Locating Ruelle–Pollicott resonances[*]

Oliver Butterley[1,**], Niloofar Kiamari[2] and Carlangelo Liverani[1]

[1] Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy
[2] Dipartimento di Matematica e Fisica, III Università di Roma, Largo San Leonardo Murialdo, 1-00146 Roma, Italy

E-mail: butterley@mat.uniroma2.it, n.kiamari@lboro.ac.uk and liverani@mat.uniroma2.it

Abstract

We study the spectrum of transfer operators associated to various dynamical systems. Our aim is to obtain precise information on the discrete spectrum. To this end we propose a unitary approach. We consider various settings where new information can be obtained following different branches along the proposed path. These settings include affine expanding Markov maps, uniformly expanding Markov maps, non-uniformly expanding or simply monotone maps, hyperbolic diffeomorphisms. We believe this approach could be greatly generalised.

Keywords: transfer operators, resonances, spectral gap
Mathematics Subject Classification numbers: 37A25.

(Some figures may appear in colour only in the online journal)

1. Introduction

Transfer operators are used widely in dynamical systems. Their first manifestations going back, at least, to the Koopman operator, and its use by John von Neumann to prove the mean ergodic

theorem. Next, the Russian school developed the spectral theory for the Koopman operator acting on L^2 and its relation to the statistical properties of the system (such as ergodicity, mixing, ...), [27]. Later attention concentrated on the adjoint of the Koopman operator, now called the Frobenius–Perron or the Ruelle–Frobenius–Perron transfer operator. First such an operator appeared after coding the system [13]. Subsequently, starting with [58, 73], the direct study of the transfer operator acting on functions and, more recently, starting with [12], acting on spaces of distributions, acquired progressively more importance[3]. This is the current focus.

Historically research was mostly focussed on the study of the peripheral spectrum (which encodes sharp, quantitative, information on ergodicity and mixing), on establishing a spectral gap (which yields the rate of mixing) (e.g. [6, 60]), and on estimates of the essential spectrum and relations with the Ruelle zeta function (which encodes information on the spectrum of periodic orbits), see [7] for a recent review. For flows or systems with a neutral direction the study is often more involved but there is still the possibility of some type of spectral gap, e.g. [16, 17, 19, 23, 29, 80].

However, recently it has become apparent the need of a much deeper and detailed understanding of the point spectrum [18, 34, 35, 37, 40, 48–50, 57]. Possibilities include identifying the point spectrum by understanding the connection to the action of the dynamics on cohomology [37] or obtaining results related to bands of spectrum for transfer operators associated to systems with a neutral direction [18, 36, 38–40]. Additionally various works investigated the possibility of an explicit description of the spectrum for analytic expanding or hyperbolic maps [10, 76, 78] (using Blaschke products), or perturbative and generic results [1, 11, 56, 68]. Clearly, a more explicit description of the spectrum is important also in applications as it provides precise quantitative information on the invariant measures, entropy, decay of correlation, variance in the CLT and so on.

Unfortunately, no general theory exists to address this issue. One exception being the Hilbert metric technique, see [60], however such an approach can yield results only for the spectral gap and they are often far from optimal (see remarks 3.2, 3.7 and 4.9), hence the need for an alternative approach. The special cases in which some results have been obtained seem to point to a general philosophy: to study the commutator between some type of differentiation and the transfer operator (e.g. see [33, 37, 50]). Although this idea is rather vague, we believe it can give rise to a general theory. In fact, it is surprising that this approach has not been explored in any systematic way, in spite of the vast literature devoted to transfer operators. Hence, the first step to substantiate our claim is an investigation of several concrete examples. This is the task of the present work.

We apply the above philosophy to different classes of dynamical system, starting from very simple ones and increasing progressively in complexity. For each example we obtain nontrivial results that illustrate the power of this approach to the problem. Although our results fall short of a general theory, we believe they suffice to argue that walking further along this path is likely to yield interesting results in a much more general setting. Let us describe our results more in detail.

In section 2 we discuss the point spectrum for a family of transfer operators of Markov piecewise affine expanding maps. This is the simplest possible, nontrivial, example. Yet, it goes a long way in illustrating our strategy.

In section 3 we address similar questions in the case of full branch piecewise smooth expanding maps of the interval. The simplest non-linear case. This is a class of maps which has been

---

[3] But see [75] for precursors of this point of view.

extensively studied and for which one could expect that all has been said already. Yet, we are able to obtain new interesting information. In particular, we concentrate on two transfer operators. The one associated to the SRB measure for which we obtain effective bounds on the spectral gap and fine informations about the spectra. The other is the operator associated to the measure of maximal entropy for which we establish a spectral gap of size at least $e^{h_{top}} - 1$. This illustrates the fact, seen also later in other examples, that the transfer operator associated to the measure of maximal entropy enjoys surprisingly large gaps.

In section 4 we study the spectral gap for the operator associated to the measure of maximal entropy for full branch monotone maps. This includes the case of maps with attracting periodic orbits. We show that the measures of maximal entropy are exponentially mixing with a rate, at least, h_{top}. We are not aware of similar results. Apart from the case of intermittent maps (when only neutral fixed points are present) for which it is known to exist a unique measure of maximal entropy which is exponentially mixing. However, even in this special case, nothing quantitatively precise was known on the speed of mixing.

Finally, in section 5, we study hyperbolic maps. We start, as an illustration, with automorphisms of the torus. This sheds some light on the difficulties involved in extending the approach to the general hyperbolic case. Next, we propose a possible solution to such difficulties: to study the spectrum of the action of the pushforward, for hyperbolic maps, on forms. This allows, for example, to study, again, the measure of maximal entropy. Once more we obtain a large gap. In particular, our approach provides alternative proofs, and a slight strengthening, of recent results by Baladi [8, theorem 2.1] and Forni [41], moreover we establish a topological interpretation of the point spectrum which should hold in more generality.

2. Affine expanding Markov maps

In this section we discuss the simplest possible case: one dimensional piecewise affine Markov maps. This allows us to show our approach in the simplest possible form and it is presented mainly for didactical purposes. Indeed, the results of this section are essentially not new (for prior results see [21, 66] [77, section 2 and appendix], and also remark 2.6).

In this setting the invariant densities can be computed easily since the Frobenius–Perron operator can be represented by a finite-dimensional matrix (see [14, chapter 9] for full details). Here we go beyond the peripheral spectrum and show that studying a particular family of matrices yields the full Ruelle–Pollicott spectrum. To this end, the smoothness of the observables is relevant. This will be a leitmotiv in the following and it is essential since it is known that even the point spectrum of the transfer operator may change drastically if one considers a class of observables that allow discontinuities (e.g. see [12, 15] and also remark 2.3).

Let $I := [0, 1]$ and let $f : I \to I$ be a *piecewise affine expanding Markov map* in the following sense: there exists a collection of disjoint open intervals $\{I_j\}_{j=1}^N = \{(p_j, p_{j+1})\}_{j=1}^N$ which form a partition of a full measure subset of I and, for all i, j,

$$\text{either} \quad f(I_i) \cap I_j = \emptyset, \quad \text{or} \quad I_j \subseteq f(I_i).$$

Moreover, we suppose that f' is constant on each I_i. Finally we suppose that there exists $\lambda > 1$ such that[4] $f'(x) \geqslant \lambda$ for all $x \in \cup_i I_i$.

The partition $\{I_i\}_{i=1}^N$ is called a *Markov partition*. Let $\mathcal{I} = \cup_{i=1}^N I_i$ be the disjoint union of the partition elements. The $N \times N$ matrix A defined by

---

[4] We consider only the transformations f which are orientation preserving. See remark 2.8 concerning the general case $|f'| \geqslant \lambda$.

$$A[i, j] = 1, \text{ if } I_j \subseteq f(I_i) \quad \text{and} \quad A[i, j] = 0, \text{ if } f(I_i) \cap I_j = \emptyset,$$

is called the *adjacency matrix*[5] of the Markov map f. For convenience let $\lambda_j := f'|_{I_j}$ and $\lambda = \min_j \lambda_j$. For any[6] $k \in \mathbb{N}_0$, let B_k be the $N \times N$ matrix defined by

$$B_k[i, j] := \lambda_j^{-k} A[j, i]. \tag{2.1}$$

If partition elements are equally sized then B_1 is left stochastic, i.e. $\sum_i B_1[i, j] = 1$ for each j. In general there exists a diagonal matrix D such that $D^{-1} B_1 D$ is left stochastic [14, section 9.3].

For simplicity, in the following theorem we additionally suppose that f is topologically transitive. This means that there exists[7] a unique f-invariant probability measure which is absolutely continuous with respect to Lebesgue (denoted μ_{SRB}) and a unique measure of maximal entropy (also known as the Bowen–Margulis measure) (denoted μ_{BM}). We let h_{top} denote the topological entropy.

Also, we use $\mathcal{C}^\infty(\mathcal{I})$ denote the set of functions on I which are $\mathcal{C}^\infty$ when restricted to each I_j.

Finally, we use σ to denote the spectrum of a matrix and $\sigma_B(\mathcal{L})$ for the spectrum of an operator $\mathcal{L}$ acting on a Banach space B.

We can now state a result concerning Ruelle–Pollicott resonances.

Theorem 2.1. *There exists a set of complex numbers $\Xi_1 = \{\xi_1, \xi_2, \ldots\}$ and, for each $\xi_i \in \Xi_1$, an associated integer[8] m_i such that, for any $\phi, \varphi \in \mathcal{C}^\infty(\mathcal{I})$ and $\epsilon > 0$ there is an asymptotic expansion*

$$\int_I \phi \cdot \varphi \circ f^n \, d\mu_{\text{SRB}} = \sum_{\xi_i \in \Xi_1 : |\xi_i| \geqslant \epsilon} \sum_{k=0}^{m_i - 1} \xi_i^n n^k C_{i,k}(\phi, \varphi) + o(\epsilon^n)$$

where $C_{i,k}(\phi, \varphi)$ are finite rank and non-zero bilinear functions of ϕ, φ.

The set Ξ_1 is equal (as a subset of $\mathbb{C}$) to $\bigcup_{l=1}^\infty \sigma(B_l)$ and the equality holds also for the total multiplicity of each eigenvalue[9].

Similarly there exists a set of complex numbers $\Xi_0 = \{\xi_1, \xi_2, \ldots\}$ and for each $\xi_i \in \Xi_0$ an associated integer m_i such that, for any $\phi, \varphi \in \mathcal{C}^\infty(\mathcal{I})$ and $\epsilon > 0$ there is an asymptotic expansion

$$\int_I \phi \cdot \varphi \circ f^n \, d\mu_{\text{BM}} = e^{-n h_{top}} \sum_{\xi_i \in \Xi_0 : |\xi_i| \geqslant \epsilon} \sum_{k=0}^{m_i - 1} \xi_i^n n^k C'_{i,k}(\phi, \varphi) + o(\epsilon^n)$$

where $C'_{i,k}(\phi, \varphi)$ are finite rank and non-zero bilinear functions of ϕ, φ. The set Ξ_0 is equal (as a subset of $\mathbb{C}$) to $\bigcup_{l=0}^\infty \sigma(B_l)$ and the equality holds also for the total multiplicity of each eigenvalue.

---

[5] It is also called the *incidence* matrix [14].

[6] We use throughout the convention $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

[7] The existence of these invariant measures is well known in this context and also follows from the results later in this section.

[8] The integer m_i is the Jordan block dimension. A given ξ_i might be repeated in Ξ_1 according to the geometric multiplicity.

[9] More can be said about the multiplicities and Jordan blocks, see theorem 2.4 and remark 2.12.

The proof of the above theorem is included towards the end of the section and follows from a significantly stronger result (theorem 2.4), described in terms of transfer operators, that needs some further preliminaries to be properly stated.

Remark 2.2. The assumption of topological transitivity means that B_1 is irreducible. Since also $D^{-1}B_1 D$ is left stochastic for some diagonal matrix D it follows that 1 is the leading eigenvalue of B_1 and this eigenvalue has multiplicity 1. Moreover $C_{1,0}(\phi, \varphi) = \int \phi \, d\mu_{SRB} \int \varphi \, d\mu_{SRB}$.

Remark 2.3. In the case where f has the form $x \mapsto \kappa x \mod 1$ for some $\kappa \in \{2, 3, \ldots\}$ we could consider f as a smooth map of the circle. In this case, restricting our attention to observables which are smooth on the circle, the set of Ruelle–Pollicott resonances would reduce[10] to $\{0\}$. However, studying these same systems for observables that are smooth on the interval, we see a much more interesting spectrum, see remark 2.13.

Observe that for any $r \geqslant 0$, the Sobolev space $W^{r,1}(I)$ is the set of all $h \in L^1(I)$ such that h and all of its weak derivatives up to the rth belong to $L^1(I)$. Consider, for any $r \geqslant 0$, the space $W^{r,1}(\mathcal{I})$ which is the set of all $h \in L^1(I)$ such that, for each i, the restriction of h to I_i is in $W^{r,1}(I_i)$. For convenience we write h' and $h^{(l)}$ to mean the weak derivative and l weak derivative respectively of h restricted to $\mathcal{I}$. For each $r \in \mathbb{N}_0$ the space $W^{r,1}(\mathcal{I})$ is a Banach space equipped with the norm

$$\|h\|_{r,1} = \sum_{l=0}^{r} \int_{\mathcal{I}} |h^{(l)}(x)| \, dx.$$

In the following, to simplify notation, we will write W_r for $W^{r,1}(\mathcal{I})$ and we will write $\|\cdot\|_r$ for $\|\cdot\|_{r,1}$. Observe that W_0 coincides with $L^1(I)$.

Since, by assumption, $f|_{I_j}$ is invertible on its range, let us call g_j its inverse ($g_j := f|_{I_j}^{-1}$). The domain of g_j is the interval $f(I_j)$ which might not be equal to the unit interval. If $f(I_j) = (0, 1)$ for all j then f is said to be a *full branch* map. We can now define our main objects of investigation: the transfer operators. For all $k \in \mathbb{N}_0$, $h \in L^1(I)$ and $x \in I_i$ we define[11]

$$\mathcal{L}_k h(x) := \sum_{y \in f^{-1}(x)} \frac{h(y)}{[f'(y)]^k} = \sum_j B_k[i, j] \, h \circ g_j(x).$$

Since f preserves the Markov partition, composition with an affine transformation preserves Sobolev space and the sum consists of a finite number of terms it follows that these operators are well defined as operators $\mathcal{L}_k : W_r \to W_r$. Similarly they are well defined, by this same formula, on $\mathcal{C}^r(\mathcal{I}) = \bigoplus_i \mathcal{C}^r(I_i)$.

Observe that $\mathcal{L}_1$ coincides with the usual transfer operator: the dual of the Koopman operator.

---

[10] This can be seen by considering the action of the dynamics on Fourier series.

[11] The second sum here is understood in the sense that, when the summands are defined on a subset of the full integral, they are extended to the full interval by taking the value zero where not defined.

We define $\mathfrak{P}_r(\mathcal{I})$ to be the set of polynomial functions[12] of degree r on each interval I_j. Since f is piecewise affine, the space $\mathfrak{P}_r(\mathcal{I})$ is invariant under $\mathcal{L}_k$ for each $r, k \in \mathbb{N}_0$. Thus, it is natural to consider the finite rank operator $\mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})}$.

Theorem 2.4. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. There exists a projector $\Pi_{k,r} : W_r \to \mathfrak{P}_r(\mathcal{I}) \subset W_r$ such that the spectral radius of $\mathcal{L}_k(1 - \Pi_{k,r})$ on W_r is not greater than $\lambda^{-(k+r-1)}$. Moreover*

$$\sigma\left(\mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})}\right) = \bigcup_{l=0}^{r} \sigma(B_{k+l})$$

and the multiplicity of each eigenvalue $\xi \in \sigma\left(\mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})}\right)$ is equal to the sum of the multiplicities of ξ as eigenvalues of B_{k+l}, $l \in \{0, \ldots, r\}$.

The remainder of this section is devoted to the proofs of the two above theorems.

Remark 2.5. This result tells nothing about the spectrum of $\mathcal{L}_k$ within the disk $\{|z| \leqslant \lambda^{-(k+r-1)}\}$ more than the fact that the essential spectrum is contained somewhere within. Indeed a full disk of essential spectrum is expected (see [26]).

Remark 2.6. As mentioned earlier, the results of this section are essential not new although our strategy differs substantially. Indeed theorem 2.4 can be deduced [77, proposition A.5] by the following argument[13]. First observe that the space $\mathcal{H}$ in [77] (which is a space of holomorphic functions) is densely and continuously embedded in W_r. Consequently the part of the spectrum of $\mathcal{L}_k$ on W_r with modulus greater that the essential spectral radius coincides with the spectrum of $\mathcal{L}_k$ on $\mathcal{H}$ with modulus greater that the essential spectral radius. Note also that the result in the reference [77, proposition A.5] doesn't require f to be orientation preserving.

The next equality is our key observation. Albeit very simple, the rest of the paper relies on it and variants thereof.

Lemma 2.7. *For all $k, r \in \mathbb{N}_0$, $h \in W_r$ and $l \in \{0, \ldots, r\}$,*

$$(\mathcal{L}_k h)^{(l)} = \mathcal{L}_{k+l} h^{(l)}.$$

Proof. Fix $k, r \in \mathbb{N}_0$. The claimed equality holds trivially in the case $l = 0$. Observe that, by chain rule, for all $x \in \mathcal{I}_i$, $h \in \mathcal{C}^\infty(\mathcal{I})$,

$$(\mathcal{L}_k h)'(x) = \sum_j \lambda_j \, B_k[i, j] h' \circ g_j(x) = \sum_j B_{k+1}[i, j] h' \circ g_j(x) = \mathcal{L}_{k+1} h'(x).$$

If we assume that, for some $l \geqslant 0$, the claimed equality holds, i.e. for all $h \in \mathcal{C}^\infty(\mathcal{I})$,

$$(\mathcal{L}_k h)^{(l)}(x) = \mathcal{L}_{k+l} h^{(l)}(x),$$

then, using the previous observation,

$$(\mathcal{L}_k h)^{(l+1)}(x) = (\mathcal{L}_{k+l} h^{(l)})'(x) = (\mathcal{L}_{k+l+1} h^{(l+1)})(x).$$

The equality for all l follows by induction. Using the density of $\mathcal{C}^\infty(\mathcal{I})$ in W_r we obtain the result for $h \in W_r$. $\qquad\qquad\square$

---

[12] Studying the action on polynomials was also used for the vertical direction in the pseudo Anosov case [37].

[13] We are grateful to the anonymous referee who brought the relevant references to our attention and who suggested this argument.

Remark 2.8. In general we could allow the λ_j to be positive or negative. If $\mathcal{L}_1$ coincides with the operator associated to the SRB measure the derivative λ_j occurs with absolute value in the formula. However, as is clear from the proof of the above lemma, when the derivative occurs as a result of differentiating the sign of the derivative remains. This means that, if we are interested in μ_{BM} then we should consider $B_k[i, j] := \lambda_j^{-k} A[j, i]$ but, if we are interested in μ_{SRB}, then we should consider $B_k[i, j] := \lambda_j^{-(k-1)} |\lambda_j|^{-1} A[j, i]$.

To proceed we now prove a set of Lasota–Yorke inequalities for the operators $\mathcal{L}_k : W_r \to W_r$. Let $\Gamma_0 := \|f'\|_{L^\infty}$ and, for all $k \in \mathbb{N}$, let $\Gamma_k := \lambda^{-(k-1)}$.

Lemma 2.9. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. For all $h \in W_r$,*

$$\|\mathcal{L}_k h\|_r \leqslant \Gamma_k \|h\|_r$$

$$\|\mathcal{L}_k h\|_r \leqslant \lambda^{-(k+r-1)} \|h\|_r + \Gamma_k \|h\|_{r-1}.$$

The first inequality also holds in the case $r = 0$.

Proof. We start by considering the case $k \in \mathbb{N}$. Let $h \in W_r$. By definition of $\| \cdot \|_r$ and lemma 2.7,

$$\|\mathcal{L}_k h\|_r = \sum_{l=0}^{r} \int_{\mathcal{I}} |(\mathcal{L}_k h)^{(l)}(x)| dx = \sum_{l=0}^{r} \int_{\mathcal{I}} |\mathcal{L}_{k+l} h^{(l)}(x)| dx$$

$$\leqslant \sum_{l=0}^{r} \lambda^{-(k+l-1)} \int_{\mathcal{I}} \mathcal{L}_1 |h^{(l)}(x)| dx.$$

Since, by the obvious change of variables, $\int_{\mathcal{I}} \mathcal{L}_1 |h^{(l)}(x)| dx = \int_{\mathcal{I}} |h^{(l)}(x)| dx$ the above implies that, for all $r \in \mathbb{N}_0$,

$$\|\mathcal{L}_k h\|_r \leqslant \sum_{l=0}^{r} \lambda^{-(k+l-1)} \int_{\mathcal{I}} |h^{(l)}(x)| dx. \tag{2.2}$$

That is, $\|\mathcal{L}_k h\|_r \leqslant \lambda^{-(k-1)} \|h\|_r$ as required to prove the first inequality. Moreover, when $r \geqslant 1$, the above (2.2) implies that (here we separate the term $l = r$ from the rest of the sum)

$$\|\mathcal{L}_k h\|_r \leqslant \lambda^{-(k+r-1)} \int_{\mathcal{I}} |h^{(r)}(x)| dx + \lambda^{-(k-1)} \sum_{l=0}^{r-1} \int_{\mathcal{I}} |h^{(l)}(x)| dx$$

$$\leqslant \lambda^{-(k+r-1)} \|h\|_r + \lambda^{-(k-1)} \|h\|_{r-1}$$

as required by the second estimate.

To conclude we must consider the case $k = 0$. First observe that, for any $h \in \mathcal{C}^\infty(\mathcal{I})$,

$$\int_{\mathcal{I}} |\mathcal{L}_0 h|(x) dx = \int_{\mathcal{I}} |\mathcal{L}_1 f' h|(x) \leqslant \int_{\mathcal{I}} \left[\mathcal{L}_1 |f'| |h| \right](x) \leqslant \|f'\|_{L^\infty} \int_{\mathcal{I}} |h|(x) dx.$$

Similar to the proof in the case $k \in \mathbb{N}$, by definition of the norm and lemma 2.7,

$$\|\mathcal{L}_0 h\|_r = \sum_{l=0}^{r} \int_{\mathcal{I}} |(\mathcal{L}_0 h)^{(l)}(x)| dx = \sum_{l=0}^{r} \int_{\mathcal{I}} |\mathcal{L}_l h^{(l)}(x)| dx.$$

This means that, for all $r \in \mathbb{N}_0$ (recall that $\Gamma_0 = \|f'\|_{L^\infty}$),

$$\|\mathcal{L}_0 h\|_r \leqslant \sum_{l=0}^{r} \lambda^{-l} \int_{\mathcal{I}} |\mathcal{L}_0 h^{(l)}(x)| \mathrm{d}x \leqslant \Gamma_0 \sum_{l=0}^{r} \int_{\mathcal{I}} |h^{(l)}(x)| \mathrm{d}x$$

and so proves the first inequality. On the other hand, now assuming that $r \geqslant 1$,

$$\|\mathcal{L}_0 h\|_r = \int_{\mathcal{I}} |\mathcal{L}_r h^{(r)}(x)| \mathrm{d}x + \sum_{l=0}^{r-1} \int_{\mathcal{I}} |\mathcal{L}_l h^{(l)}(x)| \mathrm{d}x$$

$$\leqslant \lambda^{-(r-1)} \int_{\mathcal{I}} |\mathcal{L}_1 h^{(r)}(x)| \mathrm{d}x + \sum_{l=0}^{r-1} \int_{\mathcal{I}} |\mathcal{L}_0 h^{(l)}(x)| \mathrm{d}x.$$

Consequently

$$\|\mathcal{L}_0 h\|_r \leqslant \lambda^{-(r-1)} \int_{\mathcal{I}} |h^{(r)}(x)| \mathrm{d}x + \Gamma_0 \sum_{l=0}^{r-1} \int_{\mathcal{I}} |h^{(l)}(x)| \mathrm{d}x.$$

Thus, $\|\mathcal{L}_0 h\|_r \leqslant \lambda^{-(r-1)} \|h\|_r + \Gamma_0 \sum_{l=0}^{r-1} \|h\|_{r-1}$, as required. $\qquad\square$

Lemma 2.10. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. The operator $\mathcal{L}_k$ acting on W_r has spectral radius bounded by Γ_k and essential spectral radius bounded by $\lambda^{-(k+r-1)}$.*

Proof. The first inequality of lemma 2.9 implies that the spectral radius is bounded by Γ_k. For all $r \in \mathbb{N}$, W_r is compactly embedded in W_{r-1}. This means the Lasota–Yorke inequalities of lemma 2.9 imply, by the argument of Hennion [51], based on the formula of Nussbaum [69] (see [28, appendix B] for a pedagogical illustration of the Hennion–Nussbaum theory), that the essential spectral radius of $\mathcal{L}_k$ is bounded by $\lambda^{-(k+r-1)}$. $\qquad\square$

For convenience we use the notation $\mathcal{D} : h \mapsto h'$. For any $k \in \mathbb{N}$, $\nu \in \mathbb{C}$, let $E_k(\nu)$ denote the generalised eigenspace for $\mathcal{L}_k$ associated to the eigenvalue ν. I.e. $E_k(\nu)$ is the set of h such that $(\mathcal{L}_k - \nu)^m h = 0$ for some $m \in \mathbb{N}$. An immediate consequence of lemma 2.7 is the following commutation relation: For any $l, k, m \in \mathbb{N}_0$, $\nu \in \mathbb{C}$, $h \in W_r$

$$\mathcal{D}^l \circ (\mathcal{L}_k - \nu)^m h = (\mathcal{L}_{k+l} - \nu)^m \circ \mathcal{D}^l h.$$

This in turn means that

$$\mathcal{D}^l E_k(\nu) \subset E_{k+l}(\nu). \tag{2.3}$$

Proof of the first statement of theorem 2.4. Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. According to lemma 2.10 the essential spectral radius of $\mathcal{L}_k : W_r \to W_r$ is not greater that $\lambda^{-(k+r-1)}$. Fix some arbitrarily small $\epsilon > 0$ and define $\mathcal{H}_{k,r} := \{\nu \in \sigma_{W_r}(\mathcal{L}_k), |\nu| > \lambda^{-(k+r-1)} + \epsilon\}$. For each $\nu \in \mathcal{H}_{k,r}$ let P_ν denote the associated spectral projector and hence let $\Pi_{k,r} := \sum_{\nu \in \mathcal{H}_{k,r}} P_\nu$. Consequently $\mathcal{L}_k - \mathcal{L}_k \circ \Pi_{k,r} : W_r \to W_r$ has spectral radius not greater than $\lambda^{-(k+r-1)} + \epsilon$. For any $l \in \mathbb{N}$ lemma 2.10 gives an upper bound of $\lambda^{-(l-1)}$ for the spectral radius of $\mathcal{L}_l : W_1 \to W_1$ and so $E_l(\nu) = \{0\}$ whenever $|\nu| > \lambda^{-(l-1)}$. As observed above (2.3), differentiating r times takes the generalised eigenspace $E_k(\nu)$ to the generalised eigenspace $E_{k+r}(\nu)$ of the operator $\mathcal{L}_{k+r}$. However $E_{k+r}(\nu) = \{0\}$ since $|\nu| > \lambda^{-(k+r-1)}$. This means that $E_k(\nu) \subset \mathfrak{P}_r(\mathcal{I})$ whenever $\nu \in \mathcal{H}_{k,r}$ and so we have shown that the image of $\Pi_{k,r}$ is contained in $\cup_{\nu \in \mathcal{H}_{k,r}} E_k(\nu) \subset \mathfrak{P}_r(\mathcal{I})$. The claim follows by the arbitrariness of ϵ. $\qquad\square$

We can identify $\mathbb{R}^N$ with $\mathfrak{P}_0(\mathcal{I})$, the set of functions that are constant on each partition element, in the sense that we associate the function $\sum_i a_i 1_{I_i} \in \mathfrak{P}_0(\mathcal{I})$ to each $a = (a_i) \in \mathbb{R}^N$ (1_A denotes the characteristic function of the set A).

