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Parabolic flows renormalized by partially hyperbolic maps

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Abstract
We consider nilflows on the Heisenberg nilmanifolds which are renormalized by partially hyperbolic automorphisms, i.e., parabolic flows on 3-dimensional manifolds which are renormalized by circle extensions of Anosov diffeomorphisms. The transfer operators associated to the renormalization maps, acting on anisotropic Sobolev spaces, are known to have good spectral properties (this relies on ideas which have some resemblance to representation theory but also apply to non-algebraic systems). The spectral information is used to describe the deviation of ergodic averages and solutions of the cohomological equation for the parabolic flow.

1 Introduction

Parabolic dynamical systems are roughly classified as a type of intermediate system: in hyperbolic systems orbits diverge exponentially, in elliptic systems there is no, or little to no, divergence and in parabolic systems the divergence of orbits of nearby points is polynomial in time. Examples of parabolic systems include horocycle flows on surfaces of negative curvature, flat flows on surfaces of higher genus and Heisenberg nilflows (see e.g., [14,15,19] respectively). One of the most interesting and fruitful features of these systems is the intimate connection they enjoy with certain hyperbolic systems; this connection is referred to as *renormalization*. Let $\Psi_t : M \to M$ denote the parabolic flow defined on a manifold M. The most concrete instance of renormalization, the case considered in this paper, is when there exists a map $F : M \to M$, with some degree of hyperbolicity, which conjugates the parabolic system in the following way,

$$F \circ \Psi_{\lambda t} = \Psi_t \circ F, \tag{1}$$

for all $t \in \mathbb{R}$ and $\lambda > 1$ fixed. (For a description of a more general scheme for renormalization, see [18].)

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Renormalization is very useful, perhaps even vital, to proving important properties of these parabolic systems. Two such properties are: (i) deviation of ergodic averages; (ii) regularity of solutions to the cohomological equation. Given $x \in M$, $t > 0$, the ergodic integral is defined, for $h : M \to \mathbb{R}$, as

$$H_{x,t}(h) = \int_0^t h \circ \Psi_r(x)\, dr. \tag{2}$$

Understanding the deviation of ergodic averages requires studying the behaviour, for $h : M \to \mathbb{R}$, of $H_{x,t}(h)$ as $t \to \infty$. Solutions to the cohomological equation are functions $g : M \to \mathbb{R}$ such that for a given $h : M \to \mathbb{R}$,

$$g \circ \Psi_t(x) - g(x) = \int_0^t h \circ \Psi_r(x)\, dr \tag{3}$$

for all $x \in M$, $t \geq 0$. It is then of interest to determine the regularity of the transfer function, g. (The infinitesimal version of the cohomological equation is given by $Wg = h$, where W denotes the vector field generating the flow Ψ_t.)

Notice that both the above quantities, (2) & (3), involve integrals over increasingly longer orbits. By exploiting the renormalization of the parabolic flow by a hyperbolic system, these integrals can be transformed into equivalent expressions involving integration over orbits of a fixed length. This is the elementary, yet crucial, observation that, for any $x \in M$, $t > 0$, $h \in \mathcal{C}^0(M)$, $k \in \mathbb{N}$,

$$\begin{aligned} H_{x,t}(h) &= \int_0^t h \circ F^{-k} \circ F^k \circ \Psi_r(x)\, dr \\ &= \int_0^t h \circ F^{-k} \circ \Psi_{\lambda^{-k} r} \circ F^k(x)\, dr \\ &= \int_0^{\lambda^{-k} t} \lambda^k h \circ F^{-k} \circ \Psi_u(x_k)\, du = H_{x_k, \lambda^{-k} t}(\widehat{F}^k h) \end{aligned} \tag{4}$$

where $x_k := F^k(x)$,

$$\widehat{F} : h \mapsto \lambda h \circ F^{-1} \tag{5}$$

is the transfer operator, and λ^{-1} denotes the contraction of F in the stable direction and is consequently the same factor which appears in the renormalization (1) relation (in general the leading term λ is replaced by the inverse of the stable Jacobian of F).

By considering renormalization on the bundle of invariant distributions over the moduli space and employing tools from representation theory and harmonic analysis, information on deviation of ergodic averages, and in some cases, results on regularity of solutions to the cohomological equation, were obtained for various different systems. This was done by Forni [17,19] in the case of area-preserving flows on surfaces of higher genus, Flaminio and Forni [14] in the case of the horocycle flow on hyperbolic surfaces, and again by Flaminio and Forni [15] in the case of Heisenberg nilflows. In the case of interval exchange transformations (closely related to area-preserving flows), Mousa et al. [24] were able to extend results from Forni [17] to a larger class of systems by considering renormalization based on Rauzy–Veech induction.

One of the most rewarding outcomes of the study of parabolic flows via renormalization was that it allowed for an explicit description of the obstructions to solving the cohomological equation (and obstructions to faster convergence of the ergodic integral) by extending beyond the space of functions and working instead in the space of distributions [14,18,19]. This strongly suggested that there exists a link between the distributional obstructions and the spectra of transfer operators of hyperbolic systems acting on carefully chosen anisotropic Banach spaces (for an overview of anisotropic spaces see [2] and references within).

The speculated connection between the distributions which arise studying parabolic flows and the distributions which arise studying hyperbolic systems was made solid in the work of Giulietti and Liverani [20]. They studied flows on the torus which are renormalized (in the sense of (1) but where λt is instead a more general function of x and t) by an Anosov map. The map in their setting need not be linear, it is merely required that the foliations are $\mathcal{C}^{1+\alpha}$. The key idea is to study the spectrum of the transfer operator associated to the hyperbolic map on an appropriate choice of anisotropic Banach space and then use this spectral information to answer the pertinent questions for the parabolic flow. They show that indeed the eigenvalues of this operator correspond to the deviation spectrum of the ergodic averages as well as to the obstructions to solving the cohomological equation. The authors propose this method as a way of extending known results to more general settings, for example, beyond the smooth class of systems that have been studied using representation theory.

Formalizing this connection has inspired a new wave of work on renormalization. This modified renormalization technique was applied by Faure et al. [10] to the case of area-preserving flows on surfaces of higher genus renormalized by pseudo Anosov maps. The authors are able to link the obstructions to topological properties of the surface. Adam [1] applied this technique to the case of the horocycle flow on a surface of variable negative curvature renormalized by a geodesic flow, thus obtaining a partial extension of results which were previously only obtainable for constant curvature.

In the present work we study 3-dimensional parabolic flows (Heisenberg nilflows) which are renormalized by partially hyperbolic maps (definitions given in Sect. 2, additional details in Sect. 6). The main results of this work are contained in three theorems which we will now summarise. The precise statements appear in Sects. 3, 4, 5 respectively once the required notation has been introduced. Section 2 is devoted to the description of partially hyperbolic maps and the spectral results concerning the associated transfer operators, derived by Faure and Tsujii [12]. The anisotropic Sobolev spaces used are chosen carefully so that they are both large enough to provide useful information as well as precise enough to extract explicit spectral data. As a first step we prove that

> *The transfer operator $\widehat{F}$, considered as an operator on appropriate anisotropic spaces, admits a countable spectral decomposition* (*Theorem* 1*, Sect.* 3).

This spectral decomposition is slightly unconventional in that it involves a countable sum and also that the sum converges but not necessarily in the obvious operator norm. Nevertheless this suffices for all present purposes. Section 4 is devoted to the study of parabolic flows under the assumption that they are renormalized, in the sense of (1), by a map which satisfies the spectral properties given by the theorems proved in Sects. 2 and 3. The method of studying the behaviour of ergodic averages, $H_{x,t}(h)$ relies on two key parts. First we show that the ergodic integrals are well approximated by elements of the dual of our anisotropic Banach space. Then we compose the observable over long orbits of the flow, i.e., large values of t. The renormalization gives us a method by which we can study modified ergodic integrals that contain the same information but are taken over shorter orbits or, equivalently, over shorter intervals of time.

> *If a parabolic flow is renormalized by a partially hyperbolic map which satisfies the spectral conclusions of Theorem* 1 *then we obtain a precise description of the deviation of ergodic averages* (*Theorem* 2*, Sect.* 4).

