ANISOTROPIC SPACES AND NIL-AUTOMORPHISMS

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Abstract. We introduce anisotropic Banach spaces on Heisenberg nilmanifolds and study the resonance spectrum associated to partially hyperbolic automorphisms. In this work we describe an alternative proof of the spectrum which takes advantage of geometric-style anisotropic norms.

1. Introduction and results

Let $\mathbb{H}$ be the three-dimensional Heisenberg group, that is the Lie group equal to $\mathbb{R}^3$ with the group law

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + \tfrac{1}{2}(xy' - x'y)).$$

The quotient space $M = \Gamma \backslash \mathbb{H}$, where Γ is a discrete subgroup of $\mathbb{H}$, is a compact manifold called the Heisenberg nilmanifold. Let (V, W, Z) be elements of the Lie algebra associated to $\mathbb{H}$ which satisfy the Heisenberg commutation relations $[V, W] = Z$ and $[V, Z] = [W, Z] = 0$. The object of our interest is $\Phi : M \to M$, an automorphism such that, for some $\lambda > 1$,

$$\Phi_* V = \lambda^{-1} V, \quad \Phi_* W = \lambda W, \quad \Phi_* Z = Z. \tag{1}$$

We say that such an automorphism is *partially hyperbolic* since it exhibits contraction, expansion and neutral behaviour. Let ν denote the probability measure which is inherited from the Haar measure on $\mathbb{H}$ and which is preserved by Φ. See Appendix A for the explicit construction of partially hyperbolic automorphisms. Since Φ is an isometry in the Z direction it makes sense to define, for all $N \in \mathbb{Z}$ (i.e., Fourier decomposition in the Z direction),

$$C_N^\infty(M) = \{h \in C^\infty(M) : Zh = 2\pi i N h\}. \tag{2}$$

Observe that $h \mapsto h \circ \Phi$ leaves $C_N^\infty(M)$ invariant.

Definition 1.1. Let $\Xi = \{\xi_j\}_j$ be a set of complex numbers, each with associated non-negative integer d_j. We say that Φ has *resonance spectrum* Ξ on $\mathcal{C}_N^\infty(M)$ if, for any $g, h \in \mathcal{C}_N^\infty(M)$ and for any $\epsilon > 0$,

$$\int g \cdot h \cdot \Phi^n \, d\nu = \sum_{|\lambda_j| \geq \epsilon} \sum_{k=0}^{d_j} \xi_j^n n^k C_{j,k}(g, h) + o(\epsilon^n)$$

where $C_{j,k}(g, h)$ are finite rank, bilinear and non-zero.

This definition matches the one used by Faure, Gouëzel & Lanneau [8, Definition 1.1]. The resonance spectrum of Φ on $\mathcal{C}_0^\infty(M)$ is equal to $\{0\}$ because, in the case $N = 0$, the system reduces to the study of a toral automorphism and the resonance spectrum can be shown by considering a fourier series decomposition on the torus.

It is known [11, Proposition 2.1] that, up to an automorphism of $\mathbb{H}$, every discrete subgroup Γ is of the form $\Gamma_K = \{(x, y, z) \in \mathbb{H} : x, y \in \mathbb{Z}, K \mid y\}$ for some $K \in \mathbb{N}$. Let K be thus fixed.

Theorem 1.2. *Let, as above, $\Phi : M \to M$ be a partially hyperbolic automorphism on a Heisenberg nilmanifold. For each $N \neq 0$ there exists a set of unit complex numbers $\{\mu_j\}_{j=1}^{K|N|}$ such that the resonance spectrum of Φ on $\mathcal{C}_N^\infty(M)$ is equal to*

$$\Xi_N = \left\{\lambda^{-(\frac{1}{2}+n)}\mu_j : 1 \leq j \leq K\left|N\right|, n \in \mathbb{N}_0\right\}.$$

This result isn't new, it is contained in the work of Flaminio & Forni [10] on nilflows since the automorphisms considered here correspond to periodic renormalization of nilflows and in the work of Faure & Tsujii [9] (under the name of "prequantum transfer operator", see also [7]). The first of these references has the advantage that the full flow is studied, not only the periodic case. The second of these references has the advantage that, to some degree, also the non-affine case is permitted and results obtained.

Nonetheless various related problems remain to be properly understood (see, e.g., [1, 3–5, 13, 14, 19]). In particular the possibility to extend the techniques of anisotropic spaces to the cocycle setting and to fully extend results to the non-algebraic systems. A particular advantage of the anisotropic Banach spaces is utilized in Section 5 where the full spectrum is obtained from the peripheral spectrum. This takes the place of an argument concerning the formal inverse of a given operator in the work of Flaminio & Forni (see [10, A.3]). Potentially the exploration of anisotropic Banach spaces in this setting will be an aid for further pushing the ideas to more settings.

Overview. In order to prove the theorem we will study the transfer operator on a family of anisotropic Banach spaces denoted $\mathcal{B}_N^{p,q}(M)$. Determining the resonance spectrum is equivalent to determining the spectrum of the transfer operator on $\mathcal{B}_N^{p,q}(M)$ for p, q sufficiently large. In Section 2 the anisotropic norms are introduced and various basic properties are proven. In particular it is shown that the spaces are embedded into the space of distributions (Lemma 2.2), the operators V and W are continuous on these spaces (Lemmas 2.5 & 2.6) and that the kernel of W is null (Lemma 2.7). The last fact will be useful later because we see that the operators V and W map from one eigenspace to another. This is arguably the main theme of the present strategy. In Section 3 the essential spectral radius of the transfer operator is estimated (Lemma 3.7). That the essential spectral radius can be made as small as desired means that we can then work just with isolated spectrum which has many advantages. Section 4 is devoted to studying the spectrum of the transfer operator restricted to $\ker V$ and it is shown that the spectrum is a finite set of complex numbers, all with absolute value equal to $\lambda^{-\frac{1}{2}}$ (Lemma 4.4). Using all the previously established details, in Section 5 we then show that the rest of the spectrum is a scaled version of the peripheral spectrum (Lemma 5.3).

2. Anisotropic spaces

In this section we define norms on $\mathcal{C}^\infty_N(M)$ and then define Banach spaces by completion with respect to the norms. We also explore various basic properties of these norms. The norms used are similar to the "geometric style" anisotropic norms often used for hyperbolic systems [15–17] and those used by Faure, Gouëzel & Lanneau [8] for pseudo Anosov maps.

We consider, as introduced in Section 1, $\Phi : M \to M$ and $N \in \mathbb{N}$ fixed for the remainder of this section. The three elements of the basis of the Lie algebra (V, W, Z) can each be seen as vector fields on M. We will use the same notation for the vector fields since the meaning is clear from the context. These three vector fields define a splitting of tangent space which is invariant (1) under the action of the partially hyperbolic transformations which we study. The *stable* and *unstable* bundles are respectively the ones corresponding to V and W. Each vector field defines a foliation which is naturally orientated. For each $u \in \mathbb{R}$ let

$$\varphi^V_u : M \to M$$

be defined as sliding a distance u along the V-foliation. Similarly φ^W_u and φ^Z_u are defined. In this Lie group context we could also have written $(x, y, z) \mapsto (x, y, z) * e^{uV}$ for φ^V_u, etc.

