JOURNAL OF MODERN DYNAMICS
VOLUME 1, NO. 2, 2007, 147–168

WEB SITE: http://www.math.psu.edu/jmd

SMOOTH ANOSOV FLOWS: CORRELATION SPECTRA AND STABILITY

OLIVER BUTTERLEY AND CARLANGELO LIVERANI
(Communicated by Giovanni Forni)

ABSTRACT. By introducing appropriate Banach spaces one can study the spectral properties of the generator of the semigroup defined by an Anosov flow. Consequently, it is possible to easily obtain sharp results on the Ruelle resonances and the differentiability of the SRB measure.

1. INTRODUCTION

In the last years there has been a growing interest in the dependence of the SRB measures on the parameters of the system. In particular, G.Gallavotti [11] has argued the relevance of such an issue for nonequilibrium statistical mechanics.

On a physical basis (linear response theory) one expects that the average behavior of an observable changes smoothly with parameters. Yet the related rigorous results are limited and the existence of irregular dependence on parameters (think, for example, of the quadratic family) shows that, in general, *smooth dependence* must be properly interpreted to have any chance to hold.

The only cases in which some simple rigorous results are available are smooth uniformly hyperbolic systems and some partially hyperbolic systems. In particular, Ruelle [24] has proved differentiability and has provided an explicit (in principle computable) formula for the derivative in the case of SRB measures for smooth hyperbolic diffeomorphisms. Subsequently, D.Dolgopyat has extended such results to a large class of partially hyperbolic systems [8]. More recently Ruelle has obtained similar results for Anosov flows [26]. Ruelle's proofs of the above results use the classical thermodynamic formalism and precise structural stability results which, although reasonably efficient for diffeomorphisms, produce a quite cumbersome proof in the case of flows. It should also be remarked that much of the results concerning statistical properties of dynamical systems are related to the analytical properties of the Ruelle zeta function [23, 1]. In the context of Anosov flows such properties have been first elucidated by Pollicott in [22].

Received September 21, 2006, revised December 13, 2006.
2000 *Mathematics Subject Classification:* 37C30,37D30,37M25.
Key words and phrases: Transfer operator, resonances, differentiability SRB.
We gladly acknowledge M.I.U.R. (Prin 2004028108) for support and the I.H.P., Paris, where part of this paper was written during the trimester *Time at Work*. O.B. acknowledges the support of the Marie Curie Control Training Site.

In more recent years, several authors have attempted to put forward a different approach to the study of hyperbolic dynamical systems based on the direct study of the transfer operator (see [1] for an introduction to the theory of transfer operators in dynamical systems). Starting with [28, 5] it has become clear that it is possible to construct appropriate functional spaces such that the statistical properties of the system are accurately described by the spectral data of the operator acting on such spaces. The recent papers [19, 18, 12, 2, 3, 20, 9, 7, 21, 4, 13], have shown that such an approach yields a simpler and far-reaching alternative to the more traditional point of view based on Markov partitions.

In this paper we present an application of these methods to the aforementioned issue: the differentiability properties of the SRB measure for Anosov flows. Not only are the formulae in [24] easily recovered, but higher differentiability is obtained as well, making rigorous some of the results in [25]. In addition, the method naturally yields precise information on the structure of the Ruelle resonances, extending the results in [22, 27].

Note that the same strategy can be used to prove differentiability (and obtain in principle computable formulae) for many other physically relevant quantities (at least for $\mathscr{C}^\infty$ flows) such as: Ruelle's resonances and eigendistributions, the variance in the central limit theorem (diffusion constant), the rate in the large deviations. Also, a slight generalization of the present approach that considers transfer operators with real potential would apply to general Gibbs measures. This would allow, for example, to obtain an easy alternative proof of the results in [17].

The key reason for the straightforwardness of the present approach is that, once the proper functional setting is established, the usual formal manipulations to compute the derivative are rigorously justified, making the argument totally transparent.

The spaces used here are the ones introduced in [12] although similar results could, most likely, be obtained by using the spaces introduced in [3, 4].

Recently some new results have been obtained on the stability of mixing [10]. It would be interesting to investigate the relationship between such qualitative results and the quantitative theory in this paper.

Finally, it should be remarked that the approach of the present paper is based on the study of the resolvent, rather than the semigroup, in the spirit of [20]. Nevertheless, a recent paper by M.Tsujii [29] has shown that it is possible to introduce Banach spaces that allow the direct study of the semigroup, although this is limited to the case of suspensions over an expanding endomorphism. Such an approach yields much stronger results. To construct similar spaces for flows and, possibly, other classes of partially hyperbolic systems is one of the current challenges of the field.

The plan of the paper is as follows: Section 2 details the systems we consider, introduces the norms we use and the corresponding Banach spaces, and it states the results. In Section 3 we precisely define the Banach spaces relevant for our approach and study some of their properties. In Section 4 we look at the properties of the transfer operator in this setting and discuss the spectral decomposition of its generator. In Section 5 we give results on the behavior of the part of the spectrum close to the imaginary axis, and in Section 6 we discuss specifically the behavior of the SRB measure as the dynamical system is perturbed, in the course of which the Ruelle formula for the derivative is established. In Section 7 the main dynamical inequalities are proven for the transfer operator while in Section 8 the corresponding inequalities are established for the resolvent of the generator of the flow. The paper also includes an appendix in which some necessary technical (but intuitive) facts are proven.

REMARK 1.1. *In the present paper we will use C to designate a generic constant depending only on the Dynamical Systems $(\mathscr{M}, T_t)$, while $C_{a,b,...}$ will be used for a generic constant depending also on the parameters $a, b, \ldots$. Accordingly, the actual numerical value of C may vary from one occurrence to the next.*

Acknowledgements. C.L., wishes to thank Sébastien Gouëzel and David Ruelle for many helpful suggestion and comments. In addition we are indebted to the anonymous referee for pointing out a considerable number of imprecisions and making several precious suggestions.

2. STATEMENTS AND RESULTS

Let us consider the $\mathscr{C}^\infty$ d-dimensional compact Riemannian manifold $\mathscr{M}$ and the Anosov flow $T_t \in \mathrm{Diff}(\mathscr{M}, \mathscr{M})$. In other words the following conditions are satisfied.

CONDITION 1. *T satisfies the following*

$$T_0 = \mathbf{Id},$$
$$T_p \circ T_q = T_{p+q} \quad \text{for each } p, q \in \mathbb{R}.$$

That is T_t is a flow.

CONDITION 2. *At each point $x \in \mathscr{M}$ there exists a splitting of tangent space $T_x\mathscr{M} = E^s(x) \oplus E^f(x) \oplus E^u(x)$. The splitting is continuous and invariant with respect to T_t. E^f is one-dimensional and coincides with the flow direction. In addition, for each $v \in E^f$, $DT_t v = 0 \implies v = 0$ and there exist $\lambda > 0$ such that*

$$\|DT_t v\| \le e^{-\lambda t}\|v\| \text{ for each } v \in E^s \text{ and } t \ge 0,$$
$$\|DT_{-t} v\| \le e^{-\lambda t}\|v\| \text{ for each } v \in E^u \text{ and } t \ge 0.$$

That is the flow is Anosov.[1]

[1]In general one can have a $Ce^{-\lambda t}$ instead of $e^{-\lambda t}$ in the first two inequalities, yet it is always possible to change the Riemannian structure in order to have $C = 1$ by losing a little bit of hyperbolicity (e.g., define $\langle v, w\rangle_L := \int_{-L}^{L} e^{-2\lambda'|s|}\langle DT_t v, DT_t w\rangle ds$ with $\lambda' < \lambda$ and L such that $Ce^{(\lambda'-\lambda)L} < 1$).

A smooth flow naturally defines a related vector field V. Often the vector field is a more fundamental object than the flow, we will thus put our smoothness requirement directly on the vector field.

CONDITION 3. *We assume $V \in \mathscr{C}^{r+1}$, $r > 1$.*[2] *This implies $T_t \in \mathscr{C}^{r+1}$.*

To study the statistical properties of such systems it is helpful to study the action of the dynamics on distributions. To this end let us define $\mathscr{L}_t\colon \mathscr{D}'_{r+1} \to \mathscr{D}'_{r+1}$ by[3]

$$\langle \mathscr{L}_t h, \varphi\rangle := \langle h, \varphi \circ T_t\rangle, \quad \text{for all } \varphi \in \mathscr{C}^{r+1}. \tag{2.1}$$

It is easy to see that the $\mathscr{L}_t$ are continuous.

REMARK 2.1. *Given the standard continuous embedding*[4] $\mathbf{i}\colon \mathscr{C}^r \hookrightarrow \mathscr{D}'_r$ *we can, and we will, view functions as distributions. In particular, if $h \in \mathscr{C}^r$, then it can be viewed as the density of the absolutely continuous measure $\mathbf{i}h$. In such a case a simple computation shows that, setting*

$$\widetilde{\mathscr{L}}_t h := [h \det(DT_t)^{-1}] \circ T_t^{-1}, \tag{2.2}$$

gives $\mathbf{i}\widetilde{\mathscr{L}}_t = \mathscr{L}_t \mathbf{i}$. Formula (2.2) *provides a more common expression for the transfer operator.*

Unfortunately it turns out that the spectral properties of $\mathscr{L}_t$ on the above spaces bear no clear relation to the statistical properties of the system. To establish such a connection in a fruitful way it is necessary to introduce Banach spaces that embody in their inner geometry the key properties of the system (that is, hyperbolicity).

The first step is to define appropriate norms on $\mathscr{C}^\infty(\mathscr{M},\mathbb{C})$ and then take the closure in the relative topology. The exact definition of the norms can be found in Section 3, yet let us give here a flavor of the construction.

For each $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, consider a set Σ of manifolds of roughly uniform size and *close* to the strong stable manifolds and let $\mathscr{V}$ be the set of smooth vector fields (see Section 3 for precise definitions). For each $W \in \Sigma$, $v_1, \dots, v_p \in \mathscr{V}$ and $\varphi \in \mathscr{C}_0^{p+q}(W,\mathbb{C})$ we can then define linear functionals on $\mathscr{C}^\infty(\mathscr{M},\mathbb{C})$[5] by

$$\ell_{W,v_1,\dots,v_p,\varphi}(h) := \int_W \varphi v_1 \cdots v_p h$$

and the *dual* ball by

$$\mathbb{U}_{p,q} := \left\{ \ell_{W,v_1,\dots,v_p,\varphi} \mid W \in \Sigma, |\varphi|_{\mathscr{C}_0^{q+p}} \le 1, |v_i|_{\mathscr{C}^{q+p}} \le 1 \right\}.$$

---

[2] The reason for such a condition, instead of the more natural $r > 0$, is purely technical and rests in the limitation $p \in \mathbb{N}$ for the spaces $\mathscr{B}^{p,q}$ used in the following. Most likely it could be removed either using the spaces in [3] or generalizing the present spaces.

[3] In the following we will use $\langle h,\varphi\rangle$ and $h(\varphi)$ interchangeably to designate the action of the distribution h on the smooth function φ.

[4] If $g, f \in \mathscr{C}^r$, then $\langle \mathbf{i}f, g\rangle := \int_{\mathscr{M}} fg$.

