Lecture Notes on Evolution Equations

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Contents

Notation

- $\mathbb{R} = (-\infty, \infty)$ stands for the real line, $\mathbb{R}_+$ for $[0, \infty)$, and $\mathbb{R}_+^*$ for $(0, \infty)$.
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\} = \{\pm 1, \pm 2, \dots\}$.
- For any $\tau \in \mathbb{R}$ we denote by $\lceil \tau \rceil$ and $\{\tau\}$ the integer and the fractional part of τ, respectively, defined as

$$\lceil \tau \rceil = \max\{m \in \mathbb{Z} \ : \ m \leqslant \tau\} \qquad \{\tau\} = \tau - \lceil \tau \rceil.$$

- For any $\lambda \in \mathbb{C}$, $\Re \lambda$ and $\Im \lambda$ denote the real and imaginary parts of λ, respectively.
- $|\cdot|$ stands for the norm of a Banach space X, as well as for the absolute value of a real number or the modulus of a complex number.
- Generic elements of X will be denoted by $u, v, w \dots$
- $\mathcal{L}(X)$ is the Banach space of all bounded linear operators $\Lambda : X \to X$ equipped with the uniform norm $\|\Lambda\| = \sup_{|u| \leqslant 1} |\Lambda u|$.
- For any metric space (X, d), $\mathcal{C}_b(X)$ denotes the Banach space of all bounded uniformly continuous functions $f : X \to \mathbb{R}$ with norm

$$\|f\|_{\infty, X} = \sup_{u \in X} |f(x)|.$$

  For any $f \in \mathcal{C}_b(X)$ and $\delta > 0$ we call

$$\operatorname{osc}_f(\delta) = \sup\big\{|f(x) - f(y)| \ : \ x, y \in X \,,\ d(x, y) \leqslant \delta\big\}$$

  the *oscillation* of f over sets of diameter δ.

- Given a Banach space $(X, |\cdot|)$ and a closed interval $I \subseteq \mathbb{R}$ (bounded or unbounded), we denote by $\mathcal{C}_b(I; X)$ the Banach space of all bounded uniformly continuous functions $f : I \to X$ with norm

$$\|f\|_{\infty, I} = \sup_{s \in I} |f(s)|.$$

  We denote by $\mathcal{C}_b^1(I; X)$ the subspace of $\mathcal{C}_b(I; X)$ consisting of all functions f such that the derivative

$$f'(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t}$$

  exists for all $t \in I$ and belongs to $\mathcal{C}_b(I; X)$.

- $D(A)$ denotes the domain of a linear operator $A : D(A) \subset X \to X$.
- $\Pi_\omega = \big\{\lambda \in \mathbb{C} \ : \ \Re \lambda > \omega\big\}$ for any $\omega \in \mathbb{R}$.

1 *Semigroups of bounded linear operators*

Preliminaries

Let $(X, |\cdot|)$ be a (real or complex) Banach space. We denote by $\mathcal{L}(X)$ the Banach space of all bounded linear operators $\Lambda : X \to X$ with norm

$$\|\Lambda\| = \sup_{|u|\leqslant 1} |\Lambda u|.$$

We recall that, for any given $A, B \in \mathcal{L}(X)$, the product AB remains in $\mathcal{L}(X)$ and we have that

$$\|AB\| \leqslant \|A\| \, \|B\|. \tag{1.0.1}$$

So, $\mathcal{L}(X)$ ia a Banach algebra.

Proposition 1 *Let $A \in \mathcal{L}(X)$ be such that $\|A\| < 1$. Then $(I - A)^{-1} \in \mathcal{L}(X)$ and*

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n. \tag{1.0.2}$$

Proof. We observe that the series on the right-hand side of (1.0.2) is totally convergent in $\mathcal{L}(X)$. So,

$$\Lambda := \sum_{n=0}^{\infty} A^n \in \mathcal{L}(X).$$

Moreover,

$$(I - A)\Lambda = \sum_{n=0}^{\infty} (I - A)A^n = I = \sum_{n=0}^{\infty} A^n(I - A) = \Lambda(I - A). \qquad \square$$

1.1 Uniformly continuous semigroups

Definition 1 *A semigroup of bounded linear operators on X is a map*

$$S : [0, \infty) \to \mathcal{L}(X)$$

with the following properties:

(*a*) $S(0) = I$,

(*b*) $S(t + s) = S(t)S(s)$ *for all* $t, s \geqslant 0$.

We will use the equivalent notation $\{S(t)\}_{t\geqslant 0}$ and the abbreviated form $S(t)$.

Definition 2 *The* infinitesimal generator *of a semigroup of bounded linear operators $S(t)$ is the map $A : D(A) \subset X \to X$ defined by*

$$\begin{cases} D(A) = \left\{ u \in X \ : \ \exists \lim_{t \downarrow 0} \frac{S(t)u - u}{t} \right\} \\ Au = \lim_{t \downarrow 0} \frac{S(t)u - u}{t} \qquad \forall u \in D(A) \end{cases} \tag{1.1.1}$$

Exercise 1 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a semigroup of bounded linear operators $S(t)$. Prove that

(a) $D(A)$ is a subspace of X,

(b) A is a linear operator.

Definition 3 *A semigroup $S(t)$ of bounded linear operators on X is* uniformly continuous *if*

$$\lim_{t \downarrow 0} \|S(t) - I\| = 0.$$

Proposition 2 *Let $S(t)$ be a uniformly continuous semigroup of bounded linear operators. Then there exists $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that*

$$\|S(t)\| \leqslant M e^{\omega t} \qquad \forall t \geqslant 0.$$

Proof. Let $\tau \geqslant 0$ be such that $\|S(t) - I\| \leqslant 1/2$ for all $t \in [0, \tau]$. Then

$$\|S(t)\| \leqslant \|I\| + \|S(t) - I\| \leqslant \frac{3}{2} \qquad \forall t \in [0, \tau].$$

Since every $t \geqslant 0$ can be represented as $t = \lceil t/\tau \rceil \tau + \{t/\tau\}\tau$, we have that

$$\|S(t)\| \leqslant \|S(\tau)\|^{\lceil t/\tau \rceil} \left\| S\Big(\Big\{\frac{t}{\tau}\Big\}\tau\Big) \right\| \leqslant \Big(\frac{3}{2}\Big)^{\lceil t/\tau \rceil + 1} \leqslant \Big(\frac{3}{2}\Big)^{\frac{t}{\tau} + 1} = M e^{\omega t}$$

with $M = 3/2$ and $\omega = \log(3/2)/\tau$. □

Corollary 1 *A semigroup $S(t)$ is uniformly continuous if and only if*

$$\lim_{s \to t} \|S(s) - S(t)\| = 0 \qquad \forall t \geqslant 0.$$

Example 1 let $A \in \mathcal{L}(X)$. Then

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

is a uniformly continuous semigroup of bounded linear operators on X. More precisely, the following properties hold.

(*a*) $\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ converges for all $t \geqslant 0$ and $e^{tA} \in \mathcal{L}(X)$.

Proof. Indeed, the series is totally convergent in $\mathcal{L}(X)$ because

$$\sum_{n=0}^{\infty} \Big\| \frac{t^n}{n!} A^n \Big\| \leqslant \sum_{n=0}^{\infty} \frac{t^n}{n!} \|A\|^n < \infty. \qquad \square$$

(*b*) $e^{(t+s)A} = e^{tA} e^{sA}$ for all $s, t \geqslant 0$.

Proof. We have that

$$\begin{aligned} e^{(t+s)A} &= \sum_{n=0}^{\infty} \frac{(t+s)^n}{n!} A^n = \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \binom{n}{k} t^k s^{(n-k)} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{t^k A^k}{k!} \frac{s^{(n-k)} A^{(n-k)}}{(n-k)!} \end{aligned}$$

where the last term coincides with the Cauchy product of the two series giving e^{tA} and e^{sA}. □

(*c*) $Ae^{tA} = e^{tA} A$ for all $t \geqslant 0$.

(*d*) $\|e^{tA} - I\| = \| \sum_{n=1}^{\infty} \frac{t^n}{n!} A^n \| \leqslant t\|A\| e^{t\|A\|}$ for all $t \geqslant 0$.

(*e*) $\| \frac{e^{tA} - I}{t} - A\| = \| \sum_{n=2}^{\infty} \frac{t^{n-1}}{n!} A^n \| \leqslant t\|A\|^2 e^{t\|A\|}$ for all $t \geqslant 0$.

Notice that property (*e*) shows that A is the infinitesimal generator of e^{tA}.

Theorem 1 *For any linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(*a*) *A is the infinitesimal generator of a uniformly continuous semigroup,*

(*b*) *$D(A) = X$ and $A \in \mathcal{L}(X)$.*

Proof. Example 1 shows that (*b*) ⇒ (*a*). Let us prove that (*a*) ⇒ (*b*). Let $\tau > 0$ be fixed such that

$$\Big\| I - \frac{1}{\tau} \int_0^{\tau} S(t) dt \Big\| < 1.$$

Then the bounded linear operator $\int_0^{\tau} S(t)dt$ is invertible. For all $h > 0$ we have that

$$\begin{aligned} \frac{S(h) - I}{h} \int_0^{\tau} S(t)dt &= \frac{1}{h}\Big(\int_0^{\tau} S(t+h)dt - \int_0^{\tau} S(t)dt \Big) \\ &= \frac{1}{h}\Big(\int_h^{\tau+h} S(t)dt - \int_0^{\tau} S(t)dt \Big) = \frac{1}{h}\Big(\int_{\tau}^{\tau+h} S(t)dt - \int_0^{h} S(t)dt \Big). \end{aligned}$$

Hence

$$\begin{array}{ccc} \frac{S(h)-I}{h} & = & \frac{1}{h}\Big(\int_\tau^{\tau+h} S(t)dt - \int_0^h S(t)dt\Big)\Big(\int_0^\tau S(t)dt\Big)^{-1} \\ \downarrow & h\downarrow 0 & \downarrow \\ A & = & \big(S(\tau)-I\big)\Big(\int_0^\tau S(t)dt\Big)^{-1}. \end{array}$$

This shows that $A \in \mathcal{L}(X)$. □

Let $A \in \mathcal{L}(X)$. For any $u_0 \in X$, a solution of the Cauchy problem

$$\begin{cases} u'(t) = Au(t) & t > 0 \\ u(0) = u_0 \end{cases} \tag{1.1.2}$$

is a function $u \in \mathcal{C}^1([0,\infty[;X)$ which satisfies (1.1.2) pointwise.

Proposition 3 *Problem* (1.1.2) *has a unique solution given by* $u(t) = e^{tA}u_0$.

Proof. The fact that $u(t) = e^{tA}u_0$ solves (1.1.2) follows from Example 1. Let $v \in \mathcal{C}^1([0,\infty[;X)$ be another solution of (1.1.2). Fix any $t > 0$ and set $U(s) = e^{(t-s)A}v(s)$ for all $s \in [0,t]$. Then

$$U'(s) = -Ae^{(t-s)A}v(s) + e^{(t-s)A}Av(s) = 0 \qquad \forall s \in [0,t].$$

Therefore, U is constant on $[0,t]$ by Corollary 6 of Appendix A. So, $v(t) = U(t) = U(0) = e^{tA}u_0$. □

Example 2 Consider the integral equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \int_0^1 k(x,y)u(t,y)\,dy & t > 0 \\ u(0,x) = u_0(x) \end{cases} \tag{1.1.3}$$

where $k \in L^2\big([0,1]\times[0,1]\big)$ and $u_0 \in L^2(0,1)$. Problem (1.1.3) can be recast in the abstract form (1.1.2) taking $X = L^2(0,1)$ and

$$Au(x) = \int_0^1 k(x,y)u(t,y)\,dy \qquad \forall x \in X.$$

Then Proposition 3 insures that (1.1.3) has a unique solution $u \in \mathcal{C}^1([0,\infty[;X)$ given by $u(t) = e^{tA}u_0$.

1.2 Strongly continuous semigroups

Example 3 (Translations on $\mathbb{R}$) Let $\mathcal{C}_b(\mathbb{R})$ be the Banach space of all bounded uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the uniform norm

$$|f|_{\infty,\mathbb{R}} = \sup_{x\in\mathbb{R}} |f(x)|.$$

For any $t \in \mathbb{R}_+$ define

$$\big(S(t)f\big)(x) = f(x+t) \qquad \forall f \in \mathcal{C}_b(\mathbb{R}).$$

The following holds true.

1. $S(t)$ is a semigroup of bounded linear operators on $\mathcal{C}_b(\mathbb{R})$.

2. $S(t)$ fails to be uniformly continuous.

   Proof. For any $n \in \mathbb{N}$ the function

   $$f_n(x) = e^{-nx^2} \qquad (x \in \mathbb{R})$$

   belongs to $\mathcal{C}_b(\mathbb{R})$ and has norm equal to 1. Therefore, for any $t > 0$

   $$\|S(t) - I\| \geqslant |S(t)f_n - f_n|_{\infty,\mathbb{R}} = \sup_{x\in\mathbb{R}} |e^{-n(x+t)^2} - e^{-nx^2}| \geqslant 1 - e^{-nt^2}.$$

   Since this is true for any n, we have that $\|S(t) - I\| \geqslant 1$. □

3. For all $f \in \mathcal{C}_b(\mathbb{R})$ we have that $|S(t)f - f|_{\infty,\mathbb{R}} \to 0$ as $t \downarrow 0$.

   Proof. Indeed,

   $$|S(t)f - f|_{\infty,\mathbb{R}} = \sup_{x\in\mathbb{R}} |f(x+t) - f(x)| \leqslant \mathrm{osc}_f(t) \xrightarrow{t\downarrow 0} 0$$ □

Definition 4 *A semigroup $S(t)$ of bounded linear operators on X is called* strongly continuous *(or of* class $\mathcal{C}_0$*, or even a* $\mathcal{C}_0$-semigroup*) if*

$$\lim_{t\downarrow 0} S(t)u = u \qquad \forall u \in X. \tag{1.2.1}$$

Theorem 2 *Let $S(t)$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Then there exist $\omega \geqslant 0$ and $M \geqslant 1$ such that*

$$\|S(t)\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0. \tag{1.2.2}$$

Proof. We first prove the following:

$$\exists \tau > 0 \quad \text{and} \quad M \geqslant 1 \quad \text{such that} \quad \|S(t)\| \leqslant M \quad \forall t \in [0, \tau]. \tag{1.2.3}$$

We argue by contradiction assuming there exists a sequence $t_n \downarrow 0$ such that $\|S(t_n)\| \geqslant n$ for all $n \geqslant 1$. Then, the principle of uniform boundedness implies that, for some $u \in X$, $\|S(t_n)u\| \to \infty$ as $n \to \infty$, in contrast with (1.2.1).

Now, given $t \in \mathbb{R}_+$, let $n \in \mathbb{N}$ and $\delta \in [0, \tau[$ be such that

$$t = n\tau + \delta.$$

Then, in view of (1.2.3),

$$\|S(t)\| = \|S(\delta)S(\tau)^n\| \leqslant M \cdot M^n = M \cdot (M^{1/\tau})^{n\tau} \leqslant M \cdot (M^{1/\tau})^t$$

which yields (1.2.2) with $\omega = \frac{\log M}{\tau}$. □

Corollary 2 *Let S(t) be a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Then for every $u \in X$ the map $t \mapsto S(t)u$ is continuous from $\mathbb{R}_+$ into X.*

Definition 5 *A $\mathcal{C}_0$-semigroup of bounded linear operators on X is called* uniformly bounded *if S(t) satisfies* (1.2.2) *with $\omega = 0$. If, in addition, $M = 1$, we say that $S(t)$ is a* contraction semigroup*.*

Exercise 2 Prove that the translation semigroup of Example 3 satisfies

$$\|S(t)\| = 1 \qquad \forall t \geqslant 0.$$

So, $S(t)$ is a contraction semigroup.

Exercise 3 For any fixed $p \geqslant 1$, let $X = L^p(\mathbb{R})$ and define, $\forall f \in X$,

$$\big(S(t)f\big)(x) = f(x+t) \quad \forall x \in \mathbb{R}\,, \forall t \geqslant 0. \tag{1.2.4}$$

Prove that S is $\mathcal{C}_0$-semigroup which fails to be uniformly continuous.

Solution. Suppose S is uniformly continuous and let $\tau > 0$ be such that $\|S(t) - I\| < 1/2$ for all $t \in [0, \tau]$. Then by taking $f_n(x) = n^{1/p}\chi_{[0,1/n]}(x)$ for $p < \infty$ and $n > 1/\tau$ we have that $|f_n| = 1$ and

$$|S(\tau)f_n - f_n| = \Big(\int_{\mathbb{R}} n|\chi_{[0,1/n]}(x+\tau) - \chi_{[0,1/n]}(x)|^p dx\Big)^{\frac{1}{p}} = 2^{1/p}.$$ □

Exercise 4 Given a uniformly bounded $\mathcal{C}_0$-semigroup, $\|S(t)\| \leqslant M$, define

$$|u|_S = \sup_{t \geqslant 0} |S(t)u|\,, \quad \forall u \in X. \tag{1.2.5}$$

Show that:

(a) $|\cdot|_S$ is a norm on X,

(b) $|u| \leqslant |u|_S \leqslant M|u|$ for all $u \in X$, and

(c) S is a contraction semigroup with respect to $|\cdot|_S$.

1.3 The infinitesimal generator of a $\mathcal{C}_0$-semigroup

Theorem 3 *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X, denoted by $S(t)$. Then the following properties hold true.*

(a) *For all $t \geqslant 0$*

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} S(s)u\, ds = S(t)u \qquad \forall u \in X.$$

(b) *For all $t \geqslant 0$ and $u \in X$*

$$\int_0^t S(s)u\, ds \in D(A) \quad \text{and} \quad A \int_0^t S(s)u\, ds = S(t)u - u.$$

(c) *$D(A)$ is dense in X.*

(d) *For all $u \in D(A)$ and $t \geqslant 0$ we have that $S(t)u \in D(A)$, $t \mapsto S(t)u$ is continuously differentiable, and*

$$\frac{d}{dt} S(t)u = AS(t)u = S(t)Au.$$

(e) *For all $u \in D(A)$ and all $0 \leqslant s \leqslant t$ we have that*

$$S(t)u - S(s)u = \int_s^t S(\tau)Au\, d\tau = \int_s^t AS(\tau)u\, d\tau.$$

Proof. We remind the reader that all integrals are to be understood in the Cauchy sense.

(a) This point is an immediate consequence of the strong continuity of S.

(b) For any $t \geqslant h > 0$ we have that

$$\begin{aligned} \frac{S(h)-I}{h}\Big(\int_0^t S(s)u\, ds\Big) &= \frac{1}{h}\int_0^t (S(h+s)-S(s))u\, ds \\ &= \frac{1}{h}\Big(\int_h^{t+h} S(s)u\, ds - \int_0^t S(s)u\, ds\Big) \\ &= \frac{1}{h}\Big(\int_t^{t+h} S(s)u\, ds - \int_0^h S(s)u\, ds\Big). \end{aligned}$$

Therefore, by (a),

$$\lim_{h \downarrow 0} \frac{S(h)-I}{h}\Big(\int_0^t S(s)x\, ds\Big) = S(t)x - x$$

which proves (b).

(c) This point follows from (a) and (b).

(d) For all $u \in D(A)$, $t \geqslant 0$, and $h > 0$ we have that

$$\frac{S(h) - I}{h} S(t)u = S(t) \frac{S(h) - I}{h} u \to S(t)Au \quad \text{as} \quad h \downarrow 0.$$

Therefore, $S(t)u \in D(A)$ and $AS(t)u = S(t)Au = \frac{d^+}{dt} S(t)u$. In order to prove the existence of the left derivative, observe that for all $0 < h < t$

$$\frac{S(t-h)u - S(t)u}{-h} = S(t-h) \frac{S(h) - I}{h} u.$$

Moreover, by (1.2.2),

$$\begin{aligned} &\Big| S(t-h) \frac{S(h) - I}{h} u - S(t)Au \Big| \\ &\qquad \leqslant \Big| S(t-h) \Big| \cdot \Big| \frac{S(h) - I}{h} u - S(h)Au \Big| \\ &\qquad\qquad \leqslant M e^{\omega t} \Big| \frac{S(h) - I}{h} u - S(h)Au \Big| \stackrel{h \downarrow 0}{\longrightarrow} 0. \end{aligned}$$

Therefore

$$\frac{S(t-h)u - S(t)u}{-h} \longrightarrow S(t)Au = AS(t)u \quad \text{as} \quad h \downarrow 0,$$

showing that the left and right derivatives coincide.

(e) This point follows from (d).

The proof is complete. □

Exercise 5 Show that the infinitesimal generator of the $\mathcal{C}_0$-semigroup of left translations on $\mathbb{R}$ we introduced in Exampe 3 is given by

$$\begin{cases} D(A) = \mathcal{C}_b^1(\mathbb{R}) \\ Af = f' & \forall f \in D(A). \end{cases}$$

Solution. For any $f \in C_b^1(\mathbb{R})$ we have that

$$\Big| \frac{S(t)f - f}{t} - f' \Big|_{\infty, \mathbb{R}} = \sup_{x \in \mathbb{R}} \Big| \frac{f(x+t) - f(x)}{t} - f'(x) \Big| \leqslant \mathrm{osc}_{f'}(t) \stackrel{t \downarrow 0}{\longrightarrow} 0$$

Therefore, $\mathcal{C}_b^1(\mathbb{R}) \subset D(A)$ and $Af = f'$ for all $f \in \mathcal{C}_b^1(\mathbb{R})$. Conversely, let $f \in D(A)$. Then, $Af \in \mathcal{C}_b(\mathbb{R})$ and

$$\sup_{x \in \mathbb{R}} \Big| \frac{f(x+t) - f(x)}{t} - Af(x) \Big| \stackrel{t \downarrow 0}{\longrightarrow} 0.$$

So, $f'(x)$ exists for all $x \in \mathbb{R}$ and equals $Af(x)$. Thus, $D(A) \subset \mathcal{C}_b^1(\mathbb{R})$. □

Exercise 6 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a uniformly bounded semigroup $\|S(t)\| \leqslant M$. Prove the *Laundau-Kolmogorov inequality:*

$$|Au|^2 \leqslant 4M^2\,|u|\,|A^2u| \qquad \forall u \in D(A^2), \tag{1.3.1}$$

where

$$\begin{cases} D(A^2) = \{u \in D(A) \ : \ Au \in D(A)\} \\ A^2u = A(Au)\,, \quad \forall u \in D(A^2). \end{cases} \tag{1.3.2}$$

Solution. Assume $M = 1$. For any $u \in D(A^2)$ and all $t \geqslant 0$ we have

$$\begin{aligned} \int_0^t (t-s)S(s)A^2u\,ds &= \big[(t-s)S(s)Au\big]_{s=0}^{s=t} + \int_0^t S(s)Au\,ds \\ &= -tAu + \big[S(s)u\big]_{s=0}^{s=t} = -tAu + S(t)u - u. \end{aligned}$$

Therefore, for all $t > 0$,

$$\begin{aligned} |Au| &\leqslant \frac{1}{t}\,|S(t)u-u| + \frac{1}{t}\int_0^t (t-s)|S(s)A^2u|ds \\ &\leqslant \frac{2}{t}\,|u| + \frac{t}{2}\,|A^2u|. \end{aligned} \tag{1.3.3}$$

If $A^2u = 0$, then the above inequality yields $Au = 0$ by letting $t \to \infty$. So, (1.3.1) is true in this case. On the other hand, for $A^2u \neq 0$ the function of t on the right-hand side of (1.3.3) attains its minimum at

$$t_0 = \frac{2|u|^{1/2}}{|A^2u|^{1/2}}.$$

By taking $t = t_0$ in (1.3.3) we obtain (1.3.1) once again.
(*Question:* how to treat the case of $M \neq 1$? *Hint:* remember Exercise 4.) □

Exercise 7 Use the Landau-Kolmogorov inequality to deduce the interpolation inequality

$$|f'|^2_{\infty,\mathbb{R}} \leqslant 4\,|f|_{\infty,\mathbb{R}}\,|f''|_{\infty,\mathbb{R}} \qquad \forall f \in \mathcal{C}^2_b(\mathbb{R}).$$

1.4 The Cauchy problem with a closed operator

We recall that $X \times X$ is a Banach space with norm

$$\|(u,v)\| = |u| + |v| \qquad \forall (u,v) \in X \times X.$$

Definition 6 *An operator $A : D(A) \subset X \to X$ is said to be* closed *if its graph*

$$graph(A) := \big\{(u,v) \in X \times X \ : \ u \in D(A)\,,\ v = Au\big\}$$

is a closed subset of $X \times X$.

