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Polynomial Decay of Correlations for Flows, Including Lorentz Gas Examples

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Abstract: We prove sharp results on polynomial decay of correlations for nonuniformly hyperbolic flows. Applications include intermittent solenoidal flows and various Lorentz gas models including the infinite horizon Lorentz gas.

Contents

1. Introduction

Let (Λ, μ_Λ) be a probability space. Given a measure-preserving flow $T_t : \Lambda \to \Lambda$ and observables $v, w \in L^2(\Lambda)$, we define the correlation function $\rho_{v,w}(t) = \int_\Lambda v\, w \circ T_t\, d\mu_\Lambda - \int_\Lambda v\, d\mu_\Lambda \int_\Lambda w\, d\mu_\Lambda$. The flow is *mixing* if $\lim_{t\to\infty} \rho_{v,w}(t) = 0$ for all $v, w \in L^2(\Lambda)$.

Of interest is the rate of decay of correlations, or rate of mixing, namely the rate at which $\rho_{v,w}$ converges to zero. Dolgopyat [21] showed that geodesic flows on compact surfaces of negative curvature with volume measure μ_Λ are exponentially mixing for Hölder observables v, w. Liverani [26] extended this result to arbitrary dimensional geodesic flows in negative curvature and more generally to contact Anosov flows. However, despite ongoing progress [3,5,11,12,35], exponential mixing remains poorly understood in general.

Dolgopyat [22] considered the weaker notion of *rapid mixing* (superpolynomial decay of correlations) where $\rho_{v,w}(t) = O(t^{-q})$ for sufficiently regular observables for any fixed $q \geqslant 1$, and showed that rapid mixing is 'prevalent' for Axiom A flows: it suffices that the flow contains two periodic solutions with periods whose ratio is Diophantine. Field *et al.* [23] introduced the notion of *good asymptotics* and used this to prove that amongst C^r Axiom A flows, $r \geqslant 2$, an open and dense set of flows is rapid mixing.

In [28], results on rapid mixing were obtained for nonuniformly hyperbolic semiflows, combining the rapid mixing method of Dolgopyat [22] with advances by Young [36, 37] in the discrete time setting. First results on *polynomial mixing* for nonuniformly hyperbolic semiflows ($\rho_{v,w}(t) = O(t^{-q})$ for some fixed $q > 0$) were obtained in [29]. Under certain assumptions the results in [28,29] were established also for nonuniformly hyperbolic flows. However, for polynomially mixing flows, the assumptions in [29] are overly restrictive and exclude many examples including infinite horizon Lorentz gases.

In this paper, we develop the tools required to cover systematically large classes of nonuniformly hyperbolic flows. The recent review article [30] describes the current state of the art for rapid and polynomial decay of correlations for nonuniformly hyperbolic semiflows and flows and gives a complete self-contained proof in the case of semiflows. Here we provide the arguments required to deal with flows. Our results cover all of the examples in [30].

By [28], rapid mixing holds (at least typically) for nonuniformly hyperbolic flows that are modelled as suspensions over Young towers with exponential tails [36]. See also Remark 8.5. Here we give a different proof that has a number of advantages as discussed in the introduction to [30]. Flows are modelled as suspensions over a uniformly hyperbolic map with an unbounded roof function (rather than as suspensions over a nonuniformly hyperbolic map with a bounded roof function). It then suffices to consider twisted transfer operators with one complex parameter rather than two as in [28], reducing from four to three the number of periodic orbits that need to be considered in Proposition 6.6. Also, the proof of rapid mixing only uses superpolynomial tails for the roof function, whereas [28] requires exponential tails.

Examples covered by our results on rapid mixing include finite Lorentz gases (including those with cusps, corner points, and external forcing), Lorenz attractors, and Hénon-like attractors. We refer to [30] for references and further details.

Examples discussed in [29,30] for which polynomial mixing holds include nonuniformly hyperbolic flows that are modelled as suspensions over Young towers with polynomial tails [37]. This includes intermittent solenoidal flows, see also Remark 8.6.

The key example of continuous time planar periodic infinite horizon Lorentz gases is considered at length in Sect. 9. In the finite horizon case, exponential decay of correlations for the flow was proved in [5]. In the infinite horizon case it has been conjectured [24,27] that the decay rate for the flow is $O(t^{-1})$. (An elementary argument in [6] shows that this rate is optimal; the argument is reproduced in the current context in Proposition 9.14.) We obtain the conjectured decay rate $O(t^{-1})$ for planar infinite horizon Lorentz flows in Theorem 9.1.

Remark 1.1. (a) In [29], the decay rate $O(t^{-1})$ was proved for infinite horizon Lorentz gases at the semiflow level (after passing to a suspension over a Markov extension and quotienting out stable leaves as in Sects. 3 and 6). It was claimed in [29] that this result held also in certain special cases for the Lorentz flow, and that the decay rate $O(t^{-(1-\epsilon)})$ held for all $\epsilon > 0$ in complete generality. The spurious factor of t^{ϵ} was then removed in an unpublished preprint "Decay of correlations for flows with unbounded roof function, including the infinite horizon planar periodic Lorentz gas" by the first and third authors. Unfortunately these results for flows do not apply to Lorentz gases since hypothesis (P1) in [29] is not satisfied. The situation is rectified in the current paper. (The unpublished preprint also contained correct results on statistical limit laws such as the central limit theorem for flows with unbounded roof functions. These aspects are completed and extended in [8].)
(b) A drawback of the method in this paper, already present in [22] and inherited by [28–30], is that at least one of the observables v or w is required to be C^m in the flow direction. Here m can be estimated, with difficulty, but is likely to be quite large. In the case of the infinite horizon Lorentz gas, this excludes certain physically important observables such as velocity. A reasonable project is to attempt to combine methods in this paper with the methods for (stretched) exponential decay in [5,15] to obtain the decay rate $O(t^{-1})$ for Hölder observables v and w (cf. the second open question in [30, Section 9]).

In Part I of this paper, we consider results on rapid mixing and polynomial mixing for a class of suspension flows over infinite branch uniformly hyperbolic transformations [36]. In Part II, we show how these results apply to important classes of nonuniformly hyperbolic flows including those mentioned in this introduction. The methods of proof in this paper, especially those in Part I, are fairly straightforward adaptations of those in [30]. The main new contribution of the paper (Sect. 6 together with Part II) is to develop a general framework whereby large classes of nonuniformly hyperbolic flows, including fundamental examples such as the infinite horizon Lorentz gas, are covered by these methods.

Remark 1.2. The paper has been structured to be as self-contained as possible. It does not seem possible to reduce the results on flows in Part I of this paper to the results on semiflows in [30]. Instead, it is necessary to start from scratch and to emulate, rather than apply directly, the methods in [30]. Some of the more basic estimates in [30] are applicable and are collected together at the beginning of Sect. 4 (Lemma 4.2 to Proposition 4.10) and Sect. 5 (Propositions 5.1 to 5.3), as well as in Sect. 5.2 (Propositions 5.8, 5.12 and 5.13). Also, results on nonexistence of approximate eigenfunctions in [30] are recalled in Sects. 6.2 and 8.4.

Notation We use the "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there is a constant $C > 0$ such that $a_n \leqslant Cb_n$ for all $n \geqslant 1$. There are various "universal" constants $C_1, \dots, C_5 \geqslant 1$ depending only on the flow that do not change throughout.

Part I
Mixing rates for Gibbs–Markov flows

In this part of the paper, we state and prove results on rapid and polynomial mixing for a class of suspension flows that we call Gibbs–Markov flows. These are suspensions over infinite branch uniformly hyperbolic transformations [36]. In Sect. 2, we recall material on the noninvertible version, Gibbs–Markov semiflows (suspensions over infinite branch uniformly expanding maps). In Sect. 3, we consider skew product Gibbs–Markov flows where the roof function is constant along stable leaves and state our main theorems for such flows, namely Theorem 3.1 (rapid mixing) and Theorem 3.2 (polynomial mixing). These are proved in Sects. 4 and 5 respectively. In Sect. 6, we consider an enlarged class of Gibbs–Markov flows that can be reduced to skew products and for which Theorems 3.1 and 3.2 remain valid.

We quickly review notation associated with suspension semiflows and suspension flows. Let (Y, μ) be a probability space and let $F : Y \to Y$ be a measure-preserving transformation. Let $\varphi : Y \to \mathbb{R}^+$ be an integrable roof function. Define the suspension semiflow/flow

$$F_t : Y^\varphi \to Y^\varphi, \qquad Y^\varphi = \{(y, u) \in Y \times [0, \infty) : u \in [0, \varphi(y)]\}/\sim, \tag{1.1}$$

where $(y, \varphi(y)) \sim (Fy, 0)$ and $F_t(y, u) = (y, u + t)$ computed modulo identifications. An F_t-invariant probability measure on Y^φ is given by $\mu^\varphi = \mu \times \text{Lebesgue}/\int_Y \varphi\, d\mu$.

2. Gibbs–Markov Maps and Semiflows

In this section, we review definitions and notation from [30, Section 3.1] for a class of Gibbs–Markov semiflows built as suspensions over Gibbs–Markov maps. Standard references for background material on Gibbs–Markov maps are [1, Chapter 4] and [2].

Suppose that $(\bar{Y}, \bar{\mu})$ is a probability space with an at most countable measurable partition $\{\bar{Y}_j,\ j \geqslant 1\}$ and let $\bar{F} : \bar{Y} \to \bar{Y}$ be a measure-preserving transformation. For $\theta \in (0, 1)$, define $d_\theta(y, y') = \theta^{s(y,y')}$ where the *separation time* $s(y, y')$ is the least integer $n \geqslant 0$ such that $\bar{F}^n y$ and $\bar{F}^n y'$ lie in distinct partition elements in $\{\bar{Y}_j\}$. It is assumed that the partition $\{\bar{Y}_j\}$ separates trajectories, so $s(y, y') = \infty$ if and only if $y = y'$. Then d_θ is a metric, called a *symbolic metric*.

A function $v : \bar{Y} \to \mathbb{R}$ is *d_θ-Lipschitz* if $|v|_\theta = \sup_{y \neq y'} |v(y) - v(y')|/d_\theta(y, y')$ is finite. Let $\mathcal{F}_\theta(\bar{Y})$ be the Banach space of Lipschitz functions with norm $\|v\|_\theta = |v|_\infty + |v|_\theta$.

More generally (and with a slight abuse of notation), we say that a function $v : \bar{Y} \to \mathbb{R}$ is *piecewise d_θ-Lipschitz* if $|1_{\bar{Y}_j} v|_\theta = \sup_{y, y' \in \bar{Y}_j,\, y \neq y'} |v(y) - v(y')|/d_\theta(y, y')$ is finite for all j. If in addition, $\sup_j |1_{\bar{Y}_j} v|_\theta < \infty$ then we say that v is *uniformly piecewise d_θ-Lipschitz*. Note that such a function v is bounded on partition elements but need not be bounded on $\bar{Y}$.

Definition 2.1. The map $\bar{F} : \bar{Y} \to \bar{Y}$ is called a *(full branch) Gibbs–Markov map* if

- $\bar{F}|_{\bar{Y}_j} : \bar{Y}_j \to \bar{Y}$ is a measurable bijection for each $j \geqslant 1$, and
- The potential function $\log(d\bar{\mu}/d\bar{\mu} \circ \bar{F}) : \bar{Y} \to \mathbb{R}$ is uniformly piecewise d_θ-Lipschitz for some $\theta \in (0, 1)$.

Definition 2.2. A suspension semiflow $\bar{F}_t : \bar{Y}^\varphi \to \bar{Y}^\varphi$ as in (1.1) is called a *Gibbs–Markov semiflow* if there exist constants $C_1 \geqslant 1, \theta \in (0, 1)$ such that $\bar{F} : \bar{Y} \to \bar{Y}$ is a Gibbs–Markov map, $\varphi : \bar{Y} \to \mathbb{R}^+$ is an integrable roof function with $\inf \varphi > 0$, and

$$|1_{\bar{Y}_j} \varphi|_\theta \leqslant C_1 \inf_{\bar{Y}_j} \varphi \quad \text{for all } j \geqslant 1. \tag{2.1}$$

(Equivalently, $\log \varphi$ is uniformly piecewise d_θ-Lipschitz.) It follows that $\sup_{\bar{Y}_j} \varphi \leqslant 2C_1 \inf_{\bar{Y}_j} \varphi$ for all $j \geqslant 1$.

For $b \in \mathbb{R}$, we define the operators

$$M_b : L^\infty(\bar{Y}) \to L^\infty(\bar{Y}), \qquad M_b v = e^{ib\varphi} v \circ \bar{F}.$$

Definition 2.3. A subset $Z_0 \subset \bar{Y}$ is a *finite subsystem* of $\bar{Y}$ if $Z_0 = \bigcap_{n \geqslant 0} \bar{F}^{-n} Z$ where Z is the union of finitely many elements from the partition $\{\bar{Y}_j\}$. (Note that $\bar{F}|_{Z_0} : Z_0 \to Z_0$ is a full one-sided shift on finitely many symbols.)

We say that M_b has *approximate eigenfunctions* on Z_0 if for any $\alpha_0 > 0$, there exist constants $\alpha, \xi > \alpha_0$ and $C > 0$, and sequences $|b_k| \to \infty$, $\psi_k \in [0, 2\pi)$, $u_k \in \mathcal{F}_\theta(\bar{Y})$ with $|u_k| \equiv 1$ and $|u_k|_\theta \leqslant C|b_k|$, such that setting $n_k = [\xi \ln |b_k|]$,

$$|(M_{b_k}^{n_k} u_k)(y) - e^{i\psi_k} u_k(y)| \leqslant C|b_k|^{-\alpha} \quad \text{for all } y \in Z_0, k \geqslant 1. \tag{2.2}$$

Remark 2.4. For brevity, the statement "Assume absence of approximate eigenfunctions" is the assumption that there exists at least one finite subsystem Z_0 such that M_b does not have approximate eigenfunctions on Z_0.

3. Skew Product Gibbs–Markov Flows

In this section, we recall the notion of skew product Gibbs–Markov flow [30, Section 4.1] and state our main results on mixing for such flows.

Let (Y, d) be a metric space with diam $Y \leqslant 1$, and let $F : Y \to Y$ be a piecewise continuous map with ergodic F-invariant probability measure μ. Let $\mathcal{W}^s$ be a cover of Y by disjoint measurable subsets of Y called *stable leaves*. For each $y \in Y$, let $W^s(y)$ denote the stable leaf containing y. We require that $F(W^s(y)) \subset W^s(Fy)$ for all $y \in Y$.

Let $\bar{Y}$ denote the space obtained from Y after quotienting by $\mathcal{W}^s$, with natural projection $\bar{\pi} : Y \to \bar{Y}$. We assume that the quotient map $\bar{F} : \bar{Y} \to \bar{Y}$ is a Gibbs–Markov map as in Definition 2.1, with partition $\{\bar{Y}_j\}$, separation time $s(y, y')$, and ergodic invariant probability measure $\bar{\mu} = \bar{\pi}_* \mu$.

Let $Y_j = \bar{\pi}^{-1} \bar{Y}_j$; these form a partition of Y and each Y_j is a union of stable leaves. The separation time extends to Y, setting $s(y, y') = s(\bar{\pi} y, \bar{\pi} y')$ for $y, y' \in Y$.

Next, we require that there is a measurable subset $\widetilde{Y} \subset Y$ such that for every $y \in Y$ there is a unique $\tilde{y} \in \widetilde{Y} \cap W^s(y)$. Let $\pi : Y \to \widetilde{Y}$ define the associated projection $\pi y = \tilde{y}$. (Note that $\widetilde{Y}$ can be identified with $\overline{Y}$, but in general $\pi_*\mu \neq \bar{\mu}$.)

We assume that there are constants $C_2 \geqslant 1$, $\gamma \in (0, 1)$ such that for all $n \geqslant 0$,

$$d(F^n y, F^n y') \leqslant C_2\gamma^n \quad \text{for all } y, y' \in Y \text{ with } y' \in W^s(y), \tag{3.1}$$

$$d(F^n y, F^n y') \leqslant C_2\gamma^{s(y,y')-n} \quad \text{for all } y, y' \in \widetilde{Y}. \tag{3.2}$$

Let $\varphi : Y \to \mathbb{R}^+$ be an integrable roof function with $\inf \varphi > 0$, and define the suspension flow[1] $F_t : Y^\varphi \to Y^\varphi$ as in (1.1) with ergodic invariant probability measure μ^φ.

In this section, we suppose that φ is constant along stable leaves and hence projects to a well-defined roof function $\varphi : \overline{Y} \to \mathbb{R}^+$. It follows that the suspension flow F_t projects to a quotient suspension semiflow $\overline{F}_t : \overline{Y}^\varphi \to \overline{Y}^\varphi$. We assume that $\overline{F}_t$ is a Gibbs–Markov semiflow (Definition 2.2). In particular, increasing $\gamma \in (0, 1)$ if necessary, (2.1) is satisfied in the form

$$|\varphi(y) - \varphi(y')| \leqslant C_1 \inf_{Y_j} \varphi\, \gamma^{s(y,y')} \quad \text{for all } y, y' \in \overline{Y}_j,\ j \geqslant 1. \tag{3.3}$$

We call F_t a *skew product Gibbs–Markov flow*, and we say that F_t has *approximate eigenfunctions* if $\overline{F}_t$ has approximate eigenfunctions (Definition 2.3).

Fix $\eta \in (0, 1]$. For $v : Y^\varphi \to \mathbb{R}$, define

$$|v|_\gamma = \sup_{(y,u),(y',u)\in Y^\varphi,\, y\neq y'} \frac{|v(y,u) - v(y',u)|}{\varphi(y)\{d(y,y') + \gamma^{s(y,y')}\}}, \qquad \|v\|_\gamma = |v|_\infty + |v|_\gamma,$$

$$|v|_{\infty,\eta} = \sup_{(y,u),(y,u')\in Y^\varphi,\, u\neq u'} \frac{|v(y,u) - v(y,u')|}{|u-u'|^\eta}, \qquad \|v\|_{\gamma,\eta} = \|v\|_\gamma + |v|_{\infty,\eta}.$$

(Here $|u - u'|$ denotes absolute value, with u, u' regarded as elements of $[0, \infty)$.) Let $\mathcal{H}_\gamma(Y^\varphi)$ and $\mathcal{H}_{\gamma,\eta}(Y^\varphi)$ be the spaces of observables $v : Y^\varphi \to \mathbb{R}$ with $\|v\|_\gamma < \infty$ and $\|v\|_{\gamma,\eta} < \infty$ respectively.

We say that $w : Y^\varphi \to \mathbb{R}$ is *differentiable in the flow direction* if the limit $\partial_t w = \lim_{t\to 0}(w \circ F_t - w)/t$ exists pointwise. Note that $\partial_t w = \frac{\partial w}{\partial u}$ on the set $\{(y,u) : y \in Y,\ 0 < u < \varphi(y)\}$. Define $\mathcal{H}_{\gamma,0,m}(Y^\varphi)$ to consist of observables $w : Y^\varphi \to \mathbb{R}$ that are m-times differentiable in the flow direction with derivatives in $\mathcal{H}_\gamma(Y^\varphi)$, with norm $\|w\|_{\gamma,0,m} = \sum_{j=0}^m \|\partial_t^j w\|_\gamma$.

We can now state the main theoretical results for skew product Gibbs–Markov flows.

Theorem 3.1. *Suppose that $F_t : Y^\varphi \to Y^\varphi$ is a skew product Gibbs–Markov flow such that $\varphi \in L^q(Y)$ for all $q \in \mathbb{N}$. Assume absence of approximate eigenfunctions.*

Then for any $q \in \mathbb{N}$, there exists $m \geqslant 1$ and $C > 0$ such that

$$|\rho_{v,w}(t)| \leqslant C\|v\|_\gamma \|w\|_{\gamma,0,m}\, t^{-q} \quad \textit{for all } v \in \mathcal{H}_\gamma(Y^\varphi),\ w \in \mathcal{H}_{\gamma,0,m}(Y^\varphi),\ t > 1.$$

Theorem 3.2. *Suppose that $F_t : Y^\varphi \to Y^\varphi$ is a skew product Gibbs–Markov flow such that $\mu(\varphi > t) = O(t^{-\beta})$ for some $\beta > 1$. Assume absence of approximate eigenfunctions.*

Then there exists $m \geqslant 1$ and $C > 0$ such that

$$|\rho_{v,w}(t)| \leqslant C\|v\|_{\gamma,\eta} \|w\|_{\gamma,0,m}\, t^{-(\beta-1)} \quad \textit{for all } v \in \mathcal{H}_{\gamma,\eta}(Y^\varphi),\ w \in \mathcal{H}_{\gamma,0,m}(Y^\varphi),\ t > 1.$$

[1] Strictly speaking, F_t is not always a flow since F need not be invertible. However, F_t is used as a model for various flows, and it is then a flow when φ is the first return to Y, so it is convenient to call it a flow.

Remark 3.3. Our result on polynomial mixing, Theorem 3.2, implies the result on rapid mixing, Theorem 3.1 (for a slightly more restricted class of observables). However, the proof of Theorem 3.1 plays a crucial role in the proof of Theorem 3.2, justifying the movement of certain contours of integration to the imaginary axis after the truncation step in Sect. 5.2. Hence, it is not possible to bypass Theorem 3.1 even when only polynomial mixing is of interest.

These results are proved in Sects. 4 and 5 respectively. For future reference, we mention the following estimates. Define $\varphi_n = \sum_{j=0}^{n-1} \varphi \circ F^j$.

Proposition 3.4. *Let $\eta \in (0, \beta)$. Then*
(a) $\int_Y \varphi^\eta \circ F^i 1_{\{\varphi_n > t\}}\, d\mu \leqslant (n+1) \int_Y \varphi^\eta 1_{\{\varphi > t/n\}}\, d\mu$ for all $i \geqslant 0$, $n \geqslant 1$, $t > 0$.
(b) If $\mu(\varphi > t) = O(t^{-\beta})$ for some $\beta > 1$, then $\int_Y \varphi^\eta 1_{\{\varphi > t\}}\, d\mu = O(t^{-(\beta-\eta)})$.

Proof. Writing $\varphi^\eta \circ F^i = \varphi^\eta \circ F^i 1_{\{\varphi\circ F^i > t/n\}} + \varphi^\eta \circ F^i 1_{\{\varphi \circ F^i \leqslant t/n\}}$, we compute that

$$\begin{aligned}
&\int_Y \varphi^\eta \circ F^i 1_{\{\varphi_n > t\}}\, d\mu \\
&\quad = \int_Y \varphi^\eta \circ F^i 1_{\{\varphi\circ F^i > t/n\}} 1_{\{\varphi_n > t\}}\, d\mu + \int_Y \varphi^\eta \circ F^i 1_{\{\varphi\circ F^i \leqslant t/n\}} 1_{\{\varphi_n > t\}}\, d\mu \\
&\quad \leqslant \int_Y \varphi^\eta \circ F^i 1_{\{\varphi\circ F^i > t/n\}}\, d\mu + \sum_{j=0}^{n-1} \int_Y \Big(\frac{t}{n}\Big)^\eta 1_{\{\varphi\circ F^j > t/n\}}\, d\mu \\
&\quad = \int_Y \varphi^\eta 1_{\{\varphi > t/n\}}\, d\mu + n \int_Y \Big(\frac{t}{n}\Big)^\eta 1_{\{\varphi > t/n\}}\, d\mu \leqslant (n+1) \int_Y \varphi^\eta 1_{\{\varphi > t/n\}}\, d\mu,
\end{aligned}$$

proving part (a). Part (b) is standard (see for example [30, Proposition 8.5]). □

4. Rapid Mixing for Skew Product Gibbs–Markov Flows

In this section, we consider skew product Gibbs–Markov flows $F_t : Y^\varphi \to Y^\varphi$ for which the roof function $\varphi : Y \to \mathbb{R}^+$ lies in $L^q(Y)$ for all $q \geqslant 1$. For such flows, we prove Theorem 3.1, namely that absence of approximate eigenfunctions is a sufficient condition for rapid mixing.

First, we introduce an auxiliary roof function $\breve\varphi : \bar Y \to \mathbb{R}^+$ satisfying $\inf \breve\varphi > 0$ and $\breve\varphi \leqslant \varphi$. Let $F_t : Y^{\breve\varphi} \to Y^{\breve\varphi}$ denote the corresponding Gibbs–Markov semiflow. Throughout, the notation $\mathcal{H}_\gamma(Y^{\breve\varphi})$, $\mathcal{H}_{\gamma,\eta}(Y^{\breve\varphi})$ and $\mathcal{H}_{\gamma,0,m}(Y^{\breve\varphi})$ represents spaces of observables on $Y^{\breve\varphi}$ with finite norms $\|v\|_\gamma < \infty$, $\|v\|_{\gamma,\eta} < \infty$ and $\|v\|_{\gamma,0,m}$, but with the norms weighted by the original roof function φ. In particular, we now have

$$|v|_\gamma = \sup_{(y,u),(y',u)\in Y^{\breve\varphi},\, y\neq y'} \frac{|v(y,u) - v(y',u)|}{\varphi(y)\{d(y,y') + \gamma^{s(y,y')}\}}.$$

Remark 4.1. As far as the arguments needed for Theorem 3.1 are concerned, we could assume $\varphi \equiv \breve\varphi$. In fact, the only situation where φ and $\breve\varphi$ differ is in Sect. 5. There we work with a bounded roof function $\breve\varphi = \varphi(N)$, yet keep the original roof function as a weight in the definition of the norm. Distinguishing φ and $\breve\varphi$ provides a formalism that extends our estimates immediately to the case of the bounded roof function.

Throughout this section, we suppose that $\mu(\varphi > t) = O(t^{-\beta})$ (and thus $\mu(\breve\varphi > t) = O(t^{-\beta})$) where $\beta > 1$. For notational convenience, we suppose that $\inf \breve\varphi \geqslant 1$.

4.1. Some notation and results from [30]. Let $\mathbb{H} = \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ and $\overline{\mathbb{H}} = \{s \in \mathbb{C} : \operatorname{Re} s \geqslant 0\}$. The Laplace transform $\hat{\rho}_{v,w}(s) = \int_0^\infty e^{-st}\rho_{v,w}(t)\,dt$ of the correlation function $\rho_{v,w}$ is analytic on $\mathbb{H}$.

Lemma 4.2 ([30, Lemma 6.2]). *Consider a skew product Gibbs–Markov flow with $\varphi \equiv \breve{\varphi}$. Let $v \in L^1(Y^\varphi)$, $\epsilon > 0$, $r \geqslant 1$. Suppose that*

(i) $s \mapsto \hat{\rho}_{v,w}(s)$ is continuous on $\{\operatorname{Re} s \in [0,\epsilon]\}$ and $b \mapsto \hat{\rho}_{v,w}(ib)$ is C^r on $\mathbb{R}$ for all $w \in \mathcal{H}_\gamma(Y^\varphi)$.

(ii) There exist constants $C, \alpha > 0$ such that

$$|\hat{\rho}_{v,w}(s)| \leqslant C(|b|+1)^\alpha \|w\|_\gamma \quad \textit{and} \quad |\hat{\rho}^{(j)}_{v,w}(ib)| \leqslant C(|b|+1)^\alpha \|w\|_\gamma,$$

for all $w \in \mathcal{H}_\gamma(Y^\varphi)$, $j \leqslant r$, and all $s = a + ib \in \mathbb{C}$ with $a \in [0,\epsilon]$.

Let $m = \lceil \alpha \rceil + 2$. Then there exists a constant $C' > 0$ depending only on r and α, such that

$$|\rho_{v,w}(t)| \leqslant CC'\|w\|_{\gamma,0,m}\, t^{-r} \quad \textit{for all } w \in \mathcal{H}_{\gamma,0,m}(Y^\varphi),\ t > 1.$$

□

Remark 4.3. Since $\hat{\rho}_{v,w}$ is not *a priori* well-defined on $\overline{\mathbb{H}}$, the conditions in this lemma should be interpreted in the usual way, namely that $\hat{\rho}_{v,w} : \mathbb{H} \to \mathbb{C}$ extends to a function $g : \overline{\mathbb{H}} \to \mathbb{C}$ satisfying the desired conditions (i) and (ii). The conclusion for $\rho_{v,w}$ then follows from a standard uniqueness argument.

For completeness, we provide the uniqueness argument. By [30, Corollary 6.1], the inverse Laplace transform of $\hat{\rho}_{v,w}$ can be computed by integrating along a contour in $\mathbb{H}$. Since $g \equiv \hat{\rho}_{v,w}$ on $\mathbb{H}$, we can compute the inverse Laplace transform f of g using the same contour, and we obtain $\rho_{v,w} \equiv f$. Hence $\hat{\rho}_{v,w} \equiv g$ is well-defined on $\overline{\mathbb{H}}$ and satisfies conditions (i) and (ii), so the conclusion follows from [30, Lemma 6.2].

Define $v_s(y) = \int_0^{\breve{\varphi}(y)} e^{su} v(y,u)\,du$ and $\widehat{w}(s)(y) = \int_0^{\breve{\varphi}(y)} e^{-su} w(y,u)\,du$.

Proposition 4.4 ([30, Proposition 6.3 and Corollary 8.6]). *Let $v, w \in L^\infty(Y^{\breve{\varphi}})$ with $\int_{Y^{\breve{\varphi}}} v\,d\mu^{\breve{\varphi}} = 0$. Then $\hat{\rho}_{v,w} = \sum_{n=0}^\infty \widehat{J}_n$ on $\mathbb{H}$ where $\widehat{J}_n$ is the Laplace transform of an L^∞ function $J_n : [0,\infty) \to \mathbb{R}$ for $n \geqslant 0$, and*

$$\widehat{J}_n(s) = |\varphi|_1^{-1} \int_Y e^{-s\breve{\varphi}_n} v_s\, \widehat{w}(s) \circ F^n\, d\mu \qquad \textit{for all } s \in \overline{\mathbb{H}},\ n \geqslant 1.$$

Moreover, $|J_0(t)| = O(|v|_\infty |w|_\infty t^{-(\beta-1)})$.[2] □

Let $R : L^1(\bar{Y}) \to L^1(\bar{Y})$ denote the transfer operator corresponding to the Gibbs–Markov quotient map $\bar{F} : \bar{Y} \to \bar{Y}$. So $\int_{\bar{Y}} v\, w \circ \bar{F}\, d\bar{\mu} = \int_{\bar{Y}} Rv\, w\, d\bar{\mu}$ for all $v \in L^1(\bar{Y})$ and $w \in L^\infty(\bar{Y})$. Also, for $s \in \overline{\mathbb{H}}$, define the twisted transfer operators

$$\widehat{R}(s) : L^1(\bar{Y}) \to L^1(\bar{Y}), \qquad \widehat{R}(s)v = R(e^{-s\breve{\varphi}} v).$$

Proposition 4.5. *Let $\theta \in (0,1)$ be as in Definition 2.1. There is a constant $C > 0$ such that*

$$\|R^n v\|_\theta \leqslant C \textstyle\sum_d \bar{\mu}(d) \|1_d v\|_\theta \quad \textit{for } v \in \mathcal{F}_\theta(\bar{Y}),\ n \geqslant 1,$$

where the sum is over n-cylinders $d = \bigcap_{i=0,\ldots,n-1} \bar{F}^{-i} \bar{Y}_{j_i}$, $j_0, \ldots, j_{n-1} \geqslant 1$.

[2] All series that we consider on $\mathbb{H}$ are absolutely convergent for elementary reasons. Details are given in Lemma 4.12 but are generally omitted.