We take this opportunity to describe the argument used to estimate the deviation of ergodic averages in a way that is not dependent on the exact space of distributions used in the present application and is consequently applicable to any other setting where similar spectral results are available for another Banach (or indeed Hilbert) space which satisfies a modest compatibility requirement (Lemma 2).

If a parabolic flow is renormalized by a partially hyperbolic map which satisfies the spectral conclusions of Theorem 1, then the solutions of the cohomological equation exist and are equal to the uniform limit of an explicit quantity (*Theorem 3, Sect. 5*).

This precise formula for the solutions to the cohomological equation allows for the possibility to further study the regularity of the solutions. A weight of evidence suggests that the natural extension of these ideas, following the notion of Giulietti and Liverani [20] would consequently provide information regarding the regularity question. In a different direction, Faure and Tsujii [12, §1.4] proved similar spectral results in the extension to the Grassmannian which should, building on the framework established in the present work, allow for all the analogous argument to be carried out in the case of flows along stable manifolds of partially hyperbolic maps like the ones studied in the present work but without a regularity condition on the invariant foliation.

The setting we consider is not more general than the one studied by Flaminio and Forni [15]. Indeed our setting is less general in that we only consider the nilflows which correspond to periodic points of renormalization. Theorems 2 and 3, in terms of estimates on ergodic averages and solutions of the cohomological equation, mirror precisely what was previously shown[15]. The contribution of this work is the advancement of the understanding of the fundamental and subtle connection between the asymptotic properties of nilflows and the spectral properties of partially hyperbolic dynamical systems, the refinement of the ideas for connecting parabolic dynamics and anisotropic spaces which was introduced by Giulietti and Liverani [20], and most particularly, the exploration of generalized methods based on representation theory that allow for more general and explicit results. One notable outcome of this work is the development of the correspondence between the results and methods for prequantum transfer operators (in specific cases the prequantum cat map) and the aforementioned topics of interest regarding certain parabolic systems. As far as the authors know, these methods are the only ones currently available that are able to give information about the inner bands of the spectrum. Seeing beyond the countable number of eigenvalues in the outer band is essential, particularly when studying the cohomological equation. Consequently, we expect that the spectral results and methods in Faure and Tsujii [11,13] for contact Anosov flows could prove useful for the study of properties of the horocycle flow. The fact that these methods are essential for obtaining optimal and general results is coherent given that the argument of Faure and Tsujii [12] can be seen as an extension of the representation theoretic argument to the non-linear (non-algebraic) setting; in particular, the Bargmann Transform plays a central role in the definition of the norm.

2 Circle extensions of symplectic Anosov maps

In this section we describe the setting and the relevant spectral results from the work of Faure and Tsujii [12]. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a linear Anosov diffeomorphism on the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ (f preserves the canonical area form ω). Since f is Anosov there exists an f-invariant smooth decomposition of tangent space $T_x\mathbb{T}^2 = E_u(x) \oplus E_s(x)$ and a constant $\lambda > 1$ such that $\det Df|_{E_u} = \lambda$ and $\det Df|_{E_s} = \lambda^{-1}$ (this notation denotes the Jacobian restricted to the respective subspace). The one-dimensional unitary group $U(1)$ is the multiplicative group of complex numbers of the form $e^{i\theta}$, $\theta \in \mathbb{R}$. Let M denote the $U(1)$-principal bundle over $\mathbb{T}^2$ and denote by $\pi : M \to \mathbb{T}^2$ the corresponding projection. The main object of interest is the map (called the lift),

$$F : M \to M$$

which satisfies $(\pi \circ F)(p) = (f \circ \pi)(p)$, is equivariant with respect to the $U(1)$ action and preserves the connection associated to the symplectic area form. The map $F : M \to M$ is partially hyperbolic in the sense that tangent bundle splits into three subbundles, one exponentially contracting, one exponentially expanding and a third, corresponding to the fibres of the bundle M, in which the behaviour is neutral. The stable and unstable bundles are not jointly integrable in a precise way since F preserves the canonical contact form [12, Remark 1.2.7]. Since f is a toral automorphism, the lift F is an automorphism of M. The above conditions suffice to satisfy the assumption of Faure and Tsujii [12].[1] On the other hand partially hyperbolic automorphisms of Heisenberg nilmanifolds satisfy these assumption (further details of this and the associated nilflows are given in Sect. 6). Moreover any such principal bundle over a torus is of the form of such a nilmanifold [25].

Corresponding to the quantity we require for parabolic flows (5), we consider the transfer operator $\widehat{F} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ defined as[2]

$$\widehat{F} : h \to \lambda\, h \circ F^{-1}. \tag{6}$$

The equivariance property of F in the fibres means that there is the following natural and useful decomposition of $\mathcal{C}^\infty(M)$ (the Nth element of the decomposition corresponds to the Nth Fourier mode in the fibre). For $N \in \mathbb{Z}$ let

$$\mathcal{C}_N^\infty(M) := \left\{ h \in \mathcal{C}^\infty(M) \mid h(e^{i\theta} p) = e^{iN\theta} h(p), \text{for all } p \in M, \theta \in \mathbb{R} \right\}$$

then we can define $\widehat{F}_N : \mathcal{C}_N^\infty(M) \to \mathcal{C}_N^\infty(M)$ as

$$\widehat{F}_N := \widehat{F}|_{\mathcal{C}_N^\infty(M)}\,. \tag{7}$$

Throughout this section we assume the above setting. We now recall, restricted to this setting, the pertinent results proven by Faure and Tsujii [12]. Let $\mathcal{D}'_N(M)$ denote the dual of $\mathcal{C}_N^\infty(M)$.

[12, Theorem 1.3.1] For any $N \in \mathbb{Z}$ and sufficiently large $r > 1$ there exists a Hilbert space $\mathcal{H}_N^r(M)$, satisfying

$$\mathcal{C}_N^\infty(M) \subset \mathcal{H}_N^r(M) \subset \mathcal{D}'_N(M),$$

such that $\widehat{F}_N$ extends to a bounded operator $\widehat{F}_N : \mathcal{H}_N^r(M) \to \mathcal{H}_N^r(M)$ with spectral radius bounded above by λ and essential spectral radius bounded above by $\lambda^{-(r-1)}$. Moreover 1 is in the spectrum of $\widehat{F}_0$, has multiplicity 1 and the associated eigendistribution corresponds to the invariant measure.

The Hilbert spaces $\mathcal{H}_N^r(M)$ are often called *anisotropic Sobolev spaces*.[3] There are various possibilities for such space but for this present work we use the ones constructed in the cited

[1] In the reference the terminology "prequantum transfer operator for symplectic Anosov diffeomorphism" is used for this same object.

[2] This equates to the choice of potential $V = \ln \lambda$ in the reference [12].

[3] This work fits into a diverse and ongoing line of research developing and using functional analytic methods for dynamical systems (see e.g., [1,4–8,10,11,13,20,21] and particularly [2]).

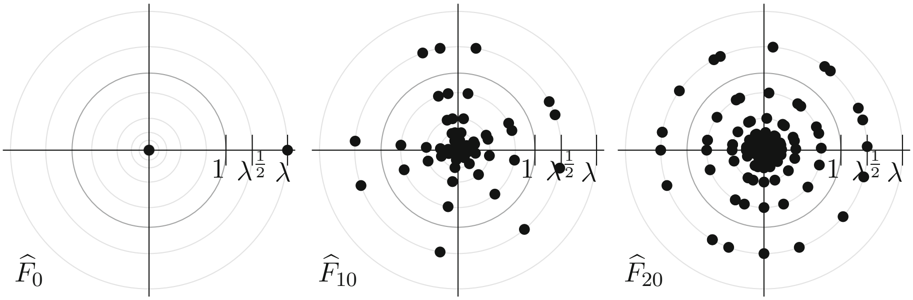

Fig. 1 Graphical representation of the spectral result which is described by [12, Theorem 1.3.4]. The eigenvalues lie on circles of radius λ, $\lambda^{1/2}$, $\lambda^{-1/2}$, $\lambda^{-3/2}$, etc. Where exactly the eigenvalues lie on each circle is merely representative in this figure but, in theory, it is possible to explicitly calculate them for the linear case [9].

reference and moreover we take advantage of the fact that, for convenience, we can take them so that they are aligned with the stable foliation. Precise details of this are postponed until Sect. 3. It is possible to obtain much more detailed information about the spectrum in this setting. Specifically, the eigenvalues of the transfer operators occur in narrow bands centred on the circles of radius $r_k = \lambda^{\frac{1}{2}-k}$ (see Fig. 1).