The following characterisation of closed operators is straightforward.

Proposition 4 *The linear operator $A : D(A) \subset X \to X$ is closed if and only if, for any sequence $\{x_n\} \subset D(A)$, the following holds:*

$$\begin{cases} u_n \to u \\ Au_n \to v \end{cases} \implies u \in D(A) \quad \textit{and} \quad Au = v. \tag{1.4.1}$$

Example 4 In the Banach space $X = \mathcal{C}_b(\mathbb{R})$, the linear operator

$$\begin{cases} D(A) = \mathcal{C}_b^1(\mathbb{R}) \\ Af = f' \qquad\qquad \forall f \in D(A). \end{cases}$$

is closed. Indeed, for any sequence $\{f_n\} \subset C_b^1(\mathbb{R})$ such that

$$\begin{cases} f_n \to f & \text{in } C_b(\mathbb{R}) \\ f_n' \to g & \text{in } C_b(\mathbb{R}), \end{cases}$$

we have that $f \in C_b^1(\mathbb{R})$ and $f' = g$.

Example 5 In the Banach space $X = \mathcal{C}([0,1])$ with the uniform norm, the linear operator

$$\begin{cases} D(A) = \mathcal{C}^1([0,1]) \\ (Af)(x) = f'(0) \qquad \forall x \in [0,1] \end{cases}$$

fails to be closed. Indeed, for any $n \geqslant 1$ let

$$f_n(x) = \frac{\sin(nx)}{n} \qquad x \in [0,1].$$

Then

$$\begin{cases} D(A) \ni f_n \to 0 & \text{in } C_b(\mathbb{R}) \\ Af_n = 1 & \forall n \geqslant 1, \end{cases}$$

in contrast with (1.4.1).

Exercise 8 Prove that if $A : D(A) \subset X \to X$ is a closed operator and $B \in \mathcal{L}(X)$, then $A + B : D(A) \subset X \to X$ is also closed. What about BA?

Exercise 9 Prove that, if $A : D(A) \subset X \to X$ is a closed operator and $f \in \mathcal{C}([a,b]; D(A))$, then

$$A \int_a^b f(t)dt = \int_a^b Af(t)dt. \tag{1.4.2}$$

Solution. Let $\pi_n = \{t_i^n\}_{i=0}^{i_n} \in \Pi(a,b)$ be such that $diam(\pi_n) \to 0$ and let $\sigma_n = \{s_i^n\}_{i=1}^{i_n} \in \Sigma(\pi_n)$. Then $\int_a^b f(t)dt \in D(A)$ and

$$\begin{cases} D(A) \ni S_{\pi_n}^{\sigma_n}(f) = \sum_{i=1}^{i_n} f(s_i^n)(t_i^n - t_{i-1}^n) \to \int_a^b f(t)dt \\ AS_{\pi_n}^{\sigma_n}(f) = \sum_{i=1}^{i_n} Af(s_i^n)(t_i^n - t_{i-1}^n) \to \int_a^b Af(t)dt \end{cases} \qquad (n \to \infty)$$

Therefore, by Proposition 4, $\int_a^b f(t)dt \in D(A)$ and (1.4.2) holds true. □

Proposition 5 *The infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(t)$ is a closed operator.*

Proof. Let $A : D(A) \subset X \to X$ be the infinitesimal generator of $S(t)$ and let $\{u_n\} \subset D(A)$ be as in (1.4.1). By Theorem 3$-(d)$ we have that, for all $t \geqslant 0$,

$$S(t)u_n - u_n = \int_0^t S(s)Au_n dx.$$

Hence, taking the limit as $n \to \infty$ and dividing by t, we obtain

$$\frac{S(t)u - u}{t} = \frac{1}{t}\int_0^t S(s)v du.$$

Passing to the limit as $t \downarrow 0$, we conclude that $Au = v$. □

Remark 1 From Proposition 5 it follows that the domain $D(A)$ of the infinitesimal generator of a $\mathcal{C}_0$-semigroup is a Banach space with the *graph norm*

$$|u|_{D(A)} = |u| + |Au| \qquad \forall u \in D(A).$$

Exercise 10 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$. Define

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ Au = \Delta u & \forall u \in D(A). \end{cases}$$

Prove that A is a closed operator on the Hilbert space $X = L^2(\Omega)$.
Solution. Let $u_i \in H^2(\Omega) \cap H_0^1(\Omega)$ be such that

$$\begin{cases} u_i \to u \\ \Delta u_i \to v \end{cases} \quad \text{in } L^2(\Omega).$$

By elliprtic regularity, we have that

$$\|u_i - u_j\|_{2.\Omega} \leqslant C\|\Delta u_i - \Delta u_j\|_{0,\Omega}$$

for some constant $C > 0$. Hence, $\{u_i\}$ is a Cauchy sequence in the Hilbert space $H^2(\Omega) \cap H_0^1(\Omega)$. So, $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\Delta u = v$. □

Given a closed operator $A : D(A) \subset X \to X$, let us consider the Cauchy problem with initial datum $u_0 \in X$

$$\begin{cases} u'(t) = Au(t) & t > 0 \\ u(0) = u_0. \end{cases} \tag{1.4.3}$$

Definition 7 *A* classical solution *of problem* (1.4.3) *is a function*

$$u \in \mathcal{C}(\mathbb{R}_+; X) \cap \mathcal{C}^1(\mathbb{R}_+^*; X) \cap \mathcal{C}(\mathbb{R}_+^*; D(A))^1$$

such that $u(0) = u_0$ *and* $u'(t) = Au(t)$ *for all* $t > 0$.

Our next result ensures the existence and uniqueness of a classical solution to (1.4.3) for initial data in $D(A)$, provided A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X.

Proposition 6 *Let* $A : D(A) \subset X \to X$ *be the infinitesimal generator of a* $\mathcal{C}_0$*-semigroup of bounded linear operators on* X, $S(t)$.

Then, for every $u_0 \in D(A)$, *problem* (1.4.3) *has a unique classical solution* $u \in \mathcal{C}^1(\mathbb{R}_+; X) \cap \mathcal{C}(\mathbb{R}_+; D(A))$ *given by* $u(t) = S(t)u_0$ *for all* $t \geqslant 0$.

Proof. The fact that $u(t) = S(t)u_0$ satisfies (1.4.3) is point (d) of Theorem 3. To show that u is the unique solution of the problem let $v \in \mathcal{C}^1(\mathbb{R}_+; X) \cap \mathcal{C}(\mathbb{R}_+; D(A))$ be any solution of (1.4.3), fix $t > 0$, and set

$$U(s) = S(t-s)v(s)\,, \quad \forall s \in [0, t].$$

Then, for all $s \in]0, t[$ we have that

$$\begin{aligned} &\frac{U(s+h) - U(s)}{h} - S(t-s)v'(s) + AS(t-s)v(s) \\ &= S(t-s-h)\frac{v(s+h) - v(s)}{h} - S(t-s)v'(s) \\ &+ \Big(\frac{S(t-s-h) - S(t-s)}{h} + AS(t-s)\Big)v(s). \end{aligned}$$

Now, point (d) of Theorem 3 immediately yields

$$\lim_{h\to 0} \frac{S(t-s-h) - S(t-s)}{h} v(s) = -AS(t-s)v(s).$$

Moreover,

$$\begin{aligned} &S(t-s-h)\frac{v(s+h) - v(s)}{h} - S(t-s)v'(s) \\ &= S(t-s-h)\Big(\frac{v(s+h) - v(s)}{h} - v'(s)\Big) \\ &\quad + \big(S(t-s-h) - S(t-s)\big)v'(s), \end{aligned}$$

[1]Here $D(A)$ is ragarded as a Banach space with the graph norm.

where

$$\big(S(t-s-h)-S(t-s)\big)v'(s)\overset{h\to 0}{\longrightarrow}0$$

by the strong continuity of $S(t)$, while

$$\Big|S(t-s-h)\Big(\frac{v(s+h)-v(s)}{h}-v'(s)\Big)\Big| \leqslant Me^{\omega(t-s-h)}\Big|\frac{v(s+h)-v(s)}{h}-v'(s)\Big|\overset{h\to 0}{\longrightarrow}0$$

in view of (1.2.2). Therefore,

$$U'(s)=-AS(t-s)v(s)+S(t-s)Av(s)=0\,,\quad \forall s\in]0,T[.$$

So, U is constant and $u(t)=U(t)=U(0)=v(t)$. □

Exercise 11 Let $S(t)$ and $T(t)$ be $\mathcal{C}_0$-semigroups with infinitesimal generators $A:D(A)\subset X\to X$ and $B:D(B)\subset X\to X$, respectively. Show that

$$A=B\quad\Longrightarrow\quad S(t)=T(t)\quad\forall t\geqslant 0.$$

Example 6 (Transport equation in $C_b(\mathbb{R})$) Returning to the left-translation semigroup on $C_b(\mathbb{R})$ of Example 3, by Proposition 6 and Exercise 5 we conclude that for each $f\in C^1_b(\mathbb{R})$ the unique solution of the problem

$$\begin{cases}\frac{\partial u}{\partial t}(t,x)=\frac{\partial u}{\partial x}(t,x) & (t,x)\in\mathbb{R}_+\times\mathbb{R}\\ u(0,x)=f(x) & x\in\mathbb{R}\end{cases}$$

is given by $u(t,x)=f(x+t)$.

1.5 Resolvent and spectrum of a closed operator

Let $A:D(A)\subset X\to X$ be a closed operator on a complex Banach space X.

Definition 8 *The* resolvent set *of A, $\rho(A)$, is the set of all $\lambda\in\mathbb{C}$ such that $\lambda I-A:D(A)\to X$ is bijective. The set $\sigma(A)=\mathbb{C}\setminus\rho(A)$ is called the* spectrum *of A. For any $\lambda\in\rho(A)$ the linear operator*

$$R(\lambda,A):=(\lambda I-A)^{-1}:X\to X$$

is called the resolvent *of A.*

Example 7 On $X=\mathcal{C}([0,1])$ with the uniform norm consider the linear operator $A:D(A)\subset X\to X$ defined by

$$\begin{cases}D(A)=\mathcal{C}^1([0,1])\\ Af=f',\quad\forall f\in D(A)\end{cases}$$

is closed (compare to Example 4). Then $\sigma(A) = \mathbb{C}$ because for any $\lambda \in \mathbb{C}$ the function $f_\lambda(x) = e^{\lambda x}$ satisfies

$$\lambda f_\lambda(x) - f'_\lambda(x) = 0 \qquad \forall x \in [0,1].$$

On the other hand, for the closed operator A_0 defined by

$$\begin{cases} D(A_0) = \big\{f \in \mathcal{C}^1([0,1]) \ : \ f(0) = 0\big\} \\ A_0 f = f', \quad \forall f \in D(A_0), \end{cases}$$

we have that $\sigma(A_0) = \varnothing$. Indeed, for any $g \in X$ the problem

$$\begin{cases} \lambda f(x) - f'(x) = g(x) \quad x \in [0,1] \\ f(0) = 0 \end{cases}$$

admits the unique solution

$$f(x) = -\int_0^x e^{\lambda(x-s)} g(s)\, dx \quad (x \in [0,1])$$

which belongs to $D(A_0)$.

Proposition 7 (properties of $R(\lambda, A)$) *Let $A : D(A) \subset X \to X$ be a closed operator on a complex Banach space X. Then the following holds true.*

(a) $R(\lambda, A) \in \mathcal{L}(X)$ for any $\lambda \in \rho(A)$.

(b) For any $\lambda \in \rho(A)$

$$AR(\lambda, A) = \lambda R(\lambda, A) - I. \tag{1.5.1}$$

(c) The resolvent identity *holds:*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A) \qquad \forall \lambda, \mu \in \rho(A). \tag{1.5.2}$$

(d) For any $\lambda, \mu \in \rho(A)$

$$R(\lambda, A) R(\mu, A) = R(\mu, A) R(\lambda, A). \tag{1.5.3}$$

Proof. Let $\lambda, \mu \in \rho(A)$.

(a) Since A is closed, so is $\lambda I - A$ and aslo $R(\lambda, A) = (\lambda I - A)^{-1}$. So, $R(\lambda, A) \in \mathcal{L}(X)$ by the closed graph theorem.

(b) This point follows from the definition of $R(\lambda, A)$.

(c) By (1.5.1) we have that

$$[\lambda R(\lambda, A) - AR(\lambda, A)]R(\mu, A) = R(\mu, A)$$

and

$$R(\lambda, A)[\mu R(\mu, A) - AR(\mu, A)] = R(\lambda, A).$$

Since $AR(\lambda, A) = R(\lambda, A)A$ on $D(A)$, (1.5.2) follows.

(d) Apply (1.5.2) to compute

$$\begin{aligned} R(\lambda, A) - R(\mu, A) &= (\mu - \lambda)R(\lambda, A)R(\mu, A) \\ R(\mu, A) - R(\lambda, A) &= (\lambda - \mu)R(\mu, A)R(\lambda, A). \end{aligned}$$

Adding the above identities side by side yields the conclusion.

The proof is complete. □

Theorem 4 (analiticity of $R(\lambda, A)$) *Let $A : D(A) \subset X \to X$ be a closed operator on a complex Banach space X. Then the resolvent set $\rho(A)$ is open in $\mathbb{C}$ and for any $\lambda_0 \in \rho(A)$ we have that*

$$|\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|} \quad \Longrightarrow \quad \lambda \in \rho(A) \tag{1.5.4}$$

and the resolvent $R(\lambda, A)$ is given by the (Neumann) series

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}. \tag{1.5.5}$$

Consequently, $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$ and

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n \, n! \, R(\lambda, A)^{n+1} \qquad \forall n \in \mathbb{N}. \tag{1.5.6}$$

Proof. For all $\lambda \in \mathbb{C}$ and $\lambda_0 \in \rho(A)$ we have that

$$\lambda I - A = \lambda_0 I - A + (\lambda - \lambda_0)I = [I - (\lambda_0 - \lambda)R(\lambda_0, A)](\lambda_0 I - A).$$

This operator is bijective if and only if $[I - (\lambda_0 - \lambda)R(\lambda_0, A)]$ is invertible, which is the case if λ satisfies (1.5.4). Then

$$R(\lambda, A) = R(\lambda_0, A)[I - (\lambda_0 - \lambda)R(\lambda_0, A)]^{-1} = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}.$$

The analyticity of $R(\lambda, A)$ and (1.5.6) follows from (1.5.5). □

Theorem 5 (integral representationof $R(\lambda, A)$) *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X, $S(t)$, and let $M \geqslant 1$ and $\omega \in \mathbb{R}$ be such that*

$$\|S(t)\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0. \tag{1.5.7}$$

Then $\rho(A)$ contains the half-plane

$$\Pi_\omega = \big\{\lambda \in \mathbb{C} \;:\; \Re\,\lambda > \omega\big\} \tag{1.5.8}$$

and

$$R(\lambda, A)u = \int_0^\infty e^{-\lambda t} S(t)u\,dt \qquad \forall u \in X\,,\ \forall \lambda \in \Pi_\omega. \tag{1.5.9}$$

Proof. We have to prove that, given any $\lambda \in \Pi_\omega$ and $u \in X$, the equation

$$\lambda v - Av = u \tag{1.5.10}$$

has a unique solution $v \in D(A)$ given by the right-hand side of (1.5.9).
Existence: observe that $v := \int_0^\infty e^{-\lambda t} S(t)u\,dt \in X$ because $\Re\,\lambda > \omega$. Moreover, for all $h > 0$,

$$\begin{aligned}
\frac{S(h)v - v}{h} &= \frac{1}{h}\Big\{\int_0^\infty e^{-\lambda t} S(t+h)u\,dt - \int_0^\infty e^{-\lambda t} S(t)u\,dt\Big\} \\
&= \frac{1}{h}\Big\{e^{\lambda h}\int_h^\infty e^{-\lambda t} S(t)u\,dt - \int_0^\infty e^{-\lambda t} S(t)u\,dt\Big\} \\
&= \frac{e^{\lambda h} - 1}{h}\, v - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} S(t)u\,dt.
\end{aligned}$$

So

$$\lim_{h\downarrow 0} \frac{S(h)v - v}{h} = \lambda v - u$$

which in turn yields that $v \in D(A)$ and (1.5.10) holds true.
Uniqueness: let $v \in D(A)$ be a solution of (1.5.10). Then

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} S(t)u\,dt &= \int_0^\infty e^{-\lambda t} S(t)(\lambda v - Av)\,dt \\
&= \lambda \int_0^\infty e^{-\lambda t} S(t)v\,dt - \int_0^\infty e^{-\lambda t} \frac{d}{dt} S(t)v\,dt = v
\end{aligned}$$

which implies that v is given by (1.5.9). □

Proposition 8 *Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed linear operators in X and suppose $B \subset A$, that is,*

$$D(B) \subset D(A) \quad \text{and} \quad Au = Bu \quad \forall x \in D(B).$$

If $\rho(A) \cap \rho(B) \neq \varnothing$, then $A = B$.

Proof. It suffices to show that $D(A) \subset D(B)$. Let $u \in D(A)$, $\lambda \in \rho(A) \cap \rho(B)$, and set

$$v = \lambda u - Au \quad \text{and} \quad w = R(\lambda, B)v.$$

Then $w \in D(B)$ and $\lambda w - Bw = \lambda u - Au$. Since $B \subset A$, $\lambda w - Bw = \lambda w - Aw$. Thus, $(\lambda I - A)(u - w) = 0$. So, $u = w \in D(B)$. □

Example 8 (Right-translation semigroup on $\mathbb{R}_+$) On the real Banach space

$$X = \{f \in \mathcal{C}_b(\mathbb{R}_+) \ : \ f(0) = 0\}$$

with the uniform norm, consider the right-translation semigroup

$$\big(S(t)f\big)(x) = \begin{cases} f(x-t) & x > t \\ 0 & x \in [0,t] \end{cases} \qquad \forall x, t \geqslant 0.$$

It is easy to check that S is a $\mathcal{C}_0$-semigroup on X with $\|S(t)\| = 1$ for all $t \geqslant 0$. In order to characterize its infinitesimal generator A, let us consider the operator $B : D(B) \subset X \to X$ defined by

$$\begin{cases} D(B) = \big\{f \in X \ : \ f' \in X\big\} \\ Bf = -f', \quad \forall f \in D(B). \end{cases}$$

We claim that:

(*i*) $B \subset A$

Proof. Let $f \in D(B)$. Then, for all $x, t \geqslant 0$ we have

$$\frac{\big(S(t)f\big)(x) - f(x)}{t} = \begin{cases} -\frac{f(x)}{t} = -f'(x_t), & 0 \leqslant x \leqslant t \\ \frac{f(x-t)-f(x)}{t} = -f'(x_t) & x \geqslant t \end{cases}$$

with $0 \leqslant x - x_t \leqslant t$. Therefore

$$\sup_{x \geqslant 0} \Big| \frac{\big(S(t)f\big)(x) - f(x)}{t} + f'(x) \Big| \leqslant \sup_{|x-y| \leqslant t} |f'(x) - f'(y)| \to 0 \quad \text{as} \quad t \downarrow 0$$

because f' is uniformly continuous. □

(*ii*) $1 \in \rho(B)$

Proof. For any $g \in X$ the unique solution f of the problem

$$\begin{cases} f \in D(B) \\ f(x) + f'(x) = g(x) \quad \forall x \geqslant 0 \end{cases}$$

is given by

$$f(x) = \int_0^x e^{s-x} g(s)\, ds \qquad (x \geqslant 0).$$ □

Since $1 \in \rho(A)$ by Proposition 5, Proposition 8 yields that $A = B$.