Proof. This follows from [30, Corollary 7.2]. □

Fix $q > 0$ with

$$\max\{1, \beta - 1\} < q < \beta.$$

Let $\eta \in (0, 1]$, $\gamma \in (0, 1)$ be as in Sect. 3. Shrinking η if needed, we may suppose without loss that

$$q + 2\eta < \beta,$$

Let $\gamma_1 = \gamma^\eta$ and increase θ if needed so that $\theta \in [\gamma_1^{1/3}, 1)$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be C^q if f is $C^{[q]}$ and $f^{([q])}$ is $(q - [q])$-Hölder. Moreover, given $g : \mathbb{R} \to [0, \infty)$ and $E \subset \mathbb{R}$, we write $|f^{(q)}(b)| \leqslant g(b)$ for $b \in E$ if for all $b, b' \in E$,

$$|f^{(k)}(b)| \leqslant g(b),\ k = 0, 1, \ldots, [q],\ \text{and}$$
$$|f^{([q])}(b) - f^{([q])}(b')| \leqslant (g(b) + g(b'))|b - b'|^{q-[q]}.$$

For $f : \overline{\mathbb{H}} \to \mathbb{R}$ and $E \subset \overline{\mathbb{H}}$, we write $|f^{(q)}(s)| \leqslant g(s)$ for $s \in E$ if $|f^{(q)}(ib)| \leqslant g(b)$ in the sense just given for $ib \in E$ and $|f^{(k)}(s)| \leqslant g(s)$ for $s \in E$, $k = 0, \ldots, [q]$. The same conventions apply to operator-valued functions on $\overline{\mathbb{H}}$.

Remark 4.6. Restricting to q as above enables us to obtain estimates for the rapid mixing and polynomially mixing situations simultaneously hence avoiding a certain amount of repetition. The trade off is that the proof of Theorem 3.1 is considerably more difficult. The reader interested only in the rapid mixing case can restrict to integer values of q with greatly simplified arguments [30, Section 7] (also see version 3 of our preprint on arxiv).

Following [30, Section 7.4], there exist constants M_0, M_1 and a scale of equivalent norms

$$\|v\|_b = \max\left\{|v|_\infty, \frac{|v|_\theta}{M_0(|b| + 1)}\right\}, \quad b \in \mathbb{R},$$

on $\mathcal{F}_\theta(\overline{Y})$ such that

$$\|\widehat{R}(s)^n\|_b \leqslant M_1 \quad \text{for all } s = a + ib \in \mathbb{C} \text{ with } a \in [0, 1] \text{ and all } n \geqslant 1. \tag{4.1}$$

Proposition 4.7. *There is a constant $C > 0$ such that*

$$\|\widehat{R}^{(q)}(s)\|_b \leqslant C \quad \textit{for all } s = a + ib \in \mathbb{C} \textit{ with } 0 \leqslant a \leqslant 1.$$

Proof. It is shown in [30, Proposition 8.7] that $\|\widehat{R}^{(q)}(s)\|_\theta \leqslant C(|b| + 1)$. Using the definition of $\| \ \|_b$, the desired estimate follows by exactly the same argument. □

Remark 4.8. Estimates such as those for $\widehat{R}^{(q)}$ in Proposition 4.7 hold equally for $\widehat{R}^{(q')}$ for all $q' < q$. We use this observation without comment throughout.

Define $\mathbb{H}_\delta = \overline{\mathbb{H}} \cap B_\delta(0)$ for $\delta > 0$. Let $\widehat{T} = (I - \widehat{R})^{-1}$. We have the key Dolgopyat estimate:

Proposition 4.9. *Assume absence of approximate eigenfunctions. Then* $\widehat{T}(s) : \mathcal{F}_\theta(\overline{Y}) \to \mathcal{F}_\theta(\overline{Y})$ *is a well-defined bounded operator for* $s \in \overline{\mathbb{H}}\setminus\{0\}$. *Moreover, for any* $\delta > 0$, *there exists* $\alpha,\ C > 0$ *such that*

$$\|\widehat{T}^{(q)}(s)\|_\theta \leqslant C|b|^\alpha \quad \textit{for all } s = a + ib \in \mathbb{C}\setminus\mathbb{H}_\delta \textit{ with } 0 \leqslant a \leqslant 1.$$

□

Proof. For the region $0 \leqslant a \leqslant 1$, $|b| \geqslant \delta$, this is explicit in [30, Corollary 8.10]. The remaining region $A = ([0,1] \times [-\delta, \delta])\setminus\mathbb{H}_\delta$ is bounded. Also, $1 \notin \operatorname{spec} \widehat{R}(s)$ for $s \in \overline{\mathbb{H}}\setminus\{0\}$ by [30, Proposition 7.8(b) and Theorem 7.10(a)]. Hence $\|\widehat{T}^{(q)}\|_\theta$ is bounded on A by Proposition 4.7. □

Proposition 4.10 ([30, Proposition 7.8 and Corollary 7.9]). *There exists* $\delta > 0$ *such that* $\widehat{R}(s) : \mathcal{F}_\theta(\overline{Y}) \to \mathcal{F}_\theta(\overline{Y})$ *has a* C^q *family of simple eigenvalues* $\lambda(s)$, $s \in \mathbb{H}_\delta$, *isolated in* spec $\widehat{R}(s)$, *with* $\lambda(0) = 1$, $\lambda'(0) = -|\varphi|_1$, $|\lambda(s)| \leqslant 1$. *The corresponding spectral projections* $P(s)$ *form a* C^q *family of operators on* $\mathcal{F}_\theta(\overline{Y})$ *with* $P(0)v = \int_{\overline{Y}} v\, d\bar{\mu}$. □

4.2. Approximation of v_s *and* $\widehat{w}(s)$. The first step is to approximate $v_s,\ \widehat{w}(s) : Y \to \mathbb{C}$ by functions that are constant on stable leaves and hence well-defined on $\overline{Y}$.

For $k \geqslant 0$, define $\Delta_k : L^\infty(Y) \to L^\infty(Y)$,

$$\Delta_k w = w \circ F^k \circ \pi - w \circ F^{k-1} \circ \pi \circ F,\ k \geqslant 1, \quad \Delta_0 w = w \circ \pi.$$

Proposition 4.11. *Let* $w \in L^\infty(Y)$. *Then*

(a) $\Delta_k w$ *is constant along stable leaves.*
(b) $\sum_{k=0}^n (\Delta_k w) \circ F^{n-k} = w \circ F^n \circ \pi$.

Proof. Part (a) is immediate from the definition and part (b) follows by induction. □

Define

$$\widehat{V}_j(s) = e^{-s\breve{\varphi}\circ F^j} \Delta_j v_s, \qquad \widehat{W}_k(s) = e^{-s\breve{\varphi}_k} \Delta_k \widehat{w}(s).$$

By Proposition 4.11(a), these can be regarded as functions $\overline{V}_j$, $\overline{W}_k$ on $\overline{Y}$. Similarly we write $\overline{\Delta_k w} \in L^\infty(\overline{Y})$.

Also, for $k \geqslant 0$, we define $E_k : L^\infty(Y) \to L^\infty(Y)$,

$$E_k w = w \circ F^k - w \circ F^k \circ \pi.$$

Lemma 4.12. *Let* $v, w \in L^\infty(Y^{\breve{\varphi}})$. *Then*

$$\hat{\rho}_{v,w} = \widehat{J}_0 + |\varphi|_1^{-1}\Big(\sum_{n=1}^\infty \widehat{A}_n + \sum_{n=1}^\infty\sum_{k=0}^{n-1} \widehat{B}_{n,k} + \sum_{j=0}^\infty\sum_{k=0}^\infty \widehat{C}_{j,k}\Big),$$

on $\mathbb{H}$, *where*

$$\widehat{A}_n(s) = \int_Y e^{-s\breve{\varphi}_n} v_s\, (E_{n-1}\widehat{w}(s)) \circ F\, d\mu,$$

$$\widehat{B}_{n,k}(s) = \int_Y e^{-s\breve{\varphi}_n\circ F^n} E_n v_s\ (\Delta_k \widehat{w}(s)) \circ F^{2n-k}\, d\mu,$$

$$\widehat{C}_{j,k}(s) = \int_{\overline{Y}} \widehat{R}(s)^{\max\{j-k-1,0\}} \widehat{T}(s) R^{j+1}\, \overline{V}_j(s)\ \overline{W}_k(s)\, d\bar{\mu}.$$

All of these series are absolutely convergent exponentially quickly, pointwise on $\mathbb{H}$.

Proof. Since this result is set in the right-half complex plane, the final statement is elementary. We sketch the arguments. Let $s \in \mathbb{C}$ with $a = \operatorname{Re} s > 0$. It is clear that $|v_s| \leqslant a^{-1}|v|_\infty e^{a\breve{\varphi}}$ and $|\widehat{w}(s)|_\infty \leqslant a^{-1}|w|_\infty$. Hence $|\widehat{A}_n(s)| \leqslant 2a^{-2}|v|_\infty|w|_\infty e^{-a(n-1)}$ and $|\widehat{B}_{n,k}(s)| \leqslant 4a^{-2}|v|_\infty|w|_\infty e^{-a(n-1)}$. Similarly, $|\widehat{V}_j(s)|_\infty \leqslant 2a^{-1}|v|_\infty$ and $|\widehat{W}_k(s)|_\infty \leqslant 2a^{-1}|w|_\infty e^{-ak}$. As an operator on $L^\infty(Y)$, we have $|\widehat{R}(s)|_\infty \leqslant e^{-a}$. Hence $|\widehat{C}_{j,k}(s)| \leqslant 4a^{-2}(1-e^{-a})^{-1}|v|_\infty|w|_\infty e^{-a\max(j-1,k)}$.

By Proposition 4.4, $\hat{\rho}_{v,w}(s) = \widehat{J}_0(s) + |\varphi|_1^{-1}\sum_{n=1}^\infty \int_Y e^{-s\breve{\varphi}_n} v_s\; \widehat{w}(s)\circ F^n\, d\mu$ for $s \in \mathbb{H}$. By Proposition 4.11(b), for each $n \geqslant 1$,

$$\begin{aligned}\int_Y e^{-s\breve{\varphi}_n} v_s\; \widehat{w}(s)\circ F^n\, d\mu &= \int_Y e^{-s\breve{\varphi}_n} v_s\; \widehat{w}(s)\circ F^{n-1}\circ\pi\circ F\, d\mu \\ &\quad + \int_Y e^{-s\breve{\varphi}_n} v_s\; (\widehat{w}(s)\circ F^{n-1} - \widehat{w}(s)\circ F^{n-1}\circ\pi)\circ F\, d\mu \\ &= \sum_{k=0}^{n-1}\int_Y e^{-s\breve{\varphi}_n} v_s\; (\Delta_k\widehat{w}(s))\circ F^{n-k-1}\circ F\, d\mu + \widehat{A}_n(s).\end{aligned}$$

Also, by Proposition 4.11(b), for each $n \geqslant 1$, $0 \leqslant k \leqslant n-1$,

$$\begin{aligned}\int_Y e^{-s\breve{\varphi}_n} v_s\; (\Delta_k\widehat{w}(s))\circ F^{n-k-1}\circ F\, d\mu &= \int_Y e^{-s\breve{\varphi}_n\circ F^n} v_s\circ F^n\; (\Delta_k\widehat{w}(s))\circ F^{2n-k}\, d\mu \\ &= \sum_{j=0}^{n}\int_Y e^{-s\breve{\varphi}_n\circ F^n}(\Delta_j v_s)\circ F^{n-j}\;\; \Delta_k\widehat{w}(s)\circ F^{2n-k}\, d\mu + \widehat{B}_{n,k}(s) \\ &= \sum_{j=0}^{n}\int_{\overline{Y}} e^{-s\breve{\varphi}_n\circ\overline{F}^j}\,\overline{\Delta_j v_s}\;\overline{\Delta_k\widehat{w}(s)}\circ\overline{F}^{n-k+j}\, d\bar{\mu} + \widehat{B}_{n,k}(s).\end{aligned}$$

Next,

$$\begin{aligned}&\int_{\overline{Y}} e^{-s\breve{\varphi}_n\circ\overline{F}^j}\,\overline{\Delta_j v_s}\;\overline{\Delta_k\widehat{w}(s)}\circ\overline{F}^{n-k+j}\, d\bar{\mu} = \int_{\overline{Y}} e^{-s\breve{\varphi}_n} R^j\,\overline{\Delta_j v_s}\;\overline{\Delta_k\widehat{w}(s)}\circ\overline{F}^{n-k}\, d\bar{\mu} \\ &= \int_{\overline{Y}} e^{-s\breve{\varphi}_{n-k}} R^j\,\overline{\Delta_j v_s}\;(e^{-s\breve{\varphi}_k}\,\overline{\Delta_k\widehat{w}(s)})\circ\overline{F}^{n-k}\, d\bar{\mu} = \int_{\overline{Y}} \widehat{R}(s)^{n-k} R^j\,\overline{\Delta_j v_s}\;\overline{W}_k(s)\, d\bar{\mu} \\ &= \int_{\overline{Y}} \widehat{R}(s)^{n-k-1} R^{j+1}(e^{-s\breve{\varphi}\circ\overline{F}^j}\,\overline{\Delta_j v_s})\;\overline{W}_k(s)\, d\bar{\mu} = \int_{\overline{Y}} \widehat{R}(s)^{n-k-1} R^{j+1}\,\overline{V}_j(s)\;\overline{W}_k(s)\, d\bar{\mu}.\end{aligned}$$

Altogether,

$$\sum_{n=1}^\infty\int_Y e^{-s\breve{\varphi}_n} v_s\,\widehat{w}(s)\circ F^n\, d\mu = \sum_{n=1}^\infty \widehat{A}_n(s) + \sum_{n=1}^\infty\sum_{k=0}^{n-1}\widehat{B}_{n,k}(s) + C(s)$$

where

$$C(s) = \sum_{n=1}^\infty\sum_{k=0}^{n-1}\sum_{j=0}^{n}\int_{\overline{Y}} \widehat{R}(s)^{n-k-1} R^{j+1}\,\overline{V}_j(s)\;\overline{W}_k(s)\, d\bar{\mu}.$$

Now

$$\sum_{n=1}^\infty\sum_{k=0}^{n-1}\sum_{j=0}^{n}\widehat{R}(s)^{n-k-1}a_j\; b_k = \sum_{0\leqslant j\leqslant k}\;\sum_{n=k+1}^\infty \widehat{R}(s)^{n-k-1}a_j\; b_k + \sum_{j>k\geqslant 0}\;\sum_{n=j}^\infty \widehat{R}(s)^{n-k-1}a_j\; b_k$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \widehat{T}(s)a_j\, b_k + \sum_{j=1}^{\infty}\sum_{k=0}^{j-1} \widehat{R}(s)^{j-k-1}\widehat{T}(s)a_j\, b_k = \sum_{j,k=0}^{\infty} \widehat{R}(s)^{\max\{j-k-1,0\}}\widehat{T}(s)a_j\, b_k.$$

This completes the proof. □

For $w \in L^\infty(Y^{\breve{\varphi}})$, we define the approximation operators

$$\widetilde{\Delta}_k w(y,u) = \begin{cases} w(F^k\pi y,u) - w(F^{k-1}\pi F y,u) & k \geqslant 1 \\ w(\pi y,u) & k = 0 \end{cases},$$

$$\widetilde{E}_k w(y,u) = w(F^k y,u) - w(F^k\pi y,u),\ k \geqslant 0,$$

for $y \in Y$, $u \in [0,\breve{\varphi}(F^k y)]$.

Proposition 4.13. *(a) Let $w \in \mathcal{H}_\gamma(Y^{\breve{\varphi}})$, $k \geqslant 0$. Then for all $y \in Y$, $u \in [0,\breve{\varphi}(F^k y)]$,*

$$|\widetilde{\Delta}_k w(y,u)| \leqslant 2C_2\gamma_1^{k-1}\|w\|_\gamma \varphi(F^k y)^\eta \quad \textit{and} \quad |\widetilde{E}_k w(y,u)| \leqslant 2C_2\gamma_1^k |w|_\gamma \varphi(F^k y)^\eta.$$

(b) Let $w \in \mathcal{H}_\gamma(Y^{\breve{\varphi}})$, $k \geqslant 0$. Then for all $y, y' \in Y$, $u \in [0,\breve{\varphi}(F^k y)] \cap [0,\breve{\varphi}(F^k y')]$,

$$|\widetilde{\Delta}_k w(y,u) - \widetilde{\Delta}_k w(y',u)| \leqslant 4C_2\gamma_1^{s(y,y')-k}|w|_\gamma \varphi(F^k y)^\eta.$$

(c) Let $w \in \mathcal{H}_{\gamma,\eta}(Y^{\breve{\varphi}})$, $k \geqslant 0$. Then for all $y \in Y$, $u, u' \in [0,\breve{\varphi}(F^k y)]$,

$$|\widetilde{\Delta}_k w(y,u) - \widetilde{\Delta}_k w(y,u')| \leqslant 2|w|_{\infty,\eta}|u-u'|^\eta.$$

Proof. (a) Clearly $|\widetilde{\Delta}_0 w(y,u)| \leqslant |w|_\infty$. By (3.1), for $k \geqslant 1$,

$$\begin{aligned} |\widetilde{\Delta}_k w(y,u)| &\leqslant |w|_\gamma \varphi(F^k y)(d(F^k\pi y, F^{k-1}\pi F y) + \gamma^{s(F^k\pi y, F^{k-1}\pi F y)}) \\ &= |w|_\gamma \varphi(F^k y) d(F^k\pi y, F^{k-1}\pi F y) \leqslant C_2\gamma^{k-1}|w|_\gamma \varphi(F^k y). \end{aligned}$$

Also, $|\widetilde{\Delta}_k w| \leqslant 2|w|_\infty$, so

$$|\widetilde{\Delta}_k w(y,u)| \leqslant 2C_2\|w\|_\gamma \min\{1,\gamma^{k-1}\varphi(F^k y)\} \leqslant 2C_2\gamma_1^{k-1}\|w\|_\gamma \varphi(F^k y)^\eta.$$

This proves the estimate for $\widetilde{\Delta}_k w$, and the estimate for $\widetilde{E}_k w$ is similar.
(b) First suppose that $k \geqslant 1$ and note by (3.2) that

$$d(F^k\pi y, F^k\pi y') \leqslant C_2\gamma^{s(y,y')-k}, \quad d(F^{k-1}\pi F y, F^{k-1}\pi F y') \leqslant C_2\gamma^{s(y,y')-k}.$$

It follows that

$$\begin{aligned} |w(F^k\pi y,u) - w(F^k\pi y',u)| &\leqslant |w|_\gamma \varphi(F^k y)(d(F^k\pi y, F^k\pi y') + \gamma^{s(F^k\pi y, F^k\pi y')}) \\ &\leqslant |w|_\gamma \varphi(F^k y)(C_2\gamma^{s(y,y')-k} + \gamma^{s(y,y')-k}) \leqslant 2C_2\gamma^{s(y,y')-k}|w|_\gamma \varphi(F^k y). \end{aligned}$$

Similarly, $|w(F^{k-1}\pi F y,u) - w(F^{k-1}\pi F y',u)| \leqslant 2C_2\gamma^{s(y,y')-k}|w|_\gamma \varphi(F^k y)$. Hence

$$\begin{aligned} |\widetilde{\Delta}_k w(y,u) - \widetilde{\Delta}_k w(y',u)| &\leqslant |w(F^k\pi y,u) - w(F^k\pi y',u)| \\ &\quad + |w(F^{k-1}\pi F y,u) - w(F^{k-1}\pi F y',u)| \\ &\leqslant 4C_2\gamma^{s(y,y')-k}|w|_\gamma \varphi(F^k y). \end{aligned}$$

Also, $|\widetilde{\Delta}_k w(y,u) - \widetilde{\Delta}_k w(y',u)| \leqslant 4|w|_\infty$, so

$$|\widetilde{\Delta}_k w(y,u) - \widetilde{\Delta}_k w(y',u)| \leqslant 4C_2\gamma_1^{s(y,y')-k}|w|_\gamma \varphi(F^k y)^\eta.$$

The case $k=0$ is the same with one term omitted.
(c) For $k \geqslant 1$,

$$|\widetilde{\Delta}_k w(y,u) - \widetilde{\Delta}_k w(y,u')| \leqslant |w(F^k\pi y,u) - w(F^k\pi y,u')| + |w(F^{k-1}\pi Fy,u) - w(F^{k-1}\pi Fy,u')| \leqslant 2|w|_{\infty,\eta}|u-u'|^\eta.$$

The case $k=0$ is the same with one term omitted. □

We end this subsection by noting for all $k \geqslant 0$ the identities

$$\Delta_k v_s(y) = \int_0^{\breve{\varphi}(F^k y)} e^{su}\widetilde{\Delta}_k v(y,u)\,du, \qquad \Delta_k \widehat{w}(s)(y) = \int_0^{\breve{\varphi}(F^k y)} e^{-su}\widetilde{\Delta}_k w(y,u)\,du,$$

$$E_k v_s(y) = \int_0^{\breve{\varphi}(F^k y)} e^{su}\widetilde{E}_k v(y,u)\,du, \qquad E_k \widehat{w}(s)(y) = \int_0^{\breve{\varphi}(F^k y)} e^{-su}\widetilde{E}_k w(y,u)\,du.$$

4.3. Estimates for A_n and $B_{n,k}$. We continue to suppose that $\mu(\varphi > t) = O(t^{-\beta})$ where $\beta > 1$, and that $q, \eta, \gamma_1, \theta$ are as in Sect. 4.1. Let $c' = 1/(2C_1)$. As shown in the proofs of Propositions 4.15 and 4.16 below, $\widehat{A}_n$ and $\widehat{B}_{n,k}$ are Laplace transforms of L^∞ functions $A_n,\ B_{n,k} : [0,\infty) \to \mathbb{R}$. In this subsection, we obtain estimates for these functions $A_n,\ B_{n,k}$.

Proposition 4.14. *There is a constant $C > 0$ such that*

$$\textstyle\int_Y \varphi\,\varphi\circ F^n 1_{\{\varphi_{n+1}>t\}}\,d\mu \leqslant Cn \int_Y \varphi 1_{\{\varphi>c't/n\}}\,d\mu \quad \text{for all } n \geqslant 1,\ t>0.$$

Proof. Since F is Gibbs–Markov, there is a constant C_0 (called C_2 in [30]) such that

$$\begin{aligned}|R(\varphi 1_{\{\varphi>c\}})|_\infty &\leqslant C_0 \sum \mu(Y_j)|1_{Y_j}\varphi|_\infty 1_{\{|1_{Y_j}\varphi|_\infty > c\}} \\ &\leqslant 2C_0C_1 \sum \mu(Y_j)\inf\nolimits_{Y_j}\varphi 1_{\{\inf_{Y_j}\varphi > c'c\}} \leqslant K \textstyle\int_Y \varphi 1_{\{\varphi>c'c\}}\,d\mu,\end{aligned}$$

where $K = 2C_0C_1$. Similarly, $|R\varphi|_\infty \leqslant K|\varphi|_1$ and $|R1_{\{\varphi>c\}}|_\infty \leqslant K\mu(\varphi > c'c)$.

Now

$$\begin{aligned}\int_Y \varphi\,\varphi\circ F^n 1_{\{\varphi_{n+1}>t\}}\,d\mu &\leqslant \sum_{j=0}^n \int_Y \varphi\circ F^n\,\varphi 1_{\{\varphi\circ F^j > t/n\}}\,d\mu \\ &= \sum_{j=0}^n \int_Y \varphi\,R^n(\varphi 1_{\{\varphi\circ F^j>t/n\}})\,d\mu = \sum_{j=0}^n \int_Y \varphi\,R^{n-j}(1_{\{\varphi>t/n\}}R^j\varphi)\,d\mu.\end{aligned}$$

For $1 \leqslant j \leqslant n-1$,

$$\begin{aligned}\Big|\textstyle\int_Y \varphi\,R^{n-j}(1_{\{\varphi>t/n\}}R^j\varphi)\,d\mu\Big| &\leqslant |\varphi|_1|R^{n-j}(1_{\{\varphi>t/n\}}R^j\varphi)|_\infty \\ &\leqslant |\varphi|_1|R^j\varphi|_\infty|R^{n-j}1_{\{\varphi>t/n\}}|_\infty \leqslant |\varphi|_1|R\varphi|_\infty|R1_{\{\varphi>t/n\}}|_\infty \leqslant K^2|\varphi|_1^2\mu(\varphi > c't/n).\end{aligned}$$

For $j = n$,

$|\int_Y \varphi\, R^{n-j}(1_{\{\varphi>t/n\}}R^j\varphi)\,d\mu| \leqslant |R\varphi|_\infty \int_Y \varphi\, 1_{\{\varphi>t/n\}}\,d\mu \leqslant K|\varphi|_1 \int_Y \varphi 1_{\{\varphi>c't/n\}}\,d\mu.$

Finally for $j = 0$,

$|\int_Y \varphi\, R^{n-j}(1_{\{\varphi>t/n\}}R^j\varphi)\,d\mu| \leqslant |\varphi|_1|R(\varphi\, 1_{\{\varphi>t/n\}})|_\infty \leqslant K|\varphi|_1 \int_Y \varphi 1_{\{\varphi>c't/n\}}\,d\mu,$

completing the proof. □

Proposition 4.15. *There is a constant* $C > 0$ *such that*

$$|A_n(t)| \leqslant Cn^\beta\gamma_1^n|v|_\infty|w|_\gamma\,(t+1)^{-(\beta-1)} \text{ for all } v \in L^\infty(Y^{\breve\varphi}),\ w \in \mathcal{H}_\gamma(Y^{\breve\varphi}),\ n \geqslant 1,\ t > 0.$$

Proof. We compute that

$$\begin{aligned}\widehat{A}_n(s) &= \int_Y e^{-s\breve\varphi_n}v_s\,(E_{n-1}\widehat{w}(s))\circ F\,d\mu \\ &= \int_Y\int_0^{\breve\varphi(y)} v(y,u)\int_0^{\breve\varphi(F^ny)} e^{-s(\breve\varphi_n(y)-u+u')}\widetilde{E}_{n-1}w(Fy,u')\,du'\,du\,d\mu \\ &= \int_Y\int_0^{\breve\varphi(y)} v(y,u)\int_{\breve\varphi_n(y)-u}^{\breve\varphi_{n+1}(y)-u} e^{-st}\widetilde{E}_{n-1}w(Fy,t-\breve\varphi_n(y)+u)\,dt\,du\,d\mu.\end{aligned}$$

Hence

$$A_n(t) = \int_Y\int_0^{\breve\varphi(y)} v(y,u)1_{\{\breve\varphi_n(y)-u<t<\breve\varphi_{n+1}(y)-u\}}\widetilde{E}_{n-1}w(Fy,t-\breve\varphi_n(y)+u)\,du\,d\mu.$$

By Proposition 4.13(a), $|\widetilde{E}_{n-1}w(Fy,t-\breve\varphi_n(y)+u)| \leqslant 2C_2\gamma_1^{n-1}|w|_\gamma\varphi^\eta(F^ny)$ and so

$|A_n(t)| \leqslant 2C_2\gamma_1^{n-1}|v|_\infty|w|_\gamma \int_Y \breve\varphi\,\varphi^\eta\circ F^n 1_{\{\breve\varphi_{n+1}>t\}}\,d\mu \ll \gamma_1^n|v|_\infty|w|_\gamma \int_Y \varphi\,\varphi\circ F^n 1_{\{\varphi_{n+1}>t\}}\,d\mu.$

The result follows from Propositions 3.4(b) and 4.14. □

Proposition 4.16. *There is a constant* $C > 0$ *such that*

$$|B_{n,k}(t)| \leqslant Cn^\beta\gamma_1^n|v|_\gamma|w|_\infty\,(t+1)^{-(\beta-1)} \text{ for all } v \in \mathcal{H}_\gamma(Y^{\breve\varphi}),\ w \in L^\infty(Y^{\breve\varphi}),\ n \geqslant 1,\ k \geqslant 0,\ t > 0.$$

Proof. We compute that

$$\begin{aligned}\widehat{B}_{n,k}(s) &= \int_Y e^{-s\breve\varphi_n\circ F^n}E_nv_s\,(\Delta_k\widehat{w}(s))\circ F^{2n-k}\,d\mu \\ &= \int_Y\int_0^{\breve\varphi(F^{2n}y)}\int_0^{\breve\varphi(F^ny)} e^{-s(\breve\varphi_n(F^ny)-u'+u)}\widetilde{E}_nv(y,u')\widetilde{\Delta}_kw(F^{2n-k}y,u)\,du'\,du\,d\mu \\ &= \int_Y\int_0^{\breve\varphi(F^{2n}y)}\int_{\breve\varphi_{n-1}(F^{n+1}y)+u}^{\breve\varphi_n(F^ny)+u} e^{-st}\widetilde{E}_nv(y,\breve\varphi_n(F^ny)-t+u)\widetilde{\Delta}_kw(F^{2n-k}y,u)\,dt\,du\,d\mu.\end{aligned}$$

Hence

$$B_{n,k}(t) = \int_Y\int_0^{\breve\varphi(F^{2n}y)} 1_{\{\breve\varphi_{n-1}(F^{n+1}y)+u<t<\breve\varphi_n(F^ny)+u\}}$$

$$\times\, \widetilde{E}_n v(y, \breve{\varphi}_n(F^n y) - t + u)\widetilde{\Delta}_k w(F^{2n-k} y, u)\, du\, d\mu.$$

By Proposition 4.13(a), $|\widetilde{E}_n v(y, \breve{\varphi}_n(F^n y) - t + u)| \leqslant 2C_2\gamma_1^n |v|_\gamma \varphi^\eta(F^n y)$. Also $|\widetilde{\Delta}_k w(F^{2n-k} y, u)| \leqslant 2|w|_\infty$. Hence

$$|B_{n,k}(t)| \leqslant 2C_2\gamma_1^n |v|_\gamma |w|_\infty \int_Y \breve{\varphi} \circ F^{2n}\, \varphi^\eta \circ F^n 1_{\{\breve{\varphi}_{n+1}\circ F^n > t\}}\, d\mu$$
$$\ll \gamma_1^n |v|_\gamma |w|_\infty \int_Y \varphi\, \varphi \circ F^n 1_{\{\varphi_{n+1} > t\}}\, d\mu.$$

The result follows from Propositions 3.4(b) and 4.14. □

4.4. Estimates for $\widehat{C}_{j,k}$. For the moment, we suppose that $\mu(\varphi > t) = O(t^{-\beta})$ where $\beta > 1$, and that q, η, γ_1, θ are as in Sect. 4.1. We fix some notation. Recall that the function $\widehat{W}_k(s) : Y \to \mathbb{C}$ can be regarded as a function $\overline{W}_k(s) : \overline{Y} \to \mathbb{R}$. The inverse Laplace transform of $\widehat{W}_k(s)$ will be denoted as $W_k(t) : Y \to \mathbb{C}$, while the inverse Laplace transform of $\overline{W}_k(s)$ will be denoted as $\breve{W}_k(t) : \overline{Y} \to \mathbb{R}$. The same notational convention applies to different functions associated to $\widehat{V}_j(s)$. First, we estimate $\breve{W}_k(t) : \overline{Y} \to \mathbb{R}$.