[5] Here, and in the following, the integrals are meant with respect to the induced Riemannian metric. Moreover, given a vector field v and a function h, by vh or $v(h)$ we mean the Lie derivative of h along v.

We can finally define the seminorms and norms we are interested in:

$$\begin{aligned}\|h\|^-_{p,q} &:= \sup_{\ell\in\mathbb{U}_{p,q}} \ell(h) \qquad \forall p\in\mathbb{N},\ q\in\mathbb{R}_+\\ \|h\|_{p,q} &:= \sup_{n\le p}\|h\|^-_{n,q} \qquad \forall p\in\mathbb{N},\ q\in\mathbb{R}_+.\end{aligned} \tag{2.3}$$

We define the spaces $\mathscr{B}^{p,q} := \overline{\mathscr{C}^\infty(\mathscr{M},\mathbb{C})}^{\|\cdot\|_{p,q}}$. Note that these spaces are equivalent to the ones defined in Section 2 of [12], the only difference being in their use: there they depend on the stable cone of an Anosov diffeomorphism, here they depend on the *strong* stable cone of an Anosov flow. Consequently we will often refer to results proved in [12].

A first relevant property of the spaces $\mathscr{B}^{p,q}$ was proved in [12, Lemma 2.1]:

LEMMA 2.2. $\|\cdot\|_{p-1,q+1} \le C_{p,q}\|\cdot\|_{p,q}$ *for each* $p\in\mathbb{N}_*$ *and* $q\in\mathbb{R}_+$. *In addition, the unit ball of* $\mathscr{B}^{p,q}$ *is relatively compact in* $\mathscr{B}^{p-1,q+1}$.

It is easy to show that $\mathscr{L}_t\colon \mathscr{B}^{p,q}\to\mathscr{B}^{p,q}$, with $p+q\le r$, is a bounded strongly continuous semigroup (Lemma 4.2); it is also uniformly bounded in t (Lemma 4.1). Accordingly, by general theory, the generator X of the semigroup is a closed operator. Clearly, the domain $D(X)\supset\mathscr{C}^{r+1}(\mathscr{M},\mathbb{C})$ and, restricted to $\mathscr{C}^{r+1}(\mathscr{M},\mathbb{C})$, X is the action of the adjoint of the vector field defining the flow, that is

$$Xh = -V(h) - h\operatorname{div}V \in \mathscr{C}^r. \tag{2.4}$$

Obviously, the spectral properties of the generator depend on the resolvent $R(z) = (z\mathbf{Id} - X)^{-1}$. It is well known (e.g., see [6]) that for uniformly bounded semigroup (Lemma 4.1) the spectrum of X is contained in $\{z\in\mathbb{C} : \Re(z)\le 0\}$. That is, for all $z\in\mathbb{C}$, $\Re(z)>0$, the resolvent $R(z)$ is a well-defined bounded operator on $\mathscr{B}^{p,q}$ and, moreover,

$$R(z)f = \int_0^\infty e^{-zt}\mathscr{L}_t f\,dt. \tag{2.5}$$

The above facts allow us to establish several facts concerning the spectrum of the generator.

THEOREM 1. *For each* $p\in\mathbb{N}$, $q\in\mathbb{R}_+$, $p+q\le r$, *the spectrum of the generator, acting on* $\mathscr{B}^{p,q}$, *in the strip* $0\ge\Re(z)>-\min\{p,q\}\lambda$ *consists only of isolated eigenvalues of finite multiplicity. Such eigenvalues correspond to the Ruelle resonances (see Remark 2.3 for more details). In addition, the eigenspace associated to the eigenvalue zero is the span of the SRB measures.*[6] *The SRB measure is unique*

[6]Here we adopt the following definition of SRB measure: a measure ν is SRB if there exists a positive Lebesgue measure open set U such that $\forall\varphi\in\mathscr{C}^0$ and Lebesgue a.e. $x\in U$

$$\frac{1}{T}\int_0^T \varphi\circ T_t(x)\,dt \to \nu(\varphi).$$

The above implies, in the present setting, all the usual properties of SRB measures (e.g., absolute continuity along weak unstable manifold) that we do not detail as they will not be used in the following. We will only use, at the end of the proof of Lemma 5.1, that the union of the basins of all the SRB measures is of full Lebesgue measure, that is: for each continuous function the forward ergodic average exists Lebesgue-a.s.

iff the eigenvalue is simple and it is mixing iff zero is the only eigenvalue on the imaginary axis.

The first statement is proven in Lemma 4.5, the second, and more, in Lemma 5.1. The above theorem extends the well-known results of Pollicott and Rugh [22, 27] to higher regularity and higher dimension. Indeed we can connect the above results to physically relevant quantities: the *correlation spectrum*.

If $f, g \in \mathscr{C}^\infty$ then one is interested in $C_{f,g}(t) := \int g \circ T_t f - \int f \int g$, where the integral may be with respect to Lebesgue measure or the SRB measure depending on whether one is observing the system in equilibrium or out of equilibrium starting from a properly prepared state.

REMARK 2.3. *A typical piece of information that can be obtained on the quantity $C_{f,g}$ is its Fourier transform*

$$\hat{C}_{f,g}(ik) := \int_0^\infty e^{-ikt} C_{f,g}(t)\,dt = \int \left(g - \int g\right) R(ik) f.$$

The above results thus imply that the quantity $\hat{C}_{f,g}$ has a meromorphic extension in the strip $0 \geq \Re(z) > -\min\{p,q\}\lambda$. In addition, in such a region, the poles (the so-called Ruelle resonances*) and their residues describe (and are described by) exactly the spectrum of X. In particular this means that the spectral data of X on the Banach spaces $\mathscr{B}^{p,q}$ are not a mathematics nicety but physically relevant quantities.*

Given such a spectral interpretation it is then easy to apply the perturbation theory of [12] and obtain our other main result.

Let us consider a family of vector fields $V_\eta := V + \eta V_1 \in \mathscr{C}^{r+1}$, $\eta \in (-1,1)$, and the associated flow $T_{\eta,t}$. Suppose, for simplicity, that $T_{0,t}$ has a unique SRB measure. The issue is to show that $T_{\eta,t}$ has a unique SRB measure μ_η as well, that such a measure is a smooth function of η and finally to establish a formula for its derivative.

Let us define $\mu_\eta^{(n)} := \frac{d^n}{d\eta^n}\mu_\eta$. In Section 6 we prove the following.

THEOREM 2. *There exists $\eta_0 > 0$ such that, if the flow $T_{0,t}$ has a unique SRB measure, then the same holds for the flows $T_{\eta,t}$ for $|\eta| \leq \eta_0$. Calling μ_η such an SRB measure the function $\eta \mapsto \mu_\eta$ belongs to $\mathscr{C}^{r-2}([-\eta_0,\eta_0],\mathscr{B}^{0,r})$. In addition, for all $\eta \in [-\eta_0,\eta_0]$, $n \leq r-2$ and $\varphi \in \mathscr{C}^r$, we have the formula*

$$\mu_\eta^{(n)}(\varphi) = \lim_{a\to 0^+} \int_0^\infty n e^{-at} \mu_\eta^{(n-1)}(V_1(\varphi \circ T_{\eta,t}))\,dt.$$

REMARK 2.4. *The convergence of the integral in the above formula is far from obvious and it is part of the statement of the Theorem. Notice that for $n = 1$ Theorem 2 yields Ruelle's result* [26] *while, for $n > 0$, it makes rigorous some of the results in* [25]. *In addition, if the operator X_η has a spectral gap (as may happen for geodesic flows in negative curvature* [19]*), then from the proof of Theorem 2 it follows that the above integral converges also for $a = 0$ and one has the formula*

$$\mu_\eta^{(n)}(\varphi) = \int_0^\infty n \mu_\eta^{(n-1)}(V_1(\varphi \circ T_{\eta,t}))\,dt.$$

3. THE BANACH SPACES

To define the norms it is convenient to consider a fixed $\mathscr{C}^{r+1}$ atlas $\{U_i,\Psi_i\}_{i=1}^N$ such that $\Psi_i U_i = B(0,4\delta)$ and $\cup_i \Psi_i^{-1}(B(0,\delta)) = \mathscr{M}$.[7] In addition, we can require $D_0\Psi_i^{-1}\{(0,u,0) \,:\, u\in\mathbb{R}^{d_u}\} = E^u(\Psi_i^{-1}(0))$, $D_0\Psi_i^{-1}\{(s,0,0) \,:\, s\in\mathbb{R}^{d_s}\} = E^s(\Psi_i^{-1}(0))$, and $\Psi_i^{-1}((s,u,t)) = T_t\Psi_i^{-1}((s,u,0))$.

Next we wish to define a set of *(strong) stable leaves.* For small enough $\rho>0$, large enough $M>0$ and $\xi\in B(0,\delta)$ let us define

$$\mathscr{F} := \{F\colon B(0,3\delta)\subset\mathbb{R}^{d_s}\to\mathbb{R}^{d_u+1} \,:\, F(0)=0;\, |F|_{\mathscr{C}^1}\le\rho;\, |F|_{\mathscr{C}^r}\le M\}.$$

Let $G_{x,F}(\xi) := x+(\xi,F(\xi))$ for each $F\in\mathscr{F}$, and $\widetilde{\Sigma} := \{G_{x,F} : x\in B(0,\delta), F\in\mathscr{F}\}$. To each $i\in\{1,\dots,N\}$, $G\in\widetilde{\Sigma}$ we associate the leaf $W_{i,G} = \{\Psi_i^{-1}G(\xi)\}_{\xi\in B(0,2\delta)}$, which form our set of stable leaves Σ, and its reduced and enlarged version $W_{i,G}^{\pm} = \{\Psi_i^{-1}G(\xi)\}_{\xi\in B(0,(2\pm1)\delta)}$.

Integrating over such leaves we can define linear functionals on $\mathscr{C}^r(\mathscr{M},\mathbb{R})$. More precisely, for each $i\in\{1,\dots,N\}$, $s\in\mathbb{N}$, $G\in\widetilde{\Sigma}$, $\varphi\in\mathscr{C}_0^0(\overline{W_{i,G}},\mathbb{C})$ and $\mathscr{C}^s$ vector fields $v_1,\dots,v_s$ defined in a neighborhood of $W_{i,G}^+$ we define

$$\ell_{i,G,\varphi,v_1,\dots,v_s}(h) := \int_{W_{i,G}} \varphi\, v_1\cdots v_s h \quad \forall\, h\in\mathscr{C}^r(\mathscr{M},\mathbb{C}).$$

We use the above functionals to define a set that can be intuitively interpreted as the unit ball of the dual of the space we wish to define. For $p\in\mathbb{N}$, $q\in\mathbb{R}_+$, let[8]

$$\mathbb{U}_{p,q} := \left\{\ell_{i,G,\varphi,v_1,\dots,v_p} \mid 1\le i\le N,\, G\in\widetilde{\Sigma},\, |\varphi|_{\mathscr{C}_0^{q+p}}\le 1,\, |v_j|_{\mathscr{C}^{q+p}}\le 1,\right\}.$$

The norms $\|\cdot\|_{p,q}$ are then defined in 2.3.