1.6 The Hille-Yosida generation theorem

Theorem 6 *Let $M \geqslant 1$ and $\omega \in \mathbb{R}$. For a linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(a) A is closed, $D(A)$ is dense in X, and

$$\rho(A) \supseteq \Pi_\omega = \{\lambda \in \mathbb{C} \ : \ \Re\lambda > \omega\} \tag{1.6.1}$$

$$\|R(\lambda, A)^k\| \leqslant \frac{M}{(\Re\lambda - \omega)^k} \qquad \forall k \geqslant 1, \forall \lambda \in \Pi_\omega \tag{1.6.2}$$

(b) A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup, $S(t)$, such that

$$\|S(t)\| \leqslant M e^{\omega t} \qquad \forall t \geqslant 0. \tag{1.6.3}$$

Proof of (b) ⇒ (a) The fact that A is closed, $D(A)$ is dense in X, and (1.6.1) holds true has already been proved, see Theorem 3-(c), Proposition 5, and Theorem 5. In order to prove (1.6.2) observe that, by using (1.5.9) to compute the k-th derivative of the resolvent of A, we obtain

$$\frac{d^k}{d\lambda^k} R(\lambda, A)u = (-1)^k \int_0^\infty t^k e^{-\lambda t} S(t)u \, dt \qquad \forall u \in X \, , \ \forall \lambda \in \Pi_\omega .$$

Therefore,

$$\left\| \frac{d^k}{d\lambda^k} R(\lambda, A) \right\| \leqslant M \int_0^\infty t^k e^{-(\Re\lambda - \omega)t} \, dt = \frac{M \, k!}{(\Re\lambda - \omega)^{k+1}}$$

where the integral is easily computed by induction. The conclusion follows recalling (1.5.6). □

Lemma 1 *Let $A : D(A) \subset X \to X$ be as in (a) of Theorem 6. Then:*

(i) For all $u \in X$

$$\lim_{n\to\infty} nR(n, A)u = u. \tag{1.6.4}$$

(ii) The Yosida Approximation *A_n of A, defined as*

$$A_n = nAR(n, A) \qquad (n \geqslant 1) \tag{1.6.5}$$

is a sequence of bounded operator on X which satisfies

$$A_n A_m = A_m A_n \qquad \forall n, m \geqslant 1 \tag{1.6.6}$$

and

$$\lim_{n\to\infty} A_n u = Au \qquad \forall u \in D(A). \tag{1.6.7}$$

(iii) *For all* $m, n > 2\omega$, $u \in D(A)$, $t \geqslant 0$ *we have that*

$$\begin{aligned} \|e^{tA_n}\| &\leqslant Me^{\frac{n\omega t}{n-\omega}} \leqslant Me^{2\omega t} && (1.6.8) \\ |e^{tA_n}u - e^{tA_m}u| &\leqslant M^2 te^{2\omega t}|A_n u - A_m u|. && (1.6.9) \end{aligned}$$

Consequently, for all $u \in D(A)$ *the sequence* $u_n(t) := e^{tA_n}u$ *is Cauchy in* $\mathcal{C}([0,T];X)$ *for any* $T > 0$.

Proof of (i): owing to (1.5.1), for any $u \in D(A)$ we have that

$$|nR(n,A)u - u| = |AR(n,A)u| = |R(n,A)Au| \leqslant \frac{M|Au|}{n-\omega} \overset{(n\to\infty)}{\longrightarrow} 0,$$

where we have used (1.6.2) with $k = 1$. Moreover, again by (1.6.2) ,

$$\|nR(n,A)\| \leqslant \frac{Mn}{n-\omega} \leqslant 2M \quad \forall n > 2\omega.$$

We claim that the last two inequalities yield the conclusion because $D(A)$ is dense in X. Indeed, let $u \in X$ and fix any $\varepsilon > 0$. Let $u_\varepsilon \in D(A)$ be such that $|u_\varepsilon - u| < \varepsilon$. Then

$$\begin{aligned} |nR(n,A)u - u| &\leqslant |nR(n,A)(u-u_\varepsilon)| + |nR(n,A)u_\varepsilon - u_\varepsilon| + |u_\varepsilon - u| \\ &< (2M+1)\varepsilon + \frac{M|Au_\varepsilon|}{n-\omega} \overset{(n\to\infty)}{\longrightarrow} (2M+1)\varepsilon. \end{aligned}$$

Since ε is arbitrary, (1.6.4) follows.
Proof of (ii): observe that $A_n \in \mathcal{L}(X)$ because

$$A_n = n^2R(n,A) - nI \qquad \forall n \geqslant 1. \tag{1.6.10}$$

Moreover, in view of (1.5.3) we have that

$$\begin{aligned} A_nA_m &= [n^2R(n,A) - nI]\,[m^2R(m,A) - mI] \\ &= [m^2R(m,A) - mI]\,[n^2R(n,A) - nI] = A_mA_n. \end{aligned}$$

Finally, owing to (1.6.4), for all $u \in D(A)$ we have that

$$A_nu = nAR(n,A)u = nR(n,A)Au \overset{(n\to\infty)}{\longrightarrow} Au.$$

Proof of (iii): recalling (1.6.10) we have that

$$e^{tA_n} = e^{-nt}\sum_{k=0}^{\infty} \frac{n^{2k}t^kR(n,A)^k}{k!}, \quad \forall t \geqslant 0.$$

Therefore, in view of (1.6.2),

$$\|e^{tA_n}\| \leqslant Me^{-nt}\sum_{k=0}^{\infty} \frac{n^{2k}t^k}{k!(n-\omega)^k} = Me^{\frac{n\omega t}{n-\omega}} \leqslant Me^{2\omega t}$$

for all $t \geqslant 0$ and $n > 2\omega$. This proves (1.6.8).

Next, observe that, for any $u \in D(A)$, $u_n(t) := e^{tA_n}u$ satisfies

$$\begin{cases} (u_n - u_m)'(t) = A_n(u_n - u_m)(t) + (A_n - A_m)u_m(t) & \forall t \geqslant 0 \\ (u_n - u_m)(0) = 0. \end{cases}$$

Therefore, for all $t \geqslant 0$ we have that

$$\begin{aligned} e^{tA_n}u - e^{tA_m}u &= \int_0^t e^{(t-s)A_n}(A_n - A_m)e^{sA_m}u\,ds \\ &= \int_0^t e^{(t-s)A_n}e^{sA_m}(A_n - A_m)u\,ds \end{aligned} \tag{1.6.11}$$

because A_n and $e^{sA_m}u$ commute in view of (1.6.6). Thus, by combining (1.6.11) and (1.6.8) we obtain

$$\begin{aligned} |e^{tA_n}u - e^{tA_m}u| &\leqslant M^2 \int_0^t e^{2\omega(t-s)}e^{2\omega s}|A_n u - A_m u|, ds \\ &\leqslant M^2\, t\, e^{2\omega t}\, |A_n u - A_m u|. \end{aligned}$$

In view of (1.6.7), the last inequality shows that $e^{tA_n}u$ is a Cauchy sequence in $\mathcal{C}([0,T];X)$ for any $T > 0$, thus completing the proof. □

Exercise 12 Use a density argument to prove that $e^{tA_n}u$ is a Cauchy sequence on all compact subsets of $\mathbb{R}_+$ for all $u \in X$.

Solution. Let $u \in X$ and fix any $\varepsilon > 0$. Let $u_\varepsilon \in D(A)$ be such that $|u_\varepsilon - u| < \varepsilon$. Then for all $m, n > 2\omega$ we have that

$$\begin{aligned} |e^{tA_n}u - e^{tA_m}u| &\leqslant |e^{tA_n}(u - u_\varepsilon)| \\ &\quad + |(e^{tA_n} - e^{tA_m})u_\varepsilon| + |e^{tA_m}(u_\varepsilon - u)| \\ &\leqslant |(e^{tA_n} - e^{tA_m})u_\varepsilon| + 2Me^{2\omega t}\varepsilon \end{aligned}$$

Since ε is arbitrary, recalling point (*iii*) above the conclusion follows. □

Proof of (a) ⇒ (b) On account of Lemma 1 and Exercise 12, we have that $e^{tA_n}u$ is a Cauchy sequence on all compact subsets of $\mathbb{R}_+$ for all $u \in X$. Consequently, the limit (uniform on all $[0,T] \subset \mathbb{R}_+$)

$$S(t)u = \lim_{n\to\infty} e^{tA_n}u, \quad \forall u \in X, \tag{1.6.12}$$

defines a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Moreover, passing to the limit as $n \to \infty$ in (1.6.8), we conclude that $\|S(t)\| \leqslant Me^{\omega t}$, $\forall t \geqslant 0$.

Let us identify the infinitesimal generator of $S(t)$. By (1.6.8), for $u \in D(A)$ we have that

$$\begin{aligned}\Big|\frac{d}{dt}\, e^{tA_n}u - S(t)Au\Big| &\leqslant |e^{tA_n}A_n u - e^{tA_n}Au| + |e^{tA_n}Au - S(t)Au| \\ &\leqslant Me^{2\omega t}|A_n u - Au| + |e^{tA_n}Au - S(t)Au| \stackrel{(n\to\infty)}{\longrightarrow} 0\end{aligned}$$

uniformly on all compact subsets of $\mathbb{R}_+$ by (1.6.12). Therefore, for all $T > 0$ and $u \in D(A)$ we have that

$$\begin{cases} e^{tA_n}u \stackrel{(n\to\infty)}{\longrightarrow} S(t)u \\ \frac{d}{dt}\, e^{tA_n}u \stackrel{(n\to\infty)}{\longrightarrow} S(t)Au \end{cases} \qquad \text{uniformly on } [0,T].$$

This implies that

$$S'(t)u = S(t)Au, \quad \forall u \in D(A)\,,\ \forall t \geqslant 0. \tag{1.6.13}$$

Now, let $B : D(B) \subset X \to X$ be the infinitesimal generator of $S(t)$. Then $A \subset B$ in view of (1.6.13). Moreover, $\Pi_\omega \subset \rho(A)$ by assumption (a) and $\Pi_\omega \subset \rho(B)$ by Proposition 5. So, on account of Proposition 8, $A = B$. □

Remark 2 The above proof shows that condition (a) in Theorem 6 can be relaxed as follows:

(a') *A is closed, $D(A)$ is dense in X, and*

$$\rho(A) \supseteq]\omega, \infty[\tag{1.6.14}$$

$$\|R(n,A)^k\| \leqslant \frac{M}{(n-\omega)^k} \quad \forall k \geqslant 1, \forall n > \omega. \tag{1.6.15}$$

Remark 3 When $M = 1$, the countably many bounds in condition (a) follow from (1.6.2) for $k = 1$, that is,

$$\|R(\lambda, A)\| \leqslant \frac{1}{\Re\lambda - \omega} \qquad \forall k \geqslant 1\,,\ \forall \lambda \in \Pi_\omega.$$

Example 9 (parabolic equations in $L^2(\Omega)$) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$. Define

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ Au = \sum_{i,j=1}^n D_j(a_{ij}D_j)u + \sum_{i=1}^n b_i D_i u + cu \quad \forall u \in D(A). \end{cases}$$

where

(H1) $a_{ij} \in \mathcal{C}^1(\overline{\Omega})$ satisfies $a_{ij} = a_{ji}$ for all $i, j = 1, \dots, n$ and

$$\sum_{i,j=1}^n a_{ij}(x)\xi_j\xi_i \geqslant \theta|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n,\ x \in \Omega$$

(H2) $b_i \in L^\infty(\Omega)$ for all $i = 1, \dots, n$ and $c \in L^\infty(\Omega)$.

In order to apply the Hille-Yosida theorem to show that A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup $S(t)$ on $L^2(\Omega)$, one can check that the following assumptions are satisfied.

1. $D(A)$ is dense in $L^2(\Omega)$.

   [This is a known property of Sobolev spaces (see, for instance, [3].)

2. A is a closed operator.

   Proof. Let $u_k \in D(A)$ be such that

   $$u_k \overset{k\to\infty}{\longrightarrow} u \quad \text{and} \quad Au_k \overset{k\to\infty}{\longrightarrow} f.$$

   Then, for all $h, k \geqslant 1$ we have that $v_{hk} := u_h - u_k$ satisfies

   $$\begin{cases} \sum_{i,j=1}^n D_j(a_{ij}D_j)v_{hk} + \sum_{i=1}^n b_i D_i v_{hk} + cv_{hk} =: f_{hk} & \text{in } \Omega \\ v_{hk} = 0 & \text{on } \partial\Omega. \end{cases}$$

   So, elliptic regularity insures that

   $$\|v_{hk}\|_{2,\Omega} \leqslant C\big(\|f_{hk}\|_{0,\Omega} + \|v_{hk}\|_{0,\Omega}\big)$$

   for some constant $C > 0$. The above inequality implies that $\{u_k\}$ is a Cauchy sequence in $D(A)$ and this yields $f = Au$. □

3. $\exists \omega \in \mathbb{R}$ such that $\rho(A) \supset]\omega, \infty[$.

   [This follows from elliptic theory (see, for instance, [3]).]

4. $\|R(\lambda, A)\| \leqslant \frac{1}{\lambda-\omega}$ for all $k \geqslant 1$ and $\lambda > \omega$.

   [This follows from elliptic theory (see, for instance, [3]).]

Then, for any $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, the function $u(t, x) = \big(S(t)u_0\big)(x)$ is the unique solution of the initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^n D_j(a_{ij}D_j)u + \sum_{i=1}^n b_i D_i u + cu & \text{in }]0, \infty[\times\Omega \\ u = 0 & \text{on }]0, \infty[\times\partial\Omega \\ u(0, x) = u_0(x) & x \in \Omega. \end{cases}$$

in the class

$$C^1\big([0, \infty); L^2(\Omega)\big) \cap \mathcal{C}\big([0, \infty); H^2(\Omega) \cap H_0^1(\Omega)\big).$$

1.7 Asymptotic behaviour of $\mathcal{C}_0$-semigroups

Let $S(t)$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on X.

Definition 9 *The number*

$$\omega_0(S) = \inf_{t>0} \frac{\log \|S(t)\|}{t} \tag{1.7.1}$$

is called the type *or* growth bound *of $S(t)$.*

Proposition 9 *The growth bound of S satisfies*

$$\omega_0(S) = \lim_{t\to\infty} \frac{\log \|S(t)\|}{t} < \infty. \tag{1.7.2}$$

Moreover, for any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\|S(t)\| \leqslant M_\varepsilon e^{(\omega_0(S)+\varepsilon)t} \qquad \forall t \geqslant 0. \tag{1.7.3}$$

Proof. The fact that $\omega_0(S) < \infty$ is a direct consequence of (1.7.1). In order to prove (1.7.2) it suffices to show that

$$\limsup_{t\to\infty} \frac{\log \|S(t)\|}{t} \leqslant \omega_0(S). \tag{1.7.4}$$

For any $\varepsilon > 0$ let $t_\varepsilon > 0$ be such that

$$\frac{\log \|S(t_\varepsilon)\|}{t_\varepsilon} < \omega_0(S) + \varepsilon. \tag{1.7.5}$$

Let us write any $t \geqslant t_\varepsilon$ as $t = nt_\varepsilon + \delta$ with $n = n(\varepsilon) \in \mathbb{N}$ and $\delta = \delta(\varepsilon) \in [0, t_\varepsilon[$. Then, by (1.2.2) and (1.7.5),

$$\|S(t)\| \leqslant \|S(\delta)\| \, \|S(t_\varepsilon)\|^n \leqslant Me^{\omega\delta} \, e^{nt_\varepsilon(\omega_0(S)+\varepsilon)} = Me^{(\omega-\omega_0(S)-\varepsilon)\delta} e^{(\omega_0(S)+\varepsilon)t}$$

which proves (1.7.3) with $M_\varepsilon = Me^{(\omega-\omega_0(S)-\varepsilon)\delta}$. Moreover, taking the logarithm of both sides of the above inequality we get

$$\frac{\log \|S(t)\|}{t} \leqslant \omega_0(S) + \varepsilon + \frac{\log M + (\omega - \omega_0(S) - \varepsilon)\delta}{t}$$

and (1.7.4) follows as $t \to \infty$. □

Definition 10 *For any operator $A : D(A) \subset X \to X$ we define the* spectral bound *of A as*

$$s(A) = \sup\{\, \Re\lambda \ : \ \lambda \in \sigma(A)\,\}.$$

Corollary 3 *Let $S(t)$ be a $\mathcal{C}_0$-semigroup on X with infinitesimal generator A. Then*

$$-\infty \leqslant s(A) \leqslant \omega_0(S) < +\infty.$$

Proof. By combining Theorem 5 and (1.7.3) we conclude that

$$\Pi_{\omega_0(S)+\varepsilon} \subset \rho(A) \qquad \forall \varepsilon > 0.$$

Therefore, $s(A) \leqslant \omega_0(S) + \varepsilon$ for all $\varepsilon > 0$. The conclusion follows. □

Example 10 For fixed $T > 0$ and $p \geqslant 1$ let $X = L^p(0,T)$ and

$$\big(S(t)f\big)(x) = \begin{cases} f(x-t) & x \in [t,T] \\ 0 & x \in [0,t) \end{cases} \quad \forall x \in [0,T]\,, \forall t \geqslant 0.$$

Then S is a $\mathcal{C}_0$-semigroup of bounded linear operators on X which satisfies $\|S(t)\| \leqslant 1$ for all $t \geqslant 0$. Moreover, observe that S is *nilpotent*, that is, we have $S(t) \equiv 0$, $\forall t \geqslant T$. Deduce that $\omega_0(S) = -\infty$. So, the spectral bound of the infinitesimal generator of $S(t)$ also equals $-\infty$.

Example 11 $\boldsymbol{(-\infty < s(A) = \omega_0(S))}$ In the Banach space

$$X = \mathcal{C}_b(\mathbb{R}_+;\mathbb{C}),$$

with the uniform norm, the left-translation semigroup

$$\big(S(t)f\big)(x) = f(x+t) \qquad \forall x,t \geqslant 0$$

is a $\mathcal{C}_0$-semigroup of contractions on X which satisfies $\|S(t)\| = 1$ (*Exercise*). Therefore

$$\omega_0(S) = 0.$$

The infinitesimal generator of $S(t)$ is given by

$$\begin{cases} D(A) = \mathcal{C}^1_b(\mathbb{R}_+;\mathbb{C}) \\ Af = f' & \qquad \forall f \in D(A). \end{cases}$$

By Theorem 5 we have that

$$\rho(A) \supset \big\{\lambda \in \mathbb{C} \;:\; \Re\lambda > 0\big\}.$$

We claim that

$$\sigma(A) \supset \big\{\lambda \in \mathbb{C} \;:\; \Re\lambda \leqslant 0\big\}.$$

Indeed, for any $\lambda \in \mathbb{C}$ the function $f_\lambda(x) := e^{\lambda x}$ satisfies $\lambda f - f' = 0$. Moreover, $f_\lambda \in D(A)$ for $\Re\lambda \leqslant 0$. Therefore

$$s(A) = 0.$$

Example 12 **($s(A) < \omega_0(S)$)** Let us denote by $\mathcal{C}_0(\mathbb{R}_+;\mathbb{C})$ the Banach space of all continuous functions $f:\mathbb{R}_+\to\mathbb{C}$ such that

$$\lim_{x\to\infty} f(x) = 0$$

with the uniform norm. Let X be the space of all functions $f\in\mathcal{C}_0(\mathbb{R}_+;\mathbb{C})$ such that

$$\|f\| := \sup_{x\in\mathbb{R}_+}|f(x)| + \int_0^\infty |f(x)|e^x dx < \infty.$$

Exercise 13 Prove that $(X,\|\cdot\|)$ is a Banach space.

Once again, the left-translation semigroup

$$\big(S(t)f\big)(x) = f(x+t) \qquad \forall x,t\geqslant 0$$

is a $\mathcal{C}_0$-semigroup of contractions on X. Indeed, for all $t\geqslant 0$

$$\begin{aligned}\|S(t)f\| &= \sup_{x\in\mathbb{R}_+}|f(x+t)| + \int_0^\infty |f(x+t)|e^x dx\\ &\leqslant \sup_{x\in\mathbb{R}_+}|f(x)| + e^{-t}\int_0^\infty |f(x)|e^x dx.\end{aligned}$$

Exercise 14 Prove that $\|S(t)\| = 1$ for all $t\geqslant 0$

Therefore

$$\omega_0(S) = 0.$$

The infinitesimal generator of $S(t)$ is given by

$$\begin{cases} D(A) = \big\{f\in X \ : \ f'\in X\big\} \\ Af = f' & \forall f\in D(A).\end{cases}$$

For any $\lambda\in\mathbb{C}$ the function $f_\lambda(x) := e^{\lambda x}$ satisfies $\lambda f - f' = 0$ and $f_\lambda\in D(A)$ for $\Re\lambda < -1$. So,

$$s(A)\geqslant -1. \tag{1.7.6}$$

We claim that

$$\rho(A)\supset\big\{\lambda\in\mathbb{C} \ : \ \Re\lambda > -1\big\}. \tag{1.7.7}$$

Indeed, a change of variables shows that, for any $g\in X$, the function

$$f(x) = \int_0^\infty e^{-\lambda t}\big(S(t)g\big)(x)dt = \int_0^\infty e^{-\lambda t} g(x+t)dt \qquad (x\geqslant 0)$$

satisfies $\lambda f - f' = g$. Consequently, if we show that $f \in X$, then $f \in D(A)$ follows and so $\lambda \in \rho(A)$. To check that $f \in X$ observe that, for all $x \geqslant 0$,

$$
\begin{aligned}
|f(x)| &\leqslant \int_0^\infty \big|e^{-\lambda t} g(x+t)\big| dt \\
&= \int_0^\infty e^{-t\Re\lambda} \big|g(x+t)\big| e^{x+t} e^{-x-t} dt \\
&= e^{-x} \int_0^\infty e^{-t(1+\Re\lambda)} e^{x+t} \big|g(x+t)\big| dt \qquad (1.7.8) \\
&\leqslant e^{-x} \int_x^\infty e^{s} \big|g(s)\big| ds
\end{aligned}
$$

which insures that $f \in \mathcal{C}_0(\mathbb{R}_+;\mathbb{C})$. Furthermore, by (1.7.8) we compute

$$
\begin{aligned}
\int_0^\infty |f(x)| e^x dx &\leqslant \int_0^\infty dx \int_0^\infty e^{-t(1+\Re\lambda)} e^{x+t} \big|g(x+t)\big| dt \\
&= \int_0^\infty e^{-t(1+\Re\lambda)} dt \int_0^\infty e^{x+t} \big|g(x+t)\big| dx \\
&\leqslant \int_0^\infty e^{-t(1+\Re\lambda)} dt \int_0^\infty e^{\tau} \big|g(\tau)\big| d\tau < \infty.
\end{aligned}
$$

From (1.7.6) and (1.7.7) it follows that $s(A) = -1 < 0 = \omega_0(S)$.

Exercise 15 Let $S(t)$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on X. Prove that $\omega_0(S) < 0$ if and only if

$$\lim_{t \to +\infty} \|S(t)\| = 0. \qquad (1.7.9)$$

Solution. One only needs to show that (1.7.9) implies that $\omega_0(S) < 0$. Let $t_0 > 0$ be such that $\|S(t_0)\| < 1/e$. For any $t > 0$ let $n \in \mathbb{N}$ be the unique integer such that

$$n t_0 \leqslant t < \big(n+1\big) t_0. \qquad (1.7.10)$$

Then

$$\|S(t)\| = \big\|S\big(nt_0\big) S\big(t - nt_0\big)\big\| \leqslant \frac{M e^{\omega(t-nt_0)}}{e^n} \leqslant \frac{M e^{\omega t_0}}{e^n}.$$

Therefore, on account of (1.7.9), we conclude that

$$
\begin{aligned}
\frac{\log \|S(t)\|}{t} &\leqslant \frac{\log \big(M e^{\omega t_0}\big)}{t} - \frac{n}{t} \\
&\leqslant \frac{\log \big(M e^{\omega t_0}\big)}{t} - \Big(\frac{1}{t_0} - \frac{1}{t}\Big) \qquad \forall t > 0.
\end{aligned}
$$

Taking the limit as $t \to +\infty$ we conclude that $\omega_0(S) < 0$. □

Exercise 16 Let $S(t)$ be the $\mathcal{C}_0$-semigroup on $L^2(\Omega)$ associated with the initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in }]0,\infty[\times\Omega \\ u = 0 & \text{on }]0,\infty[\times\partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases} \tag{1.7.11}$$

Show that $\omega_0(S) < 0$.