[12, Theorem 1.3.4] For any $\epsilon > 0$, there exists $C_\epsilon > 0$, $N_\epsilon \geq 1$, such that, for any $|N| \geq N_\epsilon$, the spectrum of $\widehat{F}_N : \mathcal{H}^r_N(M) \to \mathcal{H}^r_N(M)$ is contained within the union of bands

$$\bigcup_{k\geq 0} \{z \in \mathbb{C} : |z| \in (r_k - \epsilon, r_k + \epsilon)\}.$$

Moreover, when $r_k + \epsilon < \eta < r_{k-1} - \epsilon$ for some $\eta > 0$, $k \geq 1$ then

$$\sup_{|z|=\eta} \|(z - \widehat{F}_N)^{-1}\| \leq C_\epsilon.$$

Crucially, for our subsequent use, the resolvent bound included in this theorem is independent of $N \in \mathbb{Z}$.

Remark 1 While [12, Theorem 1.3.4] is presented for a larger class of systems, we state the above Theorem for the linear case. The general results of Faure and Tsujii [12] provide a framework for future extensions of the results in Sects. 4 and 5 to a larger class of parabolic flows.

Remark 2 The spectral results we use, concerning partially hyperbolic maps, are available in higher dimensions, however the method we use in Sect. 4 relies on the stable bundle being one dimensional (corresponding to the flow lines of the parabolic flow) in order to connect ergodic integrals to the norm.

Remark 3 In this present work we restrict ourselves to the case where the unstable Jacobian is smooth. A result of Hurder and Katok [23] says that any smooth, volume-preserving Anosov map of $\mathbb{T}^2$ with $\mathcal{C}^{1+\omega}$ invariant foliations is smoothly conjugate to a linear Anosov toral automorphism. (The notation $\mathcal{C}^{1+\omega}$ means differentiable with the derivative being in the class given by the modulus of continuity $\omega(s) = o(s\,|\log(s)|)$.) Consequently, since we already require the regularity of the unstable Jacobian, without loss of generality, we consider the linear case. This leaves open the more general case since typically [22] the invariant foliations for an Anosov map of the 2-torus are differentiable with Hölder derivative but not better. This case could be tackled using the technique of extending to the Grassmannian in just the same ways as Giulietti and Liverani [20] in the case of toral flows. Unfortunately the analogous spectral results are not yet available even though very close and suggestive results are [12, §1.4] (the Grassmannian bundle used to study the transfer operator can be centred on either the stable or unstable bundle and the available result is not the case required for the present problem).

In the setting described in this section, under the assumption that the outer band is isolated from the other bands, an estimate on the number of eigenvalues is obtained. [12, Theorem 1.3.8] Counting multiplicity, the number of eigenvalues of $\widehat{F}_N$ on the outer band $\{|z| \in (\lambda^{1/2} - \epsilon, \lambda^{1/2} + \epsilon)\}$ is equal to $N \int_M \omega + \mathrm{O}(1)$.
The spectrum of the transfer operator for the specific case of linear hyperbolic maps on $\mathbb{T}^2$ has been studied in Faure [9], and the resonances are shown to precisely lie on circles of radius r_k. In this specific setting, $\hat{F}_N$ has exactly N eigenvalues (counting multiplicity) on each circle of radius r_k. Moreover, it is shown [9, Thm. 1] that the resonances on different circles have the same phases and differ only in modulus by some power of λ. We choose to base this work on the result which is more general than required for the present setting because the primary aim is to assess the possibilities and prepare the technology for the more general case.

Remark 4 Observe that the outer band of the spectrum lies outside of the unit circle and, since $\widehat{F} = \sum_{N\in\mathbb{Z}} \widehat{F}_N$, we will inevitably have to manage a countable[4] number of eigenvalues if we want to obtain explicit properties from the spectral information.

3 Additional functional analytic information

This section is devoted to the details which are required in order to use the results from Sect. 2 for the estimates related to the deviation of ergodic averages and to the cohomological equation problem. First impressions may suggest serious issues using the spectral information from $\widehat{F}_N$ as the spectrum of $\widehat{F} = \sum_{N\in\mathbb{Z}} \widehat{F}_N$ is dense [12, Thm. 1.3.11] on each circle of radius $\lambda^{\frac{1}{2}-k}, k \in \{0, 1, 2, \ldots\}$. Nevertheless there is sufficient structure in order to deduce useful and explicit information; in particular the spectrum of $\widehat{F}_N$ can be used to produce a (countable) spectral decomposition of the operator $\widehat{F}$ on a relevant space. Additionally, adequate estimates indicating that this norm is applicable to estimating smooth ergodic integrals must be obtained.

Since $\mathcal{C}^\infty(M) = \bigoplus_{N\in\mathbb{Z}} \mathcal{C}^\infty_N(M)$ it is convenient to define, for $\kappa > 1$ (to be fixed shortly), the norm

$$\|h\|_{\mathcal{H}} := \sup_{N\in\mathbb{Z}} |N|^\kappa \, \|h_N\|_{\mathcal{H}^r_N(M)} \tag{8}$$

where $h = \sum_N h_N$ and $h_N \in \mathcal{C}^\infty_N(M)$ for each $N \in \mathbb{Z}$. Additionally we fix $r > 0$ sufficiently large, as required by [12, Thm. 1.3.1], and subsequently suppress it in the notation.

Lemma 1 *If $h \in \mathcal{C}^\infty_N(M)$ then $\|h\|_{\mathcal{H}} < \infty$.*

4 In [20, Rem. 2.15], it was written that the case of countable deviation spectrum corresponds to flows whereas the case of finite spectrum corresponds to maps. Instead we note here that the presence of a neutral direction (e.g., in the present work as well as when studying flows) is the distinguishing factor which determines the unavoidable presence of a countable number of eigenvalues in the problem.

Proof Since $\mathcal{H}$ is a generalised Sobolev space with an anisotropic weight that is bounded polynomially in frequency we know [12, (1.6.5)] there exists $\nu \in \mathbb{N}$ (indeed any ν sufficiently large suffices) such that, for any $h \in \mathcal{C}_N^\infty(M)$, there exists $C_\nu > 0$ such that $\|h_N\|_{\mathcal{H}_N^r(M)} \leq C_\nu |N|^{-\nu}$ for all $N \in \mathbb{N}$. This means, as long as ν is chosen sufficiently large, the supremum in the definition of the norm is finite. □

We define the weaker norm

$$\|h\|_{\widetilde{\mathcal{H}}} := \sup_{N\in\mathbb{Z}} |N|^{\kappa-1} \|h_N\|_{\mathcal{H}_N^r(N)} .$$

The two Banach spaces $\mathcal{H}(M) \subset \widetilde{\mathcal{H}}(M)$ are then defined as the completion of $\mathcal{C}^\infty(M)$ with respect to, respectively, the norms $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\widetilde{\mathcal{H}}}$.

Theorem 1 *For all $\eta > 0$ there exists $r > 0$ and, for each $N \in \mathbb{Z}$ there exists $K_N \in \mathbb{N}$ and a set of eigenvalues $\{\xi_{N,j}\}_{j=1}^{K_N}$, projectors $\{P_{N,j}\}_{j=1}^{K_N}$ and nilpotents $\{Q_{N,j}\}_{j=1}^{K_N}$ such that the operators $\widehat{F}_N : \mathcal{H}_N^r(P) \to \mathcal{H}_N^r(P)$ satisfy the decomposition*

$$\widehat{F}_N = \sum_{j=1}^{K_N} \xi_{N,j} P_{N,j} + Q_{N,j} + \widehat{F}_N P_{N,0}. \tag{9}$$

The projectors and nilpotents satisfy the commutation relations: $P_{N,j} P_{N,k} = \delta_{j,k}$, $P_{N,j} Q_{N,k} = Q_{N,k} P_{N,j} = \delta_{j,k} Q_{N,k}$, $Q_{N,k} Q_{N,j} = \delta_{k,j} Q_{N,k}^2$ and $P_{N,0}$ is the projector corresponding to the part of the spectrum contained within $\{|z| \leq \eta\}$. Viewing the operators defined on $\mathcal{H}_N^r(M)$ as operators on $\mathcal{H}$, the operator $\widehat{F} : \mathcal{H} \to \widetilde{\mathcal{H}}$ satisfies the decomposition

$$\widehat{F} = \sum_{N\in\mathbb{Z}} \sum_{j=1}^{K_N} \xi_{N,j} P_{N,j} + Q_{N,j} + \widehat{F} P_0 \tag{10}$$

where $P_0 = \sum_{N\in\mathbb{Z}} P_{N,0}$. Moreover: (1) *there are only a finite number of eigenvalues $\xi_{N,j}$ with absolute value greater than $\lambda^{\frac{1}{2}} + \epsilon$;* (2) *there exists $C_\eta > 0$ such that, for all $n \in \mathbb{N}$,*

$$\|\widehat{F}^n P_0\|_{\widetilde{\mathcal{H}}} \leq \eta^n C_\eta.$$

The above spectral information corresponds to the separation of the spectrum into the part outside of the circle $|z| = \eta$ and the remainder inside the circle $|z| = \eta$. We refer to η as the magnitude of the remainder term.