REMARK 3.1. *Note that if $h\in\mathscr{C}^\infty(\mathscr{M},\mathbb{C})$, $q\in\mathbb{R}_+$ and $p\in\mathbb{N}$ then $\|h\|_{p,q}\le|h|_{\mathscr{C}^p}$.*

We have the following characterization of $\mathscr{B}^{p,q}$, see [12, Proposition 4.1].

LEMMA 3.2. *The embedding $\mathbf{i}$ extends to a continuous injection from $\mathscr{B}^{p,q}$ to $\mathscr{D}'_q\subset\mathscr{D}'$, the distributions of order q.*

REMARK 3.3. *In the following we will often identify h and $\mathbf{i}h$ if this causes no confusion.*

4. THE TRANSFER OPERATOR

A first property of the transfer operators is detailed by the following lemma whose proof is the content of Section 7.

LEMMA 4.1. *If $p\in\mathbb{N}$, $q\in\mathbb{R}_+$, $p+q\le r$, $t\in\mathbb{R}_+$ and $h\in\mathscr{C}^r$ then*

$$\|\mathscr{L}_t h\|_{p,q}\le C_{p,q}\|h\|_{p,q}. \tag{4.1}$$

[7] Here, and in the following, by $\mathscr{C}^n$ we mean the Banach space obtained by closing $\mathscr{C}^\infty$ with respect to the norm $|f|_{\mathscr{C}^n} := \sup_{k\le n}|f^{(k)}|_\infty 2^{n-k}$. Such a norm has the useful property $|fg|_{\mathscr{C}^n}\le |f|_{\mathscr{C}^n}|g|_{\mathscr{C}^n}$, that is $(\mathscr{C}^n,|\cdot|_{\mathscr{C}^n})$ is a Banach algebra.

[8] By $|v_j|_{\mathscr{C}^{q+p}}\le 1$ we mean that there exists $U=\mathring{U}\supset W_{i,G}^+$ such that v_j is defined on U and $|v_j|_{\mathscr{C}^{q+p}(U)}\le 1$.

As an immediate consequence we have the following first result.

LEMMA 4.2. *The operators $\mathcal{L}_t$, restricted to $\mathcal{B}^{p,q}$, form a bounded strongly continuous semigroup on the Banach space $(\mathcal{B}^{p,q}, \|\cdot\|_{p,q})$.*

Proof. For all $h \in \mathcal{B}^{p,q}$ there exists, by definition, a sequence $\{h_n\} \subset \mathcal{C}^r$ converging to h in the $\|\cdot\|_{p,q}$ norm. By Lemma 3.2 the sequence converges in the spaces of distributions as well and, due to the continuity of $\mathcal{L}_t$, $\{\mathcal{L}_t h_n\}$ converges to $\mathcal{L}_t h$ in $\mathcal{D}'_q$. On the other hand, by Lemma 4.1, $\{\mathcal{L}_t h_n\}$ is a Cauchy sequence in $\mathcal{B}^{p,q}$, hence it converges and, by Lemma 3.2 again, it must converge to $\mathcal{L}_t h$. Thus $\mathcal{L}_t h \in \mathcal{B}^{p,q}$ and

$$\|\mathcal{L}_t h\|_{p,q} \le C_{p,q} \|h\|_{p,q} \qquad \forall\ h \in \mathcal{B}^{p,q}.$$

We have thus a semigroup of bounded operators. The strong continuity follows from the fact that for all $h \in \mathcal{C}^r$ we have

$$\lim_{t\to 0} |\mathcal{L}_t h - h|_{\mathcal{C}^r} = \lim_{t\to 0} \left| [h \det(DT_t)^{-1}] \circ T_t^{-1} - h \right|_{\mathcal{C}^r} = 0.$$

Next, for $h \in \mathcal{B}^{p,q}$ let $\{h_n\} \subset \mathcal{C}^r$ be converging to h, then, using Remark 3.1,

$$\begin{aligned}\|\mathcal{L}_t h - h\|_{p,q} &\le \|\mathcal{L}_t h_n - h_n\|_{p,q} + C_{p,q}\|h - h_n\|_{p,q} \\ &\le C_A |\mathcal{L}_t h_n - h_n|_{\mathcal{C}^r} + C_{p,q}\|h - h_n\|_{p,q}.\end{aligned}$$

Taking first n sufficiently large and then t small, one can make the right-hand side arbitrarily small, that is $\lim_{t\to 0} \|\mathcal{L}_t h - h\|_{p,q} = 0$ for all $h \in \mathcal{B}^{p,q}$. □

In addition we have the following result, proved in Section 8.

LEMMA 4.3. *If $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $p + q \le r$, $z \in \mathbb{C}$, $\Re(z) = a > 0$ then*

$$\|R(z)^n\|_{p,q} \le C_{p,q} a^{-n}.$$

If $\lambda' \in (0,\lambda)$, $p, n \in \mathbb{N}$, $q \in \mathbb{R}_+$ and $z \in \mathbb{C}$, $a := \Re(z) \ge a_0 > 0$ then

$$\|R(z)^n h\|_{p,q} \le C_{p,q,\lambda'} (a + \bar{p}\lambda')^{-n} \|h\|_{p,q} + a^{-n} C_{p,q,\lambda',a_0} |z|\, \|h\|_{p-1,q+1},$$

where $\bar{p} := \min\{p, q\}$.

The above means that the spectral radius of $R(z) \in L(\mathcal{B}^{p,q}, \mathcal{B}^{p,q})$, $\Re(z) = a > 0$, is bounded by a^{-1}, and in fact equals it if $z = a$ since $\int R(z)h = a^{-1} \int h$ implies that a^{-1} is an eigenvalue of the dual. Since Lemma 4.3 implies that $R(z)$ is a bounded operator from $\mathcal{B}^{p,q}$ to itself and since Lemma 2.2 implies that a bounded ball in the $\|\cdot\|_{p,q}$ norm is relatively compact in $\mathcal{B}^{p-1,q+1}$, we obtain:

LEMMA 4.4. *For each $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $p + q < r$, and $z \in \mathbb{C}$, $\Re(z) > 0$ the operator $R(z) : \mathcal{B}^{p,q} \to \mathcal{B}^{p-1,q+1}$ is compact.*

The above implies, via a standard argument [14], that the essential spectral radius of $R(z)$ is bounded by $(a + \lambda\bar{p})^{-1}$. This readily implies the following (see [19, Section 2] if details are needed).

LEMMA 4.5. *The spectrum $\sigma(X)$ of the generator is contained in the left half plane. The set $\sigma(X) \cap U_{\bar{p}\lambda'} := \{z \in \mathbb{C} \mid \Re(z) > -\bar{p}\lambda'\}$ consists of, at most, countably many isolated points of point spectrum with finite multiplicity.*

Thanks to the above result we can connect the spectral properties of the generator to the statistical properties of the flow. First of all, by the spectral decomposition of closed operators on Banach spaces (see [15, Sections 3.6.4 and 3.6.7]), if we select N isolated eigenvalues from the spectrum then

$$X = X_r + \sum_{j=1}^{N} (\zeta_{k_j} S_{k_j} + N_{k_j}),$$

where the operators S_k, N_k, X_r commute, the S_k, N_k are finite rank and $S_k S_j = \delta_{kj} S_k$, $N_k S_j = \delta_{kj} N_k$ and N_k is nilpotent. Finally, if the selected eigenvalues are the ones with imaginary part in the interval $[-L, L]$ for some $L > 0$, then X_r is a closed operator with spectrum contained in the set $\{z \in \mathbb{C} : \Re(z) \le -p\bar{\lambda}\} \cup \{z \in \mathbb{C} : \Re(z) \le 0 ; |Im(z)| > L\} \cup \{0\}$ where the eigenspace corresponding to zero is the union of the ranges of the S_k.

5. THE PERIPHERAL SPECTRUM

Here we analyze the meaning of the spectrum on the imaginary axis.

LEMMA 5.1. *The SRB measures belong to $\mathscr{B}^{p,q}$, $p+q \le r$; $0 \in \sigma(X)$ and it is simple iff the SRB measure is unique. Moreover, the SRB measure is mixing iff 0 is the only eigenvalue on the imaginary axis. Finally, $\sigma(X) \cap i\mathbb{R}$ is a group and the associated eigenfunctions are all measures absolutely continuous with respect to a convex combination of the SRB measures.*

Proof. If $Xh = ibh$, then $\mathscr{L}_t h = e^{ibt} h$. On the other hand there cannot be Jordan blocks, indeed if $Xf = ibf + h$, then $\frac{d}{dt} e^{-ibt} \mathscr{L}_t f = h$, thus $e^{-ibt} \mathscr{L}_t f = f + th$ which, since $\mathscr{L}_t$ is uniformly bounded (Lemma 4.1), is a contradiction.

Moreover we have[9]

$$\widetilde{S}_b := \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-ibt} \mathscr{L}_t dt = \begin{cases} 0 & \text{if } ib \text{ is not an eigenvalue} \\ S_k & \text{if } ib = \zeta_k \end{cases} \tag{5.1}$$

To prove the above note the following. If ib is not an eigenvalue,

$$\begin{aligned} \int_0^T e^{-ibt} \mathscr{L}_t &= \lim_{a\to 0} \int_0^T e^{-(a+ib)t} \mathscr{L}_t = \lim_{a\to 0} \left[R(a+ib) - \int_T^\infty e^{-(a+ib)t} \mathscr{L}_t \right] \\ &= \lim_{a\to 0} \left[R(a+ib) - e^{-(a+ib)T} \mathscr{L}_T \int_0^\infty e^{-(a+ib)t} \mathscr{L}_t \right] \\ &= \lim_{a\to 0} (\mathbf{Id} - e^{-(a+ib)T} \mathscr{L}_T) R(a+ib) \\ &= (\mathbf{Id} - e^{-ibT} \mathscr{L}_T) R(ib), \end{aligned}$$

which is uniformly bounded in T. On the other hand if $ib = \zeta_k$, then $R(a+ib) = (a+ib-\zeta_k)^{-1} S_k + R_1(a+ib)$, where $R_1(z)$ is an analytic function in a neighborhood of ib [15, 3.6.5 p. 180]. The result then follows by the same computations as above.[10]

[9] The integral must be interpreted in the strong topology.

[10] For further use note that the convergence in (5.1) takes place not only in $\mathscr{B}^{p,q}$, $p > 0$, where we have nontrivial spectral information, but also in $\mathscr{B}^{0,q}$. To see this, first notice that Lemma

Let ν be an SRB measure and let m be the Riemannian (Lebesgue) measure. By definition (cf. footnote 6) there exists an open set A such that, for each $\varphi \in \mathscr{C}^0$ and Lebesgue a.e. $x \in A$, $\frac{1}{T}\int_0^T \varphi \circ T_t(x)dt \to \nu(\varphi)$. Thus, given $h \in \mathscr{C}^\infty$, $\operatorname{supp} h \subset A$, $m(h) = 1$, $\forall \varphi \in \mathscr{C}^r$ by the Lebesgue Dominated Convergence and Fubini Theorems

$$\mu_h(\varphi) := S_0 h(\varphi) = \lim_{T\to\infty} \int_{\mathscr{M}} \frac{1}{T}\int_0^T h(x)\varphi(T_t x)dt = \nu(\varphi).$$

In view of Lemma 3.2, the above implies that $\mu_h = \nu$, that is $\nu \in \mathscr{B}^{p,q}$. In other words the SRB measures belong to the space and are eigenfunctions corresponding to the eigenvalue zero of X.