Let $r \in \mathbb{N}$. The space $(\mathbb{R}^N)^{(r+1)}$ is identified with $\mathfrak{P}_r(\mathcal{I})$ as follows. We use the notation $(a^0, a^1, \ldots, a^r) \in (\mathbb{R}^N)^{(r+1)}$ where $a^j = (a^j_1, a^j_2, \ldots, a^j_N)$ for each j. Let[14] $\mathcal{J} : \mathbb{R}^{N(r+1)} \to \mathfrak{P}_r(\mathcal{I})$,

$$\mathcal{J}(a^0, \ldots, a^r) : x \mapsto \sum_{l=0}^{r} x^l \sum_{j=1}^{N} a^l_j 1_{I_j}(x).$$

Observe that $\mathcal{J} : \mathbb{R}^{N(r+1)} \to \mathfrak{P}_r(\mathcal{I})$ is onto and invertible.

For any $k \in \mathbb{N}_0$, $r \in \mathbb{N}$ we define the $N(r+1) \times N(r+1)$ matrix

$$\mathcal{T}_{k,r} := \mathcal{J}^{-1} \circ \mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})} \circ \mathcal{J}.$$

In order to understand the spectrum of $\mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})}$ it suffices to study the spectrum of the matrix $\mathcal{T}_{k,r}$.

Lemma 2.11. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. Then $\mathcal{T}_{k,r}$ has lower block triangular form*

$$\mathcal{T}_{k,r} = \begin{pmatrix} B_k & & & 0 \\ F_{1,0} & B_{k+1} & & \\ \vdots & \vdots & \ddots & \\ F_{r,0} & F_{r,1} & \ldots & B_{k+r} \end{pmatrix},$$

where the matrices on the diagonal are the ones previously introduced in (2.1).

Proof. Fix $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. For any $l \in \{0, \ldots, r\}$ we consider $(a^0, a^1, \ldots, a^r) \in \mathbb{R}^{N(r+1)}$ and suppose that $a^j = 0$ for all $j \neq l$. This means that $\mathcal{J}(a^0, a^1, \ldots, a^r) = \mathbf{x}^l a$. We wish to compute $\mathcal{J}^{-1} \circ \mathcal{L}_k \circ \mathcal{J}(a^0, a^1, \ldots, a^r) = \mathcal{J}^{-1} \circ \mathcal{L}_k(\mathbf{x}^l a)$. For each j let $q_j := f(p_j^+)$ (i.e. $\lim_{\epsilon \to 0} f(p_j + \epsilon)$). Observe that, for all $x \in I_j$, $f(x) = \lambda_j(x - p_j) + q_j$. Consequently, for all $x \in f(I_j)$,

$$g_j(x) = (x - q_j)\lambda_j^{-1} + p_j. \tag{2.4}$$

Using this inverse,

$$\mathcal{L}_k(\mathbf{x}^l a)(x) = \sum_{i,j} 1_{I_i}(x) B_k[i,j] a_j \big(x\lambda_j^{-1} - q_j\lambda_j^{-1} + p_j\big)^l$$

$$= \sum_{i,j} 1_{I_i}(x) B_{l+k}[i,j] a_j x^l + \rho(x), \tag{2.5}$$

where $\rho \in \mathfrak{P}_{l-1}(\mathcal{I})$. I.e. $\mathcal{L}_k(\mathbf{x}^l a^l) = \mathbf{x}^l B_{k+l} a^l + \rho$, where $\rho \in \mathfrak{P}_{l-1}(\mathcal{I})$. This proves that $\mathcal{T}_{k,r}$ has lower diagonal block form and that the diagonal elements of the block matrix are the B_{k+l}. The exact form of the matrices $F_{i,j}$ which appear below the diagonal are superfluous to our present argument and we will not identify them further. $\qquad\square$

Proof of theorem 2.4. The first statement of the theorem was proven above, it remains to prove the second statement. Recall the lower triangular block form of $\mathcal{T}_{k,r}$ as shown in

---

[14] Abusing notation we will often write the same symbol for $a \in \mathbb{R}^N$ and the corresponding $a \in \mathfrak{P}_0(\mathcal{I})$ with the interpretation given by context.

lemma 2.11. We can assume without loss of generality that each B_{k+l} is in lower triangular form. If a matrix is in triangular form then the values on the diagonal are the eigenvalues repeated according to multiplicity. That each B_{k+l} is in triangular form means that the $N(r+1) \times N(r+1)$ matrix $\mathcal{T}_{k,r}$ is in triangular form. Moreover the diagonal is the union of the diagonals of the B_{k+l}. This implies the claimed correspondence of the eigenvalues of $\mathcal{T}_{k,r}$ and the union of the set of eigenvalues of the $\{B_{k+l}\}_{l=0}^{r}$, including correspondence in multiplicity. $\qquad\square$

Remark 2.12. The lower triangular block form shown in lemma 2.11 and the argument of the above proof further implies that, if some B_{k+l} has a Jordan block of dimension $m \in \mathbb{N}$, then $\mathcal{T}_{k,r}$ has a corresponding Jordan block of dimension m or greater. Indeed $\mathcal{T}_{k,r}$ has the possibility to have a Jordan block of greater dimension if a given eigenvalue appears in more than one of the B_{k+l}.

Proof of theorem 2.1. Fix $k \in \mathbb{N}_0$. For each $r \in \mathbb{N}$ consider the finite set of eigenvalues

$$\{\xi_j\}_{j=0}^{K_r} = \sigma_{W_r}(\mathcal{L}_k) \backslash \{|z| \leqslant \lambda^{-(k+r-1)}\}$$

described by theorem 2.4. We define as usual the corresponding eigen projectors $\{\Pi_j : W_r \to W_r\}_{j=0}^{K_r}$ and eigen nilpotents $\{Q_j : W_r \to W_r\}_{j=0}^{K_r}$ which satisfy the commutation relations: $\Pi_j \Pi_k = \delta_{j,k}, \Pi_j Q_k = Q_k \Pi_j = \delta_{j,k} Q_k$. Let $S_r := 1 - (\Pi_1 + \Pi_2 + \cdots + \Pi_{K_r})$ and observe that $\mathcal{L}_k S_r$ has spectral radius not greater than $\lambda^{-(k+r-1)}$. This means that the operator $\mathcal{L}_k : W_r \to W_r$ satisfies the decomposition

$$\mathcal{L}_k = \sum_{j=1}^{K_r} \left(\xi_j \Pi_j + Q_j \right) + \mathcal{L}_k S_r. \tag{2.6}$$

Further observe that each operator we define remains defined by the same formula on W_r for any r sufficiently large.

Now let us recall the connection between the transfer operators and invariant measures (see [61] or [6] for full details). For each $k \in \{0, 1\}$ there exists $h_k \in W_r$ (*the invariant density*), a probability measure ν_k (*the conformal measure*), $\gamma_k > 0$ (*the spectral radius*) and a probability measure μ_k defined as $\mu_k(\varphi) := \nu_k(h_k \varphi)$ (*the invariant measure*). Moreover $\nu_k(\mathcal{L}_k \varphi) = \gamma_k \nu_k(\varphi)$ and $\mu_k(\varphi \circ f) = \mu_k(\varphi)$.

In our present setting μ_0 is the measure of maximal entropy μ_{BM} and μ_1 is the SRB measure μ_{SRB}. Furthermore $\ln \gamma_0$ is equal to the topological entropy, $\gamma_1 = 1$ and ν_1 coincides with Lebesgue measure.

Continuing for $k \in \{0, 1\}$ we observe that

$$\int_I \phi \cdot \varphi \circ f^n \, \mathrm{d}\mu_k = \int_I (\phi \cdot h_k)(x) \cdot \varphi \circ f^n(x) \mathrm{d}\nu_k(x)$$

$$= \gamma_k^{-n} \int_I \mathcal{L}_k^n (\phi \cdot h_k)(x) \cdot \varphi(x) \mathrm{d}\nu_k(x).$$

We then combine this formula with the spectral decomposition above (2.6) to produce the asymptotic expansion required. $\qquad\square$

Remark 2.13. If f is full branch, the matrices B_{k+l} are such that all the entries in any column j is equal to $\lambda_j^{-(k+l)}$. The spectrum of this type of matrix is the union of zero and the sum of entries on different columns. Consequently theorem 2.4 implies that, outside of the disk $\{|\nu| \leqslant \lambda^{-(k+r-1)}\}$, the spectrum of $\mathcal{L}_k : W_r \to W_r$ is equal to $\{\xi_0, \ldots, \xi_{r-1}\}$ where $\xi_l := \sum_{j=1}^{N} \lambda_j^{-(k+l)}$.

- In the case $k = 0$ we obtain $\xi_0 = \sum_{j=1}^{N} \lambda_j^0 = N$;
- In the case $k = 1$ we see that $\xi_0 = \sum_{j=1}^{N} \lambda_j^{-1} = \sum_{j=1}^{N} |I_j| = 1$.

A comprehensive investigation of the resonances of $x \mapsto 2x \bmod 1$ can be found in [31, section 3]). The eigenfunctions for this map are the Bernoulli polynomials.

Remark 2.14. Observe that B_0 is the transpose of A and that, for any Markov map as considered in the present section, the logarithm of the spectral radius of B_0 is equal [20, section 2.1] to the topological entropy.

Remark 2.15. In this section we used Sobolev spaces W_r but, with a slightly more complex argument, we could equally well have worked with $C^r(\mathcal{I}) = \bigoplus_i C^r(I_i)$.

2.1. A Jordan block example

In the following we construct an example of a Markov expanding map such that B_1 has a Jordan block of dimension two. Previously Driebe [30] presented an example of a linear Markov expanding map such that, for each $n \in \mathbb{N}$, the eigenvalue 3^{-2n} has a Jordan block of dimension two. However the example is not orientation-preserving and the mechanism giving rise to the Jordan blocks appears to depend on this. On the contrary, the example constructed here is orientation-preserving.

Let $I_1 = (0, \frac{1}{4}), I_2 = (\frac{1}{4}, \frac{1}{2}), I_3 = (\frac{1}{2}, \frac{3}{4}), I_4 = (\frac{3}{4}, 1)$ and let $f : I \to I$ be as shown in figure 1, defined by

$$f(x) := \begin{cases} 3x + \dfrac{1}{4} & \text{if } x \in I_1 \\[2mm] 3\left(x - \dfrac{1}{4}\right) & \text{if } x \in I_2 \\[2mm] 2\left(x - \dfrac{1}{2}\right) & \text{if } x \in I_3 \\[2mm] 3\left(x - \dfrac{3}{4}\right) & \text{if } x \in I_4. \end{cases}$$

This means that

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}.$$

The matrix B_1 has the eigenvalues $\{-\frac{1}{3}, 0, 1\}$ and the eigenvalue $-\frac{1}{3}$ has a Jordan block of dimension two. If we let

$$a_1 := \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad a_2 := \begin{pmatrix} 3 \\ 3 \\ -6 \\ 0 \end{pmatrix}, \qquad a_3 := \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \qquad a_4 := \begin{pmatrix} 9 \\ 12 \\ 8 \\ 3 \end{pmatrix},$$
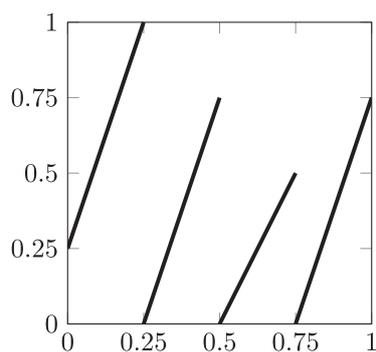
Figure 1. An orientation-preserving Markov map with Jordan block.

then $B_1 a_1 = -\frac{1}{3} a_1$, $(B_1 + \frac{1}{3} 1) a_2 = a_1$, $B_1 a_3 = 0$ and $B_1 a_4 = a_4$. In particular $\{a_1, a_2\}$ span the generalised eigenspace associated to the eigenvalue $-\frac{1}{3}$. Since the essential spectral radius of $\mathcal{L}_1$, when acting on W_r, $r \geqslant 2$, is smaller than $1/4$, then $\mathcal{L}_1$, on such spaces, has a Jordan block in the point spectrum.

Remark 2.16. Another interesting example of an affine expanding Markov map is the Baladi map studied in [25]. This is a system which exhibits non-trivial complex resonances. In the reference the connection between resonances and decay of correlation was considered and the outer set of resonances identified. Our results give a description of the full set of resonances for this system and the connection to the decay of correlations.

3. Piecewise smooth full branch expanding maps

In this section we discuss the simplest non-linear case: full branch maps. For such maps there exists already some general quantitative results on the spectral gap, e.g. [60, section 2], however they are not optimal, we will comment about the comparison case by case.

Let $f \in \mathcal{C}^r([0, 1], [0, 1])$, $r \geqslant 2$, be a full branched piecewise expanding map, $f' > \lambda > 1$. For $k \in \mathbb{N}_0$ let us consider the transfer operator

$$\mathcal{L}_k h(x) = \sum_{y \in f^{-1}(x)} \frac{h(y)}{[f'(y)]^k}. \tag{3.1}$$

Observe that $\mathcal{L}_0$ is the operator associated to the measure of maximal entropy while $\mathcal{L}_1$ is the operator associated to the SRB measure [6].[15]

For convenience, throughout this section, we denote the *distortion* by[16]

$$D_f = \left(\frac{1}{f'} \right)'.$$

The key fact we wish to leverage on, in analogy with lemma 2.7, is the formula

$$(\mathcal{L}_k h)' = \mathcal{L}_{k+1} h' + k \mathcal{L}_k (h \cdot D_f) \tag{3.2}$$

---

[15] For the measure of maximal entropy see also the beginning of section 4 where it is explained in a more general setting.

[16] Here, as in the previous section, the derivative is taken only at the smoothness points of f.

which is obtained simply by differentiating (3.1). Notice that the distortion D_f is 0 when the map is piecewise affine and this is the reason why the treatment in section 3 is more complicated than in section 2 (see lemma 2.7). Amazingly, the above formula yields several nontrivial facts. To illustrate its power we start discussing the operator $\mathcal{L}_0$.

3.1. Measure of maximal entropy

Lemma 3.1. *If f is a N covering, then $\sigma_{\mathcal{C}^1}(\mathcal{L}_0) \subset \{N\} \cup \{z \in \mathbb{C} \,:\, |z| \leqslant 1\}$. Moreover, $\sigma_{\mathcal{C}^2}(\mathcal{L}_0) \cap \{z \in \mathcal{C} \,:\, |z| \geqslant \lambda^{-1}\} = \{N\} \cup (\sigma_{\mathcal{C}^1}(\mathcal{L}_1) \cap \{z \in \mathbb{C} \,:\, |z| \geqslant \lambda^{-1}\})$.*

Proof. Note that $\mathcal{L}_0 1 = N$, so $N \in \sigma(\mathcal{L}_0)$. Since

$$\left(\frac{1}{(f^n)'}\right)' = \sum_{k=0}^{n-1} \frac{D_f \circ f^k}{(f^{n-k-1})' \circ f^{k+1}},$$

$|\left(\frac{1}{(f^n)'}\right)'| \leqslant C = \|D_f\|_\infty (1 - \lambda^{-1})^{-1}$. In particular we may use (3.2) for f^n, rather than for f and $\|\mathcal{L}_1\|_\infty \leqslant C$. Using these estimates, (3.2) and a direct computation

$$\|\mathcal{L}_0 h\|_\infty \leqslant N \|h\|_\infty,$$
$$\|(\mathcal{L}_0^n h)'\|_\infty \leqslant \|\mathcal{L}_1^n h'\|_\infty \leqslant C \|h'\|_\infty.$$

By the usual arguments the above inequalities imply that the essential spectral radius of $\mathcal{L}_0$, acting on $\mathcal{C}^1$, is at most one. On the other hand if $\mathcal{L}_0 h = \nu h$, with $|\nu| > 1$ we have, for all $n \in \mathbb{N}$,

$$\nu^n h' = (\mathcal{L}_0^n h)' = \mathcal{L}_1^n h'$$

which, since $\|\mathcal{L}_1^n\|_{L^1} = 1$, implies $|h'| = 0$, that is, h must be constant.

To conclude observe that on the one hand, if $\mathcal{L}_0 H = \nu H$, $H \in \mathcal{C}^2$, then $\mathcal{L}_1 H' = \nu H'$. On the other hand, if $\mathcal{L}_1 h = \nu h$, $h \in \mathcal{C}^1$, let $H_0(x) = \int_0^x h(y) \mathrm{d}y$ and observe that $H_0 \in \mathcal{C}^2$ and

$$((\nu - \mathcal{L}_0) H_0)' = (\nu - \mathcal{L}_1) h = 0.$$

Thus $(\nu - \mathcal{L}_0) H_0$ is a constant function. Let α denote this constant value and let $c = -(\nu - N)^{-1} \alpha$. Now let $H_c(x) = \int_0^x h(y) \mathrm{d}y + c$ and observe that

$$(\nu - \mathcal{L}_0) H_c = (\nu - \mathcal{L}_0) H_0 + (\nu - \mathcal{L}_0) c = \alpha + (\nu - N) c = 0.$$

The result follows since the essential spectrum of $\mathcal{L}_1$, when acting on $\mathcal{C}^1$, is bounded by λ^{-1}. $\qquad\square$

Remark 3.2. Note that the proof of lemma 3.1 implies that $\mathcal{L}_0 h = \int h \, \mathrm{d}\mu_{\mathrm{BM}} + Qh$, where μ_{BM} is the measure of maximal entropy, and $\|Q^n\|_{\mathcal{C}^1} \leqslant C$. That is, $\mathcal{L}_0$ has a spectral gap $N - 1$ while the Hilbert metric technique can yield, at the very best, a spectral gap $N - \lambda$, see [60].

The above shows that the spectrum of $\mathcal{L}_0$ is largely determined by the spectrum of $\mathcal{L}_1$. Hence, before continuing our investigation of the spectrum of $\mathcal{L}_0$, it is necessary to undertake an investigation of the spectrum of $\mathcal{L}_1$.

3.2. The SRB measure

Note that the vector space $\mathbb{V} = \{h \in \mathcal{C}^1 : \int_0^1 h = 0\}$ is invariant under $\mathcal{L}_1$, we can thus restrict $\mathcal{L}_1$ to $\mathbb{V}$. If we define

$$\phi(g)(x) = \int_0^x g(y)\mathrm{d}y - \int_0^1 (1-y)g(y)\mathrm{d}y = \int_0^x yg(y)\mathrm{d}y + \int_x^1 (y-1)g(y)\mathrm{d}y, \tag{3.3}$$

then $\phi : \mathcal{C}^0 \to \mathbb{V}$ and $\phi(h') = h$ for all $h \in \mathbb{V}$. Thus, for each $h \in \mathbb{V}$,

$$(\mathcal{L}_1 h)' = \mathcal{L}_2 h' + \mathcal{L}_1(\phi(h')D_f) =: \mathcal{L}_\star(h'). \tag{3.4}$$

The relevance of the operator $\mathcal{L}_\star$ rests in the next lemma.

Lemma 3.3. *If $f \in \mathcal{C}^2([0,1],[0,1])$, then the spectrum of $\mathcal{L}_1 : \mathcal{C}^1 \to \mathcal{C}^1$ satisfies*

$$\sigma_{\mathcal{C}^1}(\mathcal{L}_1) \cap \{z \in \mathbb{C} : |z| > \lambda^{-1}\} = \{1\} \cup \sigma_{\mathcal{C}^0}(\mathcal{L}_\star) \cap \{z \in \mathbb{C} : |z| > \lambda^{-1}\}.$$

Proof. It is well known that the essential spectral radius of $\mathcal{L}_1$, when acting on $\mathcal{C}^1$ is bounded by λ^{-1}, hence we can restrict ourselves to the point spectrum.

Since $\int_0^1 \varphi \mathcal{L}_1 h = \int_0^1 (\varphi \circ f) \cdot h$, it follows that the Lebesgue measure is an eigenvector, with eigenvalue one, of the dual operator, and hence $1 \in \sigma_{\mathcal{C}^1}(\mathcal{L}_1)$. In addition, $\mathbb{V} = \{h \in \mathcal{C}^1 : \int_0^1 h = 0\}$ is invariant under $\mathcal{L}_1$. It follows that if $\mathcal{L}_1 h = \nu h$, $|\nu| > \lambda^{-1}$ and $h \in \mathcal{C}^1$, then $h' \in \mathcal{C}^0$ and (3.4) implies $\mathcal{L}_\star h' = \nu h'$. On the other hand if $g \in \mathcal{C}^0$ and $\mathcal{L}_\star g = \nu g$, $|\nu| > \lambda^{-1}$, then $h = \phi(g) \in \mathbb{V}$ and

$$(\mathcal{L}_1 h - \nu h)' = \mathcal{L}_\star g - \nu g = 0.$$

Hence, there exists a constant C such that $\mathcal{L}_1 h - \nu h = C$, but integrating we have $C = 0$, thus h is an eigenvector of $\mathcal{L}_1$. $\square$

Remark 3.4. Note that the above lemma holds verbatim with $W^{1,1}$ substituted to $\mathcal{C}^1$. In the following, we find more convenient to consider the spectrum of $\mathcal{L}_1$ when acting on $W^{1,1}$.

Let

$$\tau = \lambda^{-1} + \int_0^1 \left| \left(\frac{1}{f'(y)} \right)' \right| \mathrm{d}y = \lambda^{-1} + \|D_f\|_{L^1}.$$

Lemma 3.5. *The norm of $\mathcal{L}_2$ on L^1 is bounded by λ^{-1}. The operator $\mathcal{L}_c(g) = \mathcal{L}_1(\phi(g)D_f)$, acting on L^1, is a compact operator. In addition, for all $g \in L^1$*

$$\|\phi(g)\|_{L^1} \leqslant \frac{1}{2} \|g\|_{L^1}$$

$$\|\phi(g)\|_{L^\infty} \leqslant \|g\|_{L^1}$$

$$\|\phi(g)\|_{W^{1,1}} \leqslant \frac{3}{2} \|g\|_{L^1}.$$

In particular $\mathcal{L}_\star : L^1 \to L^1$ is a well-defined operator and $\|\mathcal{L}_\star\|_{L^1} \leqslant \tau$.

Proof. Since

$$\|\mathcal{L}_2 h\|_{L^1} \leqslant \lambda^{-1} \|\mathcal{L}_1 h\|_{L^1} \leqslant \lambda^{-1} \|h\|_{L^1},$$

the first statement follows. Moreover,

$$\int_0^1 |\phi(g)(x)| \mathrm{d}x \leqslant 2 \int_0^1 |g(y)| y(1-y) \mathrm{d}y \leqslant \frac{1}{2} \|g\|_{L^1},$$

$$|\phi(g)(x)| \leqslant \int_0^1 |g(y)| \mathrm{d}y = \|g\|_{L^1}.$$

Finally, $\|\phi(g)'\|_{L^1} \leqslant \|g\|_{L^1}$ implies the last of the three inequalities and also that ϕ is compact, the compactness of $\mathcal{L}_c$ follows. $\qquad\square$

Theorem 3.6. *Let us consider* $\mathcal{L}_1$ *as an operator acting on* $W^{1,1}$, *then* $\sigma_{ess}(\mathcal{L}_1) \subset \{z \in \mathbb{C} : |z| \leqslant \lambda^{-1}\}$. *Moreover* $\sigma(\mathcal{L}_1) \backslash \{1\} \subset \{z \in \mathbb{C} : |z| \leqslant \tau\}$.

Proof. If $\nu \in \mathbb{C}$ is such that $|\nu| > \lambda^{-1}$ and $\mathcal{L}_1 h = \nu h$, for some $h \in W^{1,1}$ with $\int h = 0$, then $\mathcal{L}_\star g = \nu g$, for $g = h'$. Then, recalling lemma 3.5,

$$|\nu| \|g\|_{L^1} \leqslant \lambda^{-1} \|g\|_{L^1} + \int_0^1 \left| \left(\frac{1}{f'(y)} \right)' \right| \mathrm{d}y \, \|\phi(g)\|_{L^\infty}$$

$$\leqslant \left[\lambda^{-1} + \|D_f\|_{L^1} \right] \|g\|_{L^1}.$$

This proves the theorem since $h' = 0$ implies $h = 0$. $\qquad\square$

The above lemma provides an upper bound for the spectral gap, but it is very unsatisfactory. First, such a bound is of interest only if $\tau < 1$ (for example, in the counterexample of Keller and Rugh [56] $\tau > 1$). Second, even if $\tau < 1$, it is unclear if there exists other point spectrum outside $\{z \in \mathbb{C} : |z| \leqslant \lambda^{-1}\}$.