Proposition 4.17. *There is a constant $C > 0$ such that*

$$|\breve{W}_k(t)|_1 \leqslant C(k+1)^{\beta+1}\gamma_1^k \|w\|_\gamma\, (t+1)^{-q} \quad \textit{for all } w \in \mathcal{H}_\gamma(Y^{\breve{\varphi}}), k \geqslant 0, t > 0.$$

Proof. For all $k \geqslant 0$,

$$\widehat{W}_k(s)(y) = e^{-s\breve{\varphi}_k(y)}\Delta_k \widehat{w}(s)(y) = \int_0^{\breve{\varphi}(F^k y)} e^{-s(\breve{\varphi}_k(y)+u)}\widetilde{\Delta}_k w(y, u)\, du$$
$$= \int_{\breve{\varphi}_k(y)}^{\breve{\varphi}_{k+1}(y)} e^{-st}\widetilde{\Delta}_k w(y, t - \breve{\varphi}_k(y))\, dt.$$

Hence

$$W_k(t)(y) = 1_{\{\breve{\varphi}_k(y) < t < \breve{\varphi}_{k+1}(y)\}}\widetilde{\Delta}_k w(y, t - \breve{\varphi}_k(y)),$$

and $|W_k(t)| \leqslant 2C_2\gamma_1^{k-1}\|w\|_\gamma (\varphi \circ F^k)^\eta 1_{\{\varphi_{k+1} > t\}}$ by Proposition 4.13(a). It follows that

$$|\breve{W}_k(t)|_1 = |W_k(t)|_1 \leqslant 2C_2(k+1)\gamma_1^{k-1}\|w\|_\gamma \textstyle\int_Y \varphi^\eta 1_{\{\varphi > t/(k+1)\}}\, d\mu$$
$$\ll (k+1)^{\beta+1-\eta}\gamma_1^{k-1}\|w\|_\gamma\, (t+1)^{-(\beta-\eta)} \leqslant (k+1)^{\beta+1}\gamma_1^{k-1}\|w\|_\gamma\, (t+1)^{-q},$$

by Proposition 3.4. □

Proposition 4.18. *There exists $C > 0$ such that*

$$\|(\widehat{R}^\ell)^{(q)}(s)\|_\theta \leqslant C\ell^q(|s|+1) \quad \textit{for all } s = a + ib \in \mathbb{C} \textit{ with } a \in [0, 1] \textit{ and all } \ell \geqslant 1.$$

Proof. By Proposition 4.7, there exists a constant $M > 0$ such that $\|\widehat{R}^{(p)}(s)\|_b \leqslant M$ for all $p \leqslant q$. Also $\|\widehat{R}(s)^n\|_b \leqslant M_1$ by (4.1).

For $q \geqslant 1$, note that $(\widehat{R}^\ell)^{(q)}$ consists of ℓ^q terms (counting repetitions) of the form

$$\widehat{R}^{n_1}\widehat{R}^{(p_1)} \cdots \widehat{R}^{n_k}\widehat{R}^{(p_k)}\widehat{R}^{n_{k+1}},$$

where $n_i \geqslant 0$, $1 \leqslant p_i \leqslant q$, $n_1 + \cdots + n_{k+1} + k = \ell$, $p_1 + \cdots + p_k = q$. Since $k \leqslant q$,

$$\|\widehat{R}^{n_1}\widehat{R}^{(p_1)} \cdots \widehat{R}^{n_k}\widehat{R}^{(p_k)}\widehat{R}^{n_{k+1}}\|_b \leqslant M_1^{q+1}M^q.$$

Hence $\|(\widehat{R}^\ell)^{(q)}(s)\|_\theta \leqslant (M_0+1)(|s|+1)\|(\widehat{R}^\ell)^{(q)}(s)\|_b \ll \ell^q(|s|+1)$. □

Proposition 4.19. *Let $v \in \mathcal{H}_\gamma(Y^{\breve{\varphi}})$. Define $I_0(s) = \int_Y \int_0^{\breve{\varphi}(y)} e^{-s(\breve{\varphi}(y)-u)} v(y,u)\,du\,d\mu$. Then*

$$\textstyle\sum_{j=0}^\infty \int_Y \widehat{V}_j\,d\mu = I_0 \quad \textit{on } \overline{\mathbb{H}}.$$

Proof. For $j \geqslant 1$,

$$\begin{aligned}\int_Y \widehat{V}_j(s)\,d\mu &= \int_Y\int_0^{\breve{\varphi}(F^jy)} e^{-s(\breve{\varphi}(F^jy)-u)}(v(F^j\pi y,u) - v(F^{j-1}\pi Fy,u)\,du\,d\mu \\ &= \int_Y\int_0^{\breve{\varphi}(F^jy)} e^{-s(\breve{\varphi}(F^jy)-u)} v(F^j\pi y,u)\,du\,d\mu \\ &\qquad - \int_Y\int_0^{\breve{\varphi}(F^{j-1}y)} e^{-s(\breve{\varphi}(F^{j-1}y)-u)} v(F^{j-1}\pi y,u)\,du\,d\mu,\end{aligned}$$

while $\int_Y \widehat{V}_0(s)\,d\mu = \int_Y \int_0^{\breve{\varphi}(y)} e^{-s(\breve{\varphi}(y)-u)} v(\pi y,u)\,du\,d\mu$. Hence

$$\begin{aligned}\sum_{j=0}^J \int_Y \widehat{V}_j(s)\,d\mu &= \int_Y\int_0^{\breve{\varphi}(F^Jy)} e^{-s(\breve{\varphi}(F^Jy)-u)} v(F^J\pi y,u)\,du\,d\mu \\ &= Z_J(s) + \int_Y\int_0^{\breve{\varphi}(F^Jy)} e^{-s(\breve{\varphi}(F^Jy)-u)} v(F^Jy,u)\,du\,d\mu = Z_J(s) + I_0(s),\end{aligned}$$

where

$$Z_J(s) = \int_Y\int_0^{\breve{\varphi}(F^Jy)} e^{-s(\breve{\varphi}(F^Jy)-u)}(v(F^J\pi y,u) - v(F^Jy,u))\,du\,d\mu.$$

By (3.1),

$$|v(F^J\pi y,u) - v(F^Jy,u)| \leqslant |v|_\gamma\,\varphi(F^Jy)d(F^J\pi y, F^Jy) \leqslant C_2\gamma^J|v|_\gamma\,\varphi(F^Jy).$$

Also, $|v(F^J\pi y,u) - v(F^Jy,u)| \leqslant 2|v|_\infty$, so

$$|v(F^J\pi y,u) - v(F^Jy,u)| \leqslant 2C_2\gamma_1^J\|v\|_\gamma\,\varphi(F^Jy)^\eta.$$

Hence $|Z_J(s)| \leqslant 2C_2\gamma_1^J\|v\|_\gamma \int_Y(\varphi\circ F^J)^{1+\eta}\,d\mu = 2C_2\gamma_1^J\|v\|_\gamma\int_Y\varphi^{1+\eta}\,d\mu \to 0$ as $J\to\infty$. □

The remaining estimates in this section are not needed in Sect. 5 (except in the proof of Proposition 5.10 as explained at the time). Hence, in the remainder of the section we specialize to the rapid mixing case, so q and β are arbitrary, $\varphi = \breve{\varphi}$, and all functions previously regarded as C^q are now C^∞.

Note that

$$\min\{\gamma_1^j, \gamma_1^{s(y,y')-j}\} \leqslant \gamma_1^{\frac{1}{3}j}\gamma_1^{\frac{1}{3}s(y,y')} \leqslant \gamma_1^{\frac{1}{3}j}\theta^{s(y,y')}. \tag{4.2}$$

Proposition 4.20. *For each $r \in \mathbb{N}$, there exists $C > 0$ such that*

$$\|R^{j+1}\overline{V}_j^{(r)}(s)\|_\theta \leqslant C(|s|+1)\gamma_1^{j/3}\|v\|_\gamma \quad \textit{for all } v \in \mathcal{H}_\gamma(Y^\varphi), s \in \overline{\mathbb{H}},\ j \geqslant 0.$$

Proof. For $j \geqslant 0$,

$$\widehat{V}_j(s)(y) = e^{-s\varphi(F^j y)}\Delta_j v_s(y) = \int_0^{\varphi(F^j y)} e^{-s(\varphi(F^j y)-u)}\widetilde{\Delta}_j v(y,u)\,du.$$

Hence

$$\widehat{V}_j^{(r)}(s)(y) = (-1)^r \int_0^{\varphi(F^j y)} e^{-s(\varphi(F^j y)-u)}(\varphi(F^j y)-u)^r\widetilde{\Delta}_j v(y,u)\,du.$$

By Proposition 4.13(a), $|\widetilde{\Delta}_j v(y,u)| \leqslant 2C_2\gamma_1^{j-1}\|v\|_\gamma\varphi(F^j y)^\eta$. Hence $|\widehat{V}_j^{(r)}(s)| \leqslant 2C_2\gamma_1^{j-1}\|v\|_\gamma\,\varphi^{r+2}\circ F^j$.

Fix a $(j+1)$-cylinder d for the Gibbs–Markov map $\overline{F} : \overline{Y} \to \overline{Y}$. Since $\overline{F}^j d$ is a partition element,

$$|1_d\overline{V}_j^{(r)}(s)|_\infty \leqslant C_2\gamma_1^{j-1}\|v\|_\gamma\,|1_{\overline{F}^j d}\,\varphi^{r+2}|_\infty \leqslant (2C_1)^{r+2}C_2\gamma_1^{j-1}\|v\|_\gamma \inf_{\overline{F}^j d}\varphi^{r+2}. \tag{4.3}$$

Let $y, y' \in d$ with $\varphi(\overline{F}^j y) \geqslant \varphi(\overline{F}^j y')$. Then

$$\overline{V}_j^{(r)}(s)(y) - \overline{V}_j^{(r)}(s)(y') = (-1)^r(I_1+I_2+I_3+I_4),$$

where

$$I_1 = \int_{\varphi(F^j y')}^{\varphi(F^j y)} e^{-s(\varphi(F^j y)-u)}(\varphi(F^j y)-u)^r\widetilde{\Delta}_j v(y,u)\,du,$$

$$I_2 = \int_0^{\varphi(F^j y')} \{e^{-s(\varphi(F^j y)-u)} - e^{-s(\varphi(F^j y')-u)}\}(\varphi(F^j y)-u)^r\widetilde{\Delta}_j v(y,u)\,du,$$

$$I_3 = \int_0^{\varphi(F^j y')} e^{-s(\varphi(F^j y')-u)}\{(\varphi(F^j y)-u)^r - (\varphi(F^j y')-u)^r\}\widetilde{\Delta}_j v(y,u)\,du,$$

$$I_4 = \int_0^{\varphi(F^j y')} e^{-s(\varphi(F^j y')-u)}(\varphi(F^j y')-u)^r\{\widetilde{\Delta}_j v(y,u) - \widetilde{\Delta}_j v(y',u)\}\,du.$$

By (3.3),

$$|\varphi(F^j y) - \varphi(F^j y')| \leqslant C_1\inf_{\overline{F}^j d}\varphi\,\gamma^{s(F^j y, F^j y')} = C_1\gamma^{s(y,y')-j}\inf_{\overline{F}^j d}\varphi.$$

Hence by Proposition 4.13(a,b),

$$|\overline{V}_j^{(r)}(s)(y) - \overline{V}_j^{(r)}(s)(y')| \ll (|s|+1)\gamma_1^{s(y,y')-j}\|v\|_\gamma \inf_{\overline{F}^j d}\varphi^{r+3}.$$

At the same time, the supnorm estimate (4.3) yields

$$|\overline{V}_j^{(r)}(s)(y) - \overline{V}_j^{(r)}(s)(y')| \ll \gamma_1^j\|v\|_\gamma \inf_{\overline{F}^j d}\varphi^{r+3}.$$

Combining these estimates and using (4.2) we obtain that

$$|\overline{V}_j^{(r)}(s)(y) - \overline{V}_j^{(r)}(s)(y')| \ll (|s|+1)\gamma_1^{j/3}\theta^{s(y,y')}\|v\|_\gamma \inf_{\overline{F}^j d}\varphi^{r+3}.$$

In other words,

$$|1_d\overline{V}_j^{(r)}(s)|_\theta \ll (|s|+1)\gamma_1^{j/3}\|v\|_\gamma \inf_{\overline{F}^j d}\varphi^{r+3}.$$

Using this and (4.3), it follows by Proposition 4.5 that

$$\begin{aligned}\|R^{j+1}\overline{V}_j^{(r)}(s)\|_\theta &\ll (|s|+1)\gamma_1^{j/3}\|v\|_\gamma \textstyle\sum_d \bar{\mu}(d)\inf_d\varphi^{r+3}\circ\overline{F}^j \\ &\leqslant (|s|+1)\gamma_1^{j/3}\|v\|_\gamma \textstyle\int_{\overline{Y}}\varphi^{r+3}\circ\overline{F}^j\,d\bar{\mu} = (|s|+1)\gamma_1^{j/3}\|v\|_\gamma\int_Y\varphi^{r+3}\,d\mu,\end{aligned}$$

completing the proof. □

Define $D_{j,\ell} = \widehat{R}^\ell\widehat{T}R^{j+1}\overline{V}_j$, $j,\ell \geqslant 0$. Let δ and λ be as in Proposition 4.10, and recall that $\mathbb{H}_\delta = \overline{\mathbb{H}} \cap B_\delta(0)$.

Proposition 4.21. *For each $r \in \mathbb{N}$, there exists α, $C > 0$ such that for all $v \in \mathcal{H}_\gamma(Y^\varphi)$, $j,\ell \geqslant 0$, and all $s = a+ib \in \mathbb{C}$ with $a \in [0,1]$,*

(a) $|D_{j,\ell}^{(r)}(s)|_\infty \leqslant C(\ell+1)^r\gamma_1^{j/3}(|b|+1)^\alpha\|v\|_\gamma$ *for* $s \notin \mathbb{H}_\delta$,
(b) $|\frac{d^r}{ds^r}\{D_{j,\ell}(s) - (1-\lambda(s))^{-1}\int_Y \widehat{V}_j(s)\,d\mu\}|_\infty \leqslant C(\ell+1)^{r+1}\gamma_1^{j/3}\|v\|_\gamma$ *for* $s \in \mathbb{H}_\delta$.

Proof. Let $p \in \mathbb{N}$, $p \leqslant r$. By Propositions 4.18 and 4.20, $\|R^{j+1}\overline{V}_j^{(p)}(s)\|_\theta \ll \gamma_1^{j/3}(|b|+1)\|v\|_\gamma$, and $\|(\widehat{R}^\ell)^{(p)}(s)\|_\theta \ll (\ell+1)^r(|b|+1)$.

For $s \notin \mathbb{H}_\delta$, it follows from Proposition 4.9 that $\|\widehat{T}^{(p)}(s)\|_\theta \ll (|b|+1)^\alpha$ for some $\alpha > 0$. Combining these estimates,

$$|(\widehat{R}^\ell\widehat{T}R^{j+1}\overline{V}_j)^{(r)}(s)|_\infty \leqslant \|(\widehat{R}^\ell\widehat{T}R^{j+1}\overline{V}_j)^{(r)}(s)\|_\theta \ll (\ell+1)^r\gamma_1^{j/3}(|b|+1)^{\alpha+2}\|v\|_\gamma,$$

completing the proof of (a).

Next, suppose that $s \in \mathbb{H}_\delta$. By Proposition 4.10, $\widehat{R} = \lambda P + \widehat{R}Q$ where $P(s)$ is the spectral projection corresponding to $\lambda(s)$ and $Q(s) = I - P(s)$. By Proposition 4.10, $\lambda(s)$ is a C^∞ family of isolated eigenvalues with $\lambda(0) = 1$, $\lambda'(0) \neq 0$ and $|\lambda(s)| \leqslant 1$, and $P(s)$ is a C^∞ family of operators on $\mathcal{F}_\theta(\overline{Y})$ with $P(0)v = \int_{\overline{Y}} v\,d\bar{\mu}$. Also

$$\widehat{T} = (1-\lambda)^{-1}P + Q_1 \quad \text{on } \mathbb{H}_\delta\setminus\{0\},$$

where $Q_1 = \widehat{T}Q$ is C^∞ on $\mathbb{H}_\delta$. Hence

$$\widehat{R}^\ell\widehat{T} = (1-\lambda)^{-1}\lambda^\ell P + \widehat{R}^\ell Q_1 = (1-\lambda)^{-1}\lambda^\ell P(0) + \lambda^\ell Q_2 + \widehat{R}^\ell Q_1 \quad \text{on } \mathbb{H}_\delta\setminus\{0\},$$

where $Q_2 = (1-\lambda)^{-1}(P - P(0))$ is C^∞ on $\mathbb{H}_\delta$. Also, $(1-\lambda)^{-1}\lambda^\ell = (1-\lambda)^{-1} - (\lambda^{\ell-1} + \cdots + 1)$, so

$$D_{j,\ell} - (1-\lambda)^{-1}P(0)R^{j+1}\overline{V}_j = Q_{j,\ell} \quad \text{on } \mathbb{H}_\delta,$$

where

$$Q_{j,\ell} = \big(-(\lambda^{\ell-1} + \cdots + 1)P(0) + \lambda^\ell Q_2 + \widehat{R}^\ell Q_1\big)R^{j+1}\overline{V}_j.$$

It follows from the estimates for $R^{j+1}\overline{V}_j$ and $\widehat{R}^\ell$ that $|(\widehat{R}^\ell Q_1 R^{j+1}\overline{V}_j)^{(r)}(s)|_\infty \ll (\ell+1)^r\gamma_1^{j/3}\|v\|_\gamma$ for $s \in \mathbb{H}_\delta$. Since $|\lambda(s)| \leqslant 1$, the proof of Proposition 4.18 applies equally to λ^ℓ, so $|Q_{j,\ell}^{(r)}(s)|_\infty \ll (\ell+1)^{r+1}\gamma_1^{j/3}\|v\|_\gamma$ for $s \in \mathbb{H}_\delta$.

Finally $P(0)R^{j+1}\overline{V}_j = \int_{\overline{Y}} \overline{V}_j\, d\bar{\mu} = \int_Y \widehat{V}_j\, d\mu$ completing the proof of part (b). □

By Lemma 4.12, $\widehat{C} = \sum_{j,k=0}^\infty \widehat{C}_{j,k}$ is analytic on $\mathbb{H}$. As shown in the next result, $\widehat{C}$ extends smoothly to $\overline{\mathbb{H}}$.

Corollary 4.22. *Assume absence of eigenfunctions and let $r \in \mathbb{N}$. There exists α, $C > 0$ such that*

$$|\widehat{C}^{(r)}(s)| \leqslant C(|b|+1)^\alpha \|v\|_\gamma \|w\|_\gamma,$$

for all $s = a + ib \in \overline{\mathbb{H}}$ with $a \in [0,1]$, and all $v, w \in \mathcal{H}_\gamma(Y^\varphi)$ with $\int_{Y^\varphi} v\, d\mu^\varphi = 0$.

Proof. Let $\ell = \max\{j-k-1, 0\}$. Recall from Lemma 4.12 that $\widehat{C}_{j,k} = \int_{\overline{Y}} D_{j,\ell}\overline{W}_k\, d\bar{\mu}$. Let $p \in \mathbb{N}$, $p \leqslant r$. By Proposition 4.17, $|\check{W}_k(t)|_1 \ll (k+1)^{\beta+1}\gamma_1^k\|w\|_\gamma\,(t+1)^{-q}$, so $|\overline{W}_k^{(p)}(s)|_1 \ll (k+1)^{\beta+1}\gamma_1^k\|w\|_\gamma$. Combining this with Proposition 4.21(a),

$$|\widehat{C}_{j,k}^{(r)}(s)| \ll |b|^\alpha (k+1)^r\gamma_1^{j/3}(k+1)^{\beta+1}\gamma_1^k\|v\|_\gamma\|w\|_\gamma \quad \text{for } |b| \geqslant \delta,$$

and the proof for $|b| \geqslant \delta$ is complete.

For $|b| \leqslant \delta$, we use Proposition 4.19 to write

$$\widehat{C} = \sum_{j,k}\int_Y \Big\{D_{j,\ell} - (1-\lambda)^{-1}\int_Y \widehat{V}_j\, d\mu\Big\}\overline{W}_k\, d\mu + (1-\lambda)^{-1}I_0\sum_k\int_Y \overline{W}_k\, d\mu.$$

Proposition 4.21(b) takes care of the first term on the right-hand side, and it remains to estimate $g = (1-\lambda)^{-1}I_0$. Now

$$I_0(0) = \int_Y\int_0^{\varphi(y)} v(y,u)\, du\, d\mu = |\varphi|_1\int_{Y^\varphi} v\, d\mu^\varphi = 0, \tag{4.4}$$

so it follows from Proposition 4.10 that g is C^∞ with $|g^{(r)}(s)| \ll |v|_\infty$ on $\mathbb{H}_\delta$. □

Proof of Theorem 3.1. Recall that β and q can be taken arbitrarily large. Hence it follows from Proposition 4.4 that $\sup_{\overline{\mathbb{H}}}|\widehat{J}_0^{(r)}| \ll |v|_\infty|w|_\infty$ for all $r \in \mathbb{N}$. Similarly, by Propositions 4.15 and 4.16, $\sup_{\overline{\mathbb{H}}}|\widehat{A}_n^{(r)}| \ll n^{r+3}\gamma_1^n|v|_\infty|w|_\gamma$ and $\sup_{\overline{\mathbb{H}}}|\widehat{B}_{n,k}^{(r)}| \ll n^{r+3}\gamma_1^n|v|_\gamma|w|_\infty$. Combining these with Corollary 4.22 and substituting into Lemma 4.12, we have shown that $\hat{\rho}_{v,w} : \mathbb{H} \to \mathbb{C}$ extends to $\hat{\rho}_{v,w} : \overline{\mathbb{H}} \to \mathbb{C}$. Moreover, we have shown that for every $r \in \mathbb{N}$ there exists C, $\alpha > 0$ such that

$$|\hat{\rho}_{v,w}^{(r)}(s)| \leqslant C(|b|+1)^\alpha\|v\|_\gamma\|w\|_\gamma \quad \text{for } s = a+ib \in \mathbb{C} \text{ with } a \in [0,1],$$

for all $v, w \in \mathcal{H}_\gamma(Y^\varphi)$ with $\int_{Y^\varphi} v\, d\mu^\varphi = 0$. The result now follows from Lemma 4.2 and Remark 4.3. □

5. Polynomial Mixing for Skew Product Gibbs–Markov Flows

In this section, we consider skew product Gibbs–Markov flows $F_t : Y^\varphi \to Y^\varphi$ for which the roof function $\varphi : Y \to \mathbb{R}^+$ satisfies $\mu(\varphi > t) = O(t^{-\beta})$ for some $\beta > 1$. For such flows, we prove Theorem 3.2, namely that absence of approximate eigenfunctions is a sufficient condition to obtain the mixing rate $O(t^{-(\beta-1)})$.

If $f : \mathbb{R} \to \mathbb{R}$ is integrable, we write $f \in \mathcal{R}(a(t))$ if the inverse Fourier transform of f is $O(a(t))$. We also write $\mathcal{R}(t^{-p})$ instead of $\mathcal{R}((t+1)^{-p})$ for $p > 0$. We use the same notation for Banach space valued functions $f : \mathbb{R} \to \mathcal{B}$, writing $\|f\|_{\mathcal{B}} \in \mathcal{R}(a(t))$ if the $\mathcal{B}$-norm of the inverse Fourier transform of f is $O(a(t))$.

Proposition 5.1 ([30, Proposition 8.2]). *Let $g : \mathbb{R} \to \mathbb{R}$ be an integrable function such that $g(b) \to 0$ as $b \to \pm\infty$. If $|f^{(q)}| \leqslant g$, then $f \in \mathcal{R}(|g|_1\, t^{-q})$.* □

The convolution $f \star g$ of two integrable functions $f, g : [0, \infty) \to \mathbb{R}$ is defined to be $(f \star g)(t) = \int_0^t f(x)g(t-x)\,dx$.

Proposition 5.2 ([30, Proposition 8.4]). *Fix $b > a > 0$ with $b > 1$. Suppose that $f, g : [0, \infty) \to \mathbb{R}$ are integrable and there exist constants $C, D > 0$ such that $|f(t)| \leqslant C(t+1)^{-a}$ and $|g(t)| \leqslant D(t+1)^{-b}$ for $t \geqslant 0$. Then there exists a constant $K > 0$ depending only on a and b such that $|(f \star g)(t)| \leqslant CDK(t+1)^{-a}$ for $t \geqslant 0$.* □

Proposition 5.3. *Define $f(b) = b^{-1}(e^{-ib\varphi} - 1)$ for $b \in \mathbb{R}\setminus\{0\}$. Then there exists $C > 0$ such that $\|1_{\bar{Y}_k} f^{(q)}(b)\|_\theta \leqslant C \inf_{\bar{Y}_k} \varphi^{q+\eta} |b|^{-(1-\eta)}$ for all $b \in \mathbb{R}\setminus\{0\}$.*

Proof. This is contained in the proof of [30, Proposition 8.13]. □

5.1. Modified estimate for $R^{j+1}\overline{V}_j$. In this subsection, we improve the estimate obtained in Proposition 4.20. No auxiliary roof function is required here, so $\varphi \equiv \check{\varphi}$. As in Sect. 4.1, $q + 2\eta < \beta$.

Proposition 5.4. *There exists $C > 0$ such that*

$$\|R^{j+1}\overline{V}_j^{(q)}(ib)\|_\theta \leqslant C\gamma_1^{j/3}\|v\|_\gamma\, |b|^{-(1-\eta)},$$

for all $v \in \mathcal{H}_\gamma(Y^\varphi)$ such that v is independent of u, and all $b \neq 0$, $j \geqslant 0$.

Proof. Recall that

$$\widehat{V}_j(s) = e^{-s\varphi\circ F^j}\Delta_j v_s = \int_0^{\varphi\circ F^j} e^{-s(\varphi\circ F^j - u)}\,du\;\Delta_j v = \int_0^{\varphi\circ F^j} e^{-su}\,du\;\Delta_j v.$$

Hence $R^j\overline{V}_j(s) = \int_0^\varphi e^{-su}\,du\,R^j(\Delta_j v) = -s^{-1}(e^{-s\varphi} - 1)R^j(\Delta_j v)$. It follows that

$$R^{j+1}\overline{V}_j(ib) = iR\big(f(b)R^j(\Delta_j v)\big), \tag{5.1}$$

where $f(b) = b^{-1}(e^{-ib\varphi} - 1)$.

Let $d \in \overline{Y}$ be a j-cylinder and let $y, y' \in d$. Then the arguments in the proof of Proposition 4.13(a,b) show that

$$|\Delta_j v(y)| \ll \gamma_1^j \|v\|_\gamma\, \varphi(F^j y)^\eta, \qquad |\Delta_j v(y) - \Delta_j v(y')| \ll \gamma_1^{s(y,y')-j}\|v\|_\gamma\, \varphi(F^j y)^\eta.$$

On the other hand, $|\Delta_j v(y) - \Delta_j v(y')| \ll \gamma_1^j \|v\|_\gamma \, \varphi(F^j y)^\eta$, so by (4.2),

$$|\Delta_j v(y) - \Delta_j v(y')| \ll \gamma_1^{j/3} \theta^{s(y,y')} \|v\|_\gamma \, \varphi(F^j y)^\eta.$$

Using (3.3), it follows that

$$|1_d (1_{\overline{Y}_k} \circ \overline{F}^j) \Delta_j v|_\infty \ll \gamma_1^j \|v\|_\gamma \sup_{\overline{Y}_k} \varphi^\eta \leqslant 2C_1 \gamma_1^j \|v\|_\gamma \inf_{\overline{Y}_k} \varphi^\eta,$$

and similarly,

$$|1_d (1_{\overline{Y}_k} \circ \overline{F}^j) \Delta_j v|_\theta \ll \gamma_1^{j/3} \|v\|_\gamma \inf_{\overline{Y}_k} \varphi^\eta, \;\; \|1_d (1_{\overline{Y}_k} \circ \overline{F}^j) \Delta_j v\|_\theta \ll \gamma_1^{j/3} \|v\|_\gamma \inf_{\overline{Y}_k} \varphi^\eta.$$

By Proposition 4.5,

$$\|1_{\overline{Y}_k} R^j (\Delta_j v)\|_\theta = \|R^j \big((1_{\overline{Y}_k} \circ \overline{F}^j) \Delta_j v \big)\|_\theta \ll \gamma_1^{j/3} \|v\|_\gamma \inf_{\overline{Y}_k} \varphi^\eta.$$

Hence by Proposition 5.3,

$$\begin{aligned} \|1_{\overline{Y}_k} f^{(q)}(b) R^j (\Delta_j v)\|_\theta &\ll \inf_{\overline{Y}_k} \varphi^{q+\eta} |b|^{-(1-\eta)} \|1_{\overline{Y}_k} R^j (\Delta_j v)\|_\theta \\ &\ll \gamma_1^{j/3} \|v\|_\gamma \inf_{\overline{Y}_k} \varphi^{q+2\eta} |b|^{-(1-\eta)}. \end{aligned}$$

Applying Proposition 4.5 once more and using (5.1),

$$\begin{aligned} \|R^{j+1} \overline{V}_j^{(q)}(ib)\|_\theta = \|R\big(f^{(q)}(b) R^j (\Delta_j v)\big)\|_\theta &\ll \sum_k \bar{\mu}(\overline{Y}_k) \|1_{\overline{Y}_k} f^{(q)}(b) R^j (\Delta_j v)\|_\theta \\ &\ll \gamma_1^{j/3} \|v\|_\gamma \int_Y \varphi^{q+2\eta} \, d\mu \, |b|^{-(1-\eta)} \end{aligned}$$

as required. □

Recall that $\check{V}_j(t) : \overline{Y} \to \mathbb{R}$ denotes the inverse Laplace transform associated to $\widehat{V}_j(s) : Y \to \mathbb{C}$.

Proposition 5.5. *There is a constant C such that*

$$\|R^{j+1} \check{V}_j(t)\|_\theta \leqslant C \gamma_1^{j/3} \|v\|_{\gamma,\eta} \, (t+1)^{-q},$$

for all $v \in \mathcal{H}_{\gamma,\eta}(Y^\varphi)$ with $v(y,0) \equiv 0$ and all $j \geqslant 0$, $t > 0$.

