now, suppose $0 \leq k < n.$ Then, by our inductive assumption and (1.9.4),

$$|A^k u| \leq (2M)^{k(n-k)} |A^n u|^\frac{k}{n} |u|^{\frac{n-k}{n}} \leq (2M)^{k(n-k)} \left( (2M)^n |A^{n+1} u|^\frac{2-n}{n} |u|^\frac{1}{n+1} \right)^\frac{k}{n} |u|^{\frac{n-k}{n}} = (2M)^{k(n+1-k)} |A^{n+1} u|^\frac{k}{n+1} |u|^{\frac{n+1-k}{n+1}}.$$

The proof is complete. □

(ii) Using (1.9.3), prove that for every $n \geq 1$:

(a) $A^n$ is a closed operator.

**Solution:** proceed by induction. The conclusion is trivial for $n = 1.$ Assume it holds true for $n$ and let $\{u_k\} \subset D(A^{n+1})$ be such that

$$u_k \to u \quad \& \quad A^{n+1} u_k \to v \quad (k \to \infty).$$

Applying (1.9.4) to $w_k := A^n u_k \in D(A)$ we obtain

$$|w_k - w_h| \leq (2M)^n |A^{n+1}(u_k - u_h)|^{\frac{1}{n+1}} |u_k - u_h|^{\frac{1}{n+1}} \to 0 \quad (h, k \to \infty)$$

Therefore, for some $w \in X,$

$$w_k \to w \quad \& \quad Aw_k \to v \quad (k \to \infty).$$
Since $A$ is closed, we conclude that
\[ w \in D(A) \quad \& \quad Aw = v \quad (k \to \infty). \]

Then, by our inductive assumption, $u \in D(A^n)$ and $A^n u = w$, which implies in turn
\[ u \in D(A^{n+1}) \quad \& \quad A^{n+1} u = A w = v \quad (k \to \infty). \] □

(b) $D(A^n)$ is dense in $X$ for every $n \geq 1$.

Solution of (ii)(b): for $n = 1$ the conclusion follows from Theorem 3. Let the conclusion be true for some $n \geq 1$ and fix any $v \in X$. Then, for any $\varepsilon > 0$ there exists $u_\varepsilon \in D(A^n)$ such that $|u_\varepsilon - v| < \varepsilon$. Moreover, recalling point (a),
\[ A^n \left( \frac{1}{t} \int_0^t S(s) u_\varepsilon \, ds \right) = \frac{1}{t} \int_0^t S(s) A^n u_\varepsilon \, ds \]

Since
\[ \frac{1}{t} \int_0^t S(s) A^n u_\varepsilon \, ds \in D(A) \quad \forall t > 0 \]
we conclude that
\[ \frac{1}{t} \int_0^t S(s) u_\varepsilon \, ds \in D(A^{n+1}) \quad \forall t > 0. \]

Moreover, there exists $t_\varepsilon > 0$ such that
\[ \left| \frac{1}{t_\varepsilon} \int_0^{t_\varepsilon} S(s) u_\varepsilon \, ds - v \right| \leq \frac{1}{t_\varepsilon} \int_0^{t_\varepsilon} S(s) u_\varepsilon \, ds - u_\varepsilon + |u_\varepsilon - v| < 2\varepsilon. \] □

Generalize to the infinitesimal generator of a $C_0$-semigroup of bounded linear operators on $X$.

Exercise 20 Let $p \geq 2$. On $X = L^p(0, \pi)$ consider the operator defined by
\[ \begin{align*}
D(A) &= W^{2,p}(0, \pi) \cap W^{1,p}_0(0, \pi) \\
Af(x) &= f''(x) \quad x \in (0, \pi) \text{ a.e.}
\end{align*} \tag{1.9.5} \]

where
\[ W^{1,p}_0(0, \pi) = \{ f \in W^1_p(0, \pi) : f(0) = 0 = f(\pi) \}. \]

Since $C_c^\infty(0, \pi) \subset D(A)$, we have that $D(A)$ is dense in $X$. Show that $A$ is closed and satisfies condition $(a')$ of Remark 2 with $M = 1$ and $\omega = 0$. Theorem 6 will imply that $A$ generates a $C_0$-semigroup of contractions on $X$.

Solution. Step 1: $\sigma(A) = \{-n^2 : n \in \mathbb{N}\}$. 

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Fix any \( g \in X \). We will show that, for all \( \lambda \neq n^2 (n \geq 1) \), the Sturm-Liouville system
\[
\begin{cases}
\lambda f(x) - f''(x) = g(x), & 0 < x < \pi \\
f(0) = 0 = f(\pi)
\end{cases}
\quad (1.9.6)
\]
admits a unique solution \( f \in D(A) \). Denoting by
\[
g(x) = \sum_{n=1}^{\infty} g_n \sin(nx) \quad (x \in [0, \pi])
\]
the Fourier series of \( g \), we seek a candidate solution \( f \) of the form
\[
f(x) = \sum_{n=1}^{\infty} f_n \sin(nx) \quad (x \in [0, \pi]).
\]
In order to satisfy (1.9.6) one must have
\[(\lambda + n^2) f_n = g_n \quad \forall n \geq 1.
\]
So, for any \( \lambda \neq -n^2 \), (1.9.6) has a unique solution given by
\[
f(x) = \sum_{n=1}^{\infty} \frac{g_n}{\lambda + n^2} \sin(nx) \quad (x \in [0, \pi]).
\]
From the above representation it follows that \( f \in H^2(0, \pi) \cap H^1_0(0, \pi) \). In fact, returning to the equation in (1.9.6) one concludes that \( f \in D(A) \).

Step 2: resolvent estimate.

By multiplying both members of the equation in (1.9.6) by \(|f|^{p-2} f\) and integrating over \((0, \pi)\) one obtains, for all \( \lambda > 0 \),
\[
\lambda \int_0^{\pi} |f(x)|^p dx + (p - 1) \int_0^{\pi} |f(x)|^{p-2}|f'(x)|^2 dx = \int_0^{\pi} g(x)|f(x)|^{p-2} f(x) dx
\]
which yields
\[
|f|_p \leq \frac{1}{\lambda} |g|_p \quad \forall \lambda > 0.
\]

Step 3: conclusion.

By Proposition 6 we conclude that for each \( f \in W^{2,p}(0, \pi) \cap W^{1,p}_0(0, \pi) \) the unique solution of
\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} (t, x) = \frac{\partial^2 u}{\partial x^2} (t, x) \quad (t, x) \in \mathbb{R}_+ \times (0, \pi) \\
u(t, 0) = 0 = u(t, \pi) \quad t \geq 0 \\
u(0, x) = f(x) \quad x \in (0, \pi)
\end{cases}
\]
is given by \( u(t, x) = (S(t)f)(x) \). \(\square\)
Exercise 21. Let $S(t)$ be the $C_0$-semigroup generated by operator $A$ in (1.9.5). Prove that, for any $f \in L^p(0, \pi)$,

$$(S(t)f)(x) = \int_0^\pi K(t, x, y)f(y) \, dy, \quad \forall t \geq 0, \ x \in (0, \pi) \ a.e.$$ 

where

$$K(t, x, y) = \frac{2}{\pi} \sum_{k=1}^\infty e^{-k^2t} \sin(kx) \sin(ky).$$

Exercise 22. On $X = \{f \in C([0, \pi]) : f(0) = 0 = f(\pi)\}$ with the uniform norm, consider the linear operator $A : D(A) \subset X \to X$ defined by

$$D(A) = \{ f \in C^2([0, 1]) : f(0) = f(\pi) = 0 = f''(0) = f''(\pi) \}$$

and $Af = f''$. Let

$$A \{ \text{for all } f \in D(A) \}.$$ 

Show that $A$ generates a $C_0$-semigroup of contractions on $X$ and derive the initial-boundary value problem which is solved by such semigroup.

Solution. We only prove that $\|R(\lambda, A)\| \leq 1/\lambda$ for all $\lambda > 0$. Fix any $g \in X$ and let $f = R(\lambda, A)g$. Let $x_0 \in [0, \pi]$ be such that $|f(x_0)| = |f|_\infty$. If $f(x_0) > 0$, then $x_0 \in (0, \pi)$ is a maximum point of $f$. So, $f''(x_0) \leq 0$ and we have that

$$\lambda |f|_\infty = \lambda f(x_0) \leq \lambda f(x_0) - f''(x_0) = g(x_0) \leq \|g\|_\infty.$$ 

On the other hand, if $f(x_0) < 0$, then $x_0 \in (0, \pi)$ once again and $x_0$ is a minimum point of $f$. Thus, $f''(x_0) \geq 0$ and

$$\lambda |f|_\infty = -\lambda f(x_0) \leq -\lambda f(x_0) + f''(x_0) = -g(x_0) \leq \|g\|_\infty.$$ 

In any case, we have that $\lambda |f|_\infty \leq \|g\|_\infty$. \qed

Exercise 23. Let $(X, \| \cdot \|)$ be a separable Banach space and let $A : D(A) \subset X \to X$ be a closed operator with $\rho(A) \neq \emptyset$. Prove that $(D(A), \| \cdot \|_{D(A)})$ is also separable.

Solution. Let $\{u_n\}_{n \in \mathbb{N}}$ be dense in $X$ and let $\lambda_0 \in \rho(A)$. Fix any $v \in D(A)$ and set $w = \lambda_0 v - Av$. For arbitrary $\varepsilon > 0$ let $u_\varepsilon = u_{n_\varepsilon}$ be such that $|w - u_\varepsilon| < \varepsilon$. Then

$$|v - R(\lambda_0, A)u_\varepsilon| = |R(\lambda_0, A)(w - u_\varepsilon)| \leq \|R(\lambda_0, A)\| \varepsilon.$$ 

Moreover,

$$|Av - AR(\lambda_0, A)u_\varepsilon| = |AR(\lambda_0, A)(w - u_\varepsilon)|$$

$$\leq |\lambda_0 R(\lambda_0, A)(w - u_\varepsilon)| + |w - u_\varepsilon| \leq (\|\lambda_0\| R(\lambda_0, A)\| + 1)\varepsilon.$$ 

This shows that $\{R(\lambda_0, A)u_n\}_{n \in \mathbb{N}}$ is dense in $D(A)$. \qed
2 Dissipative operators

2.1 Definition and first properties

Let $H$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$.

**Definition 13** We say that an operator $A : D(A) \subset H \to H$ is dissipative if
\[
\Re \langle Au, u \rangle \leq 0 \quad \forall u \in D(A).
\] (2.1.1)

**Example 13** In $H = L^2(\mathbb{R}^+; \mathbb{C})$ consider the operator
\[
\left\{
\begin{array}{l}
D(A) = H^1(\mathbb{R}^+; \mathbb{C}) \\
Af(x) = f'(x) \quad x \in \mathbb{R}^+ \text{ a.e.}
\end{array}
\right.
\]
Then
\[
2\Re \langle Af, f \rangle = 2\Re \left( \int_0^\infty f'(x) \overline{f(x)} \, dx \right) \geq \int_0^\infty \frac{d}{dx} |f(x)|^2 \, dx = -|f(0)|^2 \leq 0.
\]
So, $A$ is dissipative.

**Proposition 10** An operator $A : D(A) \subset H \to H$ is dissipative if and only if for any $u \in D(A)$
\[
|\lambda I - A|u| \geq \lambda |u| \quad \forall \lambda > 0.
\] (2.1.2)

**Proof.** Let $A$ be dissipative. Then, for every $u \in D(A)$ we have that
\[
|\lambda I - A|u|^2 = \lambda^2 |u|^2 - 2\lambda \Re \langle Au, u \rangle + |Au|^2 \geq \lambda^2 |u|^2 \quad \forall \lambda > 0.
\]
Conversely, suppose $A$ satisfies (2.1.2). Then for every $\lambda > 0$ and $u \in D(A)$
\[
\lambda^2 |u|^2 - 2\lambda \Re \langle Au, u \rangle + |Au|^2 = |\lambda I - A|u|^2 \geq \lambda^2 |u|^2
\]
So, $2\lambda \Re \langle Au, u \rangle \leq |Au|^2$ which in turn yields (2.1.1) as $\lambda \to \infty$. \hfill \Box

The above characterization can be used to extend the notion of dissipative operators to a Banach space $X$.

**Definition 14** We say that an operator $A : D(A) \subset X \to X$ is dissipative if
\[
|\lambda I - A|u| \geq \lambda |u| \quad \forall u \in D(A) \quad \text{and} \quad \forall \lambda > 0.
\] (2.1.3)

**Remark 5** It follows from (2.1.3) that, if $A$ is dissipative then
\[
\lambda I - A : D(A) \to X
\]
is injective for all $\lambda > 0$. 37
Proposition 11 Let $A : D(A) \subset X \to X$ be dissipative. If
\[ \exists \lambda_0 > 0 \quad \text{such that} \quad (\lambda_0 I - A)D(A) = X, \] (2.1.4)
then the following properties hold:

(a) $\lambda_0 \in \rho(A)$ and $\|R(\lambda_0, A)\| \leq 1/\lambda_0$,

(b) $A$ is closed,

(c) $(\lambda I - A)D(A) = X$ and $\|R(\lambda, A)\| \leq 1/\lambda$ for all $\lambda > 0$.