Next, let us define $\mu := S_0 1$. The inequality

$$|\mu(\phi)| \le \lim_{T\to\infty} \frac{1}{T}\int_0^T m(|\phi| \circ T_t)dt \le |\phi|_\infty$$

shows that μ is a measure. In addition, if $Xh = ibh$ and S is the corresponding projection, since $\mathscr{C}^r$ is dense in $\mathscr{B}^{p,q}$ and $S\mathscr{C}^r$ is finite-dimensional, it follows that $S\mathscr{B}^{p,q} = S\mathscr{C}^r$. Hence there exists $f \in \mathscr{C}^r$ such that $h = Sf$. Accordingly,

$$\big|h(\varphi)\big| = \big|Sf(\varphi)\big| \le \lim_{T\to\infty} \int_{\mathscr{M}} \frac{1}{T}\int_0^T \varphi \mathscr{L}_t|f| \le |f|_\infty \mu(\varphi). \tag{5.2}$$

Therefore all the eigenfunctions corresponding to eigenvalues on the imaginary axis are measures and such measures are absolutely continuous with respect to μ and with bounded density.

Consequently, if $Xh = ibh$, then h is a measure and there exists $f \in L^\infty(\mathscr{M}, \mu)$ such that $dh = f d\mu$. But then

$$f\mu = h = e^{-ibt}\mathscr{L}_t h = e^{-ibt}\mathscr{L}_t f\mu = e^{-ibt} f \circ T_{-t}\mathscr{L}_t\mu = e^{-ibt} f \circ T_{-t}\mu,$$

hence $f \circ T_{-t} = e^{ibt} f$ μ-a.s.. The above argument shows that the peripheral spectrum of $\mathscr{L}_t$ on $\mathscr{B}^{p,q}$ is contained, with multiplicity, in the point spectrum of the Koopman operator $U_t f := f \circ T_{-t}$ acting on $L^2(\mathscr{M}, \mu)$. In fact, the two objects coincide as we are presently going to see.

Let $t \in \mathbb{R}_+$ and $f \in L^2(\mathscr{M}, \mu)$ such that $U_t f = e^{ibt} f$. Note that, since $U_t|f| = |f|$, the sets $\{x \in \mathscr{M} : |f(x)| \le L\}$ are μ-a.s. invariant. Thus we can consider, without loss of generality, the case $f \in L^\infty(\mathscr{M}, \mu)$. By the Lusin Theorem and the density of $\mathscr{C}^r$ in $\mathscr{C}^0$, for each $\varepsilon > 0$ there exists $f_\varepsilon \in \mathscr{C}^r$, $|f_\varepsilon|_\infty \le |f|_\infty$, such that $\mu(|f_\varepsilon -$

4.1 implies $\|S_0 h\|_{0,q} \le C_q \|h\|_{0,q}$ for each $h \in \mathscr{B}^{1,q}$, hence S_0 has a unique continuous extension to $\mathscr{B}^{0,q}$. Next, consider $h \in \mathscr{B}^{0,q}$. There exists $\{h_n\} \subset \mathscr{B}^{1,q}$ such that $\lim_{n\to\infty} \|h - h_n\|_{0,q} = 0$. Moreover, by Lemma 4.1, $\|T^{-1}\int_0^T \mathscr{L}_t(h_n - h)\|_{0,q} \le C_q\|h - h_n\|_{0,q}$. Thus

$$\limsup_{T\to\infty} \left\| \frac{1}{T}\int_0^T \mathscr{L}_t h - S_0 h_n \right\|_{0,q} \le C_q \|h - h_n\|_{0,q}.$$

To conclude note that the range of S_0 is finite-dimensional, hence there exists a convergent subsequence $S_0 h_{n_j}$, let $\bar{h}$ be the limit, then, taking the limit $j \uparrow \infty$ follows $S_0 h = \bar{h}$ and

$$\lim_{T\to\infty} \left\| \frac{1}{T}\int_0^T \mathscr{L}_t h - S_0 h \right\|_{0,q} = 0.$$

$f|) \le \varepsilon$. Next, let us define, for each $f \in L^2(\mathcal{M},\mu)$, $R'(z)f := \int_0^\infty e^{-zt}U_t f$. A direct computation shows that $R(z)(f\mu) = (R'(z)f)\mu$, $R'(1+ib)f = f$ and $\|f_\varepsilon\mu\|_{0,q} \le C|f|_\infty$. Accordingly, Lemma 4.3 implies

$$\|R(1+ib)^n(f_\varepsilon\mu)\|_{p,q} \le C_{p,q,\lambda',\varepsilon}(1+\lambda')^{-n} + C_{p,q,\lambda'}|f|_\infty|1+ib|,$$
$$\mu(|f - R'(1+ib)^n f_\varepsilon|) \le \mu(R'(1)^n|f - f_\varepsilon|) = \mu(|f - f_\varepsilon|) \le \varepsilon.$$

For each ε we choose n_ε such that $\|R(1+ib)^{n_\varepsilon}(f_\varepsilon\mu)\|_{p,q} \le 2C_{p,q,\lambda'}|f|_\infty|1+ib|$, so Lemma 2.2 implies that the set $\Xi := \{R(1+ib)^{n_\varepsilon}(f_\varepsilon\mu)\}$ is compact in $\mathcal{B}^{p-1,q+1}$. Let us consider a convergent subsequence ε_j, and let $\mu_f \in \mathcal{B}^{p-1,q+1}$ be the limit. Then for all $\varphi \in \mathcal{C}^{p+q}$,

$$f\mu(\varphi) = \mu(f\varphi) = \lim_{j\to\infty}\mu(R'(1+ib)^{n_{\varepsilon_j}} f_{\varepsilon_j}\cdot\varphi) = \lim_{j\to\infty}[R(1+ib)^{n_{\varepsilon_j}} f_{\varepsilon_j}\mu](\varphi) = \mu_f(\varphi).$$

The fact that the spectrum is an additive subgroup of $i\mathbb{R}$ then follows from well-known facts about positive operators [6, Section 7.4].

To conclude it suffices to prove that all the eigenfunctions of zero are SRB measure. First of all, since the range of S_0 is finite-dimensional, $S_0\mathcal{B}^{0,q+p} = S_0\mathcal{B}^{p,q}$, $\mathcal{C}^0$ is dense in $\mathcal{B}^{0,p+q}$, and remembering Footnote 10 we have $S_0\mathcal{C}^0 = S_0\mathcal{B}^{p,q}$. Hence for each $\nu \in \mathcal{B}^{p,q}$ there exists $f \in \mathcal{C}^0$ such that $\nu = S_0 f$. On the other hand, setting $f_\pm := \max\{\pm f, 0\} \in \mathcal{C}^0$, $\nu_\pm := S_0 f_\pm$ are invariant positive measures and $\nu = \nu_+ - \nu_-$, thus the range of S_0 has a base of positive probability measures. Next, we can assume, without loss of generality, that ν is an ergodic probability measure for $\{T_t\}$.[11] Then, for each $\phi \in \mathcal{C}^0$, $\phi \ge 0$, such that $\int_{\mathcal{M}} f\phi = 1$, we can define $\nu_\phi := S_0(\phi f)$. By a computation similar to (5.2), ν_ϕ is a probability measure absolutely continuous with respect to ν, hence, by ergodicity, $\nu = \nu_\phi$. Then for each $\phi \in \mathcal{C}^0$, $\phi > 0$, and $\varphi \in \mathcal{C}^q$, since Lebesgue a.e. point has forward ergodic average (see footnote 6),

$$\begin{aligned}\int_{\mathcal{M}} f\phi[\varphi^+ - \nu(\varphi)] &:= \int_{\mathcal{M}} f\phi\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi\circ T_t - \nu(\varphi)\right]\\ &= \lim_{T\to\infty}\frac{1}{T}\int_{\mathcal{M}} f\phi\int_0^T[\varphi - \nu(\varphi)]\circ T_t\\ &= S_0(f\phi)(\varphi - \nu(\varphi))\int_{\mathcal{M}} f\phi = (\nu(\varphi - \nu(\varphi))\int_{\mathcal{M}} f\phi = 0.\end{aligned}$$

Taking the sup over ϕ, the above yields $\int_{\mathcal{M}} f|\varphi^+ - \nu(\varphi)| = 0$ Accordingly, for Lebesgue almost every point in the support of f the forward average of φ is $\nu(\varphi)$, that is ν is SRB. □

[11] If not, then consider any invariant set A of positive ν measure. Since ν must be absolutely continuous with respect to μ, then the set will have positive μ measure and $\mathbf{Id}_A\frac{d\nu}{d\mu}$ is an eigenvector of U_t for each $t > 0$. Hence, by the previous discussion, $\mathbf{Id}_A\nu \in \mathcal{B}^{p,q}$. By the quasicompactness it follows that there may be only finitely many such A, hence the claim.

6. DIFFERENTIABILITY OF THE SRB MEASURES

It is possible to state precise results on the dependence of the eigenfunction on a parameter of the system. To give an idea of the possibilities let us analyze, limited to Anosov flows, a situation discussed by Ruelle in [26].

Calling $\mathscr{L}_{\eta,t}$ the transfer operator associated to the flow $T_{\eta,t}$, X_η its generator and setting $R_\eta := (z\mathbf{Id} - X_\eta)^{-1}$, it follows that the SRB measure μ_η is an eigenfunction of $R_\eta(a)$ corresponding to the eigenvalue a^{-1}. Taking $X_\eta = X + \eta X_1$ one can prove, by induction,

$$R_\eta(a) = \sum_{k=0}^{n} \eta^k [R_0(a)X_1]^k R_0(a) + \eta^{n+1}[R_0(a)X_1]^{n+1} R_\eta(a). \tag{6.1}$$

In addition, we know that a^{-1} is an isolated eigenvalue of $R_\eta(a)$. We can thus apply the perturbation theory developed in [12, Section 8] to the operator $R_\eta(a)$,[12] where we choose $\mathscr{B}^s := \mathscr{B}^{s,q+r-1-s}$ with $q \in (0,1)$ and $s \in \{0,\dots,r-1\}$. It follows that there exists $\eta_0 > 0$ such that $\mu_\eta \in \mathscr{C}^{r-2}((-\eta_0,\eta_0),\mathscr{B}^0)$. Moreover

$$\frac{d^n}{d\eta^n}\mu_\eta\Big|_{\eta=0} \in \mathscr{B}^{r-1-n}.$$

We use the natural normalization $\mu_\eta(1) = 1$ so that $\mu_\eta^{(n)}(1) = 0$. We can thus differentiate the equation $X_\eta\mu_\eta = 0$, $n \le r-2$ times with respect to η, obtaining[13]

$$X_\eta \mu_\eta^{(n)} + nX_1\mu_\eta^{(n-1)} = 0. \tag{6.2}$$

From [15, 3.6.5 p. 180] and remembering that there are no Jordan blocks we have that $R_\eta(z) = z^{-1}S_{0,\eta} + Q_\eta(z)$ where $Q_\eta(z)$ is analytic in a neighborhood of zero and $S_{0,\eta}$ is the spectral projection associated to the eigenvalue zero. In addition,

$$R_\eta(z)X_\eta = R_\eta(z)(X_\eta - z) + zR_\eta(z) = -\mathbf{Id} + zR_\eta(z).$$

Therefore

$$\lim_{z\to 0} R_\eta(z)X_\eta\mu_\eta^{(n)} = -\mu_\eta^{(n)} + S_{0,\eta}\mu_\eta^{(n)} = -\mu_\eta^{(n)}, \tag{6.3}$$

where we have used that $S_{0,\eta}\nu(\phi) = \mu_\eta(\phi)\cdot\nu(1)$ and so $S_{0,\eta}\mu_\eta^{(n)} = 0$. Combining equations (6.2) and (6.3) we may write

$$\mu_\eta^{(n)} = \lim_{z\to 0} nR_\eta(z)X_1\mu_\eta^{(n-1)} = \lim_{a\to 0^+}\int_0^\infty ne^{-at}\mathscr{L}_{\eta,t}X_1\mu_\eta^{(n-1)}dt.$$

This completes the proof of Theorem 2.