We fix $\delta > 0$, once and for all, sufficiently small that there exists a covering of M into sets of diameter not greater than δ and, subordinated to which, there exists a smooth partition of unity. For notational convenience let $I_\delta = (-\delta, \delta)$ and let $\mathcal{C}^\infty_{\mathrm{c}}(I_\delta)$ denote the set of $\mathcal{C}^\infty$ functions with support compactly contained in the interval I_δ. For $h \in \mathcal{C}^\infty_N(M)$ and $\eta \in \mathcal{C}^\infty_{\mathrm{c}}(I_\delta)$ and $m \in M$, let

$$\ell_{\eta,m}(h) = \int_{-\delta}^{\delta} \eta(u) \cdot h \circ \varphi^W_u(m)\, du. \tag{3}$$

For $p \in \mathbb{N}_0$, $q \in \mathbb{N}$ we define a norm on $\mathcal{C}^\infty_N(M)$,

$$\|h\|_{p,q} = \sup\left\{ \left|\ell_{\eta,m}(V^j h)\right| : 0 \le j \le p, m \in M, \eta \in \mathcal{C}^\infty_{\mathrm{c}}(I_\delta), \|\eta\|_{\mathcal{C}^q} \le 1 \right\}.$$

We then define the Banach space $\mathcal{B}^{p,q}_N(M)$ as the completion of $\mathcal{C}^\infty_N(M)$ with respect to this norm. Essentially this norm requires the function to be rather smooth in the V-direction and allows rather bad behaviour in the W-direction.

Suppose that $g \in \mathcal{C}^\infty(M)$ with support in a δ-neighbourhood of a point $m \in M$. Then, for notational convenience, we understand $\eta_{g,m}(h)$ to be equal to $\ell_{\eta,m}(h)$, as defined above (3), where $\eta(u) = g \circ \varphi^V_u(m)$. I.e., we can consider g on M or in local chart along the unstable curve.

These linear functionals behave continuously in the V-direction in the following sense.

Lemma 2.1. *Suppose that $m, m' \in M$ are ϵ-close and that $g \in \mathcal{C}^\infty_{-N}(M)$ has support within a δ-neighbourhood of both m and m'. Then, for all $h \in \mathcal{C}^\infty_N(M)$, $q \ge 2$,*

$$\left|\ell_{g,m}(h) - \ell_{g,m'}(h)\right| \le 2\epsilon \|g\|_{\mathcal{C}^q} \|h\|_{1,q-1}.$$

Proof. Observe that $(g \cdot h) \in \mathcal{C}_0^\infty(M)$. We have that $m' = \varphi_a^V \circ \varphi_c^Z(m)$ and so,

$$(g \cdot h) \circ \varphi_u^W(m') = (g \cdot h) \circ \varphi_u^W \circ \varphi_a^V \circ \varphi_c^Z(m) = (g \cdot h) \circ \varphi_a^V \circ \varphi_u^W(m).$$

Additionally, by definition of V,

$$(g \cdot h) \circ \varphi_a^V(\cdot) = (g \cdot h)(\cdot) + \int_0^a V(g \cdot h) \circ \varphi_s^V(\cdot)\, ds.$$

Consequently,

$$\begin{aligned}\int (g \cdot h) \circ \varphi_u^W(m')\, du = \int &(g \cdot h) \circ \varphi_u^W(m)\, du \\ &+ \int_0^a \left(\int (g \cdot Vh) \circ \varphi_s^V \circ \varphi_u^W(m)\, du \right) ds \\ &+ \int_0^a \left(\int (Vg \cdot h) \circ \varphi_s^V \circ \varphi_u^W(m)\, du \right) ds.\end{aligned}$$

The final two terms are bounded from above in absolute value by

$$|a| \, \|g\|_{\mathcal{C}^q} \|h\|_{1,q} + |a| \, \|g\|_{\mathcal{C}^q} \|h\|_{0,q-1},$$

taking advantage of the definition of the norm. □

The following lemma means that one can identify $\mathcal{B}_N^{p,q}(M)$ with a space of distributions. For any $r \in \mathbb{N}$ denote by $\mathcal{D}_N^r(M)$ the elements of the dual space $\mathcal{C}^r(M)'$ which have support in $\mathcal{C}_N^\infty(M)$.

Lemma 2.2. *Let $p \in \mathbb{N}_0$, $q \in \mathbb{N}$. There is a map $\iota : \mathcal{B}_N^{p,q}(M) \to \mathcal{D}_N^q(M)$ is continuous and injective.*

Proof. The map ι extends the canonical inclusion $\mathcal{C}_N^\infty(M) \to \mathcal{D}_N^\infty(M)$ given by $\langle \iota(h), g \rangle = \int_M h \cdot g$. Using the previously mentioned smooth partition of unity of M and then decomposing the integral into leaves of the form $\left\{ \varphi_u^W(m) : u \in (-\delta, \delta) \right\}$ we show that there exists $C > 0$ such that, for all $h \in \mathcal{C}_N^\infty(M)$,

$$\int_M h \cdot g \le C \|h\|_{p,q} \|g\|_{\mathcal{C}^p(M)}.$$

Considering the completion, this shows that any $h \in \mathcal{B}_N^{p,q}(M)$ gives a distribution on M of order at most p. That ι is injective follows in the same way as the argument of, e.g., Gouëzel & Liverani [16, Proposition 4.1]. We start by taking $h \in \mathcal{B}_N^{p,q}(M)$, $h \neq 0$. Then we deduce that there exists some $m \in M$ such that the distribution $\eta \mapsto \int_{-\delta}^{\delta} \eta(u) \cdot h \circ \varphi_u^W(m)\, du$ is non-zero. We finally use this η to construct a $\tilde{\eta} \in \mathcal{C}^\infty(M)$ such that $\langle \iota h, \tilde{\eta} \rangle \neq 0$ and hence $\iota(h) \in \mathcal{D}_N^q(M)$ is non-zero as required. □

Lemma 2.3. *Let $p, p' \in \mathbb{N}_0$, $q, q' \in \mathbb{N}$ such that $p' \le p$ and $q' \ge q$. There is a continuous inclusion $\mathcal{B}_N^{p,q}(M) \subseteq \mathcal{B}_N^{p',q'}(M)$.*

Proof. Let $h \in \mathcal{C}_N^\infty(M)$. By definition

$$\|h\|_{p',q'} = \sup \left\{ \left| \ell_{\eta,m}(V^j h) \right| : 0 \le j \le p', m \in M, \|\eta\|_{\mathcal{C}^{q'}} \le 1 \right\}.$$

Observe that $j \le p'$ implies that $j \le p$ and $\|\eta\|_{\mathcal{C}^{q'}} \le 1$ implies that $\|\eta\|_{\mathcal{C}^q} \le 1$ because $p' \le p$ and $q' \ge q$. Consequently the above term is bounded by $\|h\|_{p,q}$. By density the full result follows. □

The following compact embedding result and argument is very similar to the one appearing in other works using geometric-style anisotropic space (e.g., [16, §5] and [8, §2.2]).