Proof. We observe that point (a) follows from Remark 5 and inequality (2.1.3). As for point (b), we note that, since $R(\lambda_0, A)$ is closed, $\lambda_0 I - A$ is also closed, and therefore $A$ is closed.

Proof of (c). By point (a) the set \[ \Lambda = \{ \lambda \in ]0, \infty[ : (\lambda I - A)D(A) = X \} \]
is contained in $\rho(A)$ which is open in $\mathbb{C}$. This implies that $\Lambda$ is also open. Let us show that $\Lambda$ is closed: let $\Lambda \ni \lambda_n \to \lambda > 0$ and fix any $v \in X$. There exists $u_n \in D(A)$ such that
\[ \lambda_n u_n - Au_n = v. \] (2.1.5)

From (2.1.2) it follows that $|u_n| \leq |v|/\lambda_n \leq C$ for some $C > 0$. Again by (2.1.2),
\[ \lambda_n |u_n - u_m| \leq |\lambda_n(u_n - u_m) - A(u_n - u_m)| \]
\[ \leq |\lambda_n - \lambda_m| |u_n| + |\lambda_n u_n - Au_n - (\lambda_m u_m - Au_m)| \]
\[ \leq C|\lambda_m - \lambda_n|. \]

Therefore $\{u_n\}$ is a Cauchy sequence. Let $x \in X$ be such that $u_n \to u$. Then $Au_n \to \lambda u - v$ by (2.1.5). Since $A$ is closed, $u \in D(A)$ and $\lambda u - Au = v$. This show that $\lambda I - A$ is surjective and implies that $\lambda \in \Lambda$. Thus, $\Lambda$ is both open and closed in $]0, \infty[$. Moreover, $\Lambda \neq \emptyset$ because $\lambda_0 \in \Lambda$. So, $\Lambda = ]0, \infty[$. The inequality $\|R(\lambda, A)\| \leq 1/\lambda$ is a consequence of dissipativity. \hfill $\square$

2.2 Maximal dissipative operators

Definition 15 A dissipative operator $A : D(A) \subset X \to X$ is called maximal dissipative if $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$ (hence, for all $\lambda > 0$).

Remark 6 Let $A : D(A) \subset X \to X$ be a maximal dissipative operator and let $\overline{A} \supset A$ be a dissipative extension of $A$. Then:

(i) $\overline{A}$ is maximal dissipative ($\lambda I - \overline{A}$ is surjective since so is $\lambda I - A$);
(ii) \( \overline{A} = A \) (since both \( \rho(A) \) and \( \rho(\overline{A}) \) contain \( ]0, \infty[ \)).

**Theorem 8** Let \( X \) be a reflexive Banach space. If \( A : D(A) \subset X \to X \) is a maximal dissipative operator, then \( D(A) \) is dense in \( X \).

We give the proof for a Hilbert space. The case of a reflexive Banach space is treated in exercises 24 to 27.

**Proof.** Let \( v \in X \) be such that \( \langle v, u \rangle = 0 \) for all \( u \in D(A) \). We will show that \( v = 0 \), or
\[
\langle v, w \rangle = 0 \quad \forall w \in X.
\]

Since \((I - A)\) is surjective, the above is equivalent to
\[
0 = \langle v, u - Au \rangle \quad \forall u \in D(A).
\]

So, we need to prove that
\[
\langle v, u \rangle = 0 \quad \forall u \in D(A) \quad \implies \quad \langle v, Au \rangle = 0 \quad \forall u \in D(A). \tag{2.2.1}
\]

Let \( u \in D(A) \). Since \( nI - A \) is onto, there exists a sequence \( \{u_n\} \subset D(A) \) such that
\[
uu = nu_n - Au_n \quad \forall n \geq 1. \tag{2.2.2}
\]

Since \( Au_n = n(u_n - u) \in D(A) \), we have that \( u_n \in D(A^2) \) and
\[
Au = Au_n - \frac{1}{n} A^2 u_n \quad \text{or} \quad Au_n = \left( I - \frac{1}{n} A \right)^{-1} Au.
\]

Since \( \|(I - \frac{1}{n} A)^{-1}\| \leq 1 \) by (2.1.2), the above identity yields \( |Au_n| \leq |Au| \).

So, by (2.2.2) we obtain
\[
|u_n - u| \leq \frac{1}{n} |Au|.
\]

Therefore, \( u_n \to u \). Moreover, since \( \{Au_n\} \) is bounded, there is a subsequence \( Au_{n_k} \) such that \( Au_{n_k} \to w \). Since \( A \) is closed, \( graph(A) \) is a closed subspace of \( X \times X \). Then, \( graph(A) \) is also weakly closed and we have that \( w = Au \). Therefore,
\[
\langle v, Au \rangle = \lim_{k \to \infty} \langle v, Au_{n_k} \rangle = \lim_{k \to \infty} n_k \langle v, u_{n_k} - u \rangle
\]

and (2.2.1) follows from the vanishing of the rightmost term above. \( \square \)

**Example 14** We now show that the above density may be fail in a general Banach space. On \( X = C([0, 1]) \) with the uniform norm \( \| \cdot \|_{\infty} = \| \cdot \|_{\infty,[0,1]} \), consider the linear operator \( A : D(A) \subset X \to X \) defined by
\[
\begin{cases}
D(A) = \{ u \in C^1([0, 1]) : u(0) = 0 \} \\
Au(x) = -u'(x) \quad \forall x \in [0, 1].
\end{cases}
\]

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Then, for all $\lambda > 0$ and $f \in X$ we have that the equation $\lambda u - Au = f$ has the unique solution $u \in D(A)$ given by

$$u(x) = \int_0^x e^{\lambda(y-x)} f(y) \, dy \quad (x \in [0,1])$$

Therefore, $\lambda I - A$ is onto. Moreover,

$$\lambda |u(x)| \leq \int_0^x \lambda e^{\lambda(y-x)} \|f\|_\infty \, dy = (1 - e^{-\lambda x}) \|f\|_\infty \leq \|\lambda u - Au\|_\infty.$$ 

So, $A$ is dissipative. On the other hand, $D(A)$ is not dense in $X$ because all functions in $D(A)$ vanish at $x = 0$.

**Exercise 24** We recall that the duality set of a point $x \in X$ is defined as

$$\Phi(x) = \{ \phi \in X^* : \langle x, \phi \rangle = |x|^2 = \|\phi\|^2 \}.$$  \hspace{1cm} (2.2.3)

Observe that the Hahn-Banach theorem ensures $\Phi(x) \neq \emptyset$.

We also recall that, for all $x \in X$,

$$\partial |x| = \{ \phi \in X^* : |x + h| - |x| \geq \langle h, \phi \rangle, \forall x, h \in X \}.$$ \hspace{1cm} (2.2.4)

Prove that

$$\Phi(x) = x \partial |x| = \{ \psi \in X^* : \psi = |x| \phi, \phi \in \partial |x| \}.$$

**Exercise 25** Prove that, for any operator $A : D(A) \subset X \to X$ the following properties are equivalent:

(a) $A$ is dissipative,

(b) for all $x \in D(A)$ there exists $\phi \in \Phi(x)$ such that $\Re \langle Ax, \phi \rangle \leq 0$.

**Exercise 26** Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $C_0$-semigroup of contractions. Prove that, for all $x \in D(A)$,

$$\Re \langle Ax, \phi \rangle \leq 0 \quad \forall \phi \in \Phi(x).$$

**Exercise 27** Mimic the proof of Theorem 8 to treat the general case of a reflexive Banach space.

**Theorem 9 (Lumer-Phillips 1)** Let $A : D(A) \subset X \to X$ be a densely defined linear operator. Then the following properties are equivalent:

(a) $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions,

(b) $A$ is maximal dissipative.
Proof of (a) ⇒ (b) In view of Theorem 5, we have that $]0, \infty[ \subset \rho(A)$. So, 
$(\lambda I - A)D(A) = X$ for all $\lambda > 0$. Moreover, by the Hille-Yosida theorem for 
all $\lambda > 0$ and $v \in X$ we have that $\lambda |R(\lambda, A)v| \leq |v|$ or, setting $u = R(\lambda, A)v$,

$$\lambda |u| \leq |(\lambda I - A)u| \quad \forall u \in D(A).$$

So, $A$ is maximal dissipative.

Proof of (b) ⇒ (a) We have that:

(i) $D(A)$ is dense by hypothesis,

(ii) $A$ is closed by Proposition 11-(b),

(iii) $]0, \infty[ \subset \rho(A)$ and $\| R(\lambda, A) \| \leq 1/\lambda$ for all $\lambda > 0$ by Proposition 11-(c).

The conclusion follows by the Hille-Yosida theorem.

Example 15 (Wave equation in $L^2(\Omega)$) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^2$. For any given $f \in H^2(\Omega) \cap H^1_0(\Omega)$ and $g \in H^1_0(\Omega)$, consider the problem

$$\begin{cases}
\frac{\partial^2 u}{\partial t^2}(t, x) = \Delta u & \text{in } ]0, \infty[ \times \Omega \\
u = 0 & \text{on } ]0, \infty[ \times \partial \Omega \\
u(0, x) = f(x), \frac{\partial u}{\partial t}(0, x) = g(x) & x \in \Omega
\end{cases} \quad (2.2.5)
$$

Let $H$ be the Hilbert space $H^1_0(\Omega) \times L^2(\Omega)$ with the scalar product

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \rangle = \int_{\Omega} (Du(x) \cdot D\bar{u}(x) + v(x)\bar{v}(x)) dx.$$

Define $A : D(A) \subset H \to H$ by

$$\begin{cases}
D(A) = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \\
A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \end{pmatrix}
\end{cases} \quad (2.2.6)
$$

We will show that $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions on $H$ by checking that $A$ is maximal dissipative.

Let $\begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$. Then, integrating by parts we obtain

$$\langle A \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle = \int_{\Omega} (Du(x) \cdot Dv(x) + v(x)\Delta u(x)) dx = 0. \quad (2.2.7)$$

So, $A$ is dissipative.
Now, consider the resolvent equation

\[
\begin{aligned}
&\begin{pmatrix} u \\ v \end{pmatrix} \in D(A) \\
&\begin{pmatrix} u \\ v \end{pmatrix} - (I - A) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \in H
\end{aligned}
\] (2.2.8)

which is equivalent to the system

\[
\begin{aligned}
&u \in H^2(\Omega) \cap H^1_0(\Omega), \quad v \in H^1_0(\Omega) \\
&u - v = f \in H^1_0(\Omega) \\
&v - \Delta u = g \in L^2(\Omega).
\end{aligned}
\] (2.2.9)

Using elliptic theory (see, for instance, [3]) one can show that the boundary value problem

\[
\begin{aligned}
&u \in H^2(\Omega) \cap H^1_0(\Omega), \\
&u - \Delta u = f + g \in L^2(\Omega)
\end{aligned}
\]

has a unique solution. Then, taking \( v = u - f \in H^1_0(\Omega) \) we obtain the unique solution of problem (2.2.9). So, \( A \) is maximal dissipative and therefore \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions, \( S(t) \).

For any \( f \in H^2(\Omega) \cap H^1_0(\Omega), g \in H^1_0(\Omega) \), let \( u(t) \ (t \in \mathbb{R}_+) \) be the first component of

\[ S(t) \begin{pmatrix} f \\ g \end{pmatrix} \]

Then \( u \) is the unique solution of problem (3.2.4) in the space

\[ C^2(\mathbb{R}_+; L^2(\Omega)) \cap C^1(\mathbb{R}_+; H^1_0(\Omega)) \cap C(\mathbb{R}_+; H^2(\Omega) \cap H^1_0(\Omega)). \]

**Example 16** Consider the age-structured population model

\[
\begin{aligned}
&\frac{\partial u}{\partial t}(t, a) + \frac{\partial u}{\partial a}(t, a) + \mu(a)u(t, a) = 0, \quad a \in [0, a_1], t \geq 0 \\
u(t, 0) = \int_0^{a_1} \beta(a)u(t, a) \, da, \quad t \geq 0 \\
u(0, a) = u_0(a), \quad a \in [0, a_1].
\end{aligned}
\] (2.2.10)

which was proposed in [5]. Here, \( u(t, a) \) is the population density of age \( a \) at time \( t \), \( \mu \) is the mortality rate, \( \beta \) the birth rate, and \( a_1 > 0 \) is the maximal age. We assume that \( \mu, \beta \in C([0, a_1]), \mu, \beta \geq 0 \), and

\[
\int_0^{a_1} \beta(a)e^{-\int_0^a \mu(\rho) \, d\rho} \, da < 1.
\] (2.2.11)
In order to recast problem (2.2.10) as an evolution equation in \( H = L^2(0,a_1) \), we define the linear operator

\[
\begin{align*}
D(A) &= \left\{ u \in H^1(0,a_1) : u(0) = \int_0^{a_1} \beta(a) u(a) \, da \right\} \\
Au(a) &= -u'(a) - \mu(a) u(a) \quad (a \in [0,a_1] \text{ a.e.}).
\end{align*}
\]

(2.2.12)

We now proceed to show the following:

1. \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup on \( H \).