REMARK 6.1. *Note that the perturbation theory in* [18] *and* [12] *allows to investigate, by similar arguments, also the behavior of the other eigenvalues of X_η, with the related eigenspaces, outside the essential spectrum.*

---

[12] Such a theory applies since $R_\eta(a)$ satisfies a uniform Lasota–Yorke inequality, (6.1) allows to estimate the closeness of $R_0(a)$ and $R_\eta(a)$ in the appropriate norms and since the X_η are bounded operators from $\mathscr{B}^{p,q}$ to $\mathscr{B}^{p-1,q+1}$. In particular this means that the domain of X_η, viewed as a closed operator on $\mathscr{B}^{p,q}$, contains $\mathscr{B}^{p+1,q-1}$.

[13] Remembering again that X, X_1 are a bounded operators from $\mathscr{B}^{p,q}$ to $\mathscr{B}^{p-1,q+1}$, we can exchange X_0, X_1 with the derivative with respect to η provided that $n \le r-2$.

7. LASOTA–YORKE TYPE INEQUALITIES—THE TRANSFER OPERATOR

Here we prove Lemma 4.1. But first let us introduce some convenient notation.

REMARK 7.1. *We will use the notation $\prod_{i=1}^{n} v_i$ to write the action of many vector fields. That is*

$$\prod_{i=1}^{n} v_i h := v_1 \dots v_n h.$$

Note that this suggestive notation does not mean that the vector fields commute.

Let $W \in \Sigma$, $0 < n \le p$, and let $v_1, \dots, v_n$ be $\mathscr{C}^{q+n}$ vector fields defined on a neighborhood of W^+ with $|v_i|_{\mathscr{C}^{q+n}} \le 1$, and $\varphi \in \mathscr{C}_0^{n+q}(W)$ with $|\varphi|_{\mathscr{C}^{n+q}(W)} \le 1$. We need to estimate

$$\int_W v_1 \dots v_n (\mathscr{L}_t h) \cdot \varphi.$$

The basic idea is to decompose each v_i as a sum $v_i = w_i^u + w_i^f + w_i^s$ where w_i^s is tangent to W, w_i^f points in the flow direction and w_i^u is "almost" in the strong unstable direction. We will state precisely what we mean by "almost" in Lemma 7.4. The w_i^s may then be dealt with by an integration by parts and then noting that w_i^u, w_i^f are not expanded by DT_{-t} allows us to conclude.

We wish to look at the problem locally and so we use a partition of unity as given in the following lemma ([12, Lemma 3.3]):

LEMMA 7.2. *For any admissible leaf W and $t \in \mathbb{R}^+$, there exist leaves $W_1, \dots, W_\ell$, whose number ℓ is bounded by a constant depending only on t, such that*

1. *$T_{-t}(W) \subset \bigcup_{j=1}^{\ell} W_j^-$.*
2. *$T_{-t}(W^+) \supset \bigcup_{j=1}^{\ell} W_j^+$.*
3. *There exists a constant C (independent of W and t) such that a point of $T_{-t}W^+$ is contained in at most C sets W_j.*
4. *There exist functions $\rho_1, \dots, \rho_\ell$ of class $\mathscr{C}^{r+1}$ and compactly supported on W_j^- such that $\sum \rho_j = 1$ on $T_{-t}(W)$, and $|\rho_j|_{\mathscr{C}^{r+1}} \le C$.*

REMARK 7.3. *Note that the construction in Lemma 7.2 can be easily modified to ensure that there exists $c > 0$ such that for all $t \in \mathbb{R}_+$ and $|s - t| \le c\delta$, the leaves $T_s W_i$ and the partition $\rho_i \circ T_{-s}$ still satisfy properties (1–4).*

Given some index j, we will estimate

$$\left| \int_{T_t(W_j)} v_1 \dots v_n (\mathscr{L}_t h) \cdot \varphi \cdot \rho_j \circ T_{-t} \right|. \tag{7.1}$$

The needed decomposition of v_i is given by the following lemma whose proof can be found in Appendix A:

LEMMA 7.4. *Fix $\lambda' \in (0, \lambda)$. Let v be a vector field on a neighborhood of W^+ with $|v|_{\mathscr{C}^a} \le 1$, $a \le r$ and $t \in \mathbb{R}_+$. Then there exists $c > 0$ such that, for each j, there exists a neighborhood U_j of $\bigcup_{s \in [t-c\delta, t+c\delta]} T_t(W_j^+)$ and $\mathscr{C}^a(U_j)$ vector fields w^f, w^u and w^s satisfying, for all $|s - t| \le c\delta$:*

a. *if* $x \in T_s(W_j)$ *then* $v(x) = w^s(x) + w^f(x) + w^u(x)$.
b. *if* $x \in T_s(W_j)$ *then* $w^s(x)$ *is tangent to* $T_s(W_j)$.
c. *if* $x \in T_s(W_j)$ *then* $w^f(x)$ *is proportional to the flow direction* V.
d. $|w^s|_{\mathscr{C}^a(U_j)} \le C_t$, $|w^u|_{\mathscr{C}^a(U_j)} \le C_t$ *and* $|w^f|_{\mathscr{C}^a(U_j)} \le C_t$.
e. $|w^s \circ T_s|_{\mathscr{C}^a(W_j)} \le C$.
f. $|(T_s^* w^u)|_{\mathscr{C}^a(T_{-s}U_j)} \le Ce^{-\lambda' s}$ *and* $|w^f \circ T_s|_{\mathscr{C}^a(T_{-s}U_j)} \le C$,
where $(T_t^* w^u) = DT_t(x)^{-1} w^u(T_t x)$ *is the pullback of* w^u *by* T_t.

The fundamental remark in the following computations is that, since the commutator of two $\mathscr{C}^{n+q}$ vector fields is a $\mathscr{C}^{n+q-1}$ vector field, if we exchange two vector fields, the difference consists of terms with $n-1$ $\mathscr{C}^{n-1+q}$ vector fields, hence it can be bounded by $C_{n,q}\|\mathscr{L}_t h\|^-_{n-1,q}$. For each j in (7.1) we can then write $w_1^s + w_1^f + w_1^u$ instead of v_1 since they agree on $T_t W_j$. After that we can commute such vector fields with the vector fields v_j, $j \in \{2,\dots,n\}$, as explained above. At this point we can decompose v_2 and so until (7.1) is bounded by

$$\sum_{\sigma\in\{s,f,u\}^n} \left|\int_{T_t(W_j)} w_1^{\sigma_1}\dots w_n^{\sigma_n}(\mathscr{L}_t h)\cdot\varphi\cdot\rho_j\circ T_{-t}\right| + C_{n,q,t}\|\mathscr{L}_t h\|^-_{n-1,q}.$$

Take $\sigma \in \{s,f,u\}^n$, and let $k = \#\{i \mid \sigma_i = s\}$ and $l = \#\{i \mid \sigma_i = f\}$. Let π be a permutation of $\{1,\dots,n\}$ such that $\pi\{1,\dots,k\} = \{i \mid \sigma_i = s\}$ and $\pi\{n-l+1,\dots,n\} = \{i \mid \sigma_i = f\}$. Therefore

$$\begin{aligned}&\left|\int_{T_t(W_j)} w_1^{\sigma_1}\dots w_n^{\sigma_n}(\mathscr{L}_t h)\cdot\varphi\cdot\rho_j\circ T_{-t}\right| \le \\ &\quad\left|\int_{T_t(W_j)} \prod_{i=1}^{k} w^s_{\pi(i)} \prod_{i=k+1}^{n-l} w^u_{\pi(i)} \prod_{i=n-l+1}^{n} w^f_{\pi(i)}(\mathscr{L}_t h)\cdot\varphi\cdot\rho_j\circ T_{-t}\right| + C_{n,q,t}\|\mathscr{L}_t h\|^-_{n-1,q}.\end{aligned}$$

By definition $w_i^f(g) = \alpha_i V(g)$, where $\alpha_i \in \mathscr{C}^{n+q}$, so $w_i^f(g) = -\alpha_i Xg - \alpha_i g \operatorname{div} V$, where X, for the time being, is defined by (2.4). The terms coming from taking derivatives of the α_i or the terms involving the divergence of the vector fields are bounded by the $\|\cdot\|^-_{n-1,q}$. In particular $\|X^l h\|^-_{n-l,q} \le \|h\|^-_{n,q} + C_{n,q}\|h\|_{n-1,q}$. Hence, setting $\bar\alpha := (-1)^l \prod_{i=k+1}^{k+l}\alpha_{\pi(i)}$, for $k>0$ we have

$$\begin{aligned}&\left|\int_{T_t(W_j)} w_1^{\sigma_1}\dots w_n^{\sigma_n}(\mathscr{L}_t h)\cdot\varphi\cdot\rho_j\circ T_{-t}\right| \le \\ &\quad\left|\int_{T_t(W_j)} \prod_{i=1}^{k} w^s_{\pi(i)} \prod_{i=k+1}^{n-l} w^u_{\pi(i)} X^l(\mathscr{L}_t h)\cdot\varphi\cdot\rho_j\circ T_{-t}\cdot\bar\alpha\right| + C_{n,q,t}\|\mathscr{L}_t h\|_{n-1,q}.\end{aligned} \tag{7.2}$$

Next, we integrate by parts with respect to the vector fields $w^s_{\pi(i)}$. These vector fields are tangent to the manifold W, hence $\int_W w^s_{\pi(i)} f\cdot g = -\int_W f\cdot w^s_{\pi(i)} g + \int_W fg\cdot \operatorname{div} w^s_{\pi(i)}$. Since $w^s_{\pi(i)}$ is $\mathscr{C}^{q+n}$ and the manifold W is $\mathscr{C}^{r+1}$ with a $\mathscr{C}^{r+1}$ volume form, the divergence terms are bounded by $C_{n,q,t}\|\mathcal{L}_t h\|_{n-1,q}$. This yields

$$\left|\int_{T_t(W_j)} w_1^{\sigma_1}\dots w_n^{\sigma_n}(\mathcal{L}_t h)\cdot\varphi\cdot\rho_j\circ T_{-t}\right| \\ \le \left|\int_{T_t(W_j)}\prod_{i=k+1}^{n-l} w^u_{\pi(i)}X^l(\mathcal{L}_t h)\cdot\prod_{i=1}^{k} w^s_{\pi(i)}(\varphi\cdot\rho_j\circ T_{-t}\cdot\bar\alpha)\right| + C_{n,q,t}\|\mathcal{L}_t h\|_{n-1,q}.$$

By Lemma 7.4 it follows that $\prod_{i=1}^k w^s_{\pi(i)}(\varphi\cdot\rho_j\circ T_{-t}\cdot\bar\alpha)$ is a $\mathscr{C}^{q+n-k}$ test function while only $n-k$ vector fields act on $\mathcal{L}_t h$. Thus the above integral can be bounded by the $\|\cdot\|_{n-1,q}$ norm unless $k=0$.