Remark 5 Viewing the operator as an operator acting from a stronger to a weaker space is a convenience since this result suffices entirely for our present purposes and requires relatively weak assumptions. There is some similarity to the idea which can be used for flows [7] when oscillatory cancellation results are not as strong as could be desired. However in present work the motivation is different in the sense that good oscillatory cancellation estimates are already available [12] but now we wish to go deeper into the spectrum.

Remark 6 In reality, since our case reduces to the case of studying lifts of toral automorphisms (i.e., we are in the algebraic case), we know [9] that this "finite number of eigenvalues" in the above statement is actually a singleton consisting only of the eigenvalue at λ which corresponds to the measure of maximal entropy (in the linear case this coincides with the SRB measure).

Proof of Theorem 1 Choosing $\epsilon > 0$ small and r large (observe that $\epsilon > 0$ and $r > 0$ in [12, Theorem 1.3.4] are independent of N) the quasi compactness result of [12, Theorem 1.3.1] implies immediately that we have the spectral decomposition (9) such that the finite sum of terms covers all of the spectrum outside of the set $|z| \leq \eta$. Note that the order of the nilpotents is not greater than K_N. Choosing $\eta > 0$ slightly smaller if required, we can assume that $r_k + \epsilon < \eta < r_{k-1} - \epsilon$ and consequently $P_{N,0} = -\frac{1}{2\pi i} \int_{|z|=\eta} R_N(z)\, dz$ where $R_N(z) := (z - \widehat{F}_N)^{-1}$ is the resolvent operator.

In order to meaningfully consider iterates we need to establish control on $\|\widehat{F}_N^n P_{N,0}\|$. According to the standard formula in holomorphic functional calculus, for all $n \in \mathbb{N}$,

$$\widehat{F}_N^n P_{N,0} = -\frac{1}{2\pi i} \int_{|z|=\eta} z^n R_N(z)\, dz$$

and so the (uniform in N) control on $\sup_{|z|=\eta} \|R(z)\|$ given by Theorem [12, Theorem 1.3.4] implies that, as an operator $\widehat{F}_N^n P_{N,0} : \mathcal{H}_N^r(M) \to \mathcal{H}_N^r(M)$,

$$\|\widehat{F}_N^n P_{N,0}\| \leq \eta^n C_\epsilon.$$

The uniformity, in N, of this estimate means that the same estimate holds for $\|\widehat{F}^n P_0\|_{\mathcal{H}}$.

This is not the only issue related to the spectral representation having a countable but not finite sum: we must deal with the fact that K_N is not bounded but grows with $|N|$. Here we use the fact that we consider the full decomposition as an operator between a stronger and weaker space. We already know that K_N is bounded above by a constant which is proportional to $|N|$ (and by the number of bands being considered but this is constant once $\eta > 0$ is fixed). This, by a simple estimate of the sum and the definition of the $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ norms guarantees that the countable sum converges. □

The final functional analytic ingredient is to show that certain objects are elements of the dual and obtain good bounds for these. This is a notion of compatibility of the norm and is essential in order to use the spectral results to study the pertinent questions related to nilflows. (Such an estimate corresponds to [1, Lem. 5.11] and [20, Lem. B.4].) Given a finite length piece of stable manifold $\gamma \subset M$ (stable manifold of $F : M \to M$) we define, for any $\varphi \in \mathcal{C}_0^q(\gamma)$, $h \in \mathcal{C}^\infty(M)$,

$$H_\varphi(h) = \int_\gamma \varphi \cdot h. \tag{11}$$

Lemma 2 *For any $\nu \in \mathbb{N}$ there exists $C_\nu > 0$ such that, for all $h \in \mathcal{C}^\infty(M)$, stable curve γ of unit length, $\varphi \in \mathcal{C}_0^\nu(\gamma)$,*

$$\left|H_\varphi(h)\right| \leq C_\nu \|h\|_{\mathcal{H}} \|\varphi\|_{\mathcal{C}^\nu(\gamma)}.$$

In order to prove the lemma we first recall the pertinent details of the definition of the anisotropic Sobolev spaces and then we prove (Lemma 3) a local version of Lemma 2. Finally we show how this estimate suffices for the full estimate.

Using charts to work locally and observing that for each fixed N the behaviour in the neutral direction is predetermined, the definition of the norm [12, Def. 6.4.1] on the manifold is reduced to defining[5] the norm for functions in $\mathcal{C}^\infty(\mathbb{R}^2)$. As mentioned earlier, we can and do take the choice of using for the definition of the norm local coordinates which coincide with the stable foliation. The connection between the *local data* [12, (6.3.1)] used for defining

[5] The reference allows for higher dimension but here we restrict to the case ($d = 1$) as required in this present work. Although the reference uses interchangeable N and $\hbar$ where $\hbar = \frac{1}{2\pi N}$ here we systematically use N.

the norm and the original observable is the natural one [12, Prop. 6.3.2] using a fixed (for each N partition of unity) partition of unity [12, Prop. 6.2.2]. It is important to note that for each N an N^{th} level atlas of charts is chosen and fixed once and for all. These atlases depend on $|N|$ and the size of the charts is of order $|N|^{-1}$. The norm is therefore determined by this fixed set of atlases and partitions of unity.

The Bargmann transform $\mathcal{B}_N : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^4)$ is defined [12, Def. 3.1.1] initially on Schwartz space $\mathcal{S}(\mathbb{R}^2)$ as $(\mathcal{B}_N h)(x, \xi) = \int \overline{\phi_{x,\xi}}(y) \cdot h(y)\, dy$ where the Bargmann kernel is defined as[6]

$$\phi_{x,\xi}(y) := 4\sqrt{N} \exp\left(\pi i N \xi \cdot (2y - x) - 2\pi N |y - x|^2\right),$$

and then shown [12, Lem. 3.1.2] to extend to $L^2(\mathbb{R}^2)$.

The escape function is $\mathcal{W}_N^r(x, \xi) := W_N^r(\zeta_p, \zeta_q)$. Here (ζ_p, ζ_q) are the coordinates [12, (4.2.5)] in phase space (ξ) which are called *normal coordinates* [12, Prop. 2.2.6] and correspond to the stable and unstable dynamics. On the other hand [12, Def. 3.3.2] $W_N^r(\zeta_p, \zeta_q) := W^r(\sqrt{2\pi N}\zeta_p, \sqrt{2\pi N}\zeta_q)$ is the anisotropic weight function based on the fixed function $W^r(\zeta_p, \zeta_q)$ which is defined [12, §3.3] to be smooth and, in particular, such that $W^r(\zeta_p, 0) \approx |\zeta_p|^{-r}$ and $W^r(0, \zeta_q) \approx |\zeta_q|^r$ when $|\zeta_p|, |\zeta_q|$ are large.