Lemma 2.4. *Let $p, p' \in \mathbb{N}_0$, $q, q' \in \mathbb{N}$, $p' > p$, $q' < q$. Then, the inclusion $\mathcal{B}_N^{p',q'}(M) \subset \mathcal{B}_N^{p,q}(M)$ is compact.*

Proof. By Lemma 2.3 it suffices to prove the result for the case $p' = p+1$, $q' = q-1$ and by density it suffices to work with $h \in \mathcal{C}_N^\infty(M)$. We will show that, for each $\epsilon > 0$ sufficiently small and given $N > 0$, there exist $N_\epsilon > 0$ and a finite set $\{\rho_k\}_{k=1}^{N_\epsilon}$ of linear functionals on $\mathcal{B}_N^{p+1,q-1}(M)$ such that, for any $h \in \mathcal{C}_N^\infty(M)$,

$$\|h\|_{p,q} \leq \sup_{1\leq k\leq N_\epsilon} |\rho_k(h)| + \epsilon \|h\|_{p+1,q-1}. \tag{4}$$

This implies the claimed compactness (see, e.g., [8, Proof of Prop. 2.8]).

Fix $0 < \epsilon < \frac{\delta}{4}$. Let $\{m_k\}_{k=1}^{N_\epsilon}$ denote a finite set of points in M such that every point of M is ϵ-close to at least one of the m_k. Moreover, using the compactness of $\mathcal{C}^q$ in $\mathcal{C}^{q-1}$, choose a finite set $\{\zeta_k\}_{k=1}^{N_\epsilon}$ of functions in $\mathcal{C}_c^\infty(I_\delta)$ such that, for every η in the unit ball of $\mathcal{C}_c^q(I_\delta)$, there exists some ζ_k in the set such that $\|\eta - \zeta_k\|_{\mathcal{C}^{q-1}} \leq \epsilon$.

We need to estimate integrals of the form,

$$\int \eta(u) \cdot V^j h \circ \varphi_u^W(m)\, du, \quad 0 \leq j \leq p.$$

For any $m \in M$, there exists m_k in the chosen finite set such that m and m_k are ϵ-close. We can extend $\eta \circ \varphi_u^W(m)$ to a neighbourhood of m in such a way that we can apply Lemma 2.1. Consequently, paying a price of $\epsilon \left\|V^j h\right\|_{1,q-1} \leq \epsilon \|h\|_{p+1,q-1}$, it suffices to consider only integrals centred at some m_k.

Summarising what we have shown so far, we know the existence of a constant $C > 0$ such that

$$\left|\int \eta(u) \cdot V^j h \circ \varphi_u^W(m)\, du\right| \leq \sup_{k,\widetilde{\eta}} \left|\int \widetilde{\eta}(u) \cdot V^j h \circ \varphi_u^W(m_k)\, du\right| + \epsilon C \|h\|_{p+1,q-1}.$$

To finish, we recall that there exists some $\zeta_{k'} \in \{\zeta_k\}_{k=1}^{N_\epsilon}$ with the property that $\|\widetilde{\eta} - \zeta_{k'}\|_{\mathcal{C}^{q-1}} \leq \epsilon$. This means that

$$\left|\int \widetilde{\eta}(u) \cdot V^j h \circ \varphi_u^W(m_k)\, du\right|$$

is bounded from above by

$$\left|\int \zeta_{k'}(u) \cdot V^j h \circ \varphi_u^W(m_k)\, du\right| + \left|\int (\widetilde{\eta} - \zeta_{k'})(u) \cdot V^j h \circ \varphi_u^W(m_k)\, du\right| \leq \left|\int \zeta_{k'}(u) \cdot V^j h \circ \varphi_u^W(m_k)\, du\right| + \epsilon C \|h\|_{p+1,q-1}.$$

As required (4), we have proved that the existence of a finite number of continuous linear functional $(\rho_k)_{1\leq k\leq N_\epsilon}$ on $\mathcal{B}_N^{p,q}$. □

Lemma 2.5. *Let $p, q \in \mathbb{N}$. The operator $V : \mathcal{B}_N^{p,q}(M) \to \mathcal{B}_N^{p-1,q}(M)$ is continuous.*[1]

Proof. Let $h \in \mathcal{C}_N^\infty(M)$. Since $[V, Z] = 0$ we know that $Vh \in \mathcal{C}_N^\infty(M)$. Further observe that

$$\begin{aligned}\|Vh\|_{p-1,q} &= \sup\left\{\left|\ell_{\eta,m}(V^j(Vh))\right| : 0 \le j \le p-1, m \in M, \|\eta\|_{\mathcal{C}^q} \le 1\right\}\\ &= \sup\left\{\left|\ell_{\eta,m}(V^j h)\right| : 1 \le j \le p, m \in M, \|\eta\|_{\mathcal{C}^q} \le 1\right\}\\ &\le \|h\|_{p,q}.\end{aligned}$$

By density the full result follows. □

Lemma 2.6. *Let $p \in \mathbb{N}_0$, $q \in \mathbb{N}$. The operator $W : \mathcal{B}_N^{p,q}(M) \to \mathcal{B}_N^{p,q+1}(M)$ is continuous.*

Proof. Let $h \in \mathcal{C}_N^\infty$. We must estimate $\|Wh\|_{p,q+1}$. As such, let $\eta \in \mathcal{C}^\infty$, $\|\eta\|_{\mathcal{C}^{q+1}} \le 1$ and let $1 \le j \le p$. The commutation relations of V, W imply that $V^jW = WV^j + jV^{j-1}Z$ (in the case $j = 0$ the term $jV^{j-1}Z$ is not present). Rearranging and integrating by parts,

$$\begin{aligned}&\int \eta(u) \cdot (V^jWh) \circ \phi_u^W(m)\, du\\ &= \int \eta(u) \cdot W(V^jh) \circ \phi_u^W(m) du + j \int \eta(u) \cdot V^{j-1}Zh \circ \phi_u^W(m)\, du\\ &= -\int \eta'(u) \cdot V^jh \circ \phi_u^W(m) du + j \int \eta(u) \cdot V^{j-1}Zh \circ \phi_u^W(m)\, du.\end{aligned}$$