Next we need to analyze the case $k=0$ in more detail. For each $h\in\mathscr{C}^r$, $X^l\mathcal{L}_t h=\mathcal{L}_t X^l h=(X^l h)\circ T_{-t}\det(DT_t)^{-1}\circ T_{-t}$.[14] If we differentiate $\det(DT_t)^{-1}\circ T_{-t}$ we obtain terms bounded by $C_{n,q,t}\|X^l\mathcal{L}_t h\|_{n-l-1,q+1}\le C_{n,q,t}\|\mathcal{L}_t h\|_{n-1,q}$. Hence

$$\left|\int_{T_t(W_j)} w_1^{\sigma_1}\dots w_n^{\sigma_n}(\mathcal{L}_t h)\cdot\varphi\cdot\rho_j\circ T_{-t}\right| \\ \le \left|\int_{T_t(W_j)}\prod_{i=1}^{n-l} w^u_{\pi(i)}(X^l h)\circ T_{-t}\cdot\varphi\cdot\left[\rho_j\cdot\det(DT_t)^{-1}\right]\circ T_{-t}\cdot\bar\alpha\right| + C_{n,q,t}\|\mathcal{L}_t h\|_{n-1,q}.$$

Let $\bar w_i^u(x)=DT_t(x)^{-1}w_i^u(T_t x)$. This is a vector field on a neighborhood of W_j^+. We can then write the above integral as

$$\int_{T_t(W_j)}\left(\prod_{i=1}^{n-l}\bar w^u_{\pi(i)}X^l h\right)\circ T_{-t}\cdot\rho_j\circ T_{-t}\cdot\det(DT_t)^{-1}\circ T_{-t}\cdot\bar\alpha\cdot\varphi,$$

and, changing variables, we obtain

$$\int_{W_j}\prod_{i=1}^{n-l}\bar w^u_{\pi(i)}X^l h\cdot(\bar\alpha\varphi)\circ T_t\cdot\rho_j\cdot\det(DT_t)^{-1}\cdot J_W T_t, \tag{7.3}$$

where $J_W T_t$ is the Jacobian of $T_t\colon W_j\to W$. Note that

$$\left|(\bar\alpha\varphi)\circ T_t\right|_{\mathscr{C}^{q+n}}\le C_{p,q}\left|\varphi\right|_{\mathscr{C}^{q+n}}\le C_{p,q},$$

because of Lemma 7.4. Moreover, $\left|\bar w^u_{\pi(i)}\right|_{\mathscr{C}^{q+n}}\le C_{p,q}e^{-\lambda' t}$ (see Lemma 7.4) and so:

$$\prod_{i=k+1}^{n-l}|\bar w^u_{\pi(i)}|_{\mathscr{C}^{q+n}}\le C_{p,q}e^{-\lambda'(n-k-l)t}. \tag{7.4}$$

[14] Since for smooth ϕ holds $VT_t\phi=T_tV\phi$ and we have used (2.2).

Putting together all the above estimates we finally obtain[15]

$$\left|\int_W v_1 \dots v_n(\mathscr{L}_t h)\cdot\varphi\right| \le \sum_{\substack{0\le l\le n\\ j\le\ell}}\left|\int_{W_j} V^l \prod_{i=1}^{n-l}\bar{w}^u_{\pi(i)} h\cdot(\bar{\alpha}\varphi)\circ T_t \frac{\rho_j\cdot J_W T_t}{\det(DT_t)}\right| + C_{p,q,t}(\|\mathscr{L}_t h\|_{n-1,q}+\|h\|_{n-1,q}). \tag{7.5}$$

To conclude, we need the following distortion lemma:[16]

LEMMA 7.5 ([12] Lemma 6.2)**.** *Given $W\in\Sigma$ and leaves W_j such that $W\subset\bigcup_{j\le\ell}W_j$ and $W^+\supset\bigcup_{j\le\ell}W_j$, we have the following control:*

$$\sum_{j\le\ell}\left|J_W T_t\cdot\det(DT_t)^{-1}\right|_{\mathscr{C}^r(W_j,\mathbb{R})}\le C. \tag{7.6}$$

Lemma 7.5 together with (7.4) and (7.5) implies, for all $0<n\le p$,

$$\begin{aligned}\|\mathscr{L}_t h\|^-_{0,q} &\le C\|h\|^-_{0,q}\\ \|\mathscr{L}_t h\|^-_{n,q} &\le Ce^{-\lambda' t}\|h\|^-_{n,q}+C\|V^n h\|_{0,q+n}+C_{p,q,t}(\|\mathscr{L}_t h\|_{n-1,q}+\|h\|_{n-1,q}).\end{aligned} \tag{7.7}$$

The idea is to finish the proof by induction. For $n=0$ the first inequality of (7.7) is the same as $\|\mathscr{L}_t h\|_{0,q}\le C_{p,q}\|h\|_{0,q}$. On the other hand if $\|\mathscr{L}_t h\|_{m,q}\le C_{p,q}\|h\|_{m,q}$ for each $m\le n<p$, then the second inequality of (7.7) yields

$$\begin{aligned}\|\mathscr{L}_t h\|^-_{n+1,q} &\le Ce^{-\lambda' t}\|h\|^-_{n+1,q}+C\|X^{n+1}h\|_{0,q+n+1}+C_{p,q,t}(\|\mathscr{L}_t h\|_{n,q}+\|h\|_{n,q})\\ &\le Ce^{-\lambda' t}\|h\|^-_{n+1,q}+C\|X^{n+1}h\|_{0,q+n+1}+C_{p,q,t}\|h\|_{n,q}.\end{aligned}$$

Next, choose t_0 such that $Ce^{-\lambda' t_0}\le\sigma<1$ Then

$$\begin{aligned}\|\mathscr{L}_{t_0+t}h\|^-_{n+1,q} &\le \sigma\|\mathscr{L}_t h\|^-_{n+1,q}+C\|\mathscr{L}_t X^{n+1}h\|_{0,q+n+1}+C_{p,q}\|\mathscr{L}_t h\|_{n,q}\\ &\le \sigma\|\mathscr{L}_t h\|^-_{n+1,q}+C_{p,q}\|X^{n+1}h\|_{0,q+n+1}+C_{p,q}\|h\|_{n,q}.\end{aligned}$$

Writing t as mt_0+s, $s\in(0,t_0)$, and iterating the above equation yields

$$\begin{aligned}\|\mathscr{L}_t h\|^-_{n+1,q} &\le \sigma^m\|\mathscr{L}_s h\|^-_{n+1,q}+(1-\sigma)^{-1}C_{p,q}\left[\|X^{n+1}h\|_{0,q+n+1}+\|h\|_{n,q}\right]\\ &\le C_{p,q}\|h\|^-_{n+1,q}+C_{p,q}\|h\|_{n,q}.\end{aligned}$$

Finally we have

$$\|\mathscr{L}_t h\|_{n+1,q}\le\|\mathscr{L}_t h\|^-_{n+1,q}+\|\mathscr{L}_t h\|_{n,q}\le C_{p,q}\|h\|_{n+1,q}.$$

This completes the proof of Lemma 4.1.

[15] Where we have used again the possibility to commute the vector fields by paying an error bound in the $\|\cdot\|_{n-1,q}$ norm and we have recalled (2.4).

[16] In fact, [12] applies to hyperbolic maps, yet the proof holds also for flows with the only change of thickening $T_t W_j$ by ρ also in the flow direction.

8. LASOTA–YORKE TYPE ESTIMATES—THE RESOLVENT

In this section we prove Lemma 4.3. In order to do this note that the following may be shown by induction from equation (2.5):

$$R(z)^m h = \frac{1}{(m-1)!}\int_0^\infty t^{m-1} e^{-zt}\mathcal{L}_t h\,dt. \tag{8.1}$$

The first inequality of Lemma 4.3 follows directly from equations (4.1) and (8.1) by integration over t. Analogously we can use (4.1) to cut the domain of integration.

Indeed for each $z := a+ib$, $a \ge a_0 > 0$, $\beta \ge 16$ and $L := \frac{m\beta}{a}$, we have[17]

$$\begin{aligned}\left\|\frac{1}{(m-1)!}\int_L^\infty t^{m-1}e^{-zt}\mathcal{L}_t h\,dt\right\|_{p,q} &\le \frac{1}{(m-1)!}\int_L^\infty t^{m-1}e^{-at}C_{p,q}\|h\|_{p,q}\\ &\le C_{p,q}a^{-m}e^{-\frac{m\beta}{2}}\|h\|_{p,q}.\end{aligned} \tag{8.2}$$

Accordingly, to prove the second part of Lemma 4.3 it is sufficient to fix $n \le p$, $|v_i|_{\mathcal{C}^{q+n}} \le 1$, $|\varphi|_{\mathcal{C}_0^{n+q}} \le 1$ and estimate

$$\frac{1}{(m-1)!}\int_0^L t^{m-1}e^{-zt}\int_W v_1\dots v_n(\mathcal{L}_t h)\cdot\varphi\,dt.$$

To do so it is convenient to localize in time by introducing a smooth partition of unity $\{\phi_i\}$ of $\mathbb{R}_+$ subordinated to the partition $\{[(s-1/2)t_*,(s+3/2)t_*]\}_{s\in\mathbb{N}}$, where $t_* = c\delta$ and c is specified in Remark 7.3. In fact, it is possible to have such a partition of the form $\phi_s(t) := \phi(t-st_*)$ for some fixed function ϕ.