Using these two ingredients, the norm (the local version used within charts) is defined [12, Def. 4.4.1] as

$$\|h\|_{\mathcal{H}_N^r(\mathbb{R}^2)} := \left\|\mathcal{W}_N^r \cdot \mathcal{B}_N h\right\|_{L^2(\mathbb{R}^4)}.$$

The full norm on M is then defined by choosing local charts and using $\|\cdot\|_{\mathcal{H}_N^r(\mathbb{R}^2)}$ within those charts. I.e., (as per [12, Def. 6.4.1])

$$\|h\|_{\mathcal{H}_N^r(M)} := \left(\sum_{j=1}^{I_N} \|h_j\|^2_{\mathcal{H}_N^r(\mathbb{R}^2)}\right)^{\frac{1}{2}} \tag{12}$$

where h_j is the local data [12, (6.3.1)] (i.e., h in local coordinates for a given chart indexed by j) and I_N is the number of local charts chosen for each N.

Lemma 3 *For any $\nu \in \mathbb{N}$ there exists $C_\nu > 0$ such that, for all $N \in \mathbb{Z}$, $h \in \mathcal{C}_N^\infty(M)$, stable curve γ of unit length and $\varphi \in \mathcal{C}_0^\nu(\gamma)$,*

$$\left|H_\varphi(h_j)\right| \leq C_\nu \left\|h_j\right\|_{\mathcal{H}_N^r(\mathbb{R}^2)} \|\varphi\|_{\mathcal{C}^\nu(\gamma)}$$

where h_j is the local data corresponding to the chart identified by j in the N^{th} level atlas of charts.

Proof Working in local coordinates [12, (6.3.1)] we are considering the integral over $\tilde{\gamma} \subset \mathbb{R}^2$ where $\tilde{\gamma}$ is the projection of $\gamma \subset M$ onto $\mathbb{R}^2$. Note that the local data h_j is supported in a single chart and so, without needing to truncate γ or φ, we are working locally. To ease notation we now write h instead of h_j for the local data. We will estimate $|\tilde{\varphi}(h)|$ where $\tilde{\varphi}(h) = \int_{\tilde{\gamma}} \varphi \cdot h$. We recall the property of the Bargmann transform $\mathcal{B}_N^* \mathcal{B}_N = id$, shown in [12, Prop. 3.1.4] (note that $\mathcal{B}_N \mathcal{B}_N^*$ is not the identity but instead the orthogonal projection onto

[6] The primary reference [12] uses two operators, $\mathcal{B}_\hbar$ and $\mathcal{B}_x$. The first is defined as the Bargmann transform with a kernel similar to above (but slightly different scaling) and then the second is a scaling [12, (4.2.7)], of the first as $\mathcal{B}_x := \tilde{\sigma}^{-1} \circ \mathcal{B}_\hbar \circ \sigma$ where $\sigma h(x) := 2^{-\frac{1}{4}} h(2^{-\frac{1}{2}} x)$ and $\tilde{\sigma} v(x, \xi) := v(2^{-\frac{1}{2}} x, 2^{\frac{1}{2}} \xi)$. For our present purposes it makes sense to work directly with the required Bargmann transform which we denote $\mathcal{B}_N$ and which corresponds precisely to $\mathcal{B}_x$ of the reference.

the image of $\mathcal{B}_N$ in $L^2(\mathbb{R}^4)$). We also recall the escape function [12, Def. 3.3.2]. Supposing for a moment that $\tilde{\varphi} \in L^2(\mathbb{R}^2)$ and not a distribution, we may write

$$|\tilde{\varphi}(h)| = \langle \tilde{\varphi}, \mathcal{B}_N^* \mathcal{B}_N h \rangle_{L^2(\mathbb{R}^2)} = \left\langle \frac{1}{\mathcal{W}_N^r} \cdot \mathcal{B}_N \tilde{\varphi}, \mathcal{W}_N^r \cdot \mathcal{B}_N h \right\rangle_{L^2(\mathbb{R}^4)}.$$

By definition of the norm, $\|h\|_{\mathcal{H}_N^r(\mathbb{R}^2)} = \left\| \mathcal{W}_N^r \cdot \mathcal{B}_N h \right\|_{L^2(\mathbb{R}^4)}$ and so this would imply that

$$|\tilde{\varphi}(h)| \leq \| \tfrac{1}{\mathcal{W}_N^r} \cdot \mathcal{B}_N \tilde{\varphi} \|_{L^2(\mathbb{R}^4)} \, \|h\|_{\mathcal{H}_N^r(\mathbb{R}^2)} \, .$$

Consequently, in order to complete the estimate of the lemma, it suffices to estimate $\| \frac{1}{\mathcal{W}_N^r} \cdot \mathcal{B}_N \tilde{\varphi} \|_{L^2(\mathbb{R}^4)}$. That this estimate holds depends crucially on the anisotropic nature of the space (which is encoded in $\mathcal{W}_N^r$) and that integrating along a piece of stable curve identifies a preferred direction.

Here we take advantage of the fact that we have chosen to work with a norm defined in terms of coordinates [12, (4.2.5)] that coincide with the stable direction, i.e., that $y = (y_p, y_q)$ and that $\gamma \subset \{(y_p, y_q) : y_p \in \mathbb{R}, y_q = 0\}$. We calculate that $(\mathcal{B}_N \tilde{\varphi})(x, \xi)$ is equal to

$$\begin{aligned} &4\sqrt{N} \int \varphi(y_p) \exp \left(\pi i N \begin{pmatrix} \xi_p \\ \xi_q \end{pmatrix} \cdot \begin{pmatrix} 2y_q - x_p \\ -x_q \end{pmatrix} - 2\pi N \left| \begin{pmatrix} y_p - x_p \\ -x_q \end{pmatrix} \right|^2 \right) \, dy_p \\ &\quad = 4\sqrt{N} \int \varphi(y_p) \exp \left(\pi i N \xi_p \left(2y_p - x_p\right) - 2\pi N \left| y_p - x_p \right|^2 \right) \, dy_p \\ &\qquad \times \exp \left(-\pi i N \xi_q x_q - \pi N {x_q}^2 \right) \end{aligned}$$

The smoothness of φ, together with integration by parts along the curve γ (i.e., in coordinate y_p), implies a bound of $\left| N \xi_p \right|^{-\nu}$. Coupling this with basic estimates we obtain

$$|(\mathcal{B}_N \tilde{\varphi})(x, \xi)| \leq C_\nu \exp \left(-\pi \, |N| (x_p^2 + x_q^2) \right) \left| N \xi_p \right|^{-\nu} .$$

To finish we recall that, because of the anisotropic estimates available for the escape function [12, §3.3], $\frac{1}{\mathcal{W}_N^r}(x, \xi) \leq C \left(\left| \xi_p \right|^r |N|^{\frac{r}{2}} + \left| \xi_q \right|^{-r} |N|^{-\frac{r}{2}} \right)$. We require $\nu < r$ and thus, by multiplying the two estimates, we have shown that $\| \frac{1}{\mathcal{W}_N^r} \cdot \mathcal{B}_x \tilde{\varphi} \|_{L^2(\mathbb{R}^4)}$ is bounded, independently of N. □

Proof of Lemma 2 We now use the estimate of Lemma 3 in order to prove the main lemma. Firstly we fix N and suppose that $h \in \mathcal{H}_N^r(M)$. In order to obtain an estimate for $\left| H_\varphi(h) \right|$ we must sum over the contribution from each $\left| H_\varphi(h_j) \right|$ where the h_j are the local data for h on the chosen N^{th} level atlas of local charts on M. The key observation is that the factor that could be expected to occur due to the smaller and smaller local charts is already built into the definition of the norm and that the norm is defined in terms of summing over the contribution from each local data (12). This means that

$$\left| H_\varphi(h) \right| \leq C_\nu \, \|h\|_{\mathcal{H}_N^r(M)} \, \|\varphi\|_{\mathcal{C}^\nu(\gamma)}$$

since the estimate of Lemma 3 is independent of N. To finish the estimate we need to consider the contribution for all frequencies, i.e., for all N, and this converges without difficulty due to the choice of definition of $\|\cdot\|_{\mathcal{H}}$ (8) with appropriate κ to guarantee that the sum converges. □

4 Deviation of ergodic averages

In this section we use the information described in Sect. 3, in particular Theorem 1, to give a precise description of the deviation of ergodic averages for the parabolic flow. A similar concept for the study of solutions of the cohomological equation is postponed until Sect. 5. The standing assumption of this section is that we have a flow Ψ_t which is renormalized (1) by a partially hyperbolic map F. We further require that the transfer operator associated to F has a spectral decomposition as given in the conclusion of Theorem 1 with respect to a function space (in the extended sense) which satisfies the conclusion of Lemma 2. This section is completely independent of the precise details of the function space being used, as long as it satisfies the above mentioned properties.