Proof. For $j \geqslant 0$,

$$\widehat{V}_j(s)(y) = \int_0^{\varphi(F^j y)} e^{-s(\varphi(F^j y) - u)} \widetilde{\Delta}_j v(y,u) \, du = \int_0^{\varphi(F^j y)} e^{-st} \widetilde{\Delta}_j v(y, \varphi(F^j y) - t) \, dt,$$

so

$$V_j(t)(y) = 1_{\{\varphi(F^j y) > t\}} \widetilde{\Delta}_j v(y, \varphi(F^j y) - t).$$

Recall that $c' = 1/(2C_1)$. Fix a $(j+1)$-cylinder d. By Proposition 4.13(a), for $y \in d$,

$$\begin{aligned} |V_j(t)(y)| &\leqslant 2C_2 \gamma_1^{j-1} \|v\|_\gamma \, \varphi(F^j y)^\eta 1_{\{|1_{\overline{F}^j d} \varphi|_\infty > t\}} \\ &\leqslant 4C_1 C_2 \gamma_1^{j-1} \|v\|_\gamma \inf_{\overline{F}^j d} \varphi^\eta 1_{\{\inf_{\overline{F}^j d} \varphi > c' t\}}. \end{aligned} \tag{5.2}$$

For $y, y' \in d$,

$$|\varphi(F^j y) - \varphi(F^j y')| \leqslant C_1 \inf_{\bar{F}^j d} \varphi\, \gamma^{s(y,y')-j}.$$

so by Propositions 4.13(b,c), for $t \in [0, \varphi(F^j y)] \cap [0, \varphi(F^j y')]$,

$$\begin{aligned} |\widetilde{\Delta}_j v(y, \varphi(F^j y) - t) - \widetilde{\Delta}_j v(y', \varphi(F^j y') - t)| & \\ \leqslant 4C_2 \gamma_1^{s(y,y')-j} \|v\|_\gamma \inf_{\bar{F}^j d} \varphi^\eta + 2|v|_{\infty,\eta} |\varphi(F^j y) - \varphi(F^j y')|^\eta & \\ \ll \gamma_1^{s(y,y')-j} \|v\|_{\gamma,\eta} \inf_{\bar{F}^j d} \varphi^\eta. & \qquad (5.3) \end{aligned}$$

Similarly, for $t \in [\varphi(F^j y'), \varphi(F^j y)]$,

$$\begin{aligned} |\widetilde{\Delta}_j v(y, \varphi(F^j y) - t)| = |\widetilde{\Delta}_j v(y, \varphi(F^j y) - t) - \widetilde{\Delta}_j v(y, 0)| \leqslant 2|v|_{\infty,\eta} |\varphi(F^j y) - t|^\eta & \\ \leqslant 2|v|_{\infty,\eta} |\varphi(F^j y) - \varphi(F^j y')|^\eta \ll \gamma_1^{s(y,y')-j} |v|_{\infty,\eta} \inf_{\bar{F}^j d} \varphi^\eta. & \qquad (5.4) \end{aligned}$$

For $y, y' \in d$ with $\varphi(F^j y) \geqslant \varphi(F^j y')$,

$$V_j(t)(y) - V_j(t)(y') = \begin{cases} \widetilde{\Delta}_j v(y, \varphi(F^j y) - t) - \widetilde{\Delta}_j v(y', \varphi(F^j y') - t), & \varphi(F^j y') > t \\ \widetilde{\Delta}_j v(y, \varphi(F^j y) - t), & \varphi(F^j y) > t \geqslant \varphi(F^j y') \\ 0, & \varphi(F^j y) \leqslant t. \end{cases}$$

If $\varphi(F^j y') > t$, then using (5.3),

$$\begin{aligned} |V_j(t)(y) - V_j(t)(y')| &\ll \gamma_1^{s(y,y')-j} \|v\|_{\gamma,\eta} 1_{\{|1_{\bar{F}^j d} \varphi|_\infty > t\}} \inf_{\bar{F}^j d} \varphi^\eta \\ &\leqslant \gamma_1^{s(y,y')-j} \|v\|_{\gamma,\eta} 1_{\{\inf_{\bar{F}^j d} \varphi > c't\}} \inf_{\bar{F}^j d} \varphi^\eta. \end{aligned}$$

If $\varphi(F^j y) > t \geqslant \varphi(F^j y')$, then using (5.4),

$$|V_j(t)(y) - V_j(t)(y')| \ll \gamma_1^{s(y,y')-j} |v|_{\infty,\eta} 1_{\{\inf_{\bar{F}^j d} \varphi > c't\}} \inf_{\bar{F}^j d} \varphi^\eta.$$

Hence in all cases,

$$|V_j(t)(y) - V_j(t)(y')| \ll \gamma_1^{s(y,y')-j} \|v\|_{\gamma,\eta} 1_{\{\inf_{\bar{F}^j d} \varphi > c't\}} \inf_{\bar{F}^j d} \varphi^\eta.$$

On the other hand, by (5.2), $|V_j(t)(y) - V_j(t)(y')| \ll \gamma_1^j \|v\|_\gamma \inf_{\bar{F}^j d} \varphi^\eta 1_{\{\inf_{\bar{F}^j d} \varphi > c't\}}$. Combining these estimates and using (4.2),

$$|V_j(t)(y) - V_j(t)(y')| \ll \gamma_1^{j/3} \theta^{s(y,y')} \|v\|_{\gamma,\eta} 1_{\{\inf_{\bar{F}^j d} \varphi > c't\}} \inf_{\bar{F}^j d} \varphi^\eta.$$

Hence

$$\|1_d V_j(t)\|_\theta \ll \gamma_1^{j/3} \|v\|_{\gamma,\eta} 1_{\{\inf_{\bar{F}^j d} \varphi > c't\}} \inf_{\bar{F}^j d} \varphi^\eta.$$

By Proposition 4.5,

$$\begin{aligned} \|R^{j+1} \check{V}_j(t)\|_\theta &\ll \gamma_1^{j/3} \|v\|_{\gamma,\eta} \textstyle\sum_d \bar{\mu}(d) 1_{\{\inf_d \varphi \circ \bar{F}^j > c't\}} (\inf_d \varphi \circ \bar{F}^j)^\eta \\ &\leqslant \gamma_1^{j/3} \|v\|_{\gamma,\eta} \int_{\bar{Y}} 1_{\{\varphi \circ \bar{F}^j > c't\}} (\varphi \circ \bar{F}^j)^\eta \, d\mu = \gamma_1^{j/3} \|v\|_{\gamma,\eta} \int_Y 1_{\{\varphi > c't\}} \varphi^\eta \, d\mu. \end{aligned}$$

Now apply Proposition 3.4(b). □

Remark 5.6. Recall that $R^{j+1}\overline{V}_j(s)$ is the Laplace transform of $R^{j+1}\check{V}_j(t)$. It follows from the estimates in Propositions 5.4 and 5.5 that $R^{j+1}\check{V}_j(t)$ can be recovered as the inverse Fourier transform of $R^{j+1}\overline{V}_j(ib)$. The arguments are completely standard and hence generally omitted; they are written out carefully once in the proof of Corollary 5.11. (In the proof of Corollary 5.11, the verification is carried out in a situation where there is a bounded roof function $\varphi(N)$. The required integrability conditions are not uniform in N but this is of no consequence.)

A further convention: we often consider expressions of the form $\kappa(b)R^{j+1}\overline{V}_j(ib)$. For brevity, we refer to these as $\kappa R^{j+1}\overline{V}_j$ and regard them as functions of the real variable $b \in \mathbb{R}$.

Corollary 5.7. *Let $\kappa : \mathbb{R} \to \mathbb{R}$ be $\mathbb{C}^\infty$ with $|\kappa^{(k)}(b)| = O((b^2+1)^{-1})$ for all $k \in \mathbb{N}$. Then $\|\kappa R^{j+1}\overline{V}_j\|_\theta \in \mathcal{R}(\gamma_1^{j/3}\|v\|_{\gamma,\eta}\, t^{-q})$ for all $v \in \mathcal{H}_{\gamma,\eta}(Y^\varphi)$, $j \geqslant 0$.*

Proof. Write $v(y,u) = v_0(y) + v_1(y,u)$ where $v_0(y) = v(y,0)$. We have the corresponding decomposition $\overline{V}_j = \overline{V}_{j,0} + \overline{V}_{j,1}$. The function $g(b) = \kappa(b)|b|^{-(1-\eta)}$ is integrable and $\|(\kappa R^{j+1}\overline{V}_{j,0})^{(q)}\|_\theta \ll \gamma_1^{j/3}\|v\|_{\gamma,\eta}\, g$ by Proposition 5.4. Hence, $\|\kappa R^{j+1}\overline{V}_{j,0}\|_\theta \in \mathcal{R}(\gamma_1^{j/3}\|v\|_\gamma\, t^{-q})$ by Proposition 5.1. Also, $\kappa \in \mathcal{R}(t^{-q})$ by Proposition 5.1, so $\|\kappa R^{j+1} \overline{V}_{j,1}\|_\theta \in \mathcal{R}(\gamma_1^{j/3}\|v\|_{\gamma,\eta}\, t^{-q})$ by Propositions 5.2 and 5.5. □

5.2. Truncation. We proceed in a manner analogous to [30, Section 8.4], replacing φ by a bounded roof function. Given $N \geqslant 1$, let $Y(N) = \bigcup_{j\geqslant 1:\inf_{Y_j}\varphi\geqslant N} Y_j$. Define $\varphi(N) = N$ on $Y(N)$ and $\varphi(N) = \varphi$ elsewhere. (Unlike [30], it is not sufficient to take $\varphi(N) = \min\{\varphi, N\}$.) Note that $\varphi(N) \leqslant 2C_1N$ by (3.3).

Consider the suspension semiflows F_t and $F_{N,t}$ on Y^φ and $Y^{\varphi(N)}$ respectively. (Here, $F_{N,t}$ is computed modulo the identification $(y, \varphi(N)(y)) \sim (Fy, 0)$ on $Y^{\varphi(N)}$.) Let $\rho_{v,w}$ and $\rho_{v,w}^{\text{trunc}}$ denote the respective correlation functions. In particular, $\rho_{v,w}^{\text{trunc}}(t) = \int_{Y^{\varphi(N)}} v\, w \circ F_{N,t}\, d\mu^{\varphi(N)} - \int_{Y^{\varphi(N)}} v\, d\mu^{\varphi(N)} \int_{Y^{\varphi(N)}} w\, d\mu^{\varphi(N)}$ where the observables $v, w : Y^{\varphi(N)} \to \mathbb{R}$ are the restrictions of $v, w : Y^\varphi \to \mathbb{R}$ to $Y^{\varphi(N)}$.

Proposition 5.8 ([30, Proposition 8.19]). *There are constants C, $t_0 > 0$, $N_0 \geqslant 1$ such that*

$$|\rho_{v,w}(t) - \rho_{v,w}^{\text{trunc}}(t)| \leqslant C|v|_\infty |w|_\infty (tN^{-\beta} + N^{-(\beta-1)}),$$

for all $v, w \in L^\infty(Y^\varphi)$, $N \geqslant N_0$, $t > t_0$. □

When computing the norms of observables $v : Y^{\varphi(N)} \to \mathbb{R}$, we retain φ as the weight function in the denominator. It follows that $\|v\|_{\gamma,\eta} = \|v'\|_{\gamma,\eta}$ where v' is the extension of v by zero to Y^φ. This convention is in accordance with the formalism introduced in Sect. 4; with $\varphi(N)$ playing the role of $\breve{\varphi}$ (see Remark 4.1).

Note also that $v \in \mathcal{H}_{\gamma,\eta}(Y^\varphi)$ restricts to $v|_{Y^{\varphi(N)}} \in \mathcal{H}_{\gamma,\eta}(Y^{\varphi(N)})$ with $\|v|_{Y^{\varphi(N)}}\|_{\gamma,\eta} \leqslant \|v\|_{\gamma,\eta}$. The similar convention applies to observables $w \in \mathcal{H}_\gamma(Y^{\varphi(N)})$. However, restricting $w \in \mathcal{H}_{\gamma,0,m}(Y^\varphi)$ to $Y^{\varphi(N)}$ need not preserve smoothness in the flow direction. Below we prove:

Lemma 5.9. *Assume absence of approximate eigenfunctions. In particular, there is a finite union $Z \subset \overline{Y}$ of partition elements such that the corresponding finite subsystem Z_0 does not support approximate eigenfunctions. Choose $N_1 \geqslant |1_Z\varphi|_\infty + 3$.*

There exist $m \geqslant 1$, $C > 0$ such that

$$|\rho^{\rm trunc}_{v,w}(t)| \leqslant C\|v\|_{\gamma,\eta}\|w\|_{\gamma,0,m}\, t^{-(\beta-1)},$$

for all $v \in \mathcal{H}_{\gamma,\eta}(Y^{\varphi(N)})$, $w \in \mathcal{H}_{\gamma,0,m}(Y^{\varphi(N)})$, $N \geqslant N_1$, $t > 1$.

Proof of Theorem 3.2. Let $m \geqslant 1$, $N_1 \geqslant 3$ be as in Lemma 5.9. As discussed above, the observable $v : Y^\varphi \to \mathbb{C}$ restricts to an observable $v : Y^{\varphi(N)} \to \mathbb{C}$ with no increase in the value of $\|v\|_{\gamma,\eta}$, but restricting $w \in \mathcal{H}_{\gamma,0,m}(Y^\varphi)$ to $Y^{\varphi(N)}$ need not preserve smoothness in the flow direction. To circumvent this, following [29,30] we define an approximating observable $w_N : Y^{\varphi(N)} \to \mathbb{R}$, $N \geqslant N_1$,

$$w_N(y,u) = \begin{cases} w(y,u) & (y,u) \notin Y(N) \times [N-2,N] \\ \sum_{j=0}^{2m+1}(u-N+2)^j d_{N,j}(y) & (y,u) \in Y(N) \times [N-2,N-1] \\ w(y,u+\varphi(y)-N) & (y,u) \in Y(N) \times (N-1,N] \end{cases},$$

where the $d_{N,j}(y)$ are linear combinations of $\partial_t^i w(y,N-2)$ and $\partial_t^i w(y,\varphi(y)-1)$, $i = 0,\dots,m$, with coefficients independent of y and N uniquely specified by the requirements $\partial_t^i w_N(y,N-2) = \partial_t^i w(y,N-2)$ and $\partial_t^i w_N(y,N-1) = \partial_t^i w(y,\varphi(y)-1)$ for $i = 0,\dots,m$.[3]

It is immediate from the definitions that w_N is m-times differentiable in the flow direction. We claim that $\|w_N\|_{\gamma,0,m} \leqslant C'\|w\|_{\gamma,0,m+1}$ for some constant C' independent of N. By Lemma 5.9,

$$|\rho^{\rm trunc}_{v,w_N}(t)| \leqslant CC'\|v\|_{\gamma,\eta}\|w\|_{\gamma,0,m+1}\, t^{-(\beta-1)}.$$

Also,

$$\begin{aligned}|\rho^{\rm trunc}_{v,w}(t) - \rho^{\rm trunc}_{v,w_N}(t)| &\leqslant |v|_\infty(|w|_\infty + |w_N|_\infty)\mu^{\varphi(N)}(F_{N,t}^{-1}S_N) \\ &= |v|_\infty(|w|_\infty + |w_N|_\infty)\mu^{\varphi(N)}(S_N) \leqslant 2|v|_\infty|w|_\infty\mu(\varphi > N) \ll |v|_\infty|w|_\infty\, N^{-\beta},\end{aligned}$$

so

$$|\rho^{\rm trunc}_{v,w}(t)| \ll \|v\|_{\gamma,\eta}\|w\|_{\gamma,0,m+1}\,(t^{-(\beta-1)} + N^{-\beta}).$$

Taking $N = [t]$, the result follows directly from Proposition 5.8.

It remains to verify the claim. Fix $k \in \{0,\dots,m\}$. Let (y,u), $(y',u) \in Y(N) \times [N-2, N-1]$, where y, y' lie in the same partition element. Then

$$\begin{aligned}|\partial_t^k w_N(y,u)| &\leqslant (2m+1)!\textstyle\sum_{j=0}^{2m+1}|d_{N,j}(y)| \\ &\leqslant C\textstyle\sum_{i=0}^m(|\partial_t^i w(y,N-2)| + |\partial_t^i w(y,\varphi(y)-1)|) \leqslant 2C\|w\|_{\gamma,0,m},\end{aligned}$$

where C is a constant independent of N. Also, by (3.3), for $0 \leqslant i \leqslant m$

$$\begin{aligned}|\partial_t^i w(y,\varphi(y)-1) - \partial_t^i w(y,\varphi(y')-1)| &\leqslant |\partial_t^{i+1}w|_\infty|\varphi(y)-\varphi(y')| \\ &\leqslant C_1|\partial_t^{i+1}w|_\infty\varphi(y)\gamma^{s(y,y')}.\end{aligned}$$

[3] In fact $d_{N,j}(y) = (1/j!)\partial_t^j w(y,N-2)$ for $0 \leqslant j \leqslant m$ but the remaining formulas are messier. When $m = 1$, for instance, $d_{N,2}(y) = -3w(y,N-2) - 2\partial_t w(y,N-2) + 3w(y,\varphi(y)-1) - \partial_t w(y,\varphi(y)-1)$, $d_{N,3}(y) = 2w(y,N-2) + \partial_t w(y,N-2) - 2w(y,\varphi(y)-1) + \partial_t w(y,\varphi(y)-1)$.

Hence

$$
\begin{aligned}
|\partial_t^k w_N(y,u)| - \partial_t^k w_N(y',u)| &\leqslant (2m+1)! \textstyle\sum_{j=0}^{2m+1} |d_{N,j}(y) - d_{N,j}(y')| \\
&\leqslant C \textstyle\sum_{i=0}^{m} \big(|\partial_t^i w(y,N-2) - \partial_t^i w(y',N-2)| + |\partial_t^i w(y,\varphi(y)-1) \\
&\quad - \partial_t^i w(y',\varphi(y')-1)|\big) \\
&\leqslant 2C \textstyle\sum_{i=0}^{m} |\partial_t^i w|_\gamma \varphi(y)\{d(y,y') + \gamma^{s(y,y')}\} \\
&\quad + C \sum_{i=0}^{m} |\partial_t^i w(y,\varphi(y)-1) - \partial_t^i w(y,\varphi(y')-1)| \\
&\leqslant 2C\|w\|_{\gamma,0,m}\varphi(y)\{d(y,y') + \gamma^{s(y,y')}\} + CC_1\|w\|_{\gamma,0,m+1}\varphi(y)\gamma^{s(y,y')} \\
&\leqslant 3CC_1\|w\|_{\gamma,0,m+1}\varphi(y)\{d(y,y') + \gamma^{s(y,y')}\}.
\end{aligned}
$$

This completes the verification of the claim on the region $Y(N) \times [N-2, N-1]$ and the other regions are easier to treat. □

Our strategy for proving Lemma 5.9 is identical to that for [30, Lemma 8.20]. Let $\chi : [0,\infty) \to [0,1]$ be a C^∞ function vanishing on $[0,1]$ and replace $\rho_{v,w}^{\rm trunc}$ by $\chi\rho_{v,w}^{\rm trunc}$. Then the asymptotics of $\rho_{v,w}^{\rm trunc}$ and the smoothness of $\widehat{\rho_{v,w}^{\rm trunc}}$ are unchanged uniformly in N. Hence, as in [30, Section 6.1], we may assume without loss that $\rho_{v,w}^{\rm trunc}$ vanishes for $t \leqslant 1$.

The next step is to show that the inverse Laplace transform of $\widehat{\rho_{v,w}^{\rm trunc}}$ can be computed using the imaginary axis as the contour of integration (cf. Remark 5.6).

Proposition 5.10. *Let $N \geqslant N_1$, $v, w \in \mathcal{H}_\gamma(Y^{\varphi(N)})$. Then there exists $\epsilon > 0$, $C > 0$, $\alpha \geqslant 0$, such that $\widehat{\rho_{v,w}^{\rm trunc}}$ is continuous on $\{\operatorname{Re} s \in [0,\epsilon]\}$ and $|\widehat{\rho_{v,w}^{\rm trunc}}(s)| \leqslant C(|b|+1)^\alpha$ for all $s = a + ib$ with $a \in [0,\epsilon]$.*

Proof. In this proof, the constant C is not required to be uniform in N. Consequently, the estimates are very straightforward compared to other estimates in this section.

The desired properties for $\widehat{\rho_{v,w}^{\rm trunc}}$ hold provided they are verified for all the constituent parts in Lemma 4.12. Note that if f is integrable on $[0,\infty)$, then $\hat f$ satisfies the required properties with $\alpha = 0$. By Proposition 4.4, $|J_0(t)| \ll |v|_\infty |w|_\infty (t+1)^{-2}$ (since $\varphi(N)$ is bounded) and hence is integrable.

By definition of N_1, the truncated roof function $\varphi(N)$ coincides with φ on the subsystem Z_0, so absence of approximate eigenfunctions passes over to the truncated flow for each $N \geqslant N_1$. Also $\varphi(N)$ is bounded. Hence the estimate for $\widehat{C}$ comes from Corollary 4.22 with $r = 0$.

It remains to consider the terms A_n and B_n. Starting from the end of the proof of Proposition 4.15, we obtain

$$|A_n(t)| \leqslant 4C_1C_2N\gamma_1^{n-1}|v|_\infty|w|_\gamma \textstyle\int_Y \varphi^\eta \circ F^n 1_{\{\varphi_{n+1}>t\}}\, d\mu.$$

Hence, by Proposition 3.4, $|A_n(t)| \ll n^{\beta+1-\eta}\gamma_1^n|v|_\infty|w|_\gamma\,(1+t)^{-(\beta-\eta)}$. Similarly $|B_{n,k}(t)| \ll n^{\beta+1}\gamma_1^n|v|_\gamma|w|_\infty\,(1+t)^{-(\beta-\eta)}$. Hence $\sum_{n\geqslant 1} A_n$ and $\sum_{0\leqslant k<n<\infty} B_{n,k}$ are integrable, completing the proof. □

Choose $\psi : \mathbb{R} \to [0,1]$ to be C^∞ and compactly supported such that $\psi \equiv 1$ on a neighbourhood of zero. Let $\kappa_m(b) = (1-\psi(b))(ib)^{-m}$, $m \geqslant 2$.

Corollary 5.11. *Let $N \geqslant N_1$, $m \geqslant \alpha + 2$, $v \in \mathcal{H}_\gamma(Y^{\varphi(N)})$, $w \in \mathcal{H}_{\gamma,0,m}(Y^{\varphi(N)})$. Then*

$$\rho^{\text{trunc}}_{v,w}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(b)e^{ibt}\,\widehat{\rho^{\text{trunc}}_{v,w}}(ib)db + \frac{1}{2\pi}\int_{-\infty}^{\infty} \kappa_m(b)e^{ibt}\,\widehat{\rho^{\text{trunc}}_{v,\partial_t^m w}}(ib)db.$$

Proof. We proceed as in the proof of [30, Corollary 6.1 and Lemma 6.2]. Recall our assumption that $\rho^{\text{trunc}}_{v,w}$ vanishes for t near zero, so that

$$\widehat{\rho^{\text{trunc}}_{v,w}}(s) = s^{-m}\,\widehat{\rho^{\text{trunc}}_{v,\partial_t^m w}}(s) \quad \text{for all } s \in \mathbb{H}. \tag{5.5}$$

Using the elementary estimate $|\widehat{\rho^{\text{trunc}}_{v,w}}(a+ib)| \leqslant 4a^{-1}|(|s|^2+1)^{-1}|v|_1(|w|_\infty + |\partial_t^2 w|_\infty)$ for $s = a+ib \in \mathbb{H}$, it follows from the classical inverse Laplace transform formula that

$$\rho^{\text{trunc}}_{v,w}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\epsilon'+ib)t}\,\widehat{\rho^{\text{trunc}}_{v,w}}(\epsilon'+ib)\,db$$

for all $\epsilon' > 0$. By (5.5) and Proposition 5.10, we have $\widehat{\rho^{\text{trunc}}_{v,w}}(\epsilon'+ib) \leqslant C(|b|+1)^{-2}$ for every $b \in \mathbb{R}$ and every $\epsilon' \in [0,\epsilon)$. Hence, by continuity of $\widehat{\rho^{\text{trunc}}_{v,w}}$ and dominated convergence,

$$\begin{aligned} 2\pi\rho^{\text{trunc}}_{v,w}(t) &= \int_{-\infty}^{\infty} e^{ibt}\,\widehat{\rho^{\text{trunc}}_{v,w}}(ib)db \\ &= \int_{-\infty}^{\infty} \psi(b)e^{ibt}\,\widehat{\rho^{\text{trunc}}_{v,w}}(ib)db + \int_{-\infty}^{\infty}(1-\psi(b))e^{ibt}\,\widehat{\rho^{\text{trunc}}_{v,w}}(ib)db. \end{aligned}$$

By Proposition 5.10, equation (5.5) extends to $\overline{\mathbb{H}}\setminus\{0\}$ and the result follows. □

From now on we suppress the superscript "trunc" for sake of readability. Notation $\widehat{R}$, $\widehat{T}$ and so on refers to the operators obtained using $\varphi(N)$ instead of φ. We end this subsection by recalling some further estimates from [30]. The first is a uniform version of Proposition 4.9.

Proposition 5.12 ([30, Proposition 8.27]). *Assume absence of approximate eigenfunctions. Then there exists $m \geqslant 2$ such that*

$$\|\kappa_m(b)\widehat{T}(ib)\|_\theta \in \mathcal{R}(t^{-q}) \quad \textit{uniformly in } N \geqslant N_1.$$

□

The remaining estimates in this subsection are required when b is close to zero. By Proposition 4.10, for each $N \geqslant 1$ there exists $\delta > 0$ such that

$$\widehat{R}(ib) = \lambda(b)P(b) + \widehat{R}(ib)Q(b) \quad \text{for } |b| < \delta,$$

where λ, P and $Q = I - P$ are C^∞ on $(-\delta,\delta)$ and $\lambda(0) = 1$, $\lambda'(0) = -i|\varphi(N)|_1$ and $P(0)v = \int_{\bar{Y}} v\,d\bar{\mu}$. In fact, as shown in [30, Section 8.5], $\delta > 0$ can be chosen uniformly in N. Moreover, $\|\widehat{R}^{(q)}(ib)\|_\theta$ is bounded uniformly in N on $(-\delta,\delta)$, so λ, P, Q are C^q uniformly in N on $(-\delta,\delta)$.

Define

$$\widetilde{P}(b) = b^{-1}(P(b) - P(0)), \qquad \tilde{\lambda} = b^{-1}(1-\lambda(b)).$$

Proposition 5.13. *There exists a constant $C > 0$, uniform in $N \geqslant 1$, such that*

$$|(\tilde{\lambda}^{-1})^{(q)}(ib)|,\ \|(\tilde{\lambda}^{-1}\widetilde{P})^{(q)}(ib)\|_\theta \leqslant C|b|^{-(1-\eta)} \quad \textit{for } |b| < \delta,$$

Proof. By [30, Proposition 8.18], $\|\widetilde{P}^{(q_1)}(b)\|_\theta \ll \begin{cases} |b|^{-(1-\eta)} & q_1 < \beta - 2\eta \\ 1 & q_1 < \beta - 1 \end{cases}$. The argument in the proof of [30, Proposition 8.26] gives the same estimates for $\tilde{\lambda}^{-1}$ completing the estimates for $\tilde{\lambda}^{-1}\widetilde{P}$. □

5.3. Proof of Lemma 5.9. Let ψ and κ_m be as in Corollary 5.11 with the extra property that supp $\psi \subset (-\delta, \delta)$. By Proposition 5.1,

$$\psi,\ \kappa_m \in \mathcal{R}(t^{-p}) \quad \text{for all } p > 0, m \geqslant 2. \tag{5.6}$$

By Corollary 5.11, we need to show that $\psi(b)\hat{\rho}_{v,w}(ib) \in \mathcal{R}(\|v\|_{\gamma,\eta}\|w\|_\gamma\, t^{-(\beta-1)})$ and $\kappa_m(b)\hat{\rho}_{v,w}(ib) \in \mathcal{R}(\|v\|_{\gamma,\eta}\|w\|_\gamma\, t^{-(\beta-1)})$ for all $v \in \mathcal{H}_{\gamma,\eta}(Y^{\varphi(N)})$, $w \in \mathcal{H}_\gamma(Y^{\varphi(N)})$, uniformly in $N \geqslant N_1$.

Let $\widehat{A} = \sum_{n=1}^\infty \widehat{A}_n$, $\widehat{B} = \sum_{n=1}^\infty \sum_{k=0}^{n-1} \widehat{B}_{n,k}$, $\widehat{C} = \sum_{j,k=0}^\infty \widehat{C}_{j,k}$. Recalling Remark 5.6, we refer to functions of the type $b \mapsto \psi(b)\widehat{J}_0(ib)$ as $\psi\widehat{J}_0$. By Lemma 4.12, it remains to show that each of the terms

$$\psi\widehat{J}_0,\quad \psi\widehat{A},\quad \psi\widehat{B},\quad \psi\widehat{C};\qquad \kappa_m\widehat{J}_0,\quad \kappa_m\widehat{A},\quad \kappa_m\widehat{B},\quad \kappa_m\widehat{C},$$

lies in $\mathcal{R}(\|v\|_{\gamma,\eta}\|w\|_\gamma\, t^{-(\beta-1)})$ uniformly in $N \geqslant 1$.

By Propositions 4.4, 4.15 and 4.16, $\widehat{J}_0,\ \widehat{A},\ \widehat{B} \in \mathcal{R}(\|v\|_\gamma\|w\|_\gamma\, t^{-(\beta-1)})$. (Estimates such as these that hold even before truncation are clearly independent of N.) By (5.6) and Proposition 5.2, uniformly in $N \geqslant 1$,

$$\psi\widehat{J}_0,\ \psi\widehat{A},\ \psi\widehat{B},\ \kappa_m\widehat{J}_0,\ \kappa_m\widehat{A},\ \kappa_m\widehat{B} \in \mathcal{R}(\|v\|_\gamma\|w\|_\gamma\, t^{-(\beta-1)}).$$

Hence it remains to estimate $\psi\widehat{C}$ and $\kappa_m\widehat{C}$. The next lemma provides the desired estimates and completes the proof of Lemma 5.9 (recall that $q > \beta - 1$).

Lemma 5.14. *Assume absence of approximate eigenfunctions. Fix N_1 as in the statement of Lemma 5.9. There exists $m \geqslant 2$ such that after truncation, uniformly in $N \geqslant N_1$,*

(a) $\kappa_m\widehat{C} \in \mathcal{R}(\|v\|_{\gamma,\eta}\|w\|_\gamma\, t^{-q})$, *and*
(b) $\psi\widehat{C} \in \mathcal{R}(\|v\|_{\gamma,\eta}\|w\|_\gamma\, t^{-(\beta-1)})$,

for all $t > 1$, $v \in \mathcal{H}_{\gamma,\eta}(Y^\varphi)$, $w \in \mathcal{H}_\gamma(Y^\varphi)$.

Proof. (a) Let $\ell = \max\{j - k - 1, 0\}$ and recall that

$$\widehat{C}_{j,k} = \textstyle\int_{\bar{Y}} D_{j,\ell}\overline{W}_k\, d\bar{\mu}, \qquad D_{j,\ell} = \widehat{R}^\ell\widehat{T}R^{j+1}\overline{V}_j.$$

By Proposition 5.12, we can choose $m \geqslant 2$ such that $\|\kappa_{m-5}\widehat{T}\|_\theta \in \mathcal{R}(t^{-q})$ uniformly in $N \geqslant N_1$. Write $\kappa_m = \kappa_3\kappa_{m-5}\kappa_2$, where κ_i is C^∞, vanishes in a neighborhood of zero, and is $O(|b|^{-i})$. Then

$$|\kappa_m D_{j,\ell}|_\infty \leqslant \|\kappa_3\widehat{R}^\ell\|_\theta\|\kappa_{m-5}\widehat{T}\|_\theta\|\kappa_2 R^{j+1}\overline{V}_j\|_\theta.$$

The estimates for $\widehat{R}^\ell$ and $R^{j+1}\overline{V}_j$ in Proposition 4.18 and Corollary 5.7 hold even before truncation and hence are uniform in $N \geqslant 1$. Using (5.6) and Propositions 5.1 and 5.2,

$$\|\kappa_3 \widehat{R}^\ell\|_\theta \in \mathcal{R}((\ell+1)^\beta t^{-q}), \quad \|\kappa_2 R^{j+1}\overline{V}_j\|_\theta \in \mathcal{R}\big(\gamma_1^{j/3}\|v\|_{\gamma,\eta}\, t^{-q}\big),$$

uniformly in $N \geqslant 1$. Since $q > 1$, it follows from Proposition 5.2 that uniformly in $N \geqslant N_1$,

$$|\kappa_m D_{j,\ell}|_\infty \in \mathcal{R}\big((\ell+1)^\beta t^{-q} \star t^{-q} \star \gamma_1^{j/3}\|v\|_{\gamma,\eta}\, t^{-q}\big) \in \mathcal{R}\big((\ell+1)^\beta \gamma_1^{j/3}\|v\|_{\gamma,\eta}\, t^{-q}\big).$$

Also, $|\overline{W}_k|_1 \in \mathcal{R}\big((k+1)^{\beta+1}\gamma_1^k\|w\|_\gamma\, t^{-q}\big)$ by Proposition 4.17 and this is uniform in $N \geqslant 1$. Applying Proposition 5.2 once more, uniformly in $N \geqslant N_1$,

$$\kappa_m \widehat{C}_{j,k} \in \mathcal{R}((j+1)^\beta \gamma_1^{j/3}(k+1)^{\beta+1}\gamma_1^k\|v\|_{\gamma,\eta}\|w\|_\gamma\, t^{-q}),$$

and part (a) follows.
(b) As in the proof of Proposition 4.21, we write

$$D_{j,\ell} = (1-\lambda)^{-1}\textstyle\int_Y \widehat{V}_j\, d\mu + Q_{j,\ell},$$

where

$$Q_{j,\ell} = \big(-(\lambda^{\ell-1} + \cdots + 1)P(0) + \lambda^\ell Q_2 + \widehat{R}^\ell Q_1\big)R^{j+1}\overline{V}_j.$$

Here, $Q_2 = (1-\lambda)^{-1}(P - P(0)) = \tilde{\lambda}^{-1}\widetilde{P}$.