2. \( \rho(A) \supset [0, +\infty) \) and, for any \( \lambda > 0 \),

\[
R(\lambda, A)u(a) = \frac{U(a,0)}{1 - \int_0^{a_1} \beta(a) U(a,0) \, da} \int_0^{a_1} \beta(a) \, da \int_0^a U(a,s) u(s) \, ds
\]

\[+ \int_0^a U(a,s) u(s) \, ds, \quad a \in [0,a_1], \; u \in H, \quad (2.2.13)\]

where

\[
U(a,s) = e^{-\lambda(a-s)} - \int_s^a \mu(\rho) \, d\rho, \quad a, \; s \in [0,a_1].
\]

(2.2.14)

**Proof.** Given \( \lambda \geq 0 \) and \( v \in H \) we consider the equation

\[
\lambda u - Au = v,
\]

(2.2.15)

which is equivalent to

\[
\begin{align*}
(\lambda + \mu)u + u' &= v, \\
u(0) &= \int_0^{a_1} \beta(a) u(a) \, da.
\end{align*}
\]

(2.2.16)

If \( u \) is a solution of Eq. (2.2.15), then

\[
u(a) = U(a,0)u(0) + \int_0^a U(a,s) v(s) \, ds,
\]

(2.2.17)

where \( U \) is given by Eq. (2.2.14). Multiplying Eq. (2.2.17) by \( \beta \) and integrating with respect to \( a \) over \([0,a_1]\) yields

\[
u(0) = \int_0^{a_1} \beta(a) u(a) \, da
\]

\[
= \left( \int_0^{a_1} \beta(a) U(a,0) \, da \right) u(0) + \int_0^{a_1} \beta(a) \, da \int_0^a U(a,s) v(s) \, ds. \quad (2.2.18)
\]

From Eq. (2.2.11), we have

\[
\int_0^{a_1} \beta(a) U(a,0) \, da < 1, \quad \forall \; a \in [0,a_1];
\]

(2.2.19)
then, also from Eq. (2.2.17),
\[ u(0) = \frac{1}{1 - \int_0^{a_1} \beta(a) U(a,0) da} \int_0^{a_1} \beta(a) da \int_0^{a} U(a,s)v(s) ds. \tag{2.2.20} \]

Consequently, \( u(a) = R(\lambda, A)v(a) \) is given by Eq. (2.2.13).

Conversely, given \( v \in H \), the function
\[ u(a) = \frac{U(a,0)}{1 - \int_0^{a_1} \beta(a) U(a,0) da} \int_0^{a_1} \beta(a) da \int_0^{a} U(a,s)v(s) ds \]
\[ + \int_0^{a} U(a,s)v(s) ds, \quad a \in [0, a_1], \]
fulfills Eq. (2.2.15). □

3. For all \( u \in D(A) \)
\[ \langle Au, u \rangle \leq -\frac{1}{2} \int_0^{a_1} u^2(a) \left( 2\mu(a) - \int_0^{a_1} \beta^2(s) ds \right) da - \frac{1}{2} u(a_1)^2. \tag{2.2.21} \]

Consequently, if
\[ 2\mu(a) \geq \int_0^{a_1} \beta^2(s) ds, \quad \forall a \in [0, a_1], \tag{2.2.22} \]
then \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions on \( H \).

Proof. To show Eq. (2.2.21), we observe that, for all \( u \in D(A) \),
\[ \langle Au, u \rangle = -\int_0^{a_1} u'(a) u(a) da - \int_0^{a_1} \mu(a) u^2(a) da \]
\[ = \frac{1}{2} u(0)^2 - \frac{1}{2} u(a_1)^2 - \int_0^{a_1} \mu(a) u^2(a) da \]
\[ = \frac{1}{2} \left( \int_0^{a_1} \beta(a) u(a) da \right)^2 - \frac{1}{2} u(a_1)^2 - \int_0^{a_1} \mu(a) u^2(a) da. \]

So, by Hölder’s inequality,
\[ \langle Au, u \rangle \leq \frac{1}{2} \int_0^{a_1} \beta^2(a) da \int_0^{a_1} u^2(s) ds - \frac{1}{2} u(a_1)^2 - \int_0^{a_1} \mu(a) u^2(a) da \]
\[ = -\frac{1}{2} \int_0^{a_1} u^2(a) \left( 2\mu(a) - \int_0^{a_1} \beta^2(s) ds \right) da - \frac{1}{2} u(a_1)^2. \]

This shows that \( A \) is maximal dissipative if (2.2.22) is satisfied. In this case, the Lumer-Phillips theorem insures that \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions. □

When \( A \) and \( -A \) are maximal dissipative a stronger conclusion holds true.
**Corollary 4** Let $A : D(A) \subset X \to X$ be a densely defined linear operator. If both $A$ and $-A$ are maximal dissipative, then $A$ is the infinitesimal generator of a $C_0$-group, $G(t)$, which satisfies $\|G(t)\| = 1$ for all $t \in \mathbb{R}$.

**Proof.** By the Lumer-Phillips theorem, $A$ and $-A$ are infinitesimal generators of $C_0$-semigroups of contractions, $S_+(t)$ and $S_-(t)$ respectively. Therefore, Theorem 7 ensures that $A$ is the infinitesimal generator of a $C_0$-group, $G(t)$. Moreover, $1 = \|G(t)G(-t)\| \leq \|S_+(t)\| \|S_-(t)\| \leq 1$. Hence, $\|G(t)\| = 1$. □

**Example 17** (Wave equation continued) We return to the wave equation that was studied in Example 15. We proved that operator $A$, defined in (2.2.6), is maximal dissipative. We claim that $-A$ is maximal dissipative as well. Indeed, equation (2.2.7) implies that $-A$ is dissipative. Moreover, the resolvent equation for $-A$ takes the form

$$
\begin{cases}
    u \in H^2(\Omega) \cap H^1_0(\Omega), & v \in H^1_0(\Omega) \\
    u + v = f \in H^1_0(\Omega) \\
    v + \Delta u = g \in L^2(\Omega),
\end{cases}
$$

which can be uniquely solved arguing exactly as we did for system (2.2.9).

Then, by Corollary 4, $A$ is the infinitesimal generator of a $C_0$-group, $G(t)$, which satisfies $\|G(t)\| = 1$ for all $t \in \mathbb{R}$. So, for any $f \in H^2(\Omega) \cap H^1_0(\Omega)$, $g \in H^1_0(\Omega)$, the first component $u(t)$ ($t \in \mathbb{R}_+$) of

$$
G(t)\begin{pmatrix}
    f \\
    g
\end{pmatrix}
$$

is the unique solution of problem (3.2.4) in the space

$$
C^2(\mathbb{R}; L^2(\Omega)) \cap C^1(\mathbb{R}; H^1_0(\Omega)) \cap C(\mathbb{R}; H^2(\Omega) \cap H^1_0(\Omega)).
$$

### 2.3 The adjoint semigroup

In this section, we consider the special case when $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space. We denote by $j_X : X^* \to X$ the Riesz isomorphism, which associates with any $\phi \in X^*$ the unique element $j_X(\phi) \in X$ such that

$$
\phi(u) = \langle u, j_X(\phi) \rangle \quad \forall u \in X.
$$

We refer the reader to [4] for the treatment of a general Banach space.

**Adjoint of a linear operator**

Let $A : D(A) \subset X \to X$ be a densely defined linear operator.
Exercise 28 Prove that the set

\[ D(A^*) = \left\{ v \in X \mid \exists C \geq 0 : u \in D(A) \implies |\langle Au, v \rangle| \leq C|u| \right\} \quad (2.3.1) \]

is a subspace of \( X \) and, for any \( v \in D(A^*) \), the linear map \( u \mapsto \langle Au, v \rangle \) can be uniquely extended to a bounded linear functional \( \phi_v \in X^* \).

Solution. The fact that \( D(A^*) \) is a subspace of \( X \) is easy to show. Let \( v \in D(A^*) \), fix any \( u \in X \), and let \( u_n \in D(A) \) be such that \( u_n \xrightarrow{n \to \infty} u \). Then \( |\langle A(u_n - u_m), v \rangle| \leq C|u_n - u_m| \) which implies that \( \{\langle Au_n, v \rangle\} \) is a Cauchy sequence in \( \mathbb{R} \) and therefore converges as \( n \to \infty \). Moreover, if \( u'_n \in D(A) \) is another sequence such that \( u'_n \xrightarrow{n \to \infty} u \), then \( |\langle A(u_n - u'_n), v \rangle| \leq C|u_n - u'_n| \). Therefore, the map

\[ \phi_v(u) = \lim_{n \to \infty} \langle Au_n, v \rangle \quad (u \in X), \]

where \( \{u_n\} \) is any sequence in \( D(A) \) converging to \( u \) is well defined. Moreover, \( \phi_v \) is linear and \( |\phi_v(u)| \leq C|u| \) for all \( u \in X \). So, \( \phi_v \in X^* \).

Definition 16 The adjoint of a is the map \( A^* : D(A^*) \subset X \to X \) defined by

\[ A^*v = j_X(\phi_v) \quad \forall v \in D(A^*) \]

where \( D(A^*) \) is given by (2.3.1) and \( \phi_v \in X^* \) is the functional extending \( u \mapsto \langle Au, v \rangle \) to \( X \) (see Exercise 28).

Exercise 29 Prove that, if \( A \in \mathcal{L}(X) \), then \( A^* \in \mathcal{L}(X) \) as well and

\[ \|A\| = \|A^*\|. \quad (2.3.2) \]

Solution. Since \( A \in \mathcal{L}(X) \) we have that \( D(A^*) = X \) and

\[ \langle Au, v \rangle = \langle u, A^*v \rangle \quad \forall u, v \in X. \]

So, by the definition of \( A^* \) we have that \( \|A^*\| \leq \|A\| \). Moreover, taking \( v = Au \) in the above identity, we obtain \( |Au|^2 \leq |u| \|A^*\| |Au| \). So, \( \|A\| \leq \|A^*\| \). \( \square \)

Proposition 12 (properties of \( A^* \)) Let \( A : D(A) \subset X \to X \) be a densely defined linear operator. Then the following properties hold.

(i) \( A \) satisfies the adjoint identity

\[ \langle Au, v \rangle = \langle u, A^*v \rangle \quad \forall u \in D(A), \forall v \in D(A^*). \quad (2.3.3) \]

(ii) \( A^* : D(A^*) \subset X \to X \) is a closed linear operator.

(iii) If \( \lambda \in \rho(A) \), then \( \overline{\lambda} \in \rho(A^*) \) and \( R(\overline{\lambda}, A^*) = R(\lambda, A)^* \).
(iv) If, in addition, $A$ is closed then $D(A^*)$ is dense in $X$.