Observe that $\|\eta'\|_{\mathcal{C}^q} \le 1$. This all means that,

$$\left|\int \eta(u) \cdot (V^jWh) \circ \phi_u^W(m)\, du\right| \le \left\|V^jh\right\|_{0,q} + j\left\|V^{j-1}(Zh)\right\|_{0,q+1}$$

and so we have shown that $\|Wh\|_{p,q+1} \le (2\pi|N|p + 1)\|h\|_{p,q}$. □

Lemma 2.7. *Suppose that $N \ne 0$, $h \in \mathcal{B}_N^{p,q}(M)$ and $Wh = 0$. Then $h = 0$.*

Proof. Recall that the norm is based on linear functions of the form

$$\ell_{\eta,m}(h) = \int \eta(u) \cdot h \circ \varphi_u^W(m)\, du$$

and these linear functionals extend to $\mathcal{B}_N^{p,q}(M)$. Using the assumption $Wh = 0$ we deduce that $\ell_{\eta,m}(h) = \ell_{\eta_s,m}(h)$ where $\eta_s(u) = \eta(u - s)$ is the translation of any η. In other words, the distribution induced by h on the leaf $\{\varphi_u^W(m) : u \in \mathbb{R}\}$ is translation invariant. This means that the distribution is equal to a constant multiple of Lebesgue measure along the leaf.[2] Such leaf are dense in M. Moreover, the quantity $\ell_{\eta,m}(h)$ varies continuously as m moves in the V-direction as shown in Lemma 2.1. Consequently h is equal to a constant multiple of Lebesgue measure on the entire of M. However, since $N \ne 0$, this contradicts the oscillating behaviour of h in the Z-direction. □

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[1] In the sense that V extends to a continuous operator on these spaces and, abusing notation, we use the same symbol for the extension.

[2] Let $\varphi \in \mathcal{S}'(\mathbb{R})$ such that $\varphi(f') = 0$ for any $f \in \mathcal{S}(\mathbb{R})$. Fix $\gamma \in S(\mathbb{R})$ with $Leb(\gamma) = 1$. Set $\tilde{g}(u) = \int_{-\infty}^u g(t) - \gamma(t)Leb(g)dt$. Then $\tilde{g} \in \mathcal{S}(\mathbb{R})$. Since $\tilde{g}'(u) = g(u) - \gamma(u)Leb(g)$ and $\varphi(\tilde{g}') = 0$, $\varphi(g) = \varphi(\gamma)Leb(g)$.

A key characteristic of the present setting is that Lemma 2.7 doesn't hold when W is replaced with V. Indeed, as we will see in Section 4, the set $\{h \in \mathcal{B}_N^{p,q}(M) : Vh = 0\}$ is non-empty. This is one of the consequences of the anisotropic spaces used in the present setting and essential for this analysis.

3. Transfer operator and norm estimates

This section is devoted to proving several key properties of the transfer operator in relation to the the anisotropic norms. In particular that the transfer operator is quasi compact. Again we consider $M = \Gamma \backslash \mathbb{H}$, $N \in \mathbb{Z}$ and $\Phi : M \to M$, as introduced in Section 1, chosen and fixed for the entire section.

We consider the linear operator $\mathcal{L} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, given by

$$\mathcal{L} : h \mapsto h \circ \Phi \tag{5}$$

and which we call the *transfer operator.*

Remark. In the present affine setting it doesn't make a difference, apart from a simple scaling, if we consider the operator corresponding to the measure of maximal entropy or the one corresponding to the SRB measure. Moreover, since the system is invertible, we could consider $h \mapsto h \circ \Phi^{-1}$ and swap V and W.

Lemma 3.1. *For all $j, k \in \mathbb{N}$, $h \in \mathcal{C}_N^\infty(M)$,*

$$V^j \mathcal{L}^k h = \lambda^{-jk} \mathcal{L}^k \left(V^j h\right) \quad \text{and} \quad W^j \mathcal{L}^k h = \lambda^{jk} \mathcal{L}^k \left(W^j h\right).$$

Proof. We know that (1),

$$V \mathcal{L}^k h = V(h \circ \Phi^k) = (Vh \circ \Phi^k)\lambda^{-k} = \lambda^{-k} \mathcal{L}(Vh)$$

and so $V\mathcal{L}^k h = \lambda^{-k} \mathcal{L}^k (Vh)$. Iterating this leads to the full result. Similarly for W. □

The above result extends to all spaces on which V, W and $\mathcal{L}$ extend continuously, a fact which will soon be useful.

For convenience, for all $j \in \mathbb{N}_0$, $q \in \mathbb{N}$, we define the semi-norm on $\mathcal{C}_N^\infty(M)$,

$$|h|_{j,q} = \sup \left\{ \left| \ell_{\eta,m}(V^j h) \right| : m \in M, \eta \in \mathcal{C}_c^\infty(I_\delta), \|\eta\|_{\mathcal{C}^q} \le 1 \right\}. \tag{6}$$

This definition has the consequence that $\|h\|_{p,q} = \sup_{0 \le j \le p} |h|_{j,q}$.

The norms are defined based on $\delta > 0$. The following lemma clarifies the dependence of the norm on this choice.

Lemma 3.2. *There exists $C > 0$ such that, for all $\eta \in \mathcal{C}^\infty(\mathbb{R})$ supported in some interval $A \subset \mathbb{R}$ of length $|A| \ge 2\delta$ and for all $h \in \mathcal{C}_N^\infty(M)$, $q \in \mathbb{R}$,*

$$\left| \int \eta(u) \cdot h \circ \varphi_u^W(m) \, du \right| \le C \, |A| \, \|\eta\|_{\mathcal{C}^q} \, |h|_{0,q}.$$

Proof. Using a smooth partition of unity we write $\eta = \sum_{i=1}^{3|A|} \tilde{\eta}_i$ where each $\tilde{\eta}_i$ is supported on an interval smaller that 2δ and $\|\tilde{\eta}_i\|_{\mathcal{C}^q} \le C$ for a uniform constant $C > 0$. □

Lemma 3.3. *For all $q \in \mathbb{N}$ there exists $C > 0$ such that, for all $k \in \mathbb{N}$, $h \in \mathcal{C}_N^\infty(M)$,*

$$|\mathcal{L}^k h|_{0,q} \le C \, |h|_{0,q}.$$

Proof. Let $k \in \mathbb{N}$, $m \in M$, $h \in \mathcal{C}^\infty_N(M)$ and $\eta \in \mathcal{C}^\infty_c(I_\delta)$ such that $\|\eta\|_{\mathcal{C}^q} \le 1$. We need to estimate

$$\ell_{\eta,m}(\mathcal{L}^k h) = \int_{-\delta}^{\delta} \eta(u) \cdot \mathcal{L}^k h \circ \varphi^W_u(m)\ du.$$