Remark 3.7. For $\mathcal{L}_1$ the Hilbert metric approach yields a bound of the spectral gap given by a rather cumbersome formula. However, if one considers the limit of large λ and small D_f, then, using [60, lemma 2.3], one can check that the bound of the spectral gap cannot be better than $\lambda^{-1}(1 + 2\|D_f\|_\infty) + \|D_f\|_\infty$, which is worse than the one provided, in the same limit, by theorem 3.6. However, for large D_f the bound of theorem 3.6 is useless while [60, lemma 2.3] provides an explicit, although rather poor, bound.

Very few results are known on the existence of point spectrum with the notable exception of cases when the map has been explicitly constructed to exhibit point spectrum [56] or when one restricts the map to the class of holomorphic maps, often of a special nature, as in [10, 76, 78]. No analytical technique is available to treat $\mathcal{C}^2$ open classes of maps. On the contrary a lot of work exists on the side of numerical computation, mainly of the invariant measure but also, to some extent, of the spectrum, e.g. see [63] and references therein. While most of the numerical work does not track round off errors and hence it is unsatisfactory from the rigorous point of view, some notable exceptions use interval arithmetic and hence have the status of a proof, e.g. [3, 44, 52].

Hence, it is interesting to note that the present approach offers an alternative, possibly much more convenient, route to a numerical computation of the spectrum.

Remark 3.8. We conclude the section with a remark on how the above discussion can provide a numerical scheme to locate eigenvalues. Let $\mathcal{K} := \mathcal{L}_1(\phi(g)D_f)$, ϕ being defined in (3.3). Also, let $\alpha > 0$ and $\{\varphi_i\}_{i=1}^\infty$ be a Schauder base of $W^{1,1}$ such that, calling Π_N the projection onto span $\{\varphi_i\}_{i=1}^N$ along span $\{\varphi_i\}_{i=N+1}^\infty$, we have $\|\Pi_N\|_{L^1} \leqslant C_\sharp$ and

$$\|1 - \Pi_N\|_{W^{1,1} \to L^1} \leqslant C_\sharp N^{-\alpha}.$$

Then, to study the spectrum of $\nu - \mathcal{L}_\star$, $1 > |\nu| > \lambda^{-1}$ when acting on L^1, write

$$\nu - \mathcal{L}_\star = (\nu - \mathcal{L}_2)\left[1 - (\nu - \mathcal{L}_2)^{-1}\mathcal{K}\right] = (\nu - \mathcal{L}_2)\left[1 - \Pi_N(\nu - \mathcal{L}_2)^{-1}\Pi_N\mathcal{K} - \Delta_N\right]$$

$$= (\nu - \mathcal{L}_2)\left[1 - \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N\right]\left(1 - \left[1 - \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N\right]^{-1}\Delta_N\right),$$

$$\Delta_N = (1 - \Pi_N)(\nu - \mathcal{L}_2)^{-1}\mathcal{K} + \Pi_N(\nu - \mathcal{L}_2)^{-1}(1 - \Pi_N)\mathcal{K}.$$

Note that lemma 3.5 implies $\|\mathcal{K}\|_{L^1} + \|\mathcal{K}\|_{L^1 \to W^{1,1}} \leqslant C_\sharp$, hence

$$\|\Delta_N\|_{L^1} \leqslant C_\sharp(\lambda^{-1} - |\nu|)^{-1}N^{-\alpha}.$$

Thus ν belongs to the resolvent of $\mathcal{L}_\star$ if

$$\left\|\left[1 - \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N\right]^{-1}\Delta_N\right\|_{L^1} \leqslant C_\sharp \frac{\left\|\left[1 - \Pi_N(\nu - \mathcal{L}_2)^{-1}\Pi_N\mathcal{K}\right]^{-1}\right\|_{L^1}}{(\lambda^{-1} - |\nu|)N^\alpha} \leqslant 1.$$

Since $\Pi_N(\nu - \mathcal{L}_2)^{-1}\Pi_N\mathcal{K}$ is a finite rank operator, $\left[1 - \Pi_N(\nu - \mathcal{L}_2)^{-1}\Pi_N\mathcal{K}\right]^{-1}$ its norm can be evaluated numerically. In fact, by Neumann series, we have, for ζ small and setting $\mathcal{R}_N = \Pi_N(\nu - \mathcal{L}_2)^{-1}\Pi_N$,

$$[1 - \zeta\mathcal{R}_N\mathcal{K}]^{-1} = 1 + \zeta[1 - \zeta\mathcal{R}_N\mathcal{K}\Pi_N]^{-1}\mathcal{R}_N\mathcal{K},$$

thus, by analyticity, the same holds for $\zeta = 1$. Hence, the spectrum of $\mathcal{L}_\star$ is close to the values of ν for which $\mathcal{R}_N\mathcal{K}\Pi_N$ has eigenvalue one. Since $\mathcal{R}_N\mathcal{K}\Pi_N$ is a $N \times N$ matrix, this provides a rather quick way to determine rigorously if $\mathcal{L}_1$ has point spectrum outside the spectral radius of $\mathcal{L}_2$, aside from one.

3.3. Point spectrum

If we consider class of maps with some special features, it is possible use arguments like the ones put forward in remark 3.8 to obtain relevant information about the point spectrum without any computer assisted method.

As an example, the next theorem provides more precise information on the spectrum in a special class of maps. Note that the following approach can be generalised, here we present only the simplest application to illustrate the logic of the argument.

Theorem 3.9. *Let $I := [0, 1]$ and $f : I \to I$. Consider the partition $\{(p_i, p_{i+1})\}_{i=1}^N$ to be a partition of a full-measure subset of $[0, 1]$ such that for any $1 \leqslant i \leqslant N$, $f([p_i, p_{i+1}]) = [0, 1]$, $f \in \mathcal{C}^3([p_i, p_{i+1}], [0, 1])$, and $f'(p_i^+) = f'(p_i^-)$, $i \in \{2, \ldots, N\}$.[17] Also assume that $D_f \not\equiv 0$ and $D_f \geqslant 0$. Then[18], for $\mathcal{L}_1 : W^{2,1}([0, 1]) \to W^{2,1}([0, 1])$*

$$\sigma(\mathcal{L}_1) \subset \left\{z \in \mathbb{C} \ : \ |z| \leqslant \min\left\{1, \frac{2}{f'(1)} - \frac{1}{f'(0)}\right\}\right\} \cup \{1\}$$

$$\sigma_{ess}(\mathcal{L}_1) \subset \left\{z \in \mathbb{C} \ : \ |z| \leqslant \frac{1}{f'(1)^2}\right\}.$$

---

[17] By $g(p^+)$, $g(p^-)$ we mean the right and left limit, respectively, of the function g at the point p. Since $f'(p_i^+) = f'(p_i^-)$ there is no need to distinguish between p_i^- and p_i^+, so we will not do it anymore.

[18] Note that the following provides a spectral gap if $f'(1) \geqslant 2$.

Moreover, $\{1\}$ is a simple eigenvalue of $\mathcal{L}_1$. In addition, there exists $\mu_2 < \frac{1}{f'(1)}$ such that, setting $\Delta = \frac{1}{f'(1)} - \frac{1}{f'(0)}$, $\mu_* = \frac{1}{f'(1)}$, $\Gamma = \left(1 - \sum_{i=1}^{N} \frac{1}{f'(p_i)}\right)$ and

$$A_0 = \{a \in \mathbb{R} \,:\, \mu_2 < a < 1\}$$

$$A_1 = \left\{a + ib \in \mathbb{C} \,:\, a > \mu_*, b^2 < \frac{(a - \mu_*)\Delta}{2(1 + a^{-1}\Gamma)} \left[\sqrt{1 + 4\frac{(1 + a^{-1}\Gamma)^2(a - \mu_*)^2}{\Delta^2}} - 1\right]\right\}$$

$$A_2 = \left\{a + ib \in \mathbb{R} \,:\, a < -\mu_*, \, b^2 < \frac{(|a| - \mu_*)^2(a^2 - \mu_*^2 - \Delta|a|)}{a^2 - \mu_*^2 + \Delta|a|}\right\}$$

$$A_3 = \left\{a + ib \in \mathbb{C} \,:\, a \geqslant 0, \, |a + ib - \mu_*|^2 > \mu_*^2 + \mu_* \frac{\Delta + \sqrt{4\mu_*^2 + \Delta^2}}{2}\right\}$$

$$A_4 = \left\{a + ib \in \mathbb{C} \,:\, a < 0, \, b^2 > \mu_*\Delta + \mu_*^2 + 2|a|\mu_* - a^2\right\}$$

we have

$$\left(\cup_{i=0}^{3} A_i\right) \cap [\sigma(\mathcal{L}_1)\setminus\{1\}] = \emptyset.$$

The proof of the above theorem is a boring computation using the ideas illustrate in the previous section, so we postpone it to appendix A. Here we provide an application of the theorem to show what can be achieved with some, moderately lengthy, hand made computations. Much more could be obtained using the assistance of a computer as mentioned in remark 3.8.
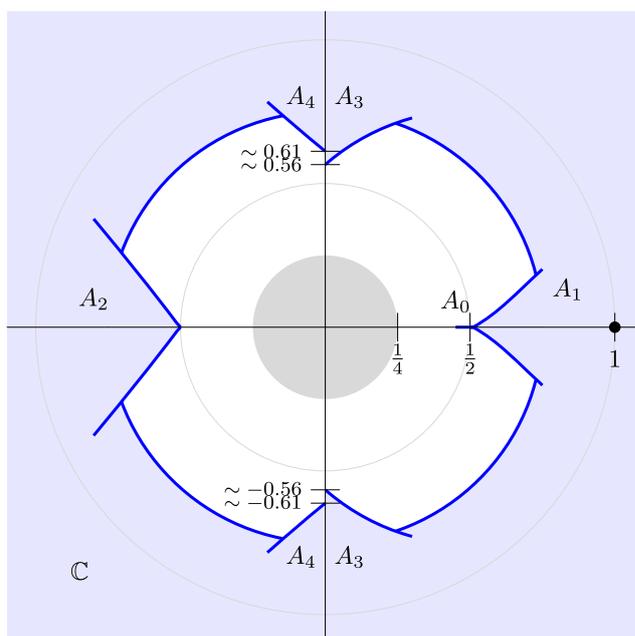
Remark 3.10. Note that if $D_f \equiv 0$, then $f'(1) = N$ and $\mathcal{L}_1$ has eigenvalue N^{-1} with eigenfunction $g(x) = x - \frac{1}{2}$. Indeed,

$$\mathcal{L}_1 g(x) = \sum_{i=1}^{N} N^{-1} g\left(\frac{x + i - 1}{N}\right) = \sum_{i=1}^{N} \frac{(x + i - 1)}{N^2} - \frac{1}{2N}$$

$$= N^{-1}\left(x + \frac{N - 1}{2} - \frac{N}{2}\right) = Ng(x).$$

By perturbation theory, see [55], it follows that such an eigenvalue survives for small distortion. However, the above theorem implies that, for perturbations satisfying theorem 3.9, one cannot make it increase more than $\frac{2}{f'(1)} - \frac{1}{f'(0)}$.

Remark 3.11. As an example consider $f(x) = 4x - x^2 \bmod 1$. In this case theorem 3.9 applies with $D_f = \frac{2}{(4 - 2x)^2} > 0$, $f'(0) = 4$, $f'(1) = 2$, $\Delta = \frac{1}{4}$, $\mu_* = \frac{1}{2}$ and some $\mu_2 < \frac{1}{2}$. Moreover $p_1 = 0$, $p_2 = 2 - \sqrt{3}$, $p_3 = 2 - \sqrt{2}$, hence $f'(p_2) = 2\sqrt{3}$, $f'(p_3) = 2\sqrt{2}$ and $\Gamma = 1 - \frac{1}{4} - \frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{2}}$. Consequently, for $\mathcal{L}_1 : W^{2,1}([0, 1]) \to W^{2,1}([0, 1])$,

$$\sigma(\mathcal{L}_1) \subset \left\{z \in \mathbb{C} \,:\, |z| \leqslant \frac{3}{4}\right\} \cup \{1\}$$

$$\sigma_{\text{ess}}(\mathcal{L}_1) \subset \left\{z \in \mathbb{C} \,:\, |z| \leqslant \frac{1}{4}\right\}.$$

Figure 2. Spectrum of the example. Theorem 3.9 implies that the disk of radius $1/4$ (central shaded region) contains the essential spectrum while the eigenvalues, except for 1, cannot belong to the exterior shaded region.

Moreover, the sets

$$A_0 = \{a \in \mathbb{R} \ : \ \mu_2 < a < 1\}$$

$$A_1 = \left\{ a + ib \in \mathbb{C} \ : \ a > \frac{1}{2}, b^2 < \frac{\left(a - \frac{1}{2}\right)}{8(1 + a^{-1}\Gamma)} \left[\sqrt{1 + 64(1 + a^{-1}\Gamma)^2 \left(a - \frac{1}{2}\right)^2} - 1 \right] \right\}$$

$$A_2 = \left\{ a + ib \in \mathbb{R} \ : \ a < -\frac{1}{2}, b^2 < \frac{\left(|a| - \frac{1}{2}\right)^2 (4a^2 - 1 - |a|)}{4a^2 - 1 + |a|} \right\}$$

$$A_3 = \left\{ a + ib \in \mathbb{C} \ : \ a \geqslant 0, |a + ib - \frac{1}{2}|^2 > \frac{5 + \sqrt{17}}{16} \right\}$$

$$A_4 = \left\{ a + ib \in \mathbb{C} \ : \ a < 0, b^2 > \frac{3}{8} + |a| - a^2 \right\}$$

contain no spectrum of $\mathcal{L}_1$ except 1. These regions are illustrated in figure 2.

3.3.1. Different operators. As a last comment on the present approach to the study of the spectrum of $\mathcal{L}_1$, let us remark that it is possible to investigate the commutation relations with different operators. As an example, let us consider the operator $A(h) = h' + \alpha h$ for some function α.

Then

$$A\mathcal{L}_1 h = \mathcal{L}_2 h' + \mathcal{L}_1(D_f h) + \mathcal{L}_1((\alpha \circ f)h) = \mathcal{L}_2(Ah) + \mathcal{L}_1\left[\left(D_f - \frac{\alpha}{f'} + \alpha \circ f\right)h\right].$$

(3.5)

In general, is not obvious what the best choice of α could be. To keep the discussion short let us consider only the special, well known, case in which $\ln f'$ is $\mathcal{C}^1$ cohomologous to a constant.[19] That is, there exists a $\mathcal{C}^1$ function B such that

$$\ln f' + B - B \circ f = c.$$

Then we can choose $\alpha = B'$ and obtain

$$A\mathcal{L}_1 h = \mathcal{L}_2(Ah).$$

Accordingly, if there exists $h \in W^{1,1}$ such that $\mathcal{L}_1 h = \nu h$, $|\nu| > \lambda^{-1}$, then $\mathcal{L}_2(Ah) = \nu Ah$, thus $Ah = 0$ (since the spectral radius of $\mathcal{L}_2$ on L^1 is bounded by λ^{-1}). This implies that $h = e^{-B}a$, $a \in \mathbb{C}$, hence

$$\mathcal{L}_1 e^{-B} = \mathcal{L}_0 e^{-c-B \circ f} = e^{-c-B}\mathcal{L}_0 1 = e^{-c}Ne^{-B}.$$

Integrating yields $e^{-c}N = 1$, hence $\nu = 1$. It follows that

$$\sigma_{W^{1,1}}(\mathcal{L}_1) \subset \{1\} \cup \{z \in \mathbb{C} \ : \ |z| \leqslant \lambda^{-1}\},$$

hence, as expected, the existence of a large spectral gap.

In the general case, one could try to minimise $D_f - \frac{\alpha}{f'} + \alpha \circ f$ in order to produce estimates that improve theorem 3.9.

4. Piecewise monotone maps

Up to now we have considered uniformly expanding systems. However much of our arguments were rather general, it is then natural to ask if one can apply the present philosophy also to non-uniformly expanding maps or even maps that expand only in some part of the phase space. We believe the answer to be affirmative and to justify such a belief we discuss one of the simplest possibilities: one dimensional piecewise monotone map. Note however that we will develop the full theory only for one dimensional full branch monotone maps (see [64] for full details on the related theory). Of course, for such more general systems one cannot expect to prove as many results as in the previous section. Yet, some interesting and novel results can be obtained. To illustrate such a fact we will discuss the operator associated to the measures of maximal entropy.

Let $\mathcal{P} = \{I_1, \ldots, I_N\}$, $N \in \mathbb{N}$, be a partition of $[0, 1]$ in the sense that the I_i are open disjoint sets and $\cup_{i=1}^N \overline{I_i} = [0, 1]$. Let $f : [0, 1] \to [0, 1]$ be a map such that $f|_{I_i}$ is strictly monotone and $f|_{I_i} \in \mathcal{C}^1$. Thus each point in $(0, 1)$ has at most N preimages. Suppose that $\Lambda = \|f'\|_\infty < \infty$. We write $\mathbb{M}_*$ for the set of maps satisfying the above properties.

---

[19] This happens if f is $\mathcal{C}^2$ conjugated to a map $f_\ell(x) = \ell x \bmod 1$, $\ell \in \mathbb{Z}$ with $|\ell| \geqslant 2$.

Remark 4.1. Note that maps in $\mathbb{M}_*$ can have attracting fixed points or attracting periodic orbits and can be multimodal.

Remark 4.2. Note that we ask only $f|_{I_i} \in \mathcal{C}^1$, rather than $f|_{I_i} \in \mathcal{C}^{1+\alpha}$ as is necessary when studying the SRB measure.

We start by studying the spectral and essential spectral radius of $\mathcal{L}_0$. For each $h \in L^1$,

$$\|\mathcal{L}_0 h\|_{L^1} \leqslant \int_0^1 \mathcal{L}_1 |f'h| = \int |f'||h| \leqslant \Lambda \|h\|_{L^1}. \tag{4.1}$$

On the other hand, for $h \in \mathrm{BV}$ and $\varphi \in \mathcal{C}^1$ we have, calling ϕ_i the inverse of $f|_{I_i}$,

$$\int_0^1 \varphi'(x) \mathcal{L}_0 h(x) x = \sum_{i=1}^N \int_0^1 \varphi'(x) 1_{f(I_i)}(x) h(\phi_i(x)) \mathrm{d}x$$

$$= \sum_{i=1}^N \int_{I_i} \varphi'(f(x)) |f'(x)| h(x) \mathrm{d}x$$

$$= \sum_{i=1}^N \int_{I_i} \frac{\mathrm{d}}{\mathrm{d}x} \left[\mathrm{sign}(f'(x)) \cdot \varphi \circ f(x) \right] h(x) \mathrm{d}x.$$

We can then define the counterm, recall that $I_i = (p_i, p_{i+1})$,

$$\psi(x) = \sum_{i=1}^n 1_{I_i}(x) \, \mathrm{sign}(f'(x)) \left\{ \varphi(f(p_i)) + \frac{\varphi(f(p_{i+1})) - \varphi(f(p_i))}{p_{i+1} - p_i}(x - p_i) \right\}$$

and $\varphi_*(x) = \mathrm{sign}(f'(x)) \varphi(f(x)) - \psi(x)$. Note that, by construction, φ_* is Lipschitz, $\|\varphi\|_{\mathcal{C}^0} \leqslant 2\|\varphi\|_{\mathcal{C}^0}$ and

$$\int_0^1 \varphi'(x) \mathcal{L}_0 h(x) x = \int_0^1 \varphi'_*(x) h(x) \mathrm{d}x + \sum_{i=1}^N \int_{I_i} \mathrm{sign}(f'(x)) \frac{\varphi(f(p_{i+1})) - \varphi(f(p_i))}{p_{i+1} - p_i} h(x) \mathrm{d}x. \tag{4.2}$$

Accordingly, there exists $C > 0$ such that

$$\|\mathcal{L}_0 h\|_{\mathrm{BV}} \leqslant 2\|h\|_{\mathrm{BV}} + C\|h\|_{L^1}. \tag{4.3}$$

If we apply the above to the map f^n, rather than f, we have, since also $f \in \mathbb{M}_*$, the Lasota–Yorke inequality

$$\|\mathcal{L}_0 h\|_{L^1} \leqslant \Lambda \|h\|_{L^1},$$
$$\|\mathcal{L}_0^n h\|_{\mathrm{BV}} \leqslant 2\|h\|_{\mathrm{BV}} + C_n \|h\|_{L^1}, \tag{4.4}$$

for each $n \in \mathbb{N}$ and some constants $C_n > 0$.

From (4.4) and Hennion's theorem [51] (see also [28, appendix B]) it follows that the spectral radius, on BV, of $\mathcal{L}_0$ is bounded by Λ while the essential spectral radius is bounded by one[20].

This establishes the first step of our strategy. Next we have to consider the commutation between the derivative and $\mathcal{L}_0$. Since BV functions have weak derivatives that are measures, this makes sense, however such measures can be rather singular. To simplify matters it seems convenient to restrict our functions to SBV (special bounded variation) functions, that is functions for which the singular part of the weak derivative is supported at most on countably many point, not a Cantor set. Recall that SBV is a closed subspace of BV [2, corollary 4.3].

Lemma 4.3. *It holds true $\mathcal{L}_0(SBV) \subset SBV$. The essential spectrum of $\mathcal{L}_0$ is bounded by one. Moreover, the eigenvectors associated to eigenvalues of modulus strictly larger than one have zero absolutely continuous part. Hence, if a power of f is a Markov map, then the eigenvectors associated to eigenvalues of modulus strictly larger than one are constant on the elements of the Markov partition.*

Proof. Let $h \in$ SBV, then we can write the weak derivative, seen as a measure, as $Dh = \sum_{a \in \mathcal{A}} \alpha_a \delta_a + \mathbb{h} \, dx$, for some $\mathbb{h} \in W^{1,1}$, a countable set $\mathcal{A}$ and numbers $\alpha_a \in \mathbb{C}$ such that $\sum_{a \in \mathcal{A}} |\alpha_a| < \infty$, see [2, corollary 3.33]. Then (4.2) implies, setting $\mathcal{A}_0 = \mathcal{A} \backslash \{p_i\}_{i=1}^N$ and $\epsilon_i = \text{sign}(f'(x))$ for $x \in I_i$,

$$\int_0^1 \varphi'(x) \mathcal{L}_0 h(x) dx = -\int_0^1 \varphi_*(x) Dh(dx) + \sum_{i=1}^N \epsilon_i \frac{\varphi(f(p_{i+1})) - \varphi(f(p_i))}{p_{i+1} - p_i} \int_{I_i} h(x) dx$$

$$= -\sum_{a \in \mathcal{A}_0} \alpha_a [\text{sign}(f'(a)) \varphi(f(a)) - \psi(a)]$$

$$- \int_0^1 \varphi \circ f(x) \, \text{sign}(f'(x)) \mathbb{h}(x) dx$$

$$+ \int_0^1 \psi(x) \mathbb{h}(x) dx + \sum_{i=1}^N \epsilon_i \frac{\varphi(f(p_{i+1})) - \varphi(f(p_i))}{p_{i+1} - p_i} \int_{I_i} h(x) dx.$$

Hence

$$D\mathcal{L}_0 h = \mathcal{L}_1 \mathbb{h}(x) dx + \sum_{a \in \mathcal{A}_0} \alpha_a \, \text{sign}(f'(a)) \delta_{f(a)}$$

$$- \sum_{i=1}^N \sum_{a \in \mathcal{A}_0} 1_{I_i}(a) \epsilon_i \left\{ \frac{p_{i+1} - a}{p_{i+1} - p_i} \delta_{f(p_i)} + \frac{a - p_i}{p_{i+1} - p_i} \delta_{f(p_{i+1})} \right\}$$

$$- \sum_{i=1}^N \epsilon_i \left\{ \delta_{f(p_i)} \int_{I_i} \frac{p_{i+1} - x}{p_{i+1} - p_i} \mathbb{h}(x) dx + \delta_{f(p_{i+1})} \int_{I_i} \frac{x - p_i}{p_{i+1} - p_i} \mathbb{h}(x) dx \right\}$$

$$- \sum_{i=1}^N \epsilon_i \frac{\delta_{f(p_{i+1})} - \delta_{f(p_i)}}{p_{i+1} - p_i} \int_{I_i} h(x) dx$$

----

[20] In (4.4) choose n_0 such that $2^{1/n_0} \leqslant \Lambda$ and iterate with steps n_0.

$$= \mathcal{L}_1 \mathbb{h}(x)\mathrm{d}x - \sum_{i=1}^{N} \epsilon_i \delta_{f(p_{i+1})} \int_{I_i} \mathbb{h}(x)\mathrm{d}x + \sum_{a \in \mathcal{A}_0} \alpha_a \operatorname{sign}(f'(a))\delta_{f(a)}$$

$$+ \sum_{i=1}^{N} \sum_{a \in \mathcal{A}_0} 1_{I_i}(a)\epsilon_i \left\{ \frac{p_{i+1} - a + \alpha_a}{p_{i+1} - p_i} \delta_{f(p_i)} + \frac{a - p_i - \alpha_a}{p_{i+1} - p_i} \delta_{f(p_{i+1})} \right\} \in \mathrm{SBV}.$$

Thus $\mathcal{L}_0$ is well defined on SBV.

The bound on the essential spectral radius follows from (4.4) and Hennion's theorem [51] (see also [28, appendix B]).

Next, assume that $\nu \in \sigma_{\mathrm{SBV}}(\mathcal{L}_0)$ and $|\nu| > 1$. Then ν must be point spectrum and there exists $h \in \mathrm{SBV}$ such that $\mathcal{L}_0 h = \nu h$. But, differentiating, this would imply that the absolutely continuous part of Dh, call it $\mathbb{h}$, satisfies $\mathcal{L}_1 \mathbb{h} = \nu \mathbb{h}$ and, since the spectral radius of $\mathcal{L}_1$ is one, this implies $\mathbb{h} = 0$.

Finally, assume that f admits a Markov partition. Then we can assume, without loss of generality, that $\{I_i = (p_i, p_{i+1})\}_{i=1}^{N}$ is the Markov partition hence, $f(p_i) = p_j$ for some j. Let $\mathcal{A}$ be the set of jumps of h, $\{\alpha_a\}_{a \in \mathcal{A}}$ the size of the jumps and set $\mathcal{A}_* = \mathcal{A} \backslash \{p_i\}_i^N$. By the Markov property if $a \in \mathcal{A}_*$ then $f^{-1}(a) \notin \{p_i\}_i^N$. Hence, the above formula implies that if $\beta_n(a) = \{b \in \mathcal{A}_* : f^n(b) = a\}$ then, for each $n \in \mathbb{N}$,

$$|\nu|^n |\alpha_a| \leqslant \sum_{b \in \beta_n(a)} |\alpha_b| \leqslant \|h\|_{\mathrm{SBV}}.$$

which is possible only if $\alpha_p = 0$. Hence h can jump only at the boundaries of the partition and it is constant inside. $\qquad\square$

The above theorem shows that, for Markov maps, the study of the eigenvalues larger than one is reduced, as in section 2, to the study of the finite dimensional matrix B_0 defined in (2.1). However, in the general case identifying all the measures of maximal entropy requires some work that, in this generality, exceeds our scopes. To give an idea of what can be done let us restrict ourselves to the simplest Markov example: full branch maps.

4.1. Full branch monotone maps

Let $\mathbb{M} = \{f \in \mathbb{M}_* : f(I_i) = (0, 1)\}$. In this case, the previous computations show that $\mathcal{L}_0 W^{1,1} \subset W^{1,1}$ and $\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_0 = \mathcal{L}_1 \frac{\mathrm{d}}{\mathrm{d}x}$. From now on we will thus work in $W^{1,1}$. For this maps we want to investigate the measure of maximal entropy. To start, note that, in this case, $\Lambda \geqslant N$ since

$$N = \sum_{i=1}^{N} |f(I_i)| = \sum_{i=1}^{N} \int_{I_i} |f'(x)|\mathrm{d}x \leqslant \Lambda \sum_{i=1}^{N} |I_i| = \Lambda,$$

where the inequality is strict if f' is not constant.

Next, let us recall some well known facts (see [20] for a review).

Lemma 4.4 ([65, theorem 1]). *For $f \in \mathbb{M}$ holds the variational principle*

$$h_{top} = \ln N = \sup_{\mu \in \mathcal{M}} h_\mu(f)$$

where $\mathcal{M}$ is the set of invariant measures of f and $h_\mu(f)$ is the Kolmogorov–Sinai entropy.

Theorem 4.5. *The operator $\mathcal{L}_0$ when acting on $W^{1,1}$ has the spectral decomposition $\mathcal{L}_0 h = N \cdot \mu_{\mathrm{BM}}(h) + Q(h)$ where $Q1 = 0$, $\mu_{\mathrm{BM}}(Q(h)) = 0$, for all $h \in W^{1,1}$, $\sigma_{W^{1,1}}(Q) \subset \{z \in \mathbb{C} : |z| \leqslant 1\}$, and μ_{BM} is a measure of maximal entropy.*

Proof. We know that if $\nu \in \sigma(\mathcal{L}_0)$ and $|\nu| > 1$, then ν is point spectrum. That is, there exist $h \in W^{1,1}$ such that $\mathcal{L}_0 h = \nu h$. But then, differentiating, we have $\mathcal{L}_1 h' = \nu h'$ where $h' \in L^1$. However, $\mathcal{L}_1$ is a contraction on L^1, hence it must be either $|\nu| \leqslant 1$, contrary to the hypothesis, or $h' = 0$. The latter implies that h is constant, hence, we can always normalise it so that $h = 1$. On the other hand $\mathcal{L}_0 1(x) = \sum_{y \in f^{-1}(x)} 1 = N$. Hence $\nu = N$ and has geometric multiplicity one. Indeed, if the geometric multiplicity is not one, then there must exists $h \in W^{1,1}$ such that $\mathcal{L}_0 h = Nh + c$ for some constant c. But then, differentiating, $\mathcal{L}_1 h' = Nh'$, so h must be constant again.

In addition, note that for each $h \in W^{1,1}$ we have

$$\|N^{-n}\mathcal{L}_0^n h\|_{W^{1,1}} \leqslant \int_0^1 N^{-n}\mathcal{L}_0^n|h| + \int_0^1 N^{-n}\mathcal{L}_1^n|h'| \leqslant \|h\|_{\mathcal{C}^0} + N^{-n}\|h\|_{W^{1,1}}$$
$$\leqslant (1 + N^{-n})\|h\|_{W^{1,1}}.$$

Hence $N^{-n}\mathcal{L}_0^n$ is uniformly bounded on $W^{1,1}$ and thus, by [32, lemma 8.8.1], N is semi-simple. It follows that the maximal eigenvalue is simple.

Accordingly, we have the spectral decomposition $\mathcal{L}_0 = N1 \otimes \mu + Q$ where Q has spectral radius smaller or equal one, $Q1 = 0$, $\mu(Qh) = 0$ for all $h \in W^{1,1}$, $\mu(1) = 1$, and μ belongs to the dual of $W^{1,1}$. It remains to prove that μ is a measure and, indeed, a measure of maximal entropy μ_{BM}.

Note that, for each $h \in W^{1,1}$,

$$|\mu(h)| = \lim_{n \to \infty}\left|\int_0^1 N^{-n}\mathcal{L}_0^n h\right| \leqslant \lim_{n \to \infty}\|h\|_\infty \int_0^1 N^{-n}\mathcal{L}_0^n 1 = \|h\|_\infty.$$

Thus μ is a measure. In addition, for each $h \in \mathcal{C}^1$ such that $h \geqslant 0$, we have

$$\mu(h) = \lim_{n \to \infty}\int_0^1 N^{-n}\mathcal{L}_0^n h \geqslant 0$$

thus μ is a positive measure and, since it is normalised, it is a probability measure. Next, note that

$$\mu(\mathcal{L}_0 h) = \lim_{n \to \infty}\int_0^1 N^{-n}\mathcal{L}_0^{n+1} h = N \lim_{n \to \infty}\int_0^1 N^{-n}\mathcal{L}_0^n h = N \cdot \mu(h).$$

It follows

$$\mu(h \circ f) = N^{-1}\mu(\mathcal{L}_0 h \circ f) = N^{-1}\mu(h\mathcal{L}_0 1) = \mu(h).$$

That is μ is an invariant measure. In addition, by the above considerations, $([0,1], f, \mu)$ is ergodic.

The proof is concluded if we show $h_\mu(f) \geqslant h_{\mathrm{top}}$. Let $\mathcal{P}_n$ denote the nth canonical dynamical refinement of the partition $\mathcal{P}$. Let $p \in \mathcal{P}_n$ and $p_-, p_+ \in \mathcal{P}_n$ be the elements on the left and the

right of p, respectively, if they exist. Let $J = p_- \cup p \cup p_+$. Let $h \in \mathcal{C}^1(\mathbb{R}, [0, 1])$ be supported in J and such that $h|_p = 1$. Then $\mathcal{L}_0^n h(x) \leqslant 3$ and

$$
\mu(p) \leqslant \mu(h) = \lim_{m \to \infty} \int_0^1 N^{-m-n} \mathcal{L}_0^{m+n} h \leqslant \lim_{m \to \infty} 3 \int_0^1 N^{-m-n} \mathcal{L}_0^m 1
$$
$$
= 3N^{-n}\mu(1) = 3N^{-n}.
$$

Accordingly, calling $p_n(x)$ the element of $\mathcal{P}_n$ which contains x, the Shannon–McMillan–Breiman theorem (e.g. see [70, section 6.2, theorem 2.3]) states that for μ almost every point

$$
h_\mu(f) \geqslant h_\mu(\mathcal{P}, f) = \lim_{n \to \infty} -\frac{1}{n} \ln \mu(p_n(x)) \geqslant \lim_{n \to \infty} \ln N^{1-1/n} = \ln N,
$$

which concludes the proof by lemma 4.4. $\qquad\qquad\square$

Remark 4.6. We do not know if μ_{BM} is unique in this case, we have just constructed a measure of maximal entropy. Look at subsection 4.2 for a case where it is easy to prove that the measure of maximal entropy is unique.

Remark 4.7. The monotone interval maps and transfer operators studied in this section fit into the framework considered in [9]. In [9] the essential spectral radius (as operators acting on BV) is obtained and consequently a spectral decomposition. Here we show that the spectral gap is large for the operator associated to the measure of maximal entropy.

We have finally the announced mixing rate estimate

Corollary 4.8. *For any $\nu > \frac{1}{N}$ there exists $C_\nu > 0$ such that, for each $h \in W^{1,1}$ and $\varphi \in L^1(\mu_{\mathrm{BM}})$*

$$
\left| \int h \, \varphi \circ f^n \, \mathrm{d}\mu_{\mathrm{BM}} - \int h \, \mathrm{d}\mu_{\mathrm{BM}} \int \varphi \circ f^n \, \mathrm{d}\mu_{\mathrm{BM}} \right| \leqslant C_\nu \nu^n \|h\|_{W^{1,1}} \|\varphi\|_{L^1(\mu_{\mathrm{BM}})}.
$$

Proof. We start assuming that $\varphi \in \mathcal{C}^1$. Then, using theorem 4.5,

$$
\left| \int h \, \varphi \circ f^n \, \mathrm{d}\mu_{\mathrm{BM}} - \int h \, \mathrm{d}\mu_{\mathrm{BM}} \int \varphi \, \mathrm{d}\mu_{\mathrm{BM}} \right|
$$
$$
= \left| \lim_{m \to \infty} \int_0^1 N^{-m}(\mathcal{L}_0^m h \, \varphi \circ f^n)(x) \, \mathrm{d}x - \int h \mathrm{d}\mu_{\mathrm{BM}} \int \varphi \, \mathrm{d}\mu_{\mathrm{BM}} \right|
$$
$$
= \left| \lim_{m \to \infty} \int_0^1 [N^{-m+n}\mathcal{L}_0^{m-n}\varphi N^{-n}\mathcal{L}_0^n h](x) \, \mathrm{d}x - \int h \, \mathrm{d}\mu_{\mathrm{BM}} \int \varphi \, \mathrm{d}\mu_{\mathrm{BM}} \right|
$$
$$
= \left| \int \varphi N^{-n} \mathcal{L}_0^n h \, \mathrm{d}\mu_{\mathrm{BM}} - \int h \, \mathrm{d}\mu_{\mathrm{BM}} \int \varphi \, \mathrm{d}\mu_{\mathrm{BM}} \right|
$$
$$
= \left| \int \varphi N^{-n} Q^n h \, \mathrm{d}\mu_{\mathrm{BM}} \right| \leqslant C_\nu \nu^n \|h\|_{W^{1,1}} \int |\varphi| \, \mathrm{d}\mu_{\mathrm{BM}}.
$$

The corollary follows by a simple approximation argument. $\qquad\qquad\square$

4.2. Non-uniformly expanding maps

Let $\mathcal{E} \subset \mathbb{M}$ the set of maps such that $f' \geqslant 1$, $f' = 1$ at finitely many points and $\Lambda = \|f'\|_\infty < \infty$.

This class of maps includes the well known Manneville–Pomeau map [62, 71].

Remark 4.9. In [24] non-uniformly expanding systems are studied and the existence of a spectral gap (and hence decay of correlations) is proven for a class of equilibrium states which includes the measure of maximal entropy. The approach in [24] is based on Hilbert metrics and, although not stated explicitly, it provides a poor estimate of the spectral gap (see remark 3.7 for similar considerations) whereas our present approach provides an explicit and close to optimal estimate.

Here, we limit ourselves to the one dimensional case to present the idea in its simpler form. It is likely that similar results can be obtained for more general non-uniformly expanding maps, e.g. the higher dimensional examples treated in [24].

Note that the maps in $\mathcal{E}$ have a basic property.

Lemma 4.10. *Any map $f \in \mathcal{E}$ is expansive.*

Proof. Let $\kappa = \min_{I \in \mathcal{P}} |I|$. For each $\delta > 0$ let $\mathcal{I}_\delta = \{[a,b] \subset [0,1] \,:\, [a,b] \subset \bar{I}, I \in \mathcal{P} \,; |b - a| \geqslant \delta\}$ and, for each $[a,b] \in \mathcal{I}_\delta$, define $\varphi(a,b) := \frac{1}{|b-a|}\int_a^b f'(\xi)\mathrm{d}\xi$. Note that, by hypothesis, $\varphi(a,b) > 1$, and since it depends continuously from a,b (which vary in a compact set) there must be $\tau_\delta > 1$ such $\varphi(a,b) \geqslant \tau_\delta$. Accordingly, $f^n(x)$ and $f^n(y)$ always belong to the same partition element we have $|f^n(x) - f^n(y)| \leqslant \kappa$ for all $n \in \mathbb{N}$ which is possible only for $x = y$. On the other hand, if for some $f^n(x)$ and $f^n(y)$ belong to two different partition element, then either $|f^n(x) - f^n(y)| \geqslant \kappa$ or they belong to contiguous elements of $\mathcal{P}$. In such a case it is easy to see that there exists δ such that either $|f^n(x) - f^n(y)| \geqslant \delta$ or $|f^{n+1}(x) - f^{n+1}(y)| \geqslant \delta$, hence the expansivity. $\qquad\square$

The above fact allows to prove that the measure of maximal entropy is unique.

Lemma 4.11. *For $f \in \mathcal{E}$ the measure of maximal entropy μ_{BM} is unique.*

Proof. Since the map is expansive, there exists a map $\Phi : [0,1] \to \{1, \ldots, d\}^\mathbb{N} =: \Sigma$ which is well defined and invertible, apart from countably many points, that conjugates f with the full shift σ. Hence, Φ induces a measurable isomorphism for each non-atomic measure. On the other hand for (Σ, σ) holds the variational principle, hence the sup of the metric entropies is the topological entropy, which is $\ln N$, and there exists a unique measure of maximal entropy. Since atomic measures have zero entropy, and since the entropy is an affine function of the measures, it follows that the sup on the measure entropies is achieved on non-atomic measures. Thus, via the isomorphism Φ and since the entropy is an invariant for measure-preserving conjugacy, it follows that measure of maximal entropy for f is unique. $\qquad\square$

5. Hyperbolic maps

For hyperbolic, or partially hyperbolic maps the situation is less clear than in the expanding case and much more remains to be understood. Yet, the ghost of a general theory seems to be present. Let us start with the simplest possible case: linear maps.

In this case it is possible to study the problem using Fourier series (see [59]), however it is interesting to develop an alternative approach that does not rely on the algebraic structure of the map and thus has the potential to be applicable in greater generality.

5.1. Automorphisms of the torus

Here we consider a linear map $f : \mathbb{T}^n \to \mathbb{T}^n$ defined by $f(x) = Ax \bmod 1$ where $A \in \mathrm{SL}(d, \mathbb{Z})$, i.e. a matrix with integer coefficient and $\det A = 1$. Let us call E^u the unstable subspace, E^s the stable one and E^c the central one.

Note that, by hypothesis f preserves the volume, thus the volume is the SRB measure. We are interested in its statistical properties, hence in the transfer operator

$$\mathcal{L}h = h \circ f^{-1}.$$

Next we introduce a norm. Let $\{v_i^s\}$, $\|v_i^s\| = 1$, be a basis of E^s and $\{v_i^u\}$, $\|v_i^u\| = 1$, be a basis of E^u and define, for each $h, \varphi \in \mathcal{C}^\infty$ and $p, q \in \mathbb{N}_0$, $\partial_i^{s/u} h = \langle v_i^{s/u}, \nabla h \rangle$, and

$$
\begin{aligned}
|\varphi|_q^s &= \sup_{0 \leqslant k \leqslant q} \sup_{i_1,\dots,i_k} \| \partial_{i_1}^s \dots \partial_{i_k}^s \varphi \|_\infty \\
\|h\|_{p,q} &= \sum_{0 \leqslant k \leqslant p} \sup_{i_1,\dots,i_k} \sup_{|\varphi|_{k+q}^s \leqslant 1} \int_{\mathbb{T}^n} \varphi \partial_{i_1}^u \dots \partial_{i_k}^u h.
\end{aligned}
\tag{5.1}
$$

We call $\mathcal{B}^{p,q}$ the completion of $\mathcal{C}^\infty$ with respect to the norms $\| \cdot \|_{p,q}$.

In the following we assume $E^u \neq \{0\}$. In addition, to simplify the exposition, we assume that A has no Jordan blocks. The general case can be treated with a slight sophistication of the following arguments. We can thus choose the v_i^u such that $Av_i^u = \lambda_i v_i^u$, with $\lambda_i \geqslant \lambda > 1$. Also let λ be such that $\|A|_{E^s}\| \leqslant \lambda^{-1}$.

Remark 5.1.   The above norms are inspired by [4]. They are one of the many possible constructions of anisotropic Banach spaces adapted to hyperbolic maps or flows, see [7] for an extensive discussion. Given the linear structure of the invariant foliations, the norms (5.1) turn out to be especially convenient and simple to deal with, hence allowing a completely self-contained discussion. In the next section, we will use instead the norms defined in [47] in order to avoid having to redevelop the all theory (e.g. the Lasota–Yorke inequality) in the style of [4], which would certainly be possible.

The following is the equivalent of [4, proposition 3.2].

Proposition 5.2.   *For each $p, q \in \mathbb{N}_0, n \in \mathbb{N}$, and $\nu \in (\lambda^{-\min\{p+1,q\}}, 1)$, there exist $A, B > 0$ such that,*

$$
\begin{aligned}
\|\mathcal{L}h\|_{p,q} &\leqslant \|h\|_{p,q} \\
\|\mathcal{L}^n h\|_{p+1,q} &\leqslant A\nu^n \|h\|_{p+1,q} + B\|h\|_{p,q+1}.
\end{aligned}
$$

Proof. Since $\langle v, \nabla(h \circ f^{-1}) \rangle = \langle Df^{-1}v, \nabla h \rangle \circ f^{-1}$. We have

$$\int_{\mathbb{T}^n} \varphi \partial^u_{i_1} \ldots \partial^u_{i_k} \mathcal{L} h = \prod_{j=1}^{k} \lambda^{-1}_{i_j} \int_{\mathbb{T}^n} \varphi \circ f \partial^u_{i_1} \ldots \partial^u_{i_k} h.$$

Since $|\varphi \circ f|^s_{k+q} \leqslant |\varphi|^s_{k+q}$, the first inequality follows.

Next, note that

$$\|h\|_{p,q} = \sup_{i_1,\ldots,i_p} \sup_{|\varphi|^s_{p+q} \leqslant 1} \int_{\mathbb{T}^n} \varphi \partial^u_{i_1} \ldots \partial^u_{i_p} h + \|h\|_{p-1,q}.$$

Thus, by the above computations,

$$\|\mathcal{L}^n h\|_{p,q} \leqslant \lambda^{-pn}\|h\|_{p,q} + \|\mathcal{L}h\|_{p-1,q}.$$

It thus suffices to consider the case $k < p$. If $|\varphi|^s_{k+q} \leqslant 1$, then, for each $\varepsilon > 0$, let φ_ε be such that $|\varphi - \varphi_\varepsilon|^s_{k+q-1} \leqslant \varepsilon$, $|\varphi - \varphi_\varepsilon|^s_{k+q} \leqslant 2$ and $|\varphi_\varepsilon|^s_{k+q+1} \leqslant C\varepsilon^{-1}$, for some fixed constant $C > 2$.[21]

$$\left| \int_{\mathbb{T}^n} \varphi \partial^u_{i_1} \ldots \partial^u_{i_k} \mathcal{L}^n h \right| \leqslant \prod_{j=1}^{k} \lambda^{-n}_{i_j} \left\{ \left| \int_{\mathbb{T}^n} (\varphi - \varphi_\varepsilon) \circ f^n \partial^u_{i_1} \ldots \partial^u_{i_k} h \right| + C\varepsilon^{-1}\|h\|_{k,q+1} \right\}$$

$$\leqslant \prod_{j=1}^{k} \lambda^{-n}_{i_j} \left\{ \max\{\varepsilon, 2\lambda^{-(k+q)n}\}\|h\|_{k,q} + C\varepsilon^{-1}\|h\|_{k,q+1} \right\}$$

$$\leqslant 2\lambda^{-(2k+q)n}\|h\|_{k,q} + \frac{C}{2}\lambda^{(k+q)n}\|h\|_{k,q+1}$$

where, in the last line, we have chosen $\varepsilon = 2\lambda^{-(k+q)n}$. Accordingly,

$$\|\mathcal{L}^n h\|_{p,q} \leqslant (\lambda^{-pn} + 2\lambda^{-qn})\|h\|_{p,q} + C\lambda^{(q+p)n}\|h\|_{k,q+1}.$$

Next, choose $n_0 \in \mathbb{N}$ such that $3\lambda^{-\min\{p,q\}n_0} \leqslant \nu^{n_0}$, write $n = kn_0 + m$ with $m < n_0$ and iterate the above equation to obtain

$$\|\mathcal{L}^n h\|_{p,q} \leqslant \frac{3\lambda^{-\min\{p,q\}n_0}}{\nu^{n_0}} \nu^n \|h\|_{p,q} + \frac{C}{1-\nu} \lambda^{(q+p)n_0}\|h\|_{k,q+1},$$

which proves the proposition. $\square$

Remark 5.3. Note that proposition 5.2 implies that the spectral radius of $\mathcal{L}$ when acting on any space $\mathcal{B}^{p,q}$ is bounded by one. On the other hand, since $\mathcal{L}1 = 1$, the spectral radius must be exactly one.

The following is the equivalent of [4, lemma 4.1], although the proof follows a different path, easier in this particular case.

---

[21] Such a function can be constructed by convolving with a mollifier in the space E^s.

Lemma 5.4. *If $E^c = \{0\}$, then, for each $p, q \in \mathbb{N}$, $\{h \in \mathcal{B}^{p,q} : \|h\|_{p,q} \leqslant 1\}$ is relatively compact in $\mathcal{B}^{p-1,q+1}$.*

Proof. Let $d_s = \dim(E^s)$ and $d_u = \dim(E^u)$. By hypothesis $d = d_s + d_u$.

Let $U : \mathbb{R}^{d_s} \to \mathbb{R}^{d_u}$ such that $\{\boldsymbol{v}^s(v) = (v, Uv)\}_{v \in \mathbb{R}^{d_s}} = E^s$ and $V : \mathbb{R}^{d_u} \to \mathbb{R}^{d_s}$ such that $\{\boldsymbol{v}^u(v) = (Vv, v)\}_{v \in \mathbb{R}^{d_u}} = E^u$.[22]

Finally, consider mollifiers $j_\varepsilon^{s/u}(x) = \varepsilon^{-d_{s/u}} j^{s/u}(\varepsilon^{-1}x)$, where $j^{s/u} \in \mathcal{C}^\infty(\mathbb{R}^{d_{s/u}}, \mathbb{R}_+)$ such that $\operatorname{supp} j^{s/u} \subset \{\|x\| \leqslant 1\}$ and $\int_{\mathbb{R}^{d_{s/u}}} j^{s/u}(x)\mathrm{d}x = 1$. Then, for each $|\varphi|_{p+q} \leqslant 1$ and $h \in \mathcal{B}^{p,q}$,

$$
\int_{\mathbb{T}^d} \mathrm{d}x\, \varphi \partial_{i_1}^u \dots \partial_{i_{p-1}}^u h = \int_{\mathbb{T}^d} \mathrm{d}x \int_{\mathbb{R}^{d_s}} \mathrm{d}v\, \varphi(x + \boldsymbol{v}^s(v)) j_\varepsilon^s(v) \partial_{i_1}^u \dots \partial_{i_{p-1}}^u h(x) + \mathcal{O}(\varepsilon \|h\|_{p,q})
$$

$$
= \int_{\mathbb{T}^d} \mathrm{d}x \int_{\mathbb{R}^{d_s}} \mathrm{d}v \int_{\mathbb{R}^{d_u}} \mathrm{d}w \varphi(x + \boldsymbol{v}^s(v) - \boldsymbol{v}^u(w))
$$

$$
\times\, j_\varepsilon^s(v) j_\varepsilon^u(w) \partial_{i_1}^u \dots \partial_{i_{p-1}}^u h(x) + \mathcal{O}(\varepsilon \|h\|_{p,q})
$$

$$
=: \int_{\mathbb{T}^d} \mathrm{d}x\, \boldsymbol{\varphi}_\varepsilon(x) \partial_{i_1}^u \dots \partial_{i_{p-1}}^u h(x) + \mathcal{O}(\varepsilon \|h\|_{p,q}).
$$

Note that $\|\boldsymbol{\varphi}_\varepsilon\|_{\mathcal{C}^{p+q+1}} \leqslant C\varepsilon^{-p-1}$ and hence for each ε there is a set $\{\phi_i\}_{i=1}^{N_\varepsilon} \subset \mathcal{C}^{p+q}$ such that, for all φ we have $\|\boldsymbol{\varphi}_\varepsilon - \phi_i\|_{\mathcal{C}^{p+q}} \leqslant \varepsilon$ for some ϕ_i. It follows that, for each ε,

$$
\|h\|_{p-1,q+1} \leqslant C_\sharp \varepsilon \|h\|_{p,q} + \sup_{i \leqslant N_\varepsilon} \left| \int_{\mathbb{T}^d} \phi_i h \right|.
$$

From the above the wanted compactness follows by a standard diagonalisation argument. $\square$

We can now define the operators $\mathcal{D}_i h = \partial_i^u h$. Then

$$
\mathcal{D}_i \mathcal{L} h = \langle Df^{-1} v_i^u, \nabla h \circ f^{-1} \rangle = \lambda_i^{-1} \langle v_i^u, h \rangle \circ f^{-1} = \lambda_i^{-1} \mathcal{L} \mathcal{D}_i h. \tag{5.2}
$$

The usefulness of these operators rests in the following lemma. This is the only place in which we use Fourier series, however the result follows essentially from the accessibility property although with a more cumbersome proof.

Lemma 5.5. *The $\mathcal{D}_i$ are bounded operators from $\mathcal{B}^{p-1,q+1}$ to $\mathcal{B}^{p,q}$. In addition, if we assume that A has no eigenvalues that are roths of unity, then if $h \in \mathcal{B}^{p,q}$, $p \geqslant 1$ and $\mathcal{D}_i h = 0$, for all i, then h is constant.*

Proof. The fact that the $\mathcal{D}_j$ are bounded operators from $\mathcal{B}^{p-1,q+1}$ to $\mathcal{B}^{p,q}$ is a direct consequence of the definition of the norms in (5.1) and integration by parts.

Next, Katznelson lemma [54, lemma 3] (applied to A^*) implies that there exists $C_0 > 0$ such that, for each $k \in \mathbb{Z}^n \setminus \{0\}$,

$$
\operatorname{dist}(k, (E^u \oplus E^c)^\perp) \geqslant C_0 \|k\|^{-n}, \tag{5.3}
$$

---

[22] We can always choose coordinates in which this is possible.

Let $\hat{h}_k$ be the Fourier coefficient of h. Suppose tha $\hat{h}_k \neq 0$ for some $k \neq 0$. Since, by hypothesis $0 = \widehat{\mathcal{D}_j h}_k = \mathrm{i}\langle v_j^u, k\rangle \hat{h}_k$, we have $\langle v_j^s, k\rangle = 0$ for all j. Thus $k \perp E^u$, contradicting (5.3). Thus h must be constant. $\qquad\square$

We are now ready to draw our conclusions.

Lemma 5.6. *If $E^c = \{0\}$, then for each $\varepsilon > 0$ and p, q large enough we have $\sigma_{\mathcal{B}^{p,q}}(\mathcal{L}) \subset \{1\} \cup \{z \in \mathbb{C} \,:\, |z| \leqslant \varepsilon\}$.*

Proof. By proposition 5.2 and lemma 5.4, together with Hennion argument [51] (or [28, appendix B]) we have that the spectrum in the considered region is only point spectrum provided $\lambda^{-\min\{p,q\}} < \varepsilon$. We thus require p, q to be such that $\lambda^{-\min\{p,q\}} < \varepsilon$.

Next, suppose that $\mathcal{L}h = \nu h$ with $|\nu| > \varepsilon$, then, for all j, (5.2) implies

$$\mathcal{L}(\mathcal{D}_j^q h) = \lambda_j^q \nu(\mathcal{D}_j^q h).$$

But since $|\lambda_j^q \nu| > 1$ it cannot be an eigenvalue of $\mathcal{L}$, thus it must be $\mathcal{D}_j^q h = 0$. But, since integrating by parts yields

$$\int_{\mathbb{T}^d} \mathcal{D}_l^q h = 0,$$

for all $0 < l \leqslant j$, lemma 5.5 implies that h must be a constant, which, in turn, implies $\nu = 1$. $\qquad\square$

Remark 5.7. Lemma 5.6, by the usual arguments, implies that $\mathcal{C}^\infty$ observables have a super-exponential decay of correlations.

Remark 5.8. It seems reasonable to expect that a similar result should hold also if E^c is not trivial, but the map is ergodic (A does not have eigenvalues that are roots of unity). However, the proof of lemma 5.4 fails in this case. To overcome this problem it may be necessary to use a different Banach space. Thus, at present, it is not clear how to apply this strategy to partially hyperbolic systems, even in the simplest case.

The nonlinear case is much more subtle even in the Anosov setting. The obvious idea would be to consider an unstable vector field w and the operator $\mathcal{D} = \langle w, \nabla h\rangle$. Unfortunately, in general, unstable vector fields are only Hölder. Hence, it is not clear if $\mathcal{D}$ is a well defined bounded operator from $\mathcal{B}^{p,q}$ to $\mathcal{B}^{p-1,q+1}$. To solve this problem one should probably use different Banach spaces (for some appropriate version of such spaces such as the ones introduced in [46], see [79] for some recent progress along these lines).

Indeed, on the one hand w is smooth along unstable manifolds, on the other hand in the stable direction is only Hölder so its derivatives must be regarded as distribution, like h, and multiplication of distributions is a rather touchy business. So the situation, although not hopeless, is rather unclear.

Such issue needs further thought. Here we limit ourselves to explore an interesting alternative: considering the external derivative d as the appropriate differential operator. This simple change of perspective yields interesting results since it seems to provide a connection with the topology of the manifold. At least, this is the situation in the following where we discuss only the simplest case: two dimensional Anosov maps.

5.2. Anosov map on two dimensional manifolds

Let M be a smooth two dimensional compact and connected Riemannian manifold and $f \in \text{Diff}^\infty(M, M)$, be a transitive Anosov map. In other words, there exists $\lambda > 1$ and two continuous strictly invariant cone fields $\mathcal{C}^s, \mathcal{C}^u$ such that, for all $x \in M$,

$$\|d_x f v\| \geqslant \lambda \|v\| \quad \forall\, v \in \mathcal{C}^u(x)$$

$$\|d_x f^{-1} v\| \geqslant \lambda \|v\| \quad \forall\, v \in \mathcal{C}^s(x).$$

Remark 5.9. According to the Franks–Newhouse theorem [43, 67], every Anosov diffeomorphism of a two-dimensional compact Riemannian manifold is topologically conjugate to a hyperbolic toral automorphism. Hence, in our case, M must be homeomorphic to $\mathbb{T}^2$. Note however that in the following the smoothness of the map plays a fundamental role, hence one cannot in general reduce the discussion to the case $\mathbb{T}^2 = \mathbb{R}^2 \backslash \mathbb{Z}^2$. It is thus convenient to argue considering M a general two dimensional manifold. This has also the advantage to emphasise the possibility of a higher dimensional extension. Indeed, we will use the Franks–Newhouse theorem only at the end of the argument (lemma 5.21), to characterise the cohomology groups.

In analogy with the previous sections, we will obtain results on the mixing properties of the measure of maximal entropy μ_{BM}.

Theorem 5.10. *The exists $r \in \mathbb{N}$, $C > 0$ and $\kappa \in (0,1)$ such that for all $g, h \in \mathcal{C}^\infty$ and $n \in \mathbb{N}$ we have*

$$\left| \int_M g \circ f^n h \, d\mu_{\text{BM}} - \int_M g \, d\mu_{\text{BM}} \int_M h \, d\mu_{\text{BM}} \right| \leqslant C \|g\|_{\mathcal{C}^r} \|h\|_{\mathcal{C}^r} e^{-h_{top}n} \kappa^n.$$

This result is a corollary of the much more precise theorem 5.12 and it is proven in section 5.6. To state theorem 5.12 we need to first introduce several objects.

5.3. The operators

The operator associated to the SRB measure is simply (e.g. see [45])

$$\mathcal{L}h(x) = (\det D_{f^{-1}(x)} f)^{-1} h \circ f^{-1}(x).$$

However, in the present context the interesting object to study seems to be the action of forms, or rather *currents*.[23] Recall that the pullback on a differential form ω by a map g is defined as

$$(g^* \omega)_x(v_1, v_2) = \omega_{g(x)}(d_x g(v_1), d_x g(v_2)).$$

If g is a diffeomorphism we can define the pushforward as $g_* \omega = (g^{-1})^* \omega$. It is then natural to define the action of the dynamics on forms as the pushforward f_*.

Let ω_0 be the Riemannian volume. Then any two form can be written as $\omega = h\omega_0$ for some function h. Then

$$
\begin{aligned}
[f_* \omega(v_1, v_2)](x) &= h \circ f^{-1}(x) \omega_0 \left((d_x f^{-1}(v_1), d_x f^{-1}(v_2)) \right)(x) \\
&= h \circ f^{-1}(x) \det(D_{f^{-1}(x)} f)^{-1} \omega_0(v_1, v_2)(x) \\
&= \left[(\mathcal{L}h \cdot \omega_0)(v_1, v_2) \right](x).
\end{aligned}
\tag{5.4}
$$

---

[23] The idea that currents are a relevant object to study in the context of the statistical properties of dynamical systems goes back, at least, to [75]. See, for example, [5, 74] for further use of k-forms in the dynamical systems context.

That is, the operator $\mathcal{L}$ is equivalent to the pushforward on two forms.

Recall that

$$d(f_* h) = f_* \, dh. \tag{5.5}$$

where, if h is a zero form, then $f_* \, dh(x) = \left[D_x f^{-1} \right]^{\mathrm{T}} (\nabla h) \circ f^{-1}(x)$.

The scalar product in $T^* M$ is canonically defined by using the canonical duality $\pi : T^* M \to T_* M$ defined by $\omega(v) = \langle \pi(\omega), v \rangle$, for all $v \in T_* M$. That is,

$$\langle \omega_1, \omega_2 \rangle = \langle \pi(\omega_1), \pi(\omega_2) \rangle = \omega_1(\pi(\omega_2)). \tag{5.6}$$

For each $x \in M$ and $v_1, v_2, w_1, w_2 \in T_x^* M$ we define

$$\langle v_1 \wedge v_2, w_1 \wedge w \rangle = \det(\langle v_i, w_j \rangle). \tag{5.7}$$

Assuming bilinearity, the above formula defines uniquely a scalar product among two-forms. Also, we define a duality from ℓ to $2 - \ell$ forms via (see [47, appendix A] for more details)

$$\langle v, w \rangle \omega_0 = (-1)^{\ell(2-\ell)} v \wedge *w = (-1)^{\ell(2-\ell)} w \wedge *v = *v \wedge w. \tag{5.8}$$

Since such a formula must hold for all ℓ-forms, the $(2 - \ell)$-forms $*w$, $*v$ are uniquely defined. The operator '$*$' is the so called *Hodge operator*.

5.4. The Banach spaces and the main result

The operators f_* have been studied for flows in [47] using appropriate Banach spaces. We use the same notation and almost the same Banach spaces defined in [47, section 3]. However, since here we consider maps rather than flows, we do not have the requirement that the forms be null in the flow direction (see [47, equation (3.5)]). Here we provide some more detail but we refer to [47] for the full story.

For any $r \in \mathbb{N}$, we assume that there exists $\delta_0 > 0$ such that, for each $\delta \in (0, \delta_0)$ and $\rho \in (0, 4)$, there exists an atlas $\{(U_\alpha, \Theta_\alpha)\}_{\alpha \in A}$, where A is a finite set, such that[24]

$$\begin{cases} \Theta(U_\alpha) = B_2(0, 30\delta\sqrt{1 + \rho^2}), \\ \cup_\alpha \Theta_\alpha^{-1}(B_2(0, 2\delta)) = M, \\ \|(\Theta_\alpha)_*\|_\infty + \|(\Theta_\alpha^{-1})_*\|_\infty \leqslant 2; \quad \|\Theta_\alpha \circ \Theta_\beta^{-1}\|_{C^r} \leqslant 2. \end{cases} \tag{5.9}$$

Fix $L_0 > 0$. For any $L > L_0$, let us define

$$\mathcal{F}_r(\rho, L) := \{ F \in \mathcal{C}^r(B_1(0, 6\delta), \mathbb{R}) : F(0) = 0;$$

$$\|DF\|_{\mathcal{C}^0(B_1(0,6\delta))} \leqslant \rho; \|F\|_{\mathcal{C}^r(B_1(0,6\delta))} \leqslant L \} \,.$$

Where the $\mathcal{C}^r$ is defined as usual, e.g. see [47, equation (3.6)]. For each $F \in \mathcal{F}_r(\rho, L)$, $x \in \mathbb{R}^2$, $\xi \in \mathbb{R}^1$, let $G_{x,F}(\xi) : B_1(0, 6\delta) \to \mathbb{R}^2$ be defined by $G_{x,F}(\xi) := x + (\xi, F(\xi))$. Let us also define

---

[24] We use the notation $B_d(x, r) = \{ y \in \mathbb{R}^d \ : \ \|y - x\| < r \}$.

$\tilde{\Sigma}(\rho, L) := \{G_{x,F} : x \in B_1(0, 2\delta), F \in \mathcal{F}_r(\rho, L)\}$. For each $\alpha \in A$ and $G \in \tilde{\Sigma}(\rho, L)$, we define the leaf

$$W_{\alpha,G} := \{\Theta_\alpha^{-1} \circ G(\xi)\}_{\xi \in B_1(0, 3\delta)}.$$

For each $\alpha \in A$, $G \in \tilde{\Sigma}(\rho, L)$, note that $W_{\alpha,G} \subset \hat{U}_\alpha := \Theta_\alpha^{-1}(B_d(0, 6\delta\sqrt{1 + \rho^2})) \subseteq U_\alpha$. Finally, we define $\Sigma_\alpha = \cup_{G \in \tilde{\Sigma}(\rho,L)} W_{\alpha,G}$. This is the set of 'almost stable' leaves that we will use to define our norms.

Given a curve $W_{\alpha,G} \in \Sigma_\alpha$, we consider $\Gamma_c^{\ell,s}(\alpha, G)$ as the $\mathcal{C}^s$ sections of the fiber bundle on $W_{\alpha,G}$ with fibers in $\wedge^\ell(T^*M)$, as defined in [47, equation (3.8)], equipped with the $\mathcal{C}^s$ norm.

Following [47, section 3] let $V^s(\alpha, G)$ be the set of uniformly $\mathcal{C}^s(U_{\alpha,G})$ vector fields, where $U_{\alpha,G}$ is any open set such that $U_\alpha \supset U_{\alpha,G} \supset W_{\alpha,G}$.

Let ω_{vol} be the volume form induced on $W_{\alpha,G}$ by the push-forward of Riemannian volume via the chart Θ_α^{-1}. Write L_ν for the Lie derivative along a vector field ν. For all $\alpha \in A$, $G \in \Sigma_\alpha$, $g \in \Gamma_c^{l,0}(\alpha, G)$, $\bar{v}^p = (v_1, \ldots, v_p) \in V^s(\alpha, G)^p$, let us define the functionals $J_{\alpha,G,g,\bar{v}^p} : C^p \to \mathbb{C}$ by[25]

$$J_{\alpha,G,g,\bar{v}^p}(h) = \int_{W_{\alpha,G}} \langle g, L_{v_1} \ldots L_{v_p} h \rangle \omega_{\text{vol}}. \tag{5.10}$$

Next, for all $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $p + q < r - 1$, $l \in \{0, 1, 2\}$, let

$$\mathbb{U}_{\rho,L,p,q,\ell} = \Big\{ J_{\alpha,G,g,\bar{v}^p} | \alpha \in A, G \in \Sigma_\alpha(\rho, L), g \in \Gamma_c^{\ell,p+q}, \nu_j \in V^{p+q},$$

$$\|g\|_{\Gamma_c^{\ell,p+q}(\alpha,G)} \leqslant 1, \|\nu_j\|_{C^{p+q}(U_{\alpha,G})} \leqslant 1 \Big\}.$$

where, for $\nu \in V^s(\alpha, G)$, $\|\nu\|_{C^s(U_{\alpha,G})} = \sup_{\alpha,i} \|\langle \nu, e_{\alpha,i} \rangle \circ \Theta_\alpha^{-1}\|_{C^s(\Theta_\alpha(U_{\alpha,G}))}$.

For all $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $\ell \in \{0, 1, 2\}$, we finally define the spaces $\mathcal{B}^{p,q,\ell}$ as the closure of the $\mathcal{C}^\infty$ ℓ forms with respect to the norm

$$\|h\|_{p,q,\ell} = \sup_{n \leqslant p} \sup_{J \in \mathbb{U}_{\rho,L,n,q,\ell}} J(h).$$

Note that $\mathcal{B}^{p,q,\ell}$ is contained in the space of ℓ $(p + q$ smooth) currents (see [47]).

Remark 5.11. Note that $(\Theta_\alpha)_*(L_\nu h) = L_{(\Theta_\alpha)_*\nu}(\Theta_\alpha)_* h$, thus, in coordinates, $L_\nu h$ will have the form $\sum_{i=1}^2 \alpha_i L_{e_i} h =: \sum_{i=1}^2 \alpha_i \partial_{x_i} h$, for some functions α_i and where $\{e_i\}_{i=1}^2$ is the standard basis of $\mathbb{R}^2$. If follows that restricting the vector fields in (5.10) to $\Theta_\alpha^* e_i$ yields an equivalent norm. Hence, to simplify notation, in the following we will use the notation $\partial^\beta h$ to designate the application of $|\beta|$ vector fields to h, where β are the usual multiindexes used in PDE.

The main result of this section consists in the following theorem which provides a rather precise characterisation of the spectrum of the action on one forms, which is well known to be related to the measure of maximal entropy and thus plays the same role of the operators $\mathcal{L}_0$ in the previous sections.

---

[25] By $\langle \cdot, \cdot \rangle$ we mean the usual scalar product between forms.

Theorem 5.12.   *For each $\varepsilon > 0$, for p, q large enough,*

$$\{e^{-h_{top}}, e^{h_{top}}\} \cup (\sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)\backslash\{z \in \mathbb{C} \, : \, |z| < \varepsilon\}) \subset \sigma_{\mathcal{B}^{p,q,1}}(f_*)$$

$$\sigma_{\mathcal{B}^{p,q,1}}(f_*) \subset \{z \in \mathbb{C} \, : \, |z| < \varepsilon\} \cup \{e^{-h_{top}}, e^{h_{top}}\} \cup \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \cup \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L}),$$

moreover $1 \notin \sigma_{\mathcal{B}^{p,q,1}}(f_*)$.

In addition, $\sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)\backslash\{z \in \mathbb{C} \, : \, |z| < \varepsilon\}$ consists only of point spectrum and there exists $\kappa \in (0, 1)$ such that $\sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \subset \{1\} \cup \{z \in \mathbb{C} \, : \, |z| < \kappa\}$. The same holds for $\sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L})$.

Remark 5.13.   In fact, we conjecture that, for $\lambda^{\min\{p,q\}} > \varepsilon^{-1}$,

$$\sigma_{\mathcal{B}^{p,q,1}}(f_*)\backslash\{z \in \mathbb{C} \, : \, |z| < \varepsilon\} = \left[\{e^{-h_{\text{top}}}, e^{h_{\text{top}}}\} \cup \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \cup \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L})\right] \backslash\{1\},$$

see remark 5.23. This would be consistent with the fact that, by duality, the spectrum of $\mathcal{L}$ equals the spectrum of f_*^{-1} and that the spectra of f_* on forms determine the Ruelle zeta function, see [72], and the latter is described in term of periodic orbits, which are the same for f and f^{-1}. Accordingly, one expects a symmetry between the spectra of f_* and f_*^{-1}.[26]

The next section is devoted to the proof of the above theorem, while in section 5.7 we present a minimalistic discussion of cohomology in the spaces $\mathcal{B}^{p,q,1}$ and section 5.8 is devoted to comments on the implications of such a Theorem and a comparison with existing results.

5.5. Proof of theorem 5.12

We start with some preliminary results establishing minimal information about Hodge duality and exterior differentials in our spaces of currents $\mathcal{B}^{p,q,\ell}$.

Lemma 5.14.   *The Hodge duality map $\Phi h := \star h = h\omega_0$, between zero forms and two forms, extends to a bounded isomorphism between $\mathcal{B}^{p,q,0}$ and $\mathcal{B}^{p,q,2}$ and $\Phi\mathcal{L} = f_*\Phi$. In particular, $\sigma_{\mathcal{B}^{p,q,2}}(f_*) = \sigma_{\mathcal{B}^{p,q,0}}(\mathcal{L})$.*

Proof.   By equation (5.8), for each smooth zero form h, $\Phi h = h\omega_0$. Thus equation (5.4) implies $\Phi\mathcal{L}h = f_*\Phi h$ for each smooth zero form. The injectivity follows since $\mathcal{B}^{p,q,0}, \mathcal{B}^{p,q,2}$ are isomorphic to a subspace of the space of currents, see [47, lemma 3.10], and the extension of Φ to the current is an isomorphism. The result then follows by proving that Φ is a bounded operator. For each multi-index α, $|\alpha| = p$, smooth two form ω and zero form h we have

$$\left|\int_W \langle \omega, \partial^\alpha \Phi(h)\rangle\right| \leqslant \sum_{\beta+\gamma=\alpha} \left|\int_W \langle \omega, \partial^\beta \omega_0\rangle \partial^\gamma h\right| \leqslant C_\sharp \|\omega\|_{\mathcal{C}^{q+p(W)}}\|h\|_{p,q,0}$$

from which the claim follows.   □

Lemma 5.15.   *The exterior derivative* d *extends to a bounded operator $\mathcal{B}^{p,q,\ell} \to \mathcal{B}^{p-1,q+1,\ell+1}$.[27] If $h \in \mathcal{B}^{p,q,0}$, $\psi \in \mathcal{C}^\infty(M, \mathbb{R}_+)$, with the interior of $\mathrm{supp}(\psi)$ connected, and*

---

[26] Note that if f_* acts on some Banach space, then here one considers f^{-1} acting on its dual, so the relation is not obvious *a priori*. Indeed, one does not necessarily expect f_*^{-1} to be a bounded operator when acting on a Banach space on which f_* is bounded.

[27] With a slight abuse of notation we will call such an extension *d* as well.

$\psi \, \mathrm{d}h = 0$, *then there exists* $c \in \mathbb{C}$ *such that* $\psi(h - c) = 0$.[28] *Finally,* $\mathrm{d}(\mathcal{B}^{p,q,0})$ *is closed in* $\mathcal{B}^{p-1,q+1,1}$.

Proof.   If h is an ℓ form, then, for each $\ell + 1$ form ω and multi-index α, $|\alpha| = p - 1$, we have that there exists a constant $C_\ell > 0$ such that

$$\left| \int_W \langle \omega, \partial^\alpha \, \mathrm{d}h \rangle \right| \leqslant C_\ell \|\omega\|_{\mathcal{C}^{p+q}(W)} \|h\|_{p,q,\ell}$$

from which it follows that $\|\mathrm{d}h\|_{p-1,q+1,\ell+1} \leqslant C_\ell \|h\|_{p,q,\ell}$.

Next, let h, ψ be such that $\psi \, \mathrm{d}h = 0$. Note that $\mathcal{B}^{p,q,0}$ is isomorphic to a subspace of the space of distributions $(\mathcal{C}^{p+q})'$, see [47, lemma 3.10]. Let $K = \operatorname{supp} \psi$ and $U = \overset{\circ}{K}$, note that U is connected by hypothesis. Thus for each smooth local function φ, $\operatorname{supp} \varphi \subset U$, and disintegration of ω along a smooth foliation $\{W_t\} \subset \Sigma$, we have[29]

$$\int_M \varphi \partial_{x_i} h = \int \mathrm{d}t \int_{W_t} \langle \frac{\varphi}{\psi} \mathrm{d}x_i, \psi \, \mathrm{d}h \rangle = 0. \tag{5.11}$$

It follows that $\partial_{x_i} h = 0$ as a distribution on U, hence $h = c$ on U, for some $c \in \mathbb{C}$. That is $\psi(h - c) = 0$ on M. From [47, lemma 3.10], again, it follows that $\psi(h - c) = 0$ as an element of $\mathcal{B}^{p,q,0}$.

To conclude the Lemma we want to prove that $\mathrm{d}(\mathcal{B}^{p,q,0})$ is closed in $\mathcal{B}^{p-1,q+1,1}$. Let us suppose that $\omega_n \to \omega$, in $\mathcal{B}^{p-1,q+1,1}$, with $\omega_n \in \mathrm{d}(\mathcal{B}^{p,q,0})$. That is, there exists $\Xi_n \in \mathcal{B}^{p,q,0}$ such that $\omega_n = \mathrm{d}\Xi_n$. Let $\widehat{\Xi}_n = \Xi_n - \int_M \Xi_n$. Then, for each function φ supported in a chart $(U_\alpha, \Theta_\alpha)$ we can write

$$\int \varphi \widehat{\Xi}_n = \int_M \left(\varphi - \int_M \varphi \right) \Xi_n.$$

Let $\bar{x}$ be such that $\int_M \varphi = \varphi(\bar{x})$, then[30]

$$\int \varphi \widehat{\Xi}_n = \int_{U_\alpha} \mathrm{d}x \int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} \varphi(\bar{x} + (x - \bar{x})t) \Xi_n(x)$$

$$= -\sum_{i=1}^2 \int_M \varphi(\bar{x} + (x - \bar{x})t) \langle (x_i - \bar{x}_i), \partial_{x_i} \Xi_n(x) \rangle.$$

Hence, setting $\Psi_t(x) = -\sum_{i=1}^2 \varphi(\bar{x} + (x - \bar{x})t)(x_i - \bar{x}_i)\mathrm{d}x_i$, we have, recalling equation (5.6),

$$\int \varphi \widehat{\Xi}_n = \int_0^1 \mathrm{d}t \int_M \langle \Psi_t, \omega_n \rangle.$$

Arguing similarly for $\partial^\alpha \widehat{\Xi}_n$ it follows that $\widehat{\Xi}_n$ is a Cauchy sequence. Let Ξ be the limit, then, by the continuity of d, $\mathrm{d}\Xi = \omega$, hence $\omega \in \mathrm{d}(\mathcal{B}^{p,q,0})$. $\square$

---

[28] This essentially says that closed anisotropic zero currents are constant. Since for zero currents being closed and being harmonic is the same, this is a little piece of Hodge theory, all that is presently needed. Yet, it would be clearly useful to develop the Hodge theory in the context of anisotropic spaces.

[29] Since the foliation is smooth the Jacobian J of the disintegration is a smooth function and $\hat{\varphi} = J\varphi$. Note that $\frac{\hat{\varphi}}{\psi}$ is a smooth function on W_t.

[30] To simplify notation we do not write explicitly the change of coordinates Ξ_α.

5.5.1. Spectral radius and essential spectral radius of f_* and $\mathcal{L}$.   The first step in the study of the operators $f_*, \mathcal{L}$ is the following.

Lemma 5.16.   *The action of f_* on ℓ-forms extends to a linear bounded operator from $\mathcal{B}^{p,q,\ell}$ to itself. With a slight abuse of notation we use f_* for the action on each $\mathcal{B}^{p,q,\ell}$. Then, there exists a constant $C_\sharp > 0$ such that*

$$\|f_*^n h\|_{0,q,0} \leqslant C_\sharp \|h\|_{0,q,0}$$

$$\|f_*^n h\|_{0,q,1} \leqslant C_\sharp e^{h_{top} n} \|h\|_{0,q,1}$$

$$\|f_*^n h\|_{p,q,0} \leqslant C_\sharp \lambda^{-np} \|h\|_{p,q,0} + C_\sharp \|h\|_{p-1,q+1,0}$$

$$\|f_*^n h\|_{p,q,1} \leqslant C_\sharp e^{h_{top} n} \lambda^{-np} \|h\|_{p,q,1} + C_\sharp e^{h_{top} n} \|h\|_{p-1,q+1,1}$$

$$\|\mathcal{L}^n h\|_{p,q,0} \leqslant C_\sharp \lambda^{-np} \|h\|_{p,q,0} + C_\sharp \|h\|_{p-1,q+1,0}.$$

Proof.   To start with let $h \in \mathcal{C}^\infty(M, \mathbb{C})$ be a function, then

$$\int_W \langle \varphi, f_*^n h \rangle = \int_W \langle \varphi, h \circ f^{-n} \rangle = \int_{f^{-n}W} \langle \varphi \circ f^n, h\lambda_n^s \rangle$$

where $\lambda_n^s(x)$ is the contraction of f^n in the direction $T_* W$ at the point x. We can divide $f^{-n}W$ in a collection $\{W_i\} \subset \Sigma$. Let $\{\vartheta_i\}$ be a smooth partition of unity subordinated to $\{W_i\}$. If $\lambda_{n,i}^s = \min_{x \in W_i} \lambda_n^s(x)$, then the usual distortion arguments implies, that for all $x \in W_i$, $C_\sharp \lambda_{n,i}^s \leqslant \lambda_n^s(s) \leqslant C_\sharp \lambda_{n,i}^s$, thus, integrating $|f^n(W_i)| = \int_{W_i} \lambda_{n,i}^s \geqslant C_\sharp \lambda_{n,i}^s \delta$, where δ is the size of the manifolds (see the beginning of section 5.4). In addition, for all $q \in \mathbb{N}$, $\|\lambda_n^s(s)\|_{\mathcal{C}^q(W_i)} \leqslant C_\sharp \lambda_{n,i}^s$. Accordingly,

$$\left| \int_W \langle \varphi, f_*^n h \rangle \right| \leqslant \sum_i \left| \int_{W_i} \langle \vartheta_i \lambda_n^s \varphi \circ f^n, h \rangle \right|$$

$$\leqslant C_\sharp \sum_i |f^n(W_i)| \delta^{-1} \|\varphi \circ f^n\|_{\mathcal{C}^q(W_i)} \|h\|_{0,q,0}$$

$$\leqslant C_\sharp \|\varphi\|_{\mathcal{C}^q(W)} \|h\|_{0,q,0}$$

which, by density, proves the first inequality of the lemma. Next, recalling (5.6), we have, for each $v = \pi(\omega), h$ a $\mathcal{C}^\infty$ one form,

$$\left| \int_W \langle \omega, f_*^n h \rangle \right| \leqslant \left| \int_W f_*^n h(v) \right| = \left| \int_W h_{f^{-n}(x)} (d_x f^{-n} v(x)) dx \right|.$$

Setting $v_n(x) = d_{f^n(x)} f^{-n} v(f^n(x))$, by the usual distortion arguments we have $\|v_n\|_{\mathcal{C}^q(W_i)} \leqslant C_\sharp (\lambda_{n,i}^s)^{-1} \|v\|_{\mathcal{C}^q(W)}$, hence

$$\left| \int_W \langle \omega, f_*^n h \rangle \right| \leqslant C_\sharp \sum_i \left| \int_{W_i} h(\vartheta_i \lambda_n^s v_n) \right| \leqslant C_\sharp \sum_i |W_i| \delta^{-1} \|h\|_{0,q,1}$$

$$\leqslant C_\sharp e^{h_{top} n} \|\omega\|_{\mathcal{C}^q} \|h\|_{0,q,1}$$

where, in the last line, we have used $|f^{-n}W| \sim e^{h_{top}n}$, see [47, appendix D]. Taking the sup on W and ω the second inequality of the lemma follows.

The next two inequalities are proven, similarly, as done in [47, lemma 4.7], while the last follows by [45, lemma 2.2] taking into account lemma 5.14. □

We are now able to obtain a first information on the peripheral spectrum.

Lemma 5.17. *For p, q large enough, the spectra of f_* on $\mathcal{B}^{p,q,0}$ and on $\mathcal{B}^{p,q,2}$ are contained in $\{1\} \cup \{z \in \mathbb{C} \ : \ |z| < \kappa\}$ for some $\kappa < 1$. The eigenvectors associated to the eigenvalue 1 are the constant function 1 and the measure μ_{SRB} respectively.*

Proof. By lemma 5.14 the action of f_* on $\mathcal{B}^{p,q,2}$ is conjugated to the action of $\mathcal{L}$ on $\mathcal{B}^{p,q,0}$, thus they have the same spectrum. But [45] implies that there exists $\kappa \in (0, 1)$ such that $\sigma_{\mathcal{B}^{p,q,0}}(\mathcal{L}) \subset \{1\} \cup \{z \in \mathbb{C} \ : \ |z| \leqslant \kappa\}$ and one is a simple eigenvalue. This proves the Lemma for $\mathcal{B}^{p,q,2}$.

Let us discuss $\mathcal{B}^{p,q,0}$. Lemma 5.16 and Hennion's theorem [51] (or see [28, appendix B]) imply that the radius of the essential spectrum of f_* acting on $\mathcal{B}^{p,q,0}$ is at most λ^{-1} while the spectral radius is one, moreover the operator is power bounded. Accordingly, if f_* has no eigenvalue on the unit circle apart from 1 and 1 is a simple eigenvalue, then there exists a κ that satisfies the Lemma. Thus, we need only study eigenvalues of the form $e^{i\theta}$. Since the operator is power bounded, there cannot be a (non-trivial) Jordan block associated to such a maximal eigenvalue, hence their geometric and algebraic multiplicity coincide. Hence, we have the spectral decomposition

$$f_* = \sum_j e^{i\theta_j}\Pi_j + Q$$

where $\theta_j \in \mathbb{R}$, $\Pi_i\Pi_j = \delta_{ij}\Pi_j$, $\Pi_jQ = Q\Pi_j = 0$ and $\|Q^n\|_{p,q,0} \leqslant C_\sharp\kappa^n$. Suppose that $e^{i\theta} \in \sigma_{\mathcal{B}^{p,q,0}}(f_*)$, $\theta \in \mathbb{R}\backslash\{0\}$, then there exists $h \in \mathcal{B}^{p,q,0}\backslash\{0\}$ such that $f_*h = e^{i\theta}h$, and, by the spectral decomposition, there exists $h_0 \in \mathcal{C}^\infty$ such that

$$h = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} e^{-i\theta k} f_*^k h_0.$$

It follows that, for all $\varphi \in \mathcal{C}^\infty(M, \mathbb{C})$,[31]

$$\int_M \varphi h\omega_0 = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} e^{-i\theta k}\int_M \varphi f_*^k h_0\omega_0 = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} e^{-i\theta k}\int_M \varphi \cdot h_0 \circ f^{-k}\omega_0.$$

But f^{-1} is also a transitive Anosov diffeomorphism with its SRB measure, call it μ_{SRB}^-, then

$$\lim_{k\to\infty}\int_M \varphi \cdot h_0 \circ f^{-k}\omega_0 = \int_M \varphi\omega_0 \int_M h\mu_{SRB}^-$$

which implies

$$\int_M \varphi h\omega_0 = 0$$

---

[31] Note that the integral on M can be decomposed as an integral over elements of Σ, which are continuous functionals in the $\mathcal{B}^{p,q}$ norms, hence we can exchange the limit with the integral.

and since $\mathcal{B}^{p,q,0}$ is a space of distributions, see [47], it follows $h = 0$ contrary to the hypothesis. We are left with the case $\theta = 0$, that is the eigenvalue 1. Since $1 \circ f^{-1} = 1$, one is an eigenvalue, we want to prove that it is simple. Let $f_* h = h$, and let h_0 as before,

$$\left| \int_M \varphi h \omega_0 \right| \leqslant \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_M |\varphi| \cdot |h_0| \circ f^{-k} \omega_0 \leqslant \|\varphi\|_{L^1(\omega_0)} \|h_0\|_{\mathcal{C}^0}.$$

Thus $h \in L^\infty(\omega_0)$ (the dual of $L^1(\omega_0)$), and $h = h \circ f^{-1}$, ω_0-almost surely. On the other hand, since ω_0 is ergodic for the Anosov map, it follows that h is almost surely constant. Thus 1 is simple point spectrum for f_* acting on $\mathcal{B}^{p,q,0}$. $\qquad\square$

5.5.2. The peripheral spectrum of f_* acting on $\mathcal{B}^{p,q,1}$.
Here, we start a more in depth study of $\sigma_{\mathcal{B}^{p,q,1}}(f_*)$.

Lemma 5.18.　*The spectrum of f_* on $\mathcal{B}^{p,q,1}$ contains $e^{h_{top}}$, which is also the spectral radius. In addition, the essential spectral radius is bounded by $e^{h_{top}} \lambda^{-\min\{p,q\}}$. The eigenvector associated to $e^{h_{top}}$ is the Margulis measure and, together with the dual eigenvector defines the measure of maximal entropy.*

Proof.　The statement on the spectral radius and essential spectrum follows from lemma 5.16 and Hennion's theorem [51] (see also [28, appendix B]). Next, if ν is an eigenvalue of f_*, $|\nu| = e^{h_{top}}$, then, by lemma 5.16, $\nu^{-n} f_*$ is power bounded, hence it cannot be associated to a Jordan block. Let $h \in \mathcal{B}^{p,q,1}$ be an eigenvalue, then, by lemma 5.16 again, $\|h\|_{p,q,1} \leqslant C_\sharp \|h\|_{0,p+q,1}$.

Next, let $\bar{v}^u(x) \in \mathcal{C}(x)$ be a smooth normalised vector field. Then, for each $\omega \in \mathcal{C}^{p+q}$, $W \in \Sigma$ and $n \in \mathbb{N}$ let $v = \pi(\omega)$ and $v = w + v^u$ where v belongs to the tangent space of W and $v^u(x) = \alpha(f^{-n}(x)) \mathrm{d} f^n_{f^{-n}(x)} \bar{v}^u(f^{-n}(x))$. Note that $\|\alpha\|_{\mathcal{C}^{p+q}(f^{-n}(W))} \leqslant \lambda^{-n} C_\sharp \|\omega\|_{\mathcal{C}^{p+q}}$. Thus,

$$\left| \int_W \langle \omega, h \rangle \right| = \left| \int_W \langle \omega, \nu^{-n} f_*^n h \rangle \right| = \left| \nu^{-n} \int_W f_*^n h(v) \right|$$

$$\leqslant \left| \int_W \nu^{-n} h_{f^{-n}(x)}(\mathrm{d}_x f^{-n} w(x)) \mathrm{d}x \right| + \left| \int_W \nu^{-n} \alpha(f^{-n}(x)) h_{f^{-n}(x)} \left(\bar{v}^u(f^{-n}(x)) \mathrm{d}x \right| \right.$$

$$\leqslant C_\sharp \sum_i \int_{W_i} \vartheta_i(x) \nu^{-n} h_x(\bar{w}_i(x)) \mathrm{d}x + C_\sharp \lambda^{-n} \|h\|_{0,p+q,1} \|v\|_{\mathcal{C}^{p+q}}$$

where $\bar{w}_i$ belongs to the tangent space of W_i and $\|\bar{w}_i\|_{\mathcal{C}^{p+q}} \leqslant \|w\|_{\mathcal{C}^0} + \lambda^{-n} \|w\|_{\mathcal{C}^{p+q}}$. Thus,

$$\left| \int_W \langle \omega, h \rangle \right| \leqslant C_\sharp |f^{-n} W| \delta^{-1} e^{-h_{top} n} \|h\|_{0,p+q,1} \|w\|_{\mathcal{C}^0} + C_\sharp \lambda^{-n} \|h\|_{0,p+q,1} \|v\|_{\mathcal{C}^{p+q}}$$

$$\leqslant C_\sharp \delta^{-1} \|h\|_{0,p+q,1} \|\omega\|_{\mathcal{C}^0} + C_\sharp \lambda^{-n} \|h\|_{0,p+q,1} \|\omega\|_{\mathcal{C}^{p+q}}$$

where, in the last line, we have used the estimate on the growth of invariant manifolds, see [47, appendix C] for details. Taking the limit $n \to \infty$ and the sup in W and ω yields

$$\|h\|_{0,0,1} \leqslant C_\sharp \|h\|_{p,q,1}.$$

Next, let v^s be the normalised stable direction. Then, setting $\bar{\phi}(x) = \ln |\mathrm{d}_{f(x)} f^{-1} v^s(f(x))|$,

$$\mathrm{d}_x f^{-n} v^s(x) = e^{\sum_{k=0}^{n-1} \bar{\phi} \circ f^{-n+k}(x)} v^s(f^{-n}(x)).$$

We can then define the transfer operator

$$\mathcal{L}_* g(x) = g \circ f^{-1}(x) e^{\bar{\phi}(x)}. \tag{5.12}$$

Defining the map $\Gamma : \mathcal{B}^{0,0,1} \to \mathcal{M}$, the space of signed measures, by $\Gamma(h) = h(v^s)\omega_0$, we have

$$\int_M f_*(h)(gv^s) = \int_M g\mathcal{L}_*\Gamma(h).$$

In [46] is proven that $\mathcal{L}_*$ is has maximal eigenvalue $e^{-h_{top}}$ and the associated eigenvector is the Margulis measure. This concludes the lemma. $\qquad\square$

5.5.3. Deeper in the spectrum of f_* acting on $\mathcal{B}^{p,q,1}$. By lemma 5.15 we can extend the de Rham cohomology to the currents in the spaces $\mathcal{B}^{p,q,\ell}$. In other words we can call *closed* the elements $\omega \in \mathcal{B}^{p,q,\ell}$ such that $d\omega = 0$ and *exact* the ones for which it exists $\alpha \in \mathcal{B}^{p+1,q-1,\ell-1}$ such that $\omega = d\alpha$.

Remark 5.19. *By equation* (5.5) *and lemma* 5.15 *it follows that f_* sends closed currents into closed currents and exact currents into exact currents. Hence f_* induces an action in cohomology (of the $\mathcal{B}^{p,q,\ell}$ currents), let us call it $f_\sharp$.*

The next result shows that such a cohomology (let us call it *anisotropic cohomology*) is relevant to our problem.

Lemma 5.20. *If $\nu \in \sigma_{\mathcal{B}^{p,q,1}}(f_*)$, and $\omega \in \mathcal{B}^{p,q,1}$ are such that $f_*\omega = \nu\omega$ and $|\nu| > e^{h_{top}}\lambda^{-\min\{p,q\}}$, then either ω is not exact or $\nu \in \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)\setminus\{1\}$. Moreover, $\sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)\setminus\{1\} \subset \sigma_{\mathcal{B}^{p,q,1}}(f_*)$. If $\nu \in (\sigma_{\mathcal{B}^{p,q,1}}(f_*)\setminus\sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L}))\cup\{1\}$, then for each $\omega \in \mathcal{B}^{p,q,1}$ such that $f_*\omega = \nu\omega$ we have $d\omega = 0$.*

Proof. To start with note that, by lemma 5.18, ν must belong to the point spectrum.

Let $\nu \in (\sigma_{\mathcal{B}^{p,q,1}}(f_*)\setminus\sigma_{\mathcal{B}^{p,q,0}}(f_*))\cup\{1\}$ and $\omega \in \mathcal{B}^{p,q,1}$ such that $f_*\omega = \nu\omega$ and suppose that ω is exact. Thus, there exists $h \in \mathcal{B}^{q-1,p+1,0}$ such that $dh = \omega$. This implies

$$\nu\, dh = f_*\, dh = d f_* h.$$

That is $d(f_* h - \nu h) = 0$. It follows by lemma 5.15 that $f_* h = \nu h + c$. By a change of variable it follows that the dual $(f_*)'$ of f_* is given by the transfer operator $\mathcal{L}_{f^{-1}}$ associated to the map f^{-1}. Since f^{-1} is Anosov as well lemmata 5.16 and 5.17 apply and the measure μ^-_{SRB} associated to f^{-1} belongs to the dual of $\mathcal{B}^{p,q,0}$. Since $f_*1 = 1$ and the space $\mathbb{V} = \{h : \int_M h\, d\mu^-_{\text{SRB}} = 0\}$ is invariant for f_*, it is natural to write $h = \alpha + g$ with $\alpha \in \mathbb{C}$ and $g \in \mathbb{V}$. Then, we have

$$c + \nu\alpha + \nu g = \alpha + f_* g.$$

Applying μ^-_{SRB} to the above implies $c = \alpha(1 - \nu)$, hence $\nu g = f_* g$. The only possibility is then $\nu = 1$ but the associated eigenvector would be $1 \notin \mathbb{V}$, it follows $g = 0$. But then $\omega = dh = d\alpha = 0$. Hence, ω cannot be exact. The inclusion of the spectra is obvious.

If $f_*\omega = \nu\omega$ and $d\omega = h\omega_0$, by lemma 5.14 we have $\mathcal{L}h = \nu h$. Accordingly, either $\nu \in \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L})$ or $h = 0$, that is $d\omega = 0$. On the other hand, if $\nu = 1$, then $h\omega_0 = \mu_{\text{SRB}}$. Hence,

$d\omega = \mu_{\mathrm{SRB}}$ and

$$\int_M \mu_{\mathrm{SRB}} = \int_M d\omega = 0$$

which is impossible since μ_{SRB} is a positive measure. Accordingly, it must be $d\omega = 0$, that is, again, the form is closed. $\qquad\square$

To conclude we need a theory of anisotropic de Rham cohomology, such a general theory goes a beyond our present scopes so we will develop only the minimal version needed here. This is contained in section 5.7, and in particular in lemma 5.26 which states that the anisotropic cohomology of one forms is isomorphic to standard de Rham cohomology. In particular, this implies that the vector space of the equivalence classes is finite dimensional, hence $f_\sharp$, defined in remark 5.19, has only point spectrum, let us call Ω the spectrum of $f_\sharp$ when acting on one forms.

Next, we want to identify Ω. As stated in remark 5.9 this is the only place where we use that our map is topologically conjugated to the linear model.

Lemma 5.21. *We have* $\Omega = \{e^{-h_{top}}, e^{h_{top}}\}$.

Proof. Lemma 5.26 implies that the anisotropic de Rham cohomology for one forms is a topological invariant, hence so is $f_\sharp$. Since our map is conjugated to a linear model (see [53, section 2.6]), $f_\sharp$ is conjugated to the action of the linear model on homology. The lemma follows by a direct computation, see [53, section 3.2-e] for details. $\qquad\square$

The following lemma concludes the proof of theorem 5.12.

Lemma 5.22. *For each $\varepsilon > 0$, if p, q are large enough, we have*

$$\left[\Omega \cup \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \setminus \left(\{1\} \cup \{z \in \mathbb{C} \,:\, |z| < \varepsilon\}\right)\right] \subset \sigma_{\mathcal{B}^{p,q,1}}(f_*)$$

$$\sigma_{\mathcal{B}^{p,q,1}}(f_*) \subset \left[\{z \in \mathbb{C} \,:\, |z| < \varepsilon\} \cup \Omega \cup \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \cup \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L})\right] \setminus \{1\}.$$

Proof. Lemma 5.18 implies that if p, q are large enough we have to worry only about point spectrum.

Thus, if $\nu \in \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)$ then there exists $\theta \in \mathcal{B}^{p+1,q-1,0}$ such that $f_*\theta = \nu\theta$. This implies that $f_* d\theta = \nu\, d\theta$ so either $d\theta = 0$, but then by lemma 5.15 we have h constant and $\nu = 1$, or $\nu \in \sigma_{\mathcal{B}^{p,q,1}}(f_*)$.

If $\nu \in \Omega$, the spectrum of $f_\sharp$ (defined in remark 5.19), then it means that there exists $\omega \in \mathcal{B}_0^{p,q,1}$ and $\psi \in \mathcal{B}^{p+1,q-1,0}$ such that $f_*\omega = \nu\omega + d\psi$, that is $f_\sharp[\omega] = \nu[\omega]$, where $[\omega] \neq 0$ is the equivalence class of ω. If $\nu \notin \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)$, we can define $\theta = (\nu - f_*)^{-1}\psi$ and

$$(\nu - f_*)d\theta = d\psi.$$

But then $f_*(\omega + d\theta) = \nu(\omega + d\theta)$ which implies $\nu \in \sigma_{\mathcal{B}_0^{p,q,1}}(f_*)$ unless $\omega + d\theta = 0$. But the latter possibility would imply that ω is exact, that is $[\omega] = 0$, contrary to the assumption. This proves the first inclusion of the lemma.

To prove the second inclusion note that if $f_*\omega = \nu\omega$, $\omega \in \mathcal{B}^{p,q,1}$ and $\nu \notin \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L}) \setminus \{1\}$, then the last part of the lemma 5.20 implies $d\omega = 0$. Then $f_\sharp[\omega] = \nu[\omega]$, thus either $\nu \in \Omega$ or

$[\omega] = 0$, i.e. ω is exact. But if $\nu \notin \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)\setminus\{1\}$ the first part of lemma 5.20 implies that ω is not exact, hence $[\omega] \neq 0$. Since lemma 5.21 implies $1 \notin \Omega$, the Lemma follows.   □

Remark 5.23. It is conceivable that lemma 5.22 could be upgraded to an equality. Indeed, suppose that for a two current $\int_M \omega = 0$ implies that there exist a one current θ such that $\omega = d\theta$.[32] Then if $f_*\omega = \nu\omega$, $\nu \neq 1$ and $\omega \not\equiv 0$, we have $\int_M \omega = 0$ thus we can write $\omega = d\theta$ and $d(\nu\theta - f_*\theta) = 0$. Thus $\nu\theta - f_*\theta = \psi$ with d66666$\psi = 0$. Hence, if $\nu \notin \sigma_{\mathcal{B}^{p,q,1}}(f_*)$, we have $\theta = (\nu - f_*)^{-1}\psi$. Since $d(z - f_*)^{-1}\psi$ is a meromorphic function and for large z the von Neumann expansion implies the it is zero, we have $d\theta = 0$, a contradiction. Hence the second inclusion of the lemma 5.22 would be an equality.

5.6. Application to the measure of maximal entropy

In this section we prove theorem 5.10.

Lemma 5.18 implies that there exists $\ell_\star \in (\mathcal{B}^{p,q,1})'$ and $h_\star \in \mathcal{B}^{p,q,1}$ such that $f_* h_\star = e^{h_{\text{top}}} h_\star$ and $\ell_\star(f_*\omega) = e^{h_{\text{top}}}\ell_\star(\omega)$, for all $\omega \in \mathcal{B}^{p,q,1}$. In addition, $\ell_\star(\varphi h_\star) = \mu_{\text{BM}}(\varphi)$. Lemmata 5.22 and 5.17 imply that the rest of the spectrum is contained in $\{z \in \mathbb{C} \; : \; |z| < \kappa\}$ for some $\kappa \in (0, 1)$. It follows that the spectral decomposition $f_* = e^{h_{\text{top}}} h_\star \otimes \ell_\star + \mathcal{Q}$ with $\ell_\star\mathcal{Q} = 0$, $\mathcal{Q}h_\star = 0$, $\ell(h) = 1$ and $\|\mathcal{Q}^n\|_{p,1} \leqslant C_\sharp \kappa^n$. Also note that the multiplication by a smooth function is a bounded operator. Thus

$$\int_M g \circ f^n h \, d\mu_{\text{BM}} = \ell_\star(g \circ f^n h h_\star) = e^{-n h_{\text{top}}}\ell_\star(f_*^n(g \circ f^n h h_\star)) = e^{-n h_{\text{top}}}\ell_\star(g f_*^n(h h_\star))$$

$$= \ell_\star(g h_\star)\ell_*(h h_\star) + e^{-n h_{\text{top}}}\ell_\star(g \mathcal{Q}^n(h h_\star)).$$

It follows that, for r large enough,

$$\left| \int_M g \circ f^n h \, d\mu_{\text{BM}} - \int_M g \, d\mu_{\text{BM}} \int_M h \, d\mu_{\text{BM}} \right| \leqslant C_\sharp \|g\|_{\mathcal{C}^r} \|h\|_{\mathcal{C}^r} e^{-n h_{\text{top}}} \kappa^n.$$

5.7. Anisotropic de Rham cohomology

While to develop a theory of anisotropic de Rham cohomology as well as the relative Hodge theory may certainly be of interest, in this section we will develop only the bare minimum necessary to our needs and we will keep the arguments as elementary as possible.

Without loss of generality we can, and will, assume that there exist *good covers* $\{U_\alpha^+\}$, and $\{U_\alpha\}$ such that $U_\alpha^+ \supset \overline{U}_\alpha$, and a partition of unity $\{\psi_\alpha\}$ subordinated to $\{U_\alpha\}$. Also let $\{\psi_\alpha^+\}$ be such that $\text{supp}(\psi_\alpha^+) \subset U_\alpha^+$ and $\psi_\alpha^+|_{U_\alpha} = 1$.[33]

Lemma 5.24. *If $h \in \mathcal{B}^{p,q,1}$ is closed then, for each α, there exists $H_\alpha, H_\alpha^+ \in \mathcal{B}^{p+1,q-1,0}$ such that $dH_\alpha = h\psi_\alpha + H_\alpha^+ d\psi_\alpha$ and $H_\alpha = H_\alpha^+ \psi_\alpha$.*
Proof. For each U_α let us choose $x_\alpha \in \Theta_\alpha(U_\alpha^+ \setminus \text{supp} \, \psi_\alpha^+)$. We start assuming that h is a smooth one form and we define, for all $x \in U_\alpha$,

$$\bar{H}_\alpha(x) = \int_0^1 \Theta_{\alpha*} h_{x_\alpha(1-t)+tx}(x - x_\alpha)dt. \tag{5.13}$$

---

[32] This is equivalent to studying the cohomology for two forms.
[33] Recall that a good cover is a cover such that, for each collection $\mathcal{A}$ of indexes, $\cap_{\alpha \in \mathcal{A}} U_\alpha$ is contractible.

Also, for simplicity of notation, we confuse h and $\Theta_{\alpha*}h =: \sum_i h_i \, dx_i$ and set $\gamma_{lin,x}(t) = x_\alpha(1 - t) + tx$. Then

$$
\begin{aligned}
\partial_{x_i} \bar{H}_\alpha(x) &= \sum_{k=1}^{2} \int_0^1 \left[t(\partial_{x_i} h_k) \circ \gamma_{\mathrm{lin},x}(t)(x - x_\alpha)_k + h_i \circ \gamma_{\mathrm{lin},x}(t) \right] \\
&= \sum_{k=1}^{2} \int_0^1 t(\partial_{x_k} h_i) \circ \gamma_{\mathrm{lin},x}(t)(x - x_\alpha)_k + h_i \circ \gamma_{\mathrm{lin},x}(t) \\
&\quad - \int_0^1 t \, \mathrm{d}h(x - x_\alpha, e_i) \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \left[th_i \circ \gamma_{\mathrm{lin},x}(t) \right] - \int_0^1 t \, \mathrm{d}_{\gamma_{\mathrm{lin},x}(t)} h(x - x_\alpha, e_i) \\
&= h_i(x) - \int_0^1 t \, \mathrm{d}_{\gamma_{\mathrm{lin},x}(t)} h(x - x_\alpha, e_i).
\end{aligned}
\tag{5.14}
$$

Thus, if h is a closed form, then we have $d\bar{H}_\alpha = h$.

Next, let $\gamma \in \Sigma_\alpha$ and $\varphi \in \mathcal{C}^q(\gamma)$, and set $H_\alpha = \psi_\alpha \bar{H}_\alpha$, $\varphi_\alpha = \varphi \psi_\alpha \circ \gamma$, then

$$
\begin{aligned}
\int_\gamma \varphi \cdot H_\alpha &= \sum_{i=1}^{2} \int_a^b \mathrm{d}s \int_0^1 \mathrm{d}t \, \varphi_\alpha(s) \langle dx_i, \Theta_{\alpha,*}h \rangle (x_\alpha(1 - t) + t\gamma(s)) \\
&\quad \cdot (\gamma(s) - x_\alpha)_i \\
&= \sum_{i=1}^{2} \int_0^1 \mathrm{d}t \, t^{-1} \int_{ta}^{tb} \mathrm{d}s \, \varphi_\alpha(t^{-1}s) \langle dx_i, \Theta_{\alpha,*}h \rangle (x_\alpha(1 - t) + t\gamma(t^{-1}s)) \\
&\quad \cdot (\gamma(t^{-1}s) - x_\alpha)_i.
\end{aligned}
$$

If we define $\gamma_t(s) = x_\alpha(1 - t) + t\gamma(t^{-1}s)$, then $\gamma_t'(s) = \gamma'(t^{-1}s) \in \mathcal{C}^s$, and setting $\bar{\varphi}_{\alpha,t} = \sum_{i=1}^{2} \varphi_\alpha(t^{-1}s)(\gamma(t^{-1}s) - x_\alpha)_i dx_i$, we have, for some $c_\alpha \in (0, 1)$,

$$
\int_\gamma \varphi \cdot H_\alpha = \int_{c_\alpha}^1 \mathrm{d}t \, t^{-1} \int_{\gamma_t} \langle \bar{\varphi}_{\alpha,t}, h \rangle.
\tag{5.15}
$$

Equation (5.15) implies that H_α is a continuous functional of h hence it can be extended to all $h \in \mathcal{B}^{0,q,1}$. By the same scheme we can define $H_\alpha^+ = \psi_\alpha^+ \bar{H}_\alpha$ when $h \in \mathcal{B}^{0,q,1}$. Next, setting $x_{t,s} := x_\alpha(1 - t) + t\gamma(s)$ and using (5.14), we have

$$
\int_\gamma \varphi \partial_{x_i} H_\alpha = \int_\gamma \varphi \psi_\alpha \langle dx_i, h \rangle + \int_0^1 \mathrm{d}t \int_{\gamma_t} \langle \varphi_{\alpha,t}, *dx_i \rangle * dh + \int_\gamma \langle \varphi \, dx_i, d\psi_\alpha \rangle H_\alpha^+.
\tag{5.16}
$$

Hence, if h is closed, $dH_\alpha = \psi_\alpha h + H_\alpha^+ \, d\psi_\alpha$. If $h \in \mathcal{B}^{1,q,1}$ is closed, then there exist smooth forms h_n that converge to h. Moreover, by lemmata 5.15 and 5.14 it follows that $dh_n \to 0$ in $\mathcal{B}^{0,q+1,2}$, hence equation (5.16) implies

$$
\|H_{\alpha,n} - H_{\alpha,m}\|_{1,q-1,0} \leqslant \|h_n - h_m\|_{0,q,1} + C_\sharp \|dh_n - dh_m\|_{0,q+1,2},
$$

thus $H_{\alpha,n}$ is a Cauchy sequence in $\mathcal{B}^{1,q-1,1}$. Analogously, one can prove that $H_{\alpha,n}^+$ is Cauchy and, calling H_α, H_α^+ the limits, we have $dH_\alpha = \psi_\alpha h + H_\alpha^+ \, d\psi_\alpha$.

Similar arguments show that if $h \in \mathcal{B}^{p,q,1}$ and closed then $H_\alpha, H_\alpha^+ \in \mathcal{B}^{p+1,q-1,0}$ and $\mathrm{d}H_\alpha = \psi_\alpha h + H_\alpha^+ \, \mathrm{d}\psi_\alpha$. $\qquad\square$

Lemma 5.25. *There exist constants $c_{\alpha,\beta} \in \mathbb{C}$ such that, for all α, β,*

$$\psi_\alpha \psi_\beta [H_\alpha^+ - H_\beta^+ + c_{\alpha,\beta}] = 0.$$

Proof. By lemma 5.24 follows

$$\begin{aligned}
\mathrm{d}([H_\alpha^+ - H_\beta^+]\psi_\alpha\psi_\beta) &= \mathrm{d}(H_\alpha\psi_\beta - H_\beta\psi_\alpha) \\
&= H_\alpha^+ \psi_\beta \, \mathrm{d}\psi_\alpha + H_\alpha \, \mathrm{d}\psi_\beta - H_\beta^+ \psi_\alpha \, \mathrm{d}\psi_\beta + H_\beta \, \mathrm{d}\psi_\alpha \\
&= [H_\alpha^+ - H_\beta^+]\mathrm{d}(\psi_\alpha\psi_\beta).
\end{aligned}$$

This implies $\psi_\alpha\psi_\beta \, \mathrm{d}[H_\alpha^+ - H_\beta^+] = 0$ and the lemma follows thanks to the last assertion of lemma 5.15. $\qquad\square$

This fact allows to obtain our basic result.

Lemma 5.26. *The anisotropic de Rham cohomology for one forms is isomorphic to the standard de Rham cohomology.*

Proof. The first task is to understand when $h \in \mathcal{B}^{p,q,1}$ is exact. Let $\bar{c} = (c_\alpha) \in \mathbb{C}^N$, where $N = \sharp\{U_\alpha\}$, and define $H(\bar{c}) = \sum_\alpha (H_\alpha^+ + c_\alpha)\psi_\alpha$. If h is exact, then there exists $\theta \in \mathcal{B}^{p+1,q-1,0}$ such that $\mathrm{d}\theta = h$ but then[34]

$$\psi_\alpha \, \mathrm{d}(\theta - H_\alpha^+) = 0.$$

Then lemma 5.15 implies that there exists c_α such that $\psi_\alpha(\theta - H_\alpha^+ - c_\alpha) = 0$, hence for such a collection of constants $\bar{c} = \{c_\alpha\}$ we have $\theta = H(\bar{c})$. It follows h is exact if and only if it is possible to choose $\bar{c}$ so that $\mathrm{d}H(\bar{c}) = h$.

To start with we have thus to compute

$$\mathrm{d}H(\bar{c}) = \sum_\alpha \psi_\alpha h + \sum_\alpha (H_\alpha^+ + c_\alpha)\mathrm{d}\psi_\alpha = h + \sum_{\alpha,\beta}(H_\alpha^+ + c_\alpha)\psi_\beta \, \mathrm{d}\psi_\alpha. \qquad (5.17)$$

Accordingly, if

$$(H_\alpha^+ + c_\alpha - H_\beta^+ - c_\beta)\psi_\beta \, \mathrm{d}\psi_\alpha = 0, \qquad (5.18)$$

then,

$$\begin{aligned}
\sum_{\alpha,\beta}(H_\alpha^+ + c_\alpha)\psi_\beta \, \mathrm{d}\psi_\alpha &= \sum_{\alpha,\beta}(H_\beta^+ + c_\beta)\psi_\beta \, \mathrm{d}\psi_\alpha \\
&= \sum_\beta (H_\beta^+ + c_\beta)\psi_\beta \, \mathrm{d}\left(\sum_\alpha \psi_\alpha\right) = 0,
\end{aligned}$$

---

[34] Note that lemma 5.24 implies that $\psi_\alpha \, \mathrm{d}H_\alpha^+ = h\psi_\alpha$.

and, recalling equation (5.17), $dH(\bar{c}) = h$. To conclude note that the problem is now reduced to the study of the Čech cohomology $\check{H}^1(\mathcal{U}, \mathbb{C})$ where $\mathcal{U} = \{U_\alpha\}$. Indeed, a one-cochain f is a one-cocycle iff for each two-simplex $(U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2})$ holds:[35]

$$f(U_{\alpha_1}, U_{\alpha_2}) - f(U_{\alpha_0}, U_{\alpha_2}) + f(U_{\alpha_0}, U_{\alpha_1}) = 0 \tag{5.19}$$

while it is a coboundary if there exists a 0-cochain g such that for all one-simplex $(U_{\alpha_0}, U_{\alpha_1})$ holds

$$f(U_0, U_1) = g(U_0) - g(U_1). \tag{5.20}$$

Accordingly, we can interpret the constants $\bar{c} = \{c_\alpha\}$ as 0-cochain and the constants $\bar{C} = \{c_{\alpha,\beta}\}$, in lemma 5.25, as a one-cochain. Then lemma 5.25 implies that $\bar{C}$ must be a one-cocycle. To see it, given any two-simplex $\{U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2}\}$ consider any smooth function φ such that its support is strictly contained in $U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}$, then, by lemma 5.25 and the definition of $\{\psi_\alpha\}$,

$$0 = \int_M \varphi \left[H_{\alpha_1}^+ - H_{\alpha_2}^+ + c_{\alpha_1,\alpha_2} - H_{\alpha_0}^+ + H_{\alpha_2}^+ - c_{\alpha_0,\alpha_2} + H_{\alpha_0}^+ - H_{\alpha_1}^+ + c_{\alpha_0,\alpha_1} \right]$$

$$= \int_M \varphi \left[c_{\alpha_1,\alpha_2} - c_{\alpha_0,\alpha_2} + c_{\alpha_0,\alpha_1} \right]$$

which implies $c_{\alpha_1,\alpha_2} - c_{\alpha_0,\alpha_2} + c_{\alpha_0,\alpha_1} = 0$ by the arbitrariness of φ.

On the other hand equation (5.18) is satisfied iff $\bar{C}$ is a one-coboundary. To see this, let $\{U_{\alpha_0}, U_{\alpha_1}\}$ be a one-simplex. We can assume w.l.o.g. that $\psi_{\alpha_0} d\psi_{\alpha_1} \neq 0$ otherwise ψ_{α_1} would be constant different from zero and one on $\mathrm{supp}(\psi_{\alpha_0})$. But then for each sufficiently small θ such that $\mathrm{supp}(\theta) \subset \mathrm{supp}(\psi_{\alpha_0})$ the set $\{\tilde{\psi}_\alpha\} := \{\psi_\alpha\}_{\alpha \notin \{\alpha_0, \alpha_1\}} \cup \{\psi_{\alpha_1} - \theta, \psi_{\alpha_0} + \theta\}$ would still be a partition of unity subordinated to $\mathcal{U}$ and one can choose θ such that $\tilde{\psi}_{\alpha_0} d\tilde{\psi}_{\alpha_1} \neq 0$. We can then find an open set $U \subset U_{\alpha_0} \cap U_{\alpha_1}$ such that $\psi_{\alpha_0} d\psi_{\alpha_1} \neq 0$ in U. Then, using equation (5.18) multiplied by $\varphi(\psi_{\alpha_0} d\psi_{\alpha_1})^{-1}$ and the statement of lemma 5.25 multiplies by $\varphi(\psi_{\alpha_0} \psi_{\alpha_1})^{-1}$, for each φ supported in U we have

$$0 = \int_M \varphi \left[H_{\alpha_0}^+ + c_{\alpha_0} - H_{\alpha_1}^+ - c_{\alpha_1} - H_{\alpha_0}^+ + H_{\alpha_1}^+ - c_{\alpha_0,\alpha_1} \right] = \int_M \varphi \left[c_{\alpha_0} - c_{\alpha_1} - c_{\alpha_0,\alpha_1} \right]$$

which, by the arbitrariness of φ implies $c_{\alpha_0,\alpha_1} = c_{\alpha_0} - c_{\alpha_1}$.

The above discussion implies that h is exact if and only if $\bar{C}$ is a one-coboundary. This implies the $\mathcal{B}^{p,q,1}$ cohomology is isomorphic to the Čech cohomology, which is isomorphic to the de Rham cohomology. $\qquad\square$

5.8. Conclusion and comparisons

While the results for the simple case studied in section 2 are fully satisfactory, the results in sections 3–5 are still partial. Indeed, we show that the preset approach yields rather sharp results for the operator associated to the measure of maximal entropy, but less information is obtained, e.g. for the operator associated to the SRB measure. It is possible that considering the commutation of different operators with the transfer operator more information can be obtained, but this requires further work.

---

[35] Recall that $\{U_{\alpha_0}, \ldots, U_{\alpha_q}\}$ is a q-simplex if $\cap_{i=0}^q U_{\alpha_i} \neq \emptyset$ while a q-cochain is a function from the q-simplex to $\mathbb{C}$.

Also, in sections 3 and 4 we consider only one dimensional maps, yet the present approach seems amenable to extension to the higher dimensional setting. In particular, the arguments of section 5 should allow to considerably improve [24], at least for small potentials.

In the case of two dimensional hyperbolic maps, presented in section 5, our approach reproduces in a unified manned all the known results. Theorems 5.10 and 5.12 are a refinement of [8, corollary 2.5], which contains slightly stronger results than [41]. In addition, for the application to toral parabolic flows, we can obtain the exact equivalent of [8, corollary 2.3] which is sharper than the corresponding results in [41]. Indeed, if h_t is the unit speed flow along the stable manifold of an Anosov map f then our results yield (see [48] for details)

$$\left| \int_0^T g \circ h_t(x)\mathrm{d}t - T\mu_{\mathrm{top}}(g) \right| \leqslant C_\sharp \|g\|_\infty$$

which implies that the ergodic average either grows linearly, or g is a cocycle. (See also [22] for a very recent and short proof of a logarithmic bound in a more general setting.)

We have thus seen that the present approach both reproduces the results in [8], and enlightens the connection with the action in cohomology (already present, in some form, in [41, 42]).

In conclusion, the present strategy unifies and refines the existing results in all the cases we have presented. In addition, it appears amenable to further generalisation. In particular, it seems possible to extend it to the higher dimensional case.

Another promising direction would be to apply it to Anosov flows where some hints of the relevance of some type of cohomology already exists (e.g. see [79]). Along the same lines, it is reasonable that our ideas can yield relevant results if applied to pseudo-Anosov and partially hyperbolic maps.

Appendix A. Proof of theorem 3.9

Before proving theorem 3.9 we need a few preliminary lemmata.

In this case it is convenient to define $\psi(g)(x) = \int_0^x g(y)\mathrm{d}y$ and

$$\mathcal{L}_+ g = \mathcal{L}_2 g + \mathcal{L}_1(D_f \cdot \psi(g)).$$

Note that $\mathcal{L}_+$ is a positive operator: if $g \geqslant 0$, then

$$\mathcal{L}_+ g \geqslant \mathcal{L}_1(D_f \cdot \psi(g)) \geqslant 0 \tag{A.1}$$

(here we use the assumption that $D_f \geqslant 0$). This facilitates the study of its spectrum. There is an obvious connection with the operator we are interested in:

$$\mathcal{L}_\star g = \mathcal{L}_+ g - (\mathcal{L}_1 D_f) \cdot \int_0^1 (1-y)g(y)\mathrm{d}y, \tag{A.2}$$

that is $\mathcal{L}_\star$ is a rank one perturbation of $\mathcal{L}_+$.

Before proceeding further we need some information on $\mathcal{L}_+$. Recall that, as defined in the theorem, $\mu_* = \frac{1}{f'(1)}$.

Lemma A.1. *The spectral radius of $\mathcal{L}_+$, acting on L^1, is μ_*. Moreover, μ_* is an eigenvalue of $\mathcal{L}'_+$ (the dual operator to $\mathcal{L}_+$), acting on L^∞ with eigenvector given by the constant function one.*

Proof. Note that, for all $g \in L^1$,

$$
\int_0^1 1 \cdot \mathcal{L}_+ g(y) \mathrm{d}y = \int_0^1 \left[\frac{g(y)}{f'(y)} + \left(\frac{1}{f'(y)} \right)' \psi(g)(y) \right] \mathrm{d}y = \int_0^1 \left(\frac{\psi(g)}{f'} \right)' (y) \, \mathrm{d}y
$$

$$
= \frac{1}{f'(1)} \psi(g)(1) - \frac{1}{f'(0)} \psi(g)(0) = \frac{1}{f'(1)} \int_0^1 1 \cdot g(y) \mathrm{d}y. \tag{A.3}
$$

Hence, $\frac{1}{f'(1)}$ is an eigenvalue of the dual of $\mathcal{L}_+$ and hence it belongs to the spectrum of $\mathcal{L}_+$. The lemma follows since

$$
\int_0^1 |\mathcal{L}_+ g(y)| \, \mathrm{d}y \leqslant \int_0^1 \mathcal{L}_+ |g|(y) \mathrm{d}y = \frac{1}{f'(1)} \int_0^1 |g(y)| \mathrm{d}y. \tag{A.4}
$$

$\square$

Note that the above lemma implies that the space $\mathbb{V}_0 = \{ h \in L^1 \mid \int_0^1 h = 0 \}$ is invariant under $\mathcal{L}_+$. However, this does not give much information on the spectrum. To learn more it is convenient to study the operator $\mathcal{L}_+$ acting on $W^{1,1}$.

Lemma A.2. *For all $g \in W^{1,1}$ we have*

$$
\| \mathcal{L}_+ g \|_{L^1} \leqslant \mu_* \| g \|_{L^1}
$$

$$
\| \mathcal{L}_+ g \|_{W^{1,1}} \leqslant \mu_*^2 \| g \|_{W^{1,1}} + (3 \| D_f \|_\infty + \| D_f' \|_{L^1} + \| D_f^2 \|_{L^1} + \mu_*) \| g \|_{L^1}.
$$

Proof. The first inequality follows from (A.4). Next, for each $g \in W^{1,1}$, using again (3.2), we have

$$
(\mathcal{L}_+ g)' = \mathcal{L}_3 g' + 3 \mathcal{L}_2 D_f g + \mathcal{L}_2 D_f' \psi(g) + \mathcal{L}_1 D_f^2 \psi(g). \tag{A.5}
$$

Thus (note that $D_f \geqslant 0$ implies that $f'' \leqslant 0$ and so $f'(0) \geqslant f(x) \geqslant f'(1)$), using also the estimates of lemma 3.5,

$$
\| (\mathcal{L}_+ g)' \|_{L^1} \leqslant \mu_*^2 \| g' \|_{L^1} + (3 \| D_f \|_\infty + \| D_f' \|_{L^1} + \| D_f^2 \|_{L^1}) \| g \|_{L^1}.
$$

The lemma follows using again (A.4). $\square$

Lemma A.3. μ_* *is a simple eigenvalue of $\mathcal{L}_+$. Moreover, h_+, the eigenvector associated to μ_*, is strictly positive, i.e. $h_+ > 0$. In addition, there exists $\mu_1 < \mu_*$ such that $\sigma_{W^{1,1}}(\mathcal{L}_+) \subset \{\mu_*\} \cup \{z \in \mathbb{C} : |z| \leqslant \mu_1\}$.*

Proof. Lemma A.2 and [51] (see also [28, appendix B]) imply that the essential spectrum of $\mathcal{L}_+$ is contained in a disk of size μ_*^2. Thus, since μ_* is an eigenvalue of $\mathcal{L}_+'$, μ_* must be an eigenvalue of $\mathcal{L}_+$. Moreover, lemma A.2 implies that $\{\mu_*^n \mathcal{L}_+^n\}$ is uniformly bounded when acting on $W^{1,1}$, hence by [32, lemma 8.8.1] μ_* is a semi-simple eigenvalue (no Jordan blocks).

Let $h_+ \in W^{1,1} \setminus \{0\} \subset \mathcal{C}^0$ be a corresponding eigenvector. Next, suppose that $\mathcal{L}_+ g = \mu_* \mathrm{e}^{\mathrm{i}\theta} g$, then $\mu_* |g| \leqslant \mathcal{L}_+ |g|$, but then

$$
\int_0^1 \mathcal{L}_+ |g| - \mu_* |g| = 0
$$

thus $\mu_*|g| = \mathcal{L}_+|g|$. Accordingly, we can assume that $h_+ \geqslant 0$. But then it must be $h_+ > 0$. Indeed, if there exists $\bar{x}$ such that $h_+(\bar{x}) = 0$, then, calling y the maximal element in $f^{-1}(\bar{x})$,

$$0 = \mu_* h_+(\bar{x}) \geqslant \frac{1}{f'(y)}\left(\frac{1}{f'}\right)'(y)\int_0^y h_+.$$

Hence $h_+(x) = 0$ for all $x \leqslant y$. Iterating the argument we have that $h_+(x) = 0$ for all $x < 1$, and, by continuity, $h_+ \equiv 0$, contrary to the assumption. Accordingly, if there exists another h such that $\mathcal{L}_+ h = \mu_* h$, then it cannot be zero anywhere otherwise $|h|$, which is also an eigenvector, would be identically zero. But then there exists $\alpha \in \mathbb{R}$ such that $\alpha h_+ - |h|$ has a zero and hence, since $\alpha h_+ - |h|$ also is an eigenvalue, $h = \alpha h_+$. Hence, μ_* is simple (the corresponding eigenspace has dimension one).

Therefore, if $e^{i\theta}\mu_* g = \mathcal{L}_+ g$ then there must exist $\vartheta \in \mathcal{C}^0$ such that $g = e^{i\vartheta}h_+$. It follows

$$0 = \mu_* h_+ - e^{-i\theta-i\vartheta}\mathcal{L}_+(e^{i\vartheta}h_+) = \mathcal{L}_+ h_+ - e^{-i\theta-i\vartheta}\mathcal{L}_+(e^{i\vartheta}h_+)$$

$$= \mathcal{L}_2\left[1 - e^{-i\theta-i\vartheta\circ f+i\vartheta}\right]h_+ + \mathcal{L}_1 D_f\left[\psi(g) - e^{-i\theta-i\vartheta\circ f}\psi(e^{i\vartheta}h_+)\right].$$

Taking the real part and integrating yields

$$0 = \int_0^1 \frac{1 - \cos[\theta + \vartheta \circ f - \vartheta]}{f'}h_+ + \int_0^1 \mathrm{d}x\, D_f(x)\int_0^x \mathrm{d}y\left[1 - \cos\left[\theta + \vartheta \circ f(x) - \vartheta(y)\right]h_+(y).$$

Since both terms are positive, the only possibility is $\theta + \vartheta \circ f(x) - \vartheta(y) = k\pi$. This implies that ϑ is constant and hence g is proportional to h_+, hence it must be $\theta = 0$. This proves that μ_* is the only peripheral eigenvalue and the spectral gap. $\qquad\square$

We can now conclude our argument.

Proof of theorem 3.9.   Equation (A.2) and lemma A.2 imply the bound on the essential spectral radius.

Since f' is continuous on $[0, 1]$ we know that

$$\int_0^1 |(\frac{1}{f'(y)})'|\mathrm{d}y = \frac{1}{f'(1)} - \frac{1}{f'(0)} =: \Delta.$$

This means that the first statement of the theorem follows from theorem 3.6 where

$$\tau = \lambda^{-1} + \int_0^1 \left|\left(\frac{1}{f'(y)}\right)'\right|\mathrm{d}y = \frac{2}{f'(1)} - \frac{1}{f'(0)} = \Delta + \mu_*.$$

It remains to show the absence of eigenvalues in the sets $A_0, \ldots, A_4$. By equation (A.2) we have that if $\mathcal{L}_* g = zg$ then

$$(z - \mathcal{L}_+)g = -\mathcal{L}_1 D_f \int_0^1 (1 - y)g(y)\mathrm{d}y. \tag{A.6}$$

Recall $\mu_1 < \mu_*$ from lemma A.3. Let $|z| > \mu_1$ and suppose, for sake of contradiction, that the right-hand side of the above equation is zero. This would mean, by lemma A.3, that $g = h_+$ and $z = \mu_*$. However this would then imply that the integral on the right was strictly positive and, since $D_f \not\equiv 0$, would contradict the assumption. This means that the right-hand side of the above equation cannot be zero. Moreover, $z \neq \mu_*$ since, if it were, $\int_0^1(\mu_* - \mathcal{L}_+)g = 0$ by

lemma A.1, and this would imply that the right-hand side of (A.6) is zero, contrary to what we have seen.

It follows that, possibly after a normalisation, for $|z| > \mu_1$ we can write $g = (z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f$, and substituting in (A.6) we have

$$(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f(x) = -(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f(x) \int_0^1 (1 - y)(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f(y)\mathrm{d}y.$$

Accordingly, if we define

$$\Xi(z) := 1 + \int_0^1 (1 - y)(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f(y)\mathrm{d}y \tag{A.7}$$

we have that z is an eigenvalue of $\mathcal{L}_\star$ if and only if $\Xi(z) = 0$. In the following we will repeatedly use the following facts: By lemma A.1, the spectral radius of $\mathcal{L}_+$ is μ_*, and, for each $\varphi \in L^1$, $\int_0^1 \mathcal{L}_+ \varphi = \mu_* \int_0^1 \varphi$, while, by definition, $\int_0^1 \mathcal{L}_1 \varphi = \int_0^1 \varphi$. Also, by (A.1), $\mathcal{L}_+$ is a positive operator and so is, obviously, $\mathcal{L}_1$. Moreover we set $\int_0^1 D_f = \Delta$.

Since by lemma A.1 we have $\|\mathcal{L}_+\|_{L^1} \leqslant \mu_*$, for $|z| \geqslant \mu_*$ we can do the trivial estimate

$$\left| \int_0^1 (1 - y)(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f(y)\mathrm{d}y \right| \leqslant \sum_{n=0}^\infty \int_0^1 (1 - y)|z|^{-n-1}\mathcal{L}_+^n \mathcal{L}_1 D_f(y)\mathrm{d}y$$

$$\leqslant \sum_{n=0}^\infty \int_0^1 |z|^{-n-1}\mathcal{L}_+^n \mathcal{L}_1 D_f(y)\mathrm{d}y = \frac{\Delta}{|z| - \mu_*}. \tag{A.8}$$

The above implies that $\Xi(z) \neq 0$ for all $|z| > \tau$, which we know already. Hence, to gain further informations we have to analyse (A.8) more in depth.

For $z > \mu_*$,

$$\int_0^1 (1 - y)(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f(y)\mathrm{d}y = \sum_{n=0}^\infty \int_0^1 (1 - y)z^{-n-1}\mathcal{L}_+^n \mathcal{L}_1 D_f(y)\mathrm{d}y > 0.$$

which implies $\Xi(z) > 1$, hence non zero. Moreover, by lemma A.3 we get the spectral representation $\mathcal{L}_+ h = \mu_* h_+ \int_0^1 h + Qh$ where for all $\mu \in (\mu_1, \mu_*)$ there exists C_μ such that $\|Q^n\|_{W^{1,1}} \leqslant C_\mu \mu^n$. Thus, for $z \in (\mu, \mu_*)$,

$$\int_0^1 (1 - y)(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f(y)\mathrm{d}y = \int_0^1 (1 - y)(z - \mu_*)^{-1}h_+(y)\mathrm{d}y \left(\int_0^1 \mathcal{L}_1 D_f \right)$$

$$+ \sum_{n=0}^\infty \int_0^1 (1 - y)z^{-n-1}Q^n \mathcal{L}_1 D_f(y)\mathrm{d}y.$$

Thus, for some $C > 0$,

$$\left| \Xi(z) - 1 - (z - \mu_*)^{-1}\left[\frac{1}{f'(1)} - \frac{1}{f'(0)} \right] \int_0^1 (1 - y)h_+ \right| \leqslant C_\mu C(|z| - \mu)^{-1}$$

Hence, there exists $\mu_2 \in (\mu_1, \mu_*)$ such that $\Xi(z) = 0$ has no solution for $z > \mu_2$.[36] This establishes the fact that the set A_0 does not belong to the spectrum.

To study non positive z, note that $\overline{\Xi(z)} = \Xi(\bar{z})$ and that $(z - \mathcal{L}_+)^{-1} - (\bar{z} - \mathcal{L}_+)^{-1} = (\bar{z} - z)(z - \mathcal{L}_+)^{-1}(\bar{z} - \mathcal{L}_+)^{-1}$. Hence, if $z = a + ib$, we have

$$\Im(\Xi(z)) = ib \int_0^1 (1-y)(\bar{z} - \mathcal{L}_+)^{-1}(z - \mathcal{L}_+)^{-1} \mathcal{L}_1 D_f(y) dy$$

$$= ib \int_0^1 (1-y)([a - \mathcal{L}_+]^2 + b^2)^{-1} \mathcal{L}_1 D_f(y) dy$$

$$\Re(\Xi(z)) = 1 + \int_0^1 (1-y)(a - \mathcal{L}_+)([a - \mathcal{L}_+]^2 + b^2)^{-1} \mathcal{L}_1 D_f(y) dy. \qquad (A.9)$$

Note that if $a > \mu_*$, then, by lemma A.1 and (A.1), $[a - \mathcal{L}_+]^{-1} = \sum_{n=0}^{\infty} a^{-n-1} \mathcal{L}_+^n$ is a positive operator. Hence, if $b \neq 0$, $\Xi(z) = 0$ implies

$$0 = 1 + \int_0^1 (1-y)(1 + [a - \mathcal{L}_+]^{-2} b^2)^{-1} [a - \mathcal{L}_+]^{-1} \mathcal{L}_1 D_f(y) dy$$

$$= 1 + \int_0^1 (1-y)[a - \mathcal{L}_+]^{-1} \mathcal{L}_1 D_f(y) dy$$

$$- \int_0^1 (1-y)(1 + [a - \mathcal{L}_+]^{-2} b^2)^{-1} [a - \mathcal{L}_+]^{-3} b^2 \mathcal{L}_1 D_f(y) dy. \qquad (A.10)$$

Next we want to estimate the two integrals on the right-hand side of (A.10). Let us start with the first integral, assuming $a > \mu_*$,[37]

$$\int_0^1 (1-y)[a - \mathcal{L}_+]^{-1} \mathcal{L}_1 D_f(y) dy \geqslant a^{-1} \int_0^1 (1-y) \mathcal{L}_1 D_f(y) dy$$

$$= a^{-1} \sum_{i=1}^N \int_{p_i}^{p_{i+1}} (1 - f(y)) \left(\frac{1}{f'(y)} \right)' dy$$

$$= a^{-1} \left(1 - \sum_{i=1}^N \frac{1}{f'(p_i)} \right) =: a^{-1} \Gamma.$$

To estimate the second integral note that, for each $g \geqslant 0$ and $|b| < a - \mu_*$, we can write

$$(1 + [a - \mathcal{L}_+]^{-2} b^2)^{-1} g = \sum_{n=0}^{\infty} (-1)^n [a - \mathcal{L}_+]^{-2n} b^{2n} g$$

$$\leqslant \sum_{n=0}^{\infty} [a - \mathcal{L}_+]^{-4n} b^{4n} g. \qquad (A.11)$$

---

[36] With some further work one could estimate μ_2, but we believe the above suffices to show how to proceed.
[37] Note that in this computation we are sacrificing optimality of the result to the simplicity of the formulae.

Hence

$$\int_0^1 (1-y)(1 + [a - \mathcal{L}_+]^{-2}b^2)^{-1}[a - \mathcal{L}_+]^{-3}b^2\mathcal{L}_1 D_f(y)dy$$

$$\leqslant \sum_{n=0}^\infty \int_0^1 [a - \mathcal{L}_+]^{-4n-3}b^{4n+2}\mathcal{L}_1 D_f(y)dy$$

$$= \sum_{n=0}^\infty [a - \mu_*]^{-4n-3}b^{4n+2}\Delta = \frac{(a - \mu_*)b^2\Delta}{(a - \mu_*)^4 - b^4}.$$

The above implies that $\Xi(z) = 0$ has no solution if $a > \mu_*$, $|b| < a - \mu_*$ and

$$1 + a^{-1}\Gamma > \frac{b^2(a - \mu_*)\Delta}{(a - \mu_*)^4 - b^4}$$

which is implied by $a > \mu_*$ and

$$b^2 < \frac{(a - \mu_*)\Delta}{2(1 + a^{-1}\Gamma)}\left[\sqrt{1 + 4\frac{(1 + a^{-1}\Gamma)^2(a - \mu_*)^2}{\Delta^2}} - 1\right]. \tag{A.12}$$

This establishes that A_1 is disjoint from the spectrum (apart from 1).

If $a < -\mu_*$, we cannot easily use positivity arguments since

$$\mathfrak{R}(\Xi(z)) = 1 - \int_0^1 (1-y)(|a| + \mathcal{L}_+)([|a| + \mathcal{L}_+]^2 + b^2)^{-1}\mathcal{L}_1 D_f(y)dy.$$

We are thus left with the cruder estimate, $|(|a| + \mathcal{L}_+)^{-1}g| \leqslant \sum_{n=0}^\infty |a|^{-n-1}\mathcal{L}_+^n|g| = (|a| - \mathcal{L}_+)^{-1}|g|$. Hence, for $|b| \leqslant |a| - \mu_*$ and recalling the computation for $b = 0$, we have

$$\int_0^1 (1-y)(|a| + \mathcal{L}_+)([|a| + \mathcal{L}_+]^2 + b^2)^{-1}\mathcal{L}_1 D_f(y)dy$$

$$\leqslant \int_0^1 (1-y)(|a| + \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f(y)dy + \sum_{n=1}^\infty \int_0^1 b^{2n}[|a| - \mathcal{L}_+]^{-2n-1}\mathcal{L}_1 D_f(y)dy.$$

The above decomposition allows to still use some positivity argument for the first term after the inequality. Indeed,

$$\int_0^1 (1-y)(|a| + \mathcal{L}_+)^{-1}\mathcal{L}_1 D_f(y)dy$$

$$= \sum_{n=0}^\infty |a|^{-1-n}(-1)^n \int_0^1 (1-y)\mathcal{L}_+^n\mathcal{L}_1 D_f(y)dy$$

$$\leqslant \sum_{n=0}^\infty |a|^{-1-2n} \int_0^1 \mathcal{L}_+^{2n}\mathcal{L}_1 D_f(y)dy$$

$$\leqslant \sum_{n=0}^\infty |a|^{-1-2n}\mu_*^{2n}\Delta = \frac{|a|\Delta}{a^2 - \mu_*^2}.$$

Accordingly,

$$\int_0^1 (1-y)(|a| + \mathcal{L}_+)([|a| + \mathcal{L}_+]^2 + b^2)^{-1} \mathcal{L}_1 D_f(y) dy$$

$$\leqslant \frac{|a|\Delta}{a^2 - \mu_*^2} + \frac{\Delta b^2}{(|a| - \mu_*)[(|a| - \mu_*)^2 - b^2]}.$$

which implies that $\Xi(x) = 0$ has no solutions if $a < -\mu_*$ and

$$b^2 < \frac{(|a| - \mu_*)^2(a^2 - \mu_*^2 - \Delta|a|)}{a^2 - \mu_*^2 + \Delta|a|}. \tag{A.13}$$

This establishes that A_2 is disjoint from the spectrum.

On the other hand, for $b \neq 0$, the equation (A.9) and $\Xi(z) = 0$ imply also

$$0 = 1 - \int_0^1 (1-y)\mathcal{L}_+([a - \mathcal{L}_+]^2 + b^2)^{-1} \mathcal{L}_1 D_f(y) dy. \tag{A.14}$$

If $|z|^2 > 2|a|\mu_*$, then

$$\int_0^1 (1-y)\mathcal{L}_+(|z|^2 - 2a\mathcal{L}_+ + \mathcal{L}_+^2)^{-1} \mathcal{L}_1 D_f(y) dy$$

$$= \int_0^1 (1-y)\mathcal{L}_+(1 + (|z|^2 - 2a\mathcal{L}_+)^{-1}\mathcal{L}_+^2)^{-1}(|z|^2 - 2a\mathcal{L}_+)^{-1} \mathcal{L}_1 D_f(y) dy.$$

Note that if $a \geqslant 0$, then $(|z|^2 - 2a\mathcal{L}_+)^{-1}$ is a positive operator, hence

$$\int_0^1 (1-y)\mathcal{L}_+([a - \mathcal{L}_+]^2 + b^2)^{-1} \mathcal{L}_1 D_f(y) dy$$

$$= \sum_{n=0}^\infty (-1)^n \int_0^1 (1-y)\mathcal{L}_+(|z|^2 - 2a\mathcal{L}_+)^{-n-1}\mathcal{L}_+^{2n}\mathcal{L}_1 D_f(y) dy$$

$$\leqslant \sum_{n=0}^\infty \int_0^1 \mathcal{L}_+(|z|^2 - 2a\mathcal{L}_+)^{-2n-1}\mathcal{L}_+^{4n}\mathcal{L}_1 D_f(y) dy$$

$$= \sum_{n=0}^\infty \mu_*(|z|^2 - 2a\mu_*)^{-2n-1}\mu_*^{4n}\Delta = \frac{\mu_*\Delta(|z|^2 - 2a\mu_*)}{(|z|^2 - 2a\mu_*)^2 - \mu_*^4}.$$

It follows that equation (A.14) has no solution if $a > 0$, $|z - \mu_*| > \mu_*$ and

$$\mu_*\Delta(|z - \mu_*|^2 - \mu_*^2) \leqslant (|z - \mu_*|^2 - \mu_*^2)^2 - \mu_*^4,$$

that is $a > 0$ and

$$|z - \mu_*|^2 > \mu_*^2 + \mu_* \frac{\Delta + \sqrt{4\mu_*^2 + \Delta^2}}{2}. \tag{A.15}$$

This establishes that A_3 is disjoint from the spectrum.

If $a < 0$, again it is harder to take advantage of the positivity, hence, for simplicity, we content ourselves with a cruder estimate

$$
\int_0^1 (1-y)\mathcal{L}_+(|z|^2 - 2a\mathcal{L}_+ + \mathcal{L}_+^2)^{-1}\mathcal{L}_1 D_f(y)\mathrm{d}y
$$

$$
= \sum_{n=0}^\infty |z|^{-2-2n} \int_0^1 (1-y)\mathcal{L}_+(2a\mathcal{L}_+ - \mathcal{L}_+^2)^n \mathcal{L}_1 D_f(y)\mathrm{d}y
$$

$$
\leqslant \sum_{n=0}^\infty |z|^{-2-2n} \int_0^1 \mathcal{L}_+(\mathcal{L}_+^2 + 2|a|\mathcal{L}_+)^n \mathcal{L}_1 D_f(y)\mathrm{d}y
$$

$$
= \sum_{n=0}^\infty |z|^{-2-2n} \mu_*(\mu_*^2 + 2|a|\mu_*)^n \Delta = \frac{\mu_*\Delta}{|z|^2 - \mu_*^2 - 2|a|\mu_*}.
$$

It follows that equation (A.14) has no solution if

$$
b^2 > \mu_*\Delta + \mu_*^2 + 2|a|\mu_* - a^2. \tag{A.16}
$$

Finally, this establishes that A_4 is disjoint from the spectrum.

Collecting (A.12), (A.13), (A.15) and (A.16) completes the proof of theorem 3.9. $\square$

Remark A.4.   The previous proof it is not optimal. We refrain form walking this road further as our purpose was only to show that the study of the spectrum can be reduced to the study of a concrete function (A.7), which plays the role of a determinant but may be easier to study than the dynamical determinant.

References

[1] Adam A 2017 Generic non-trivial resonances for Anosov diffeomorphisms *Nonlinearity* **30** 1146–64
[2] Ambrosio L, Fusco N and Pallara D 2000 *Functions of Bounded Variation and Free Discontinuity Problems* (*Oxford Mathematical Monographs*) (New York: Oxford University Press)
[3] Bahsoun W, Galatolo S, Nisoli I and Niu X 2016 Rigorous approximation of diffusion coefficients for expanding maps *J. Stat. Phys.* **163** 1486–503
[4] Bahsoun W and Liverani C 2020 Anosov diffeomorphisms, anisotropic BV spaces and regularity of foliations (arXiv:2003.06811)
[5] Baillif M and Baladi V 2005 Kneading determinants and spectra of transfer operators in higher dimensions: the isotropic case *Ergod. Theor. Dynam. Syst.* **25** 1437–70
[6] Baladi V 2000 Positive transfer operators and decay of correlations *Advanced Series in Nonlinear Dynamics* vol 16 (Singapore: World Scientific)
[7] Baladi V 2018 *Dynamical zeta functions and dynamical determinants for hyperbolic maps* (*A functional approach. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*) vol 68 (Berlin: Springer)
[8] Baladi V 2021 There are no deviations for the ergodic averages of the Giulietti–Liverani horocycle flows on the two-torus *Ergod. Theor. Dynam. Syst.* 1–14
[9] Baladi V and Keller G 1990 Zeta functions and transfer operators for piecewise monotone transformations *Commun. Math. Phys.* **127** 459–77
[10] Bandtlow O F, Just W and Slipantschuk J 2017 Spectral structure of transfer operators for expanding circle maps *Ann. Inst. Henri Poincaré C. Anal. Non Linéaire* **34** 31–43

[11]   Bandtlow O F and Naud F 2019 Lower bounds for the Ruelle spectrum of analytic expanding circle
         maps *Ergod. Theor. Dynam. Syst.* **39** 289–310
[12]   Blank M, Keller G and Liverani C 2002 Ruelle–Perron–Frobenius spectrum for Anosov maps
         Nonlinearity **15** 1905–73
[13]   Bowen R 1975 *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms* (*Lecture
         Notes in Mathematics*) vol 470 (Berlin: Springer)
[14]   Boyarsky A and Góra P 1997 *Laws of Chaos* (*Invariant Measures and Dynamical Systems in One
         Dimension. Probability and its Applications*) (Boston, MA: Birkhäuser)
[15]   Brini F, Siboni S, Turchetti G and Vaienti S 1997 Decay of correlations for the automorphism of
         the torus $\mathbb{T}^2$ *Nonlinearity* **10** 1257–68
[16]   Butterley O 2016 A note on operator semigroups associated to chaotic flows *Ergod. Theor. Dynam.
         Syst.* **36** 1396–408 (Corrigendum: 36:1409–1410, 2016)
[17]   Butterley O and Eslami P 2017 Exponential mixing for skew products with discontinuities *Trans.
         Am. Math. Soc.* **369** 783–803
[18]   Butterley O and Simonelli L D 2020 Parabolic flows renormalized by partially hyperbolic maps
         Boll. Unione Mat. Ital. **13** 341–60
[19]   Butterley O and War K 2020 Open sets of exponentially mixing Anosov flows *J. Eur. Math. Soc.* **22**
         2253–85
[20]   Buzzi J 2010 Entropy theory on the interval *École de Théorie Ergodique* (*Sémin. Congr.*) vol 20
         Soc. Math., France, Paris pp 39–82
[21]   Byon Chol S, Yoshitake N, Okamoto H and Mori H 1984 Correlations and spectra of an intermittent
         chaos near its onset point *J. Stat. Phys.* **36** 367–400
[22]   Carrand J 2020 Logarithmic bounds for ergodic averages of constant type rotation number flows on
         the torus: a short proof (arXiv:2012.07481)
[23]   Castorrini R and Liverani C 2020 Quantitative statistical properties of two-dimensional partially
         hyperbolic systems (arXiv:2007.05602)
[24]   Castro A and Varandas P 2013 Equilibrium states for non-uniformly expanding maps: decay of
         correlations and strong stability *Ann. Inst. Henri Poincaré C. Anal. Non Linéaire* **30** 225–49
[25]   Collet P and Eckmann J-P 2004 Liapunov multipliers and decay of correlations in dynamical
         systems *J. Stat. Phys.* **115** 217–54
[26]   Collet P and Isola S 1991 On the essential spectrum of the transfer operator for expanding Markov
         maps *Commun. Math. Phys.* **139** 551–7
[27]   Cornfeld I P, Fomin S V and Sinai Y G 1982 *Ergodic Theory* Translated from the Russian by
         A B Sosinskii. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of
         Mathematical Sciences] vol 245 (New York: Springer)
[28]   Demers M, Kiamari N and Liverani C 2021 *Transfer operators in Hyperbolic Dynamics An
         introduction. 33 Colloquio Brasilero de Matematica* (Rio de Janeiro: IMPA) p 252
[29]   Dolgopyat D 1998 On decay of correlations in Anosov flows *Ann. Math.* **147** 357–90
[30]   Driebe D J 1997 Jordan blocks in a one-dimensional Markov map *Comput. Math. Appl.* **34** 103–9
[31]   Driebe D J 1999 *Fully Chaotic Maps and Broken Time Symmetry* (Dordrecht: Kluwer)
[32]   Dunford N and Schwartz J T 1958 *Linear Operators* vol 1 (New York: Interscience (Wiley-
         Interscience))
[33]   Dyatlov S, Faure F and Guillarmou C 2015 Power spectrum of the geodesic flow on hyperbolic
         manifolds *Anal. PDE* **8** 923–1000
[34]   Dyatlov S and Zworski M 2016 Dynamical zeta functions for Anosov flows via microlocal analysis
         Ann. Sci. École Norm. Sup. **49** 543–77
[35]   Dyatlov S and Zworski M 2017 Ruelle zeta function at zero for surfaces *Invent. Math.* **210** 211–29
[36]   Faure F 2007 Prequantum chaos: resonances of the prequantum cat map *J. Mod. Dyn.* **1** 255–85
[37]   Faure F, Gouëzel S and Lanneau E 2018 Ruelle spectrum of linear pseudo-Anosov maps *Journal
         de l'École polytechnique — Mathématiques* **6** 811–77
[38]   Faure F and Tsujii M 2013 Band structure of the Ruelle spectrum of contact Anosov flows *C. R.
         Math.* **351** 385–91
[39]   Faure F and Tsujii M 2015 *Prequantum Transfer Operator for Symplectic Anosov Diffeomorphism*
         (*Astérisque v.375*) vol 375 (Paris, France: Société Mathématique de France)
[40]   Faure F and Tsujii M 2017 The semiclassical zeta function for geodesic flows on negatively curved
         manifolds *Invent. Math.* **208** 851–998
[41]   Forni G 2020 On the equidistribution of unstable curves for pseudo-Anosov diffeomorphisms of
         compact surfaces (arXiv:2007.03144)

[42] Forni G 2020 Ruelle resonances from cohomological equations (arXiv:2007.03116)

[43] Franks J 1970 Anosov diffeomorphisms *Proc. Symp. Pure Math.* 61–93

[44] Galatolo S and Nisoli I 2016 Rigorous computation of invariant measures and fractal dimension for maps with contracting fibers: 2D Lorenz-like maps *Ergod. Theor. Dynam. Syst.* **36** 1865–91

[45] Gouëzel S and Liverani C 2006 Banach spaces adapted to Anosov systems *Ergod. Theor. Dynam. Syst.* **26** 189–217

[46] Gouëzel S and Liverani C 2008 Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties *J. Differ. Geom.* **79** 433–77

[47] Giulietti P, Liverani C and Pollicott M 2013 Anosov flows and dynamical zeta functions *Ann. Math.* **178** 687–773

[48] Giulietti P and Liverani C 2019 Parabolic dynamics and anisotropic Banach spaces *J. Eur. Math. Soc.* **21** 2793–858

[49] Guedes Bonthonneau Y, Guillarmou C, Hilgert J and Weich T 2020 Ruelle-Taylor resonances of Anosov actions (arXiv:2007.14275)

[50] Guillarmou C and Faure F 2018 Horocyclic invariance of Ruelle resonant states for contact Anosov flows in dimension 3 *Math. Res. Lett.* **25** 1405–27

[51] Hennion H 1993 Sur un théorème spectral et son application aux noyaux lipchitziens *Proc. Am. Math. Soc.* **118** 627–34

[52] Ippei O 2011 Computer-assisted verification method for invariant densities and rates of decay of correlations *SIAM J. Appl. Dyn. Syst.* **10** 788–816

[53] Katok H and Hasselblatt B 1995 Introduction to the modern theory of dynamical systems. With a supplementary chapter by Katok and Leonardo Mendoza *Encyclopedia of Mathematics and its Applications* vol 54 (Cambridge: Cambridge University Press)

[54] Katznelson Y 1971 Ergodic automorphisms of $\mathbb{T}^n$ are Bernoulli shifts *Isr. J. Math.* **10** 186–95

[55] Keller G and Liverani C 1999 Stability of the spectrum for transfer operators *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **28** 141–52

[56] Keller G and Rugh H H 2004 Eigenfunctions for smooth expanding circle maps *Nonlinearity* **17** 1723–30

[57] Küster B and Weich T 2020 Pollicott–Ruelle resonant states and Betti numbers *Commun. Math. Phys.* **378** 917–41

[58] Lasota A and Yorke J A 1973 On the existence of invariant measures for piecewise monotonic transformations *Trans. Am. Math. Soc.* **186** 481

[59] Lind D A 1982 Dynamical properties of quasihyperbolic toral automorphisms *Ergod. Theor. Dynam. Syst.* **2** 49–68

[60] Liverani C 1995 Decay of correlations *Ann. Math.* **142** 239–301

[61] Liverani C, Saussol B and Vaienti S 1998 Conformal measure and decay of correlation for covering weighted systems *Ergod. Theor. Dynam. Syst.* **18** 1399–420

[62] Liverani C, Saussol B and Vaienti S 1999 A probabilistic approach to intermittency *Ergod. Theor. Dynam. Syst.* **19** 671–85

[63] Liverani C 2001 Rigorous numerical investigation of the statistical properties of piecewise expanding maps. A feasibility study *Nonlinearity* **14** 463–90

[64] Milnor J and Thurston W 1988 On iterated maps of the interval *Dynamical Systems* (*(College Park, MD, 1986–87), Lecture Notes in Mathematics*) vol 1342 (Berlin: Springer) pp 465–563

[65] Misiurewicz M and Szlenk W 1980 Entropy of piecewise monotone mappings *Stud. Math.* **67** 45–63

[66] Mori H, So B-C and Ose T 1981 Time-correlation functions of one-dimensional transformations *Prog. Theor. Phys.* **66** 1266–83

[67] Newhouse S E 1970 On codimension one Anosov diffeomorphisms *Am. J. Math.* **92** 761–70

[68] Naud F 2012 The Ruelle spectrum of generic transfer operators *Discrete Contin. Dyn. Syst.* A **32** 2521–31

[69] Nussbaum R D 1970 The radius of the essential spectrum *Duke Math. J.* **37** 473–8

[70] Petersen K 1989 *Ergodic Theory* (*Cambridge Studies in Advanced Mathematics*) Corrected Reprint of the 1983 Original vol 2 (Cambridge: Cambridge University Press)

[71] Pomeau Y and Manneville P 1980 Intermittent transition to turbulence in dissipative dynamical systems *Commun. Math. Phys.* **74** 189–97

[72] Ruelle D 1976 Zeta-functions for expanding maps and Anosov flows *Invent. Math.* **34** 231–42

[73] Ruelle D 1989 The thermodynamic formalism for expanding maps *Commun. Math. Phys.* **125** 239–62

[74] Ruelle D 1990 An extension of the theory of Fredholm determinants *Publications Mathématiques de l'Institut des Hautes Scientifiques* **72** 175–93

[75] Ruelle D and Sullivan D 1975 Currents, flows and diffeomorphisms *Topology* **14** 319–27

[76] Slipantschuk J, Bandtlow O F and Just W 2013 Analytic expanding circle maps with explicit spectra *Nonlinearity* **26** 3231–45

[77] Slipantschuk J, Bandtlow O F and Just W 2013 On the relation between Lyapunov exponents and exponential decay of correlations *J. Phys. A: Math. Theor.* **46** 075101

[78] Slipantschuk J, Bandtlow O F and Just W 2017 Complete spectral data for analytic Anosov maps of the torus *Nonlinearity* **30** 2667–86

[79] Tsujii M 2018 On cohomological theory of dynamical zeta functions (arXiv:1805.11992)

[80] Tsujii M and Zhang Z 2020 Smooth mixing Anosov flows in dimension three are exponential mixing (arXiv:2006.04293)