Solution. We know from Example 9 that the infinitesimal generator of $S(t)$ is the operator A defined by

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ Au = \Delta u & \forall u \in D(A). \end{cases}$$

For $u_0 \in D(A)$, let $u(t,x) = \big(S(t)u_0\big)(x)$. Then u satisfies (1.7.11). So

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega |u(t,x)|^2 dx\Big) = -\frac{1}{2}\int_\Omega |Du(t,x)|^2 dx \qquad \forall t > 0.$$

Moreover, by Poincaré's inequality we have that

$$\int_\Omega |u(t,x)|^2 dx \leqslant c(\Omega) \int_\Omega |Du(t,x)|^2 dx.$$

Therefore,

$$\frac{d}{dt}\,|u(t)|^2 \leqslant -\frac{2}{c(\Omega)}\,|u(t)|^2$$

which ensures, by Gronwall's lemma, that

$$|u(t)| \leqslant e^{-t/c(\Omega)}|u_0| \qquad \forall t > 0.$$

By a density argument, one concludes that the above inequality holds true for any $u_0 \in L^2(\Omega)$, so that $\omega_0(S) \leqslant -1/c(\Omega)$. □

1.8 Strongly continuous groups

Definition 11 *A strongly continuous group, or a $\mathcal{C}_0$-group, of bounded linear operators on X is a map $G : \mathbb{R} \to \mathcal{L}(X)$ with the following properties:*

(*a*) $G(0) = I$ *and* $G(t+s) = G(t)G(s)$ *for all* $t, s \in \mathbb{R}$,

(*b*) *for all* $u \in X$

$$\lim_{t\to 0} G(t)u = u. \tag{1.8.1}$$

Definition 12 *The infinitesimal generator of a $\mathcal{C}_0$-group of bounded linear operators on X, $G(t)$, is the map $A : D(A) \subset X \to X$ defined by*

$$\begin{cases} D(A) = \left\{ u \in X \ : \ \exists \lim_{t\to 0} \frac{S(t)u-u}{t} \right\} \\ Au = \lim_{t\to 0} \frac{S(t)u-u}{t} \qquad\qquad \forall u \in D(A) \end{cases}$$

Theorem 7 *Let $M \geqslant 1$ and $\omega \geqslant 0$. For a linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(a) *A is the infinitesimal generator of a $\mathcal{C}_0$-group, $G(t)$, such that*

$$\|G(t)\| \leqslant M e^{\omega|t|} \qquad \forall t \in \mathbb{R}. \tag{1.8.2}$$

(b) *A and $-A$ are the infinitesimal generators of $\mathcal{C}_0$-semigroups, $S_+(t)$ and $S_-(t)$) respectively, satisfying*

$$\|S_\pm(t)\| \leqslant M e^{\omega t} \qquad \forall t \geqslant 0. \tag{1.8.3}$$

(c) *A is closed, $D(A)$ is dense in X, and*

$$\rho(A) \supseteq \left\{ \lambda \in \mathbb{C} \ : \ |\Re\lambda| > \omega \right\} \tag{1.8.4}$$

$$\|R(\lambda, A)^k\| \leqslant \frac{M}{(|\Re\lambda| - \omega)^k} \quad \forall k \geqslant 1, \forall |\Re\lambda| > \omega \tag{1.8.5}$$

Remark 4 Let A and $S_\pm(t)$ be as in point (b) above. We claim that

(i) $S_+(t)S_-(s) = S_-(s)S_+(t)$ for all $s, t \geqslant 0$,

(ii) $S_+(t)^{-1} = S_-(t)$ for all $t \geqslant 0$.

Indeed, recall that

$$S_+(t) = \lim_{n\to\infty} e^{tA_n}, \qquad S_-(t) = \lim_{n\to\infty} e^{tB_n}$$

where

$$A_n = nAR(n, A), \qquad B_n = -nAR(n, -A) = nAR(-n, A)$$

are the Yosida approximations of A and $-A$, respectively. Since A_n and B_m commute in view of (1.5.3), so do e^{tA_n} and e^{tB_m} and (i) holds true.

Consequently,

$$S(t) := S_+(t)S_-(t) \qquad (t \geqslant 0)$$

is also a $\mathcal{C}_0$-semigroup and, for all $u \in D(A) = D(-A)$, we have that

$$\frac{S(t)u - u}{t} = S_+(t)\,\frac{S_-(t)u - u}{t} + \frac{S_+(t)u - u}{t} \overset{t\downarrow 0}{\longrightarrow} -Au + Au = 0.$$

So, $\frac{d}{dt} S(t)u = 0$ for all $t \geqslant 0$. Hence, $S(t)u = u$ for all $t \geqslant 0$ and $u \in D(A)$. By density, $S(t)u = u$ for all $x \in X$, which yields $S_+(t)^{-1} = S_-(t)$. □

Proof of (a) ⇒ (b) Define, for all $t \geqslant 0$,

$$S_+(t) = G(t) \quad \text{and} \quad S_-(t) = G(-t).$$

Then it can be checked that $S_\pm(t)$ is $\mathcal{C}_0$-semigroup satisfying (1.8.3). Moreover, observing that

$$\frac{S_-(t)u - u}{t} = \frac{G(-t)u - u}{t} = -G(-t)\frac{G(t)u - u}{t},$$

it is easy to show that $\pm A$ is the infinitesimal generator of $S_\pm(t)$. □

Proof of (b) ⇒ (c) By the Hille-Yosida theorem we conclude that A is closed, $D(A)$ is dense in X, and

$$\rho(A) \supseteq \Pi_\omega = \{\lambda \in \mathbb{C} \;:\; \Re\lambda > \omega\}$$
$$\|R(\lambda, A)^k\| \leqslant \frac{M}{(\Re\lambda - \omega)^k} \quad \forall k \geqslant 1,\ \forall \lambda \in \Pi_\omega.$$

Since

$$(\lambda I + A)^{-1} = -(-\lambda I - A)^{-1}, \tag{1.8.6}$$

we have that $-\rho(A) = \rho(-A) \supseteq \Pi_\omega$, or

$$\rho(A) \supseteq -\Pi_\omega = \{\lambda \in \mathbb{C} \;:\; \Re\lambda < -\omega\},$$

and

$$\|R(\lambda, A)^k\| = \|R(-\lambda, -A)^k\| \leqslant \frac{M}{(-\Re\lambda - \omega)^k} \quad \forall k \geqslant 1,\ \forall \lambda \in -\Pi_\omega. \qquad \square$$

Proof of (c) ⇒ (a) Recalling (1.8.6), by the Hille-Yosida theorem it follows that $\pm A$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup, $S_\pm(t)$, satisfying (1.8.3). For all $u \in X$ define

$$G(t)u = \begin{cases} S_+(t)u & (t \geqslant 0) \\ S_-(-t)u & (t < 0). \end{cases}$$

Then, it follows that (1.8.1) and (1.8.2) hold true, and A is the infinitesimal generator of $G(t)$. Let us check that $G(t+s) = G(t)G(s)$ for all $t \geqslant 0$ and all $s \leqslant 0$ such that $t + s \geqslant 0$. Recalling point (ii) of Remark 4, we have that

$$G(t)G(s) = S_+(t)S_-(-s) = S_+(t+s)S_+(-s)S_+(-s)^{-1} = G(t+s). \qquad \square$$

1.9 Additional exercises

Exercise 17 Let S be $\mathcal{C}_0$-semigroup of bounded linear operators on X and let $K \subset X$ be compact. Prove that for every $t_0 \geqslant 0$

$$\lim_{t \to t_0} \sup_{u \in K} \big|S(t)u - S(t_0)u\big| = 0\,. \tag{1.9.1}$$

Solution. We may assume $S \in \mathcal{G}(M, 0)$ for some $M >$) without loss of generality. Let $t_0 > 0$ and fix any $\varepsilon > 0$. Since K is totally bounded, there exist $u_1, \dots, u_{N_\varepsilon} \in X$ such that

$$K \subset \bigcup_{n=1}^{N_\varepsilon} B\Big(u_n, \frac{\varepsilon}{M}\Big).$$

Moreover, there exists $\tau > 0$ such that

$$|t - t_0| < \tau \quad \Longrightarrow \quad \big|S(t)u_n - S(t_0)u_n\big| < \varepsilon \qquad \forall n = 1, \dots, N_\varepsilon.$$

Thus, for all $|t - t_0| < \tau$ we have that, if $u \in K$ is such that $u \in B\big(u_n, \frac{\varepsilon}{M}\big)$, then

$$\begin{aligned}
&\big|S(t)u - S(t_0)u\big| \\
&\quad\leqslant \big|S(t)u - S(t)u_n\big| + \big|S(t)u_n - S(t_0)u_n\big| + \big|S(t_0)u_n - S(t_0)u\big| \\
&\quad\leqslant 2M|u - u_n| + \varepsilon < 3\varepsilon.
\end{aligned}$$

So, the limit of $\big|S(t)u - S(t_0)u\big|$ as $t \to t_0$ is uniform on K. □

Exercise 18 Let $A : D(A) \subset X \to X$ be a closed operator satisfying (1.6.2) but suppose $D(A)$ fails to be dense in X. In the Banach space $Y := \overline{D(A)}$, define the operator B, called the *part of A in Y*, by

$$\begin{cases} D(B) = \big\{u \in D(A) \;:\; Au \in Y\big\} \\ Bu = Au \qquad \forall u \in D(B). \end{cases}$$

Prove that B is the infinitesimal generator of a $\mathcal{C}_0$-semigroup on Y.

Solution. $R(\lambda, A)(Y) \subset D(B)$ for all $\lambda \in \mathbb{C}$ such that $\Re\lambda > \omega$. Indeed, owing to (1.5.1) for all $u \in D(A)$ we have that

$$\lim_{n\to\infty} nR(n, A)u = \lim_{n\to\infty} \big\{R(n, A)Au + u\big\} = u. \tag{1.9.2}$$

Since $\|nR(n, A)\|$ is bounded, (1.9.2) holds true for all $u \in Y$. Hence, $D(B)$ is dense in Y. So, B satisfies in Y all the assumptions of Theorem 6. □

Exercise 19 Let X be a Banach space and let $A : D(A) \subset X \to X$ be the infinitesimal generator of a uniformly bounded semigroup. Define, for $n \geqslant 1$,

$$D(A^n) := \{u \in D(A^{n-1}) \ : \ A^{n-1}u \in D(A)\}.$$

(i) Prove the following extension of the Landau-Kolmogorov inequality (1.3.1):

$$|A^k u| \leqslant (2M)^{k(n-k)} \, |A^n u|^{\frac{k}{n}} \, |u|^{\frac{n-k}{n}} \quad \forall u \in D(A^n), \ \forall 0 \leqslant k \leqslant n \quad (1.9.3)$$

Solution: proceed by induction. The conclusion is trivial for $n = 1$. Assume (1.9.3) holds true for n and let $u \in D(A^{n+1})$. Then, in view of (1.3.1), we have that

$$\begin{aligned}|A^n u| &\leqslant 2M \, |A^{n+1}u|^{\frac{1}{2}} \, |A^{n-1}u|^{\frac{1}{2}} \\ &\leqslant 2M \, |A^{n+1}u|^{\frac{1}{2}} \Big((2M)^{n-1} |A^n u|^{\frac{n-1}{n}} \, |u|^{\frac{1}{n}} \Big)^{\frac{1}{2}} \\ &= (2M)^{\frac{n+1}{2}} \, |A^{n+1}u|^{\frac{1}{2}} |A^n u|^{\frac{n-1}{2n}} \, |u|^{\frac{1}{2n}}.\end{aligned}$$

Therefore,

$$|A^n u| \leqslant (2M)^n \, |A^{n+1}u|^{\frac{n}{n+1}} |u|^{\frac{1}{n+1}}, \qquad (1.9.4)$$

which is (1.9.3) for $n+1$ with $k = n$. Now, suppose $0 \leqslant k < n$. Then, by our inductive assumption and (1.9.4),

$$\begin{aligned}|A^k u| &\leqslant (2M)^{k(n-k)} \, |A^n u|^{\frac{k}{n}} \, |u|^{\frac{n-k}{n}} \\ &\leqslant (2M)^{k(n-k)} \Big((2M)^n \, |A^{n+1}u|^{\frac{n}{n+1}} |u|^{\frac{1}{n+1}} \Big)^{\frac{k}{n}} |u|^{\frac{n-k}{n}} \\ &= (2M)^{k(n+1-k)} |A^{n+1}u|^{\frac{k}{n+1}} |u|^{\frac{n+1-k}{n+1}}.\end{aligned}$$

The proof is complete. □

(ii) Using (1.9.3), prove that for every $n \geqslant 1$:

(a) A^n is a closed operator.

Solution: proceed by induction. The conclusion is trivial for $n = 1$. Assume it holds true for n and let $\{u_k\} \subset D(A^{n+1})$ be such that

$$u_k \to u \qquad \& \qquad A^{n+1}u_k \to v \qquad (k \to \infty).$$

Applying (1.9.4) to $w_k := A^n u_k \in D(A)$ we obtain

$$|w_k - w_h| \leqslant (2M)^n \, |A^{n+1}(u_k - u_h)|^{\frac{n}{n+1}} |u_k - u_h|^{\frac{1}{n+1}} \to 0 \quad (h, k \to \infty)$$

Therefore, for some $w \in X$,

$$w_k \to w \qquad \& \qquad Aw_k \to v \qquad (k \to \infty).$$

Since A is closed, we conclude that

$$w \in D(A) \qquad \& \qquad Aw = v \qquad (k \to \infty).$$

Then, by our inductive assumptiom, $u \in D(A^n)$ and $A^n u = w$, which implies in turn

$$u \in D(A^{n+1}) \qquad \& \qquad A^{n+1}u = Aw = v \qquad (k \to \infty). \qquad \square$$

(b) $D(A^n)$ is dense in X for every $n \geqslant 1$.

Solution of (ii)(b): for $n = 1$ the conclusion follows from Theorem 3. Let the conclusion be true for some $n \geqslant 1$ and fix any $v \in X$. Then, for any $\varepsilon > 0$ there exists $u_\varepsilon \in D(A^n)$ such that $|u_\varepsilon - v| < \varepsilon$. Moreover, recalling point (a),

$$A^n\Big(\frac{1}{t}\int_0^t S(s)u_\varepsilon\, ds\Big) = \frac{1}{t}\int_0^t S(s)A^n u_\varepsilon\, ds$$

Since

$$\frac{1}{t}\int_0^t S(s)A^n u_\varepsilon\, ds \in D(A) \qquad \forall t > 0$$

we conclude that

$$\frac{1}{t}\int_0^t S(s)u_\varepsilon\, ds \in D(A^{n+1}) \qquad \forall t > 0.$$

Moreover, there exists $t_\varepsilon > 0$ such that

$$\Big|\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} S(s)u_\varepsilon\, ds - v\Big| \leqslant \Big|\frac{1}{t_\varepsilon}\int_0^{t_\varepsilon} S(s)u_\varepsilon\, ds - u_\varepsilon\Big| + |u_\varepsilon - v| < 2\varepsilon. \ \square$$

Generalize to the infinitesimal generator of a $\mathcal{C}_0$-semigroup of bounded linear operators on X.

Exercise 20 Let $p \geqslant 2$. On $X = L^p(0,\pi)$ consider the operator defined by

$$\begin{cases} D(A) = W^{2,p}(0,\pi) \cap W_0^{1,p}(0,\pi) \\ Af(x) = f''(x) & x \in (0,\pi) \text{ a.e.} \end{cases} \tag{1.9.5}$$

where

$$W_0^{1,p}(0,\pi) = \big\{ f \in W^{1,p}(0,\pi) \ : \ f(0) = 0 = f(\pi) \big\}.$$

Since $\mathcal{C}_c^\infty(0,\pi) \subset D(A)$, we have that $D(A)$ is dense in X. Show that A is closed and satisfies condition (a') of Remark 2 with $M = 1$ and $\omega = 0$. Theorem 6 will imply that A generates a $\mathcal{C}_0$-semigroup of contractions on X.

Solution. Step 1: $\sigma(A) = \{-n^2 \ : \ n \in \mathbb{N}\}$.

Fix any $g \in X$. We will show that, for all $\lambda \neq n^2 (n \geqslant 1)$, the Sturm-Liouville system

$$\begin{cases} \lambda f(x) - f''(x) = g(x), & 0 < x < \pi \\ f(0) = 0 = f(\pi) \end{cases} \tag{1.9.6}$$

admits a unique solution $f \in D(A)$. Denoting by

$$g(x) = \sum_{n=1}^{\infty} g_n \sin(nx) \qquad (x \in [0, \pi])$$

the Fourier series of g, we seek a candidate solution f of the form

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(nx) \qquad (x \in [0, \pi]).$$

In order to satisfy (1.9.6) one must have

$$(\lambda + n^2) f_n = g_n \qquad \forall n \geqslant 1.$$

So, for any $\lambda \neq -n^2$, (1.9.6) has a unique solution given by

$$f(x) = \sum_{n=1}^{\infty} \frac{g_n}{\lambda + n^2} \sin(nx) \qquad (x \in [0, \pi]).$$

From the above representation it follows that $f \in H^2(0, \pi) \cap H^1_0(0, \pi)$. In fact, returning to the equation in (1.9.6) one concludes that $f \in D(A)$.

Step 2: resolvent estimate.

By multiplying both members of the equation in (1.9.6) by $|f|^{p-2} f$ and integrating over $(0, \pi)$ one obtains, for all $\lambda > 0$,

$$\lambda \int_0^{\pi} |f(x)|^p dx + (p-1) \int_0^{p} |f(x)|^{p-2} |f'(x)|^2 dx = \int_0^{\pi} g(x) |f(x)|^{p-2} f(x)\, dx$$

which yields

$$|f|_p \leqslant \frac{1}{\lambda} |g|_p \qquad \forall \lambda > 0.$$

Step 3: conclusion.

By Proposition 6 we conclude that for each $f \in W^{2,p}(0, \pi) \cap W^{1,p}_0(0, \pi)$ the unique solution of

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) & (t, x) \in \mathbb{R}_+ \times (0, \pi) \\ u(t, 0) = 0 = u(t, \pi) & t \geqslant 0 \\ u(0, x) = f(x) & x \in (0, \pi) \end{cases}$$

is given by $u(t, x) = (S(t) f)(x)$. □

Exercise 21 Let $S(t)$ be the $\mathcal{C}_0$-semigroup generated by operator A in (1.9.5). Prove that, for any $f \in L^p(0,\pi)$,

$$(S(t)f)(x) = \int_0^\pi K(t,x,y)f(y)\,dy\,, \quad \forall t \geqslant 0,\ x \in (0,\pi) \text{ a.e.}$$

where

$$K(t,x,y) = \frac{2}{\pi}\sum_{k=1}^\infty e^{-k^2 t}\sin(kx)\sin(ky).$$

Exercise 22 On $X = \{f \in \mathcal{C}([0,\pi]) \ : \ f(0) = 0 = f(\pi)\}$ with the uniform norm, consider the linear operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \{f \in \mathcal{C}^2([0,1]) \ : \ f(0) = f(\pi) = 0 = f''(0) = f''(\pi)\} \\ Af = f'', \quad \forall f \in D(A). \end{cases}$$

Show that A generates a $\mathcal{C}_0$-semigroup of contractions on X and derive the initial-boundary value problem which is solved by such semigroup.

Solution. We only prove that $\|R(\lambda,A)\| \leqslant 1/\lambda$ for all $\lambda > 0$. Fix any $g \in X$ and let $f = R(\lambda,A)g$. Let $x_0 \in [0,\pi]$ be such that $|f(x_0)| = |f|_\infty$. If $f(x_0) > 0$, then $x_0 \in (0,\pi)$ is a maximum point of f. So, $f''(x_0) \leqslant 0$ and we have that

$$\lambda|f|_\infty = \lambda f(x_0) \leqslant \lambda f(x_0) - f''(x_0) = g(x_0) \leqslant |g|_\infty.$$

On the other hand, if $f(x_0) < 0$, then $x_0 \in (0,\pi)$ once again and x_0 is a minimum point of f. Thus, $f''(x_0) \geqslant 0$ and

$$\lambda|f|_\infty = -\lambda f(x_0) \leqslant -\lambda f(x_0) + f''(x_0) = -g(x_0) \leqslant |g|_\infty.$$

In any case, we have that $\lambda|f|_\infty \leqslant |g|_\infty$. □

Exercise 23 Let $\big(X, |\cdot|\big)$ be a separable Banach space and let $A : D(A) \subset X \to X$ be a closed operator with $\rho(A) \neq \emptyset$. Prove that $\big(D(A), |\cdot|_{D(A)}\big)$ is also separable.

Solution. Let $\{u_n\}_{n\in\mathbb{N}}$ be dense in X and let $\lambda_0 \in \rho(A)$. Fix any $v \in D(A)$ and set $w = \lambda_0 v - Av$. For arbitrary $\varepsilon > 0$ let $u_\varepsilon = u_{n_\varepsilon}$ be such that $|w - u_\varepsilon| < \varepsilon$. Then

$$|v - R(\lambda_0,A)u_\varepsilon| = |R(\lambda_0,A)(w - u_\varepsilon)| \leqslant \|R(\lambda_0,A)\|\varepsilon.$$

Moreover,

$$\begin{aligned} |Av - AR(\lambda_0,A)u_\varepsilon| &= |AR(\lambda_0,A)(w-u_\varepsilon)| \\ &\leqslant |\lambda_0 R(\lambda_0,A)(w-u_\varepsilon)| + |w-u_\varepsilon| \leqslant \big(|\lambda_0|\|R(\lambda_0,A)\| + 1\big)\varepsilon. \end{aligned}$$

This shows that $\{R(\lambda_0,A)u_n\}_{n\in\mathbb{N}}$ is dense in $D(A)$. □

2 *Dissipative operators*

2.1 Definition and first properties

Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$.

Definition 13 *We say that an operator $A : D(A) \subset H \to H$ is* dissipative *if*

$$\Re \langle Au, u\rangle \leqslant 0 \qquad \forall u \in D(A). \tag{2.1.1}$$

Example 13 In $H = L^2(\mathbb{R}_+;\mathbb{C})$ consider the operator

$$\begin{cases} D(A) = H^1(\mathbb{R}_+;\mathbb{C}) \\ Af(x) = f'(x) \qquad\qquad x \in \mathbb{R}_+ \text{ a.e.} \end{cases}$$

Then

$$2\Re \langle Af, f\rangle = 2\Re\Big(\int_0^\infty f'(x)\overline{f(x)}\,dx\Big) = \int_0^\infty \frac{d}{dx}|f(x)|^2\,dx = -|f(0)|^2 \leqslant 0.$$

So, A is dissipative.