We observe that $\Phi^{-k} \circ \varphi^W_u = \varphi^W_{\lambda^k u} \circ \Phi^{-k}$ and so

$$\mathcal{L}^k h \circ \varphi^W_u = h \circ \Phi^{-k} \circ \varphi^W_u = h \circ \varphi^W_{\lambda^k u} \circ \Phi^{-k}.$$

Changing variables in the integral ($s = \lambda^k u$),

$$\ell_{\eta,m}(\mathcal{L}^k h) = \lambda^k \int_{-\lambda^k\delta}^{\lambda^k\delta} \eta(\lambda^{-k}s) \cdot h \circ \varphi^W_s(m)\ ds.$$

We apply Lemma 3.2 and note that $s \mapsto \eta(\lambda^{-k}s)$ is supported on the interval $(-\lambda^k\delta, \lambda^k\delta)$. Consequently $\left|\ell_{\eta,m}(\mathcal{L}^k h)\right| \le C \left|h\right|_{0,q}$. □

We observe that $|h|_{j,q} = \left|V^j h\right|_{0,q}$. Hence the estimate of the above lemma implies also that $\left|\mathcal{L}^k h\right|_{j,q} \le C \left|h\right|_{j,q}$ for all $j \in \mathbb{N}$. This suffices to show that the operator $\mathcal{L}$ extends to a continuous operator on $\mathcal{B}^{p,q}_N(M)$.

In the following lemma we take advantage of the contraction in the V-direction.

Lemma 3.4. *There exists $C > 0$ such that, for all $j, q, k \in \mathbb{N}$, $h \in \mathcal{C}^\infty_N(M)$,*

$$|\mathcal{L}^k h|_{j,q} \le C\lambda^{-jk} \left|h\right|_{j,q}.$$

Proof. By definition of the norm and Lemma 3.1,

$$\begin{aligned} |\mathcal{L}^k h|_{j,q} &= |V^j \mathcal{L}^k h|_{0,q} \\ &= \lambda^{-jk} \left|\mathcal{L}^k (V^j h)\right|_{0,q}. \end{aligned}$$

Using also the estimate of Lemma 3.3,

$$|\mathcal{L}^k h|_{j,q} \le C\lambda^{-jk} \left|V^j h\right|_{0,q} = C\lambda^{-jk} \left|h\right|_{j,q}. \qquad \square$$

The above lemma means that $\left\|\mathcal{L}^k h\right\|_{p,q} \le C\lambda^{-jk} \left\|h\right\|_{p,q} + \sup_{j<p} \left\|\mathcal{L}^k h\right\|_{j,q}$. We now need to get a useful estimate for $\left\|\mathcal{L}^k h\right\|_{j,q}$ when $0 \le j \le p-1$. We do this in the following lemma, using the expansion in the W-direction.

Lemma 3.5. *There exists $C > 0$ such that, for all $j, q, k \in \mathbb{N}$, $h \in \mathcal{C}^\infty_N(M)$,*

$$|\mathcal{L}^k h|_{j,q} \le C\lambda^{-qk} \left|h\right|_{j,q} + C_k \left|h\right|_{j,q+1}.$$

Proof. By Lemma 3.1 it suffices to prove the case when $j = 0$ because $|\mathcal{L}^k h|_{j,q} = |V^j \mathcal{L}^k h|_{0,q} = \lambda^{-j} \left|\mathcal{L}^k V^j h\right|_{0,q}$. Let $k \in \mathbb{N}$, $m \in M$, $h \in \mathcal{C}^\infty_N(M)$ and $\eta \in \mathcal{C}^\infty_c(I_\delta)$ such that $\|\eta\|_{\mathcal{C}^q} \le 1$. We need to estimate

$$\ell_{\eta,m}(\mathcal{L}^k h) = \int \eta(u) \cdot \mathcal{L}^k h \circ \varphi^W_u(m)\ du.$$

For $\epsilon > 0$ we fix ρ_ϵ an ϵ-sized mollifier and define the smoothed version $\eta_\epsilon = \eta \star \rho_\epsilon$. This construction means that

$$\|\eta - \eta_\epsilon\|_{\mathcal{C}^{q-1}} \le C\epsilon, \quad \|\eta_\epsilon\|_{\mathcal{C}^q} \le C, \quad \|\eta_\epsilon\|_{\mathcal{C}^{q+1}} \le C\epsilon^{-1}.$$

We choose $\epsilon = \lambda^{-qk}$. We write

$$\ell_{\eta,m}(\mathcal{L}^k h) = \int \eta_\epsilon(u) \cdot \mathcal{L}^k h \circ \varphi_u^W(m)\ du + \int (\eta - \eta_\epsilon)(u) \cdot \mathcal{L}^k h \circ \varphi_u^W(m)\ du.$$

For the first integral we have the estimate

$$\left| \int \eta_\epsilon(u) \cdot \mathcal{L}^k h \circ \varphi_u^W(m)\ du \right| \le \|\eta_\epsilon\|_{\mathcal{C}^{q+1}} \left|\mathcal{L}^k h\right|_{0,q+1} \le C\epsilon^{-1} \left|\mathcal{L}^k h\right|_{0,q+1}.$$

For the second integral, let $\tilde{\eta}(u) = (\eta - \eta_\epsilon)(\lambda^{-k}u)$ (similar to the proof of Lemma 3.3).

$$\begin{aligned} \|\tilde{\eta}\|_{\mathcal{C}^q} &\le \|\tilde{\eta}\|_{\mathcal{C}^q} + \|\tilde{\eta}\|_{\mathcal{C}^{q-1}} \\ &\le \lambda^{-qk} \|\eta - \eta_\epsilon\|_{\mathcal{C}^q} + \|\eta - \eta_\epsilon\|_{\mathcal{C}^{q-1}} \\ &\le C\lambda^{-qk} + C\epsilon. \end{aligned}$$

Again we use the estimate of Lemma 3.2. Summing the above two estimates completes the proof of the lemma. □

Lemma 3.6. *Let $p \in \mathbb{N}_0$, $q \in \mathbb{N}$. The operator $\mathcal{L}$ extends to a continuous operator on $\mathcal{B}_N^{p,q}(M)$. Moreover there exists $C > 0$ such that, for every $h \in \mathcal{B}_N^{p,q}(M)$, $k \in \mathbb{N}$,*

$$\|\mathcal{L}^k h\|_{p,q} \le C \|h\|_{p,q}.$$

Furthermore, for any $\tilde{\lambda} \in (1, \lambda)$, there exists $C > 0$ such that, for every $h \in \mathcal{B}_N^{p,q}(M)$, $k \in \mathbb{N}$,

$$\|\mathcal{L}^k h\|_{0,q} \le C\tilde{\lambda}^{-qk} \|h\|_{p,q},$$

and, when $p \ge 1$,

$$\|\mathcal{L}^k h\|_{p,q} \le C\tilde{\lambda}^{-\min(p,q)k} \|h\|_{p,q} + C \|h\|_{p-1,q+1}.$$