**Proof of (i).** Let $u \in D(A), v \in D(A^*)$ and let $\phi_v \in X^*$ be the functional extending $u \mapsto \langle Au, v \rangle$ to $X$. Then
\[
\langle Au, v \rangle = \phi_v(u) = \langle u, j_X(\phi_v) \rangle = \langle u, A^*v \rangle
\]

**Proof of (ii).** Now, to prove that $A^*$ is closed, let $\{v_n\} \subset D(A^*)$ and $v, w \in X$ be such that
\[
v_n \to v \quad (n \to \infty)
\]
\[
A^*v_n \to w
\]
Then $\{A^*v_n\}$ is bounded, say $|A^*v_n| \leq C$. So, recalling (2.3.3), we have that
\[
|\langle u, A^*v_n \rangle| = C|u| \quad \forall u \in D(A)
\]
This yields
\[
|\langle u, v \rangle| \leq C|u| \quad \forall u \in D(A)
\]
which in turn implies that $v \in D(A^*)$. Moreover
\[
\langle Au, v \rangle = \lim_{n \to \infty} \langle Au, v_n \rangle = \langle u, w \rangle \quad \forall u \in D(A).
\]
Thus, $\langle u, A^*v - w \rangle = 0$ for all $u \in D(A)$. Since $D(A)$ is dense, $A^*v = w$. □

**Proof of (iii).** Let $\lambda \in \rho(A)$. From the definition of the adjoint we have that
\[
(\lambda I - A)^* = \overline{\lambda I} - A^*.
\]
Aiming to prove that $\overline{\lambda I} \in \rho(A^*)$, first we show that $\overline{\lambda I} - A^*$ is injective. If $(\overline{\lambda I} - A^*)v = 0$ for some $v \in D(A^*)$, then
\[
0 = \langle u, (\overline{\lambda I} - A^*)v \rangle = \langle (\lambda I - A)u, v \rangle \quad \forall u \in D(A).
\]
Since $\lambda I - A$ is surjective, the above identity implies that $v = 0$. So, $\overline{\lambda I} - A^*$ is injective. Next, observe that, for all $v \in X$ and $u \in D(A)$,
\[
\langle u, v \rangle = \langle R(\lambda, A)(\lambda I - A)u, v \rangle = \langle (\lambda I - A)u, R(\lambda, A)^*v \rangle,
\]
yielding $R(\lambda, A)^*v \in D((\lambda I - A)^*) = D(\overline{\lambda I} - A^*) = D(A^*)$ and
\[
(\overline{\lambda I} - A^*) R(\lambda, A)^*v = v \quad \forall v \in X.
\]
On the other hand, if $u \in X$ and $v \in D(A^*)$, then
\[
\langle u, v \rangle = \langle (\lambda I - A)R(\lambda, A)u, v \rangle = \langle R(\lambda, A)u, (\overline{\lambda I} - A^*)v \rangle.
\]
Therefore,
\[
R(\lambda, A)^*(\overline{\lambda I} - A^*)v = v \quad \forall v \in D(A^*).
\]
(2.3.4) and (2.3.5) imply that $\overline{X} \in \rho(A^*)$ and $R(\overline{X}, A^*) = R(\lambda, A^*)$.

Proof of (iv). We argue by contradiction assuming the existence of $u_0 \neq 0$ such that $\langle u_0, v \rangle = 0$ for every $v \in D(A^*)$. Then $(0, u_0) \notin \text{graph}(A)$, which is a closed subspace of $X \times X$. From the Hahn-Banach theorem it follows that there exist $v_1, v_2 \in X$ such that the associated hyperplane in $X \times X$ separates $\text{graph}(A)$ and the point $(0, u_0)$, that is,

$$\langle u, v_1 \rangle - \langle Au, v_2 \rangle = 0 \quad \forall u \in D(A) \quad \text{and} \quad \langle 0, v_1 \rangle - \langle u_0, v_2 \rangle \neq 0$$

But the first identity implies that $v_2 \in D(A^*)$, which in turn yields $\langle u_0, v_2 \rangle = 0$, in contrast with the second equation above. So, $D(A^*) = X$. □

The Lumer-Phillips theorem

By introducing dissipativity of the adjoint of $A$ we can replace maximality in the Lumer-Phillips theorem.

**Theorem 10 (Lumer-Phillips 2)** Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator. If $A$ and $A^*$ are dissipative, then $A$ is the infinitesimal generator of a contraction semigroup on $X$.

**Proof.** In view of Theorem 9, it suffices to show that $]0, \infty[ \subset \rho(A)$. Since $\lambda I - A$ is one-to-one for any $\lambda > 0$, one just has to check that

$$(\lambda I - A)D(A) = X \quad \forall \lambda > 0.$$ 

Step 1: $(\lambda I - A)D(A)$ is dense in $X$ for every $\lambda > 0$.

Let $v \in X$ be such that $\langle \lambda u - Au, v \rangle = 0 \quad \forall u \in D(A)$. The identity $\langle Au, v \rangle = \lambda \langle u, v \rangle$ yields $v \in D(A^*)$ and the fact that

$$\langle u, \lambda v - A^*v \rangle = 0,$$

first for all $u \in D(A)$ and then, by density, for all $u \in X$. So, $\lambda v - A^*v = 0$. Since, being dissipative, $\lambda I - A^*$ is also one-to-one, we conclude that $v = 0$.

Step 2: $\lambda I - A$ is surjective for every $\lambda > 0$.

Fix any $v \in X$. By Step 1, there exists $\{u_n\} \subset D(A)$ such that

$$\lambda u_n - Au_n =: v_n \to v \quad \text{as} \quad n \to \infty.$$ 

By (2.1.2) we deduce that, for all $n, m \geq 1$,

$$|u_n - u_m| \leq \frac{1}{\lambda} |v_n - v_m|.$$
which insures that \( \{u_n\} \) is a Cauchy sequence in \( X \). Therefore, there exists \( u \in X \) such that
\[
\begin{align*}
  u_n &\to u \\
  Au_n &= \lambda u_n - v_n \to \lambda u - v \quad (n \to \infty)
\end{align*}
\]
Since \( A \) is closed, \( u \in D(A) \) and \( \lambda u - Au = v \). \[\square\]

The adjoint semigroup

In order to make further progress we have to better understand the relationship between the adjoint, \( S(t)^* \), of a \( C_0 \)-semigroup of bounded linear operators on \( X \) and the adjoint, \( A^* \), of its infinitesimal generator.

**Theorem 11** Let \( S(t) \) be a \( C_0 \)-semigroup of bounded linear operators on \( X \) with infinitesimal generator \( A : D(A) \subset X \to X \). Then \( S(t)^* \) is a \( C_0 \)-semigroup of bounded linear operators on \( X \), called the adjoint semigroup, whose infinitesimal generator is \( A^* \), the adoint of \( A \).

**Proof.** We observe first that properties \((a)\) and \((b)\) of the definition of a semigroup are easy to check. Moreover, in view of the bound (1.2.2) and Exercise 29 we have that \( S(t)^* \) satisfies the growth condition
\[
\|S(t)^*\| \leq Me^{\omega t} \quad \forall t \geq 0 \quad (2.3.6)
\]
with the same constants \( M, \omega \) as \( S(t) \). Hereafter, we assume \( \omega \geq 0 \).

Aiming to prove that \( S(t)^* \) is strongly continuous we observe that, for all \( u \in X \) and \( v \in D(A^*) \),
\[
|\langle u, S(t)^*v - v \rangle| = |\langle S(t)u - u, v \rangle| = \left| \int_0^t \langle AS(s)u, v \rangle ds \right|
\]
\[
= \left| \int_0^t \langle S(s)u, A^*v \rangle ds \right| = \left| \int_0^t \langle u, S(s)^*A^*v \rangle ds \right|. \quad (2.3.7)
\]
Therefore, on account of (2.3.6),
\[
|S(t)^*v - v| \leq Mte^{\omega t}|A^*v| \quad \forall v \in D(A^*).
\]
This implies that \( \lim_{t \downarrow 0} S(t)^*v = v \) first for every \( v \in D(A^*) \) and then for all \( v \in X \) thanks to (2.3.6) since \( D(A^*) \) is dense in \( X \) by Proposition 12.

Finally, we show that \( A^* \) is the infinitesimal generator of the adjoint semigroup. Denote by \( B : D(B) \subset X \to X \) the infinitesimal generator of \( S(t)^* \). Owing to (2.3.7), for every \( v \in D(A^*) \) we have that
\[
\frac{S(t)^*v - v}{t} = \frac{1}{t} \int_0^t S(s)^*A^*v ds \xrightarrow{t \downarrow 0} A^*v.
\]
Therefore, \( A^* \subset B \). Moreover, \( \rho(A) \cap \rho(B) \neq \emptyset \) because \( \Pi_\omega \subset \rho(A^*) \) by Theorem 12 and \( \Pi_\omega \subset \rho(B) \) by (2.3.6) and Proposition 5. So, \( A^* = B \). \[\square\]
**Self-adjoint operators and Stone’s theorem**

**Definition 17** A densely defined linear operator \( A : D(A) \subseteq X \to X \) is called:

(a) symmetric if \( A \subseteq A^* \), that is,
\[
D(A) \subseteq D(A^*) \quad \text{and} \quad Au = A^*u \quad \forall u \in D(A).
\]

(b) self-adjoint if \( A = A^* \). 

**Remark 7**

1. Observe that a symmetric operator \( A \) is self-adjoint if and only if \( D(A) \subseteq D(A^*) \).
2. In view of Proposition 12, any self-adjoint operator is closed.
3. If \( A \in \mathcal{L}(X) \), then \( A \) is self-adjoint if and only if \( A \) is symmetric.

**Example 18** In \( X = L^2(0,1; \mathbb{C}) \), consider the linear operator
\[
\begin{cases}
D(A) = H^1_0(0,1; \mathbb{C}) \\
Au(x) = iu'(x) \quad x \in [0,1] \text{ a.e.}
\end{cases}
\]

Then, \( A \) is densely defined and symmetric. Indeed, for all \( u,v \in D(A) \),
\[
\langle Au,v \rangle = i \int_0^1 u'(x) \overline{v(x)} \, dx \quad (2.3.8)
\]
\[
= [iu(x)\overline{v(x)}]_{x=1}^{x=0} - i \int_0^1 u(x)\overline{v'(x)} \, dx = \langle u, Av \rangle.
\]

On the other hand, \( A \) fails to be self-adjoint because, as we show next,
\[
D(A^*) \supseteq H^1(0,1; \mathbb{C}),
\]
so that \( D(A) \subsetneq D(A^*) \). Indeed, integrating by parts as in (2.3.8), for all \( v \in H^1(0,1; \mathbb{C}) \) and \( u \in H^1_0(0,1; \mathbb{C}) \) we have that
\[
\langle Au,v \rangle = \left| -i \int_0^1 u(x)\overline{v'(x)} \, dx \right| \leq |u|_2 |v'|_2. \quad \square
\]

**Proposition 13** Let \( A : D(A) \subseteq X \to X \) be a densely defined closed linear operator such that \( \rho(A) \cap \mathbb{R} \neq \emptyset \). If \( A \) is symmetric, then \( A \) is self-adjoint.

**Proof.** We prove that \( D(A^*) \subseteq D(A) \) in two steps. Fix any \( \lambda \in \rho(A) \cap \mathbb{R} \).

**Step 1:** \( R(\lambda, A) = R(\lambda, A)^* \)
Since $R(\lambda, A) \in \mathcal{L}(X)$, in view of Exercise 29 it suffices to show that
\[
\langle R(\lambda, A)u, v \rangle = \langle u, R(\lambda, A)v \rangle \quad \forall u, v \in X.
\]
Fix any $u, v \in X$ and set
\[
x = R(\lambda, A)u \quad \text{and} \quad y = R(\lambda, A)v
\]
so that $x, y \in D(A)$ and
\[
\lambda x - Ax = u \quad \text{and} \quad \lambda y - Ay = v.
\]
Since $A$ is symmetric, we have that
\[
\langle R(\lambda, A)u, v \rangle = \langle x, v \rangle = \langle x, \lambda y - Ay \rangle = \langle \lambda x - Ax, y \rangle = \langle u, R(\lambda, A)v \rangle.
\]

**Step 2:** $D(A^*) \subset D(A)$

Let $u \in D(A^*)$ and set $x = \lambda u - A^*u$. Observe that, for all $v \in D(A)$,
\[
\langle x, v \rangle = \langle \lambda u - A^*u, v \rangle = \langle u, \lambda v - Av \rangle.
\]
Now, take any $w \in X$ and let $v = R(\lambda, A)w$. Then the above identity yields
\[
\langle x, R(\lambda, A)w \rangle = \langle u, w \rangle \quad \forall w \in X.
\]
So, by Step 1 we conclude that $u = R(\lambda, A)^*x = R(\lambda, A)x \in D(A)$. □

The following is another interesting spectral property of self-adjoint operators.

**Proposition 14** If $A : D(A) \subset X \to X$ is self-adjoint then
\[
\rho(A) \supset \{ \lambda \in \mathbb{C} : \exists \lambda \neq 0 \}
\]
Consequently, $\sigma(A)$ is real.

**Proof.** For every $u \in D(A)$ we have that
\[
\langle Au, u \rangle = \overline{\langle Au, u \rangle} \in \mathbb{R}. \quad (2.3.9)
\]
Therefore, $|\langle \lambda u - Au, u \rangle| \geq |\lambda| |u|^2$ which in turn yields
\[
|\lambda u - Au| \geq |\lambda| |u| \quad \forall x \in D(A). \quad (2.3.10)
\]

The last inequality ensures that $\lambda I - A$ is an injective operator with closed range for all $\lambda \in \mathbb{C}$ with $\exists \lambda \neq 0$. Let us show that $(\lambda I - A)D(A)$ is dense in $X$ for any such $\lambda$. Suppose there exists $v \neq 0$ such that
\[
\langle \lambda u - Au, v \rangle = 0 \quad \forall u \in D(A).
\]
Then $v \in D(A^*) = D(A)$ and we have that
\[ \langle u, \overline{\lambda} v - Av \rangle = 0 \quad \forall u \in D(A). \]
Since $D(A)$ is dense in $X$, this implies that $\overline{\lambda} v - Av = 0$. Then, by (2.3.9), $\overline{\lambda} = \lambda \in \mathbb{R}$ contradicting $\exists \lambda \neq 0$.

The following is an immediate consequence of Theorem 10.

**Corollary 5 (Lumer-Phillips 3)** Let $A : D(A) \subset X \to X$ be a densely defined linear operator. If $A$ is self-adjoint and dissipative, then $A$ is the infinitesimal generator of a contraction semigroup on $X$.

**Example 19** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^2$. Define
\begin{align*}
D(A) &= H^2 \cap H^1_0(\Omega; \mathbb{C}) \\
Au(x) &= \Delta u(x) - V(x)u(x) \quad x \in \Omega \text{ a.e.} \tag{2.3.11}
\end{align*}
where we assume $V \in L^\infty(\Omega, \mathbb{R})$. Let us check that $A$ is self-adjoint in $L^2(\Omega; \mathbb{C})$. Indeed, integration by parts insures that $A$ is symmetric. So, by Proposition 13, it suffices to check that $\rho(A) \cap \mathbb{R} \neq \emptyset$. We claim that, for $\lambda \in \mathbb{R}$ large enough, for any $h \in L^2(\Omega; \mathbb{C})$ the problem
\begin{align*}
w &\in H^2 \cap H^1_0(\Omega; \mathbb{C}) \\
(\lambda + V(x))w(x) - \Delta w(x) &= h(x) \quad (x \in \Omega \text{ a.e.}) \tag{2.3.12}
\end{align*}
has a unique solution. Equivalently, by setting $f = \Re h$, $g = \Im h \in L^2(\Omega)$ and $u = \Re w$, $v = \Im w$, we have to prove solvability for the boundary value problems
\begin{align*}
u &\in H^2 \cap H^1_0(\Omega) \\
(\lambda + V)u - \Delta u &= f \\
v &\in H^2 \cap H^1_0(\Omega) \\
(\lambda + V)v - \Delta v &= g
\end{align*}
The latter is a well-established fact in elliptic theory (see, e.g. [3]). On the other hand, operator $A$ fails to be dissipative, in general.

**Exercise 30** Prove that operator $A$ in Example 19 is dissipative if
\[ \|V\|_\infty \leq 1/C_\Omega, \]
where $C_\Omega$ is the Poincaré constant of $\Omega$.

The following property of self-adjoint operators is very useful. We recall that an operator $U \in \mathcal{L}(X)$ is unitary if $UU^* = U^*U = I$.

**Theorem 12 (Stone)** Let $X$ be a complex Hilbert space. For any densely defined linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:

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(a) $A$ is self-adjoint,
(b) $iA$ is the infinitesimal generator of a $C_0$-group of unitary operators.

**Proof of (a) $\Rightarrow$ (b)**

Since $A$ is self-adjoint, $A$ is closed and we have that

$$\langle Au, u \rangle = \langle u, A^* u \rangle = \langle u, Au \rangle \quad \forall u \in D(A).$$

Thus, $\langle Au, u \rangle$ is real, so that

$$\Re \langle iAu, u \rangle = 0 \quad \forall u \in D(A).$$

The above identity implies that both $iA$ and $-iA$ are dissipative operators. Since

$$\langle iAu, v \rangle = i \langle u, Av \rangle = \langle u, -iAv \rangle \quad \forall u, v \in D(A),$$

we have that $(iA)^* = -iA$. So, by Theorem 10 we deduce that $\pm iA$ is the infinitesimal generator of a $C_0$-semigroup of contractions that we denote by $e^{\pm iAt}$. Then, by Theorem 7, $iA$ generates a $C_0$ group, $G(t)$. Such a group is unitary because for any $t \geq 0$ we have that

$$G(t)^{-1} = G(-t) = e^{-iAt} = e^{(iA)^* t} = (e^{iAt})^* = G(t)^*,$$

while, for any $t < 0$,

$$G(t)^{-1} = e^{iA|t|} = e^{iA^*|t|} = (e^{-iA}|t|)^* = G(|t|)^* = G(t)^*.$$  \[\square\]

**Proof of (b) $\Rightarrow$ (a)**

Let $iA$ be the infinitesimal generator of a $C_0$-group of unitary operators on $X$, say $G(t)$. Then, for all $u \in D(A)$, we have that

$$iAu = \lim_{t \to 0} \frac{G(t)u - u}{t} = -\lim_{t \to 0} \frac{G(-t)u - u}{t} = -\lim_{t \to 0} \frac{G(t)^*u - u}{t} = -(iA)^* u = iA^* u.$$

Thus, $u \in D(A^*)$ and $Au = A^* u$. By running the above computation backwards, we conclude that $D(A^*) \subseteq D(A)$. Therefore, $A$ is self-adjoint. \[\square\]

**Example 20 (Schrödinger equation in a bounded domain)**

Let us consider the initial-boundary value problem

$$\begin{cases}
\frac{1}{i} \frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) - V(x)u(t, x) & (t, x) \in \mathbb{R} \times \Omega \\
u(t, x) = 0 & t \in \mathbb{R}, \ x \in \partial \Omega \\
u(0, x) = u_0(x) & x \in \Omega
\end{cases} \tag{2.3.13}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class $C^2$ and $V \in L^\infty(\Omega)$. In Example 19, we have already checked that the operator $A$, defined in (2.3.11), is self-adjoint on $L^2(\Omega; \mathbb{C})$. Therefore, by Theorem 12 we conclude that, for any $u_0 \in H^2 \cap H^1_0(\Omega; \mathbb{C})$, problem (2.3.13) has a unique solution

$$u \in C^1(\mathbb{R}; L^2(\Omega; \mathbb{C})) \cap C(\mathbb{R}; H^2 \cap H^1_0(\Omega; \mathbb{C})).$$  \[\square\]
The Cauchy problem with a self-adjoint operator

In this section, we will see that the homogeneous Cauchy problem with initial datum \( u_0 \in X \)

\[
\begin{aligned}
  u'(t) &= Au(t) \quad t > 0 \\
  u(0) &= u_0.
\end{aligned}
\] (2.3.14)

can be solved in a strict sense without requiring \( u_0 \) to be in \( D(A) \) if \( A \) is a self-adjoint and dissipative.

We begin with an interpolation result of interest in its own right.

**Lemma 2** Let \( A : D(A) \subset X \rightarrow X \) be a self-adjoint dissipative operator and let \( u \in H^1(0,T;X) \cap L^2(0,T;D(A)) \) be such that \( u(0) = 0 \). Then the function

\[
t \mapsto \langle Au(t), u(t) \rangle
\]

is absolutely continuous on \([0,T]\) and

\[
\frac{d}{dt} \langle Au(t), u(t) \rangle = 2 \Re \langle u'(t), Au(t) \rangle \quad (a.e. \ t \in [0,T]).
\] (2.3.15)

**Proof.** Define \( U_n(t) = \langle A_n u(t), u(t) \rangle \ (t \in [0,T]) \), where \( A_n = nAR(n,A) \) is the Yosida approximation of \( A \). Then \( U_n \) is absolutely continuous on \([0,T]\) and

\[
\frac{d}{dt} \langle A_n u(t), u(t) \rangle = 2 \Re \langle u'(t), A_n u(t) \rangle \quad (a.e. \ t \in [0,T])
\]
or

\[
\langle A_n u(t), u(t) \rangle = 2 \Re \int_0^t \langle u'(s), A_n u(s) \rangle ds \quad \forall t \in [0,T].
\] (2.3.16)

Now, since for a.e. \( t \in [0,T] \)

\[
A_n u(t) = nR(n,A)Au(t) \xrightarrow{n \rightarrow \infty} Au(t)
\]

\[
|A_n u(t)| \leq |Au(t)|,
\]

we can pass to the limit as \( n \rightarrow \infty \) in (2.3.16) by Lebesgue’s theorem to obtain

\[
\langle Au(t), u(t) \rangle = 2 \Re \int_0^t \langle u'(s), Au(s) \rangle ds \quad \forall t \in [0,T].
\]

This shows that \( t \mapsto \langle Au(t), u(t) \rangle \) is absolutely continuous on \([0,T]\) and yields (2.3.15). \( \square \)

**Theorem 13** Let \( A : D(A) \subset X \rightarrow X \) be a self-adjoint dissipative operator. Then \( S(t)u \in D(A) \) for all \( u \in X \) and \( t > 0 \). Moreover, for every \( T > 0 \) the following inequality holds

\[
4 \int_0^T t|AS(t)u|^2 dt - 2T\langle AS(T)u, S(T)u \rangle + |S(T)u|^2 \leq |u|^2 \quad \forall u \in X.
\] (2.3.17)
Observe that all the terms on the left side of (2.3.17) are nonnegative, so that each of them is bounded by $|u|^2$.

**Proof.** For any $n \geq 1$ define

$$u_n = nR(n, A)u, \quad v_n(t) = S(t)u_n, \quad v(t) = S(t)u.$$ 

Since $u_n \in D(A)$, we have that $v_n'(t) = Av_n(t)$ for all $t \geq 0$. So,

$$\frac{1}{2} \frac{d}{dt} |v_n(t)|^2 = \langle v_n'(t), v_n(t) \rangle = \langle Av_n(t), v_n(t) \rangle$$

Therefore,

$$|v_n(t)|^2 - 2 \int_0^t \langle Av_n(s), v_n(s) \rangle ds = |u_n|^2 \quad \forall t \geq 0. \tag{2.3.18}$$

Similarly, for all $t \geq 0$ we have that

$$t|v_n'(t)|^2 = t\langle Av_n(t), v_n'(t) \rangle = \frac{1}{2} \frac{d}{dt} \left(t\langle Av_n(t), v_n(t) \rangle\right) - \frac{1}{2} \langle Av_n(t), v_n(t) \rangle.$$ 

Integrating the above identity over $[0, T]$ yields, by (2.3.18),

$$2 \int_0^T t|v_n'(t)|^2 dt - T\langle Av_n(T), v_n(T) \rangle = - \int_0^T \langle Av_n(t), v_n(t) \rangle dt$$

or

$$4 \int_0^T t|Av_n(t)|^2 dt - 2T\langle Av_n(T), v_n(T) \rangle + |v_n(T)|^2 \leq |u|^2 \tag{2.3.19}$$

since $\|nR(n, A)\| \leq 1$. The last inequality implies that, for any $\varepsilon \in [0, T[$, $\{v_n\}$ is bounded in $L^2(\varepsilon, T; D(A))$. Therefore, there exists a weakly convergent subsequence $\{v_{n_k}\}$ in $L^2(\varepsilon, T; D(A))$. On the other hand, $v_{n_k} \to v$ uniformly on $[0, T]$. So, $v \in L^2(\varepsilon, T; D(A))$ for any $\varepsilon \in [0, T[$, which in turn yields $S(t)u \in D(A)$ for a.e. $t > 0$—hence for all $t > 0$! Moreover,

$$Av_n(t) = nR(n, A)AS(t)u \xrightarrow{n \to \infty} AS(t)u \quad \forall t > 0.$$ 

Taking the limit as $n \to \infty$ in (2.3.19), by Fatou’s lemma we get (2.3.17). □

The above result can be used to introduce an intermediate space between $X$ and $D(A)$, namely the interpolation space $[X, D(A)]_{1/2}$, such that $t \mapsto S(t)u_0$ belongs to $H^1(0, T; X) \cap L^2(0, T; D(A))$ whenever $u_0 \in [D(A), X]_{1/2}$. We give a brief account of such a construction, referring the reader to [1] for more.
Proposition 15 Let $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator. Then, for any $u \in X$, the functions

$$t \mapsto -\langle AS(t)u, S(t)u \rangle$$

and

$$t \mapsto -\frac{1}{t} \int_0^t \langle AS(s)u, S(s)u \rangle \, ds$$

are both nonincreasing on $]0, \infty[$.

Proof. Since $S(t)u \in D(A)$ for every $t > 0$ by Theorem 13, we have that

$$0 \leq 2|AS(t)u|^2 = \frac{d}{dt} \langle AS(t)u, S(t)u \rangle \quad \forall t > 0.$$ 

This shows that $t \mapsto -\langle AS(t)u, S(t)u \rangle$ is nondecreasing on $]0, \infty[$. The other conclusion is a consequence of the general fact which is proven below. □

Lemma 3 Let $f$ be a nonnegative nonincreasing function on $]0, \infty[$. Then $t \mapsto \frac{1}{t} \int_0^t f(s) \, ds$ is nonincreasing on $]0, \infty[$.