By Proposition 4.19,

$$\widehat{C} = \textstyle\sum_{j,k}\int_Y D_{j,\ell}\overline{W}_k\, d\mu = \sum_{j,k}\int_Y Q_{j,\ell}\overline{W}_k\, d\mu + (1-\lambda)^{-1}I_0\sum_k\int_Y \overline{W}_k\, d\mu,$$

where $I_0(s) = \int_Y \int_0^{\varphi(y)} e^{-s(\varphi(y)-u)}v(y,u)\, du\, d\mu$.

Choose ψ_1 to be C^∞ with compact support such that $\psi_1 \equiv 1$ on supp ψ. By Proposition 5.13 we know that $\|Q_2^{(q)}(ib)\|_\theta \leqslant C|b|^{-(1-\eta)}$. Proposition 4.18 tells us that $\|(\widehat{R}^\ell)^{(q)}(ib)\|_\theta \leqslant C\ell^q(|b|+1)$ and, by standard perturbation theory, a similar estimate holds for $\lambda^\ell(b)$. Using also Corollary 5.7, this means that, uniformly in $N \geqslant 1$,

$$|\psi\lambda^\ell Q_2 R^{j+1}\overline{V}_j|_\infty \in \mathcal{R}\big((\ell+1)^\beta\gamma_1^{j/3}\|v\|_{\gamma,\eta}\, t^{-q}\big).$$

The other terms in $Q_{j,\ell}$ are simpler and we obtain that $|\psi Q_{j,\ell}|_\infty \in \mathcal{R}\big((\ell+1)^\beta\gamma_1^{j/3}\|v\|_{\gamma,\eta}\, t^{-q}\big)$. Hence by Proposition 4.17, uniformly in $N \geqslant 1$,

$$\psi\sum_{j,k}\int_Y Q_{j,\ell}\overline{W}_k\, d\mu \in \mathcal{R}(\|v\|_{\gamma,\eta}\|w\|_\gamma\, t^{-q}), \qquad \sum_k\int_Y \overline{W}_k\, d\mu \in \mathcal{R}\big(\|w\|_\gamma\, t^{-q}\big).$$

To complete the proof, it remains to estimate $\psi(1-\lambda)^{-1}I_0$. Recall from (4.4) that $I_0(0) = 0$, so $(1-\lambda)^{-1}I_0 = \tilde{\lambda}^{-1}\widehat{I}_1$ where

$$\widehat{I}_1(s) = s^{-1}(I_0(s) - I_0(0)) = s^{-1}\int_Y\int_0^{\varphi(y)}(e^{-s(\varphi(y)-u)} - 1)v(y,u)\, du\, d\mu,$$

with inverse Laplace transform $I_1(t) = -\int_Y\int_0^{\varphi(y)} 1_{\{\varphi(y)>t+u\}}v(y,u)\, du\, d\mu$. Proposition 3.4(b) implies that $|I_1(t)| \leqslant |v|_\infty\int_Y \varphi 1_{\{\varphi>t\}}\, d\mu \ll |v|_\infty\, t^{-(\beta-1)}$, uniformly in

$N \geqslant 1$, and by the arguments in the proof of Corollary 5.11 (see also Remark 5.6), $I_1(t)$ is the inverse Fourier transform of $\widehat{I_1}(ib)$. Hence, $\widehat{I_1} \in \mathcal{R}(|v|_\infty t^{-(\beta-1)})$. Combining this with Proposition 5.13, we obtain that

$$\psi(1-\lambda)^{-1}I_0 = \psi\tilde{\lambda}^{-1}\widehat{I_1} \in \mathcal{R}(t^{-q} \star |v|_\infty t^{-(\beta-1)}) \in \mathcal{R}(|v|_\infty t^{-(\beta-1)}),$$

uniformly in $N \geqslant 1$. □

6. General Gibbs–Markov Flows

In this section, we assume the setup from Sect. 3 but we drop the requirement that φ is constant along stable leaves.

In Sect. 6.1, we introduce a criterion, condition (H), that enables us to reduce to the skew product Gibbs–Markov flows studied in Sects. 3, 4 and 5. This leads to an enlarged class of Gibbs–Markov flows for which we can prove results on mixing rates (Theorem 6.4 below). Our strategy for proving Theorem 6.4 is that we introduce an auxiliary skew product Gibbs–Markov flow for which the results of the previous sections apply. Then we construct a measure-preserving semiconjugacy between the two suspension flows. This way we can relate the decay rates, for an appropriate class of observables on the Gibbs–Markov flow, to the decay rates of observables on the skew product Gibbs–Markov flow. Condition (H) plays a crucial role in our arguments. In Sect. 6.2, we recall criteria for absence of approximate eigenfunctions based on periodic data.

6.1. Condition (H). Let $F : Y \to Y$ be a map as in Sect. 3 with quotient Gibbs–Markov map $\bar{F} : \bar{Y} \to \bar{Y}$, and define $\widetilde{Y}_j = Y_j \cap \widetilde{Y}$. Let $\varphi : Y \to \mathbb{R}^+$ be an integrable roof function with $\inf \varphi > 1$ and associated suspension flow $F_t : Y^\varphi \to Y^\varphi$.

We no longer assume that φ is constant along stable leaves. Instead of condition (3.3) we require that

$$|\varphi(y) - \varphi(y')| \leqslant C_1 \inf_{Y_j}\varphi\, \gamma^{s(y,y')} \quad \text{for all } y, y' \in \widetilde{Y}_j,\ j \geqslant 1. \tag{6.1}$$

(Clearly, if φ is constant along stable leaves, then conditions (3.3) and (6.1) are identical.)

Recall that $\pi : Y \to \widetilde{Y}$ is the projection along stable leaves. Define

$$\chi(y) = \textstyle\sum_{n=0}^{\infty}(\varphi(F^n\pi y) - \varphi(F^n y)),$$

for all $y \in Y$ such that the series converges absolutely. We assume

(H) (a) The series converges almost surely on Y and $\chi \in L^\infty(Y)$.
(b) There are constants $C_3 \geqslant 1$, $\gamma \in (0,1)$ such that

$$|\chi(y) - \chi(y')| \leqslant C_3(d(y,y') + \gamma^{s(y,y')}) \quad \text{for all } y, y' \in Y.$$

When conditions (6.1) and (H) are satisfied, we call F_t a *Gibbs–Markov flow*. (If φ is constant along stable leaves then $\chi = 0$, so every skew product Gibbs–Markov flow is a Gibbs–Markov flow.)

Since $\inf \varphi > 0$, it follows that $\varphi_n = \sum_{j=0}^{n-1}\varphi \circ F^j \geqslant 4|\chi|_\infty + 1$ for all n sufficiently large. For simplicity we suppose from now on that $\inf \varphi \geqslant 4|\chi|_\infty + 1$ (otherwise, replace F by F^n).

Define

$$\tilde{\varphi} = \varphi + \chi - \chi \circ F. \tag{6.2}$$

Note that $\inf \tilde\varphi \geqslant \inf \varphi - 2|\chi|_\infty \geqslant 1$ and $\int_Y \tilde\varphi\, d\mu = \int_Y \varphi\, d\mu$, so $\tilde\varphi : Y \to \mathbb{R}^+$ is an integrable roof function. Hence we can define the suspension flow $\widetilde{F}_t : Y^{\tilde\varphi} \to Y^{\tilde\varphi}$. Also, a calculation shows that $\tilde\varphi(y) = \sum_{n=0}^{\infty}(\varphi(F^n\pi y) - \varphi(F^n\pi Fy))$, so $\tilde\varphi$ is constant along stable leaves and we can define the quotient roof function $\bar\varphi : \overline{Y} \to \mathbb{R}^+$ with quotient semiflow $\overline{F}_t : \overline{Y}^{\tilde\varphi} \to \overline{Y}^{\tilde\varphi}$.

In the remainder of this section, we prove that $\widetilde{F}_t$ is a skew product Gibbs–Markov flow (and hence $\overline{F}_t$ is a Gibbs–Markov semiflow), and show that (super)polynomial decay of correlations for $\widetilde{F}_t$ is inherited by F_t.

Proposition 6.1. *Let $F_t : Y^\varphi \to Y^\varphi$ be a Gibbs–Markov flow. Then $\widetilde{F}_t : Y^{\tilde\varphi} \to Y^{\tilde\varphi}$ is a skew product Gibbs–Markov flow.*

Proof. We verify that the setup in Sect. 3 holds. All the conditions on the map $F : Y \to Y$ are satisfied by assumption. Hence it suffices to check that $\tilde\varphi$ satisfies condition (3.3).

Let $y, y' \in \widetilde{Y}_j$ for some $j \geqslant 1$. By (3.2), $d(y, y') \leqslant C_2\gamma^{s(y,y')}$ and $d(Fy, Fy') \leqslant C_2\gamma^{s(y,y')-1}$. By (H)(b), $|\chi(y) - \chi(y')| \leqslant 2C_2C_3\gamma^{s(y,y')}$ and $|\chi(Fy) - \chi(Fy')| \leqslant 2C_2C_3\gamma^{s(y,y')-1}$. Hence by (6.1) and (6.2),

$$|\tilde\varphi(y) - \tilde\varphi(y')| \leqslant |\varphi(y) - \varphi(y')| + |\chi(y) - \chi(y')| + |\chi(Fy) - \chi(Fy')| \ll \inf_{Y_j}\varphi\, \gamma^{s(y,y')}.$$

Also, $\inf_{Y_j}\varphi \leqslant \inf_{Y_j}\tilde\varphi + 2|\chi|_\infty \leqslant \inf_{Y_j}\tilde\varphi + \frac{1}{2}\inf \varphi \leqslant \inf_{Y_j}\tilde\varphi + \frac{1}{2}\inf_{Y_j}\varphi$. Hence $\inf_{Y_j}\varphi \leqslant 2\inf_{Y_j}\tilde\varphi$ and $|\tilde\varphi(y) - \tilde\varphi(y')| \ll \inf_{Y_j}\tilde\varphi\, \gamma^{s(y,y')}$ as required. □

Corollary 6.2. *There is a constant $C > 0$ such that*

$$|\varphi(y) - \varphi(y')| \leqslant C\inf_{Y_j}\varphi\{d(y, y') + d(Fy, Fy') + \gamma^{s(y,y')}\} \quad \textit{for all } y, y' \in Y_j,\ j \geqslant 1.$$

Proof. Let $\tilde y = \widetilde{Y} \cap W^s(y)$, $\tilde y' = \widetilde{Y} \cap W^s(y')$. Since $\tilde\varphi$ is constant along stable leaves, it follows as in the proof of Proposition 6.1 that

$$|\tilde\varphi(y) - \tilde\varphi(y')| = |\tilde\varphi(\tilde y) - \tilde\varphi(\tilde y')| \ll \inf_{Y_j}\varphi\, \gamma^{s(\tilde y,\tilde y')} = \inf_{Y_j}\varphi\, \gamma^{s(y,y')}.$$

Hence by (6.2) and (H)(b)

$$\begin{aligned}|\varphi(y) - \varphi(y')| &\leqslant |\tilde\varphi(y) - \tilde\varphi(y')| + |\chi(Fy) - \chi(Fy')| + |\chi(y) - \chi(y')| \\ &\ll \inf_{Y_j}\varphi\{\gamma^{s(y,y')} + d(Fy, Fy') + \gamma^{s(Fy,Fy')} + d(y, y')\}.\end{aligned}$$

The result follows since $\gamma^{s(Fy,Fy')} = \gamma^{-1}\gamma^{s(y,y')}$. □

Next, we relate the two suspension flows $F_t : Y^\varphi \to Y^\varphi$ and $\widetilde{F}_t : Y^{\tilde\varphi} \to Y^{\tilde\varphi}$. Note that $(y, \varphi(y))$ is identified with $(Fy, 0)$ in the first flow and $(y, \tilde\varphi(y))$ is identified with $(Fy, 0)$ in the second flow. Define

$$\begin{aligned} &g_+ : Y^\varphi \to Y^{\tilde\varphi}, \qquad g_+(y, u) = (y, u + \chi(y) + |\chi|_\infty), \\ &g_- : Y^{\tilde\varphi} \to Y^\varphi, \qquad g_-(y, u) = (y, u - \chi(y) + |\chi|_\infty), \end{aligned}$$

computed modulo identifications. Using (6.2) and the identifications on $Y^{\tilde\varphi}$,

$$\begin{aligned} g_+(y, \varphi(y)) = (y, \varphi(y) + \chi(y) + |\chi|_\infty) &= (y, \tilde\varphi(y) + \chi(Fy) + |\chi|_\infty) \\ &\sim (Fy, \chi(Fy) + |\chi|_\infty) = g_+(Fy, 0), \end{aligned}$$

so g_+ respects the identification on Y^φ and hence is well-defined. It follows easily that $g_+ : Y^\varphi \to Y^{\tilde\varphi}$ is a measure-preserving semiconjugacy between the two suspension flows. Similarly, g_- is well-defined and $g_- \circ g_+ = F_{2|\chi|_\infty} : Y^\varphi \to Y^\varphi$.

Given observables $v, w : Y^\varphi \to \mathbb{R}$, let $\tilde v = v \circ g_-$, $\tilde w = w \circ g_- : Y^{\tilde\varphi} \to \mathbb{R}$. When speaking of $\mathcal{H}_\gamma(Y^{\tilde\varphi})$ and so on, we use the metric $d_1(y, y') = d(y, y')^\eta$ on Y instead of d. Let $\gamma_1 = \gamma^\eta$.

Let $\mathcal{H}^*_{\gamma,\eta}(Y^\varphi) = \{v : Y^\varphi \to \mathbb{R} : \|v\|^*_{\gamma,\eta} < \infty\}$ and $\mathcal{H}^*_{\gamma,0,m}(Y^\varphi) = \{w : Y^\varphi \to \mathbb{R} : \|w\|^*_{\gamma,0,m} < \infty\}$ where

$$\|v\|^*_{\gamma,\eta} = \|v\|_{\gamma,\eta} + \|v \circ F_{2|\chi|_\infty}\|_{\gamma,\eta}, \qquad \|w\|^*_{\gamma,0,m} = \|w\|_{\gamma,0,m} + \|w \circ F_{2|\chi|_\infty}\|_{\gamma,0,m}.$$

Lemma 6.3. *Let $v \in \mathcal{H}^*_{\gamma,\eta}(Y^\varphi)$, $w \in \mathcal{H}^*_{\gamma,0,m}(Y^\varphi)$, for some $m \geqslant 1$. Then $\tilde v \in \mathcal{H}_{\gamma_1,\eta}(Y^{\tilde\varphi})$, $\tilde w \in \mathcal{H}_{\gamma_1,0,m}(Y^{\tilde\varphi})$, and $\|\tilde v\|_{\gamma_1,\eta} \leqslant 4C_3\|v\|^*_{\gamma,\eta}$, $\|\tilde w\|_{\gamma_1,0,m} \leqslant 2C_3\|w\|^*_{\gamma,0,m}$.*

Proof. We have $\tilde v(y, u) = v(y, u - \chi(y) + |\chi|_\infty)$. It is immediate that $|\tilde v|_\infty \leqslant |v|_\infty$.

Now let (y, u), $(y', u) \in Y^{\tilde\varphi}$. Suppose without loss that $\chi(y) \geqslant \chi(y')$. First, we consider the case $u - \chi(y) + |\chi|_\infty \leqslant \varphi(y)$, $u - \chi(y') + |\chi|_\infty \leqslant \varphi(y')$. By (H)(b) and the definition of $\|v\|_{\gamma,\eta}$,

$$\begin{aligned}|\tilde v(y,u) - \tilde v(y',u)| &\leqslant \big|v(y, u - \chi(y) + |\chi|_\infty) - v(y', u - \chi(y) + |\chi|_\infty)\big| \\ &\quad + \big|v(y', u - \chi(y) + |\chi|_\infty) - v(y', u - \chi(y') + |\chi|_\infty)\big| \\ &\leqslant |v|_\gamma \varphi(y)(d(y,y') + \gamma^{s(y,y')}) + |v|_{\infty,\eta}|\chi(y) - \chi(y')|^\eta \\ &\leqslant 2|v|_\gamma \tilde\varphi(y)(d(y,y') + \gamma^{s(y,y')}) + |v|_{\infty,\eta} C_3 (d(y,y') + \gamma^{s(y,y')})^\eta \\ &\leqslant 2C_3\|v\|_{\gamma,\eta}\tilde\varphi(y)(d_1(y,y') + \gamma_1^{s(y,y')}).\end{aligned}$$

Second, we consider the case $u \geqslant \chi(y) + |\chi|_\infty \geqslant \chi(y') + |\chi|_\infty$. Then we can write $g_-(y, u) = F_\sigma(y, u - \chi(y) - |\chi|_\infty)$, $g_-(y', u) = F_\sigma(y', u - \chi(y') - |\chi|_\infty)$ where $\sigma = 2|\chi|_\infty$, so

$$|\tilde v(y,u) - \tilde v(y',u)| = \big|v \circ F_\sigma(y, u - \chi(y) - |\chi|_\infty) - v \circ F_\sigma(y', u - \chi(y') - |\chi|_\infty)\big|.$$

Proceeding as in the first case,

$$|\tilde v(y,u) - \tilde v(y',u)| \leqslant 2C_3\|v \circ F_\sigma\|_{\gamma,\eta}\tilde\varphi(y)(d_1(y,y') + \gamma_1^{s(y,y')}).$$

This leaves the case $u < \chi(y) + |\chi|_\infty \leqslant 2|\chi|_\infty$ and $u \geqslant \min\{\varphi(y) + \chi(y) - |\chi|_\infty, \varphi(y') + \chi(y') - |\chi|_\infty\} \geqslant \inf\varphi - 2|\chi|_\infty$. This is impossible since $\inf\varphi > 4|\chi|_\infty$. Hence

$$|\tilde v(y,u) - \tilde v(y',u)| \leqslant 2C_3\|v\|^*_{\gamma,\eta}\tilde\varphi(y)(d_1(y,y') + \gamma_1^{s(y,y')}) \quad \text{for all } (y,u), (y',u) \in Y^{\tilde\varphi},$$

so $\|\tilde v\|_{\gamma_1} \leqslant 2C_3\|v\|^*_{\gamma,\eta}$.

The estimate for $|\tilde v|_{\infty,\eta}$ splits into cases similarly. Let $0 \leqslant u < u' \leqslant \tilde\varphi(y)$. Then

$$|\tilde v(y,u) - \tilde v(y,u')| \leqslant \begin{cases} |v|_{\infty,\eta}|u - u'|^\eta & u' - \chi(y) + |\chi|_\infty \leqslant \varphi(y) \\ |v \circ F_\sigma|_{\infty,\eta}|u - u'|^\eta & u \geqslant \chi(y) + |\chi|_\infty \end{cases}.$$

This leaves the case $u' - \chi(y) + |\chi|_\infty > \varphi(y)$ and $u < \chi(y) + |\chi|_\infty$. But then $u' - u > \varphi(y) + 2\chi(y) > \varphi(y) - 2|\chi|_\infty > \frac{1}{2}\varphi(y) \geqslant \frac{1}{2}$, so we obtain $|\tilde v(y,u) - \tilde v(y,u')| \leqslant 2|v|_\infty \leqslant 4|v|_\infty|u - u'|^\eta$. Hence $|\tilde v|_{\infty,\eta} \leqslant 4\|v\|^*_{\gamma,\eta}$ completing the estimate for $\|\tilde v\|_{\gamma_1,\eta}$. The calculation for $\tilde w$ is similar. □

We say that a Gibbs–Markov flow has approximate eigenfunctions if this is the case for $\widetilde{F}_t$ (equivalently $\overline{F}_t$).

Theorem 6.4. *Suppose that $F_t : Y^\varphi \to Y^\varphi$ is a Gibbs–Markov flow such that $\mu(\varphi > t) = O(t^{-\beta})$ for some $\beta > 1$. Assume absence of approximate eigenfunctions. Then there exists $m \geqslant 1$ and $C > 0$ such that*

$$|\rho_{v,w}(t)| \leqslant C\|v\|^*_{\gamma,\eta}\|w\|^*_{\gamma,0,m}\, t^{-(\beta-1)} \quad \textit{for all } v \in \mathcal{H}^*_{\gamma,\eta}(Y^\varphi),\ w \in \mathcal{H}^*_{\gamma,0,m}(Y^\varphi),\ t > 1.$$

Proof. Since g_+ is a measure-preserving semiconjugacy and $g_- \circ g_+ = F_{2|\chi|_\infty}$,

$$\begin{aligned}\int_{Y^\varphi} v\, w \circ F_t\, d\mu^\varphi &= \int_{Y^\varphi} v \circ g_- \circ g_+\, w \circ g_- \circ g_+ \circ F_t\, d\mu^\varphi \\ &= \int_{Y^\varphi} \tilde{v} \circ g_+\, \tilde{w} \circ \widetilde{F}_t \circ g_+\, d\mu^\varphi = \int_{Y^{\tilde{\varphi}}} \tilde{v}\, \tilde{w} \circ \widetilde{F}_t\, d\mu^{\tilde{\varphi}}\end{aligned}$$

where $\widetilde{F}_t$ does not possess approximate eigenfunctions. Note also that $\mu(\tilde{\varphi} > t) = O(t^{-\beta})$. By Lemma 6.3, $\tilde{v} \in \mathcal{H}_{\gamma_1,\eta}(Y^{\tilde{\varphi}})$, $\tilde{w} \in \mathcal{H}_{\gamma_1,0,m}(Y^{\tilde{\varphi}})$.

By Theorem 3.2, we can choose $m \geqslant 1$ such that $|\rho_{v,w}(t)| = |\int_{Y^{\tilde{\varphi}}} \tilde{v}\, \tilde{w} \circ \widetilde{F}_t\, d\mu^{\tilde{\varphi}} - \int_{Y^{\tilde{\varphi}}} \tilde{v}\, d\mu^{\tilde{\varphi}} \int_{Y^{\tilde{\varphi}}} \tilde{w}\, d\mu^{\tilde{\varphi}}| \ll \|\tilde{v}\|_{\gamma_1,\eta}\|\tilde{w}\|_{\gamma_1,0,m}\, t^{-(\beta-1)} \leqslant 8C_3^2\|v\|^*_{\gamma,\eta}\|w\|^*_{\gamma,0,m}\, t^{-(\beta-1)}$. □

6.2. Periodic data and absence of approximate eigenfunctions. In this subsection, we recall the relationship between periodic data and approximate eigenfunctions and review two sufficient conditions to rule out the existence of approximate eigenfunctions. We continue to assume that F_t is a Gibbs–Markov flow as in Sect. 6.1.

Define $\varphi_n = \sum_{j=0}^{n-1} \varphi \circ F^j$. Similarly, define $\tilde{\varphi}_n$ and $\bar{\varphi}_n$. If y is a periodic point of period p for F (that is, $F^p y = y$), then y is periodic of period $\mathcal{L} = \varphi_p(y)$ for F_t (that is, $F_{\mathcal{L}} y = y$). Recall that $\bar{\pi} : Y \to \overline{Y}$ is the quotient projection.

Proposition 6.5. *Suppose that there exist approximate eigenfunctions on $Z_0 \subset \overline{Y}$. Let α, C, b_k, n_k be as in Definition 2.3. If $y \in \bar{\pi}^{-1} Z_0$ is a periodic point with $F^p y = y$ and $F_{\mathcal{L}} y = y$ where $\mathcal{L} = \varphi_p(y)$, then*

$$\operatorname{dist}(b_k n_k \mathcal{L} - p\psi_k, 2\pi\mathbb{Z}) \leqslant C(\inf \varphi)^{-1}\mathcal{L}|b_k|^{-\alpha} \quad \textit{for all } k \geqslant 1. \tag{6.3}$$

Proof. Define $\bar{y} = \bar{\pi} y \in Z_0$ and note that $\overline{F}^p \bar{y} = \overline{F}^p \bar{\pi} y = \bar{\pi} F^p y = \bar{y}$. By (6.2),

$$\bar{\varphi}_p(\bar{y}) = \tilde{\varphi}_p(y) = \varphi_p(y) + \chi(y) - \chi(F^p y) = \mathcal{L}.$$

Now $(M_b^p v)(\bar{y}) = e^{ib\bar{\varphi}_p(\bar{y})} v(\overline{F}^p \bar{y}) = e^{ib\mathcal{L}} v(\bar{y})$. Hence substituting $\bar{y}$ into (2.2), we obtain $|e^{ib_k n_k \mathcal{L}} - e^{ip\psi_k}| \leqslant Cp|b_k|^{-\alpha}$. Also $\mathcal{L} = \varphi_p(y) \geqslant p \inf \varphi$. □

The following Diophantine condition is based on [22, Section 13]. (Unlike in [22], we have to consider periods corresponding to three periodic points instead of two.)

Proposition 6.6. *Let $y_1, y_2, y_3 \in \bigcup Y_j$ be fixed points for F, and let $\mathcal{L}_i = \varphi(y_i)$, $i = 1, 2, 3$, be the corresponding periods for F_t. Let $Z_0 \subset \overline{Y}$ be the finite subsystem corresponding to the three partition elements containing $\bar{\pi} y_1, \bar{\pi} y_2, \bar{\pi} y_3$.*

If $(\mathcal{L}_1 - \mathcal{L}_3)/(\mathcal{L}_2 - \mathcal{L}_3)$ is Diophantine, then there do not exist approximate eigenfunctions on Z_0.

Proof. Using Proposition 6.5, the proof is identical to that of [30, Proposition 5.3]. □

The condition in Proposition 6.6 is satisfied with probability one but is not robust. Using the notion of *good asymptotics* [23], we obtain an open and dense condition.

Proposition 6.7. *Let $Z_0 \subset \overline{Y}$ be a finite subsystem. Let $y_0 \in \bar{\pi}^{-1}Z_0$ be a fixed point for F with period $\mathcal{L}_0 = \varphi(y_0)$ for the flow. Let $y_N \in \bar{\pi}^{-1}Z_0$, $N \geqslant 1$, be a sequence of periodic points with $F^N y_N = y_N$ such that their periods $\mathcal{L}_N = \varphi_N(y_N)$ for the flow F_t satisfy*

$$\mathcal{L}_N = N\mathcal{L}_0 + \kappa + E_N\gamma^N \cos(N\omega + \omega_N) + o(\gamma^N),$$

where $\kappa \in \mathbb{R}$, $\gamma \in (0,1)$ are constants, $E_N \in \mathbb{R}$ is a bounded sequence with $\liminf_{N\to\infty} |E_N| > 0$*, and either (i) $\omega = 0$ and $\omega_N \equiv 0$, or (ii) $\omega \in (0,\pi)$ and $\omega_N \in (\omega_0 - \pi/12, \omega_0 + \pi/12)$ for some ω_0. (Such a sequence of periodic points is said to have* good asymptotics.*)*

Then there do not exist approximate eigenfunctions on Z_0.

Proof. Using Proposition 6.5, the proof is identical to that of [30, Proposition 5.5]. □

By [23], for any finite subsystem Z_0, the existence of periodic points with good asymptotics in $\bar{\pi}^{-1}Z_0$ is a C^2-open and C^∞-dense condition. Although [23] is set in the uniformly hyperbolic setting, the construction applies directly to the current set up as we now explain. Assume that (Y, d) is a Riemannian manifold. Let $\bar{Z}_1$ and $\bar{Z}_2$ be two of the partition elements in Z and set $Z_j = \operatorname{Int} \bar{\pi}^{-1}\bar{Z}_j$ for $j = 1, 2$. Assume that Z_1, Z_2 are submanifolds of Y and that F and φ are C^r when restricted to $Z_1 \cup Z_2$ for some $r \geqslant 2$.

Let $y_0 \in Z_1$ be a fixed point for F and choose a transverse homoclinic point in Z_2. Following [23], we construct a sequence of N-periodic points y_N, $N \geqslant 1$, for F with orbits lying in $Z_1 \cup Z_2$. The sequence automatically has good asymptotics except that in exceptional cases it may be that $\liminf_{N\to\infty} |E_N| = 0$. By [23], the liminf is positive for a C^2 open and C^r dense set of roof functions φ.

Combining this construction with Proposition 6.7, it follows that nonexistence of approximate eigenfunctions holds for an open and dense set of smooth Gibbs–Markov flows.

Part II
Mixing rates for nonuniformly hyperbolic flows

In this part of the paper, we show how the results for suspension flows in Part I can be translated into results for nonuniformly hyperbolic flows defined on an ambient manifold. In Sect. 7, we show how this is done under the assumption that condition (H) from Sect. 6 is valid. In Sect. 8, we describe a number of situations where condition (H) is satisfied. This includes all the examples considered here and in [30]. In Sect. 9, we consider in detail the planar infinite horizon Lorentz gas.

7. Nonuniformly Hyperbolic Flows and Suspension Flows

In this section, we describe a class of nonuniformly hyperbolic flows $T_t : M \to M$ that have most of the properties required for T_t to be modelled by a Gibbs–Markov flow. (The remaining property, condition (H) from Sect. 6, is considered in Sect. 8.)

In Sect. 7.1, we consider a class of nonuniformly hyperbolic transformations $f : X \to X$ modelled by a Young tower [36,37], making explicit the conditions from [36] that are needed for this paper. In Sect. 7.2, we consider flows that are Hölder suspensions over such a map f and show how to model them, subject to condition (H), by a Gibbs–Markov flow. In Sect. 7.3, we generalise the Hölder structures in Sect. 7.2 to ones that are dynamically Hölder.

In applications, f is typically a first-hit Poincaré map for the flow T_t and hence is invertible. Invertibility is used in Proposition A.1 but not elsewhere, so many of our results do not rely on injectivity of f.

7.1. Nonuniformly hyperbolic transformations $f : X \to X$. Let $f : X \to X$ be a measurable transformation defined on a metric space (X, d) with $\operatorname{diam} X \leqslant 1$. We suppose that f is nonuniformly hyperbolic in the sense that it is modelled by a Young tower [36,37]. We recall the metric parts of the theory; the differential geometry part leading to an SRB or physical measure does not play an important role here.

Product structure Let Y be a measurable subset of X. Let $\mathcal{W}^s$ be a collection of disjoint measurable subsets of X (called "stable leaves") and let $\mathcal{W}^u$ be a collection of disjoint measurable subsets of X (called "unstable leaves") such that each collection covers Y. Given $y \in Y$, let $W^s(y)$ and $W^u(y)$ denote the stable and unstable leaves containing y.

We assume that for all $y, y' \in Y$, the intersection $W^s(y) \cap W^u(y')$ consists of precisely one point, denoted $z = W^s(y) \cap W^u(y')$, and that $z \in Y$. Also we suppose there is a constant $C_4 \geqslant 1$ such that

$$d(y, z) \leqslant C_4 d(y, y') \quad \text{for all } y, y' \in Y,\ z = W^s(y) \cap W^u(y'). \tag{7.1}$$

Induced map Next, let $\{Y_j\}$ be an at most countable measurable partition of Y such that $Y_j = \bigcup_{y \in Y_j} W^s(y) \cap Y$ for all $j \geqslant 1$. Also, fix $\tau : Y \to \mathbb{Z}^+$ constant on partition elements such that $f^{\tau(y)} y \in Y$ for all $y \in Y$. Define $F : Y \to Y$ by $Fy = f^{\tau(y)} y$. Let μ be an ergodic F-invariant probability measure on Y and suppose that τ is integrable. (It is not assumed that τ is the first return time to Y.)