In the first part of this section we will estimate precisely the ergodic integral (2) $H_{x,t}(h)$ and prove the following.

Theorem 2 *Let $\epsilon > 0$, $\beta < 1$. Let $\{\xi_j\}_{j=1}^{\infty}$ and $\{P_j\}_{j=1}^{\infty}$ denote the countable set of eigenvalues and corresponding projectors given by the spectral decomposition of Theorem 1 such that, letting*

$$\alpha_j := \frac{\ln|\xi_j|}{h_{\text{top}}} \in (\beta, 1),$$

there exists $J \in \mathbb{N}$ such that $|\xi_j| \leq \lambda^{\frac{1}{2}+\epsilon}$ for all $j \geq J$ and that the magnitude of the remainder term is bounded by $e^{\beta h_{top}}$.
(i) *There exist linear functionals $\ell_{x,j}^t(\cdot)$ such that, for all $x \in M$, $t > 1$, and $h \in \mathcal{C}^\infty(M)$,*

$$H_{x,t}(h) = t\int_M h\,dm + \sum_{j=1}^{\infty} t^{\alpha_j}(\ln t)^{d_j}\,\ell_{x,j}^t(h) + \mathrm{O}\left(\mathcal{K}_\beta(t)\,\|h\|_{\mathcal{C}^r}\right)$$

where

$$\mathcal{K}_\beta(t) := \begin{cases} t^\beta & \text{if } \beta > 0 \\ \ln t & \text{if } \beta = 0 \\ 1 & \text{if } \beta < 0. \end{cases}$$

The linear functionals are bounded uniformly in t, x, j and are related to the projectors in the sense that if $h \in \ker(P_j)$ then $\ell_{x,j}^t(h) = 0$.
(ii) *In particular, if $h \in \ker(P_j)$ for all $j \leq J$ then there exists $C > 0$ such that, for all $t \geq 0$,*

$$|H_{x,t}(h)| \leq C t^{\frac{1}{2}+\epsilon}.$$

The proof of this theorem follows after several useful lemmas and includes the precise definition of the linear functionals. Observe that, unlike [3] in the case of Giulietti–Liverani [20] horocycle flows on the two-torus, the resonances in this setting really exist as guaranteed by the estimate of the total multiplicity of eigenvalues lying on the unit circle [12, Theorem 1.3.8]. Recall that the transfer operator (6) was defined as $\hat{F} : h \mapsto \lambda h \circ F^{-1}$. The basic idea is that we use the renormalization (1) to transform the ergodic integral (2) $H_{x,t}(h) = \int_0^t h \circ \Psi_r(x)\,dr$ as shown in the computation (4). In this way the long time behaviour is understood by studying iterates of the map; it is natural to take $k = k(t)$ such that $\lambda^k t$ is of the same scale for all t. In order to compute estimates in terms of the spectral information of the transfer operator, we want to work with continuous functionals so we would like to have smooth versions of ergodic integrals. To this end, for any $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ with compact support, $x \in M$, and $h \in C^0(M)$, let

$$H_{x,\varphi}(h) := \int_{\mathbb{R}} \varphi(t) \cdot h \circ \Psi_t(x)\, dt. \tag{13}$$

Note that the smooth version of the ergodic integral (13) corresponds to the functional (11) where the integral is taken over a stable curve. Lemma 2 means that this quantity is well controlled by the functional analytic results we have. In the following lemma, we combine the renormalization (1) with the smoothed version of the ergodic integral (2).

Lemma 4 *Suppose that $k \in \mathbb{Z}$. Then, for all $h \in \mathcal{C}^0(M)$, $x \in M$,*

$$H_{x,\varphi}(h) = H_{\tilde{x},\tilde{\varphi}}(\hat{F}^k h)$$

where $\tilde{\varphi}(t) := \varphi(\lambda^k t)$ and $\tilde{x} := F^k(x)$.

Proof Using the renormalization (1)

$$\begin{aligned} H_{x,\varphi}(h) &= \int_{\mathbb{R}} \varphi(r) \cdot h \circ F^{-k} \circ F^k \circ \Psi_r(x)\, dr \\ &= \int_{\mathbb{R}} \varphi(r) \cdot h \circ F^{-k} \circ \Psi_{\lambda^{-k} r} \circ F^k(x)\, dr \\ &= \int_{\mathbb{R}} \varphi(\lambda^k u) \cdot \left(\lambda^k h \circ F^{-k}\right) \circ \Psi_u(F^k(x))\, du. \end{aligned}$$

Since $\lambda^k h \circ F^{-k} = \hat{F}^k h$ this completes the proof of the lemma. □

The next step is to relate the original ergodic integrals (2) with smooth versions.

Lemma 5 *Suppose that $T > 0$. Let $N = \left\lfloor \frac{\ln T}{\ln 4} \right\rfloor$ and for all $k \in \mathbb{N}$, $|k| \le N$, let $m_k := \lfloor (\ln T - |k| \ln 4) / (\ln \lambda) \rfloor$. There exists a set of functions $\{\tilde{\varphi}_k\}_k \subset \mathcal{C}^\infty(\mathbb{R}, [0,1])$, each supported on an interval of unit length with the property that $\|\tilde{\varphi}_k\|_{\mathcal{C}^q} \le C_q$ (independent of T), such that, for every $x \in P$ and every $h \in \mathcal{C}^\infty(M)$, letting $x_k = F^{m_k} x$,*

$$H_{x,T}(h) = \sum_{k=-N}^{N} H_{x_k,\tilde{\varphi}_k}(\hat{F}^{m_k} h) + \mathrm{O}(\|h\|_{\mathcal{C}^0(M)}).$$

Prior to proving this Lemma, it is convenient to introduce a partition of unity which "zooms in" on the end points of an interval. The construction is for an interval of length $T > 0$, and the number of components of the partition is determined by $n \in \mathbb{N}$. **Step 1:** Fix, once and for all, a smooth step function $\eta \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ such that $\eta(t) = 0$ whenever $t \le 0$ and $\eta(t) = 1$ whenever $t \ge 1$. **Step 2:** For all $k \in \{0, 2, \ldots, n-1\}$ let

$$\eta_k(t) := \eta\left(\tfrac{8t - 4^{-k} T}{4^{-k} T}\right). \tag{14}$$

Step 3: Let $\varphi_0(t) := \eta_0(t) - \eta_0(T - t)$. For all $k \in \{1, 2, \ldots, n-1\}$, let $\varphi_k := \eta_k - \eta_{k-1}$ and let $\varphi_n := 1 - \eta_{n-1}$; for all $k \in \{1, 2, \ldots, n\}$, let $\varphi_{-k}(t) := \varphi_k(T - t)$. We denote by $\{\varphi_k\}_{k=-n}^n$ the zooming partition of unity. Observe that φ_n and φ_{-n} differ from the other φ_k in that they are only one-sided bumps whereas all the other φ_k are compactly supported smooth functions. This construction has the following properties.

Lemma 6 *Let $\{\varphi_k\}_{k=-n}^n$ be the zooming partition of unity for $[0, T]$ with $n \in \mathbb{N}$ components. Then*

(1) $\sum_{k=-n}^{n} \varphi_k(t) = 1$ *for all* $t \in [0, T]$*;*
(2) The support of φ_k *is contained in an interval of length* $4^{-|k|}T$*;*
(3) Let $\sigma_k : t \mapsto t4^{-|k|}T$*. For each* $q \in \mathbb{N}$ *there exists* $C_q > 0$*, independent of* T*, such that* $\|\varphi_k \circ \sigma_k\|_{\mathcal{C}^q} \leq C_q$ *for all* $|k| \leq n$*.*

Proof That (1) holds is immediate from the telescoping construction. For (2), observe that φ_0 is supported on $(\frac{T}{8}, \frac{7T}{8})$. In the cases $0 < k < n$ the function φ_k is supported on the interval $(\frac{1}{8}4^{-k}T, 4^{-k}T)$ and the length of this interval is bounded by $4^{-|k|}T$. The same bound also holds for $k = n$. By symmetry when k is negative the supports satisfy these same bounds. The control on the derivatives, (3) is a consequence of the scaling used in the definition (14) of the functions $\|\varphi_k \circ \sigma_k\|_{\mathcal{C}^q} \leq 8^q \|\eta\|_{\mathcal{C}^q} \leq C_q$. □

We now use the above partition of unity for the postponed proof.