Proposition 10 *An operator $A : D(A) \subset H \to H$ is dissipative if and only if for any $u \in D(A)$*

$$|(\lambda I - A)u| \geqslant \lambda |u| \qquad \forall \lambda > 0. \tag{2.1.2}$$

Proof. Let A be dissipative. Then, for every $u \in D(A)$ we have that

$$|(\lambda I - A)u|^2 = \lambda^2|u|^2 - 2\lambda\Re\langle Au, u\rangle + |Au|^2 \geqslant \lambda^2|u|^2 \qquad \forall \lambda > 0.$$

Conversely, suppose A satisfies (2.1.2). Then for every $\lambda > 0$ and $u \in D(A)$

$$\lambda^2|u|^2 - 2\lambda\Re\langle Au, u\rangle + |Au|^2 = |(\lambda I - A)u|^2 \geqslant \lambda^2|u|^2$$

So, $2\lambda\Re\langle Au, u\rangle \leqslant |Au|^2$ which in turn yields (2.1.1) as $\lambda \to \infty$. □

The above characterization can be used to extend the notion of dissipative operators to a Banach space X.

Definition 14 *We say that an operator $A : D(A) \subset X \to X$ is* dissipative *if*

$$|(\lambda I - A)u| \geqslant \lambda|u| \qquad \forall u \in D(A) \quad \textit{and} \quad \forall \lambda > 0. \tag{2.1.3}$$

Remark 5 It follows from (2.1.3) that, if A is dissipative then

$$\lambda I - A : D(A) \to X$$

is injective for all $\lambda > 0$.

Proposition 11 *Let $A : D(A) \subset X \to X$ be dissipative. If*

$$\exists \lambda_0 > 0 \quad \textit{such that} \quad (\lambda_0 I - A)D(A) = X, \tag{2.1.4}$$

then the following properties hold:

(a) $\lambda_0 \in \rho(A)$ *and* $\|R(\lambda_0, A)\| \leqslant 1/\lambda_0$,

(b) A *is closed,*

(c) $(\lambda I - A)D(A) = X$ *and* $\|R(\lambda, A)\| \leqslant 1/\lambda$ *for all* $\lambda > 0$.

Proof. We observe that point (a) follows from Remark 5 and inequality (2.1.3). As for point (b), we note that, since $R(\lambda_0, A)$ is closed, $\lambda_0 I - A$ is also closed, and therefore A is closed.
Proof of (c). By point (a) the set

$$\Lambda = \big\{\lambda \in]0, \infty[\, : \, (\lambda I - A)D(A) = X\big\}$$

is contained in $\rho(A)$ which is open in $\mathbb{C}$. This implies that Λ is also open. Let us show that Λ is closed: let $\Lambda \ni \lambda_n \to \lambda > 0$ and fix any $v \in X$. There exists $u_n \in D(A)$ such that

$$\lambda_n u_n - A u_n = v. \tag{2.1.5}$$

From (2.1.2) it follows that $|u_n| \leqslant |v|/\lambda_n \leqslant C$ for some $C > 0$. Again by (2.1.2),

$$\begin{aligned} \lambda_m |u_n - u_m| &\leqslant |\lambda_m(u_n - u_m) - A(u_n - u_m)| \\ &\leqslant |\lambda_m - \lambda_n| \, |u_n| + |\lambda_n u_n - A u_n - (\lambda_m u_m - A u_m)| \\ &\leqslant C|\lambda_m - \lambda_n|. \end{aligned}$$

Therefore $\{u_n\}$ is a Cauchy sequence. Let $x \in X$ be such that $u_n \to u$. Then $Au_n \to \lambda u - v$ by (2.1.5). Since A is closed, $u \in D(A)$ and $\lambda u - Au = v$. This show that $\lambda I - A$ is surjective and implies that $\lambda \in \Lambda$. Thus, Λ is both open and closed in $]0, \infty[$. Moreover, $\Lambda \neq \varnothing$ because $\lambda_0 \in \Lambda$. So, $\Lambda =]0, \infty[$. The inequality $\|R(\lambda, A)\| \leqslant 1/\lambda$ is a consequence of dissipativity. □

2.2 Maximal dissipative operators

Definition 15 *A dissipative operator $A : D(A) \subset X \to X$ is called* maximal dissipative *if $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$ (hence, for all $\lambda > 0$).*

Remark 6 Let $A : D(A) \subset X \to X$ be a maximal dissipative operator and let $\overline{A} \supset A$ be a dissipative extension of A. Then:

(i) $\overline{A}$ is maximal dissipative ($\lambda I - \overline{A}$ is surjective since so is $\lambda I - A$);

(ii) $\overline{A} = A$ (since both $\rho(A)$ and $\rho(\overline{A})$ contain $]0, \infty[$).

Theorem 8 *Let X be a reflexive Banach space. If $A : D(A) \subset X \to X$ is a maximal dissipative operator, then $D(A)$ is dense in X.*

We give the proof for a Hilbert space. The case of a reflexive Banach space is treated in exercises 24 to 27.

Proof. Let $v \in X$ be such that $\langle v, u \rangle = 0$ for all $u \in D(A)$. We will show that $v = 0$, or

$$\langle v, w \rangle = 0 \qquad \forall w \in X.$$

Since $(I - A)$ is surjective, the above is equivalent to

$$0 = \langle v, u - Au \rangle \qquad \forall u \in D(A).$$

So, we need to prove that

$$\langle v, u \rangle = 0 \qquad \forall u \in D(A) \quad \Longrightarrow \quad \langle v, Au \rangle = 0 \qquad \forall u \in D(A). \tag{2.2.1}$$

Let $u \in D(A)$. Since $nI - A$ is onto, there exists a sequence $\{u_n\} \subset D(A)$ such that

$$nu = nu_n - Au_n \qquad \forall n \geqslant 1. \tag{2.2.2}$$

Since $Au_n = n(u_n - u) \in D(A)$, we have that $u_n \in D(A^2)$ and

$$Au = Au_n - \frac{1}{n} A^2 u_n \quad \text{or} \quad Au_n = \Big(I - \frac{1}{n} A\Big)^{-1} Au.$$

Since $\|(I - \frac{1}{n} A)^{-1}\| \leqslant 1$ by (2.1.2), the above identity yields $|Au_n| \leqslant |Au|$. So, by (2.2.2) we obtain

$$|u_n - u| \leqslant \frac{1}{n} |Au|.$$

Therefore, $u_n \to u$. Moreover, since $\{Au_n\}$ is bounded, there is a subsequence Au_{n_k} such that $Au_{n_k} \rightharpoonup w$. Since A is closed, $graph(A)$ is a closed subspace of $X \times X$. Then, $graph(A)$ is also weakly closed and we have that $w = Au$. Therefore,

$$\langle v, Au \rangle = \lim_{k\to\infty} \langle v, Au_{n_k} \rangle = \lim_{k\to\infty} n_k \langle v, u_{n_k} - u \rangle$$

and (2.2.1) follows from the vanishing of the rightmost term above. □

Example 14 We now show that the above density may be fail in a general Banach space. On $X = \mathcal{C}([0, 1])$ with the uniform norm $\|\cdot\|_\infty = \|\cdot\|_{\infty,[0,1]}$, consider the linear operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \big\{u \in \mathcal{C}^1([0, 1]) \ : \ u(0) = 0\big\} \\ Au(x) = -u'(x) \qquad\qquad\qquad \forall x \in [0, 1]. \end{cases}$$

Then, for all $\lambda > 0$ and $f \in X$ we have that the equation $\lambda u - Au = f$ has the unique solution $u \in D(A)$ given by

$$u(x) = \int_0^x e^{\lambda(y-x)} f(y)\, dy \quad (x \in [0,1])$$

Therefore, $\lambda I - A$ is onto. Moreover,

$$\lambda |u(x)| \leqslant \int_0^x \lambda e^{\lambda(y-x)} \|f\|_\infty \, dy = (1 - e^{-\lambda x}) \|f\|_\infty \leqslant \|\lambda u - Au\|_\infty.$$

So, A is dissipative. On the other hand, $D(A)$ is not dense in X because all functions in $D(A)$ vanish at $x = 0$.

Exercise 24 We recall that the duality set of a point $x \in X$ is defined as

$$\Phi(x) = \{\phi \in X^* \ : \ \langle x, \phi \rangle = |x|^2 = \|\phi\|^2\}. \tag{2.2.3}$$

Observe that the Hahn-Banach theorem ensures $\Phi(x) \neq \varnothing$.

We also recall that, for all $x \in X$,

$$\partial |x| = \{\phi \in X^* \ : \ |x + h| - |x| \geqslant \langle h, \phi \rangle \, , \ \forall x, h \in X\}. \tag{2.2.4}$$

Prove that

$$\Phi(x) = x \partial |x| = \{\psi \in X^* \ : \ \psi = |x| \phi \, , \ \phi \in \partial |x|\}.$$

Exercise 25 Prove that, for any operator $A : D(A) \subset X \to X$ the following properties are equivalent:

(a) A is dissipative,

(b) for all $x \in D(A)$ there exists $\phi \in \Phi(x)$ such that $\Re \langle Ax, \phi \rangle \leqslant 0$.

Exercise 26 Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions. Prove that, for all $x \in D(A)$,

$$\Re \langle Ax, \phi \rangle \leqslant 0 \qquad \forall \phi \in \Phi(x).$$

Exercise 27 Mimic the proof of Theorem 8 to treat the general case of a reflexive Banach space.

Theorem 9 (Lumer-Phillips 1) *Let $A : D(A) \subset X \to X$ be a densely defined linear operator. Then the following properties are equivalent:*

(a) A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions,

(b) A is maximal dissipative.

Proof of (a) ⇒ (b) In view of Theorem 5, we have that $]0, \infty[\subset \rho(A)$. So, $(\lambda I - A)D(A) = X$ for all $\lambda > 0$. Moreover, by the Hille-Yosida theorem for all $\lambda > 0$ and $v \in X$ we have that $\lambda|R(\lambda, A)v| \leqslant |v|$ or, setting $u = R(\lambda, A)v$,

$$\lambda|u| \leqslant |(\lambda I - A)u| \qquad \forall u \in D(A).$$

So, A is maximal dissipative. □

Proof of (b) ⇒ (a) We have that:

(i) $D(A)$ is dense by hypothesis,

(ii) A is closed by Proposition 11-(b),

(iii) $]0, \infty[\subset \rho(A)$ and $\|R(\lambda, A)\| \leqslant 1/\lambda$ for all $\lambda > 0$ by Proposition 11-(c).

The conclusion follows by the Hille-Yosida theorem. □

Example 15 (Wave equation in $L^2(\Omega)$) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$. For any given $f \in H^2(\Omega) \cap H_0^1(\Omega)$ and $g \in H_0^1(\Omega)$, consider the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t, x) = \Delta u & \text{in }]0, \infty[\times\Omega \\ u = 0 & \text{on }]0, \infty[\times\partial\Omega \\ u(0, x) = f(x)\,,\ \frac{\partial u}{\partial t}(0, x) = g(x) & x \in \Omega \end{cases} \tag{2.2.5}$$

Let H be the Hilbert space $H_0^1(\Omega) \times L^2(\Omega)$ with the scalar product

$$\Big\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \Big\rangle = \int_\Omega \big(Du(x) \cdot D\bar{u}(x) + v(x)\bar{v}(x)\big)dx.$$

Define $A : D(A) \subset H \to H$ by

$$\begin{cases} D(A) = \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega) \\ A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \end{pmatrix} \end{cases} \tag{2.2.6}$$

We will show that A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions on H by checking that A is maximal dissipative.

Let $\begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A})$. Then, integrating by parts we obtain

$$\Big\langle A\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \Big\rangle = \int_\Omega \big(Du(x) \cdot Dv(x) + v(x)\Delta u(x)\big)dx = 0\,. \tag{2.2.7}$$

So, A is dissipative.

Now, consider the resolvent equation

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A}) \\ \\ (I - A)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \in H \end{cases} \tag{2.2.8}$$

which is equivalent to the system

$$\begin{cases} u \in H^2(\Omega) \cap H_0^1(\Omega), \quad v \in H_0^1(\Omega) \\ u - v = f \in H_0^1(\Omega) \\ v - \Delta u = g \in L^2(\Omega). \end{cases} \tag{2.2.9}$$

Using elliptic theory (see, for instance, [3]) one can show that the boundary value problem

$$\begin{cases} u \in H^2(\Omega) \cap H_0^1(\Omega), \\ u - \Delta u = f + g \in L^2(\Omega) \end{cases}$$

has a unique solution. Then, taking $v = u - f \in H_0^1(\Omega)$ we obtain the unique solution of problem (2.2.9). So, A is maximal dissipative and therefore A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions, $S(t)$.

For any $f \in H^2(\Omega) \cap H_0^1(\Omega)$, $g \in H_0^1(\Omega)$, let $u(t)\,(t \in \mathbb{R}_+)$ be the first component of

$$S(t)\begin{pmatrix} f \\ g \end{pmatrix}$$

Then u is the unique solution of problem (3.2.4) in the space

$$\mathcal{C}^2\big(\mathbb{R}_+; L^2(\Omega)\big) \cap \mathcal{C}^1\big(\mathbb{R}_+; H_0^1(\Omega)\big) \cap \mathcal{C}\big(\mathbb{R}_+; H^2(\Omega) \cap H_0^1(\Omega)\big).$$

Example 16 Consider the age-structured population model

$$\begin{cases} \frac{\partial u}{\partial t}(t,a) + \frac{\partial u}{\partial a}(t,a) + \mu(a)u(t,a) = 0, & a \in [0,a_1], t \geqslant 0 \\ u(t,0) = \int_0^{a_1} \beta(a)u(t,a)\,da, & t \geqslant 0 \\ u(0,a) = u_0(a). & a \in [0,a_1]\,. \end{cases} \tag{2.2.10}$$

which was proposed in [5]. Here, $u(t,a)$ is the population density of age a at time t, μ is the mortality rate, β the birth rate, and $a_1 > 0$ is the maximal age. We assume that $\mu, \beta \in C\big([0,a_1]\big)$, $\mu, \beta \geqslant 0$, and

$$\int_0^{a_1} \beta(a) e^{-\int_0^a \mu(\rho)\,d\rho}\,da < 1\,. \tag{2.2.11}$$

In order to recast problem (2.2.10) as an evolution equation in $H = L^2(0, a_1)$, we define the linear operator

$$\begin{cases} D(A) = \Big\{u \in H^1(0, a_1) : \ u(0) = \int_0^{a_1} \beta(a)u(a)\, da\Big\} \\ Au(a) = -u'(a) - \mu(a)u(a) \qquad \big(a \in [0, a_1] \text{ a.e.}\big)\,. \end{cases} \tag{2.2.12}$$

We now proceed to show the following:

1. A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup on H.
2. $\rho(A) \supset [0, +\infty)$ and, for any $\lambda > 0$,

$$R(\lambda, A)u(a) = \frac{U(a,0)}{1 - \int_0^{a_1}\beta(a)U(a,0)da} \int_0^{a_1} \beta(a)\, da \int_0^a U(a,s)u(s)\, ds + \int_0^a U(a,s)u(s)\, ds, \qquad a \in [0, a_1],\ u \in H, \tag{2.2.13}$$

where

$$U(a,s) = e^{-\lambda(a-s) - \int_s^a \mu(\rho)\, d\rho}, \quad a,\ s \in [0, a_1]\,. \tag{2.2.14}$$

Proof. Given $\lambda \geqslant 0$ and $v \in H$ we consider the equation

$$\lambda u - Au = v, \tag{2.2.15}$$

which is equivalent to

$$\begin{cases} (\lambda + \mu)u + u' = v, \\ u(0) = \int_0^{a_1}\beta(a)u(a)\, da\,. \end{cases} \tag{2.2.16}$$

If u is a solution of Eq. (2.2.15), then

$$u(a) = U(a,0)u(0) + \int_0^a U(a,s)v(s)\, ds, \tag{2.2.17}$$

where U is given by Eq. (2.2.14). Multiplying Eq. (2.2.17) by β and integrating with respect to a over $[0, a_1]$ yields

$$\begin{aligned} u(0) &= \int_0^{a_1} \beta(a)u(a)\, da \\ &= \Big(\int_0^{a_1} \beta(a)U(a,0)\, da\Big)u(0) + \int_0^{a_1}\beta(a)\, da \int_0^a U(a,s)v(s)\, ds\,. \end{aligned} \tag{2.2.18}$$

From Eq. (2.2.11), we have

$$\int_0^{a_1} \beta(a)U(a,0)\, da < 1, \quad \forall\, a \in [0, a_1], \tag{2.2.19}$$

then, also from Eq. (2.2.17),

$$u(0) = \frac{1}{1 - \int_0^{a_1} \beta(a) U(a,0)\, da} \int_0^{a_1} \beta(a)\, da \int_0^a U(a,s) v(s)\, ds. \quad (2.2.20)$$

Consequently, $u(a) = R(\lambda, A) v(a)$ is given by Eq. (2.2.13).

Conversely, given $v \in H$, the function

$$u(a) = \frac{U(a,0)}{1 - \int_0^{a_1} \beta(a) U(a,0)\, da} \int_0^{a_1} \beta(a)\, da \int_0^a U(a,s) v(s)\, ds \\ + \int_0^a U(a,s) v(s)\, ds, \quad a \in [0, a_1],$$

fulfills Eq. (2.2.15). □

3. For all $u \in D(A)$

$$\langle Au, u \rangle \leqslant -\frac{1}{2} \int_0^{a_1} u^2(a) \left(2\mu(a) - \int_0^{a_1} \beta^2(s) ds \right) da - \frac{1}{2}\, u(a_1)^2. \quad (2.2.21)$$

Consequently, if

$$2\mu(a) \geqslant \int_0^{a_1} \beta^2(s) ds, \quad \forall\, a \in [0, a_1], \quad (2.2.22)$$

then A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions on H.

Proof. To show Eq. (2.2.21), we observe that, for all $u \in D(A)$,

$$\begin{aligned} \langle Au, u \rangle &= -\int_0^{a_1} u'(a)\, u(a)\, da - \int_0^{a_1} \mu(a)\, u^2(a)\, da \\ &= \tfrac{1}{2}\, u(0)^2 - \tfrac{1}{2}\, u(a_1)^2 - \int_0^{a_1} \mu(a)\, u^2(a)\, da \\ &= \tfrac{1}{2} \left(\int_0^{a_1} \beta(a)\, u(a)\, da \right)^2 - \tfrac{1}{2}\, u(a_1)^2 - \int_0^{a_1} \mu(a)\, u^2(a)\, da. \end{aligned}$$

So, by Hölder's inequality,

$$\begin{aligned} \langle Au, u \rangle &\leqslant \frac{1}{2} \int_0^{a_1} \beta^2(a)\, da \int_0^{a_1} u^2(s)\, ds - \frac{1}{2}\, u(a_1)^2 - \int_0^{a_1} \mu(a)\, u^2(a)\, da \\ &= -\frac{1}{2} \int_0^{a_1} u^2(a) \left(2\mu(a) - \int_0^{a_1} \beta^2(s) ds \right) da - \frac{1}{2}\, u(a_1)^2. \end{aligned}$$

This shows that A is maximal dissipative if (2.2.22) is satisfied. In this case, the Lumer-Phillips theorem insures that A is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions. □

When A and $-A$ are maximal dissipative a stronger conclusion holds true.

Corollary 4 *Let $A : D(A) \subset X \to X$ be a densely defined linear operator. If both A and $-A$ are maximal dissipative, then A is the infinitesimal generator of a $\mathcal{C}_0$-group, $G(t)$, which satisfies $\|G(t)\| = 1$ for all $t \in \mathbb{R}$.*

Proof. By the Lumer-Phillips theorem, A and $-A$ are infinitesimal generators of $\mathcal{C}_0$-semigroups of contractions, $S_+(t)$ and $S_-(t)$ respectively. Therefore, Theorem 7 ensures that A is the infinitesimal generator of a $\mathcal{C}_0$-group, $G(t)$. Moreover, $1 = \|G(t)G(-t)\| \leqslant \|S_+(t)\| \, \|S_-(t)\| \leqslant 1$. Hence, $\|G(t)\| = 1$. □

Example 17 (Wave equation continued) We return to the wave equation that was studied in Example 15. We proved that operator A, defined in (2.2.6), is maximal dissipative. We claim that $-A$ is maximal dissipative as well. Indeed, equation (2.2.7) implies that $-A$ is dissipative. Moreover, the resolvent equation for $-A$ takes the form

$$\begin{cases} u \in H^2(\Omega) \cap H_0^1(\Omega), \qquad v \in H_0^1(\Omega) \\ u + v = f \in H_0^1(\Omega) \\ v + \Delta u = g \in L^2(\Omega)\,, \end{cases}$$

which can be uniquely solved arguing exactly as we did for system (2.2.9).

Then, by Corollary 4, A is the infinitesimal generator of a $\mathcal{C}_0$-group, $G(t)$, which satisfies $\|G(t)\| = 1$ for all $t \in \mathbb{R}$. So, for any $f \in H^2(\Omega) \cap H_0^1(\Omega)$, $g \in H_0^1(\Omega)$, the first component $u(t)$ ($t \in \mathbb{R}_+$) of

$$G(t)\begin{pmatrix} f \\ g \end{pmatrix}$$

is the unique solution of problem (3.2.4) in the space

$$\mathcal{C}^2\big(\mathbb{R}; L^2(\Omega)\big) \cap \mathcal{C}^1\big(\mathbb{R}; H_0^1(\Omega)\big) \cap \mathcal{C}\big(\mathbb{R}; H^2(\Omega) \cap H_0^1(\Omega)\big).$$

2.3 The adjoint semigroup

In this section, we consider the special case when $\big(X, \langle \cdot, \cdot \rangle\big)$ is a Hilbert space. We denote by $j_X : X^* \to X$ the Riesz isomorphism, which associates with any $\phi \in X^*$ the unique element $j_X(\phi) \in X$ such that

$$\phi(u) = \langle u, j_X(\phi) \rangle \qquad \forall u \in X.$$

We refer the reader to [4] for the treatment of a general Banach space.

Adjoint of a linear operator

Let $A : D(A) \subset X \to X$ be a densely defined linear operator.

Exercise 28 Prove that the set

$$D(A^*) = \Big\{ v \in X \;\Big|\; \exists C \geqslant 0 \;:\; u \in D(A) \quad \Longrightarrow \quad |\langle Au, v\rangle| \leqslant C|u| \Big\} \tag{2.3.1}$$

is a subspace of X and, for any $v \in D(A^*)$, the linear map $u \mapsto \langle Au, v\rangle$ can be uniquely extended to a bounded linear functional $\phi_v \in X^*$.