As in Sect. 3, we suppose that $F(W^s(y)) \subset W^s(Fy)$ for all $y \in Y$. Let $\bar{Y}$ denote the space obtained from Y after quotienting by $\mathcal{W}^s$, with natural projection $\bar{\pi} : Y \to \bar{Y}$. We assume that the quotient map $\bar{F} : \bar{Y} \to \bar{Y}$ is a Gibbs–Markov map as in Definition 2.1, with partition $\{\bar{Y}_j\}$ and ergodic invariant probability measure $\bar{\mu} = \bar{\pi}_* \mu$. Let $s(y, y')$ denote the separation time on $\bar{Y}$.

Contraction/expansion Let $Y_j = \bar{\pi}^{-1} \bar{Y}_j$; these form a partition of Y and each Y_j is a union of stable leaves. The separation time extends to Y, setting $s(y, y') = s(\bar{\pi} y, \bar{\pi} y')$ for $y, y' \in Y$.

We assume that there are constants $C_2 \geqslant 1$, $\gamma \in (0, 1)$ such that for all $n \geqslant 0$, $y, y' \in Y$,

$$d(f^n y, f^n y') \leqslant C_2 \gamma^{\psi_n(y)} d(y, y') \quad \text{for all } y' \in W^s(y), \tag{7.2}$$

$$d(f^n y, f^n y') \leqslant C_2 \gamma^{s(y,y') - \psi_n(y)} \quad \text{for all } y' \in W^u(y), \tag{7.3}$$

where $\psi_n(y) = \#\{j = 1, \ldots, n : f^j y \in Y\}$ is the number of returns of y to Y by time n. Note that conditions (3.1) and (3.2) are special cases of (7.2) and (7.3) where $\tilde{Y}$ can be chosen to be any fixed unstable leaf. In particular, all the conditions on F in Sects. 3 and 6 are satisfied.

In Sects. 7.3, 8.4 and 9, we make use of the condition

$$F(W^u(y) \cap Y_j) = W^u(Fy) \cap Y \quad \text{for all } y \in Y_j,\ j \geqslant 1. \tag{7.4}$$

Remark 7.1. Further hypotheses in [36] ensure the existence of SRB measures on $\bar{Y}$, Y and X. These assumptions are not required here and no special properties of μ and $\bar{\mu}$ (other than the properties mentioned above) are used.

Remark 7.2. The abstract setup in [36] essentially satisfies all of the assumptions above. However condition (7.2) is stated in the slightly weaker form $d(f^n y, f^n y') \leqslant C_2\gamma^{\psi_n(y)}$. As pointed out in [20], the stronger form (7.2) is satisfied in all known examples where the weaker form holds.

Condition (7.4) is not stated explicitly in [36] but is an automatic consequence of the set up therein provided $f : X \to X$ is injective. We provide the details in Proposition A.1. In the examples considered in this paper and in [30], the map f is a first return map for a flow and hence is injective, so condition (7.4) is not very restrictive.

Condition (7.4) is also used in [30, Section 5.2] but is stated there in a slightly different form. In [30], the subspace X is not needed (and hence not mentioned) and the stable and unstable disks $W^s(y)$, $W^u(y)$ are replaced by their intersections with Y. Hence the condition $F(W^u(y) \cap Y_j) \supset W^u(Fy)$ for $y \in Y_j$ in [30, Section 5.2] becomes $F(W^u(y) \cap Y_j) \supset W^u(Fy) \cap Y$ for $y \in Y_j$ in our present notation and hence holds by (7.4).

Proposition 7.3. *$d(f^n y, f^n y') \leqslant C_2C_4(\gamma^{\psi_n(y)}d(y, y') + \gamma^{s(y,y')-\psi_n(y)})$ for all $y, y' \in Y$, $n \geqslant 0$.*

Proof. Let $z = W^s(y) \cap W^u(y')$. Note that $s(z, y') = s(y, y')$ and $\psi_n(z) = \psi_n(y)$. Hence

$$\begin{aligned} d(f^n y, f^n y') \leqslant d(f^n y, f^n z) + d(f^n z, f^n y') &\leqslant C_2(\gamma^{\psi_n(y)}d(y, z) + \gamma^{s(z,y')-\psi_n(z)}) \\ &\leqslant C_2C_4(\gamma^{\psi_n(y)}d(y, y') + \gamma^{s(y,y')-\psi_n(y)}), \end{aligned}$$

as required. □

7.2. Hölder flows and observables. Let $T_t : M \to M$ be a flow defined on a metric space (M, d) with diam $M \leqslant 1$. Fix $\eta \in (0, 1]$.

Given $v : M \to \mathbb{R}$, define $|v|_{C^\eta} = \sup_{x\neq x'} |v(x) - v(x')|/d(x, x')^\eta$ and $\|v\|_{C^\eta} = |v|_\infty + |v|_{C^\eta}$. Let $C^\eta(M) = \{v : M \to \mathbb{R} : \|v\|_{C^\eta} < \infty\}$. Also, define $|v|_{C^{0,\eta}} = \sup_{x\in M,\, t>0} |v(T_t x) - v(x)|/t^\eta$ and let $C^{0,\eta}(M) = \{v : M \to \mathbb{R} : |v|_\infty + |v|_{C^{0,\eta}} < \infty\}$. (Such observables are Hölder in the flow direction.)

We say that $w : M \to \mathbb{R}$ is *differentiable in the flow direction* if the limit $\partial_t w = \lim_{t\to 0}(w \circ T_t - w)/t$ exists pointwise. Define $\|w\|_{C^{\eta,m}} = \sum_{j=0}^{m} \|\partial_t^j w\|_{C^\eta}$ and let $C^{\eta,m}(M) = \{w : M \to \mathbb{R} : \|w\|_{C^{\eta,m}} < \infty\}$.

Let $X \subset M$ be a Borel subset and define $C^\eta(X)$ using the metric d restricted to X. We suppose that $T_{h(x)}x \in X$ for all $x \in X$, where $h : X \to \mathbb{R}^+$ lies in $C^\eta(X)$ and $\inf h > 0$. In addition, we suppose that for any $D_1 > 0$ there exists $D_2 > 0$ such that

$$d(T_t x, T_t x') \leqslant D_2 d(x, x')^\eta \quad \text{for all } t \in [0, D_1],\, x, x' \in M. \tag{7.5}$$

Define $f : X \to X$ by $fx = T_{h(x)}x$. We suppose that f is a nonuniformly hyperbolic transformation as in Sect. 7.1, with induced map $F = f^\tau : Y \to Y$ and so on.

Define $h_\ell = \sum_{j=0}^{\ell-1} h \circ f^j$. We define the induced roof function

$$\varphi = h_\tau : Y \to \mathbb{R}^+, \qquad \varphi(y) = \sum_{\ell=0}^{\tau(y)-1} h(f^\ell y).$$

Note that $h \leqslant \varphi \leqslant |h|_\infty \tau$ so $\varphi \in L^1(Y)$ and $\inf \varphi > 0$. Define the suspension flow $F_t : Y^\varphi \to Y^\varphi$ as in (1.1).

To deduce rates of mixing for nonuniformly hyperbolic flows from the corresponding result for Gibbs–Markov flows, Theorem 6.4, we need to verify that

(i) Condition (6.1) holds.
(ii) Condition (H) from Sect. 6 holds.
(iii) Regular observables on M lift to regular observables on Y^φ.

Ingredients (i) and (ii) guarantee that the suspension flow $F_t : Y^\varphi \to Y^\varphi$ is a Gibbs–Markov flow and ingredient (iii) ensures that Theorem 6.4 applies to the appropriate observables on M.

In the remainder of this subsection, we deal with ingredients (i) and (iii). First, we verify that φ satisfies condition (6.1). Let $d_1(y, y') = d(y, y')^\eta$ and $\gamma_1 = \gamma^\eta$.

Proposition 7.4. *Let $y, y' \in Y_j$ for some $j \geqslant 1$ and let $\ell = 0, \dots, \tau(y) - 1$. Then*

$$|h_\ell(y) - h_\ell(y')| \leqslant C_2 C_4 |h|_\eta\, \ell (d_1(y, y') + \gamma_1^{s(y,y')}).$$

Moreover,

$$|\varphi(y) - \varphi(y')| \leqslant 2C_2^2 C_4 (\inf h)^{-1} |h|_\eta \inf_{Y_j} \varphi\, \gamma_1^{s(y,y')} \quad \textit{for all } y, y' \in \widetilde{Y}_j,\ j \geqslant 1.$$

Proof. Note that $\psi_\ell(y) = 0$, so by Proposition 7.3,

$$d(f^\ell y, f^\ell y') \leqslant C_2 C_4 (d(y, y') + \gamma^{s(y,y')}). \tag{7.6}$$

Hence

$$\begin{aligned} |h_\ell(y) - h_\ell(y')| &\leqslant \sum_{j=0}^{\ell-1} |h(f^j y) - h(f^j y')| \\ &\leqslant |h|_\eta \sum_{j=0}^{\ell-1} d(f^j y, f^j y')^\eta \leqslant C_2 C_4 |h|_\eta\, \ell (d_1(y, y') + \gamma_1^{s(y,y')}), \end{aligned}$$

establishing the estimate for h_ℓ. Also, $\tau(y) \leqslant (\inf h)^{-1} \inf 1_{Y_j} \varphi$, so taking $\ell = \tau(y)$ and using (7.3) with $n = 0$, we obtain the estimate for φ. □

Next we deal with ingredient (iii) assuming (ii). Define $\pi : Y^\varphi \to M$ as $\pi(y, u) = T_u y$.

Proposition 7.5. *Suppose that the function $\chi : Y \to \mathbb{R}$ satisfies condition (H).*

Then observables $v \in C^\eta(M) \cap C^{0,\eta}(M)$ lift to observables $\tilde{v} = v \circ \pi : Y^\varphi \to \mathbb{R}$ that lie in $\mathcal{H}^*_{\gamma_2,\eta}(Y^\varphi)$ where $\gamma_2 = \gamma^{\eta^2}$ and the metric d on Y is replaced by the metric $d_2(y, y') = d(y, y')^{\eta^2}$.

For $m \geqslant 1$, observables $w \in C^{\eta,m}(M)$ lift to observables $\tilde{w} = w \circ \pi \in \mathcal{H}^*_{\gamma_2,0,m}(Y^\varphi)$.

Moreover, there is a constant $C > 0$ such that $\|\tilde{v}\|^*_{\gamma_2,\eta} \leqslant C(\|v\|_{C^\eta} + \|v\|_{C^{0,\eta}})$ and $\|\tilde{w}\|^*_{\gamma_2,0,m} \leqslant C\|w\|_{C^{\eta,m}}$.

Proof. Let $\sigma = 2|\chi|_\infty$. We show that $\|\tilde{v} \circ F_\sigma\|_{\gamma_2,\eta} \ll \|v\|_{C^\eta} + \|v\|_{C^{0,\eta}}$. The same calculation with $\sigma = 0$ shows that $\|\tilde{v}\|_{\gamma_2,\eta} \ll \|v\|_{C^\eta} + \|v\|_{C^{0,\eta}}$, so $\|\tilde{v}\|^*_{\gamma_2,\eta} \ll \|v\|_{C^\eta} + \|v\|_{C^{0,\eta}}$, We take $D_1 = |h|_\infty + 2|\chi|_\infty$ with corresponding value of D_2 in (7.5)

Let (y, u), $(y', u) \in Y^\varphi$ with $y, y' \in Y_j$ for some $j \geqslant 1$. There exists $\ell, \ell' \in \{0, \dots, \tau(y) - 1\}$ such that

$$u \in [h_\ell(y), h_{\ell+1}(y)] \cap [h_{\ell'}(y'), h_{\ell'+1}(y')].$$

Suppose without loss that $\ell \leqslant \ell'$. Then

$$u = h_\ell(y) + r = h_\ell(y') + r',$$

where $r \in [0, |h|_\infty]$ and $r' = u - h_\ell(y') \geqslant u - h_{\ell'}(y') \geqslant 0$. Note that $T_u y = T_r T_{h_\ell(y)} y = T_r f^\ell y$. Hence $\tilde{v}(y, u) = v(T_r f^\ell y)$ and so $\tilde{v} \circ F_\sigma(y, u) = v(T_{\sigma+r} f^\ell y)$. Similarly, $T_u y' = T_{r'} f^\ell y'$ and $\tilde{v} \circ F_\sigma(y', u) = v(T_{\sigma+r'} f^\ell y')$. Also, $\sigma + r \in [0, D_1]$. By (7.5) and (7.6),

$$\begin{aligned}|v(T_{\sigma+r} f^\ell y) - v(T_{\sigma+r} f^\ell y')| &\leqslant |v|_{C^\eta} d(T_{\sigma+r} f^\ell y, T_{\sigma+r} f^\ell y')^\eta \leqslant D_2^\eta |v|_{C^\eta} d(f^\ell y, f^\ell y')^{\eta^2} \\ &\ll |v|_{C^\eta} (d_2(y, y') + \gamma_2^{s(y,y')}).\end{aligned}$$

Since $u \geqslant h_\ell(y') \geqslant \ell \inf h$, it follows from Proposition 7.4 that

$$\begin{aligned}|v(T_{\sigma+r} f^\ell y') - v(T_{\sigma+r'} f^\ell y')| &\leqslant |v|_{C^{0,\eta}} |r - r'|^\eta = |v|_{C^{0,\eta}} |h_\ell(y) - h_\ell(y')|^\eta \\ &\ll |v|_{C^{0,\eta}}\, \ell (d_2(y, y')^\eta + \gamma_2^{s(y,y')}) \leqslant (\inf h)^{-1} |v|_{C^{0,\eta}}\, u (d_2(y, y')^\eta + \gamma_2^{s(y,y')}).\end{aligned}$$

Hence

$$\begin{aligned}|\tilde{v} \circ F_\sigma(y, u) - \tilde{v} \circ F_\sigma(y', u)| &= |v(T_{\sigma+r} f^\ell y) - v(T_{\sigma+r'} f^\ell y')| \\ &\ll (|v|_{C^\eta} + |v|_{C^{0,\eta}})(u + 1)(d_2(y, y') + \gamma_2^{s(y,y')})\end{aligned}$$

whenever $s(y, y') \geqslant 1$. For $s(y, y') = 0$, we have the estimate $|\tilde{v} \circ F_\sigma(y, u) - \tilde{v} \circ F_\sigma(y', u)| \leqslant 2|v|_\infty = 2|v|_\infty \gamma_2^{s(y,y')} \ll |v|_\infty \varphi(y)(d_2(y, y') + \gamma_2^{s(y,y')})$, so in all cases we obtain

$$\begin{aligned}|\tilde{v} \circ F_\sigma(y, u) - \tilde{v} \circ F_\sigma(y', u)| &\ll (\|v\|_{C^\eta} + |v|_{C^{0,\eta}})(u + 1)(d_2(y, y') + \gamma_2^{s(y,y')}) \\ &\leqslant 2(\|v\|_{C^\eta} + |v|_{C^{0,\eta}})\varphi(y)(d_2(y, y') + \gamma_2^{s(y,y')}).\end{aligned}$$

Also,

$$|\tilde{v} \circ F_\sigma(y, u) - \tilde{v} \circ F_\sigma(y, u')| = |v(T_{\sigma+u} y) - v(T_{\sigma+u'} y) \leqslant |v|_{C^{0,\eta}} |u - u'|^\eta,$$

so $\|\tilde{v} \circ F_\sigma\|_{\gamma_2.\eta} \ll |v|_{C^0} + |v|_{C^{0,\eta}}$ as required. □

7.3. Dynamically Hölder flows and observables. The Hölder assumptions in Sect. 7.2 can be replaced by dynamically Hölder as follows. We continue to assume that $\inf h > 0$.

Definition 7.6. The roof function h, the flow T_t and the observable v are *dynamically Hölder* if $v \in C^{0,\eta}(M)$ for some $\eta \in (0, 1]$ and there is a constant $C \geqslant 1$ such that for all $y, y' \in Y_j$, $j \geqslant 1$,

(a) $|h(f^\ell y) - h(f^\ell y')| \leqslant C(d(y, y')^\eta + \gamma^{s(y,y')})$ for all $0 \leqslant \ell \leqslant \tau(y) - 1$.
(b) For every $u \in [0, \varphi(y)] \cap [0, \varphi(y')]$, there exist $t, t' \in \mathbb{R}$ such that $|t - t'| \leqslant C(u + 1)(d(y, y')^\eta + \gamma^{s(y,y')})$, and setting $z = W^s(y) \cap W^u(y')$,

$$\max\left\{|v(T_u y) - v(T_t z)|\,,\ |v(T_u y') - v(T_{t'} z)|\right\} \leqslant C(u + 1)(d(y, y')^\eta + \gamma^{s(y,y')}).$$

Also, we replace the assumption $w \in C^{\eta,m}(M)$ by the condition that $\partial_t^k w$ lies in $C^{0,\eta}(M)$ and satisfies (b) for all $k = 0, \dots, m$.

Remark 7.7. In the proof of Proposition 7.5, we showed that $|v(T_u y) - v(T_u y')| = |\tilde v(y, u) - \tilde v(y', u)| \ll (u + 1)(d(y, y')^\eta + \gamma^{s(y,y')})$ (for modified d and γ) under the old hypotheses. Hence, taking $t = t' = u$, we see that Definition 7.6 is indeed a relaxed version of the conditions in Sect. 7.2.

It is easily verified that condition (6.1) remains valid under the more relaxed assumption on h in Definition 7.6(a). Also, it follows as in the proof of Proposition 7.5 that $|\tilde v(y, u) - \tilde v(y, u')| \leqslant |v|_{C^{0,\eta}} |u - u'|^\eta$.

Next we estimate $|\tilde v(y, u)-\tilde v(y', u)|$ and $|\tilde v\circ F_\sigma(y, u)-\tilde v\circ F_\sigma(y', u)|$ for $(y, u),\ (y', u) \in Y^\varphi$, where $\sigma = 2|\chi|_\infty$. If $s(y, y') = 0$, then $|\tilde v(y, u) - \tilde v(y', u)|$, $|\tilde v \circ F_\sigma(y, u) - \tilde v \circ F_\sigma(y', u)| \ll |v|_\infty\, \varphi(y)(d_2(y, y) + \gamma_2^{s(y,y')})$ as in the proof of Proposition 7.5. Hence we can suppose that $y, y' \in Y_j$ for some $j \geqslant 1$. Set $z = W^s(y) \cap W^u(y')$ and choose t, t' as in Definition 7.6(b). Then

$$\begin{aligned}|\tilde v(y, u) - \tilde v(y', u)| &= |v(T_u y) - v(T_u y')|\\ &\leqslant |v(T_u y) - v(T_t z)| + |v(T_{t'} z) - v(T_u y')| + |v(T_t z) - v(T_{t'} z)|\\ &\leqslant 4C\varphi(y)(d(y, y')^\eta + \gamma^{s(y,y')}) + |v|_{C^{0,\eta}} |t - t'|^\eta \ll \varphi(y)(d_2(y, y') + \gamma_2^{s(y,y')}).\end{aligned}$$

Hence $|\tilde v(y, u) - \tilde v(y', u)| \ll \varphi(y)(d_2(y, y')^\eta + \gamma_2^{s(y,y')})$ for all $(y, u),\ (y', u) \in Y^\varphi$, and so $\tilde v \in \mathcal{H}_{\gamma_2,\eta}(Y^\varphi)$.

To proceed, we recall that $z = W^s(y) \cap W^u(y')$, so $Fz = W^s(Fy) \cap W^u(Fy')$ by (7.4). Hence

$$d(Fy, Fy') \leqslant d(Fy, Fz) + d(Fz, Fy') \ll d(y, z) + \gamma^{s(y,y')} \leqslant C_4 d(y, y') + \gamma^{s(y,y')}. \tag{7.7}$$

To control $\tilde v \circ F_\sigma(y, u) - \tilde v \circ F_\sigma(y', u)$, we assume without loss that $\varphi(y) \geq \varphi(y')$, and distinguish three cases.

If $u + \sigma < \varphi(y')$, we argue as in the bound for $\tilde v(y, u) - \tilde v(y', u)$.

If $u + \sigma \geq \varphi(y)$, then there exists $0 \leq \bar u \leq \sigma$ and $\bar u' \geqslant \bar u$ such that $T_{u+\sigma} y = T_{\bar u} F y$ and $T_{u+\sigma} y' = T_{\bar u'} F y'$. By Corollary 6.2 and (7.7),

$$|\bar u - \bar u'| = |\varphi(y) - \varphi(y')| \ll \varphi(y)(d(y, y') + \gamma^{s(y,y')})$$

and so

$$|v(T_{\bar u} F y') - v(T_{\bar u'} F y')| \ll \varphi(y)(d_2(y, y') + \gamma_2^{s(y,y')}).$$

On the other hand, choosing $\bar{t}$ and $\bar{t}'$ for $\bar{u}$ as in Definition 7.6(b), we get

$$\begin{aligned}
&|v(T_{\bar{u}}Fy) - v(T_{\bar{u}}Fy')| \\
&\quad \leqslant |v(T_{\bar{u}}Fy) - v(T_{\bar{t}}Fz)| + |v(T_{\bar{t}'}Fz) - v(T_{\bar{u}}Fy')| + |v(T_{\bar{t}}Fz) - v(T_{\bar{t}'}Fz)| \\
&\quad \leqslant 2C(\bar{u}+1)(d(Fy,Fy')^\eta + \gamma^{s(Fy,Fy')}) + |v|_{C^{0,\eta}}|\bar{t} - \bar{t}'|^\eta \ll d_2(y,y') + \gamma_2^{s(y,y')}
\end{aligned}$$

where we have used (7.7) and $\bar{u} \leqslant \sigma$. Hence

$$\begin{aligned}
|\tilde{v} \circ F_\sigma(y,u) - \tilde{v} \circ F_\sigma(y',u)| &\leqslant |v(T_{\bar{u}}Fy) - v(T_{\bar{u}}Fy')| + |v(T_{\bar{u}}Fy') - v(T_{\bar{u}'}Fy')| \\
&\ll \varphi(y)(d_2(y,y') + \gamma_2^{s(y,y')}).
\end{aligned}$$

Finally, if $\varphi(y') \le u + \sigma < \varphi(y)$, there exist $0 < u_1,\ u_2 \le \varphi(y) - \varphi(y')$ such that $Fy = T_{u_1}T_{u+\sigma}y$ and $T_{u+\sigma}y' = T_{u_2}Fy'$. Using again Corollary 6.2 and (7.7),

$$\begin{aligned}
&|\tilde{v} \circ F_\sigma(y,u) - \tilde{v} \circ F_\sigma(y',u)| = |v(T_{u+\sigma}y) - v(T_{u+\sigma}y')| \\
&\quad \le |v(T_{u+\sigma}y) - v(Fy)| + |v(Fy) - v(Fy')| + |v(Fy') - v(T_{u+\sigma}y')| \\
&\quad = |v(T_{u+\sigma}y) - v(T_{u_1+u+\sigma}y)| + |v(Fy) - v(Fy')| + |v(Fy') - v(T_{u_2}Fy')| \\
&\quad \ll \varphi(y)(d_2(y,y') + \gamma_2^{s(y,y')}).
\end{aligned}$$

This completes the verification that $\tilde{v} \in \mathcal{H}^*_{\gamma_2,\eta}(Y^\varphi)$. A similar argument shows that $\tilde{w} \in \mathcal{H}^*_{\gamma_2,0,m}(Y^\varphi)$, completing the verification that Proposition 7.5 holds under the modified assumptions.

8. Condition (H) for Nonuniformly Hyperbolic Flows

In this section, we consider various classes of nonuniformly hyperbolic flows for which condition (H) in Sect. 6 can be satisfied. We are then able to apply Theorem 6.4 to obtain results that superpolynomial and polynomial mixing applies to such flows as follows:

Corollary 8.1. *Let $T_t : M \to M$ be a nonuniformly hyperbolic flow as in Sect. 7.2 and assume that condition (H) is satisfied. Then*
(a) $F_t : Y^\varphi \to Y^\varphi$ is a Gibbs–Markov flow.
(b) Suppose that $\mu(\varphi > t) = O(t^{-\beta})$ for some $\beta > 1$ and assume absence of approximate eigenfunctions for F_t. Then there exists $m \geqslant 1$ and $C > 0$ such that

$$|\rho_{v,w}(t)| \leqslant C(\|v\|_{C^\eta} + \|v\|_{C^{0,\eta}})\|w\|_{C^{\eta,m}}\, t^{-(\beta-1)},$$

for all $v \in C^\eta(M) \cap C^{0,\eta}(M)$, $w \in C^{\eta,m}(M)$, $t > 1$.

Proof. Part (a) follows from the discussion in Sect. 7.2 (so ingredient (i) is automatic and ingredient (ii) is now assumed).

As described in Sect. 6.1, there is a measure-preserving conjugacy from F_t to T_t, so part (b) is immediate from Theorem 6.4 combined with Proposition 7.5. □

The analogous result holds for nonuniformly hyperbolic flows and observables satisfying the dynamically Hölder conditions in Sect. 7.3.

We verify condition (H) for three classes of flows. In Sect. 8.1, we consider roof functions with bounded Hölder constants. In Sect. 8.2, we consider flows for which there is exponential contraction along stable leaves. In Sect. 8.3, we consider flows with an invariant Hölder stable foliation. These correspond to the situations mentioned in [30, Section 4.2].

Also, in Sect. 8.4, we briefly review the temporal distance function and a criterion for absence of approximate eigenfunctions.

8.1. Roof functions with bounded Hölder constants. We assume a "bounded Hölder constants" condition on φ, namely that for all $y, y' \in Y$,

$$|\varphi(y) - \varphi(y')| \leqslant C_1 d(y, y') \quad \text{for all } y' \in W^s(y), \tag{8.1}$$

$$|\varphi(y) - \varphi(y')| \leqslant C_1 \gamma^{s(y,y')} \quad \text{for all } y' \in W^u(y), s(y, y') \geqslant 1. \tag{8.2}$$

This leads directly to an enhanced version of (6.1):

Proposition 8.2. $|\varphi(y)-\varphi(y')| \leqslant C_1 C_4 (d(y, y')+\gamma^{s(y,y')})$ *for all* $y, y' \in Y$, $s(y, y') \geqslant 1$.

Proof. Let $z = W^s(y) \cap W^s(y')$. Then

$$\begin{aligned}|\varphi(y) - \varphi(y')| &\leqslant |\varphi(y) - \varphi(z)| + |\varphi(z) - \varphi(y')| \\ &\leqslant C_1(d(y, z) + \gamma^{s(z,y')}) \leqslant C_1 C_4 (d(y, y') + \gamma^{s(y,y')}),\end{aligned}$$

as required. □

Lemma 8.3. *If conditions* (8.1) *and* (8.2) *are satisfied, then condition (H) holds.*

Proof. By (7.2) and (8.1), for all $y \in Y$, $n \geqslant 0$,

$$|\varphi(F^n \pi y) - \varphi(F^n y)| \leqslant C_1 d(F^n \pi y, F^n y) \leqslant C_1 C_2 \gamma^n d(\pi y, y) \leqslant C_1 C_2 \gamma^n.$$

It follows that

$$|\chi(y)| \leqslant \textstyle\sum_{n=0}^{\infty} |\varphi(F^n \pi y) - \varphi(F^n y)| \leqslant C_1 C_2 (1-\gamma)^{-1}.$$

Hence $|\chi|_\infty \leqslant C_1 C_2 (1-\gamma)^{-1}$ and condition (H)(a) is satisfied.

Next, let $y, y' \in Y$, and set $N = [\frac{1}{2} s(y, y')]$, $\gamma_1 = \gamma^{1/2}$. Write

$$\chi(y) - \chi(y') = A(\pi y, \pi y') - A(y, y') + B(y) - B(y'),$$

where

$$A(p, q) = \sum_{n=0}^{N-1} (\varphi(F^n p) - \varphi(F^n q)), \qquad B(p) = \sum_{n=N}^{\infty} (\varphi(F^n \pi p) - \varphi(F^n p)).$$

By the calculation for $|\chi|_\infty$, we obtain $|B(p)| \leqslant C_1 C_2 (1-\gamma)^{-1} \gamma^N$ for all $p \in Y$. Also, $\gamma^N \leqslant \gamma^{-1} \gamma^{\frac{1}{2} s(y,y')} = \gamma^{-1} \gamma_1^{s(y,y')}$, so $B(p) = O(\gamma_1^{s(y,y')})$ for $p = y, y'$.

For $n \leqslant N-1$ we have $s(F^n y, F^n y') \geqslant 1$, so by Propositions 7.3 and 8.2,

$$|\varphi(F^n y) - \varphi(F^n y')| \leqslant C_1 C_4 (d(F^n y, F^n y') + \gamma^{s(y,y')-n}) \leqslant C(\gamma^n d(y, y') + \gamma^{s(y,y')-n}),$$

where $C = 2C_4^2 C_1 C_2$. Hence

$$\begin{aligned}|A(y, y')| &\leqslant \sum_{n=0}^{N-1} |\varphi(F^n y) - \varphi(F^n y')| \leqslant C \sum_{n=0}^{N-1} (\gamma^n d(y, y') + \gamma^{s(y,y')-n}) \\ &\leqslant C(1-\gamma)^{-1}(d(y, y') + \gamma^{s(y,y')-N}) \leqslant C(1-\gamma)^{-1}(d(y, y') + \gamma_1^{s(y,y')}).\end{aligned}$$

Similarly for $A(\pi y, \pi y')$. Hence $|\chi(y) - \chi(y')| \ll d(y, y') + \gamma_1^{s(y,y')}$, so (H)(b) holds. □

8.2. Exponential contraction along stable leaves. In this subsection, we suppose that $h \in C^\eta(X)$ and that f is exponentially contracting along stable leaves:

$$d(f^n y, f^n y') \leqslant C_2\gamma^n d(y, y') \quad \text{for all } n \geqslant 0 \text{ and all } y, y' \in Y \text{ with } y' \in W^s(y). \tag{8.3}$$

Note that this strengthens condition (7.2). Proposition 7.3 becomes

$$d(f^n y, f^n y') \leqslant C_2C_4(\gamma^n d(y, y') + \gamma^{s(y,y')-\psi_n(y)}) \quad \text{for all } n \geqslant 0,\ y, y' \in Y. \tag{8.4}$$

Lemma 8.4. *If condition* (8.3) *is satisfied, then condition (H) holds.*

Proof. Let $\gamma_1 = \gamma^\eta$, $\gamma_2 = \gamma_1^{1/2}$. We verify condition (H) with γ_2 and $d_1(y, y') = d(y, y')^\eta$, using the equivalent definition for χ,

$$\chi(y) = \textstyle\sum_{n=0}^{\infty}(h(f^n\pi y) - h(f^n y)).$$

By (8.3),

$$|\chi(y)| \leqslant \textstyle\sum_{n=0}^{\infty} |h|_\eta d(f^n\pi y, f^n y)^\eta \leqslant C_2|h|_\eta \sum_{n=0}^{\infty} \gamma_1^n d_1(\pi y, y) \leqslant C_2|h|_\eta(1-\gamma_1)^{-1}.$$

Hence $|\chi|_\infty \leqslant C_2|h|_\eta(1-\gamma_1)^{-1}$ and condition (H)(a) is satisfied.