Proof of Lemma 5 Let $N := \lfloor \frac{\ln T}{\ln 4} \rfloor$. Using the zooming partition of unity, Lemma 6, we write

$$\begin{aligned} H_{x,T}(h) &= \int_0^T h \circ \Psi_t(x)\, dt \\ &= \sum_{k=-(N-1)}^{N-1} H_{x,\varphi_k}(h) + \int_{\mathbb{R}} (\varphi_{-N} + \varphi_N)(t) \cdot h \circ \Psi_t(x)\, dt. \end{aligned}$$

The final integral term is bounded by $2 \cdot 4^{-N} T \|h\|_{\mathcal{C}^0(M)} \leq 2\|h\|_{\mathcal{C}^0(M)}$. For the smooth ergodic integrals in the sum we apply Lemma 4 and obtain

$$H_{x,\varphi_k}(h) = H_{x_k,\tilde{\varphi}_k}(\hat{F}^{m_k} h)$$

where $\tilde{\varphi}_k(t) := \varphi_k(\lambda^{m_k} t)$, $x_k = F^{m_k}(x)$. Observe that, if we set $m_k := \lfloor (\ln T - |k| \ln 4) / (\ln \lambda) \rfloor$, then

$$\lambda^{-1} T 4^{-|k|} < \lambda^{m_k} \leq T 4^{-|k|}.$$

Consequently, $\|\tilde{\varphi}_k\|_{\mathcal{C}^q} \leq C_q$ for some C_q which does not depend on k, n, or T as per (3) of Lemma 6. □

Proof of Theorem 2 We use the spectral decomposition of Theorem 1 to obtain

$$\hat{F}^m h = \sum_{j=0}^{\infty} \left(\xi_j P_j + Q_j\right)^m h + R_m h.$$

where $R_m = \widehat{F}^m P_0$ and so $\|R_m h\|_{\mathcal{H}} \leq \eta^n C_\eta \|h\|_{\mathcal{H}}$. By Lemma 5 we know that $H_{x,t}(h) = \sum_{k=-N}^{N} H_{x_k,\tilde{\varphi}_k}\left(\hat{F}^{m_k} h\right) + \mathrm{O}(\|h\|_{\mathcal{C}^0})$ and so

$$\begin{aligned} H_{x,t}(h) = &\sum_{k=-N}^{N} \sum_{j=0}^{\infty} H_{x_k,\tilde{\varphi}_k}\left(\left(\xi_j P_j + Q_j\right)^{m_k} h\right) \\ &+ \sum_{k=-N}^{N} H_{x_k,\tilde{\varphi}_k}(R_{m_k} h) + \mathrm{O}(\|h\|_{\mathcal{C}^0}). \end{aligned}$$

Recall that $m_k = \left\lfloor \frac{\ln t - |k| \ln 4}{\ln \lambda} \right\rfloor$ and so $\eta^{m_k} \leq C \exp\left(\frac{\ln \eta}{\ln \lambda}(\ln t - |k| \ln 4)\right) = Ct^{\beta} 4^{-\beta|k|}$ where $\beta = \ln \eta / h_{\text{top}}$. Combining Lemma 2 and the estimate for R_p from Theorem 1,

$$\sum_{k=-N}^{N} H_{x_k,\tilde{\varphi}_k}(R_{m_k} h) \leq \sum_{k=-N}^{N} C\eta^{m_k} \|h\|_{\mathcal{H}}$$
$$\leq \sum_{k=-N}^{N} Ct^{\beta} 4^{-\beta|k|} \|h\|_{\mathcal{H}} \leq C' \mathcal{K}_\beta(t) \|h\|_{\mathcal{H}}$$

where we have the bound

$$\mathcal{K}_\beta(t) := \begin{cases} t^\beta & \text{if } \beta > 0 \\ \ln t & \text{if } \beta = 0 \\ 1 & \text{if } \beta < 0 \end{cases}$$

as per the statement of the theorem. Let $\{d_j\}_j$ be the integers such that $Q_j^{d_j} = 0$ and $Q_j^{d_j - 1} \neq 0$ or $d_j = 0$. Recall that we defined the linear functionals

$$\ell^t_{x,j}(h) := t^{-\alpha_j} (\ln t)^{-d_j} \sum_{k=-N}^{N} H_{x_k,\tilde{\varphi}_k}\left((\xi_j P_j + Q_j)^{m_k} h \right).$$

It remains to show that $\left|\ell^t_{x,j}(h)\right|$ is bounded in $t \geq 0$. This follows from computing,

$$\left| H_{x_k,\tilde{\varphi}_k}\left((\xi_j P_j + Q_j)^{m_k} h \right) \right| \leq \xi_j^{m_k} {m_k}^{d_j} \|\tilde{\varphi}_k\|_{\mathcal{C}^1} \|h\|_1$$
$$\leq Ct^{\frac{\ln \xi_j}{\ln \lambda}} \xi_j^{-|k| \frac{\ln 4}{\ln \lambda}} (\ln t)^{d_j} |\ln \lambda|^{d_j} \|h\|_1$$
$$\leq Ct^{\alpha_j} (\ln t)^{d_j}$$

where $1 \leq \eta < \left|\xi_j\right| \leq \lambda$, $\ln \lambda = h_{\text{top}}$ is a constant, and $\|\tilde{\varphi}_k\|_{\mathcal{C}^1}$ is uniformly bounded (from Lemma 5). This concludes the proof of (i). To prove (ii), observe that if $h \in \ker(P_j)$ then $\ell^t_{x,j}(h) = 0$. So if $h \in \ker(P_j)$ for each $j \leq J$, $H_{x,t}(h) \leq C \left(\|h\|_{\mathcal{C}^0} + t^\beta \|h\|_{\mathcal{C}^r} \right) \leq C_h t^\beta$. □

5 The cohomological equation

In this section we show the existence of a solution to the cohomological equation. I.e., given a function h on M we wish to find (if it exists) g such that $g \circ \Psi_t(x) - g(x) = \int_0^t h \circ \Psi_r(x)\, dr$ for all $x \in M$, $t \geq 0$. If h is such that the ergodic integral $\int_0^t h \circ \Psi_s(x)\, ds$ is uniformly bounded in $t > 0$ then, by the Gottschalk–Hedlund Theorem, h is a coboundary. The argument we use now is an independent proof of this in this particular setting and also gives us the possibility for further investigation. Considering the differential version of the cohomological equation $Wg = h$ where W represents differentiation with respect to the vector field associated to to the parabolic flow, we can rephrase our objective as finding an inverse to the operator W.

The method we use here follows closely the one used by Giulietti and Liverani [20]. Since we only consider the linear case we can continue with the same operator as used for the deviation of ergodic averages whereas they were obliged to study the action of the dynamics acting on one-forms. In places the present formulation is somewhat different, particularly in that we will show that there is a formal inverse to the operator W that is determined by a countable sum. Expressing the coboundary in this way enables us to more easily explore the regularity. Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^+, [0, 1])$ such that $\chi(s) = 1$ for $s \in [0, \frac{1}{2}]$ and $\chi(s) = 0$ for $s \in [1, \infty)$. Let $\varphi(t) := \chi(t/\lambda) - \chi(t)$ and observe that this is a bump function with support contained within $(\frac{1}{2}, \lambda)$. For all $h \in \mathcal{C}^\infty(M)$ we define pointwise (this is a type of local integrating operator),

$$(Kh)(x) := \int \varphi(s) \cdot h \circ \Psi_s(x)\, ds \tag{15}$$

Observe that, by Lemma 2, this quantity is well defined on the anisotropic Sobolev spaces. For any $k \in \mathbb{N}$, $h \in \mathcal{C}^\infty(M)$ we again define pointwise

$$\widetilde{G}h(x) := \int \chi(s) \cdot h \circ \Psi_s(x)\, ds, \quad (G_k h)(x) := \left(K \widehat{F}^k h\right)(F^k x).$$

Theorem 3 *Let $\{P_j\}_{j \in \mathcal{J}}$ be a countable set of projectors given by Theorem 1 for the case when we choose $\eta < 1$. Suppose that $h \in \ker(P_j)$ for all $j \in \mathcal{J}$. Then the sum $\sum_{k=0}^\infty G_k h(x)$ converges uniformly for $x \in M$. Moreover h is a coboundary for the flow $\{\Psi_t\}_t$ with transfer function*

$$g(x) = -\widetilde{G}h(x) - \sum_{k=0}^{\infty} G_k h(x).$$

Remark 7 In a certain sense the formal inverse to differentiation by W is the operator defined as $G := -\widetilde{G} - \sum_{k=0}^\infty G_k$.

In order to prove Theorem 3, we first prove that the sum $\sum_{k=0}^\infty G_k h(x)$ converges.

Proof Since, by assumption, $h \in \ker(P_j)$ for all $j \in \mathcal{J}$, then the estimate of Lemma 2, together with Theorem 2, implies that

$$|(G_k h)(x)| = \left|\left(K \widehat{F}^k h\right)(F^k x)\right| \leq C \left\|\hat{F}^k h\right\| \leq C\eta^k \|h\|.$$

The sum is therefore bounded by a geometric sum and consequently converges. □

Lemma 7 *For all $n \in \mathbb{N}$, $x \in M$ and $h \in \mathcal{C}^\infty(M)$,*

$$\int \chi(\lambda^{-n} t) \cdot h \circ \Psi_t(x)\, dt = \widetilde{G}h(x) + \sum_{k=0}^{n-1} G_k h(x).$$

Proof Let $t_j := \lambda^j t$ for $j \in \mathbb{N}$. We observe that, since $\chi(\lambda^{-n} t) = \chi(s) + \sum_{k=0}^{n-1} \left[\chi(\lambda^{-(k+1)} t) - \chi(\lambda^{-k} t)\right]$,

$$\begin{aligned}
\int \chi(\lambda^{-n} t) \cdot h \circ \Psi_t(x)\, dt &= \sum_{j=0}^{n-1} \int \left[\chi(\lambda^{-(k+1)} t) - \chi(\lambda^{-k} t)\right] h \circ \Psi_t(x)\, dt \\
&\quad + \int \chi(s) \cdot h \circ \Psi_s(x)\, ds \\
&= \sum_{k=0}^{n-1} \int \varphi(\lambda^{-k} t) \cdot h \circ \Psi_t(x)\, dt + \widetilde{G}h(x).
\end{aligned}$$

Furthermore, using the renormalization of Lemma 4,

$$\int \varphi(\lambda^{-k}t)\cdot h\circ\Psi_t(x)\,dt = \int \varphi(t)\cdot(\widehat{F}^k h)\circ\Psi_t(\widehat{F}^k x)\,dt = G_k h(x).$$

□

Having established that the limit exists we will now complete the proof of Theorem 3 by showing that this limit is, in fact, the object we are looking for.

Proof of second part of Theorem 3 We show that g satisfies $g\circ\Psi_W^t(x) - g(x) = \int_0^t h\circ\Psi_W^s(x)\,ds$. For convenience let

$$g_n(x) := \widetilde{G}h(x) + \sum_{k=0}^{n-1} G_k h(x).$$

For $t \in (0, \frac{1}{2})$ (by the semigroup property this suffices to prove the result for all $t > 0$), Lemma 7 implies that

$$\begin{aligned} g_n(x) - g_n(\Psi_t x) &- \int_0^t h\circ\Psi_W^s(x)\,ds \\ &= \int_{\mathbb{R}^+} \left[\chi(\lambda^{-n}s) - 1_{[0,t]} - \chi(\lambda^{-n}(s-t))\right] h\circ\Psi_W^s(x)\,ds = H_{x,\eta_n}(h) \end{aligned}$$

where we define $\eta_n := \chi(\lambda^{-n}s) - (1_{[0,t]}(s) + \chi(\lambda^{-n}(s-t)))$. Note that $\mathrm{Supp}(\eta_n) \subset (\frac{\lambda^n}{2}, \lambda^n + t)$. Using the renormalization of the smooth ergodic integral (4), scaling by a factor λ^n, we can write

$$H_{x,\eta_n}(h) = \int_{s=t}^{\infty} [\chi(s) - \chi(s-t)]\cdot(\widehat{F}^n)\circ\Psi_s(F^n x)\,ds$$

which, by the repeatedly used argument, exponentially small in n. □

6 Heisenberg Nilflows

This section is devoted to a concise overview of Heisenberg nilmanifolds and nilflows. For $E \in \mathbb{N}$, a 3-dimensional Heisenberg nilmanifold $M = \Gamma_E\backslash N$, is a compact quotient of the Heisenberg group N by a lattice Γ_E,

$$N := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},$$

$$\Gamma_E := \left\{ \begin{pmatrix} 1 & p & \frac{r}{E} \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : p, q, r \in \mathbb{Z} \right\}.$$

It is common to express elements of N in terms of the entries above the diagonal; in this form, the group operation is given by[7]

[7] Folland [16] refers to this as the symplectic Heisenberg group. It is also common to see the polarized Heisenberg group, given by the group law $(x, y, z) * (a, b, c) = (x+a, y+b, z+c+xb)$, which corresponds directly to matrix multiplication. The two groups are equivalent; the map $(x, y, z) \to (x, y, z + \frac{1}{2}xy)$ gives an isomorphism between them.

$$(x, y, z) * (a, b, c) = \left(x + a, y + b, z + c + \frac{1}{2}(xb - ya)\right).$$

(Observe that $xb - ya$ is the canonical symplectic form on $\mathbb{R}^2$.) We are in the setting described in Sect. 2 since M is a nontrivial circle bundle over $\mathbb{T}^2$ (see e.g., [16, §1.10]). That it is a bundle over the torus can be seen by the fact that the group action is Abelian in the first two coordinates. The circle fibre is given by the third coordinate and it is a non-trivial bundle due to the factor $xb - ya$. The standard basis of the Lie Algebra of N is given by the elements

$$\left\{X_0 = \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix},\ Y_0 = \begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix},\ Z_0 = \begin{pmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}\right\}$$

which satisfy the commutation relations

$$[X_0, Y_0] = Z_0 \text{ and } [X_0, Z_0] = [Y_0, Z_0] = 0. \tag{16}$$

The natural probability measure, μ, on M is inherited from the Haar measure on N. Nilflows on M are given by the right action of one-parameter subgroups of N, and thus, preserve μ. When the projected flow is an irrational linear flow on $\mathbb{T}^2$, the nilflow is uniquely ergodic and minimal (see, e.g., [15]). For $(x, y, z) \in M$, consider a partially hyperbolic automorphism of N given by

$$F(x, y, z) = (A(x, y), z) \tag{17}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix with integer entries, determinant 1, and eigenvalues λ, λ^{-1} for some $\lambda^{-1} \in (0, 1)$. F induces a diffeomorphism of $M = \Gamma \setminus N$ when $F(\Gamma) = \Gamma$. Note that F preserves the symplectic form, and thus, can be considered in the framework of Faure [9] and Faure and Tsujii [12] from Sect. 2. We describe the relevant parabolic system that is renormalized by the automorphism (17) by constructing a flow in the stable direction. Consider an element of the Lie Algebra, $W = \alpha X_0 + \beta Y_0$, where α, β are the non-zero entries of the normalised eigenvector associated to λ. We consider the flow generated by W, i.e., the right action of the subgroup of N given by[8]

$$\{\Psi_t^W\}_{t\in\mathbb{R}} := \{\exp(tW)\}_{t\in\mathbb{R}}. \tag{18}$$

A simple calculation gives the following renormalization with the partially hyperbolic automorphism,

$$F \circ \Psi_{\lambda t}^W = \Psi_t^W \circ F. \tag{19}$$

Combining Theorem (1) with the renormalization (19), we can estimate the growth of ergodic integrals, in this case, of the form

$$\int_0^t h \circ \Psi_r^W(x)\, dx$$

by applying Theorem 2. We also get existence of a coboundary from Theorem 3.

[8] We can compute explicitly the formula $\exp tW = (\alpha t, \beta t, \frac{1}{2}\alpha\beta t^2)$.

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