Solution. The fact that $D(A^*)$ is a subspace of X is easy to show. Let $v \in D(A^*)$, fix any $u \in X$, and let $u_n \in D(A)$ be such that $u_n \overset{n\to\infty}{\longrightarrow} u$. Then $|\langle A(u_n - u_m), v\rangle| \leqslant C|u_n - u_m|$ which implies that $\{\langle Au_n, v\rangle\}$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges as $n \to \infty$. Moreover, if $u_n' \in D(A)$ is another sequence such that $u_n' \overset{n\to\infty}{\longrightarrow} u$, then $|\langle A(u_n - u_n'), v\rangle| \leqslant C|u_n - u_n'|$. Therefore, the map

$$\phi_v(u) = \lim_{n\to\infty} \langle Au_n, v\rangle \qquad (u \in X),$$

where $\{u_n\}$ is any sequence in $D(A)$ converging to u is well defined. Moreover, ϕ_v is linear and $|\phi_v(u)| \leqslant C|u|$ for all $u \in X$. So, $\phi_v \in X^*$. □

Definition 16 *The adjoint of A is the map $A^* : D(A^*) \subset X \to X$ defined by*

$$A^* v = j_X(\phi_v) \qquad \forall v \in D(A^*)$$

where $D(A^*)$ is given by (2.3.1) and $\phi_v \in X^*$ is the functional extending $u \mapsto \langle Au, v\rangle$ to X (see Exercise 28).

Exercise 29 Prove that, if $A \in \mathcal{L}(X)$, then $A^* \in \mathcal{L}(X)$ as well and

$$\|A\| = \|A^*\|. \tag{2.3.2}$$

Solution. Since $A \in \mathcal{L}(X)$ we have that $D(A^*) = X$ and

$$\langle Au, v\rangle = \langle u, A^* v\rangle \qquad \forall u, v \in X.$$

So, by the definition of A^* we have that $\|A^*\| \leqslant \|A\|$. Moreover, taking $v = Au$ in the above identity, we obtain $|Au|^2 \leqslant |u|\,\|A^*\|\,|Au|$. So, $\|A\| \leqslant \|A^*\|$. □

Proposition 12 (properties of A^*) *Let $A : D(A) \subset X \to X$ be a densely defined linear operator. Then the following properties hiold.*

(i) A satisfies the adjoint identity

$$\langle Au, v\rangle = \langle u, A^* v\rangle \qquad \forall u \in D(A), \forall v \in D(A^*). \tag{2.3.3}$$

(ii) $A^* : D(A^*) \subset X \to X$ is a closed linear operator.

(iii) If $\lambda \in \rho(A)$, then $\overline{\lambda} \in \rho(A^*)$ and $R(\overline{\lambda}, A^*) = R(\lambda, A)^*$.

(iv) If, in addition, A is closed then $D(A^*)$ is dense in X.

Proof of (i). Let $u \in D(A), v \in D(A^*)$ and let $\phi_v \in X^*$ be the functional extending $u \mapsto \langle Au, v\rangle$ to X. Then

$$\langle Au, v\rangle = \phi_v(u) = \langle u, j_X(\phi_v)\rangle = \langle u, A^*v\rangle \qquad \square$$

Proof of (ii). Now, to prove that A^* is closed, let $\{v_n\} \subset D(A^*)$ and $v, w \in X$ be such that

$$\begin{cases} v_n \to v \\ A^*v_n \to w \end{cases} \qquad (n \to \infty)$$

Then $\{A^*v_n\}$ is bounded, say $|A^*v_n| \leqslant C$. So, recalling (2.3.3), we have that

$$|\langle Au, v_n\rangle| = |\langle u, A^*v_n\rangle| \leqslant C|u| \qquad \forall u \in D(A)$$

This yields

$$|\langle Au, v\rangle| \leqslant C|u| \qquad \forall u \in D(A)$$

which in turn implies that $v \in D(A^*)$. Moreover

$$\langle Au, v\rangle = \lim_{n\to\infty} \langle Au, v_n\rangle = \langle u, w\rangle \qquad \forall u \in D(A).$$

Thus, $\langle u, A^*v - w\rangle = 0$ for all $u \in D(A)$. Since $D(A)$ is dense, $A^*v = w$. $\square$

Proof of (iii). Let $\lambda \in \rho(A)$. From the definition of the adjoint we have that

$$(\lambda I - A)^* = \overline{\lambda} I - A^*.$$

Aiming to prove that $\overline{\lambda} \in \rho(A^*)$, first we show that $\overline{\lambda} I - A^*$ is injective. If $(\overline{\lambda} I - A^*)v = 0$ for some $v \in D(A^*)$, then

$$0 = \langle u, (\overline{\lambda} I - A^*)v\rangle = \langle (\lambda I - A)u, v\rangle \qquad \forall u \in D(A).$$

Since $\lambda I - A$ is surjective, the above identity implies that $v = 0$. So, $\overline{\lambda} I - A^*$ is injective. Next, observe that, for all $v \in X$ and $u \in D(A)$,

$$\langle u, v\rangle = \langle R(\lambda, A)(\lambda I - A)u, v\rangle = \langle (\lambda I - A)u, R(\lambda, A)^*v\rangle,$$

yielding $R(\lambda, A)^*v \in D((\lambda I - A)^*) = D(\overline{\lambda} I - A^*) = D(A^*)$ and

$$(\overline{\lambda} I - A^*)R(\lambda, A)^*v = v \qquad \forall v \in X. \tag{2.3.4}$$

On the other hand, if $u \in X$ and $v \in D(A^*)$, then

$$\langle u, v\rangle = \langle (\lambda I - A)R(\lambda, A)u, v\rangle = \langle R(\lambda, A)u, (\overline{\lambda} I - A^*)v\rangle.$$

Therefore,

$$R(\lambda, A)^*(\overline{\lambda} I - A^*)v = v \qquad \forall v \in D(A^*). \tag{2.3.5}$$

(2.3.4) and (2.3.5) imply that $\overline{\lambda} \in \rho(A^*)$ and $R(\overline{\lambda}, A^*) = R(\lambda, A)^*$. $\square$

Proof of (iv). We argue by contradiction assuming the existence of $u_0 \neq 0$ such that $\langle u_0, v\rangle = 0$ for every $v \in D(A^*)$. Then $(0, u_0) \notin graph(A)$, which is a closed subspace of $X \times X$. Fron the Hahn-Banach theorem it follows that there exist $v_1, v_2 \in X$ such that the associated hyperplane in $X \times X$ separates $graph(A)$ and the point $(0, u_0)$, that is,

$$\langle u, v_1\rangle - \langle Au, v_2\rangle = 0 \qquad \forall u \in D(A) \qquad \text{and} \qquad \langle 0, v_1\rangle - \langle u_0, v_2\rangle \neq 0$$

But the first identity implies that $v_2 \in D(A^*)$, which in turn yields $\langle u_0, v_2\rangle = 0$, in contrast with the second equation above. So, $\overline{D(A^*)} = X$. $\square$

The Lumer-Phillips theorem

By introducing dissipativity of the adjoint of A we can replace maximality in the Lumer-Phillips theorem.

Theorem 10 (Lumer-Phillips 2) *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator. If A and A^* are dissipative, then A is the infinitesimal generator of a contraction semigroup on X.*

Proof. In view of Theorem 9, it suffices to show that $]0, \infty[\subset \rho(A)$. Since $\lambda I - A$ is one-to-one for any $\lambda > 0$, one just has to check that

$$(\lambda I - A)D(A) = X \qquad \forall \lambda > 0.$$

Step 1: *$(\lambda I - A)D(A)$ is dense in X for every $\lambda > 0$.*
Let $v \in X$ be such that

$$\langle \lambda u - Au, v\rangle = 0 \qquad \forall u \in D(A).$$

The identity $\langle Au, v\rangle = \lambda\langle u, v\rangle$ yields $v \in D(A^*)$ and the fact that

$$\langle u, \lambda v - A^* v\rangle = 0,$$

first for all $u \in D(A)$ and then, by density, for all $u \in X$. So, $\lambda v - A^* v = 0$. Since, being dissipative, $\lambda I - A^*$ is also one-to-one, we conclude that $v = 0$.

Step 2: *$\lambda I - A$ is surjective for every $\lambda > 0$.*
Fix any $v \in X$. By Step 1, there exists $\{u_n\} \subset D(A)$ such that

$$\lambda u_n - Au_n =: v_n \to v \quad \text{as} \quad n \to \infty.$$

By (2.1.2) we deduce that, for all $n, m \geqslant 1$,

$$|u_n - u_m| \leqslant \frac{1}{\lambda} |v_n - v_m|$$

which insures that $\{u_n\}$ is a Cauchy sequence in X. Therefore, there exists $u \in X$ such that

$$\begin{cases} u_n \to u \\ Au_n = \lambda u_n - v_n \to \lambda u - v \end{cases} \qquad (n \to \infty)$$

Since A is closed, $u \in D(A)$ and $\lambda u - Au = v$. □

The adjoint semigroup

In order to make further progress we have to better understand the relationship between the adjoint, $S(t)^*$, of a $\mathcal{C}_0$-semigroup of bounded linear operators on X and the adjoint, A^*, of its infinitesimal generator.

Theorem 11 *Let $S(t)$ be a $\mathcal{C}_0$-semigroup of bounded linear operators on X with infinitesimal generator $A : D(A) \subset X \to X$. Then $S(t)^*$ is a $\mathcal{C}_0$-semigroup of bounded linear operators on X, called the* adjoint semigroup*, whose infinitesimal generator is A^*, the adoint of A.*

Proof. We observe first that properties (a) and (b) of the definition of a semigroup are easy to check. Moreover, in view of the bound (1.2.2) and Exercise 29 we have that $S(t)^*$ satisfies the growth condition

$$\|S(t)^*\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0 \tag{2.3.6}$$

with the same constants M, ω as $S(t)$. Hereafter, we assume $\omega \geqslant 0$.

Aiming to prove that $S(t)^*$ is strongly continuous we observe that, for all $u \in X$ and $v \in D(A^*)$,

$$\begin{aligned} |\langle u, S(t)^*v - v\rangle| = |\langle S(t)u - u, v\rangle| &= \Big|\int_0^t \langle AS(s)u, v\rangle ds\Big| \\ &= \Big|\int_0^t \langle S(s)u, A^*v\rangle ds\Big| = \Big|\int_0^t \langle u, S(s)^*A^*v\rangle ds\Big|. \end{aligned} \tag{2.3.7}$$

Therefore, on account of (2.3.6),

$$|S(t)^*v - v| \leqslant Mte^{\omega t}|A^*v| \qquad \forall v \in D(A^*).$$

This implies that $\lim_{t\downarrow 0} S(t)^*v = v$ first for every $v \in D(A^*)$ and then for all $v \in X$ thanks to (2.3.6) since $D(A^*)$ is dense in X by Proposition12.

Finally, we show that A^* is the infinitesimal generator of the adjoint semigroup. Denote by $B : D(B) \subset X \to X$ the infinitesimal generator of $S(t)^*$. Owing to (2.3.7), for every $v \in D(A^*)$ we have that

$$\frac{S(t)^*v - v}{t} = \frac{1}{t}\int_0^t S(s)^*A^*v\, ds \overset{t\downarrow 0}{\longrightarrow} A^*v.$$

Therefore, $A^* \subset B$. Moreover, $\rho(A) \cap \rho(B) \neq \varnothing$ because $\Pi_\omega \subset \rho(A^*)$ by Theorem 12 and $\Pi_\omega \subset \rho(B)$ by (2.3.6) and Proposition 5. So, $A^* = B$. □

Self-adjoint operators and Stone's theorem

Definition 17 *A densely defined linear operator $A : D(A) \subset X \to X$ is called:*

(*a*) symmetric *if $A \subset A^*$, that is,*

$$D(A) \subset D(A^*) \quad \text{and} \quad Au = A^*u \qquad \forall u \in D(A).$$

(*b*) self-adjoint *if $A = A^*$.*

Remark 7
1. Observe that a symmetric operator A is self-adjoint if and only if $D(A) \subseteq D(A^*)$.
2. In view of Proposition 12, any self-adjoint operator is closed.
3. If $A \in \mathcal{L}(X)$, then A is self-adjoint if and only if A is symmetric.

Example 18 In $X = L^2(0,1;\mathbb{C})$, consider the linear operator

$$\begin{cases} D(A) = H_0^1(0,1;\mathbb{C}) \\ Au(x) = i\,u'(x) \qquad x \in [0,1] \ a.e. \end{cases}$$

Then, A is densely defined and symmetric. Indeed, for all $u, v \in D(A)$,

$$\begin{aligned} \langle Au, v\rangle &= i\int_0^1 u'(x)\overline{v(x)}\,dx \qquad (2.3.8) \\ &= \big[iu(x)\overline{v(x)}\big]_{x=0}^{x=1} - i\int_0^1 u(x)\overline{v'(x)}\,dx = \langle u, Av\rangle. \end{aligned}$$

On the other hand, A fails to be self-adjoint because, as we show next,

$$D(A^*) \supseteq H^1(0,1;\mathbb{C}),$$

so that $D(A) \subsetneq D(A^*)$. Indeed, integrating by parts as in (2.3.8), for all $v \in H^1(0,1;\mathbb{C})$ and $u \in H_0^1(0,1;\mathbb{C})$ we have that

$$\big|\langle Au, v\rangle\big| = \Big| -i\int_0^1 u(x)\overline{v'(x)}\,dx\Big| \leqslant |u|_2|v'|_2. \qquad \square$$

Proposition 13 *Let $A : D(A) \subset X \to X$ be a densely defined closed linear operator such that $\rho(A) \cap \mathbb{R} \neq \varnothing$. If A is symmetric, then A is self-adjoint.*

Proof. We prove that $D(A^*) \subset D(A)$ in two steps. Fix any $\lambda \in \rho(A) \cap \mathbb{R}$.

Step 1: $\boxed{R(\lambda, A) = R(\lambda, A)^*}$

Since $R(\lambda, A) \in \mathcal{L}(X)$, in view of Exercise 29 it suffices to show that

$$\langle R(\lambda, A)u, v\rangle = \langle u, R(\lambda, A)v\rangle \qquad \forall u, v \in X.$$

Fix any $u, v \in X$ and set

$$x = R(\lambda, A)u \quad \text{and} \quad y = R(\lambda, A)v$$

so that $x, y \in D(A)$ and

$$\lambda x - Ax = u \quad \text{and} \quad \lambda y - Ay = v.$$

Since A is symmetric, we have that

$$\langle R(\lambda, A)u, v\rangle = \langle x, v\rangle = \langle x, \lambda y - Ay\rangle = \langle \lambda x - Ax, y\rangle = \langle u, R(\lambda, A)v\rangle.$$

Step 2: $\boxed{D(A^*) \subset D(A)}$
Let $u \in D(A^*)$ and set $x = \lambda u - A^*u$. Observe that, for all $v \in D(A)$,

$$\langle x, v\rangle = \langle \lambda u - A^*u, v\rangle = \langle u, \lambda v - Av\rangle.$$

Now, take any $w \in X$ and let $v = R(\lambda, A)w$. Then the above identity yields

$$\langle x, R(\lambda, A)w\rangle = \langle u, w\rangle \qquad \forall w \in X.$$

So, by Step 1 we conclude that $u = R(\lambda, A)^*x = R(\lambda, A)x \in D(A)$. □The following is another interesting spectral property of self-adjoint operators.

Proposition 14 *If $A : D(A) \subset X \to X$ is self-adjoint then*

$$\rho(A) \supset \{\lambda \in \mathbb{C} \ : \ \Im\lambda \neq 0\}$$

Consequently, $\sigma(A)$ is real.

Proof. For every $u \in D(A)$ we have that

$$\langle Au, u\rangle = \overline{\langle Au, u\rangle} \in \mathbb{R}. \tag{2.3.9}$$

Therefore, $|\langle \lambda u - Au, u\rangle| \geqslant |\Im\lambda|\, |u|^2$ which in turn yields

$$|\lambda u - Au| \geqslant |\Im\lambda|\, |u| \qquad \forall x \in D(A). \tag{2.3.10}$$

The last inequality ensures that $\lambda I - A$ is an injective operator with closed range for all $\lambda \in \mathbb{C}$ with $\Im\lambda \neq 0$. Let us show that $(\lambda I - A)D(A)$ is dense in X for any such λ. Suppose there exists $v \neq 0$ such that

$$\langle \lambda u - Au, v\rangle = 0 \qquad \forall u \in D(A).$$

Then $v \in D(A^*) = D(A)$ and we have that

$$\langle u, \overline{\lambda} v - Av \rangle = 0 \qquad \forall u \in D(A).$$

Since $D(A)$ is dense in X, this implies that $\overline{\lambda} v - Av = 0$. Then, by (2.3.9), $\overline{\lambda} = \lambda \in \mathbb{R}$ contradicting $\Im\lambda \neq 0$. □

The following is an immediate consequence of Theorem 10.

Corollary 5 (Lumer-Phillips 3) *Let $A : D(A) \subset X \to X$ be a densely defined linear operator. If A is self-adjoint and dissipative, then A is the infinitesimal generator of a contraction semigroup on X.*

Example 19 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$. Define

$$\begin{cases} D(A) = H^2 \cap H_0^1(\Omega; \mathbb{C}) \\ Au(x) = \Delta u(x) - V(x)u(x) \quad x \in \Omega \text{ a.e.} \end{cases} \tag{2.3.11}$$

where we assume $V \in L^\infty(\Omega, \mathbb{R})$. Let us check that A is self-adjoint in $L^2(\Omega; \mathbb{C})$. Indeed, integration by parts insures that A is symmetric. So, by Proposition 13, it suffices to check that $\rho(A) \cap \mathbb{R} \neq \varnothing$. We claim that, for $\lambda \in \mathbb{R}$ large enough, for any $h \in L^2(\Omega; \mathbb{C})$ the problem

$$\begin{cases} w \in H^2 \cap H_0^1(\Omega; \mathbb{C}) \\ (\lambda + V(x))w(x) - \Delta w(x) = h(x) \quad (x \in \Omega \text{ a.e.}) \end{cases} \tag{2.3.12}$$

has a unique solution. Equivalently, by setting $f = \Re h$, $g = \Im h \in L^2(\Omega)$ and $u = \Re w$, $v = \Im w$, we have to prove solvability for the boundary value problems

$$\begin{cases} u \in H^2 \cap H_0^1(\Omega) \\ (\lambda + V)u - \Delta u = f \end{cases} \qquad \text{and} \qquad \begin{cases} v \in H^2 \cap H_0^1(\Omega) \\ (\lambda + V)v - \Delta v = g \end{cases}$$

The latter is a well-established fact in elliptic theory (see, e.g. [3]). On the other hand, operator A fails to be dissipative, in general.

Exercise 30 Prove that operator A in Example 19 is dissipative if

$$\|V\|_\infty \leqslant 1/C_\Omega,$$

where C_Ω is the Poincaré constant of Ω.

The following property of self-adjoint operators is very useful. We recall that an operator $U \in \mathcal{L}(X)$ is *unitary* if $UU^* = U^*U = I$.

Theorem 12 (Stone) *Let X be a complex Hilbert space. For any densely defined linear operator $A : D(A) \subset X \to X$ the following properties are equivalent:*

(a) A is self-adjoint,

(b) iA is the infinitesimal generator of a $\mathcal{C}_0$-group of unitary operators.

<u>*Proof of* (a) ⇒ (b)</u> Since A is self-adjoint, A is closed and we have that

$$\langle Au, u\rangle = \langle u, A^*u\rangle = \langle u, Au\rangle = \overline{\langle Au, u\rangle} \qquad \forall u \in D(A).$$

Thus, $\langle Au, u\rangle$ is real, so that

$$\Re\langle iAu, u\rangle = 0 \qquad \forall u \in D(A).$$

The above identity implies that both iA and $-iA$ are dissipative operators. Since

$$\langle iAu, v\rangle = i\langle u, Av\rangle = \langle u, -iAv\rangle \qquad \forall u, v \in D(A),$$

we have that $(iA)^* = -iA$. So, by Theorem 10 we deduce that $\pm iA$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup of contractions that we denote by $e^{\pm iAt}$. Then, by Theorem 7, iA generates a $\mathcal{C}_0$ group, $G(t)$. Such a group is unitary because for any $t \geqslant 0$ we have that

$$G(t)^{-1} = G(-t) = e^{-iAt} = e^{(iA)^*t} = (e^{iAt})^* = G(t)^*,$$

while, for any $t < 0$,

$$G(t)^{-1} = e^{iA|t|} = e^{iA^*|t|} = e^{(-iA)^*|t|} = (e^{-iA|t|})^* = G(-|t|)^* = G(t)^*. \qquad \square$$

<u>*Proof of* (b) ⇒ (a)</u> Let iA be the infinitesimal generator of a $\mathcal{C}_0$-group of unitary operators on X, say $G(t)$. Then, for all $u \in D(A)$, we have that

$$\begin{aligned} iAu &= \lim_{t\to 0} \frac{G(t)u - u}{t} = -\lim_{t\to 0} \frac{G(-t)u - u}{t} = -\lim_{t\to 0} \frac{G(t)^*u - u}{t} = \\ &= -(iA)^*u = iA^*u. \end{aligned}$$

Thus, $u \in D(A^*)$ and $Au = A^*u$. By running the above computation backwards, we conclude that $D(A^*) \subseteq D(A)$. Therefore, A is self-adjoint. $\square$

Example 20 (Schrödinger equation in a bounded domain) Let us consider the initial-boundary value problem

$$\begin{cases} \frac{1}{i}\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - V(x)u(t,x) & (t,x) \in \mathbb{R}\times\Omega \\ u(t,x) = 0 & t \in \mathbb{R}\,,\ x \in \partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases} \tag{2.3.13}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class $\mathcal{C}^2$ and $V \in L^\infty(\Omega)$. In Example 19, we have already checked that the operator A, defined in (2.3.11), is self-adjoint on $L^2(\Omega;\mathbb{C})$. Therefore, by Theorem 12 we conclude that, for any $u_0 \in H^2 \cap H^1_0(\Omega;\mathbb{C})$, problem (2.3.13) has a unique solution

$$u \in \mathcal{C}^1\big(\mathbb{R}; L^2(\Omega;\mathbb{C})\big) \cap \mathcal{C}\big(\mathbb{R}; H^2 \cap H^1_0(\Omega;\mathbb{C})\big). \qquad \square$$

The Cauchy problem with a self-adjoint operator

In this section, we will see that the homogeneous Cauchy problem with initial datum $u_0 \in X$

$$\begin{cases} u'(t) = Au(t) & t > 0 \\ u(0) = u_0. \end{cases} \tag{2.3.14}$$

can be solved in a strict sense without requiring u_0 to be in $D(A)$ if A is a self-adjoint and dissipative.

We begin with an interpolation result of interest in its own right.