Proof. Observe that for any $0 < t < t'$ we have that

$$f(t') \leq \frac{1}{(t' - t)} \int_t^{t'} f(s) \, ds \leq f(t) \leq \frac{1}{t} \int_0^t f(s) \, ds. \quad (2.3.20)$$

This yields

$$\frac{1}{t'} \int_0^{t'} f(s) \, ds = \frac{1}{t} \int_0^t f(s) \, ds + \left( \frac{1}{t'} - \frac{1}{t} \right) \int_0^t f(s) \, ds + \frac{1}{t} \int_t^{t'} f(s) \, ds = \frac{1}{t} \int_0^t f(s) \, ds + \frac{t' - t}{t'} \left\{ \frac{1}{t'} \int_t^{t'} f(s) \, ds - \frac{1}{t} \int_0^t f(s) \, ds \right\} \leq \frac{1}{t} \int_0^t f(s) \, ds,$$

where we have made repeated use of (2.3.20). The conclusion follows. □

In view of the above proposition, we have that, for any $u \in X$,

$$\lim_{T \downarrow 0} \frac{1}{T} \int_0^T \langle AS(t)u, S(t)u \rangle \, dt = \sup_{T>0} \frac{1}{T} \int_0^T \langle AS(t)u, S(t)u \rangle \, dt$$

Definition 18 (interpolation space $[D(A), X]_{1/2}$) For any $u \in u$ we set

$$|u|_{1/2}^2 = \lim_{T \downarrow 0} \frac{1}{T} \int_0^T \langle AS(t)u, S(t)u \rangle \, dt$$

and we define

$$[D(A), X]_{1/2} = \{ u \in u : |u|_{1/2} < \infty \}. \quad (2.3.21)$$
It is easy to see that \([D(A), X]_{1/2}\) is a subspace of \(X\) containing \(D(A)\) and \(\|x\|_{1/2} = |x| + |x|_{1/2}\) is a norm on \([D(A), X]_{1/2}\).

**Theorem 14** Let \(A : D(A) \subset X \to X\) be a self-adjoint dissipative operator. Then
\[
\int_0^\infty |AS(t)u|^2 dt \leq \frac{1}{2} |u|_{1/2}^2 \quad \forall u \in [D(A), X]_{1/2}.
\]

**Proof.** Fix any \(\varepsilon > 0\) and let \(T_\varepsilon \in ]0, \varepsilon[\) be such that
\[
-\langle AS(T_\varepsilon)u, S(T_\varepsilon)u \rangle < |u|_{1/2}^2 + \varepsilon.
\]
Set \(v(t) = S(t)u\) and integrate the identity \(|Av(y)|^2 = \langle Av(t), v'(t) \rangle\) over \([T_\varepsilon, T]\) for any fixed \(T > \varepsilon\) to obtain
\[
\int_{T_\varepsilon}^T |Av(t)|^2 dt = \frac{1}{2} \langle Av(T), v(T) \rangle - \frac{1}{2} \langle Av(T_\varepsilon), v(T_\varepsilon) \rangle < |u|_{1/2}^2 + \varepsilon.
\]
This implies the conclusion as \(\varepsilon \downarrow 0\) and \(T \uparrow \infty\). \qed

**Example 21** On \(X = L^2(0, \pi)\) let \(A : D(A) \subset X \to X\) be the operator
\[
\begin{cases}
D(A) = H^2(0, \pi) \cap H^1_0(0, \pi) \\
Af(x) = f''(x) & x \in (0, \pi) \text{ a.e.}
\end{cases}
\]
We know that \(A\) is self-adjoint and dissipative. We now show that
\[
[D(A), X]_{1/2} = H^1_0(0, \pi). \tag{2.3.22}
\]
Let us fix \(f \in H^1_0(0, \pi)\) and consider its Fourier series
\[
f(x) = \sum_{n=1}^\infty f_n \sin(nx) \quad (x \in [0, \pi]).
\]
By Parseval’s identity we have that
\[
\sum_{n=1}^\infty n^2 |f_n|^2 = \frac{2}{\pi} \int_0^\pi |f'(x)|^2 dx.
\]
Moreover,
\[
S(t)f(x) = \sum_{n=1}^\infty e^{-nt}f_n \sin(nx) \quad (x \in [0, \pi]).
\]
and
\[ \text{AS}(t)f(x) = -\sum_{n=1}^{\infty} n^2 e^{-n^2 t} f_n \sin(nx) \quad (x \in [0, \pi]). \]

Therefore,
\[ \langle \text{AS}(t)f, S(t)f \rangle = -\sum_{n=1}^{\infty} n^2 e^{-n^2 t} |f_n|^2 \int_0^{\pi} \sin^2(nx) \, dx \]
\[ = -\frac{\pi}{2} \sum_{n=1}^{\infty} n^2 e^{-n^2 t} |f_n|^2. \]

Hence, recalling that \( 1 - e^{-x} \leq x \) for all \( x \in \mathbb{R} \), we deduce that
\[ -\frac{1}{T} \int_0^T \langle \text{AS}(t)f, S(t)f \rangle \, 2dt = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1 - e^{-2n^2 T}}{2T} |f_n|^2 \]
\[ \leq \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 |f_n|^2 = \int_0^{\pi} |f'(x)|^2 dx. \]

The last inequality implies that \( H^1_0(0, \pi) \subset [D(A), X]_{1/2} \). The proof of the converse inclusion is left to the reader as an Exercise.

**Hint.** Use (2.3.23) to give a lower bound for
\[ \lim_{T \downarrow 0} -\frac{1}{T} \int_0^T \langle \text{AS}(t)f, S(t)f \rangle \, 2dt. \]

**Exercise 31** Use Theorem 13 to show that, for any self-adjoint dissipative operator \( A : D(A) \subset X \rightarrow X \), the following holds:

(a) \( S(t)u \in D(A^n) \) for all \( t > 0 \), all \( u \in X \), and all \( n \in \mathbb{N} \);

(b) for all \( u \in X \)
\[ |\text{AS}(t)u| \leq \frac{|u|}{t^{1/2}} \quad \forall t > 0. \]

**Solution.** To prove (a) it suffices to observe that for all \( t > 0 \) and \( u \in X \),
\[ \text{AS}(t)u = S(t/2) \text{AS}(t/2)u \in D(A) \quad \implies \quad S(t)u \in D(A^2). \]

The general case follows by induction.

Next, using the dissipativity of \( A \) we obtain
\[ \frac{d}{dt} |\text{AS}(t)u|^2 = 2 \langle A^2 S(t)u, \text{AS}(t)u \rangle \leq 0. \]

Thus, \( t \mapsto |\text{AS}(t)u|^2 \) is nonincreasing. So, (2.3.17) yields
\[ 2t^2 |\text{AS}(t)u|^2 \leq 4 \int_0^t s |\text{AS}(s)u|^2 ds \leq 4 \int_0^t s |\text{AS}(s)u|^2 ds \leq |u|^2. \]

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The inhomogeneous Cauchy problem

In this chapter, we assume that \((X, \langle \cdot, \cdot \rangle)\) is a separable Hilbert space and denote by \(\{e_j\}_{j \in \mathbb{N}}\) a complete orthonormal system in \(X\).

We study the Cauchy problem

\[
\begin{aligned}
    u'(t) &= Au(t) + f(t) \\
    u(0) &= u^0,
\end{aligned}
\]

(3.0.1)

where \(f \in L^2(0, T; X)\) and \(A : D(A) \subset X \to X\) is the infinitesimal generator of a \(C_0\)-semigroup on \(X\), \(S(t)\), which satisfies the growth condition (1.6.3). For the extension of this theory to a general Banach space, we refer the reader to the classic monograph by Pazy [4] or the more recent text [2].

3.1 Notions of solution

Let \(u^0 \in X\) and \(f \in L^2(0, T; X)\).

Definition 19 (Mild solutions) The function \(u \in C([0, T]; X)\) defined by

\[
    u(t) = S(t)u^0 + \int_0^t S(t-s)f(s) \, ds
\]

(3.1.1)

is called the mild solution of (3.0.1).

Observe that the convolution term in formula (3.1.1) for the solution \(u\) is well-defined in view of Proposition 22 in Appendix B.

Theorem 15 (Approximation of mild solutions) Let \(u \in C([0, T]; X)\) be the mild solution of (3.0.1) and suppose \(f \in C([0, T]; X)\). Then, the sequence \(u_n := nR(n, A)u\), defined for all \(n > \omega\), satisfies

\[
    u_n \in H^1(0, T; X) \cap L^2(0, T; D(A)) \quad \text{and} \quad u_n \xrightarrow{n \to \infty} u \quad \text{in} \ C([0, T]; X).
\]

Proof. Let \(u\) be given by (3.1.1) and define

\[
\begin{aligned}
    u_n(t) &= nR(n, A)u(t) \\
    f_n(t) &= nR(n, A)f(t) \quad \forall n \in \mathbb{N}, n > \omega \\
    u_0^n &= nR(n, A)u^0
\end{aligned}
\]

where \(\omega \geq 0\) is such that (1.6.3) holds true. Then

\[
u_n(t) = S(t)u_0^n + \int_0^t S(t-s)f_n(s) \, ds \quad (t \in [0, T]).\]
Since $u_n^0 \in D(A)$ and $f_n \in C([0, T]; D(A))$, by Proposition 21 and 22 below we conclude that
\[
  u_n \in H^1(0, T; X) \cap L^2(0, T; D(A)) \quad \text{and} \quad \begin{cases} 
    u'_n - Au_n = f_n \\
    u_n(0) = u_n^0.
  \end{cases}
\]
Moreover, invoking Lemma 1 we conclude that $u_n^0 \to u^0$ as $n \to \infty$ while
\[
  f_n(t) \overset{(n \to \infty)}{\longrightarrow} f(t) \quad \text{and} \quad |f_n(t)| \leq \frac{Mn}{n - \omega} |f(t)| \quad (\text{for all } t \in [0, T])
\]
Therefore,
\[
  \sup_{t \in [0, T]} |u_n(t) - u(t)| \leq Me^{\omega T} \left( |u_n^0 - u^0| + \int_0^T |f_n(s) - f(s)| \, ds \right) \overset{(n \to \infty)}{\longrightarrow} 0.
\]
The conclusion follows. \qed

**Definition 20 (Strict solutions)** A function $u \in H^1(0, T; X) \cap L^2(0, T; D(A))$ is a **strict solution** of (3.0.1) if $u(0) = u^0$ and
\[
  u'(t) = Au(t) + f(t) \quad (t \in [0, T] \text{ a.e.})
\]
Observe that Theorem 15 guarantees that the mild solution of (3.0.1) is the uniform limit of the strict solutions of the approximate problems
\[
  \begin{cases} 
    u'_n - Au_n = f_n \\
    u_n(0) = u_n^0.
  \end{cases}
\]
Let $u^0 \in X$ and $f \in C([0, T]; X)$. 

**Definition 21 (Classical solutions)** A classical solution of (3.0.1) is a function $u \in C([0, T]; X)$ such that
\begin{enumerate}
  \item[(a)] $u \in C^1([0, T]; X) \cap C([0, T]; D(A))$;
  \item[(b)] $u(0) = u^0$;
  \item[(c)] $u'(t) = Au(t) + f(t)$ for all $t \in [0, T]$.
\end{enumerate}
We now show that any classical solution coincides with the mild solution.

**Proposition 16** Let $u$ be a classical solution of (3.0.1). Then $u$ equals the mild solution given by (3.1.1).

**Proof.** Let $u$ be a classical solution of (3.0.1). Then, for any fixed $t \in [0, T]$ we have that $s \mapsto S(t - s)u(s)$ is continuous on $[0, t]$, differentiable on $]0, t[$, and
\[
  \frac{d}{ds} \left( S(t - s)u(s) \right) = S(t - s)f(s) \quad (s \in ]0, t[).
\]
By integrating over $[0, t]$ we deduce that $u$ is given by (3.1.1). \qed
3.2 Regularity

Our first result guarantees that the mild solution of (3.0.1) is classical when \( f \) has better “space regularity”.

**Theorem 16** Let \( u^0 \in D(A) \) and let \( f \in L^2(0,T;D(A)) \cap C([0,T];X) \). Then the mild solution \( u \) of problem (3.0.1) is classical. Moreover,

\[
    u \in C^1([0,T];X) \cap C([0,T];D(A)). \tag{3.2.1}
\]

We begin the proof by studying the case of \( u^0 = 0 \).

**Lemma 4** For any \( f \in L^2(0,T;D(A)) \cap C([0,T];X) \) define

\[
    F_A(t) = \int_0^t S(t-s)f(s)\,ds \quad (t \in [0,T]). \tag{3.2.2}
\]

Then \( F_A \in C^1([0,T];X) \cap C([0,T];D(A)) \) and

\[
    F'_A(t) = AF_A(t) + f(t) \quad \forall t \in [0,T]. \tag{3.2.3}
\]

**Proof.** Since \( f \in L^2(0,T;D(A)) \) we have that, for any \( t \in [0,T] \),

\[
    A \int_0^t S(t-s)f(s)\,ds = \int_0^t S(t-s)Af(s)\,ds.
\]

So, \( F_A \in C([0,T];D(A)) \) on account of Proposition 22.

Next, in order to prove that \( F_A \in C^1([0,T];X) \), fix \( t \in [0,T] \) and let \( 0 < h < T - t \). Then

\[
    \frac{F_A(t+h) - F_A(t)}{h} = \frac{1}{h} \left\{ \int_0^{t+h} S(t+h-s)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds \right\} = \frac{S(h) - I}{h} F_A(t) + \frac{1}{h} \int_t^{t+h} S(t+h-s)f(s)\,ds.
\]

Now,

\[
    \lim_{h \downarrow 0} \frac{S(h) - I}{h} F_A(t) = AF_A(t)
\]

because \( F_A \in C([0,T];D(A)) \). Also,

\[
    \lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} S(t+h-s)f(s)\,ds = f(t)
\]

because \( f \in C([0,T];X) \). Therefore, \( F_A \) is of class \( C^1([0,T];X) \) and satisfies (3.2.3). \( \square \)
Proof of Theorem 16. Let $u$ be the mild solution of problem (3.0.1). Then

$$u(t) = S(t)u^0 + F_A(t) \quad \forall t \in [0, T],$$

where $F_A$ is defined in (3.2.2). The conclusion follows from Theorem 3 and Lemma 4. □

We will now show a similar result if $f$ has better “time regularity”.

Theorem 17 Let $u^0 \in D(A)$ and let $f \in H^1(0, T; X)$. Then the mild solution $u$ of problem (3.0.1) is classical and satisfies (3.2.1).

The proof is similar to the one above. One has just to replace Lemma 4 with the following one.

Lemma 5 For any $f \in H^1(0, T; X)$ let $F_A$ be defined as in (3.2.2). Then $F_A \in C([0, T]; D(A))$ and

$$F_A'(t) = AF_A(t) + f(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds \quad (t \in [0, T]).$$

Proof. Since $F_A$ can be rewritten as

$$F_A(t) = \int_0^t S(s)f(t-s)ds \quad (t \in [0,T]),$$

by differentiating the integral we conclude that

$$F'_A(t) = S(t)f(0) + \int_0^t S(s)f'(t-s)ds$$

$$= S(t)f(0) + \int_0^t S(t-s)f'(s)ds \quad \forall t \in [0,T].$$

Now, Proposition 22 implies that $F_A \in C([0,T]; X)$. Moreover, returning to definition (3.2.2), for all $t \in [0,T]$ we also have that

$$F_A(t) = \lim_{h \downarrow 0} \frac{1}{h} \left\{ \int_0^{t+h} S(t+h-s)f(s)ds - \int_0^t S(t-s)f(s)ds \right\}$$

$$= \lim_{h \downarrow 0} \left\{ \frac{S(h) - I}{h} F_A(t) + \frac{1}{h} \int_t^{t+h} S(t+h-s)f(s)ds \right\}. $$

Since $H^1(0,T;X) \subset C([0,T]; X)$, we have that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} S(t+h-s)f(s)ds = f(t).$$

The above identity implies that $F_A(t) \in D(A)$ and

$$AF_A(t) = F'_A(t) - f(t) \quad \forall t \in [0,T].$$

Consequently, $F_A \in C([0,T]; D(A))$ and the proof is complete. □
Example 22 In general, the mild solution of (3.0.1) fails to be classical assuming just $f \in C([0,T]; X)$. Indeed, let $w \in X \setminus D(A)$ and take $f(t) = S(t)w$. Then the mild solution of (3.0.1) with $u^0 = 0$ is given by

$$u(t) = tS(t)w \quad \forall t \geq 0$$

which fails to be differentiable for $t > 0$. □

Exercise 32 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^2$. Give conditions on $f \in L^2([0,T] \times \Omega), u^0 : \Omega \to \mathbb{R}$, and $u^1 : \Omega \to \mathbb{R}$ which guarantee the existence and uniqueness of the classical solution to inhomogeneous wave equation

$$\begin{cases}
\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u + f(t,x) & \text{in } ]0, \infty[ \times \Omega \\
u = 0 & \text{on } ]0, \infty[ \times \partial \Omega} \\
u(0,x) = u^0(x), \frac{\partial u}{\partial t}(0,x) = u^1(x) & x \in \Omega
\end{cases}$$

(3.2.4)

Solution. Let $A$ be defined as in Example 15. Then, applying Theorem 16 and Theorem 17 we conclude that the above problem has a unique classical solution if

(i) $(u^0, u^1) \in D(A)$, that is, $u^0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $u^1 \in H^1_0(\Omega)$;

(ii) $f$ satisfies any of the following conditions

(a) $f \in C([0,T]; L^2(\Omega))$, $\frac{\partial f}{\partial x} \in L^2([0,T] \times \Omega)$, and $f(t, \cdot)_{|\partial \Omega} = 0$, or

(b) $\frac{\partial f}{\partial t} \in L^2([0,T] \times \Omega)$. □

For special classes of generators, one can show that mild solutions are strict under rather weak conditions.

Theorem 18 Let $A : D(A) \subset X \to X$ be a densely defined self-adjoint dissipative operator. Then, for any $u^0 \in [D(A), X]_{1/2}$ and $f \in C([0,T]; X)$, the mild solution $u$ of problem (3.0.1) is strict.

As above, we begin the proof by studying the case of $u^0 = 0$.

Lemma 6 Let $A : D(A) \subset X \to X$ be a densely defined self-adjoint dissipative operator. For any $f \in C([0,T]; X)$ let $F_A$ be defined as in (3.2.2). Then $F_A \in H^1(0,T; X) \cap L^2(0,T; D(A))$ and

$$F'_A(t) = AF_A(t) + f(t) \quad \text{a.e. } t \in [0,T].$$

Moreover, $t \mapsto \langle AF_A(t), F_A(t) \rangle$ is absolutely continuous

$$\frac{d}{dt} \langle AF_A(t), F_A(t) \rangle = 2 \Re \langle F'_A(t), AF_A(t) \rangle \quad \text{a.e. } t \in [0,T],$$

and

$$\|AF_A\|_2 \leq \|f\|_2.$$
Proof. Define
\[ f_n(t) = nR(n, A)f(t) \quad \text{and} \quad F_n(t) = nR(n, A)F_A(t) \quad \forall t \in [0, T]. \]
Then \( f_n \in C([0, T]; D(A)) \) for every \( n \) and
\[ F_n(t) = \int_a^t S(t-s)f_n(s)\,ds \quad (t \in [0, T]). \]
Owing to Lemma 4, we have that \( F_n \in C^1([0, T]; X) \cap C([0, T]; D(A)) \) and
\[ F'_n(t) = AF_n(t) + f_n(t) \quad \forall t \in [0, T]. \quad (3.2.8) \]
Moreover,
\[ 2\int_0^t \Re \langle F'_n(s), AF_n(s) \rangle \,ds = \langle AF_n(t), F_n(t) \rangle \leq 0 \quad \forall t \in [0, T] \]
because \( A \) is dissipative. Therefore, by multiplying each member of (3.2.8) by \( 2AF_n(t) \), taking real parts, and integrating over \([0, T]\) we obtain
\[ 2\int_0^T |AF_n(t)|^2\,dt \leq -2\int_0^T \Re \langle f_n(t), AF_n(t) \rangle \,dt \leq \int_0^T \left( |f_n(t)|^2 + |AF_n(t)|^2 \right)\,dt. \]
Hence
\[ \int_0^T |AF_n(t)|^2\,dt \leq \int_0^T |f_n(t)|^2\,dt \leq \int_0^T |f(t)|^2\,dt. \]
Thus, \( \{F_n\}_n \) is bounded in \( H^1(0, T; X) \cap L^2(0, T; D(A)) \). Therefore, there exists a subsequence \( \{F_{n_k}\}_k \) and a function \( F_\infty \) such that
\[ F_{n_k} \xrightarrow{n \to \infty} F_\infty \quad \text{in} \quad H^1(0, T; X) \cap L^2(0, T; D(A)). \]
Recalling that \( F_{n_k} \xrightarrow{n \to \infty} F_A \) in \( C([0, T]; X) \) by Theorem 15, we conclude that \( F_\infty = F_A \in H^1(0, T; X) \cap L^2(0, T; D(A)) \).
Now, fix any \( g \in L^2(0, T; X) \). Then, taking the product of each member of (3.2.8)—for \( n = n_k \)—with \( g \) we have that
\[ \int_0^T \langle F'_{n_k}(t), g(t) \rangle \,dt = \int_0^T \langle AF_{n_k}(t) + f_{n_k}(t), g(t) \rangle \,dt. \]
So, in the limit as \( n \to \infty \),
\[ \int_0^T \langle F'_A(t) - AF_A(t) - f(t), g(t) \rangle \,dt = 0 \quad \forall g \in L^2(0, T; X) \]
which in turn yields $F_A'(t) = AF_A(t) + f(t)$ for a.e. $t \in [0, T]$. 

Proof of Theorem 18. Let $u$ be the mild solution of problem (3.0.1). Then

$$u(t) = u^0(t) + F_A(t) \quad \forall t \in [0, T],$$

where

(i) $u^0(t) := S(t)u^0$ belongs to $H^1(0, T; X) \cap L^2(0, T; D(A))$ and satisfies $\frac{d}{dt} u^0(t) = Au^0(t)$ for every $t > 0$ thanks to Theorem 13 and Theorem 14;

(ii) $F_A$, defined in (3.2.2), belongs to $H^1(0, T; X) \cap L^2(0, T; D(A))$ and satisfies (3.2.5) owing to Lemma 6.

The conclusion by combining (i) and (ii).

Example 23 We can use Theorem 18 to study the problem

$$\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x) & (t, x) \in (0, T) \times (0, \pi) \text{ a.e.} \\
u(t, 0) = 0 = u(t, \pi) & t \in (0, T) \\
u(0, x) = u^0(x) & x \in (0, \pi).
\end{cases} \quad (3.2.9)$$

Recalling Example 21, we conclude that for all

$$f \in C([0, T]; L^2(0, \pi)) \quad \text{and} \quad u^0 \in H^1_0(0, \pi)$$

problem (3.2.9) has a unique solution $u$. In particular, such a solution satisfies:

$$\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2} \in L^2((0, T) \times (0, \pi)) \quad \text{and} \quad t \mapsto u(t, \cdot) \in H^1_0(0, \pi) \text{ is continuous.}$$
4 Appendix A: Cauchy integral on $C([a, b]; X)$

We recall the construction of the Riemann integral for a continuous function $f : [a, b] \rightarrow X$, where $X$ is a Banach space and $-\infty < a < b < \infty$.

Let us consider the family of partitions of $[a, b]$

$$
\Pi(a, b) = \left\{ \pi = \{ t_i \}_{i=0}^n : n \geq 1, a = t_0 < t_1 < \ldots < t_n = b \right\}
$$

and define

$$
diam(\pi) = \max_{1 \leq i \leq n} (t_i - t_{i-1}) \quad (\pi \in \Pi(a, b)).
$$

For any $\pi \in \Pi(a, b), \pi = \{ t_i \}_{i=0}^n$, we set

$$
\Sigma(\pi) = \left\{ \sigma = (s_1, \ldots, s_n) : s_i \in [t_{i-1}, t_i], 1 \leq i \leq n \right\}.
$$

Finally, for any $\pi \in \Pi(a, b), \pi = \{ t_i \}_{i=0}^n$, and $\sigma \in \Sigma(\pi), \sigma = (s_1, \ldots, s_n)$, we define

$$
S_\sigma^n(f) = \sum_{i=1}^n f(s_i)(t_i - t_{i-1}).
$$

**Theorem 19** The limit

$$
\lim_{diam(\pi) \downarrow 0} S_\sigma^n(f) =: \int_a^b f(t)dt
$$

exists uniformly for $\sigma \in \Sigma(\pi)$.

**Lemma 7** For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\pi, \pi' \in \Pi(a, b)$ with $\pi \subseteq \pi'$ we have that

$$
diam(\pi) < \delta \quad \Longrightarrow \quad |S_\sigma^n(f) - S_{\sigma'}^n(f)| < \varepsilon
$$

for all $\sigma \in \Sigma(\pi)$ and $\sigma' \in \Sigma(\pi')$.