Proof. By density, it suffices to prove the inequality for $h \in \mathcal{C}_N^\infty(M)$. Let $p, q \in \mathbb{N}$. Combining the estimates of Lemma 3.4 and Lemma 3.5 gives the following estimate.

$$\begin{aligned} \|\mathcal{L}^k h\|_{p,q} &\le \left|\mathcal{L}^k h\right|_{p,q} + \sup_{0 \le j \le p-1} \left|\mathcal{L}^k h\right|_{j,q} \\ &\le C\lambda^{-pk} |h|_{p,q} + C\lambda^{-qk} \|h\|_{p-1,q} + C_k \|h\|_{p-1,q+1} \\ &\le C\lambda^{-\min(p,q)k} \|h\|_{p,q} + C_k \|h\|_{p-1,q+1}. \end{aligned}$$

The uniform in k estimate of Lemma 3.3, together with iterating the above estimate allows the removal of the k dependence in the second term. □

Lemma 3.7. *Let $N \in \mathbb{Z}$, $p, q \in \mathbb{N}$. The operator $\mathcal{L} : \mathcal{B}_N^{p,q}(M) \to \mathcal{B}_N^{p,q}(M)$ has spectral radius not greater than 1 and essential spectral radius not greater than $\lambda^{-\min\{p,q\}}$. Moreover the spectral radius of $\mathcal{L} : \mathcal{B}_0^{p,q}(M) \to \mathcal{B}_0^{p,q}(M)$ is equal to 1.*

Proof. The estimates on the spectral and essential spectral radius follow from Lemma 3.6 and Lemma 2.4 by Hennion's argument [18]. The observation that constant functions are contained within $\mathcal{C}_0^\infty(M)$ and are invariant under $\mathcal{L}$ implies that the spectral radius of $\mathcal{L} : \mathcal{B}_0^{p,q}(M) \to \mathcal{B}_0^{p,q}(M)$ is equal to 1. □

4. Peripheral spectrum

This section is devoted to determining the spectrum of the transfer operator when restricted to the kernel of V. The motive for this will become clear in Section 5, particularly Lemma 5.1. We will see that the kernel consists of a finite number of eigenvalues of absolute value $\lambda^{\frac{1}{2}}$. As before we consider $\Phi : M \to M$, as introduced in Section 1 and $N \in \mathbb{Z} \setminus \{0\}$, chosen and fixed for the entire section. For convenience let,

$$\mathcal{I}_N = \{h \in \mathcal{B}_N^{p,q}(M) : Vh = 0\}.$$

Subsequently we will see that $\mathcal{I}_N$ does not depend on p, q, so justifying this choice of notation. Recall that, according to Lemma 2.2, the anisotropic spaces are continuously embedded in the space of distributions. To proceed it is convenient to show that the space $\mathcal{B}_N^{p,q}(M)$ is sufficiently large that we see the "invariant distributions" which were identified by Flaminio & Forni [10]. Recall, as introduced in Section 1, $\mathcal{D}_N^1(M)$ denotes the space of distributions on $\mathcal{C}^1(M)$, supported on $\mathcal{C}_N^1(M)$.

Lemma 4.1. *Suppose that $\theta \in \mathcal{D}_N^1(M)$ and that $\theta(Vh) = 0$ for all $h \in \mathcal{C}^\infty(M)$. Then, for every $p \in \mathbb{N}_0$, $q \in \mathbb{N}$ there exists $g \in \mathcal{B}_N^{p,q}(M)$ such that $\theta = \iota g$.*

Proof. Since, for all $q \in \mathbb{N}$, $\mathcal{B}_N^{p,q}(M) \subset \mathcal{B}_N^{p,1}(M)$ (Lemma 2.3), it suffices to prove the case $q = 1$. Moreover it suffices to work locally in $\mathbb{R}^2$. Consequently, without loss of generality, we work with the seminorm,

$$|h|_{p,1} = \sup_{\eta,x} \left| \int_{-\delta}^{\delta} \tfrac{\partial^p h}{\partial x_1^p}(x_1, x_2 + u) \cdot \eta(u)\ du \right|$$

where the supremum is over $\eta \in \mathcal{C}_c^1(\mathbb{R})$, $\|\eta\|_{\mathcal{C}^1} \leq 1$ and $x = (x_1, x_2) \in \mathbb{R}^2$. Moreover we assume that $\theta(\frac{\partial h}{\partial x_1}) = 0$ for all $h \in \mathcal{C}^\infty(\mathbb{R}^2)$. Let $\rho \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}_+)$ be the standard mollifier (in particular ρ is supported on $\{|x| < 1\}$ and $\int \rho(x)\ dx = 1$). For $\epsilon > 0$, let $\rho_\epsilon(x) = \epsilon^{-2}\rho(\epsilon^{-1}x)$. The smoothed version of a function is defined as $h_\epsilon(x) = \int \rho_\epsilon(x - y) \cdot h(y)\ dy$. (See, e.g., [6, §4] for standard details related to the mollifier and smoothing which we will use in the remainder of this proof.) Additionally, for $x \in \mathbb{R}^2$, let $\rho_{\epsilon,x}(y) = \rho_\epsilon(y - x)$. Observe that $\rho_{\epsilon,x}$ is an ϵ-sized "bump" centred at x. Let $g_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^2)$ be defined as

$$g_\epsilon(x) = \theta(\rho_{\epsilon,x}).$$

We will show that g_ϵ converges to $g \in \mathcal{B}_N^{p,1}(M)$ and satisfies the required properties. That $\theta(\frac{\partial h}{\partial x_1}) = 0$ implies that $\frac{\partial g_\epsilon}{\partial x_1} = 0$. Furthermore, observe that, for any $h \in \mathcal{C}^\infty(\mathbb{R}^2)$,

$$\begin{aligned} \langle g_\epsilon, h \rangle &= \int \theta(\rho_{\epsilon,x}) \cdot h(x)\ dx = \int \theta(\rho_{\epsilon,x} \cdot h(x))\ dx \\ &= \theta\left(\int \rho_{\epsilon,x} \cdot h(x)\ dx \right) = \theta\left(h_\epsilon\right) \end{aligned}$$

since $\left(\int \rho_{\epsilon,x} \cdot h(x)\ dx\right)(y) = \int \rho_\epsilon(y - x) \cdot h(x)\ dx = h_\epsilon(y)$. Consequently $\langle g_\epsilon, h \rangle - \theta(h) = \theta(h_\epsilon - h)$. Since $h_\epsilon \to h$ this means that $\langle g_\epsilon, \cdot \rangle \to \theta(\cdot)$ in the space of distributions of order 1. Using the invariance of g_ϵ in the first coordinate we need only consider the case $p = 0$ and,

$$\int_{-\delta}^{\delta} g_\epsilon(x_1, x_2 + u) \cdot \eta(u)\ du = \iint_{\mathbb{R}^2} g_\epsilon(y_1, y_2) \cdot \rho(y_1) \cdot \eta(y_2 - x_2)\ dy_2 dy_1.$$