Lemma 2 *Let $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator and let $u \in H^1(0,T;X) \cap L^2(0,T;D(A))$ be such that $u(0) = 0$. Then the function*

$$t \mapsto \langle Au(t), u(t)\rangle$$

is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\langle Au(t), u(t)\rangle = 2\Re\langle u'(t), Au(t)\rangle \qquad (a.e.\ t \in [0,T]). \tag{2.3.15}$$

Proof. Define $U_n(t) = \langle A_n u(t), u(t)\rangle$ $(t \in [0,T])$, where $A_n = nAR(n,A)$ is the Yosida approximation of A. Then U_n is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\langle A_n u(t), u(t)\rangle = 2\Re\langle u'(t), A_n u(t)\rangle \qquad (\text{a.e. } t \in [0,T])$$

or

$$\langle A_n u(t), u(t)\rangle = 2\Re\int_0^t \langle u'(s), A_n u(s)\rangle ds \qquad \forall t \in [0,T]. \tag{2.3.16}$$

Now, since for a.e. $t \in [0,T]$

$$\begin{array}{rcl} A_n u(t) & = & nR(n,A)Au(t) \overset{(n\to\infty)}{\longrightarrow} Au(t) \\ |A_n u(t)| & \leqslant & |Au(t)|, \end{array}$$

we can pass to the limit as $n \to \infty$ in (2.3.16) by Lebesgue's theorem to obtain

$$\langle Au(t), u(t)\rangle = 2\Re\int_0^t \langle u'(s), Au(s)\rangle ds \qquad \forall t \in [0,T].$$

This shows that $t \mapsto \langle Au(t), u(t)\rangle$ is absolutely continuous on $[0,T]$ and yields (2.3.15). □

Theorem 13 *Let $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator. Then $S(t)u \in D(A)$ for all $u \in X$ and $t > 0$. Moreover, for every $T > 0$ the following inequality holds*

$$4\int_0^T t|AS(t)u|^2 dt - 2T\langle AS(T)u, S(T)u\rangle + |S(T)u|^2 \leqslant |u|^2 \quad \forall u \in X. \tag{2.3.17}$$

Observe that all the terms on the left side of (2.3.17) are nonnegative, so that each of them is bounded by $|u|^2$.

Proof. For any $n \geqslant 1$ define

$$u_n = nR(n,A)u\,, \quad v_n(t) = S(t)u_n\,, \quad v(t) = S(t)u.$$

Since $u_n \in D(A)$, we have that $v_n'(t) = Av_n(t)$ for all $t \geqslant 0$. So,

$$\frac{1}{2}\frac{d}{dt}|v_n(t)|^2 = \langle v_n'(t), v_n(t)\rangle = \langle Av_n(t), v_n(t)\rangle$$

Therefore,

$$|v_n(t)|^2 - 2\int_0^t \langle Av_n(s), v_n(s)\rangle\, ds = |u_n|^2 \qquad \forall t \geqslant 0. \tag{2.3.18}$$

Similarly, for all $t \geqslant 0$ we have that

$$t|v_n'(t)|^2 = t\langle Av_n(t), v_n'(t)\rangle = \frac{1}{2}\frac{d}{dt}\Big(t\langle Av_n(t), v_n(t)\rangle\Big) - \frac{1}{2}\langle Av_n(t), v_n(t)\rangle.$$

Integrating the above identity over $[0,T]$ yields, by (2.3.18),

$$\begin{aligned} 2\int_0^T t|v_n'(t)|^2\, dt - T\langle Av_n(T), v_n(T)\rangle &= -\int_0^T \langle Av_n(t), v_n(t)\rangle\, dt \\ &= \frac{1}{2}\big(|u_n|^2 - |v_n(T)|^2\big) \end{aligned}$$

or

$$4\int_0^T t|Av_n(t)|^2\, dt - 2T\langle Av_n(T), v_n(T)\rangle + |v_n(T)|^2 \leqslant |u|^2 \tag{2.3.19}$$

since $\|nR(n,A)\| \leqslant 1$. The last inequality implies that, for any $\varepsilon \in]0,T[$, $\{v_n\}$ is bounded in $L^2(\varepsilon,T;D(A))$. Therefore, there exists a weakly convergent subsequence $\{v_{n_k}\}$ in $L^2(\varepsilon,T;D(A))$. On the other hand, $v_{n_k} \to v$ uniformly on $[0,T]$. So, $v \in L^2(\varepsilon,T;D(A))$ for any $\varepsilon \in]0,T[$, which in turn yields $S(t)u \in D(A)$ for a.e. $t > 0$—hence for all $t > 0$! Moreover,

$$Av_n(t) = nR(n,A)AS(t)u \overset{n\to\infty}{\longrightarrow} AS(t)u \qquad \forall t > 0.$$

Taking the limit as $n \to \infty$ in (2.3.19), by Fatou's lemma we get (2.3.17). □

The above result can be used to introduce an intermediate space between X and $D(A)$, namely the interpolation space $[X, D(A)]_{1/2}$, such that $t \mapsto S(t)u_0$ belongs to $H^1(0,T;X) \cap L^2(0,T;D(A))$ whenever $u_0 \in [D(A), X]_{1/2}$. We give a brief account of such a construction, referring the reader to [1] for more.

Proposition 15 *Let $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator. Then, for any $u \in X$, the functions*

$$t \mapsto -\langle AS(t)u, S(t)u\rangle$$

and

$$t \mapsto -\frac{1}{t}\int_0^t \langle AS(s)u, S(s)u\rangle\, ds$$

are both nonincreasing on $]0, \infty[$.

Proof. Since $S(t)u \in D(A)$ for every $t > 0$ by Theorem 13, we have that

$$0 \leqslant 2|AS(t)u|^2 = \frac{d}{dt}\langle AS(t)u, S(t)u\rangle \qquad \forall t > 0.$$

This shows that $t \mapsto -\langle AS(t)u, S(t)u\rangle$ is nondecreasing on $]0, \infty[$. The other conclusion is a consequence of the general fact which is proven below. □

Lemma 3 *Let f be a nonnegative nonincreasing function on $]0, \infty[$. Then $t \mapsto \frac{1}{t}\int_0^t f(s)\, ds$ is nonincreasing on $]0, \infty[$.*

Proof. Observe that for any $0 < t < t'$ we have that

$$f(t') \leqslant \frac{1}{(t'-t)}\int_t^{t'} f(s)\, ds \leqslant f(t) \leqslant \frac{1}{t}\int_0^t f(s)\, ds. \tag{2.3.20}$$

This yields

$$\begin{aligned}
\frac{1}{t'}\int_0^{t'} f(s)\, ds &= \frac{1}{t}\int_0^t f(s)\, ds + \Big(\frac{1}{t'} - \frac{1}{t}\Big)\int_0^t f(s) + \frac{1}{t'}\int_t^{t'} f(s)\, ds \\
&= \frac{1}{t}\int_0^t f(s)\, ds + \frac{t'-t}{t'}\Big\{\frac{1}{(t'-t)}\int_t^{t'} f(s)\, ds - \frac{1}{t}\int_0^t f(s)\, ds\Big\} \\
&\leqslant \frac{1}{t}\int_0^t f(s)\, ds + \frac{t'-t}{t'}\big\{f(t) - f(t)\big\} \leqslant \frac{1}{t}\int_0^t f(s)\, ds,
\end{aligned}$$

where we have made repeated use of (2.3.20). The conclusion follows. □

In view of the above proposition, we have that, for any $u \in X$,

$$\lim_{T \downarrow 0} -\frac{1}{T}\int_0^T \langle AS(t)u, S(t)u\rangle dt = \sup_{T>0} -\frac{1}{T}\int_0^T \langle AS(t)u, S(t)u\rangle dt$$

Definition 18 (interpolation space $[D(A), X]_{1/2}$) *For any $u \in u$ we set*

$$|u|^2_{1/2} = \lim_{T \downarrow 0} -\frac{1}{T}\int_0^T \langle AS(t)u, S(t)u\rangle dt$$

and we define

$$[D(A), X]_{1/2} = \Big\{u \in u \ : \ |u|_{1/2} < \infty\Big\}. \tag{2.3.21}$$

It is easy to see that $[D(A), X]_{1/2}$ is a subspace of X containing $D(A)$ and

$$\|x\|_{1/2} = |x| + |x|_{1/2}$$

is a norm on $[D(A), X]_{1/2}$.

Theorem 14 *Let $A : D(A) \subset X \to X$ be a self-adjoint dissipative operator. Then*

$$\int_0^\infty |AS(t)u|^2 dt \leqslant \frac{1}{2}\,|u|_{1/2}^2 \qquad \forall u \in [D(A), X]_{1/2}.$$

Proof. Fix any $\varepsilon > 0$ and let $T_\varepsilon \in]0, \varepsilon[$ be such that

$$-\langle AS(T_\varepsilon)u, S(T_\varepsilon)u\rangle < |u|_{1/2}^2 + \varepsilon.$$

Set $v(t) = S(t)u$ and integrate the identity $|Av(y)|^2 = \langle Av(t), v'(t)\rangle$ over $[T_\varepsilon, T]$ for any fixed $T > \varepsilon$ to obtain

$$\int_{T_\varepsilon}^T |Av(t)|^2 dt = \frac{1}{2}\,\langle Av(T), v(T)\rangle - \frac{1}{2}\,\langle Av(T_\varepsilon), v(T_\varepsilon)\rangle < |u|_{1/2}^2 + \varepsilon.$$

This implies the conclusion as $\varepsilon \downarrow 0$ and $T \uparrow \infty$. □

Example 21 On $X = L^2(0, \pi)$ let $A : D(A) \subset X \to X$ be the operator

$$\begin{cases} D(A) = H^2(0, \pi) \cap H_0^1(0, \pi) \\ Af(x) = f''(x) & x \in (0, \pi) \text{ a.e.} \end{cases}$$

We know that A is self-adjoint and dissipative. We now show that

$$[D(A), X]_{1/2} = H_0^1(0, \pi). \tag{2.3.22}$$

Let us fix $f \in H_0^1(0, \pi)$ and consider its Fourier series

$$f(x) = \sum_{n=1}^\infty f_n \sin(nx) \qquad (x \in [0, \pi]).$$

By Parseval's identity we have that

$$\sum_{n=1}^\infty n^2 |f_n|^2 = \frac{2}{\pi} \int_0^\pi |f'(x)|^2 dx.$$

Moreover,

$$S(t)f(x) = \sum_{n=1}^\infty e^{-n^2 t} f_n \sin(nx) \qquad (x \in [0, \pi]).$$

and

$$AS(t)f(x) = -\sum_{n=1}^{\infty} n^2 e^{-n^2 t} f_n \sin(nx) \qquad (x \in [0, \pi]).$$

Therefore,

$$\begin{aligned} \langle AS(t)f, S(t)f \rangle &= -\sum_{n=1}^{\infty} n^2 e^{-2n^2 t} |f_n|^2 \int_0^{\pi} \sin^2(nx)\, dx \\ &= -\frac{\pi}{2} \sum_{n=1}^{\infty} n^2 e^{-2n^2 t} |f_n|^2. \end{aligned}$$

Hence, recalling that $1 - e^{-x} \leqslant x$ for all $x \in \mathbb{R}$, we deduce that

$$\begin{aligned} -\frac{1}{T} \int_0^T \langle AS(t)f, S(t)f \rangle\, 2dt &= \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1 - e^{-2n^2 T}}{2T} |f_n|^2 \qquad (2.3.23) \\ &\leqslant \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 |f_n|^2 = \int_0^{\pi} |f'(x)|^2 dx. \end{aligned}$$

The last inequality implies that $H_0^1(0, \pi) \subset [D(A), X]_{1/2}$. The proof of the converse inclusion is left to the reader as an Exercise.

Hint. Use (2.3.23) to give a lower bound for

$$\lim_{T \downarrow 0} -\frac{1}{T} \int_0^T \langle AS(t)f, S(t)f \rangle\, 2dt. \qquad \square$$

Exercise 31 Use Theorem 13 to show that, for any self-adjoint dissipative operator $A : D(A) \subset X \to X$, the following holds:

(a) $S(t)u \in D(A^n)$ for all $t > 0$, all $u \in X$, and all $n \in \mathbb{N}$;

(b) for all $u \in X$

$$|AS(t)u| \leqslant \frac{|u|}{t\sqrt{2}} \qquad \forall t > 0.$$

Solution. To prove (a) it suffices to observe that for all $t > 0$ and $u \in X$,

$$AS(t)u = S(t/2)AS(t/2)u \in D(A) \quad \Longrightarrow \quad S(t)u \in D(A^2).$$

The general case follows by induction.

Next, using the dissipativity of A we obtain

$$\frac{d}{dt}|AS(t)u|^2 = 2\langle A^2 S(t)u, AS(t)u \rangle \leqslant 0.$$

Thus, $t \mapsto |AS(t)u|^2$ is nonincreasing. So, (2.3.17) yields

$$2t^2 |AS(t)u|^2 = 4\int_0^t s|AS(t)u|^2 ds \leqslant 4\int_0^t s|AS(s)u|^2 ds \leqslant |u|^2. \qquad \square$$

3 *The inhomogeneous Cauchy problem*

In this chapter, we assume that $\big(X,\langle\cdot,\cdot\rangle\big)$ is a separable Hilbert space and denote by $\{e_j\}_{j\in\mathbb{N}}$ a complete orthonormal system in X.

We study the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t) \\ u(0) = u^0, \end{cases} \tag{3.0.1}$$

where $f \in L^2(0,T;X)$ and $A : D(A) \subset X \to X$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup on X, $S(t)$, which satisfies the growth condition (1.6.3). For the extension of this theory to a general Banach space, we refer the reader to the classic monograph by Pazy [4] or the more recent text [2].

3.1 Notions of solution

Let $u^0 \in X$ and $f \in L^2(0,T;X)$.

Definition 19 (Mild solutions) *The function $u \in \mathcal{C}\big([0,T];X\big)$ defined by*

$$u(t) = S(t)u^0 + \int_0^t S(t-s)f(s)\,ds \tag{3.1.1}$$

is called the mild solution *of* (3.0.1)*.*

Observe that the convolution term in formula (3.1.1) for the solution u is well-defined in view of Proposition 22 in Appendix B.

Theorem 15 (Approximation of mild solutions) *Let $u \in \mathcal{C}\big([0,T];X\big)$ be the mild solution of* (3.0.1) *and suppose $f \in \mathcal{C}\big([0,T];X\big)$. Then, the sequence $u_n := nR(n,A)u$, defined for all $n > \omega$, satisfies*

$$u_n \in H^1(0,T;X) \cap L^2(0,T;D(A)) \quad \textit{and} \quad u_n \overset{(n\to\infty)}{\longrightarrow} u \ \textit{ in } \ \mathcal{C}\big([0,T];X\big).$$

Proof. Let u be given by (3.1.1) and define

$$\begin{cases} u_n(t) = nR(n,A)u(t) \\ f_n(t) = nR(n,A)f(t) \\ u_n^0 = nR(n,A)u^0 \end{cases} \qquad \forall n \in \mathbb{N}\,,\ n > \omega$$

where $\omega \geqslant 0$ is such that (1.6.3) holds true. Then

$$u_n(t) = S(t)u_n^0 + \int_0^t S(t-s)f_n(s)\,ds \quad (t \in [0,T]).$$

Since $u_n^0 \in D(A)$ and $f_n \in \mathcal{C}([0,T];D(A))$, by Proposition 21 and 22 below we conclude that

$$u_n \in H^1(0,T;X) \cap L^2(0,T;D(A)) \quad \text{and} \quad \begin{cases} u_n' - Au_n = f_n \\ u_n(0) = u_n^0. \end{cases}$$

Moreover, invoking Lemma 1 we conclude that $u_n^0 \to u^0$ as $n \to \infty$ while

$$f_n(t) \overset{(n\to\infty)}{\longrightarrow} f(t) \quad \text{and} \quad |f_n(t)| \leqslant \frac{Mn}{n-\omega}\,|f(t)| \quad (\text{for all } t \in [0,T])$$

Therefore,

$$\sup_{t\in[0,T]} |u_n(t) - u(t)| \leqslant Me^{\omega T}\Big(|u_n^0 - u^0| + \int_0^T |f_n(s) - f(s)|\,ds\Big) \overset{(n\to\infty)}{\longrightarrow} 0.$$

The conclusion follows. □

Definition 20 (Strict solutions) *A function* $u \in H^1(0,T;X) \cap L^2(0,T;D(A))$ *is a* strict solution *of* (3.0.1) *if* $u(0) = u^0$ *and*

$$u'(t) = Au(t) + f(t) \qquad (t \in [0,T] \ \text{a.e.})$$

Observe that Theorem 15 guarantees that the mild solution of (3.0.1) is the uniform limit of the strict solutions of the approximate problems

$$\begin{cases} u_n' - Au_n = f_n \\ u_n(0) = u_n^0. \end{cases}$$

Let $u^0 \in X$ and $f \in \mathcal{C}\big([0,T];X\big)$.

Definition 21 (Classical solutions) *A* classical solution *of* (3.0.1) *is a function* $u \in \mathcal{C}\big([0,T];X\big)$ *such that*

(*a*) $u \in \mathcal{C}^1\big(]0,T];X\big) \cap \mathcal{C}\big(]0,T];D(A)\big)$;

(*b*) $u(0) = u^0$;

(*c*) $u'(t) = Au(t) + f(t)$ *for all* $t \in]0,T]$.

We now show that any classical solution coincides with the mild solution.

Proposition 16 *Let* u *be a classical solution of* (3.0.1). *Then* u *equals the mild solution given by* (3.1.1).

Proof. Let u be a classical solution of (3.0.1). Then, for any fixed $t \in]0,T]$ we have that $s \mapsto S(t-s)u(s)$ is continuous on $[0,t]$, differentiable on $]0,t[$, and

$$\frac{d}{ds}\big(S(t-s)u(s)\big) = S(t-s)f(s) \qquad (s \in]0,t[).$$

By integrating over $[0,t]$ we deduce that u is given by (3.1.1). □

3.2 Regularity

Our first result guarantees that the mild solution of (3.0.1) is classical when f has better "space regularity".

Theorem 16 *Let $u^0 \in D(A)$ and let $f \in L^2(0,T;D(A)) \cap \mathcal{C}(]0,T];X)$. Then the mild solution u of problem* (3.0.1) *is classical. Moreover,*

$$u \in \mathcal{C}^1\big([0,T];X\big) \cap \mathcal{C}\big([0,T];D(A)\big). \tag{3.2.1}$$

We begin the proof by studying the case of $u^0 = 0$.

Lemma 4 *For any $f \in L^2(0,T;D(A)) \cap \mathcal{C}(]0,T];X)$ define*

$$F_A(t) = \int_0^t S(t-s)f(s)\,ds \quad (t \in [0,T]). \tag{3.2.2}$$

Then $F_A \in \mathcal{C}^1\big([0,T];X\big) \cap \mathcal{C}\big([0,T];D(A)\big)$ and

$$F_A'(t) = AF_A(t) + f(t) \qquad \forall t \in [0,T]. \tag{3.2.3}$$

Proof. Since $f \in L^2(0,T;D(A))$ we have that, for any $t \in [0,T]$,

$$A\int_0^t S(t-s)f(s)\,ds = \int_0^t S(t-s)Af(s)\,ds.$$

So, $F_A \in \mathcal{C}\big([0,T];D(A)\big)$ on account of Proposition 22.

Next, in order to prove that $F_A \in \mathcal{C}^1\big([0,T];X\big)$, fix $t \in [0,T[$ and let $0 < h < T-t$. Then

$$\begin{aligned}\frac{F_A(t+h)-F_A(t)}{h} &= \frac{1}{h}\Big\{\int_0^{t+h} S(t+h-s)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds\Big\}\\ &= \frac{S(h)-I}{h}F_A(t) + \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds.\end{aligned}$$

Now,

$$\lim_{h\downarrow 0}\frac{S(h)-I}{h}F_A(t) = AF_A(t)$$

because $F_A \in \mathcal{C}\big([0,T];D(A)\big)$. Also,

$$\lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds = f(t)$$

because $f \in \mathcal{C}(]0,T];X)$. Therefore, F_A is of class $\mathcal{C}^1\big([0,T];X\big)$ and satisfies (3.2.3). □

Proof of Theorem 16. Let u be the mild solution of problem (3.0.1). Then

$$u(t) = S(t)u^0 + F_A(t) \qquad \forall t \in [0,T],$$

where F_A is defined in (3.2.2). The conclusion follows from Theorem 3 and Lemma 4. □

We will now show a similar result if f has better "time regularity".

Theorem 17 *Let $u^0 \in D(A)$ and let $f \in H^1(0,T;X)$. Then the mild solution u of problem* (3.0.1) *is classical and satisfies* (3.2.1).

The proof is similar to the one above. One has just to replace Lemma 4 with the following one.

Lemma 5 *For any $f \in H^1(0,T;X)$ let F_A be defined as in* (3.2.2). *Then $F_A \in \mathcal{C}^1\big([0,T];X\big) \cap \mathcal{C}\big([0,T];D(A)\big)$ and*

$$F_A'(t) = AF_A(t) + f(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds \quad (t \in [0,T]).$$

Proof. Since F_A can be rewritten as

$$F_A(t) = \int_0^t S(s)f(t-s)ds \quad (t \in [0,T]),$$

by differentiating the integral we conclude that

$$\begin{aligned} F_A'(t) &= S(t)f(0) + \int_0^t S(s)f'(t-s))ds \\ &= S(t)f(0) + \int_0^t S(t-s)f'(s)ds \qquad \forall t \in [0,T]. \end{aligned}$$

Now, Proposition 22 implies that $F_A \in \mathcal{C}^1\big([0,T];X\big)$. Moreover, returning to definition (3.2.2), for all $t \in [0,T]$ we also have that

$$\begin{aligned} F_A'(t) &= \lim_{h\downarrow 0} \frac{1}{h}\Big\{ \int_0^{t+h} S(t+h-s)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds \Big\} \\ &= \lim_{h\downarrow 0} \Big\{ \frac{S(h)-I}{h} F_A(t) + \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds \Big\}. \end{aligned}$$

Since $H^1(0,T;X) \subset \mathcal{C}([0,T];X)$, we have that

$$\lim_{h\downarrow 0} \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds = f(t).$$

The above identity implies that $F_A(t) \in D(A)$ and

$$AF_A(t) = F_A'(t) - f(t) \qquad \forall t \in [0,T].$$

Consequently, $F_A \in \mathcal{C}\big([0,T];D(A)\big)$ and the proof is complete. □

Example 22 In general, the mild solution of (3.0.1) fails to be classical assuming just $f \in \mathcal{C}([0,T];X)$. Indeed, let $w \in X \setminus D(A)$ and take $f(t) = S(t)w$. Then the mild solution of (3.0.1) with $u^0 = 0$ is given by

$$u(t) = tS(t)w \qquad \forall t \geqslant 0$$

which fails to be differentiable for $t > 0$. □

Exercise 32 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^2$. Give conditions on $f \in L^2([0,T]\times\Omega)$, $u^0 : \Omega \to \mathbb{R}$, and $u^1 : \Omega \to \mathbb{R}$ which guarantee the existence and uniqueness of the classical solution to inhomogeneous wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u + f(t,x) & \text{in }]0,\infty[\times\Omega \\ u = 0 & \text{on }]0,\infty[\times\partial\Omega \\ u(0,x) = u^0(x)\,,\ \frac{\partial u}{\partial t}(0,x) = u^1(x) & x \in \Omega \end{cases} \tag{3.2.4}$$

Solution. Let A be defined as in Example 15. Then, applying Theorem 16 and Theorem 17 we conclude that the above problem has a unique classical solution if

(i) $(u^0, u^1) \in D(A)$, that is, $u^0 \in H^2(\Omega) \cap H^1_0(\Omega)$ and $u^1 \in H^1_0(\Omega)$;

(ii) f satisfies any of the following conditions

- (a) $f \in \mathcal{C}\big([0,T];L^2(\Omega)\big)$, $\frac{\partial f}{\partial x} \in L^2([0,T]\times\Omega)$, and $f(t,\cdot)_{|\partial\Omega} = 0$, or
- (a) $\frac{\partial f}{\partial t} \in L^2([0,T]\times\Omega)$. □

For special classes of generators, one can show that mild solutions are strict under rather weak conditions.