Next, let $y, y' \in Y$ and set $N = [\frac{1}{2}s(y, y')]$. Write $\chi(y) - \chi(y') = A(\pi y, \pi y') - A(y, y') + B(y) - B(y')$, where

$$A(p, q) = \sum_{n=0}^{N-1}(h(f^n p) - h(f^n q)), \qquad B(p) = \sum_{n=N}^{\infty}(h(f^n\pi p) - h(f^n p)).$$

By the calculation for $|\chi|_\infty$, we obtain $|B(p)| \leqslant C_2|h|_\eta(1-\gamma_1)^{-1}\gamma_1^N$ for all $p \in Y$. Also, $\gamma_1^N \leqslant \gamma_1^{-1}\gamma_1^{\frac{1}{2}s(y,y')} = \gamma_1^{-1}\gamma_2^{s(y,y')}$, so $B(p) = O(\gamma_2^{s(y,y')})$ for $p = y,\ y'$.

Finally, by (8.4) using that $\psi_n \leqslant n$,

$$\begin{aligned}
|A(y, y')| &\leqslant |h|_\eta \sum_{n=0}^{N-1} d(f^n y, f^n y')^\eta \leqslant C_2C_4|h|_\eta \sum_{n=0}^{N-1}(\gamma_1^n d_1(y, y') + \gamma_1^{s(y,y')-n}) \\
&\leqslant C_2C_4|h|_\eta(1-\gamma_1)^{-1}(d_1(y, y') + \gamma_1^{s(y,y')-N}) \\
&\leqslant C_2C_4|h|_\eta(1-\gamma_1)^{-1}(d_1(y, y') + \gamma_2^{s(y,y')}).
\end{aligned}$$

Similarly for $A(\pi y, \pi y')$. Hence $|\chi(y) - \chi(y')| \ll d_1(y, y') + \gamma_2^{s(y,y')}$, so (H)(b) holds. □

Remark 8.5. In cases where h lies in $C^\eta(X)$ and the dynamics on X is modelled by a Young tower with exponential tails (so $\mu_X(\tau > n) = O(e^{-ct})$ for some $c > 0$), it is immediate that $\varphi \in L^q(Y)$ for all q and that condition (8.3) is satisfied. Assuming absence of approximate eigenfunctions, we obtain rapid mixing for such flows.

8.3. Flows with an invariant Hölder stable foliation. Let $T_t : M \to M$ be a Hölder nonuniformly hyperbolic flow as in Sect. 7.2 and let Λ be an attractor for this flow. We suppose that (M, d) is a Riemannian manifold, that X is a C^2 embedded cross-section for the flow and that $Y \subset X \cap \Lambda$ is a set with hyperbolic product structure for the return map. We assume that the flow possesses a T_t-invariant Hölder stable foliation $\mathcal{W}^{ss}$ in a neighbourhood of Λ (a sufficient condition for this to hold is that Λ is a partially hyperbolic attracting set with a DT_t-invariant dominated splitting $T_\Lambda M = E^{ss} \oplus E^{cu}$, see [4]). We also assume that diam Y can be chosen arbitrarily small. In this subsection, we show how to use the stable foliation $\mathcal{W}^{ss}$ for the flow to show that χ is Hölder, hence verifying the hypotheses in Sect. 6.1.

Remark 8.6. As discussed in [30, Section 4.2(iii)], this framework includes (not necessarily Markovian) intermittent solenoidal flows, and yields polynomial decay $O(t^{-(\beta-1)})$ for any prescribed $\beta > 1$. These results are optimal by [31] in the Markovian case and by [9] in general.

Remark 8.7. The results of this subsection concerning flows with an invariant Hölder stable foliation are *not* required for the study of billiard systems in Sect. 9.

First, we show that if $W^s(y)$ and $W^{ss}(y)$ coincide for all $y \in Y$, then $F_t : Y^\varphi \to Y^\varphi$ is already a skew product (so $\chi = 0$). (We remind the reader that $\mathcal{W}^s$ is the collection of stable leaves of the hyperbolic product structure on X and that $\mathcal{W}^{ss}$ is the collection of strong stable manifolds for the flow on M.)

Proposition 8.8. *Suppose that $W^s(y)$ and $W^{ss}(y)$ coincide for all $y \in Y$. Then φ is constant along stable leaves $W^s(y)$, $y \in Y$.*

Proof. For $y_0 \in Y$,

$$\{T_{\varphi(y)}y : y \in W^{ss}(y_0)\} = \{Fy : y \in W^s(y_0)\} = FW^s(y_0) \subset W^s(Fy_0) = W^{ss}(Fy_0).$$

But setting $t_0 = \varphi(y_0)$,

$$\{T_{t_0}y : y \in W^{ss}(y_0)\} = T_{t_0}W^{ss}(y_0) \subset W^{ss}(T_{t_0}y_0) = W^{ss}(Fy_0).$$

Hence $\varphi|_{W^{ss}(y_0)} \equiv \varphi(y_0)$. □

We can choose a d_u-dimensional ball $\widetilde{X} \subset W^u(y_0) \subset X$ for some $y_0 \in Y$ such that the strong stable manifold $W^{ss}(x)$ is uniformly large (in the sense of containing a d_s-dimensional ball of uniform radius) for all $x \in \widetilde{X}$. We define the new cross-section to the flow $X^* = \bigcup_{x \in \widetilde{X}} W^{ss}(x)$. Shrinking Y if necessary, there exists a neighbourhood (within X) of Y which we denote by $\check{X}$ and a unique continuous function $\zeta : \check{X} \to \mathbb{R}$ with $|\zeta| \leqslant \frac{1}{2} \inf \varphi$ such that $\zeta|_{\widetilde{X}} \equiv 0$ and $\{T_{\zeta(x)}(x) : x \in \check{X}\} \subset X^*$. Moreover, ζ is Hölder since X is C^2 embedded in M and X^* is Hölder by the assumption on the regularity of the stable foliation $\mathcal{W}^{ss}$. Let $Y^* = \{T_{\zeta(y)}(y) : y \in Y\}$. Define the new roof function

$$\varphi^* : Y^* \to \mathbb{R}^+, \qquad \varphi^*(T_{\zeta(y)}y) = \varphi(y) + \zeta(Fy) - \zeta(y).$$

We observe that φ^* is the return time for the flow T_t to the cross-section X^*, restricted to Y^*.

Lemma 8.9. *Under the above assumption on $\mathcal{W}^{ss}$, condition (H) holds.*

Proof. We show that $\chi = -\zeta$. The result follows since ζ is Hölder.

Let $n \geqslant 0$, $y \in Y$. By Proposition 8.8 applied to $\varphi^* : Y^* \to \mathbb{R}^+$, we have that $\varphi^*(T_{\zeta(F^n\pi y)}F^n\pi y) = \varphi^*(T_{\zeta(F^n y)}F^n y)$. Hence by definition of φ^*,

$$\varphi(F^n\pi y) - \varphi(F^n y) = \zeta(F^n\pi y) - \zeta(F^n y) + \zeta(F^{n+1}y) - \zeta(F^{n+1}\pi y).$$

Let η be the Hölder exponent of ζ. By (7.2), $|\varphi(F^n\pi y) - \varphi(F^n y)| \leqslant 2C_2|\zeta|_\eta(\gamma^\eta)^n$ so the series $\chi(y) = \sum_{n=0}^\infty(\varphi(F^n\pi y) - \varphi(F^n y))$ converges absolutely. Moreover,

$$\begin{aligned}\chi(y) &= \lim_{N\to\infty}\sum_{n=0}^{N-1}(\varphi(F^n\pi y) - \varphi(F^n y)) \\ &= \lim_{N\to\infty}\big(\zeta(\pi y) - \zeta(y) + \zeta(F^N y) - \zeta(F^N\pi y)\big) = \zeta(\pi y) - \zeta(y).\end{aligned}$$

Finally, $\zeta(\pi y) = 0$ since $\zeta|_{\tilde{X}} \equiv 0$. □

8.4. Temporal distance function. Dolgopyat [22, Appendix] showed that for Axiom A flows a sufficient condition for absence of approximate eigenfunctions is that the range of the temporal distance function has positive lower box dimension. This was extended to nonuniformly hyperbolic flows in [29,30]. Here we recall the main definitions and result.

We assume that condition (H) holds, so that the suspension flow $Y^\varphi \to Y^\varphi$ is a Gibbs–Markov flow (and hence conjugate to a skew product flow). We also assume the dynamically Hölder setup from Sect. 7.3. In particular, the Poincaré map $f : X \to X$ is nonuniformly hyperbolic as in Sect. 7.1 and Y has a local product structure. Also we assume that the roof function φ has bounded Hölder constants along unstable leaves, so condition (8.2) is satisfied.

Let $y_1, y_4 \in Y$ and set $y_2 = W^s(y_1) \cap W^u(y_4)$, $y_3 = W^u(y_1) \cap W^s(y_4)$. Define the *temporal distance function* $D : Y \times Y \to \mathbb{R}$,

$$D(y_1, y_4) = \sum_{n=-\infty}^{\infty}\Big(\varphi(F^n y_1) - \varphi(F^n y_2) - \varphi(F^n y_3) + \varphi(F^n y_4)\Big).$$

It follows from the construction in [30, Section 5.3] (which uses (7.4) and (8.2)) that inverse branches $F^n y_i$ for $n \leqslant -1$ can be chosen so that D is well-defined.

Lemma 8.10 ([30, Theorem 5.6]). *Let $Z_0 = \bigcap_{n=0}^\infty F^{-n}Z$ where Z is a union of finitely many elements of the partition $\{Y_j\}$. Let $\overline{Z}_0$ denote the corresponding finite subsystem of $\overline{Y}$. If the lower box dimension of $D(Z_0 \times Z_0)$ is positive, then there do not exist approximate eigenfunctions on $\overline{Z}_0$.* □

Remark 8.11. For Axiom A attractors, Z_0 can be taken to be connected and D is continuous, so absence of approximate eigenfunctions is ensured whenever D is not identically zero. For nonuniformly hyperbolic flows, where the partition $\{Y_j\}$ is countably infinite, Z_0 is a Cantor set of positive Hausdorff dimension [29, Example 5.7]. In general it is not clear how to use this property since D is generally at best Hölder. However for flows with a contact structure, a formula for D in [25, Lemma 3.2] can be exploited and the lower box dimension of $D(Z_0 \times Z_0)$ is indeed positive, see [29, Example 5.7]. The arguments in [29, Example 5.7] apply to general Gibbs–Markov flows with a contact structure. A special case of this is the Lorentz gas examples considered in Sect. 9.

9. Billiard Flows Associated to Infinite Horizon Lorentz Gases

In this section we show that billiard flows associated to planar infinite horizon Lorentz gases satisfy the assumptions of Sect. 8.1. In particular, we prove decay of correlations with decay rate $O(t^{-1})$.

Background material on infinite horizon Lorentz gases is recalled in Sect. 9.1 and the decay rate $O(t^{-1})$ is proved in Sect. 9.2. In Sect. 9.3, we show that the same decay rate holds for semidispersing Lorentz flows and stadia. In Sect. 9.4, we show that the decay rate is optimal for the examples considered in this section.

9.1. Background on the infinite horizon Lorentz gas. We begin by recalling some background on billiard flows; for further details we refer to the monograph [16].

Let $\mathbb{T}^2$ denote the two dimensional flat torus, and let us fix finitely many disjoint convex scatterers $S_k \subset \mathbb{T}^2$ with C^3 boundaries of nonvanishing curvature. The complement $Q = \mathbb{T}^2 \setminus \bigcup S_k$ is the billiard domain, and the billiard dynamics are that of a point particle that performs uniform motion with unit speed inside Q, and specular reflections—angle of reflection equals angle of incidence—off the scatterers, that is, at the boundary ∂Q. The resulting billiard flow is $T_t : M \to M$, where the phase space $M = Q \times \mathbb{S}^1$ is a Riemannian manifold, and T_t preserves the (normalized) Lebesgue measure μ_M (often called Liouville measure in the literature).

There is a natural Poincaré section $X = \partial Q \times [-\pi/2, \pi/2] \subset M$ corresponding to collisions (with outgoing velocities), which gives rise to the billiard map denoted by $f : X \to X$, with absolute continuous invariant probability measure μ_X. The time until the next collision, the free flight function $h : X \to \mathbb{R}^+$, is defined to be $h(x) = \inf\{t > 0 : T_t x \in X\}$. The Lorentz gas has *finite horizon* if $h \in L^\infty(X)$ and *infinite horizon* if h is unbounded.

In the finite horizon case, [5] recently proved exponential decay of correlations. In this section, we prove

Theorem 9.1. *Let $\eta \in (0, 1]$. In the infinite horizon case, there exists $m \geqslant 1$ such that $\rho_{v,w}(t) = O(t^{-1})$ for all $v \in C^\eta(M) \cap C^{0,\eta}(M)$ and $w \in C^{\eta,m}(M)$ (and more generally for the class of observables defined in Corollary 9.6 below).*

Let us fix some terminology and notations. The billiard map $f : X \to X$ is discontinuous, with singularity set $\mathcal{S}$ corresponding to the preimages of grazing collisions. Here, $\mathcal{S}$ is the closure of a countable union of smooth curves, $X \setminus \mathcal{S}$ consists of countably many connected components $X_m, m \geqslant 1$, and $f|_{X_m}$ is C^2. If $x, x' \in X_m$ for some $m \geqslant 1$, then, in particular, x, x' and fx, fx' lie on the same scatterer (even when the configuration is unfolded to the plane). Throughout our exposition, $d(x, x')$ denotes the Euclidean distance of the two points, i.e. the distance that is generated by the Riemannian metric on X (or M).

It follows from geometric considerations in the infinite horizon case that $\mu_X(h > t) = O(t^{-2})$. Moreover, as the trajectories are straight lines, we have

$$|h(x) - h(x')| \leqslant d(x, x') + d(fx, fx') \quad \text{for all } x, x' \in X_m, m \geqslant 1; \text{ and} \tag{9.1}$$

$$d(T_t x, T_{t'} x) \leqslant |t - t'| \quad \text{for all } x \in X \text{ and } t, t' \in [0, h(x)). \tag{9.2}$$

The billiard maps considered here (both finite and infinite horizon) have uniform contraction and expansion even for f. There exist stable and unstable manifolds of positive length for almost every $x \in X$, which we denote by $W^s(x)$ and $W^u(x)$ respectively, and there exist constants $C_2 \geqslant 1$, $\gamma \in (0,1)$ such that for all $x, x' \in X$, $n \geqslant 0$,

$$d(f^n x, f^n x') \leqslant C_2 \gamma^n d(x, x') \quad \text{for } x' \in W^s(x). \tag{9.3}$$

$$d(x, x') \leqslant C_2 \gamma^n d(f^n x, f^n x') \quad \text{for } f^n x' \in W^u(f^n x). \tag{9.4}$$

This follows from the uniform hyperbolicity properties of f, see in particular [16, Formula (4.19)].

Furthermore, there is a constant $C_5 \geqslant 1$ such that for $x, x' \in X$,

$$d(T_t x, T_t x') \leqslant C_5 d(x, x') \quad \text{for } x' \in W^s(x), t \in [0, h(x)] \cap [0, h(x')]. \tag{9.5}$$

$$d(T_{-t} x, T_{-t} x') \leqslant C_5 d(x, x') \quad \text{for } x' \in W^u(x), t \in [0, h(f^{-1}x)] \cap [0, h(f^{-1}x')]. \tag{9.6}$$

To verify (9.5), note that $d(x, x')$ consists of a position and a velocity component. In course of the free flight, the velocities do not change, while for $x' \in W^s(x)$, the position component can only shrink as stable manifolds correspond to converging wavefronts. A similar argument applies to (9.6).

Remark 9.2. (a) In the remainder of the section—and in particular in the proof of Proposition 9.5 below—we apply (9.1) repeatedly, but always in the case when either $x' \in W^s(x)$, or $fx' \in W^u(fx)$. As all iterates $f^n, n \geqslant 0$ are smooth on local stable manifolds (while all iterates $f^{-n}, n \geqslant 0$ are smooth on local unstable manifolds), both of these conditions imply $x, x' \in X_m$ for some $m \geqslant 1$.
(b) For larger values of t than those in (9.5), we note that $d(T_t x, T_t x')$ may grow large *temporarily*: it can happen that one of the trajectories has already collided with some scatterer, while the other has not, hence even though the two points are close in position, the velocities differ substantially. Similar comments apply to (9.6). This phenomenon is the main reason why we require the notion of dynamically Hölder flows T_t in Definition 7.6.

In [36], Young constructs a subset $Y \subset X$ and an induced map $F = f^\tau : Y \to Y$ that possesses the properties discussed in Sect. 7.1 including (7.4). The tails of the return time $\tau : Y \to \mathbb{Z}^+$ are exponential, i.e. $\mu(\tau > n) = O(e^{-cn})$ for some $c > 0$. Moreover, the construction can be carried out so that diam Y is as small as desired. This is proved in [36] for the finite horizon, and in [14] for the infinite horizon case. We mention that (7.2) and (7.3) follow from (9.3) and (9.4), respectively, while (7.1) holds as the stable and the unstable manifolds are uniformly transversal, see [16, Formulas (4.13) and (4.21)].

Proposition 9.3. *For all $y, y' \in Y_j$, $j \geqslant 1$, and all $0 \leqslant \ell \leqslant \tau(y) - 1$,*

$$|h(f^\ell y) - h(f^\ell y')| \leqslant 2C_2^2 C_4 \gamma^{-1} (\gamma^\ell d(y, y') + \gamma^{\tau(y)-\ell} \gamma^{s(y,y')}).$$

Proof. Let $z = W^s(y) \cap W^u(y')$. By (7.4), $Fz \in W^u(Fy')$. By (9.3) and (9.4), for $0 \leqslant \ell \leqslant \tau(y)$,

$$d(f^\ell y, f^\ell y') \leqslant d(f^\ell y, f^\ell z) + d(f^\ell z, f^\ell y') \leqslant C_2(\gamma^\ell d(y, z) + \gamma^{\tau(y)-\ell} d(Fz, Fy')).$$

Using also (7.1) and (7.3),

$$d(f^\ell y, f^\ell y') \leqslant C_2(\gamma^\ell C_4 d(y, y') + \gamma^{\tau(y)-\ell} C_2 \gamma^{s(y,y')-1}).$$

Hence by (9.1), for $\ell \leqslant \tau(y) - 1$,

$$|h(f^\ell y) - h(f^\ell y')| \leqslant d(f^\ell y, f^\ell y') + d(f^{\ell+1} y, f^{\ell+1} y') \ll \gamma^\ell d(y, y') + \gamma^{\tau(y)-\ell} \gamma^{s(y,y')},$$

as required. □

Define the induced roof function $\varphi = \sum_{\ell=0}^{\tau-1} h \circ f^\ell$. Using (7.3), it is immediate from Proposition 9.3 that φ has bounded Hölder constants in the sense of Sect. 8.1:

Corollary 9.4. *Conditions* (8.1) *and* (8.2) *hold.*

Proof. If $y' \in W^s(y)$, then $s(y, y') = \infty$ so $|\varphi(y) - \varphi(y')| \ll d(y, y')$ by Proposition 9.3. If $y' \in W^u(y)$, then $d(y, y') \leqslant C_2 \gamma^{s(y,y')}$ by (7.3), so $|\varphi(y) - \varphi(y')| \ll \gamma^{s(y,y')}$ by Proposition 9.3. □

Proposition 9.5. *For* $\operatorname{diam} Y$ *sufficiently small, there exist an integer* $n_0 \geqslant 1$ *and a constant* $C > 0$ *such that for all* $y, y' \in Y$, $s(y, y') \geqslant n_0$, *and all* $u \in [0, \varphi(y)] \cap [0, \varphi(y')]$, *there exist* $t, t' \in \mathbb{R}$ *such that*

$$|t - u| \leqslant C d(y, y'), \qquad d(T_u y, T_t z) \leqslant C d(y, y'),$$
$$|t' - u| \leqslant C \gamma^{s(y,y')}, \qquad d(T_u y', T_{t'} z) \leqslant C \gamma^{s(y,y')},$$

where $z = W^s(y) \cap W^u(y')$.

Proof. Define $h_\ell(y) = \sum_{j=0}^{\ell-1} h(f^j y)$ for $y \in Y$, $0 \leqslant \ell \leqslant \tau(y)$. By Proposition 9.3, there is a constant $C > 0$ such that

$$|h_\ell(y) - h_\ell(y')| \leqslant \sum_{j=0}^{\tau(y)-1} |h(f^j y) - h(f^j y')| \leqslant C(d(y, y') + \gamma^{s(y,y')}), \tag{9.7}$$

for all $y, y' \in Y_j$, $j \geqslant 1$ (which is equivalent to $s(y, y') \geqslant 1$) and all $0 \leqslant \ell \leqslant \tau(y)$.

Now consider $y, y' \in Y$ with $s(y, y') \geqslant n_0$, and $u \in [0, \varphi(y)] \cap [0, \varphi(y')]$. Let $z = W^s(y) \cap W^u(y')$.

Choosing t. By (7.1), $d(y, z) \leqslant C_4 d(y, y')$. Also, $s(y, z) = \infty$. We can shrink Y if necessary so that $C \operatorname{diam} Y < \inf h$.

Write $T_u y = T_r f^\ell y$ where $0 \leqslant \ell \leqslant \tau(y) - 1$ and $r \in [0, h(f^\ell y))$. (When $u = \varphi(y)$, we take $\ell = \tau(y) - 1$, $r = h(f^\ell y)$.) Similarly, write $T_u z = T_{r'} f^{\ell'} z$. Note that $u = h_\ell(y) + r = h_{\ell'}(z) + r'$.

First we show that $|\ell - \ell'| \leqslant 1$. If $\ell \geqslant \ell' + 1$, then by (9.7),

$$\begin{aligned}(\ell - \ell' - 1) \inf h &\leqslant h_\ell(z) - h_{\ell'+1}(z) \leqslant h_\ell(y) - h_{\ell'+1}(z) + h_\ell(z) - h_\ell(y) \\ &\leqslant h_\ell(y) - h_{\ell'}(z) - h(f^{\ell'} z) + C \operatorname{diam} Y \\ &= r' - r - h(f^{\ell'} z) + C \operatorname{diam} Y \leqslant C \operatorname{diam} Y < \inf h,\end{aligned}$$

hence $\ell \leqslant \ell' + 1$. Similarly, if $\ell' \geqslant \ell + 1$, then $(\ell' - \ell - 1) \inf h \leqslant h_{\ell'}(y) - h_{\ell+1}(y) < \inf h$, which implies $\ell' \leqslant \ell + 1$. Thus we indeed have $|\ell - \ell'| \leqslant 1$.

If $\ell = \ell'$, then we take $t = u$. By (9.7),

$$|r - r'| = |h_\ell(y) - h_\ell(z)| \leqslant C d(y, z) \leqslant C C_4 d(y, y').$$

By (9.3), $d(f^\ell y, f^\ell z) \leqslant C_2 d(y,z) \leqslant C_2 C_4 d(y,y')$. Without loss, $r \leqslant r'$, so by (9.2) and (9.5)

$$d(T_u y, T_t z) = d(T_r f^\ell y, T_{r'} f^\ell z) \leqslant d(T_r f^\ell y, T_r f^\ell z) + d(T_r f^\ell z, T_{r'} f^\ell z)$$
$$\leqslant C_5 d(f^\ell y, f^\ell z) + |r - r'| \ll d(y, y').$$

If $\ell' = \ell - 1$, then we take $t = u + r + s$ where $s = h(f^{\ell-1}z) - r' \geqslant 0$. Then $T_u y = T_r f^\ell y$ and $T_t z = T_{r+s} T_{r'} f^{\ell-1} z = T_{r+h(f^{\ell-1}z)} f^{\ell-1} z = T_r f^\ell z$.

Note that $u = h_\ell(y) + r = h_\ell(z) - s$, hence $r + s = h_\ell(z) - h_\ell(y) \leqslant C d(y,z)$ by (9.7). In particular, $|t - u| = r + s \leqslant C C_4 d(y, y')$. Also $0 \leqslant r \leqslant r + s \leqslant C \operatorname{diam} Y \leqslant \inf h$. Hence by (9.3) and (9.5),

$$d(T_u y, T_t z) = d(T_r f^\ell y, T_r f^\ell z) \leqslant C_5 d(f^\ell y, f^\ell z) \leqslant C_2 C_5 d(y,z) \leqslant C_2 C_4 C_5 d(y, y').$$

The argument for $\ell' = \ell + 1$ is analogous.

Choosing t'. This goes along similar lines. We can shrink $\operatorname{diam} Y$ and increase n_0 so that $C(C_2 + 1)(\operatorname{diam} Y + \gamma^{n_0}) < \inf h$. Note that $s(z, y') = s(y, y') \geqslant n_0 \geqslant 1$.

Since $s(z, y') \geqslant 1$, it follows from (7.4) that $Fz \in W^u(Fy')$. Write $T_u z = T_{-r} f^{-\ell} Fz$ where $0 \leqslant \ell \leqslant \tau(y) - 1$ and $r \in [0, h(f^{-(\ell+1)} Fz))$. Similarly write $T_u y' = T_{-r'} f^{-\ell'} F y'$. Note that $u = h_{\tau(y)-\ell}(z) - r = h_{\tau(y)-\ell'}(y') - r'$.

Again, we show that $|\ell - \ell'| \leqslant 1$. If $\ell \geqslant \ell' + 1$, by (9.7),

$$(\ell - \ell' - 1) \inf h \leqslant h_{\tau(y)-\ell'-1}(y') - h_{\tau(y)-\ell}(y')$$
$$\leqslant h_{\tau(y)-\ell'-1}(y') - h_{\tau(y)-\ell}(z) + C(\operatorname{diam} Y + \gamma^{n_0})$$
$$= r' - r - h(f^{\tau(y)-\ell'-1} y') + C(\operatorname{diam} Y + \gamma^{n_0}) \leqslant C(\operatorname{diam} Y + \gamma^{n_0}) < \inf h,$$

hence $\ell \leqslant \ell' + 1$. Similarly, if $\ell' \geqslant \ell + 1$, then $(\ell' - \ell - 1) \inf h \leqslant h_{\tau(y)-\ell-1}(z) - h_{\tau(y)-\ell'}(z) < \inf h$ so $\ell' \leqslant \ell + 1$. Thus we have $|\ell - \ell'| \leqslant 1$.

If $\ell = \ell'$, then we take $t' = u$. It follows from (7.3) and (9.7) that

$$|r - r'| = |h_{\tau(y)-\ell}(y') - h_{\tau(y)-\ell}(z)| \leqslant C(d(y', z) + \gamma^{s(y',z)}) \leqslant C(C_2 + 1)\gamma^{s(y',z)}.$$

Also, by (7.3) and (9.4),

$$d(f^{-\ell} F y', f^{-\ell} F z) \leqslant C_2 d(Fy', Fz) \leqslant C_2^2 \gamma^{-1} \gamma^{s(y',z)}.$$

Without loss, $r' \leqslant r$, so by (7.3), (9.2) and (9.6),

$$d(T_u y', T_u z) = d(T_{-r'} f^{-\ell} F y', T_{-r} f^{-\ell} F z)$$
$$\leqslant d(T_{-r'} f^{-\ell} F y', T_{-r'} f^{-\ell} F z) + d(T_{-r'} f^{-\ell} F z, T_{-r} f^{-\ell} F z)$$
$$\leqslant C_5 d(f^{-\ell} F y', f^{-\ell} F z) + |r - r'| \ll \gamma^{s(y',z)} = \gamma^{s(y,y')}.$$

If $\ell = \ell' - 1$, then we take $t' = u - r' - s$ where $s = h(f^{-(\ell-1)} F z) - r \geqslant 0$. Then $T_u y' = T_{-r'} f^{-\ell'} F y'$ and $T_{t'} z = T_{-r'-s} T_u z = T_{-r'} f^{-\ell'} F z$.

Note that $u = h_{\tau(y)-\ell'}(y') - r' = h_{\tau(y)-\ell'}(z) + s$, hence $r' + s = h_{\tau(y)-\ell'}(y') - h_{\tau(y)-\ell'}(z) \leqslant C(C_2 + 1)\gamma^{s(y,y')}$ by (9.7). In particular, $|t' - u| = r' + s \ll \gamma^{s(y,y')}$. Also, $0 \leqslant r' \leqslant r' + s \leqslant C(C_2 + 1)\gamma^{n_0} \leqslant \inf h$. Hence by (7.3), (9.4) and (9.6),

$$d(T_u y', T_{t'} z) = d(T_{-r'} f^{-\ell'} F y', T_{-r'} f^{-\ell'} F z) \leqslant C_5 d(f^{-\ell'} F y', f^{-\ell'} F z) \ll \gamma^{s(y,y')}.$$

The argument for $\ell = \ell' + 1$ is analogous. □

Corollary 9.6. *Let $v \in C^{0,\eta}(M)$, $w \in C^{0,m}(M)$ such that $\partial_t^k w \in C^{0,\eta}(M)$, for all $k = 0, \ldots, m$. Suppose also that there is a constant $C > 0$ such that $|v(x) - v(x')| \leqslant Cd(x,x')^\eta$ and $|\partial_t^k w(x) - \partial_t^k w(x')| \leqslant Cd(x,x')^\eta$ for all $x, x' \in M$ of the form $x = T_u y$, $x' = T_u y'$ where $y, y' \in Y_j$ for some $j \geqslant 1$, $u \in [0, \varphi(y)]$, $u' \in [0, \varphi(y')]$, and for all $k = 0, \ldots, m$. Then h, T_t, v and w are dynamically Hölder in the sense of Definition 7.6.*

Proof. Condition (a) of Definition 7.6 follows from Proposition 9.3. To check condition (b), we distinguish two cases. If $s(y, y') < n_0$, we may take $t = t' = u$ and use that $|v(x) - v(x')| \leqslant 2|v|_\infty \ll \gamma^{n_0}$ for any $x, x' \in M$. If $s(y, y') \geqslant n_0$, Proposition 9.5 applies and, along with Formulas (9.1)–(9.6), implies Definition 7.6(b) . □

9.2. Tail estimate for φ and completion of the Proof of Theorem 9.1. Since

$$\mu_X(x \in X : h(x) > t) = O(t^{-2}) \tag{9.8}$$

$$\mu(y \in Y : \tau(y) > n) = O(e^{-cn}) \quad \text{for some } c > 0, \tag{9.9}$$

a standard argument shows that $\mu(\varphi > t) = O((\log t)^2 t^{-2})$. In fact, we have

Proposition 9.7. $\mu(\varphi > t) = O(t^{-2})$.

The crucial ingredient for proving Proposition 9.7 is due to Szász and Varjú [34].

Lemma 9.8. ([34, Lemma 16], [18, Lemma 5.1]) *There are constants $p, q > 0$ with the following property. Define*

$$X_b(m) = \big\{x \in X : [h(x)] = m \;\; \text{and} \;\; h(T^j x) > m^{1-q} \;\; \text{for some } j \in \{1, \ldots, b \log m\}\big\}.$$

Then for any b sufficiently large there is a constant $C = C(b) > 0$ such that

$$\mu_X(X_b(m)) \leqslant C m^{-p} \mu_X(x \in X : [h(x)] = m) \;\; \text{for all } m \geqslant 1.$$

□

For $b > 0$, define

$$Y_b(n) = \{y \in Y : \tau(y) \leqslant b \log n \;\; \text{and} \;\; \max_{0 \leqslant \ell < \tau(y)} h(T^\ell y) \leqslant \tfrac{1}{2} n \;\; \text{and} \;\; \varphi(y) \geqslant n\}.$$

Corollary 9.9. *For b sufficiently large, $\mu(Y_b(n)) = o(n^{-2})$.*

Proof. Fix p and q as in Lemma 9.8. Also fix b sufficiently large.

Let $y \in Y_b(n)$. Define $h_1(y) = \max_{0 \leqslant \ell < \tau(y)} h(f^\ell y)$ and choose $\ell_1(y) \in \{0, \ldots, \tau(y) - 1\}$ such that $h_1(y) = h(f^{\ell_1(y)} y)$. Define $h_2(y) = \max_{0 \leqslant \ell < \tau(y),\, \ell \neq \ell_1(y)} h(f^\ell y)$. Then $h_1(y)$ and $h_2(y)$ are the two largest free flights $h \circ f^\ell$ during the iterates $\ell = 0, \ldots, \tau(y) - 1$.