The "test function" in this integral, $(y_1, y_2) \mapsto \rho(y_1) \cdot \eta(y_2 - x_2)$ has uniformly bounded $\mathcal{C}^1$ norm. Consequently the convergence in the space of distributions proves that the norm is uniformly bounded and so g_ϵ converges in the anisotropic norm $|\cdot|_{p,1}$. □

Lemma 4.2. *$\mathcal{I}_N$ has dimension $K\,|N|$.*

Proof. Let $\mathcal{J}_N$ denote the space of V-invariant distributions, i.e., the set of $D \in \mathcal{C}^\infty(M)'$ such that $D(Vh) = 0$ for all $h \in \mathcal{C}^\infty(M)$ and supported on $\mathcal{C}^\infty_N(M)$. It is known [10, Proposition 4.4] that $\mathcal{J}_N$ has dimension $K\,|N|$ and comprises of distributions of Sobolev order $\frac{1}{2}$. We know (Lemma 2.2) that $\iota\mathcal{I}_N$ is a subset of $\mathcal{C}^\infty(M)'$ and that (Lemma 4.1) $\iota\mathcal{I}_N$ contains $\mathcal{J}_N$. Consequently the dimension of $\mathcal{I}_N$ is equal to the dimension of $\mathcal{J}_N$. □

Lemma 4.3. *$\lambda^{-1/2}\mathcal{L}$ is an isometry on $\mathcal{I}_N$.*

Proof. Lemma 4.2 tells that $\mathcal{I}_N$ is finite dimensional and every norm is equivalent. Consequently this result follows from the known result [10, Proposition 4.8] even though the reference uses a different norm. □

Lemma 4.4. *There exists a set of $K\,|N|$ unit complex numbers $\{\mu_j\}$ such that the spectrum of $\mathcal{L}$ restricted to $\mathcal{I}_N$ is $\{\lambda^{-\frac{1}{2}}\mu_j\}_{j=1}^{K|N|}$, repeated according to algebraic multiplicity.*

Proof. This result follows from the fact that $\mathcal{L}$ is an isometry (Lemma 4.3) on the subspace $\mathcal{I}_N$ which has dimension $K\,|N|$ (Lemma 4.2). □

5. Zooming in

This section is devoted to the part of the argument which allows us to derive the full spectrum of $\mathcal{L} : \mathcal{B}^{p,q}_N(M) \to \mathcal{B}^{p,q}_N(M)$, $N \neq 0$ from the peripheral spectrum proven in Section 4. As before, $M = \Gamma\diagdown\mathbb{H}$ and $N \in \mathbb{Z} \setminus \{0\}$ are considered chosen and fixed for the entire section.

We will repeatedly take advantage of the lemma of Baladi & Tsujii [2, Lemma A.1] which tells us when the point spectrum of an operator considered on two different Banach spaces will coincide. We repeat here the full statement.

Lemma ([2, Lemma A.1])**.** *Let B be a separated topological linear space and let B_1 and B_2 be Banach spaces that are continuously embedded in B. Suppose further that there is a subspace $B_0 \subset B_1 \cap B_2$ that is dense both in the Banach spaces B_1 and B_2. Let $L : B \to B$ be a continuous linear map, which preserves the subspaces B_0, B_1 and B_2. Suppose that the restriction of L to B_1 and B_2 are bounded operators whose essential spectral radii are both strictly smaller than $\rho > 0$. Then the eigenvalues of $L : B_1 \to B_1$ and $L : B_2 \to B_2$ in $\{z \in \mathbb{C} : |z| > \rho\}$ coincide. Furthermore the corresponding generalized eigenspaces coincide and are contained in the intersection $B_1 \cap B_2$.*

In our setting B_0 is the space of $\mathcal{C}^\infty$ functions whilst B is the space of distributions (Lemma 2.2). The following tells us that the outer part of the spectrum is given by Lemma 4.4.

Lemma 5.1. *Let $p, q \in \mathbb{N}$. The spectrum of $\mathcal{L} : \mathcal{B}^{p,q}_N(M) \to \mathcal{B}^{p,q}_N(M)$, $N \neq 0$, restricted to $\{|z| > \lambda^{-1}\}$ is equal to the spectrum of $\mathcal{L}|_{\mathcal{I}_N}$.*

Proof. Suppose that z is an eigenvalue of $\mathcal{L} : \mathcal{B}^{p,q}_N(M) \to \mathcal{B}^{p,q}_N(M)$ and that $|z| > \lambda^{-1}$. Consequently there exists $h_z \in \mathcal{B}^{p,q}_N(M)$ such that $\mathcal{L}h_z = zh_z$. By Lemma 3.1,

$$\lambda V \mathcal{L} h_z = \lambda z (V h_z) = \mathcal{L}(V h_z).$$

This means that, either λz is an eigenvalue of $\mathcal{L} : \mathcal{B}^{p-1,q}_N(M) \to \mathcal{B}^{p-1,q}_N(M)$ or that $Vh_z = 0$. However $|\lambda z| > 1$ but the spectral radius of $\mathcal{L}$ is at most 1 (Lemma 3.7). This means that $Vh_z = 0$, i.e., $h_z \in \mathcal{I}_N$.

For the other direction, observe that Lemma 4.4 in particular implies that the spectrum of $\mathcal{L}|_{\mathcal{I}_N}$ is a subset of $\{|z| = \lambda^{-1/2}\}$. □

Lemma 5.2. *Let $p, q, k \in \mathbb{N}$, $p, q > k$. The spectrum of $\mathcal{L} : \mathcal{B}^{p,q}_N(M) \to \mathcal{B}^{p,q}_N(M)$, $N \neq 0$, restricted to $\{z : \lambda^{-(k+1)} < |z| \leq \lambda^{-k}\}$ is contained within $\{z : \lambda^k z \in \mathrm{Spec}(\mathcal{L}|_{\mathcal{I}_N})\}$.*

Proof. We take advantage of the fact that the point spectrum is independent on the Banach space [2, Lemma A.1]. We will prove, by induction, that for all $k \in \mathbb{N}_0$,

$$\text{if } z \in \mathrm{Spec}(\mathcal{L}) \text{ and } \lambda^{-(k+1)} < |z| \leq \lambda^{-k}, \quad \text{then } \lambda^k z \in \mathrm{Spec}(\mathcal{L}|_{\mathcal{I}_N}).$$

The case $k = 0$ is given by Lemma 5.1. We now suppose the statement is already proven for k. Suppose that z is an eigenvalue of $\mathcal{L}$ and that $\lambda^{-(k+2)} < |z| \leq \lambda^{-(k+1)}$. Let $h_z \neq 0$ be such that $\mathcal{L}h_z = zh_z$. By Lemma 3.1,

$$\lambda V \mathcal{L} h_z = \lambda z (V h_z) = \mathcal{L}(V h_z).$$

This means that, either λz is an eigenvalue of $\mathcal{L}$ or $Vh_z = 0$. In the second case, $h_z \in \ker(V)$ and so, by Lemma 5.1, $|z| > \lambda^{-1}$ but this contradicts the fact that $|z| \leq \lambda^{-(k+1)} \leq \lambda^{-1}$. This means that the first case is the only possibility. Consequently we know that $\lambda z \in \mathrm{Spec}(\mathcal{L})$ and $\lambda^{-(k+1)} < |\lambda z| \leq \lambda^{-k}$. We then apply the inductive hypothesis and conclude. □

Next we take advantage of the operator W in order to upgrade Lemma 5.2 to equality. The technique we will use has some similarity to the argument used for pseudo Anosov maps [8]. It is very different to the technique of Flaminio & Forni [10, A.3] who took advantage of a basis of Hermite functions to write a formal inverse of V and then further argue for the "iterated invariant distributions" (see also [13]).