**Proof.** Since $f$ is uniformly continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t, s \in [a, b]$

$$
|t - s| < \delta \quad \Longrightarrow \quad |f(t) - f(s)| < \frac{\varepsilon}{b - a}.
$$

(4.0.1)

Let

$$
\begin{align*}
\pi = \{ t_i \}_{i=0}^n \quad &\sigma = (s_1, \ldots, s_n) \\
\pi' = \{ t'_j \}_{j=0}^m \quad &\sigma' = (s'_1, \ldots, s'_m)
\end{align*}
$$

be such that $\pi \subseteq \pi'$ and $diam(\pi) < \delta$. Then there exist positive integers

$$
0 = j_0 < j_1 < \cdots < j_n = m
$$
such that $t'_{ji} = t_i$ for all $i = 0, \ldots, n$. For any such $i$, it holds that

$$t_i - t_{i-1} = t'_{ji} - t'_{ji-1} = \sum_{j=ji-1+1}^{ji} (t'_j - t'_{j-1}).$$

Then

$$S^n_\pi(f) - S^n_{\pi'}(f) = \sum_{i=1}^{n} f(s_i)(t_i - t_{i-1}) - \sum_{j=1}^{m} f(s'_j)(t'_j - t'_{j-1})$$

$$= \sum_{i=1}^{n} \sum_{j=ji-1+1}^{ji} (f(s_i) - f(s'_j))(t'_j - t'_{j-1}).$$

Since for all $i = 1, \ldots, n$ we have that $s_i, s'_j \in [t_{i-1}, t_i]$ $\forall j_{i-1} + 1 \leq j \leq ji,$

from (4.0.1) it follows that

$$\left| S^n_\pi(f) - S^n_{\pi'}(f) \right| \leq \sum_{i=1}^{n} \sum_{j=ji-1+1}^{ji} |f(s_i) - f(s'_j)|(t'_j - t'_{j-1})$$

$$\leq \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \varepsilon.$$

The proof is complete.

Proof of Theorem 19. For any given $\varepsilon > 0$ let $\delta$ be as in Lemma 7. Let $\pi, \pi' \in \Pi(a, b)$ be such that $\text{diam}(\pi) < \delta$ and $\text{diam}(\pi') < \delta$. Finally, let $\sigma \in \Sigma(\pi)$ and $\sigma' \in \Sigma(\pi')$. Define $\pi'' = \pi \cup \pi'$ and fix any $\sigma'' \in \Sigma(\pi'')$. Then

$$\left| S^n_\pi(f) - S^n_{\pi'}(f) \right| \leq \left| S^n_\pi(f) - S^n_{\pi''}(f) \right| + \left| S^n_{\pi''}(f) - S^n_{\pi'}(f) \right| < 2\varepsilon.$$

This completes the proof since $\varepsilon$ is arbitrary.

Proposition 17 For any $f, g \in C([a, b]; X)$ and $\lambda \in \mathbb{C}$ we have that

$$\int_{a}^{b} (f(t) + g(t))dt = \int_{a}^{b} f(t)dt + \int_{a}^{b} g(t)dt$$

$$\int_{a}^{b} \lambda f(t)dt = \lambda \int_{a}^{b} f(t)dt$$

$$\left| \int_{a}^{b} f(t)dt \right| \leq \int_{a}^{b} |f(t)|dt.$$
Moreover, for any $\phi \in X^*$ we have that
\[ \langle \phi, \int_a^b f(t) dt \rangle = \int_a^b \langle \phi, f(t) \rangle dt. \] (4.0.2)
and, for any $\Lambda \in \mathcal{L}(X)$ we have that
\[ \Lambda \int_a^b f(t) dt = \int_a^b \Lambda f(t) dt. \] (4.0.3)

Proof. Exercise. \qed

**Proposition 18** For any $f \in C^1([a,b]; X)$ we have that
\[ \int_a^b f'(t) dt = f(b) - f(a) \] (4.0.4)

Proof. By (4.0.2) above, for any $\phi \in X^*$ we have that
\[ \langle \phi, \int_a^b f'(t) dt \rangle = \int_a^b \langle \phi, f'(t) \rangle dt. \]
On the other hand, the function $t \mapsto \langle \phi, f(t) \rangle$ is continuously differentiable on $[a,b]$ with derivative equal to $\langle \phi, f'(t) \rangle$. Therefore, for any $\phi \in X^*$,
\[ \int_a^b \langle \phi, f'(t) \rangle dt = \langle \phi, f(b) - f(a) \rangle. \]
Since $X^*$ separates points, the above identity yields (4.0.4). \qed

**Corollary 6** Let $f \in C^1([a,b]; X)$ be such that $f'(t) = 0$ for all $t \in [a,b]$. Then $f$ is constant.
5 Appendix B: Lebesgue integral on $L^2(a, b; H)$

Let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and let $\{e_j\}_{j \in \mathbb{N}}$ be a complete orthonormal system in $H$.

The Hilbert space $L^2(a, b; H)$

**Definition 22** A function $f : [a, b] \to H$ is said to be Borel (resp. Lebesgue) measurable if so is the scalar function $t \mapsto \langle f(t), u \rangle$ for every $u \in H$.

**Remark 8** Let $f : [a, b] \to H$.

1. Since, for any $x \in H$,
   \[ \langle f(t), x \rangle = \sum_{j=1}^{\infty} \langle f(t), e_j \rangle \langle x, e_j \rangle \quad (t \in [a, b]), \]
   we conclude that $f$ is Borel (resp. Lebesgue) measurable if and only if so is the scalar function $t \mapsto \langle f(t), e_j \rangle$ for every $j \in \mathbb{N}$.

2. Since
   \[ |f(t)|^2 = \sum_{j=1}^{\infty} |\langle f(t), e_j \rangle|^2 \quad (t \in [a, b]), \]
   we have that, if $f$ is Borel (resp. Lebesgue) measurable, then so is the scalar function $t \mapsto \|f(t)\|$.

**Definition 23** We denote by $L^2(a, b; H)$ the space of all Lebesgue measurable functions $f : [a, b] \to H$ such that
   \[ \|f\|_2 := \left( \int_a^b |f(t)|^2 dt \right)^{1/2} < \infty, \]
   where two functions $f$ and $g$ are identified if $f(t) = g(t)$ for a.e. $t \in [a, b]$.

**Proposition 19** $L^2(a, b; H)$ is a Hilbert space with the hermitian product
   \[ (f|g)_0 = \int_a^b \langle f(t), g(t) \rangle dt \quad (f, g \in L^2(a, b; H)). \]

**Proof.** We only prove completeness. Let $\{f_n\}$ be a Cauchy sequence in $L^2(a, b; H)$. Then $\{f_n\}$ is bounded:
   \[ \|f_n\|_2^2 = \int_a^b \sum_{j=1}^{\infty} |\langle f_n(t), e_j \rangle|^2 dt \leq M \quad \forall n \in \mathbb{N} \quad (5.0.1) \]
Moreover, for any $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that, for all $m, n \geq \nu$,

$$
\|f_n - f_m\|_2^2 = \int_a^b \sum_{j=1}^{\infty} |\langle f_n(t) - f_m(t), e_j \rangle|^2 \, dt \leq \varepsilon \tag{5.0.2}
$$

Therefore, $t \mapsto \langle f_n(t), e_j \rangle$ is a Cauchy sequence in $L^2(a, b)$ for all $j \in \mathbb{N}$. So, there exists functions $\phi_j \in L^2(a, b)$ such that $\langle f_n(\cdot), e_j \rangle \to \phi_j$ in $L^2(a, b)$ for all $j \in \mathbb{N}$. Thus, by Fatou’s lemma,

$$
\int_a^b \sum_{j=1}^{\infty} |\phi_j(t)|^2 \, dt \leq M \quad \text{and} \quad \int_a^b \sum_{j=1}^{\infty} |\langle f_n(t), e_j \rangle - \phi_j(t)\rangle|^2 \, dt \leq \varepsilon \quad (\forall n \geq \nu).
$$

So, we conclude that

$$
f(t) := \sum_{j=1}^{\infty} \phi_j(t)e_j \in H \quad t \in [a, b] \text{ a.e.}
$$

as well as $f \in L^2(a, b; H)$ and

$$
\int_a^b |f_n(t) - f(t)|^2 \, dt \leq \varepsilon \quad (\forall n \geq \nu),
$$

or, $f_n \to f$ in $L^2(a, b; H)$.

**Remark 9** For any $f \in L^2(a, b; H)$ we have that

$$
\sum_{j=1}^{\infty} \int_a^b |\langle f(t), e_j \rangle| \, dt \leq (b-a) \sum_{j=1}^{\infty} \int_a^b |\langle f(t), e_j \rangle|^2 \, dt < \infty.
$$

Therefore

$$
\sum_{j=1}^{\infty} e_j \int_a^b \langle f(t), e_j \rangle \, dt \in H.
$$

**Definition 24** For any $f \in L^2(a, b; H)$ we define

$$
\int_a^b f(t) \, dt = \sum_{j=1}^{\infty} e_j \int_a^b \langle f(t), e_j \rangle \, dt.
$$

**Proposition 20** For any $f \in L^2(a, b; H)$ the following properties hold true.

(a) For any $x \in H$ we have that

$$
\langle x, \int_a^b f(t) \, dt \rangle = \int_a^b \langle x, f(t) \rangle \, dt
$$
(b) \[ \left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt \]

(c) For any \( \Lambda \in \mathcal{L}(H) \) we have that
\[ \Lambda \left( \int_a^b f(t) dt \right) = \int_a^b \Lambda f(t) dt . \]

Proof. Exercise □

**Proposition 21** Let \( A : D(A) \subset H \to H \) be a closed linear operator with \( \rho(A) \neq \emptyset \). Then for any \( f \in L^2(a,b; D(A)) \) we have that
\[ \int_a^b f(t) dt \in D(A) \quad \text{and} \quad A \left( \int_a^b f(t) dt \right) = \int_a^b A f(t) dt . \]

Proof. Exercise (hint: recall that, in view of Exercise 23, \( D(A) \) is separable with respect to the graph norm). □

**Proposition 22** Let \( A : D(A) \subset H \to H \) be the infinitesimal generator of a \( \mathcal{C}_0 \)-semigroup on \( H \), \( S(t) \), which satisfies the growth condition (1.6.3). Then, for any \( f \in L^2(a,b;H) \),

(a) for any \( t \in [a,b] \) the function \( s \mapsto S(t-s)f(s) \) belongs to \( L^2(a,t;H) \), and

(b) the function
\[ F_A(t) = \int_a^t S(t-s)f(s) ds \quad (t \in [a,b]) \]
belongs to \( C([a,b];H) \).

Proof. In order to check measurability for \( s \mapsto S(t-s)f(s) \) it suffices to observe that, for all \( u \in H \) and a.e. \( s \in [0,t] \),
\[ \langle S(t-s)f(s), u \rangle = \langle f(s), S(t-s)^*u \rangle = \sum_{j=1}^{\infty} \langle f(s), e_j \rangle \overline{\langle S(t-s)^*u, e_j \rangle} . \]
Since \( s \mapsto \langle S(t-s)^*u, e_j \rangle \) is continuous and \( s \mapsto \langle f(s), e_j \rangle \) is measurable for all \( j \in \mathbb{N} \), the measurability of \( s \mapsto S(t-s)f(s) \) follows. Moreover, by (1.6.3) we have that
\[ |S(t-s)f(s)| \leq M e^{\omega(t-s)} |f(s)| \quad (s \in [a,t] \text{ a.e.}), \]
which completes the proof of (a).
In order to prove point (b), fix \( t \in ]a, b[ \) and let \( t_n \to t \). Fix \( \delta \in ]0, t - a[ \) and let \( n_\delta \in \mathbb{N} \) be such that \( t_n > t - \delta \) for all \( n \geq n_\delta \). Then we have that

\[
|F_A(t_n) - F_A(t)| \\
\leq \int_a^{t - \delta} \left| \left[ S(t_n - s)f(s) - S(t - s) \right] f(s) \right| ds \\
+ \int_{t - \delta}^{t_n} \left| S(t_n - s)f(s) \right| ds + \int_{t - \delta}^{t} \left| S(t - s)f(s) \right| ds.
\]

To complete the proof it suffices to observe that

\[
\lim_{n \to \infty} \int_a^{t - \delta} \left| \left[ S(t_n - s)f(s) - S(t - s) \right] f(s) \right| ds = 0
\]

by the dominated convergence theorem, while the remaining terms on the right-hand side of the above inequality are small with \( \delta \). □

**The Sobolev space** \( H^1(a, b; H) \)

**Definition 25** \( H^1(a, b; H) \) is the space of all functions \( u \in C([a, b]; H) \) such that

(a) \( u'(t) \) exists for a.e. \( t \in [a, b] \);

(b) \( u' \in L^2(a, b; H) \);

(c) \( u(t) - u(a) = \int_a^t u'(s)ds \quad t \in [a, b] \) a.e.

**Remark 10** \( H^1(a, b; H) \) is a Hilbert space with the scalar product

\[
(u|v)_1 = \int_a^b \left[ \langle u(t), v(t) \rangle + \langle u'(t), v'(t) \rangle \right] dt \quad (u, v \in H^1(a, b; H)).
\]
Bibliography