We begin by showing that these two flight times have comparable length. Indeed, let $m_i = [h_i]$, $i = 1, 2$. Then $n \leqslant \varphi \leqslant h_1 + (\tau - 1)h_2 \leqslant n/2 + (b \log n)h_2$. Hence

$$\frac{n}{2b \log n} - 1 \leqslant m_2 \leqslant m_1 \leqslant \frac{n}{2}. \tag{9.10}$$

In particular, $m_1 > m_2^{1-q}$ and $m_2 > m_1^{1-q}$ for large n.

Choose $\ell_2(y) \in \{0, \dots, \tau(y)-1\}$ such that $\ell_2(y) \neq \ell_1(y)$ and $h_2(y) = h(f^{\ell_2(y)}y)$. We can suppose without loss that $\ell_1(y) < \ell_2(y)$. For large n, it follows from (9.10) that $f^{\ell_1(y)}y \in X_b(m_1(y))$. Hence

$$Y_b(n) \subset f^{-\ell} X_b(m) \quad \text{for some } \ell < b \log n, m \geqslant n/(2b\log n) - 1,$$

and so

$$\mu(Y_b(n)) \ll \mu_X(Y_b(n) \times 0) \leqslant b \log n \sum_{m \geqslant n/(2b\log n)-1} \mu_X(X_b(m)).$$

By Lemma 9.8 and (9.8),

$$\begin{aligned} \mu(Y_b(n)) &\ll \log n \sum_{m \geqslant n/(2b\log n)-1} m^{-p} \mu_X(x \in X : [h(x)] = m) \\ &\ll \log n (n/\log n)^{-(2+p)} = o(n^{-2}), \end{aligned}$$

as required. □

Proof of Proposition 9.7. Define the tower $\Delta = \{(y, \ell) \in Y \times \mathbb{Z} : 0 \leqslant \ell \leqslant \tau(y) - 1\}$ with probability measure $\mu_\Delta = \mu \times \text{counting}/\bar{\tau}$ where $\bar{\tau} = \int_Y \tau \, d\mu$. Recall that $\mu_X = \pi_* \mu_\Delta$ where $\pi(y, \ell) = f^\ell y$.

Write $\max_{0 \leqslant \ell < \tau(y)} h(f^\ell y) = h(f^{\ell_1(y)} y)$ where $\ell_1(y) \in \{0, \dots, \tau(y) - 1\}$. Then

$$\begin{aligned} &\mu\{y \in Y : \max_{0 \leqslant \ell < \tau(y)} h(f^\ell y) > n/2\} = \bar{\tau}\mu_\Delta\{(y, 0) \in \Delta : h(f^{\ell_1(y)}y) > n/2\} \\ &= \bar{\tau}\mu_\Delta\{(y, \ell_1(y)) : h(f^{\ell_1(y)}y) > n/2\} = \bar{\tau}\mu_\Delta\{(y, \ell_1(y)) : h \circ \pi(y, \ell_1(y)) > n/2\} \\ &\leqslant \bar{\tau}\mu_\Delta\{p \in \Delta : h \circ \pi(p) > n/2\} = \bar{\tau}\mu_X\{x \in X : h(x) > n/2\}, \end{aligned}$$

and so $\mu\{y \in Y : \max_{0 \leqslant \ell < \tau(y)} h(T^\ell y) > n/2\} = O(n^{-2})$ by (9.8). Hence it follows from Corollary 9.9 that

$$\mu\{y \in Y : \tau(y) \leqslant b \log n \ \text{ and } \ \varphi(y) \geqslant n\} = O(n^{-2}).$$

Finally, by (9.9), $\mu(\tau > b \log n) = O(n^{-bc}) = o(n^{-2})$ for any $b > 2/c$ and so $\mu(\varphi \geqslant n) = O(n^{-2})$ as required. □

It follows from Lemma 8.3 and Corollary 9.4 that condition (H) is satisfied. Hence by Corollary 8.1(a), the suspension flow $F_t : Y^\varphi \to Y^\varphi$ is a Gibbs–Markov flow as defined in Sect. 6. By Proposition 9.7, $\mu(\varphi > t) = O(t^{-2})$. By Corollary 9.6, the flows and observables are dynamically Hölder (Definition 7.6). Hence it follows from Corollary 8.1(b) that absence of approximate eigenfunctions implies decay rate $O(t^{-1})$.

Finally, we exclude approximate eigenfunctions. By Corollary 9.4, condition (8.2) holds and hence the temporal distortion function $D : Y \times Y \to \mathbb{R}$ is defined as in Sect. 8.4. Let $\bar{Z}_0 \subset \bar{Y}$ be a finite subsystem and let $Z_0 = \bar{\pi}^{-1}\bar{Z}_0$. The presence of a contact structure implies by Remark 8.11 that the lower box dimension of $D(Z_0 \times Z_0)$ is positive. Hence absence of approximate eigenfunctions follows from Lemma 8.10.

9.3. Semi-dispersing Lorentz flows and stadia. In this subsection we discuss two further classes of billiard flows and show that the scheme presented above can be adapted to cover these examples, resulting in Theorem 9.13.

Semi-dispersing Lorentz flows are billiard flows in the planar domain obtained as $R \setminus \bigcup S_k$ where R is a rectangle and the $S_k \subset R$ are finitely many disjoint convex scatterers with C^3 boundaries of nonvanishing curvature. By the unfolding process – tiling the plane with identical copies of R, and reflecting the scatterers S_k across the sides of all these rectangles – an infinite periodic configuration is obtained, which can be regarded as an infinite horizon Lorentz gas.

Bunimovich stadia are convex billiard domains enclosed by two semicircular arcs (of equal radii) connected by two parallel line segments. An unfolding process could reduce the bounces on the parallel line segments to long flights in an unbounded domain, however, there is another quasi-integrable effect here corresponding to sequences of consecutive collisions on the same semi-circular arc.

Both of these examples have been extensively studied in the literature, see for instance [7,10,16,17,29], and references therein. A common feature of the two examples is that the billiard map itself is not uniformly hyperbolic; however, there is a geometrically defined first return map which has uniform expansion rates. As before, the billiard domain is denoted by Q, and the billiard flow is $T_t : M \to M$ where $M = Q \times \mathbb{S}^1$. However, this time we prefer to denote the natural Poincaré section $\partial Q \times [-\pi/2, \pi/2] \subset M$ by $\widetilde{X}$, the corresponding billiard map as $\tilde{f} : \widetilde{X} \to \widetilde{X}$, and the free flight function as $\tilde{h} : \widetilde{X} \to \mathbb{R}^+$ where $\tilde{h}(\tilde{x}) = \inf\{t > 0 : T_t\tilde{x} \in \widetilde{X}\}$. Then, as mentioned above, there is a subset $X \subset \widetilde{X}$ such that the first return map of $\tilde{f}$ to X has good hyperbolic properties. We denote this first return map by $f : X \to X$. The corresponding free flight function $h : X \to \mathbb{R}^+$ is given by $h(x) = \inf\{t > 0 : T_t x \in X\}$. Let us, furthermore, introduce the discrete return time $\tilde{r} : X \to \mathbb{Z}^+$ given by $\tilde{r}(x) = \min\{n \geqslant 1 : \tilde{f}^n x \in X\}$.

In the case of the semi-dispersing Lorentz flow, X corresponds to collisions on the scatterers S_k. In the case of the stadium, X corresponds to first bounces on semi-circular arcs, that is, $x \in X$ if x is on one of the semi-circular arcs, but $\tilde{f}^{-1}x$ is on another boundary component (on the other semi-circular arc, or on one of the line segments).

The following properties hold. Unless otherwise stated, standard references are [16, Chapter 8] and [17]. As in Sect. 9.1, $d(x, x')$ always denotes the Euclidean distance of the two points, generated by the Riemannian metric.

- There is a countable partition $X \setminus \mathcal{S} = \bigcup_{m=1}^{\infty} X_m$ such that $f|_{X_m}$ is C^2 and $\tilde{r}|_{X_m}$ is constant for any $m \geqslant 1$. We refer to the partition elements X_m with $\tilde{r}|_{X_m} \geqslant 2$ as *cells*; these are of two different types:
  - *Bouncing cells* are present both in the semi-dispersing billiard examples and in stadia. For these, one iteration of $f|_{X_m}$ consists of several consecutive reflections on the flat boundary components, that is, the line segments. By the above mentioned unfolding process, these reflections reduce to trajectories along straight lines in the associated unbounded table.
  - *Sliding cells* are present only in stadia. For these, one iteration of $f|_{X_m}$ consists of several consecutive collisions on the same semi-circular arc.
- $\inf h > 0$, and $\sup \tilde{h} < \infty$, however, there is no uniform upper bound on h, and no uniform lower bound for $\tilde{h}$.
- $f : X \to X$ is uniformly hyperbolic in the sense that stable and unstable manifolds exist for almost every x, and Formulas (9.3) and (9.4) hold. This follows from the uniform expansion rates of f, see [16, Formula (8.22)].

- If $x, x' \in X_m$ where X_m is a bouncing cell, in the associated unfolded table the flow trajectories until the first return to X are straight lines, hence (9.1) follows. If $x, x' \in X_m$ and X_m is a sliding cell, the induced roof function is uniformly Hölder continuous with exponent $1/4$, as established in the proof of [7, Theorem 3.1]. The same geometric reasoning applies to $\tilde{h}_k(x) = \tilde{h}(x) + \tilde{h}(\tilde{f}x) + \cdots + \tilde{h}(\tilde{f}^{k-1}x)$ as long as $k \leqslant \tilde{r}(x)$. Summarizing, we have

$$|\tilde{h}_k(x) - \tilde{h}_k(x')| \ll d(x, x')^{1/4} + d(fx, fx')^{1/4} \tag{9.11}$$

for $x, x' \in X_m$, $m \geqslant 1$ and $k \leqslant \tilde{r}(x) - 1$. In particular, $|h(x) - h(x')| \ll d(x, x')^{1/4} + d(fx, fx')^{1/4}$.
- (9.2) has to be relaxed to

$$d(T_t\tilde{x}, T_{t'}\tilde{x}) \leqslant |t - t'| \quad \text{for all } \tilde{x} \in \widetilde{X} \text{ and } t, t' \in [0, \tilde{h}(\tilde{x})). \tag{9.12}$$

- (9.5) has to be relaxed to the following two formulas:

$$d(T_t\tilde{x}, T_t\tilde{x}') \ll d(\tilde{x}, \tilde{x}') \quad \text{for } \tilde{x} \in \widetilde{X}, \tilde{x}' \in W^s(\tilde{x}),\ t \in [0, \tilde{h}(\tilde{x})) \cap [0, \tilde{h}(\tilde{x}')); \tag{9.13}$$

$$d(\tilde{f}^k x, \tilde{f}^k x') \ll d(x, x') \quad \text{for } x \in X, x' \in W^s(x),\ 0 \leqslant k. \tag{9.14}$$

Similarly, (9.6) has to be relaxed to

$$d(T_{-t}\tilde{x}, T_{-t}\tilde{x}') \ll d(\tilde{x}, \tilde{x}') \quad \text{for } \tilde{x} \in \widetilde{X}, \tilde{x}' \in W^u(\tilde{x}),\ t \in [0, \tilde{h}(\tilde{f}^{-1}\tilde{x})) \cap [0, \tilde{h}(\tilde{f}^{-1}\tilde{x}')); \tag{9.15}$$

$$d(\tilde{f}^{-k} x, \tilde{f}^{-k} x') \ll d(x, x') \quad \text{for } x \in X, x' \in W^u(x),\ 0 \leqslant k. \tag{9.16}$$

To verify (9.16), let us note first that $d(x, x')$ consists of a position and a velocity component, and in course of a free flight velocities do not change. Now the mechanism of hyperbolicity for stadia is defocusing, see, for instance, [16, Figure 8.1], which guarantees that for $x' \in W^u(x)$, the position component of $d(x, x')$ in course of the free flight is dominated by the position component at the end of the free flight. (9.14) holds for analogous reasons. To verify (9.15), by uniform hyperbolicity of f (in particular Formula (9.4), see above), it is enough to consider how $\tilde{f}$ evolves unstable vectors between two consecutive applications of f, ie. within a series of sliding or bouncing collisions. On the one hand, again by the defocusing mechanism, $\tilde{f}$ does not contract the p-length of unstable vectors, see [16, Section 8.2]. On the other hand, for an unstable vector, the ratio of the Euclidean and the p-length is $\sqrt{1 + \mathcal{V}^2} / \cos\varphi$, where $\mathcal{V}$ is the slope of the unstable vector in the standard billiard coordinates, and φ is the collision angle, see [16, Formula (8.21)]. Now $|\mathcal{V}|$ is uniformly bounded away from ∞, see Formula [16, Formula (8.18)], while $\cos\varphi$ is constant in course of a sequence of consecutive sliding or bouncing collisions. (9.13) holds by an analogous argument.
- The map $f : X \to X$ can be modeled by a Young tower with exponential tails. In particular, there exists a subset $Y \subset X$ and an induced map $F = f^\tau : Y \to Y$ that possesses the properties discussed in Sect. 7.1 including (7.4). The tails of the return time $\tau : Y \to \mathbb{Z}^+$ are exponential, i.e. $\mu(\tau > n) = O(e^{-cn})$ for some $c > 0$.[4]

[4] It is important to note that here τ is the return time to Y in terms of f; the return time in terms of $\tilde{f}$ has polynomial tails.

Moreover, the construction can be carried out so that diam Y is as small as desired. The existence of the Young tower satisfying these properties is established in [17]. As in Sect. 9.1, we introduce the induced roof function $\varphi = \sum_{\ell=0}^{\tau-1} h \circ f^\ell$.

- By construction, for $y, y' \in Y_j$, $j \geqslant 1$ and $\ell \leqslant \tau$ fixed, $f^\ell y$ and $f^\ell y'$ always belong to the same cell of X.

Let us introduce $\hat{\gamma} = \gamma^{1/4}$ and $\bar{d}(y, y') = d(y, y')^{1/4}$. The following version of Proposition 9.3 holds.

Proposition 9.10. *For all* $y, y' \in Y_j$, $j \geqslant 1$, *and all* $0 \leqslant \ell \leqslant \tau(y) - 1$,

$$|h(f^\ell y) - h(f^\ell y')| \ll \hat{\gamma}^\ell (\bar{d}(y, y') + \hat{\gamma}^{\tau(y)-\ell} \hat{\gamma}^{s(y,y')}).$$

Proof. The proof of Proposition 9.3 applies, using (9.11) instead of (9.1). □

This readily implies

Corollary 9.11. *Conditions* (8.1) *and* (8.2) *hold, with* γ *replaced by* $\hat{\gamma}$, *and* $d(y, y')$ *replaced by* $\bar{d}(y, y')$. □

The adapted version of Proposition 9.5 reads as follows.

Proposition 9.12. *For* diam Y *sufficiently small, there exist an integer* $n_0 \geqslant 1$ *and a constant* $C > 0$ *such that for all* $y, y' \in Y$, $s(y, y_0) \geqslant n_0$, *and all* $u \in [0, \varphi(y)] \cap [0, \varphi(y')]$, *there exist* $t, t' \in \mathbb{R}$ *such that*

$$|t - u| \leqslant C\bar{d}(y, y'), \qquad d(T_u y, T_t z) \leqslant C\bar{d}(y, y'),$$
$$|t' - u| \leqslant C\hat{\gamma}^{s(y,y')}, \qquad d(T_u y', T_{t'} z) \leqslant C\hat{\gamma}^{s(y,y')},$$

where $z = W^s(y) \cap W^u(y')$.

Proof. First, (9.7) can be updated as

$$|h_\ell(y) - h_\ell(y')| \leqslant \sum_{j=0}^{\tau(y)-1} |h(f^j y) - h(f^j y')| \ll \bar{d}(y, y') + \hat{\gamma}^{s(y,y')}, \tag{9.17}$$

for $0 \leqslant \ell \leqslant \tau(y)$.

Fix $y, y' \in Y_j$ for some $j \geqslant 1$, and $u \in [0, \varphi(y)] \cap [0, \varphi(y')]$. We will focus on choosing the appropriate t and obtaining the relevant estimates. The choice of t' is analogous. Recall the notation $\bar{d}(y, z) = d(y, z)^{1/4}$ and note that $\bar{d}(y, z) \ll \bar{d}(y, y')$.

First adjustment. As in the proof of Proposition 9.5, we arrive at $T_u y = T_r f^\ell y$ and $T_{t_1} z = T_{r_1} f^\ell z$ for the same $0 \leqslant \ell \leqslant \tau(y) - 1$, and such that $|u - t_1| \ll \bar{d}(y, z)$ and $|r - r_1| \ll \bar{d}(y, z)$. Indeed, a priori we have $T_u y = T_r f^\ell y$ and $T_u z = T_{r'} f^{\ell'} z$, where, as $\inf h > 0$, shrinking diam Y if needed, (9.17) implies $|\ell - \ell'| \leqslant 1$. If $\ell = \ell'$, then let $t_1 = u$, $r_1 = r'$, and $|r - r_1| \ll \bar{d}(y, z)$ follows from (9.17). If $\ell' = \ell - 1$, then $T_u z = T_{-r^*} f^\ell z$, where $r^* = h(f^{\ell-1} z) - r' \in [0, h(f^{\ell-1} z)]$. Note that $u = h_\ell(y) + r = h_\ell(z) - r^*$, hence $r + r^* = h_\ell(z) - h_\ell(y) \ll \bar{d}(y, z)$. Let $t_1 = u + r + r^*$, so that $|t_1 - u| \ll \bar{d}(y, z)$ and $r_1 = r$ as $T_{t_1} z = T_r f^\ell z$. Note that we do not claim anything about $d(T_u y, T_{t_1} z)$ at this point.

Second adjustment. For brevity, introduce $\hat{y} = f^\ell y$ and $\hat{z} = f^\ell z$. We have

$$T_u y = T_r \hat{y} = T_s \tilde{f}^k \hat{y}, \qquad T_{t_1} z = T_{r_1} \hat{z} = T_{s'} \tilde{f}^{k'} \hat{z},$$

for some $0 \leqslant k, k' \leqslant \tilde{r}(\hat{y}) - 1$ (note that $\tilde{r}(\hat{y}) = \tilde{r}(\hat{z})$), $s \in [0, \tilde{h}(\tilde{f}^k \hat{y}))$ and $s' \in [0, \tilde{h}(\tilde{f}^{k'} \hat{z}))$. Note that by (9.13), (9.14) and (9.3), for any $0 \leqslant k \leqslant \tilde{r}(\hat{y}) - 1$, we have

$$d(\tilde{f}^k \hat{y}, \tilde{f}^k \hat{z}) \ll d(\hat{y}, \hat{z}) \ll d(y, z), \quad \text{hence} \quad |\tilde{h}_k(\hat{y}) - \tilde{h}_k(\hat{z})| \ll \bar{d}(y, z), \tag{9.18}$$

where we have used (9.11). We distinguish three cases: $k = k'$, $k > k'$ and $k < k'$.

If $k = k'$, (9.18) along with $|r - r_1| \ll \bar{d}(y, z)$ implies $|s - s'| \ll \bar{d}(y, z)$. But then, again by (9.18), (9.13) and (9.14), we have

$$d(T_u y, T_{t_1} z) = d(T_s \tilde{f}^k \hat{y}, T_{s'} \tilde{f}^k \hat{z}) \ll \bar{d}(y, z).$$

As $|u - t_1| \ll \bar{d}(y, z)$, we can fix $t = t_1$.

If $k > k'$, we prefer to represent our points as

$$T_u y = T_r \hat{y} = T_s \tilde{f}^k \hat{y}, \qquad T_{t_1} z = T_{r_1} \hat{z} = T_{-s_1} \tilde{f}^k \hat{z}$$

for some $s_1 > 0$. Now by (9.18) and as $|r - r_1| \ll \bar{d}(y, z)$, we have $s + s_1 \ll \bar{d}(y, z)$. Define

$$s_2 = \min(s, \tilde{h}(\tilde{f}^k \hat{z})/2, \tilde{h}(\tilde{f}^k \hat{y})/2), \quad r_2 = s_2 + s_1 + r_1, \quad t = s_2 + s_1 + t_1.$$

Then $T_t z = T_{s_2} \tilde{f}^k \hat{z}$, where $s_2 \in [0, \tilde{h}(\tilde{f}^k \hat{y})) \cap [0, \tilde{h}(\tilde{f}^k \hat{z}))$ and

$$|s - s_2| \leqslant s \leqslant s + s_1 \ll \bar{d}(y, z).$$

Hence

$$d(T_u y, T_t z) = d(T_s \tilde{f}^k \hat{y}, T_{s_2} \tilde{f}^k \hat{z}) \leqslant d(T_{s_2} \tilde{f}^k \hat{y}, T_{s_2} \tilde{f}^k \hat{z}) + d(T_s \tilde{f}^k \hat{y}, T_{s_2} \tilde{f}^k \hat{y}),$$

where $d(T_s \tilde{f}^k \hat{y}, T_{s_2} \tilde{f}^k \hat{y}) \ll \bar{d}(y, z)$ by (9.12), while $d(T_{s_2} \tilde{f}^k \hat{y}, T_{s_2} \tilde{f}^k \hat{z}) \leqslant \bar{d}(y, z)$ by (9.13), (9.14) and (9.18). Hence $d(T_u y, T_t z) \ll \bar{d}(y, z)$, as desired. On the other hand $|t - t_1| = s_1 + s_2 \leqslant s_1 + s \ll \bar{d}(y, z)$, and as we have already controlled $|t_1 - u|$, we have $|t - u| \ll \bar{d}(y, z)$.

The case when $k < k'$ can be treated analogously. The choice of t' goes along similar lines, so we omit the details. □

Theorem 9.13. *Consider a semi-dispersing Lorentz flow or the billiard flow in a Bunimovich stadium. Let $\eta \in (0, 1]$. There exists $m \geqslant 1$ such that $\rho_{v,w}(t) = O(t^{-1})$ for all $v \in C^\eta(M) \cap C^{0,\eta}(M)$ and $w \in C^{\eta,m}(M)$ (and more generally for the class of observables defined in Corollary 9.6).*

Proof. It follows from Lemma 8.3 and Corollary 9.11 that condition (H) is satisfied. Hence by Corollary 8.1(a), the suspension flow $F_t : Y^\varphi \to Y^\varphi$ is a Gibbs–Markov flow as defined in Sect. 6. The conclusions of Corollary 9.6 follow from Propositions 9.10 and 9.12. Hence the flows and observables are dynamically Hölder (Definition 7.6).

For the tail estimate on φ, introduce $\tilde{\tau} : Y \to \mathbb{Z}^+$, $\tilde{\tau}(y) = \min\{n \geqslant 1 : \tilde{f}^n y \in Y\}$. Note that $\sup \tilde{h} < \infty$, and $\varphi(y) = \sum_{k=0}^{\tilde{\tau}(y)-1} \tilde{h}(\tilde{f}^k y) \leqslant \tilde{\tau}(y) \sup \tilde{h}$. Also it is shown in [18] (both for the semi-dispersing examples and for stadia) that $\mu(\tilde{\tau} > n) = O(n^{-2})$. Hence $\mu(\varphi > t) \leqslant \mu(\tilde{\tau} \sup \tilde{h} > t) = O(t^{-2})$.

Finally, to exclude approximate eigenfunctions, we may appeal as at the end of Sect. 9.2 to the contact structure which the billiard examples have in common. The result now follows from Corollary 8.1(b). □

9.4. Lower bounds. In this subsection, we show that it is impossible to improve on the error rate $O(t^{-1})$ for infinite horizon Lorentz gases, semidispersing Lorentz flows, and Bunimovich stadia. The following result is based on [6, Corollary 1.3].

Proposition 9.14. *Let $v \in L^2(M)$ with $\int_M v\, d\mu_M = 0$. Suppose that $\rho_{v,v}(t) = o(t^{-1})$. Then $|\int_0^t v \circ T_s\, ds|_2 = o((t \log t)^{1/2})$.*

Proof. Let $v_t = \int_0^t v \circ T_s\, ds$. Then

$$\int_M v_t^2\, d\mu_M = \int_0^t \int_0^t \int_M v \circ T_r\, v \circ T_s\, d\mu_M\, dr\, ds = 2\int_0^t \int_0^s \int_M v\, v \circ T_{s-r}\, d\mu_M\, dr\, ds$$
$$= 2\int_0^t \int_0^s \rho_{v,v}(r)\, dr\, ds = 2\int_0^t \int_r^t \rho_{v,v}(r)\, ds\, dr \leqslant 2t \int_0^t \rho_{v,v}(r)\, dr.$$

By the assumption on $\rho_{v,v}$, we obtain $|v_t|_2^2 = o(t \log t)$. □

In the case of the planar infinite horizon Lorentz gas, Szász and Varjú [34] showed that $(t \log t)^{-1/2} \int_0^t v \circ T_s\, ds$ converges in distribution to a nondegenerate normal distribution for typical Hölder mean zero observables v. The result applies equally to semidispersing Lorentz flows. Similarly, in the case of Bunimovich stadia by Bálint and Gouëzel [6, Corollary 1.6]. In particular, $(t \log t)^{-1/2} |\int_0^t v \circ T_s\, ds|_2 \not\to 0$. Hence by Proposition 9.14, an upper bound of the type $o(t^{-1})$ is impossible and so the upper bound in Theorems 9.1 and 9.13 is optimal.

Remark 9.15. There is also the possibility of obtaining an asymptotic expression of the form

$$\rho_{v,w}(t) = ct^{-1} + O(t^{-(2-\epsilon)}), \tag{9.19}$$

($\epsilon > 0$ arbitrarily small, $c > 0$) for certain classes of observables v, w. Such results are obtained in [31] in cases where there is a first return to a uniformly hyperbolic map $f : X \to X$. The first return map in the examples considered here is nonuniformly hyperbolic, modelled by a Young tower with exponential tails, so [31] does not apply directly. In a recent preprint, [13] have announced the existence of a uniformly hyperbolic first return. This combined with [31] may yield the asymptotic (9.19). (Interestingly, the class of observables in (9.19) would be disjoint from the class of observables covered by Proposition 9.14.)

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A. Markov Property

The purpose of this appendix is to show that equality (7.4) holds in the abstract framework of Young [36]. Often this equality is assumed without further comment (see, e.g., [32, (Y1) p. 8538], [19, (P2) p. 51]) but we prefer to take the opportunity here to show that it follows directly from Young's original assumptions.

In this appendix, we use the notation of Young [36] since the result is general and independent of the rest of this article. Hence our notation here is consistent with Young [36] but differs from the other sections of this article where we used notation consistent with [30]. In particular:

- The dynamical system studied is $f : M \to M$ (the corresponding notation is $f : X \to X$ in the other sections of this article);
- The set with hyperbolic product structure is Λ (this was denoted by Y in the other sections of this article);
- Local stable and unstable manifolds for points $y \in \Lambda$ are denoted as $\gamma^s = \gamma^s(y)$ and $\gamma^u = \gamma^u(y)$, respectively. In the other sections of this article, these are denoted by W^s and W^u, respectively. In other words, following the notation of [36], in the appendix, we write $\Lambda = \left(\bigcup \gamma^u\right) \cap \left(\bigcup \gamma^s\right) \subset M$ for the set with the hyperbolic product structure, which is denoted as $Y = \left(\bigcup W^s\right) \cap \left(\bigcup W^u\right) \subset X$ in the other sections of our paper;
- The return time of f to Y is denoted as $R : \Lambda \to \mathbb{Z}^+$ in the appendix (in the other sections, this is $\tau : Y \to \mathbb{Z}^+$);
- The s-subsets which make a partition of Λ are denoted as $\Lambda_i \subset \Lambda$ in the appendix (in the other sections, these are denoted as $Y_i \subset Y$).
- The return time R takes a constant value on each of the Λ_i, which, following again [36], are denoted as R_i in this appendix. That is, $R|_{\Lambda_i} = R_i$.

With these notations, the equality (7.4) reads as $f^{R_i}(\gamma^u(x) \cap \Lambda_i) = \gamma^u(f^{R_i}x) \cap \Lambda$.

Proposition A.1. *Let $f : M \to M$ be an injective transformation satisfying the abstract set up in* [36, Section 1]*: specifically, the proof below uses (P1), the second part of (P2), property (iii) of the separation time s_0, and (P4)(a).*

Let $x \in \Lambda_i$, $i \geqslant 1$. Then $f^{R_i}(\gamma^u(x) \cap \Lambda_i) = \gamma^u(f^{R_i}x) \cap \Lambda$.

Proof. Before starting the actual proof, let us summarize the essence of the argument. The Markov condition [36, (P2)] requires full crossings of Λ but, on its own, appears to also allow multiple full crossings. However we show that, in combination with the other assumptions, only single full crossings are permitted.

It follows from injectivity of f and hence f^{R_i}, as well as (P2), that

$$f^{R_i}(\gamma^u(x) \cap \Lambda_i) = f^{R_i}\gamma^u(x) \cap f^{R_i}\Lambda_i \supset \gamma^u(f^{R_i}x) \cap f^{R_i}\Lambda_i. \tag{A.1}$$

Recall from (P1) that we have the local product structure $\Lambda = \left(\bigcup_{k \in K^u} \gamma^u_k\right) \cap \left(\bigcup_{\ell \in K^s} \gamma^s_\ell\right)$. By (P2), $f^{R_i}\Lambda_i$ is a u-subset of Λ which means that $f^{R_i}\Lambda_i = \left(\bigcup_{k \in K^u_i} \gamma^u_k\right) \cap \left(\bigcup_{\ell \in K^s} \gamma^s_\ell\right)$ for some subset $K^u_i \subset K^u$. Hence $\gamma^u_k \cap \Lambda = \gamma^u_k \cap \left(\bigcup_{\ell \in K^s} \gamma^s_\ell\right) = \gamma^u_k \cap f^{R_i}\Lambda_i$ for all $k \in K^u_i$. Also, $\gamma^u_k \cap f^{R_i}\Lambda_i = \emptyset$ for all $k \notin K^u_i$.

Now, $\gamma^u(f^{R_i}x) \cap f^{R_i}\Lambda_i \neq \emptyset$ (it contains $f^{R_i}x$) so it follows from the above considerations that $\gamma^u(f^{R_i}x) \cap \Lambda = \gamma^u(f^{R_i}x) \cap f^{R_i}\Lambda_i$. Combining this with (A.1),

$$f^{R_i}(\gamma^u(x) \cap \Lambda_i) \supset \gamma^u(f^{R_i}x) \cap \Lambda. \tag{A.2}$$

It remains to prove the reverse inclusion, so suppose that $y \in \gamma^u(x) \cap \Lambda_i$. By (P1), there exists $z^* \in \gamma^u(f^{R_i}x) \cap \gamma^s(f^{R_i}y) \subset \Lambda$. By (A.2), $z^* = f^{R_i}z$ for some $z \in \gamma^u(x) \cap \Lambda_i$. Since z^* and $f^{R_i}y$ lie in the same stable disk it follows from property (iii) of the separation time that $s_0(z^*, f^{R_i} y) = \infty$. Using property (iii) once more, $s_0(z, y) \geqslant s_0(z^*, f^{R_i} y) = \infty$. But $z \in \gamma_u(x) = \gamma_u(y)$ so (P4)(a) implies that $d(z, y) \leqslant C\alpha^{s_0(z,y)} = 0$. Hence $f^{R_i} y = f^{R_i} z = z^* \in \gamma^u(f^{R_i} x)$. This shows that $f^{R_i}(\gamma^u(x) \cap \Lambda_i) \subset \gamma^u(f^{R_i} x) \cap \Lambda$ completing the proof. □

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