Theorem 18 *Let $A : D(A) \subset X \to X$ be a densely defined self-adjoint dissipative operator. Then, for any $u^0 \in [D(A), X]_{1/2}$ and $f \in \mathcal{C}([0,T];X)$, the mild solution u of problem* (3.0.1) *is strict.*

As above, we begin the proof by studying the case of $u^0 = 0$.

Lemma 6 *Let $A : D(A) \subset X \to X$ be a densely defined self-adjoint dissipative operator. For any $f \in \mathcal{C}(]0,T];X)$ let F_A be defined as in* (3.2.2)*. Then $F_A \in H^1(0,T;X) \cap L^2(0,T;D(A))$ and*

$$F_A'(t) = AF_A(t) + f(t) \qquad a.e.\ t \in [0,T]. \tag{3.2.5}$$

Moreover, $t \mapsto \langle AF_A(t), F_A(t)\rangle$ is absolutely continuous

$$\frac{d}{dt}\langle AF_A(t), F_A(t)\rangle = 2\Re\langle F_A'(t), AF_A(t)\rangle \qquad a.e.\ t \in [0,T], \tag{3.2.6}$$

and

$$\|AF_A\|_2 \leqslant \|f\|_2. \tag{3.2.7}$$

Proof. Define

$$f_n(t) = nR(n, A)f(t) \quad \text{and} \quad F_n(t) = nR(n, A)F_A(t) \qquad \forall t \in [0, T].$$

Then $f_n \in \mathcal{C}(]0, T]; D(A))$ for every n and

$$F_n(t) = \int_a^t S(t-s)f_n(s)\, ds \quad (t \in [0, T]).$$

Owing to Lemma 4, we have that $F_n \in \mathcal{C}^1([0, T]; X) \cap \mathcal{C}([0, T]; D(A))$ and

$$F_n'(t) = AF_n(t) + f_n(t) \qquad \forall t \in [0, T]. \tag{3.2.8}$$

Moreover,

$$2\int_0^t \Re \langle F_n'(s), AF_n(s)\rangle ds = \langle AF_n(t), F_n(t)\rangle \leqslant 0 \qquad \forall t \in [0, T]$$

because A is dissipative. Therefore, by multiplying each member of (3.2.8) by $2AF_n(t)$, taking real parts, and integrating over $[0, T]$ we obtain

$$\begin{aligned} 2\int_0^T |AF_n(t)|^2 dt &\leqslant -2\int_0^T \Re \langle f_n(t), AF_n(t)\rangle dt \\ &\leqslant \int_0^T \Big(|f_n(t)|^2 + |AF_n(t)|^2\Big) dt. \end{aligned}$$

Hence

$$\int_0^T |AF_n(t)|^2 dt \leqslant \int_0^T |f_n(t)|^2 dt \leqslant \int_0^T |f(t)|^2 dt.$$

Thus, $\{F_n\}_n$ is bounded in $H^1(0, T; X) \cap L^2(0, T; D(A))$. Therefore, there exists a subsequence $\{F_{n_k}\}_k$ and a function F_∞ such that

$$F_{n_k} \overset{(n\to\infty)}{\rightharpoonup} F_\infty \ \text{ in } \ H^1(0, T; X) \cap L^2(0, T; D(A)).$$

Recalling that $F_{n_k} \overset{(n\to\infty)}{\longrightarrow} F_A$ in $\mathcal{C}\big([0, T]; X\big)$ by Theorem 15, we conclude that $F_\infty = F_A \in H^1(0, T; X) \cap L^2(0, T; D(A))$.

Now, fix any $g \in L^2(0, T; X)$. Then, taking the product of each member of (3.2.8)—for $n = n_k$—with g we have that

$$\int_0^T \langle F_{n_k}'(t), g(t)\rangle\, dt = \int_0^T \langle AF_{n_k}(t) + f_{n_k}(t), g(t)\rangle\, dt.$$

So, in the limit as $n \to \infty$,

$$\int_0^T \langle F_A'(t) - AF_A(t) - f(t), g(t)\rangle\, dt = 0 \qquad \forall g \in L^2(0, T; X)$$

which in turn yields $F'_A(t) = AF_A(t) + f(t)$ for a.e. $t \in [0, T]$. □

Proof of Theorem 18. Let u be the mild solution of problem (3.0.1). Then

$$u(t) = u^0(t) + F_A(t) \qquad \forall t \in [0, T],$$

where

(*i*) $u^0(t) := S(t)u^0$ belongs to $H^1(0, T; X) \cap L^2(0, T; D(A))$ and satisfies $\frac{d}{dt}\, u^0(t) = Au^0(t)$ for every $t > 0$ thanks to Theorem 13 and Theorem 14;

(*ii*) F_A, defined in (3.2.2), belongs to $H^1(0, T; X) \cap L^2(0, T; D(A))$ and satisfies (3.2.5) owing to Lemma 6.

The conclusion by combining (*i*) and (*ii*). □

Example 23 We can use Therem 18 to study the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(t, x) & (t, x) \in (0, T) \times (0, \pi) \text{ a.e.} \\ u(t, 0) = 0 = u(t, \pi) & t \in (0, T) \\ u(0, x) = u^0(x) & x \in (0, \pi). \end{cases} \tag{3.2.9}$$

Recalling Example 21 , we conclude that for all

$$f \in \mathcal{C}\big([0, T]; L^2(0, \pi)\big) \quad \text{and} \quad u^0 \in H^1_0(0, \pi)$$

problem (3.2.9) has a unique solution u. In particular, such a solution satisfies:

$$\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2} \in L^2\big((0, T) \times (0, \pi)\big) \quad \text{and} \quad t \mapsto u(t, \cdot) \in H^1_0(0, \pi) \text{ is continuous.}$$

4 Appendix A: Cauchy integral on $\mathcal{C}([a,b];X)$

We recall the construction of the Riemann integral for a continuous function $f:[a,b]\to X$, where X is a Banach space and $-\infty<a<b<\infty$.

Let us consider the family of partitions of $[a,b]$

$$\Pi(a,b)=\Big\{\pi=\{t_i\}_{i=0}^n \ :\ n\geqslant 1,\ a=t_0<t_1<\cdots<t_n=b\Big\}$$

and define

$$diam(\pi)=\max_{1\leqslant i\leqslant n}(t_i-t_{i-1})\qquad (\pi\in\Pi(a,b)).$$

For any $\pi\in\Pi(a,b),\pi=\{t_i\}_{i=0}^n$, we set

$$\Sigma(\pi)=\Big\{\sigma=(s_1,\dots,s_n)\ :\ s_i\in[t_{i-1},t_i],\ 1\leqslant i\leqslant n\Big\}.$$

Finally, for any $\pi\in\Pi(a,b),\pi=\{t_i\}_{i=0}^n$, and $\sigma\in\Sigma(\pi),\sigma=(s_1,\dots,s_n)$, we define

$$S_\pi^\sigma(f)=\sum_{i=1}^n f(s_i)(t_i-t_{i-1}).$$

Theorem 19 *The limit*

$$\lim_{diam(\pi)\downarrow 0} S_\pi^\sigma(f)=:\int_a^b f(t)dt$$

exists uniformly for $\sigma\in\Sigma(\pi)$.

Lemma 7 *For any $\varepsilon>0$ there exists $\delta>0$ such that for all $\pi,\pi'\in\Pi(a,b)$ with $\pi\subseteq\pi'$ we have that*

$$diam(\pi)<\delta\quad\Longrightarrow\quad \big|S_\pi^\sigma(f)-S_{\pi'}^{\sigma'}(f)\big|<\varepsilon$$

for all $\sigma\in\Sigma(\pi)$ and $\sigma'\in\Sigma(\pi')$.

Proof. Since f is uniformly continuous, for any $\varepsilon>0$ there exists $\delta>0$ such that for all $t,s\in[a,b]$

$$|t-s|<\delta\quad\Longrightarrow\quad |f(t)-f(s)|<\frac{\varepsilon}{b-a}\,. \tag{4.0.1}$$

Let

$$\begin{cases}\pi=\{t_i\}_{i=0}^n\,, & \sigma=(s_1,\dots,s_n)\\ \pi'=\{t'_j\}_{j=0}^m\,, & \sigma'=(s'_1,\dots,s'_m)\end{cases}$$

be such that $\pi\subseteq\pi'$ and $diam(\pi)<\delta$. Then there exist positive integers

$$0=j_0<j_1<\cdots<j_n=m$$

such that $t'_{j_i} = t_i$ for all $i = 0, \ldots, n$. For any such i, it holds that

$$t_i - t_{i-1} = t'_{j_i} - t'_{j_{i-1}} = \sum_{j=j_{i-1}+1}^{j_i} (t'_j - t'_{j-i}).$$

Then

$$\begin{aligned} S_\pi^\sigma(f) - S_{\pi'}^{\sigma'}(f) &= \sum_{i=1}^n f(s_i)(t_i - t_{i-1}) - \sum_{j=1}^m f(s'_j)(t'_j - t'_{j-1}) \\ &= \sum_{i=1}^n \sum_{j=j_{i-1}+1}^{j_i} \big(f(s_i) - f(s'_j)\big)(t'_j - t'_{j-1}) \end{aligned}$$

Since for all $i = 1, \ldots, n$ we have that

$$s_i,\, s'_j \in [t_{i-1}, t_i] \qquad \forall j_{i-1} + 1 \leqslant j \leqslant j_i,$$

from (4.0.1) it follows that

$$\begin{aligned} \big|S_\pi^\sigma(f) - S_{\pi'}^{\sigma'}(f)\big| &\leqslant \sum_{i=1}^n \sum_{j=j_{i-1}+1}^{j_i} \big|f(s_i) - f(s'_j)\big|(t'_j - t'_{j-1}) \\ &\leqslant \frac{\varepsilon}{b-a} \sum_{i=1}^n (t_i - t_{i-1}) = \varepsilon. \end{aligned}$$

The proof is complete. □

Proof of Theorem 19. For any given $\varepsilon > 0$ let δ be as in Lemma 7. Let $\pi, \pi' \in \Pi(a,b)$ be such that $diam(\pi) < \delta$ and $diam(\pi') < \delta$. Finally, let $\sigma \in \Sigma(\pi)$ and $\sigma' \in \Sigma(\pi')$. Define $\pi'' = \pi \cup \pi'$ and fix any $\sigma'' \in \Sigma(\pi'')$. Then

$$\big|S_\pi^\sigma(f) - S_{\pi'}^{\sigma'}(f)\big| \leqslant \big|S_\pi^\sigma(f) - S_{\pi''}^{\sigma''}(f)\big| + \big|S_{\pi''}^{\sigma''}(f) - S_{\pi'}^{\sigma'}(f)\big| < 2\varepsilon.$$

This completes the proof since ε is arbitrary. □

Proposition 17 *For any $f, g \in \mathcal{C}([a,b];X)$ and $\lambda \in \mathbb{C}$ we have that*

$$\int_a^b \big(f(t) + g(t)\big)dt = \int_a^b f(t)dt + \int_a^b g(t)dt$$

$$\int_a^b \lambda f(t)dt = \lambda \int_a^b f(t)dt$$

$$\Big|\int_a^b f(t)dt\Big| \leqslant \int_a^b |f(t)|dt.$$

Moreover, for any $\phi \in X^*$ *we have that*

$$\left\langle \phi, \int_a^b f(t)dt \right\rangle = \int_a^b \langle \phi, f(t) \rangle dt. \tag{4.0.2}$$

and, for any $\Lambda \in \mathcal{L}(X)$ *we have that*

$$\Lambda \int_a^b f(t)dt = \int_a^b \Lambda f(t)dt. \tag{4.0.3}$$

Proof. Exercise. □

Proposition 18 *For any* $f \in \mathcal{C}^1([a,b];X)$ *we have that*

$$\int_a^b f'(t)dt = f(b) - f(a) \tag{4.0.4}$$

Proof. By (4.0.2) above, for any $\phi \in X^*$ we have that

$$\left\langle \phi, \int_a^b f'(t)dt \right\rangle = \int_a^b \langle \phi, f'(t) \rangle dt.$$

On the other hand, the function $t \mapsto \langle \phi, f(t) \rangle$ is continuously differentiable on $[a,b]$ with derivative equal to $\langle \phi, f'(t) \rangle$. Therefore, for any $\phi \in X^*$,

$$\int_a^b \langle \phi, f'(t) \rangle dt = \langle \phi, f(b) - f(a) \rangle.$$

Since X^* separates points, the above identity yields (4.0.4). □

Corollary 6 *Let* $f \in \mathcal{C}^1([a,b];X)$ *be such that* $f'(t) = 0$ *for all* $t \in [a,b]$. *Then* f *is constant.*

5 *Appendix B: Lebesgue integral on* $L^2(a, b; H)$

Let $\big(H, \langle\cdot,\cdot\rangle\big)$ be a separable Hilbert space and let $\{e_j\}_{j\in\mathbb{N}}$ be a complete orthonormal system in H.

The Hilbert space $L^2(a, b; H)$

Definition 22 *A function $f : [a, b] \to H$ is said to be Borel (resp. Lebesgue) measurable if so is the scalar function $t \mapsto \langle f(t), u\rangle$ for every $u \in H$.*

Remark 8 Let $f : [a, b] \to H$.

1. Since, for any $x \in H$,

$$\langle f(t), x\rangle = \sum_{j=1}^{\infty} \langle f(t), e_j\rangle\, \overline{\langle x, e_j\rangle} \qquad (t \in [a, b]),$$

   we conclude that f is Borel (resp. Lebesgue) measurable if and only if so is the scalar function $t \mapsto \langle f(t), e_j\rangle$ for every $j \in \mathbb{N}$.

2. Since

$$|f(t)|^2 = \sum_{j=1}^{\infty} \big|\langle f(t), e_j\rangle\big|^2 \qquad (t \in [a, b]),$$

   we have that, if f is Borel (resp. Lebesgue) measurable, then so is the scalar function $t \mapsto \|f(t)\|$.

Definition 23 *We denote by $L^2(a, b; H)$ the space of all Lebesgue measurable functions $f : [a, b] \to H$ such that*

$$\|f\|_2 := \Big(\int_a^b |f(t)|^2 dt\Big)^{\frac{1}{2}} < \infty\,,$$

where two functions f and g are identified if $f(t) = g(t)$ for a.e. $t \in [a, b]$.

Proposition 19 *$L^2(a, b; H)$ is a Hilbert space with the hermitian product*

$$(f|g)_0 = \int_a^b \langle f(t), g(t)\rangle dt \qquad (f, g \in L^2(a, b; H)).$$

Proof. We only prove completeness. Let $\{f_n\}$ be a Cauchy sequence in $L^2(a, b; H)$. Then $\{f_n\}$ is bounded:

$$\|f_n\|_2^2 = \int_a^b \sum_{j=1}^{\infty} \big|\langle f_n(t), e_j\rangle\big|^2 dt \leqslant M \qquad \forall n \in \mathbb{N} \tag{5.0.1}$$

Moreover, for any $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that, for all $m, n \geqslant \nu$,

$$\|f_n - f_m\|_2^2 = \int_a^b \sum_{j=1}^{\infty} \left|\langle f_n(t) - f_m(t), e_j \rangle\right|^2 dt \leqslant \varepsilon \tag{5.0.2}$$

Therefore, $t \mapsto \langle f_n(t), e_j \rangle$ is a Cauchy sequence in $L^2(a, b)$ for all $j \in \mathbb{N}$. So, there exists functions $\phi_j \in L^2(a, b)$ such that $\langle f_n(\cdot), e_j \rangle \to \phi_j$ in $L^2(a, b)$ for all $j \in \mathbb{N}$. Thus, by Fatou's lemma,

$$\int_a^b \sum_{j=1}^{\infty} \left|\phi_j(t)\right|^2 dt \leqslant M \quad \text{and} \quad \int_a^b \sum_{j=1}^{\infty} \left|\langle f_n(t), e_j \rangle - \phi_j(t)\right|^2 dt \leqslant \varepsilon \ (\forall n \geqslant \nu).$$

So, we conclude that

$$f(t) := \sum_{j=1}^{\infty} \phi_j(t) e_j \in H \qquad t \in [a, b] \text{ a.e.}$$

as well as $f \in L^2(a, b; H)$ and

$$\int_a^b |f_n(t) - f(t)|^2 dt \leqslant \varepsilon \quad (\forall n \geqslant \nu),$$

or, $f_n \to f$ in $L^2(a, b; H)$. □

Remark 9 For any $f \in L^2(a, b; H)$ we have that

$$\sum_{j=1}^{\infty} \Big| \int_a^b \langle f(t), e_j \rangle dt \Big|^2 \leqslant (b - a) \sum_{j=1}^{\infty} \int_a^b \left|\langle f(t), e_j \rangle\right|^2 dt < \infty.$$

Therefore

$$\sum_{j=1}^{\infty} e_j \int_a^b \langle f(t), e_j \rangle dt \in H.$$

Definition 24 *For any $f \in L^2(a, b; H)$ we define*

$$\int_a^b f(t) dt = \sum_{j=1}^{\infty} e_j \int_a^b \langle f(t), e_j \rangle dt\,.$$

Proposition 20 *For any $f \in L^2(a, b; H)$ the following properties hold true.*

(a) For any $x \in H$ we have that

$$\Big\langle x, \int_a^b f(t) dt \Big\rangle = \int_a^b \langle x, f(t) \rangle dt$$

$$(b) \qquad \Big|\int_a^b f(t)dt\Big| \leqslant \int_a^b |f(t)|dt$$

(*c*) *For any* $\Lambda \in \mathcal{L}(H)$ *we have that*

$$\Lambda\Big(\int_a^b f(t)dt\Big) = \int_a^b \Lambda f(t)dt\,.$$

Proof. Exercise □

Proposition 21 *Let* $A : D(A) \subset H \to H$ *be a closed linear operator with* $\rho(A) \neq \emptyset$. *Then for any* $f \in L^2(a,b;D(A))$ *we have that*

$$\int_a^b f(t)dt \in D(A) \quad \text{and} \quad A\Big(\int_a^b f(t)dt\Big) = \int_a^b Af(t)dt\,.$$

Proof. Exercise (*hint:* recall that, in view of Exercise 23, $D(A)$ is separable with respect to the graph norm). □

Proposition 22 *Let* $A : D(A) \subset H \to H$ *be the infinitesimal generator of a* $\mathcal{C}_0$*-semigroup on* H, $S(t)$, *which satisfies the growth condition* (1.6.3). *Then, for any* $f \in L^2(a,b;H)$,

(*a*) *for any* $t \in [a,b]$ *the function* $s \mapsto S(t-s)f(s)$ *belongs to* $L^2(a,t;H)$, *and*

(*b*) *the function*

$$F_A(t) = \int_a^t S(t-s)f(s)\,ds \qquad (t \in [a,b])$$

belongs to $\mathcal{C}\big([a,b];H\big)$.

Proof. In order to check measurability for $s \mapsto S(t-s)f(s)$ it suffices to observe that, for all $u \in H$ and a.e. $s \in [0,t]$,

$$\langle S(t-s)f(s),u\rangle = \langle f(s),S(t-s)^*u\rangle = \sum_{j=1}^{\infty}\langle f(s),e_j\rangle\,\overline{\langle S(t-s)^*u,e_j\rangle}.$$

Since $s \mapsto \langle S(t-s)^*u,e_j\rangle$ is continuous and $s \mapsto \langle f(s),e_j\rangle$ is measurable for all $j \in \mathbb{N}$, the measurability of $s \mapsto S(t-s)f(s)$ follows. Moreover, by (1.6.3) we have that

$$|S(t-s)f(s)| \leqslant Me^{\omega(t-s)}|f(s) \qquad (s \in [a,t] \text{ a.e.}),$$

which completes the proof of (*a*).

In order to prove point (b), fix $t \in]a,b[$ and let $t_n \to t$. Fix $\delta \in]0, t-a[$ and let $n_\delta \in \mathbb{N}$ be such that $t_n > t - \delta$ for all $n \geqslant n_\delta$. Then we have that

$$\begin{aligned}
&\big|F_A(t_n) - F_A(t)\big| \\
&\quad \leqslant \int_a^{t-\delta} \big|\big[S(t_n - s)f(s) - S(t-s)\big]f(s)\big|\, ds \\
&\qquad + \int_{t-\delta}^{t_n} \big|S(t_n - s)f(s)\big|\, ds + \int_{t-\delta}^{t} \big|S(t-s)f(s)\big|\, ds.
\end{aligned}$$

To complete the proof it suffices to observe that

$$\lim_{n\to\infty} \int_a^{t-\delta} \big|\big[S(t_n - s)f(s) - S(t-s)\big]f(s)\big|\, ds = 0$$

by the dominated convergence theorem, while the remaining terms on the right-hand side of the above inequality are small with δ. □

The Sobolev space $H^1(a,b;H)$

Definition 25 *$H^1(a,b;H)$ is the space of all functions $u \in \mathcal{C}\big([a,b];H\big)$ such that*

(a) $u'(t)$ exists for a.e. $t \in [a,b]$;

(b) $u' \in L^2(a,b;H)$;

(c) $u(t) - u(a) = \int_a^t u'(s)ds \qquad t \in [a,b]$ a.e.

Remark 10 $H^1(a,b;H)$ is a Hilbert space with the scalar product

$$(u|v)_1 = \int_a^b \big[\langle u(t), v(t)\rangle + \langle u'(t), v'(t)\rangle\big]\, dt \qquad (u, v \in H^1(a,b;H)).$$

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