We will use the notation of Section 7 and the formula (7.5) where the families of submanifolds are chosen for each $t = st_*$, $s \in \mathbb{N}$, according to Lemma 7.2 and for $t \ne st_*$ the families of submanifolds are constructed as described in Remark

---

[17] Indeed, setting $I(m) := \int_L^\infty t^m e^{-at}$, integrating by parts yields $I(m) = L^m a^{-1}e^{-aL} + ma^{-1}I(m-1)$. Hence, by induction, we can prove the formula

$$\frac{1}{(m-1)!}I(m-1) = \sum_{j=0}^{m-1}\frac{L^j}{a^{m-j}j!}e^{-aL} = a^{-m}\sum_{j=0}^{m-1}\frac{m^j\beta^j}{j!}e^{-m\beta} \le a^{-m}\sum_{j=0}^{m-1}\left(\frac{m}{j}e\right)^j\beta^j e^{-m\beta},$$

since $j! \ge j^j e^{-j}$. Next, since the maximum of $\left(\frac{m}{j}e\right)^j$ is achieved for $j=m$, hence $\left(\frac{m}{j}e\right)^j \le e^m$,

$$\frac{1}{(m-1)!}I(m-1) \le \frac{a^{-m}\beta^m}{\beta-1}e^{-m(\beta-1)} \le Ca^{-m}e^{-m\frac{\beta}{2}}.$$

7.3. We can then write, for each $s \in \mathbb{N}$ and setting $t_s := st_* - t$,[18]

$$\left|\int_0^L t^{m-1} e^{-zt} \phi_s(t) \int_W v_1 \dots v_n(\mathscr{L}_t h) \cdot \varphi dt\right| \le \sum_{\substack{0\le l\le n\\ j\le \ell}} \left|\int_0^L \frac{t^{m-1}\phi_s(t)}{e^{zt}} \int_{T_{t_s} W_j} V^l \prod_{i=1}^{n-l} \bar{w}^u_{\pi(i)} h \cdot \frac{(\bar{\alpha}\varphi)\circ T_t \cdot \rho_j \circ T_{t_s} \cdot J_W T_t}{\det(DT_t)}\right| + C_{p,q,L} L^m m^{-1} \|h\|_{n-1,q}, \tag{8.3}$$

where we have used equations (7.5), (7.7). Changing variables and using the Fubini Theorem on all the right hand side integrals and setting $t_s^+ := st_* + t$ yields

$$\int_{W_j}\int_{\mathbb{R}} (t_s^+)^{m-1} e^{-zt_s^+} \phi(t) V^l \left(\left(\prod_{i=1}^{n-l} \bar{w}^u_{\pi(i)} h\right)\circ T_t\right) \frac{(\bar{\alpha}\varphi)\circ T_{st_*} \cdot \rho_j \cdot J_W T_{st_*}}{\det(DT_{t_s})^{-1}\circ T_t}.$$

For $l \neq 0$, we can integrate by parts, since $V(\Psi\circ T_t) = \frac{d}{dt}\Psi\circ T_t$, obtaining

$$C\|h\|_{n-1,q}|z|\int_{\mathbb{R}_+} t^{m-1} e^{-at}\phi_s(t) \le C\|h\|_{n-1,q}|z|a^{-m}. \tag{8.4}$$

For $l = 0$ and $n = p$, remembering (7.4), we have

$$\sum_{\substack{s\in\mathbb{N}\\ j\le\ell}} \frac{1}{(m-1)!}\left|\int_0^L t^{m-1} e^{-zt}\phi_s(t)\int_{W_j}\prod_{i=1}^{p} \bar{w}^u_{\pi(i)} h\cdot(\bar{\alpha}\varphi)\circ T_t \cdot \frac{\rho_j \cdot J_W T_t}{\det(DT_t)}\right| \le \frac{C_{p,q}}{(m-1)!}\int_{\mathbb{R}_+} t^{m-1} e^{-(a+\lambda' p)t}\|h\|^-_{p,q} \le C_{p,q}(a+\lambda' p)^{-m}\|h\|^-_{p,q}. \tag{8.5}$$

In the case $l = 0$, $n < p$ we use a regularization trick to obtain the desired decay in the norm. Since the composition with T_t decreases the derivatives one can take advantage of such a fact by smoothing the test function.

For $\varepsilon \le \delta$ and $\bar{\varphi} \in \mathscr{C}^a_0(W,\mathbb{R})$, let $\mathbb{A}_\varepsilon\bar{\varphi} \in \mathscr{C}^{a+1}_0(W^+,\mathbb{R})$ be obtained by convolving $\bar{\varphi}$ with a $\mathscr{C}^\infty$ mollifier whose support is of size ε. We will use the following standard result.

LEMMA 8.1. *For each $n \in \mathbb{N}$, $q \in \mathbb{R}_+$ and $\bar{\varphi} \in \mathscr{C}^{q+n}$,*

$$|\mathbb{A}_\varepsilon\bar{\varphi}|_{\mathscr{C}^{q+n}} \le C|\bar{\varphi}|_{\mathscr{C}^{q+n}}, \quad |\mathbb{A}_\varepsilon\bar{\varphi}|_{\mathscr{C}^{q+1+n}} \le C\varepsilon^{-1}|\bar{\varphi}|_{\mathscr{C}^{q+n}}, \quad |\mathbb{A}_\varepsilon\bar{\varphi} - \bar{\varphi}|_{\mathscr{C}^{q+n}} \le C\varepsilon|\bar{\varphi}|_{\mathscr{C}^{q+n+1}}.$$

Hence, setting $\Delta\varphi = (\varphi - \mathbb{A}_\varepsilon\varphi)\circ T_t$, the action of T_t on the derivatives yields $|\Delta\varphi|_{\mathscr{C}^{q+n}} \le Ce^{-\lambda(q+n)t}$, provided one chooses $\varepsilon \le Ce^{-\lambda(q+n)t}$. Thus, using (7.4) as

[18]By construction, the manifolds $\{W_j\}$ in the formula (8.3) depend on s but not on t.

well, we have

$$(8.6)\quad \sum_{\substack{s\in\mathbb{N}\\ j\le\ell}}\frac{1}{(m-1)!}\left|\int_0^L t^{m-1}e^{-zt}\phi_s(t)\int_{W_j}\prod_{i=1}^n \bar w^u_{\pi(i)}h\cdot\varphi\circ T_t\cdot\frac{\rho_j\cdot J_W T_t}{\det(DT_t)}\right|$$
$$\le\sum_{\substack{s\in\mathbb{N}\\ j\le\ell}}\int_0^L\frac{t^{m-1}e^{-zt}\phi_s(t)}{(m-1)!}\left|\int_{W_j}\prod_{i=1}^n\bar w^u_{\pi(i)}h\cdot\frac{\Delta\varphi\cdot\rho_j\cdot J_W T_t}{\det(DT_t)}\right|$$
$$+\frac{C_{p,q,a_0,L}L^m\|h\|^-_{n,q+1}}{m!}$$
$$\le C_{p,q}(a+\lambda(q+n))^{-m}\|h\|_{p,q}+C_{p,q,a_0,L}\frac{L^m}{m!}\|h\|^-_{n,q+1}.$$

Collecting equations (8.2), (8.3), (8.4), (8.5) and (8.6) yields, for each $n\le p$,

$$\|R(z)^m h\|^-_{n,q}\le C_{p,q}\left[a^{-m}e^{-\frac{m\beta}{2}}+(a+\lambda' p)^{-m}+(a+\lambda q)^{-m}\right]\|h\|_{n,q}$$
$$+(a^{-m}|z|+C_{p,q,L}m^{-1})\|h\|_{n-1,q}+C_{p,q,a_0,L}\frac{L^m}{m!}\|h\|_{n-1,q+1}.$$

To conclude it is convenient to introduce, for each $0<A<1$, the equivalent weighted norms[19]

$$\|h\|_{p,q,A}:=\sum_{n\le p}A^n\|h\|^-_{n,q}.$$

Using such a norm we can write

$$\|R(z)^m h\|_{p,q,A}\le C_{p,q}\left[a^{-m}e^{-\frac{m\beta}{2}}+(a+\lambda' p)^{-m}+(a+\lambda q)^{-m}\right]\|h\|_{p,q,A}$$
$$+A(a^{-m}|z|+C_{p,q,L}m^{-1})\|h\|_{p,q,A}+C_{p,q,a_0,L}\frac{L^m}{m!}\|h\|_{p-1,q+1,A}.$$

For each $\lambda''<\lambda'$, calling $\bar p:=\min\{p,q\}$, there exists $m_a\in\mathbb{N}$, e.g., $m_a=C_{\lambda'',p,q}\,a$ will do, such that $C_{p,q}(a+\lambda'\bar p)^{-m_a}\le\frac14(a+\lambda''\bar p)^{-m_a}$. Choosing then β, and hence L, large enough[20] and A small enough we have

$$\|R(z)^{m_a}h\|_{p,q,A}\le(a+\lambda''\bar p)^{-m_a}\|h\|_{p,q,A}+C_{p,q,a_0}a^{-m_a}|z|\|h\|_{p-1,q+1,A},$$

which can be iterated to yield the desired estimate (given the equivalence of the norms).

APPENDIX A.

Proof of Lemma 7.4. Our aim is to write the vector field as $v=w^s+w^u+w^f$. We start by making a $\mathscr{C}^{r+1}$ change of variables in the charts[21] so that W_j^+ and W^+ are subsets of $\mathbb{R}^{d_s}\times\{0\}\times\{0\}$ while $T_t(s,u,\tau)=(s,u,\tau+t)$. In addition, having chosen $z\in W_j$, we can assume, without loss of generality, that $E^u(z)=\{(0,0,u)$:

[19]The advantage of using weighted norms has been pointed out to us by Sébastien Gouëzel.

[20]For example, $\beta\ge 2\lambda\bar p a^{-1}$ will do, notice that this choice implies that L can be chosen uniformly bounded with respect to a.

[21]A point in the charts will be written as $(s,\tau,u)\in\mathbb{R}^d$ with $s\in\mathbb{R}^{d_s}$, $\tau\in\mathbb{R}$ and $u\in\mathbb{R}^{d_u}$.

$u \in \mathbb{R}^{d_u}\}$ and $E^u(T_t z) = \{(0,0,u) \ : \ u \in \mathbb{R}^{d_u}\}$. We can then consider the foliation $E = \{E(s,\tau,u)\}$ of a neighborhood of W_j^+ made by the leaves $E(s,\tau,u) := \{(s,\tau,u+v) \ : \ v \in \mathbb{R}^{d_u}; |v| \le \delta\}$ and define the foliation $F = T_t E$.

The idea is to first define the splitting on $T_{t+s}W_j^+$ and then extend it to a neighborhood. We thus define the splitting on $\{(s,\tau,0)\}$ as follows: $\langle w^s, (0,\tau,u)\rangle = 0$, for each $u \in \mathbb{R}^{d_u}$, $\tau \in \mathbb{R}$; w^f is in the flow direction; w^u belongs to the tangent spaces of the leaves of the foliation F.