Lemma 5.3. *Let $p, q \in \mathbb{N}$. The spectrum of $\mathcal{L} : \mathcal{B}^{p,q}_N(M) \to \mathcal{B}^{p,q}_N(M)$, $N \neq 0$, restricted to $\{z : \lambda^{-(k+1)} < |z| \leq \lambda^{-k}\}$ is equal to $\{z : \lambda^k z \in \mathrm{Spec}(\mathcal{L}|_{\mathcal{I}_N})\}$.*

Proof. In light of Lemma 5.2, we need only prove that z in the spectrum implies that $\lambda^{-1}z$ is also in the spectrum. For this we take advantage of the fact, established in Lemma 2.7, that $Y : \mathcal{B}^{p,q}_N(M) \to \mathcal{B}^{p,q+1}_N(M)$ is injective. Let $h_z \neq 0$ be such that $\mathcal{L}h_z = zh_z$. By Lemma 3.1,

$$\lambda^{-1} W \mathcal{L} h_z = \lambda^{-1} z (W h_z) = \mathcal{L}(W h_z).$$

Here again we take advantage of the fact that the point spectrum is independent on the Banach space [2, Lemma A.1]. By Lemma 2.7 $Wh_z \neq 0$ and so $\lambda^{-1}z$ is in the spectrum. □

Appendix A. Partially hyperbolic automorphisms

In this section we discuss the explicit form of partially hyperbolic automorphisms. For our present purposes we say that an automorphism $\Phi : M \to M$ is *partially hyperbolic with neutral centre* if the tangent bundle admits a splitting $TM = E_s \oplus E_c \oplus E_u$ into one dimensional sub-bundles such that: E_c corresponds to Z and Φ is an orientation-preserving isometry on this bundle; Φ is uniformly contracting on E_s and uniformly expanding on E_u. In this section, to match with the pertinent references, we work in *polarised* coordinates, i.e., $\mathbb{H}^{\mathrm{pol}}$ is equal to $\mathbb{R}^3$ with the group law

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy').$$

We recall the following result which describes the structure of such automorphisms on Heisenberg nil-manifolds.

Lemma A.1 (Shi [20,21])**.** *Let $M = \Gamma \diagdown \mathbb{H}^{\mathrm{pol}}$. The function $\Phi : M \to M$ is an automorphism, partially hyperbolic with neutral centre, if and only if, it has the form, $\Phi = \exp \circ \phi \circ \exp^{-1}$ where*[3]

$$\phi = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ \frac{ac}{2} + \ell & \frac{bd}{2} + m & 1 \end{pmatrix} \tag{7}$$

for some $a, b, c, d, \ell, m \in \mathbb{Z}$ such that $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has eigenvalues λ, λ^{-1} for some $\lambda > 1$. Equivalently,

$$\begin{aligned} \Phi(x, y, z) &= (ax + by, cx + dy, z + \tau(x, y)), \\ \tau(x, y) &= \tfrac{ac}{2}x^2 + bcxy + \tfrac{bd}{2}y^2 + \left(\tfrac{ac}{2} + \ell\right)x + \left(\tfrac{bd}{2} + m\right)y. \end{aligned} \tag{8}$$

Sketch of proof. For full details of these calculations consult the work of Shi [20,21], here we present an overview. There is a one-to-one correspondence between the automorphisms of the Lie algebra and the automorphisms of $\mathbb{H}$ [12, Proposition 1.21]. The Lie bracket must be preserved by any automorphism of the Lie algebra which gives several restrictions on the possible form. The matrix (8) represents the associated automorphism of the Lie algebra. Applying the exponential function, the formula of the automorphism on $\mathbb{H}$ can then be deduced. Further observing that the automorphism must preserve the lattice Γ and be an orientation preserving isometry in Z gives the final restrictions on the form of the automorphism. □

An automorphism associated to $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, the classic Anosov map of the torus, is given by Lemma A.1 with the coefficients $a = 2, b = c = d = 1, l = m = 0$.

Let $\Phi : M \to M$ be an automorphism, partially hyperbolic with neutral centre. I.e., $a, b, c, d, \ell, m \in \mathbb{Z}$ are fixed and satisfy the requirements detailed

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[3]We identify the Lie algebra with $\mathbb{R}^3$ using the basis $\{X, Y, Z\}$ where,

$$X = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), \quad Y = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right), \quad Z = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right).$$

in Lemma A.1. Let $\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}\beta\\ -\alpha\end{smallmatrix}\right)$ be the normalised eigenvectors of $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$. Without loss of generality, we suppose that $\alpha, \beta > 0$ are such that $\alpha^2+\beta^2 = 1$ and

$$\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right) = \lambda\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right), \quad \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix}\beta\\ -\alpha\end{smallmatrix}\right) = \lambda^{-1}\left(\begin{smallmatrix}\beta\\ -\alpha\end{smallmatrix}\right).$$

Let

$$V = \alpha X + \beta Y + \gamma Z, \quad W = -\beta X + \alpha Y + \gamma' Z$$

where

$$\gamma = \tfrac{1}{\lambda-1}\left(\alpha(\tfrac{ac}{2}+\ell) + \beta(\tfrac{bd}{2}+m)\right),$$
$$\gamma' = \tfrac{1}{1-\lambda^{-1}}\left(\beta(\tfrac{ac}{2}+\ell) - \alpha(\tfrac{bd}{2}+m)\right).$$

Lemma A.2. *Let V, W be defined as above. Then $[V,W] = Z$, $[V,Z] = [W,Z] = 0$. I.e., $\{V,W,Z\}$ is a Heisenberg frame. Moreover*

$$\Phi_* V = \lambda V, \quad \Phi_* W = \lambda^{-1} W.$$

Proof. The commutation relations follow from the commutation relations for X, Y, Z, together with the chosen normalization. The two equalities are verified using the matrix form of the automorphism of the Lie algebra (7). □

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