To verify that the splitting satisfies the wanted properties we need to write the differential of T_t in the chosen coordinates. For each x in a neighborhood of W_j^+, by the requirement that the flow direction is mapped into the flow direction, it follows that

$$DT_t(x) = \begin{pmatrix} A_t(x) & 0 & B_t(x) \\ a_t(x) & 1 & b_t(x) \\ C_t(x) & 0 & D_t(x) \end{pmatrix}.$$

Moreover, if $x \in W_j^+$, then it must be $a_t(x) = 0$; $C_t(x) = 0$ and, finally $b_t(z) = 0$ and $B_t(z) = 0$. In addition, due to the uniform hyperbolicity of the flow, we have that, for each $x \in W_j^+$, $\|A_t(x)\| \le Ce^{-\lambda t}$, while $\|(B_t(x)u, \langle b_t(x), u\rangle, D_t(x)u)\| \ge Ce^{\lambda t}\|u\|$ for each x in a neighborhood of W_j.[22] Notice as well that the size of the neighborhood we are interested in can be chosen arbitrarily, thus, by continuity, we can assume $\|C_t\|_{\mathscr{C}^r} + \|a_t\|_{\mathscr{C}^r}$ arbitrarily small.[23] Finally, since the foliation F is close to the unstable direction, $\|B_t(x)u\| + |\langle b_t(x)u\rangle| \le \frac{1}{2}\|D_t(x)u\|$, for all $u \in \mathbb{R}^{d_u}$.

By construction the tangent space to the leaves of the foliation F has the form $\{(B_t(x)D_t(x)^{-1}u, \langle b_t(x), D_t(x)^{-1}u\rangle, u) \ : \ u \in \mathbb{R}^{d_u}\}$. Accordingly, if we write $v =: (v_s, v_f, v_u)$, we have

$$\begin{aligned} w^s &= (v_s - (B_t D_t^{-1}) \circ T_{-t} v_u, 0, 0) \\ w^f &= (0, v_f - (b_t D_t^{-1}) \circ T_{-t} v_u, 0) \\ w^u &= ((B_t D_t^{-1}) \circ T_{-t} v_u, (b_t D_t^{-1}) \circ T_{-t} v_u, v_u). \end{aligned} \tag{A.1}$$

By construction such vector fields satisfy points (**a**-**d**) of Lemma 7.4; moreover they belong to $\mathscr{C}^r(T_t(W_j))$. To estimate the $\mathscr{C}^r$ norm we must study the $\mathscr{C}^r$ norm of $U_t(x) := B_t(x)D_t(x)^{-1}$ and $\beta_t(x) := b_t(x)D_t(x)^{-1}$.[24]

To do so it is convenient to break up the trajectory into pieces of finite length t_0 and, at all the points $T_{kt_0}x$, introduce the same type of coordinates already defined. By the hyperbolicity assumption, given $\lambda' \in (0,\lambda)$, it is possible to choose $t_0 \le C$ so that $nt_0 = t$ and $\|D_{t_0}(T_{kt_0}x)^{-1}\| \le e^{-\lambda' t_0}$, $\|A_{t_0}(T_{kt_0}x)\| \le e^{-\lambda' t}$, $\|T_{kt_0}x - T_{kt_0}y\| \le e^{-\lambda' t}\|T_{(k-1)t_0}x - T_{(k-1)t_0}y\|$ for each $k \le n$ and $x, y \in W_j$. Accordingly,

[22] The latter follows from the possibility to choose δ small enough so that all the tangent spaces to the foliations E lie in the unstable cone.

[23] Given a function A with values in the matrices we define $\|A\|_{\mathscr{C}^n} := \sup_k \sum_j |A_{kj}|_{\mathscr{C}^n}$. Such a definition has the useful consequence that if $A = BD$, then $\|A\|_{\mathscr{C}^n} \le \|B\|_{\mathscr{C}^n}\|D\|_{\mathscr{C}^n}$.

[24] Note that, within a chart, the matrices do not depend on x_f

since $D_{(k+1)t_0}(x) = D_{t_0}(T_{kt_0}x)D_{kt_0}(x)$,

$$\|D_t^{-1}\|_{\mathscr{C}^r} = \|D_{nt_0}^{-1}\|_{\mathscr{C}^r} \le (e^{-\lambda' t_0} + Ce^{-\lambda'(n-1)t_0})\|D_{(n-1)t_0}^{-1}\|_{\mathscr{C}^r} \le Ce^{-\lambda' t}. \tag{A.2}$$

Next, notice that

$$\begin{pmatrix} A_{(k+1)t_0}(x) & 0 & B_{(k+1)t_0}(x) \\ 0 & 1 & b_{(k+1)t_0}(x) \\ 0 & 0 & D_{(k+1)t_0}(x) \end{pmatrix} = \begin{pmatrix} A_{t_0}(T_{kt_0}x)A_{kt_0}(x) & 0 & A_{t_0}(T_{kt_0}x)B_{kt_0}(x) + B_{t_0}(T_{kt_0}x)D_{kt_0}(x) \\ 0 & 1 & b_{t_0}(T_{kt_0}x)D_{kt_0}(x) + b_{kt_0}(x) \\ 0 & 0 & D_{t_0}(T_{kt_0}x)D_{kt_0}(x) \end{pmatrix}.$$

Thus, setting $U_k := B_{kt_0}D_{kt_0}^{-1}$, we get

$$U_{k+1} = A_{t_0}(T_{kt_0}x)U_kD_{t_0}(T_{kt_0}x)^{-1} + B_{t_0}(T_{kt_0}x)D_{t_0}(T_{kt_0}x)^{-1}.$$

Hence,

$$\|U_n\|_{\mathscr{C}^r} \le (e^{-\lambda' t_0} + e^{-\lambda'(n-1)t_0})^2\|U_{n-1}\|_{\mathscr{C}^r} + C.$$

Iterating the above equation yields $\|U_t\|_{\mathscr{C}^r(W_j)} \le C$. A similar argument gives $\|\beta_t\|_{\mathscr{C}^r(W_j)} \le C$. Applying the above estimates to (A.1) yields $|w^s \circ T_t|_{\mathscr{C}^a(W_j)} \le C$, $|w^u \circ T_t|_{\mathscr{C}^a(W_j)} \le C$ and $|w^f \circ T_t|_{\mathscr{C}^a(W_j)} \le C$, which proves (**e**).

To tackle (**f**) we need to extend the vector fields smoothly; this is easily done by taking them constant along the leaves of F. Since on W_j we have $DT_t^{-1}w^u \circ T_t = (0,0,D_t^{-1}v^u \circ T_t)$ and $w^f \circ T_t = (0, v^f \circ T_t - bD_t^{-1}v^u \circ T_t, 0)$, the above estimates imply $|T_t^* w^u|_{\mathscr{C}^a(W_j)} \le Ce^{-\lambda' t}$ and $|w^f \circ T_t|_{\mathscr{C}^a(W_j)} \le C$. Since the vector fields have been extended by keeping them constant on the leaves of F, if follows that their preimages are constant along the leaf of E, that is they do not depend on u. This means that the above bounds on the norms does not increase when they are considered on the neighborhood $T_{-t}U_j$, hence point (**f**). □

REFERENCES

[1] Viviane Baladi. *Positive Transfer Operators and Decay of Correlations*, volume 16 of *Advanced Series in Nonlinear Dynamics*. World Scientific, Singapore, 2000.

[2] Viviane Baladi. Anisotropic Sobolev spaces and dynamical transfer operators: C^∞ foliations. In *Algebraic and topological dynamics*, volume 385 of *Contemp. Math.*, pages 123–135. Amer. Math. Soc., Providence, RI, 2005.

[3] Viviane Baladi and Masato Tsujii. Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms. Preprint, to appear Ann. Inst. Fourier, 2005.

[4] Viviane Baladi and Masato Tsujii. Dynamical determinants and spectrum for hyperbolic diffeomorphisms. Preprint, 2006.

[5] M.Blank, G.Keller and C.Liverani, *Ruelle–Perron–Frobenius spectrum for Anosov maps*, Nonlinearity, **15** (2002), n.6, 1905–1973.

[6] E.B.Davies, *One-Parameter semigroups*, Academic Press, London, (1980).

[7] M.Demers, C.Liverani, *Stability of statistical properties in two-dimensional piecewise hyperbolic maps*, preprint (2006)

[8] D.Dolgopyat, *On differentiability of SRB states for partially hyperbolic systems*, Invent. Math. **155** (2004), no. 2, 389–449.

[9] F.Faure, N.Roy, *Ruelle–Pollicott resonances for real analytic hyperbolic map*, Nonlinearity **19** (2006), 1233–1252.

[10] M.Field, I.Melbourne and A.Török, *Stability of mixing and rapid mixing for hyperbolic flows*. Annals of Mathematics, to appear

[11] G.Gallavotti, *Chaotic hypotesis: Onsager reciprocity and fluctuation-dissipation theorme*, Journal of Statistical Physics, **84** (1996), 899–926.

[12] S.Gouëzel and C.Liverani, *Banach spaces adapted to Anosov systems*, Ergodic Theory and Dynamical Systems, **26** (2006), 1, 189–217.

[13] S.Gouëzel and C.Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties*, preprint (2006).

[14] H.Hennion, *Sur un théorème spectral et son application aux noyaux Lipschitziens*, Proc. Amer. Math. Soc., **118** (1993), 627–634.

[15] T.Kato, *Perturbation theory for linear operators*, Springer (1966).

[16] A.Katok and B.Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press (1995).

[17] A.Katok, G.Knieper, M.Pollicott, H.Weiss, *Differentiability of entropy for Anosov and geodesic flows* Bull. Amer. Math. Soc. (N.S.) **22** (1990), no. 2, 285–293.

[18] G.Keller, C.Liverani, *Stability of the spectrum for transfer operators*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (4) Vol. **XXVIII** (1999), 141–152.

[19] C.Liverani, *On Contact Anosov flows*, Annals of Mathematics, **159** (2004), 1275–1312.

[20] C.Liverani, *Fredholm determinants, Anosov maps and Ruelle resonances*, Discrete and Continuous Dynamical Systems, **13** (2005), no. 5, 1203–1215.

[21] C.Liverani, M. Tsujii, *Zeta functions and dynamical systems*, Nonlinearity **19** (2006), no. 10, 2467–2473.

[22] M.Pollicott, *On the rate of mixing of Axiom A flows*, Invent. Math. **81** (1985), no. 3, 413–426.

[23] D.Ruelle, *Zeta-functions for expanding maps and Anosov flows*, Invent. Math. **34** (1976), no. 3, 231–242.

[24] D.Ruelle, *Differentiation of SRB States*, Communications in Mathematical Pysics, **187** (1997), 227–241.

[25] D.Ruelle, *Nonequilibrium statistical mechanics near equilibrium: computing higher-order terms*, Nonlinearity **11** (1998), no. 1, 5–18.

[26] D.Ruelle, *Differentiation of SRB states for hyperbolic flows*, `http://arxiv.org/abs/math.DS/0408097`

[27] H-H.Rugh, *Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems*, Ergodic Theory Dynam. Systems **16** (1996), no. 4, 805–819.

[28] H-H.Rugh, *The correlation spectrum for hyperbolic analytic maps*, Nonlinearity **5** (1992), no. 6, 1237–1263.

[29] M. Tsujii, *Decay of correlations in suspension semiflows of angle-multiplying maps*, preprint, `http://arxiv.org/pdf/math.DS/0606433`.

OLIVER BUTTERLEY <oliver.butterley@imperial.ac.uk>: Imperial College London, South Kensington Campus, SW7 2AZ London, UK

CARLANGELO LIVERANI <liverani@mat.uniroma2.it>: Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy