Kazhdan–Lusztig R-polynomials, combinatorial invariance, and hypercube decompositions[1]

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Abstract

We show how the R-polynomials of the symmetric groups can be computed, in a poset-theoretic way, from canonical hypercube decompositions. This involves a new combinatorial concept, which we call a shortcut. We conjecture that the same formula holds for a certain class of combinatorially defined hypercube decompositions. We also study the behavior of these concepts under the operation of taking the direct product of two Bruhat intervals, and characterize the shortcuts of the canonical hypercube decompositions. Our main conjecture implies the Combinatorial Invariance Conjecture.

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1 Introduction

Kazhdan–Lusztig polynomials are fundamental objects in Lie theory, representation theory, and the geometry of Schubert varieties (see, e.g., [24], [2], [13], and also [3], [4], [21], and the references cited there). These polynomials $P_{u,v}(q)$, one for each Bruhat interval $[u, v]$ in a Coxeter group W, were introduced by Kazhdan and Lusztig in [23] and soon found applications in several contexts.

Among others, the combinatorial aspects of Kazhdan–Lusztig polynomials have received attention from the start ([25]). From a combinatorial point of view, one of the most intriguing open problems about Kazhdan–Lusztig polynomials is what is usually referred to as the Combinatorial Invariance Conjecture (CIC for short), formulated by Lusztig (G. Lusztig, private communications, 1980, see also [16, Rem. 7.31]), namely that the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ depends only on the isomorphism class of the Bruhat interval $[u, v]$ as a poset. The conjecture is known to hold in various special cases. For instance, it has been proved for intervals of rank ≤ 4 (see, e.g., [16, 7.31], and [4, Chap. 5, Exercises 7 and 8]), intervals of rank ≤ 8 in type A and ≤ 6 in types B and D ([22]), for intervals in type $\tilde{A}_2$ ([14]), for intervals which are lattices ([16, 7.23], [8]), and for intervals starting from the identity ([11]). We refer the reader to [10] for a complete account of the cases in which the CIC is known to be true, and further details. On the other hand, it is worth noting that certain entirely poset-theoretical generalizations of the CIC (e.g., for zircons and diamonds) fail ([12], [26]).

With the help of certain machine learning models, recently Blundell, Buesing, Davies, Veličković, and Williamson ([7], see also [15]) proposed for the first time a *constructive* conjectural procedure to compute the Kazhdan–Lusztig polynomial of a Bruhat interval in a symmetric group starting just from the interval as an abstract poset. This procedure is stated in terms of a new combinatorial concept, that they call a hypercube decomposition. The authors prove that their procedure is correct for a class of hypercube decompositions derived from maximal parabolic subgroups (the *canonical* ones, see Remark 4.2 below for the precise definition) thereby obtaining a new formula for the computation of the Kazhdan–Lusztig polynomials, and conjecture that it is correct for any hypercube decomposition. Their conjecture implies the CIC. A generalization of the formula in [7] was recently given by Gurevich and Wang ([19]).

The R-polynomial of two elements in a Coxeter group, also introduced by Kazhdan and Lusztig in [23], is equivalent to the Kazhdan–Lusztig polynomial of the two elements in the sense that if the R-polynomials of all the subintervals of a Bruhat interval $[u, v]$ are known, then the Kazhdan–Lusztig polynomials of all the subintervals of $[u, v]$ are known, and conversely. It is easy to see that the CIC is equivalent to the analogous conjecture for the R-polynomials.

Our aim in this work is to study the relationships between R-polynomials and hypercube decompositions. More precisely, we define two new combinatorial concepts, namely the set of shortcuts of an element in a Bruhat interval, and that of a join hypercube decomposition, and a new algebraic one, namely that of an algebraically calculating element in a Bruhat interval, and show that the shortcuts of any algebraically calculating hypercube decomposition can be used to compute, in a new and explicit way, the R-polynomial of the interval. We conjecture that the natural generalization of our formula holds for any join hypercube decomposition. This conjecture implies the CIC. We also give three equivalent characterizations of the shortcuts of the canonical hypercube decompositions, and study the behavior of hypercube decompositions, join hypercube decompositions, and their shortcuts, with respect to the direct product of intervals. Many open problems arise naturally from our work.

Some advantages in considering R-polynomials instead of Kazhdan–Lusztig polynomials are that the corresponding formulas are simpler and have dual counterparts. Indeed, one can define "upper"and "lower"versions of the concepts of shortcuts and (join) hypercube decompositions, and obtain upper and lower versions of our formulas.

After the first version of this article started circulating, the preprint [1] appeared on the arXiv. In it, the authors propose a conjecture similar to our own, that also implies the CIC. These two conjectures were obtained independently. We discuss the similarities and differences between them in §6.

The organization of the paper is as follows. In the next section, we recall some definitions, notation, and results that are used in the sequel. In §3, we introduce the main new combinatorial concept of this work, namely that of the shortcuts of a Bruhat interval with respect to an element in the interval. We then show how these can be used to compute in a new way the R-polynomial of a Bruhat interval in Coxeter groups of type A (Corollary 3.10). In §4, we characterize the shortcuts of a Bruhat interval in the symmetric group with respect to the canonical hypercube decompositions. More precisely, we give an algebraic characterization (Corollary 4.3), and two combinatorial ones (Theorem 4.5). In particular, we show that they can be characterized in a simple way in terms of the maps that arise in the definition of hypercube decompositions. In §5 we introduce the concept of a join hypercube decomposition and show that this class of elements is strictly smaller than that of the hypercube decompositions while still containing the canonical ones, and study shortcuts, and (join) hypercube decompositions of Bruhat intervals which are the direct product (as posets) of two smaller Bruhat intervals. In §6, we present our main conjecture (Conjecture 6.1) namely that the formula obtained for the R-polynomial of a Bruhat interval in the symmetric group in terms of shortcuts of the canonical decompositions (Corollary 3.10) holds for any join hypercube decomposition, and the evidence that we have in its favor. We also discuss the relationship of this conjecture with [1, Conj. 1.2] and with [7, Conj. 3.8] as well as some other open problems arising from this work.

2 Preliminaries

This section reviews the background material that is needed in the rest of this work. We follow [28, Chapter 3], [4] and [20] for undefined notation and terminology concerning partially ordered sets and Coxeter groups.

Given a Coxeter system (W, S), we denote by e the identity of W, by ℓ the length function of W with respect to S, and by T the set $\{wsw^{-1} \colon\ w \in W,\ s \in S\}$ of *reflections* of W. Given $u, v \in W$, we let $\ell(u, v) := \ell(v) - \ell(u)$ for short. The *Bruhat graph* of W is the directed graph $B(W)$ on vertex set W where $u \to v$ if and only if $vu^{-1} \in T$ and $\ell(u) < \ell(v)$. For short, we also write $u \xrightarrow{t} v$, if $t = vu^{-1}$. The *Bruhat order* on W, denoted $\leq$, is the partial order where $u \leq v$ provided that there is a directed path from u to v in $B(W)$. We denote by $B(u, v)$ the directed graph induced by $B(W)$ on the interval $[u, v] := \{z \in W : u \leq z \leq v\}$.

Recall (see [18, §2]) that a *reflection ordering* on (W, S) is a total order $\preceq$ on T such that if (W', S') is a dihedral reflection subgroup of W (so $W' =< J >$ for some $J \subseteq T$ and $S' = \{t \in T : \{r \in T \cap W' : \ell(rt) < \ell(t)\} = \{t\}\}$ has cardinality 2) then either

$$a \preceq aba \preceq ababa \preceq \cdots \preceq babab \preceq bab \preceq b$$

or

$$a \succeq aba \succeq ababa \succeq \cdots \succeq babab \succeq bab \succeq b$$

where $\{a,b\} = S'$. Reflection orderings always exist, and in fact there are many (we refer the reader to [4, §5.2] for further information about reflection orderings).

Let $x, y \in W$, $x \leq y$, and $\Gamma = (x = x_0 \to x_1 \to \cdots \to x_r = y)$ be a (directed) path from x to y in $B(W)$. We define r to be the *length* of Γ, denoted $\ell(\Gamma)$, and $\{x_i : i \in [0,r]\}$ to be the *support* of Γ, denoted $\mathrm{supp}(\Gamma)$. Furthermore, we define the *distance from x to y*, denoted $\mathrm{d}(x,y)$, as the distance from x to y in $B(W)$, i.e. $\min\{\ell(\Gamma) : \Gamma \text{ is a path from } x \text{ to } y\}$. Note that $\mathrm{d}(x,y) \leq \ell(x,y)$ and $\mathrm{d}(x,y) \equiv \ell(x,y) \pmod 2$. Given a reflection ordering $\preceq$, we say that a path $\Gamma = \big(x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{r-1}} x_{r-1} \xrightarrow{t_r} x_r\big)$ is *increasing (with respect to $\preceq$)* if $t_1 \preceq t_2 \preceq \cdots \preceq t_r$.

With any Coxeter system (W,S), Kazhdan and Lusztig in [23] associated two families of polynomials $\{P_{u,v}\}_{u,v \in W} \subseteq \mathbb{Z}[q]$ and $\{R_{u,v}\}_{u,v\in W} \subseteq \mathbb{Z}[q]$ which are now known as the Kazhdan–Lusztig and R-polynomials, respectively, of W. We do not define them here and instead refer the reader to [4, §5]. The $\widetilde{R}$-polynomials are a rescaling of the R-polynomials. More precisely (see, e.g., [4, Prop. 5.3.1]), the polynomial $\widetilde{R}_{u,v}(q)$ is the unique polynomial with natural coefficients satisfying

$$R_{u,v}(q) = q^{\frac{\ell(u,v)}{2}} \widetilde{R}_{u,v}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}).$$

The following combinatorial interpretation of the $\widetilde{R}$-polynomials is due to Dyer (see [18] and also [4, Thm. 5.3.4]).

Theorem 2.1. *Let (W,S) be a Coxeter system and $\preceq$ be a reflection ordering. Given $u, v \in W$, with $u \leq v$, we have*

$$\widetilde{R}_{u,v}(q) = \sum q^{\ell(\Gamma)},$$

where the sum is over all increasing paths Γ from u to v.

The next result was proved by Blanco [6, Prop. 3.9], and also follows from the previous result and [16, Prop. 7.20].

Proposition 2.2. *Let (W,S) be a Coxeter system and $\preceq$ be a reflection ordering. Let $u, v \in W$ and $k \in \mathbb{N}$. If there exists a path from u to v of length k, then there exists an increasing path from u to v of length k.*

Let S_n be the symmetric group. We write the elements of S_n in one-line notation (so $u = [a_1, \ldots, a_n]$, or simply $u = a_1 \cdots a_n$, means that $u(i) = a_i$ for all $i \in [n]$) as well as in disjoint cycle-form, omitting to write the 1-cycles. It is well known (see, e.g., [4, Prop. 1.5.4]) that S_n, with respect to the generating set $S = \{s_1, \ldots, s_{n-1}\}$, where $s_i = (i, i+1)$ for $i = 1, \ldots, n-1$, is a Coxeter group of type A_{n-1}. The corresponding set of reflections is the set of transpositions $T = \{(i,j) : 1 \leq i < j \leq n\}$, and the corresponding length function is the number of inversions, so

$$\ell(u) = |\{(i,j) \in [n]^2 : i < j \text{ and } u(i) > u(j)\}|$$

for all $u \in S_n$. This implies that, if $u \in S_n$ and $(i,j) \in T$, with $i < j$, then $u \to (i,j)u$ in the Bruhat graph if and only if $u^{-1}(i) < u^{-1}(j)$. For $u \in S_n$ and $i \in [n]$, we let

$$\{u^{i,1}, \ldots, u^{i,i}\}_< = \{u(1), \ldots, u(i)\}\,. \tag{1}$$

Here and below, the index $<$ means that $\{u^{i,1}, \ldots, u^{i,i}\} = \{u(1), \ldots, u(i)\}$ and $u^{i,1} < \cdots < u^{i,i}$. The following result is a well known characterization of the Bruhat order of S_n (see, e.g., [4, Thm. 2.6.3]) and is usually called the "tableau criterion".

Theorem 2.3. *Let $u, v \in S_n$. Then $u \leq v$ if and only if $u^{i,j} \leq v^{i,j}$ for all $1 \leq j \leq i \leq n-1$.*

For S_n, the reverse lexicographic order $(n, n-1) \prec (n, n-2) \prec \cdots \prec (n, 1) \prec (n-1, n-2) \prec \cdots \prec (2,1)$ is a reflection ordering.

For $A, B \subseteq \mathbb{N}$ with $|A| = |B| = r$, we find it convenient to write $A \leq B$ if $a_i \leq b_i$ for $i = 1, \ldots, r$, where $\{a_1, \ldots, a_r\}_< = A$ and $\{b_1, \ldots, b_r\}_< = B$. The following property, whose simple verification is omitted, is useful in what follows.

Lemma 2.4. *If $A, B \subseteq \mathbb{N}$, $a \in \mathbb{N} \setminus A$, and $b \in \mathbb{N} \setminus B$ are such that $|A| = |B|$, $A \leq B$ and $a \leq b$, then $A \cup \{a\} \leq B \cup \{b\}$.*

Given a set E, let $\mathcal{P}(E)$ denote the directed Boolean algebra on E, i.e. the directed graph having the power set of E as vertex set and where $I \to J$ if I is obtained from J by removing one element.

Let p be an element in a Coxeter group W. Let E be a set of edges of $B(W)$ having this same target p. Following [7], we say that E *spans a hypercube* if there exists a unique embedding of directed graphs $\theta_E : \mathcal{P}(E) \to W$ sending the directed edge $E \setminus \{\alpha\} \to E$ to α, for all $\alpha \in E$. Furthermore, E *spans a hypercube cluster* if every subset E' of E consisting of edges with pairwise incomparable sources (with respect to Bruhat order) spans a hypercube. Note that, if E as above spans a hypercube, and $\alpha \in E$, then $\theta_E(\emptyset) \to \theta_{E\setminus\{\alpha\}}(\emptyset)$. In the rest of this work, if there is no danger of confusion, we will simply write θ instead of θ_E.

Let $u, v \in W$ and $z \in [u, v]$. After [7], we say that the interval $[z, v]$ (or just z, for short) is an *upper hypercube decomposition* of $[u, v]$ provided that:

1. $[z, v]$ is *diamond complete* (with respect to $[u, v]$), meaning that, if there exist $x \in [u, v]$ and $a_1, a_2, y \in [z, v]$, $a_1 \neq a_2$, such that $x \to a_1 \to y$, $x \to a_2 \to y$, then $x \in [z, v]$;

2. for all $p \in [z, v]$, the set $E^p = \{x \to p : x \notin [z, v]\}$ spans a hypercube cluster.

For example, if $W = S_5$, $u = 21354$, and $v = 52341$, then one can check (preferably with the aid of a computer) that the upper hypercube decompositions of $[u, v]$ are $23451, 23514, 24153, 25134, 31452, 31524, 41253$, and 51234.

Similarly, taking the dual versions of the above definitions, we obtain the concept of a *lower hypercube decomposition*. Note that, if z is an upper hypercube decomposition of $[u, v]$, then z^{-1} is an upper hypercube decomposition of $[u^{-1}, v^{-1}]$ and, if W is finite, $z\, w_0$ and $w_0\, z$ are lower hypercube decompositions of $[vw_0, uw_0]$ and $[w_0v, w_0u]$, respectively, where w_0 is the longest element of W.

Note that a lower hypercube decomposition is called simply a hypercube decomposition in [7]. Note also that we adopt the convention that arrows in the Bruhat graph go from the shorter element to the longer one, which is opposite to the convention in [7].

3 Shortcuts and $\widetilde{R}$-polynomials

In this section, we introduce the main new combinatorial concept of this work, namely that of the shortcuts of a Bruhat interval in a Coxeter group with respect to an element of the interval. We then show how the shortcuts with respect to some specific elements, that always exist in a Bruhat interval $[u, v]$ in a symmetric group, can be used to compute, in a new and entirely combinatorial (i.e., poset-theoretic) way, the $\widetilde{R}$-polynomial of u, v. This is achieved through the related concepts of an algebraically calculating element and an R-element of a Bruhat interval. In this section, we define the "upper" versions of these concepts and obtain the "upper" versions of the results. The lower ones are entirely analogous.

Definition 3.1. Let W be a Coxeter group, and $u, v \in W$. Given an element $z \in [u, v]$, we let

$$W^z_{[u,v]} := \{p \in [z, v] : \mathrm{supp}(\Gamma) \cap [z, v] = \{p\} \text{ for all paths } \Gamma \text{ from } u \text{ to } p \text{ of length } \mathrm{d}(u, p)\},$$

and

$$\widetilde{R}^z_{u,v}(q) := \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \widetilde{R}_{p,v}(q).$$

We call the elements in $W^z_{[u,v]}$ the *(upper) shortcuts of* $[u, v]$ *with respect to* z. Furthermore, we say that an element z in $[u, v]$ is an *(upper) R-element for* $[u, v]$ (or just an *R-element* when the interval $[u, v]$ is clear from the context) if $\widetilde{R}^z_{u,v} = \widetilde{R}_{u,v}$. In particular, $W^u_{[u,v]} = \{u\}$ and u is an R-element of $[u, v]$. Note that, if z is an R-element for $[u, v]$, then, by well known properties of the R-polynomials (see, e.g., [4, Chap. 5, Ex. 10]), z^{-1} is an R-element for $[u^{-1}, v^{-1}]$ and, if W is finite, $w_0 z$ and $z\, w_0$ are R-elements for $[w_0 v, w_0 u]$ and $[v w_0, u w_0]$, respectively.

Remark 3.2. Note that the subset $W^z_{[u,v]}$ can be defined equivalently in the following recursive way:

i) $z \in W^z_{[u,v]}$;

ii) if $x \in [z, v] \setminus \{z\}$, then $x \in W^z_{[u,v]}$ if and only if

$$d(u, x) < \min\{d(u, y) + d(y, x)\},$$

where the minimum is taken over the set $\{y \in W^z_{[u,v]} : y < x\}$.

For example, if $W = S_5$, $u = 21354$, $v = 52341$, and $z = 25134$, then $W^z_{[u,v]} = \{25134, 25314\}$ (see also Fig. 1).

Remark 3.3. By definition, the polynomial $\widetilde{R}^z_{u,v}$ depends only on the poset structure of the interval $[u, v]$, on z, and on the $\widetilde{R}$-polynomials of intervals that are strictly contained in $[u, v]$. If one had a combinatorial recipe (i.e. a recipe depending only on the poset structure of the interval $[u, v]$) to detect an R-element $z \neq u$, the CIC would follow.

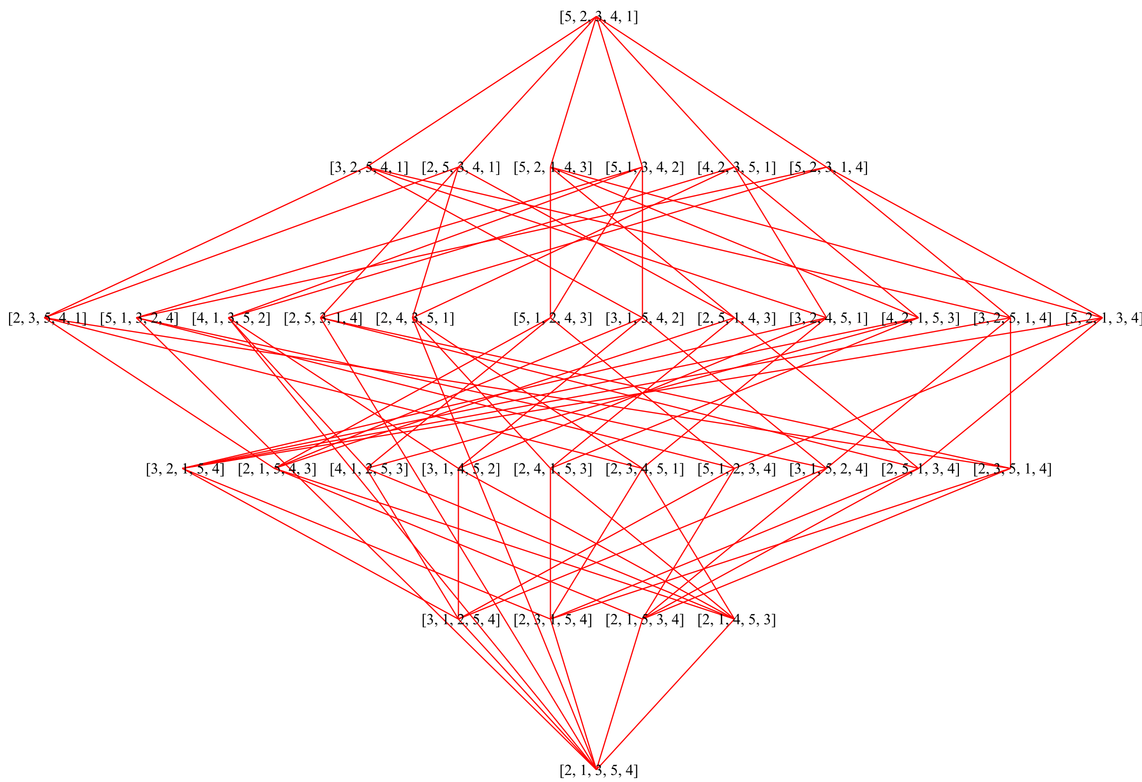

Figure 1: The directed graph $B(21354, 52341)$.

The next results shows, in particular, that every element in a lattice is an R-element.

Proposition 3.4. *Let W be a Coxeter group, and $u, v \in W$. If there are no $x, y \in [u, v]$ such that $x \to y$ and $\ell(x, y) > 1$ (for example, if $[u, v]$ is a lattice), then every element in $[u, v]$ is an R-element.*

Proof. It is well known (see [8, Theorem 6.3]) that the $\widetilde{R}$-polynomials satisfy

$$\widetilde{R}_{a,b} = q^{\ell(a,b)} \tag{2}$$

whenever there are no $c, d \in [a, b]$ such that $c \to d$ and $\ell(c, d) > 1$. Let $z \in [u, v]$. By (2), we have to show $\widetilde{R}^z_{u,v} = q^{\ell(u,v)}$. In our hypotheses, every path from x to y has length $\mathrm{d}(x, y) = \ell(x, y)$, for all $x, y \in [u, v]$. Hence $W^z_{[u,v]} = \{z\}$ and $\widetilde{R}^z_{u,v} = q^{\ell(u,z)} \widetilde{R}_{z,v}$. Again by (2), $\widetilde{R}_{z,v} = q^{\ell(z,v)}$ and the proof is completed. □

In order to find R-elements, we introduce the following property.

Definition 3.5. Let W be a Coxeter group, and $u, v \in W$. An element $z \in [u, v]$ is *algebraically calculating for* $[u, v]$ (or just *algebraically calculating* for short) provided that there exists a reflection ordering of W satisfying the following two properties:

1. every reflection that labels an edge $a \to b$, with $a \notin [z, v]$ and $b \in [z, v]$, is smaller than each reflection that labels an edge $c \to d$ with both c and d in $[z, v]$;

2. given an increasing path Γ from u to an element $p \in [z, v]$ such that $\mathrm{supp}(\Gamma) \cap [z, v] = \{p\}$, we have:

   (a) $\ell(\Gamma) = \mathrm{d}(u, p)$;

   (b) Γ is the only increasing path from u to p that satisfies $\mathrm{supp}(\Gamma) \cap [z, v] = \{p\}$;

   (c) $p \in W^z_{[u,v]}$.

The next result shows that being algebraically calculating is a stronger property than being an R-element.

Proposition 3.6. *Let W be a Coxeter group, and $u, v \in W$. Every algebraically calculating element z is an R-element, i.e.*

$$\widetilde{R}_{u,v}(q) = \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \widetilde{R}_{p,v}(q).$$

Proof. Let z be an algebraically calculating element for $[u, v]$ and fix a reflection ordering satisfying the properties in Definition 3.5. We use Theorem 2.1 with this reflection ordering, so we want to show that

$$\sum_{\Gamma} q^{\ell(\Gamma)} = \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \widetilde{R}_{p,v},$$

where the first sum is over all increasing paths Γ from u to v.

Let Γ be an increasing path from u to v. Denote by p the smallest element in $\mathrm{supp}(\Gamma) \cap [z, v]$, and by Γ_p the path from u to p given by the truncation of Γ at p. By properties (2a) and (2c) of Definition 3.5, we have $\ell(\Gamma_p) = \mathrm{d}(u, p)$ and $p \in W^z_{[u,v]}$. By property (2b) of Definition 3.5, the truncation at p of every increasing path Δ from u to v such that the smallest element of $\mathrm{supp}(\Delta) \cap [z, v]$ is p coincides with Γ_p. Furthermore, by property (1) of Definition 3.5, the concatenation of Γ_p with every increasing path from p to v gives an increasing path from u to v. Hence $\sum q^{\ell(\Delta)} = q^{\mathrm{d}(u,p)}\widetilde{R}_{p,v}$ where the sum is over all increasing paths Δ from u to v such that p is the smallest element in $\mathrm{supp}(\Delta) \cap [z,v]$.

It remains to show that, for every $p \in W^z_{[u,v]}$, there exists an increasing path Γ from u to v such that p is the smallest element in $\mathrm{supp}(\Gamma) \cap [z,v]$. By Proposition 2.2, there exists an increasing path Γ' from u to p of length $\mathrm{d}(u,p)$. By the definition of $W^z_{[u,v]}$, we have $\mathrm{supp}(\Gamma') \cap [z,v] = \{p\}$. Furthermore, by property (1) of Definition 3.5, the path Γ' can be completed to an increasing path from u to v by concatenating it with any increasing path from p to v (there exists at least one such path since $\widetilde{R}_{p,v} \neq 0$). □

Algebraically calculating elements and R-elements may or may not exist. For example, in a dihedral interval of rank 3 both coatoms are algebraically calculating elements, dihedral intervals of rank 4 or 5 have no algebraically calculating elements, but both elements of rank 2 are R-elements, dihedral intervals of rank ≥ 6 have no R-elements.

The following result estabishes the existence of algebraically calculating elements for any interval $[u,v]$ in the symmetric group. An algebraically calculating element is the minimum of the intersection of $[u,v]$ with the coset of v of a certain standard parabolic subgroup. Note that, in an arbitrary Coxeter group, the intersection of any interval with any coset of a standard parabolic subgroup is itself an interval (see [27]).

Theorem 3.7. *Let W be a Coxeter group of type A_{n-1}, and $u, v \in W$. Let $z = \min([u,v] \cap W_{S\setminus\{s_{n-1}\}}v)$, i.e. z is the smallest element in $[u,v]$ such that $z^{-1}(n) = v^{-1}(n)$. Then z is algebraically calculating.*

Proof. Consider the reflection ordering defined after Theorem 2.3. Let $R_n := \{(n,k) : k \in [n-1]\}$. Note that every transposition in R_n is smaller than each transposition in $T \setminus R_n$. Hence property (1) of Definition 3.5 holds since every transposition that labels an edge $a \to b$ with $a \notin [z,v]$ and $b \in [z,v]$ belongs to R_n and every transposition that labels an edge $c \to d$ with $c, d \in [z,v]$ belongs to $T \setminus R_n$.

Let Γ be an increasing path from u to an element p in $[z,v]$ such that $\mathrm{supp}(\Gamma) \cap [z,v] = \{p\}$. Since the last label belongs to R_n and Γ is increasing, all labels of Γ

belong to R_n: more precisely, there exist $i_1, i_2, \ldots, i_k \in [n-1]$, with $i_1 < i_2 < \cdots < i_k$, such that

$$\Gamma = \big(u = x_0 \stackrel{(n,i_k)}{\longrightarrow} x_1 \stackrel{(n,i_{k-1})}{\longrightarrow} \cdots \stackrel{(n,i_2)}{\longrightarrow} x_{k-1} \stackrel{(n,i_1)}{\longrightarrow} x_k = p\big). \tag{3}$$

Since $(n, i_1)(n, i_2) \cdots (n, i_k) = (i_k, i_{k-1}, \ldots, i_1, n)$, we have $p = (i_k, i_{k-1}, \ldots, i_1, n)u$ and hence $\mathrm{d}(u, p) = k$ since a $(k+1)$-cycle cannot be obtained as a product of less than k transpositions. So $\ell(\Gamma) = \mathrm{d}(u, p)$, i.e. property (2a) of Definition 3.5 holds.

Similarly, if

$$\Gamma' = \big(u = y_0 \stackrel{(n,j_k)}{\longrightarrow} y_1 \stackrel{(n,j_{k-1})}{\longrightarrow} \cdots \stackrel{(n,j_2)}{\longrightarrow} y_{k-1} \stackrel{(n,j_1)}{\longrightarrow} y_k = p\big)$$

is an increasing path from u to p such that $\mathrm{supp}(\Gamma) \cap [z, v] = \{p\}$, then $p = (j_k, j_{k-1}, \ldots, j_1, n)u$ and then $\Gamma' = \Gamma$, i.e. property (2b) of Definition 3.5 holds.

We now prove property (2c) of Definition 3.5. We know that (3) holds and that $i_1 < i_2 < \cdots < i_k$. Since $x_0 < x_1 < \cdots < x_k$, we have $u^{-1}(i_1) < u^{-1}(i_2) < \cdots < u^{-1}(i_k)$, i.e. the one line notation of u is of the form:

$$u = [\cdots i_1 \cdots i_2 \cdots \quad \cdots i_{k-1} \cdots i_k \cdots n \cdots],$$

and hence

$$p = [\cdots n \cdots i_1 \cdots \quad \cdots i_{k-2} \cdots i_{k-1} \cdots i_k \cdots].$$

Let $\Delta = \big(u = u_0 \stackrel{(a_1,b_1)}{\longrightarrow} u_1 \stackrel{(a_2,b_2)}{\longrightarrow} \cdots \stackrel{(a_k,b_k)}{\longrightarrow} u_k = p\big)$ be a path from u to p of length $\mathrm{d}(u, p) = k$. We claim that $a_j, b_j \in \{i_1, \ldots, i_k, n\}$ for all $j = 1, \ldots, k$. Indeed, in order to get to p, each element i_j with $j \in [k]$ has to move to the right. Since every transposition only moves one element to the right, and the path Δ is of minimal length, namely k, every transposition in the path Δ has to move one element i_j to the right, for some $j \in [k]$. If there is at least one transposition in Γ of the form (a, i_j) for some $a \notin \{i_1, \ldots, i_k, n\}$ then a is moved to the left and therefore we have one more element to move to the right in order to get to p, which is impossible in a path of minimal length. In particular, $a_k, b_k \in \{i_1, \ldots, i_k, n\}$. Since $u_{k-1} < p$, we conclude that $(u_{k-1})^{-1}(n) \neq p^{-1}(n)$, so $u_{k-1} \notin [z, v]$. Hence $p \in W^z_{[u,v]}$. □

Remark 3.8. If $\min([u, v] \cap W_{S \setminus \{s_{n-1}\}} v) = u$, i.e. $u^{-1}(n) = v^{-1}(n)$, then Theorem 3.7 holds also for $z = \min([u, v] \cap W_{S \setminus \{s_{n-1}, s_{n-2}, \ldots, s_{d-1}\}} v)$, where $d = \max\{i \in [n] : u^{-1}(i) \neq v^{-1}(i)\}$. The proof and statement are, mutatis mutandis, the same as above.

Remark 3.9. It is easy to check that Theorem 3.7 holds also for $z = \min([u,v] \cap W_{S\setminus\{s_1\}}v)$, i.e. for the smallest element in $[u,v]$ such that $z^{-1}(1) = v^{-1}(1)$, as well as for $z = \min([u,v] \cap vW_{S\setminus\{s_{n-1}\}})$, i.e. for the smallest element in $[u,v]$ such that $z(n) = v(n)$, and for $z = \min([u,v] \cap vW_{S\setminus\{s_1\}})$, i.e. for the smallest element in $[u,v]$ such that $z(1) = v(1)$.

The existence of lower versions of the results in this section reflects the fact that the $\widetilde{R}$-polynomials are invariant under the antiautomorpism of the Bruhat graph given by multiplication by w_0 (while the Kazhdan–Lusztig polynomials are not). The next result collects the formulas for the $\widetilde{R}$-polynomial of a Bruhat interval $[u,v]$ in the symmetric group obtained from Proposition 3.6, Theorem 3.7, and the preceding considerations. By Corollary 4.3 below, the first one of these can be seen as a poset-theoretical formulation of Corollary 3.9 of [9].

Corollary 3.10. *Let W be a Coxeter group of type A_{n-1}, and $u, v \in W$.*

- *Let* $z \in \{\min([u,v]\cap W_{S\setminus\{s_{n-1}\}}v), \min([u,v]\cap W_{S\setminus\{s_1\}}v), \min([u,v]\cap vW_{S\setminus\{s_{n-1}\}}), \min([u,v] \cap vW_{S\setminus\{s_1\}})\}$. *Then*

$$\widetilde{R}_{u,v} = \sum_{p\in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \widetilde{R}_{p,v}.$$

- *Let* $z \in \{\max([u,v]\cap W_{S\setminus\{s_{n-1}\}}u), \max([u,v]\cap W_{S\setminus\{s_1\}}u), \max([u,v]\cap uW_{S\setminus\{s_{n-1}\}}), \max([u,v] \cap uW_{S\setminus\{s_1\}})\}$. *Then*

$$\widetilde{R}_{u,v} = \sum_{p\in W_z^{[u,v]}} q^{\mathrm{d}(p,v)} \widetilde{R}_{u,p},$$

where

$$W_z^{[u,v]} = \{p \in [u,z] : \mathrm{supp}(\Gamma)\cap[u,z] = \{p\} \textit{ for all paths } \Gamma \textit{ from } p \textit{ to } v \textit{ of length } \mathrm{d}(p,v)\}.$$

4 Shortcuts and canonical hypercube decompositions

In this section we give algebraic and combinatorial characterizations of the shortcuts of the canonical hypercube decompositions of Bruhat intervals in the symmetric groups. In particular, we show that they can be characterized in a simple way also using the maps that are used to define hypercube decompositions.

Let W be a Coxeter group of type A_{n-1}, and $u, v \in W$. Let $z = \min([u,v] \cap W_{S\setminus\{s_{n-1}\}}v)$, and suppose $z \neq u$. Let $p \in [z,v]$, E' be a subset of $E^p = \{x \to p : x \notin [z,v]\}$ consisting of edges with incomparable sources, and $t_1, \ldots, t_r \in T$ be the labels of the edges in E'. Since $p \in [z,v]$, $t_k\, p \notin [z,v]$, and $t_k\, p < p$, there is $a_k \in [n-1]$ such that $t_k = (n, a_k)$ and $p^{-1}(n) < p^{-1}(a_k)$ for $k = 1, \ldots, r$. We may assume that $p^{-1}(a_1) < \cdots < p^{-1}(a_r)$. Since $\{t_1 p, \ldots, t_r p\}$ is an antichain, we have $a_1 < \cdots < a_r$ (for if $a_k > a_{k+1}$ for some $k \in [r-1]$ then $t_{k+1}p < t_k p$). Let $\theta : \mathcal{P}([r]) \to B(u,v)$ be the map sending a subset A of $[r]$ to

$$\theta(A) = (a_{i_1}, \ldots, a_{i_s}, n)\, p, \tag{4}$$

where $\{i_1, \ldots, i_s\}_< = [r] \setminus A$ (so $\theta([r]) = p$). Recall that we write $\{i_1, \ldots, i_s\}_<$ for the set $\{i_1, \ldots, i_s\}$ with $i_1 < i_2 < \cdots < i_s$. The next result is (the dual version of) Theorem 5.1 of [7].

Theorem 4.1. *Let W be a Coxeter group of type A_{n-1}, and $u, v \in W$ with $u \leq v$. Let $z = \min([u,v] \cap W_{S\setminus\{s_{n-1}\}}v)$. Then z is an upper hypercube decomposition and the maps θ are its embeddings.*

Remark 4.2. Theorem 4.1 holds not only for $\min([u,v] \cap W_{S\setminus\{s_{n-1}\}}v)$, but also for $\min([u,v] \cap W_{S\setminus\{s_1\}}v)$, $\min([u,v] \cap vW_{S\setminus\{s_{n-1}\}})$, and $\min([u,v] \cap vW_{S\setminus\{s_1\}}v)$. We find it convenient to call these four (possibly not mutually distinct) elements the *canonical* upper hypercube decompositions for the interval $[u,v]$ in the symmetric group.

The next result answers a natural question, namely, it determines the shortcuts of the canonical hypercube decomposition.

Corollary 4.3. *Let W be a Coxeter group of type A_{n-1}, $u, v \in W$, and $z = \min([u,v] \cap W_{S\setminus\{s_{n-1}\}}v)$. Then the following are equivalent:*

i) $p \in W^z_{[u,v]}$;

ii) $p = (i_k, i_{k-1}, \ldots, i_1, n)\, u$ *for some* $0 < i_1 < i_2 < \cdots < i_k < n$ *such that* $v^{-1}(n) = u^{-1}(i_1) < u^{-1}(i_2) < \cdots < u^{-1}(i_k) < u^{-1}(n)$.

Proof. Assume first that $p \in W^z_{[u,v]}$. Then, by definition, all Bruhat paths Γ from u to p of length $d(u,p)$ are such that $\mathrm{supp}(\Gamma) \cap [z,v] = \{p\}$. In particular, by Proposition 2.2, there is at least one such path Γ that is increasing with respect to the reflection ordering defined after Theorem 2.3. Then, reasoning exactly as in the proof of Theorem 3.7 we conclude that ii) holds.

Conversely, if ii) holds then the path Γ defined as in (3) is such that $\mathrm{supp}(\Gamma) \cap [z,v] = \{p\}$, and Γ is increasing with respect to the reflection ordering just considered. But, by Theorem 3.7, z is algebraically calculating with respect to that reflection ordering, so $p \in W^z_{[u,v]}$. □

Remark 4.4. Similarly, there are analogous versions of Corollary 4.3 for the other canonical hypercube decompositions. More precisely, the corresponding shortcuts are all the elements of the form $(1, i_1, \ldots, i_k)u$ for some $1 < i_1 < \cdots < i_k \leq n$ such that $u^{-1}(1) < u^{-1}(i_1) < \cdots < u^{-1}(i_k) = v^{-1}(1)$ if $z = \min([u,v] \cap W_{S\setminus\{s_1\}}v)$, all the elements of the form $(i_1, \ldots, i_k, a)u$ for some $a < i_1 < \cdots < i_k \leq n$ such that $u^{-1}(a) < u^{-1}(i_1) < \cdots < u^{-1}(i_k) = n$, where $a := v(n)$, if $z = \min([u,v] \cap vW_{S\setminus\{s_{n-1}\}})$, and all the elements of the form $(i_k, \ldots, i_1, a)u$ for some $1 \leq i_1 < \cdots < i_k < a$ such that $1 = u^{-1}(i_1) < \cdots < u^{-1}(i_k) < u^{-1}(a)$, where $a := v(1)$, if $z = \min([u,v] \cap vW_{S\setminus\{s_1\}})$.

Suppose that an element z in $[u,v]$ is an upper hypercube decomposition of $[u,v]$. Given $p \in [z,v]$, let $E^p = \{x \to p : x \notin [z,v]\}$ and denote by $(E^p)_{\min}$ the subset of E^p consisting of the arrows with minimal sources. We let $\theta_p : \mathcal{P}((E^p)_{\min}) \to W$ be the only embedding of directed graphs sending the directed edge $(E^p)_{\min} \setminus \{\alpha\} \to (E^p)_{\min}$ to α, for all $\alpha \in (E^p)_{\min}$. While the characterization in Corollary 4.3 is algebraic, the following characterizations of the shortcuts of the canonical hypercube decomposition(s) are entirely combinatorial.

Theorem 4.5. *Let W be a Coxeter group of type A_{n-1}, $u, v \in W$, $u \leq v$, $z = \min([u,v] \cap W_{S\setminus\{s_{n-1}\}}v)$, and $p \in [z,v]$. Then the following are equivalent:*

i) $p \in W^z_{u,v}$;

ii) $\theta_p(\emptyset) = u$;

iii) $p \in W^z_{u,v}$ *and there are* $\mathrm{d}(u,p)!$ *paths from* u *to* p *of length* $\mathrm{d}(u,p)$.

Proof. We first assume that ii) holds and show that i) holds. Let $t_1, \dots, t_r \in T$ be the labels of the edges in $(E^p)_{\min}$. Since $p \in [z,v]$, $t_k\, p \notin [z,v]$, and $t_k\, p < p$, there is $a_k \in [n-1]$ such that $t_k = (n, a_k)$ and $p^{-1}(n) < p^{-1}(a_k)$, for $k = 1, \dots, r$. We may assume $p^{-1}(a_1) < \cdots < p^{-1}(a_r)$. Since $\{t_1 p, \dots, t_r p\}$ is an antichain, we have $a_1 < \cdots < a_r$ (for if $a_k > a_{k+1}$ for some $k \in [r-1]$ then $t_{k+1}p < t_k p$). By Theorem 4.1,

$$\theta_p(A) = (a_{i_1}, \dots, a_{i_s}, n)\, p \tag{5}$$

for all $A \subseteq [r]$, where $\{i_1, \dots, i_s\}_< = [r] \setminus A$. Therefore, $u = \theta_p(\emptyset) = (a_1, \dots, a_r, n)p$ so, by Corollary 4.3, we have $p \in W^z_{u,v}$.

We now assume that i) holds and show that ii) holds. By Corollary 4.3, there are $a_1, \dots, a_r \in [n-1]$ ($r \in [n-1]$) such that $a_1 < \cdots < a_r$, $u^{-1}(a_1) < \cdots < u^{-1}(a_r) < u^{-1}(n)$, and $p = (n, a_r, \dots, a_1)\, u$. We claim that the labels of the edges in $(E^p)_{\min}$ are $(n, a_1), \dots, (n, a_r)$. Let $k \in [r]$. It is clear that $(n, a_k)\, p < p$ and that $(n, a_k)\, p \notin [z,v]$. Let $j \in [n]$ and $i \in [r]$ be such that $u^{-1}(a_i) \le j < u^{-1}(a_{i+1})$ (where $a_{r+1} = n$). Then

$$\{((n,a_k)p)(1), \dots, ((n,a_k)p)(j)\} = \begin{cases} (\{u(1), \dots, u(j)\} \setminus \{a_i\}) \cup \{a_k\}, & \text{if } 1 \le i \le k, \\ (\{u(1), \dots, u(j)\} \setminus \{a_i\}) \cup \{n\}, & \text{if } k < i \le r, \end{cases}$$

so $\{u(1), \dots, u(j)\}_< \le \{((n,a_k)p)(1), \dots, ((n,a_k)p)(j)\}_<$, and hence $u \le (n, a_k)\, p$. Now let $t \in E^p \setminus \{(n,a_1), \dots, (n,a_r)\}$. There is $a \in [n-1] \setminus \{a_1, \dots, a_r\}$ such that $t = (n, a)$ and $p^{-1}(n) < p^{-1}(a)$. Let $i \in [r]$ be such that $u^{-1}(a_i) < p^{-1}(a) < u^{-1}(a_{i+1})$ (where $a_{r+1} = n$). Then $a > a_i$ (for if $a < a_i$ then

$$\{((n,a)p)(1), \dots, ((n,a)p)(u^{-1}(a_i))\}_< < \{u(1), \dots, u(u^{-1}(a_i))\}_<$$

so $u \not\le (n,a)\, p$) and hence $tp = (n,a)p \ge (n, a_i)p$. Finally, let $1 \le i < j \le r$. By Lemma 2.4

$$\{((n,a_i)p)(1), \dots, ((n,a_i)p)(u^{-1}(a_1))\}_< < \{((n,a_j)p)(1), \dots, ((n,a_j)p)(u^{-1}(a_1))\}_<$$

while

$$\{((n,a_j)p)(1), \dots, ((n,a_j)p)(u^{-1}(a_{i+1}))\}_< < \{((n,a_i)p)(1), \dots, ((n,a_i)p)(u^{-1}(a_{i+1}))\}_<$$

so $(n, a_i)p$ and $(n, a_j)p$ are incomparable in Bruhat order. This proves our claim, which implies $\theta_p(\emptyset) = u$.

Finally, if i) holds, then by Corollary 4.3 there are $a_1, \ldots, a_r \in [n-1]$ ($r \in [n-1]$) such that $a_1 < \cdots < a_r$, $u^{-1}(a_1) < \cdots < u^{-1}(a_r) < u^{-1}(n)$, and $p = (n, a_r, \ldots, a_1)\, u$. But then, reasoning exactly as in the proof of property (2c) in the proof of Theorem 3.7, we conclude that there are $\mathrm{d}(u,p)!$ paths from u to p of length $\mathrm{d}(u,p)$, so iii) holds. □

It is natural to wonder if Theorem 4.5 holds for any hypercube decomposition (see also Problem 6.4).

5 Direct Products

In this section we define the second new combinatorial concept of this work, namely join (and meet) hypercube decompositions, and study (join) hypercube decompositions, shortcuts, and R-elements, of Bruhat intervals which are direct product (as posets) of two smaller Bruhat intervals. Our results show that these concepts are very well behaved under this operation, and have implications for some of the conjectures in the next section.

We begin with the following simple (and probably well known) result about the Bruhat graph of an interval which is the direct product of two other ones.

Lemma 5.1. *Let (W,S), (W_1,S_1), and (W_2,S_2) be three Coxeter systems and $u, v \in W$, $u_i, v_i \in W_i$ for $i = 1,2$ be such that $[u,v] \simeq [u_1,v_1] \times [u_2,v_2]$ (isomorphic as posets). Let $x \mapsto (x_1,x_2)$ be an isomorphism and $x, y \in [u,v]$. Then the following are equivalent:*

i) $x \to y$;

ii) either $x_1 = y_1$ and $x_2 \to y_2$, or $x_2 = y_2$ and $x_1 \to y_1$.

Proof. Consider the direct product $W_1 \times W_2$ of the two Coxeter systems and the interval $[(u_1,u_2),(v_1,v_2)]$ in this direct product. The Subword property (see [4, Theorem 2.2.2]) implies $[(u_1,u_2),(v_1,v_2)] = [u_1,v_1] \times [u_2,v_2]$ as posets. Therefore $[u,v] \simeq [(u_1,u_2),(v_1,v_2)]$ as posets. By [17, Proposition 3.3], this implies that the directed graphs induced on these intervals by the Bruhat graphs of (W,S) and $(W_1 \times W_2, S_1 \cup S_2)$ are also equal. Therefore, $x \to y$ in the Bruhat graph of (W,S) if and only if $(x_1,x_2) \to (y_1,y_2)$ in the Bruhat graph of $W_1 \times W_2$. But, from the definition of the direct product of two Coxeter systems, we have $(x_1,x_2) \to (y_1,y_2)$

in the Bruhat graph of $W_1 \times W_2$ if and only if either $x_1 = y_1$ and $x_2 \to y_2$, or $x_2 = y_2$ and $x_1 \to y_1$, as claimed. □

The next result shows that hypercube decompositions behave very well under direct products.

Theorem 5.2. *Let (W, S), (W_1, S_1), and (W_2, S_2) be three Coxeter systems and $u, v \in W$, $u_i, v_i \in W_i$ for $i = 1, 2$ be such that $[u, v] \simeq [u_1, v_1] \times [u_2, v_2]$ (isomorphic as posets). Let $x \mapsto (x_1, x_2)$ be an isomorphism and $z \in [u, v]$. Then the following are equivalent:*

i) z is an upper hypercube decomposition of $[u, v]$;

ii) z_i is an upper hypercube decomposition of $[u_i, v_i]$, for $i = 1, 2$.

Proof. Since the concept of a hypercube decomposition of an interval depends only on the structure of the interval as a poset (which determines its structure as a directed graph by [17, Proposition 3.3]), we may suppose that $W = W_1 \times W_2$ and $[u, v] = [(u_1, u_2), (v_1, v_2)]$.

Suppose first that z_1 and z_2 are upper hypercube decompositions of $[u_1, v_1]$ and $[u_2, v_2]$, respectively.

We begin by showing that $[(z_1, z_2), (v_1, v_2)]$ is diamond complete. Let $(x_1, x_2), (a_1, a_2), (b_1, b_2), (y_1, y_2) \in [u, v]$ be such that $(x_1, x_2) \to (a_1, a_2) \to (y_1, y_2)$, $(x_1, x_2) \to (b_1, b_2) \to (y_1, y_2)$, and $(a_1, a_2), (b_1, b_2), (y_1, y_2) \in [(z_1, z_2), (v_1, v_2)]$ with $(a_1, a_2) \neq (b_1, b_2)$. By Lemma 5.1, one of these cases must hold:

- $x_1 \to a_1 \to y_1$, $x_1 \to b_1 \to y_1$, and $x_2 = a_2 = b_2 = y_2$,
- $x_2 \to a_2 \to y_2$, $x_2 \to b_2 \to y_2$, and $x_1 = a_1 = b_1 = y_1$,
- $x_1 \to a_1 = y_1$, $x_1 = b_1 \to y_1$, $x_2 = a_2 \to y_2$, and $x_2 \to b_2 = y_2$,
- $x_1 = a_1 \to y_1$, $x_1 \to b_1 = y_1$, $x_2 \to a_2 = y_2$, and $x_2 = b_2 \to y_2$.

In the first two cases, $(x_1, x_2) \in [(z_1, z_2), (v_1, v_2)]$ holds since z_1 and z_2 are upper hypercube decompositions of $[u_1, v_1]$ and $[u_2, v_2]$. In the second two cases, $(x_1, x_2) \in [(z_1, z_2), (v_1, v_2)]$ trivially holds.

Let now $(p_1, p_2) \in [(z_1, z_2), (v_1, v_2)]$. Let, for brevity, $E^{(p_1,p_2)} = \{(x_1, x_2) \in [(u_1, u_2), (v_1, v_2)] : (x_1, x_2) \not\geq (z_1, z_2) \text{ and } (x_1, x_2) \to (p_1, p_2)\}$. It is clear from Lemma 5.1 that

$$E^{(p_1,p_2)} = E^{p_1} \times \{p_2\} \cup \{p_1\} \times E^{p_2}$$

where $E^{p_i} = \{x_i \in [u_i, v_i] : x_i \not\geq z_i \text{ and } x_i \to p_i\}$, for $i = 1, 2$. Let $A \subseteq E^{(p_1,p_2)}$ be an antichain, say $A = \{(a_1, p_2), \ldots, (a_r, p_2)\} \cup \{(p_1, b_1), \ldots, (p_1, b_s)\}$. Let $A_1 = \{a_1, \ldots, a_r\}$ and $A_2 = \{b_1, \ldots, b_s\}$. Then A_i is an antichain in $[u_i, v_i]$, for $i = 1, 2$. Since z_1 and z_2 are upper hypercube decompositions, there are unique embeddings of directed graphs $\theta_i : \mathcal{P}(A_i) \to B(u_i, v_i)$, such that $\theta_i(A_i) = p_i$, for $i = 1, 2$, $\theta_1(A_1 \setminus \{a_j\}) = a_j$ for all $j \in [r]$, and $\theta_2(A_2 \setminus \{b_k\}) = b_k$ for all $k \in [s]$. We define a map $\theta : \mathcal{P}(A) \to B((u_1, u_2), (v_1, v_2))$ by letting

$$\theta(B_1 \times \{p_2\} \cup \{p_1\} \times B_2) = (\theta_1(B_1), \theta_2(B_2))$$

for all $B_1 \subseteq A_1$ and all $B_2 \subseteq A_2$. Clearly, $\theta(A) = (p_1, p_2)$, $\theta(A \setminus \{(a_j, p_2)\}) = (a_j, p_2)$ for all $j \in [r]$, and $\theta(A \setminus \{(p_1, b_k)\}) = (p_1, b_k)$ for all $k \in [s]$, and, by Lemma 5.1, the map θ is an embedding of directed graphs. Let $\widetilde{\theta} : \mathcal{P}(A) \to B((u_1, u_2), (v_1, v_2))$ be an embedding of directed graphs such that $\widetilde{\theta}(A) = (p_1, p_2)$, $\widetilde{\theta}(A \setminus \{(a_j, p_2)\}) = (a_j, p_2)$ for all $j \in [r]$, and $\widetilde{\theta}(A \setminus \{(p_1, b_k)\}) = (p_1, b_k)$ for all $k \in [s]$. We need to prove that $\widetilde{\theta} = \theta$. Given $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$, we use induction on $r + s - |B_1 \cup B_2|$ to show that $\widetilde{\theta}(B_1 \times \{p_2\} \cup \{p_1\} \times B_2) = \theta(B_1 \times \{p_2\} \cup \{p_1\} \times B_2)$. This is clear if $|B_1 \cup B_2| \geq r + s - 1$, so assume $|B_1 \cup B_2| \leq r + s - 2$. Suppose first $|B_1| \leq r - 2$. Let $a_i, a_j \in A_1 \setminus B_1$, $i \neq j$. By the induction hypothesis, we have

$$\begin{aligned}
\widetilde{\theta}\big((B_1 \times \{p_2\} \cup \{p_1\} \times B_2) \cup \{(a_i, p_2)\}\big) &= \theta\big((B_1 \times \{p_2\} \cup \{p_1\} \times B_2) \cup \{(a_i, p_2)\}\big) \\
&= \big(\theta_1(B_1 \cup \{a_i\}), \theta_2(B_2)\big)
\end{aligned}$$

and similarly for a_j. Let $(x_1, x_2) = \widetilde{\theta}(B_1 \times \{p_2\} \cup \{p_1\} \times B_2)$. Since $\widetilde{\theta}$ is an embedding of directed graphs, we have that $(x_1, x_2) \to (\theta_1(B_1 \cup \{a_i\}), \theta_2(B_2))$ and $(x_1, x_2) \to (\theta_1(B_1 \cup \{a_j\}), \theta_2(B_2))$. If $x_2 \to \theta_2(B_2)$ then, by Lemma 5.1, $\theta_1(B_1 \cup \{a_i\}) = x_1 = \theta_1(B_1 \cup \{a_j\})$, which is a contradiction since $i \neq j$ and θ_1 is an embedding. Hence $x_2 = \theta_2(B_2)$, $x_1 \to \theta_1(B_1 \cup \{a_i\})$ and $x_1 \to \theta_1(B_1 \cup \{a_j\})$. This shows that $x_1 \to \theta_1(B_1 \cup \{a_k\})$ holds for any $a_k \in A_1 \setminus B_1$. Since $p \in [z, v]$ and z is an upper hypercube decomposition of $[u, v]$, we conclude that $x_1 = \theta_1(B_1)$. Hence

$$\widetilde{\theta}(B_1 \times \{p_2\} \cup \{p_1\} \times B_2) = (x_1, x_2) = (\theta_1(B_1), \theta_2(B_2)) = \theta(B_1 \times \{p_2\} \cup \{p_1\} \times B_2).$$

Similarly if $|B_2| \leq s-2$. Finally, suppose that $|B_1| = r-1$ and $|B_2| = s-1$, say $B_1 = A_1 \setminus \{a_j\}$ and $B_2 = A_2 \setminus \{b_k\}$, for some $j \in [r]$ and $k \in [s]$. Then $\theta(B_1 \times \{p_2\} \cup \{p_1\} \times B_2) = (a_j, b_k)$. Let $(x_1, x_2) = \widetilde{\theta}(B_1 \times \{p_2\} \cup \{p_1\} \times B_2)$. Hence $(x_1, x_2) \to (a_j, p_2)$ and $(x_1, x_2) \to (p_1, b_k)$, which, by Lemma 5.1, implies that $x_1 = a_j$ and $x_2 = b_k$, so $\widetilde{\theta}(B_1 \times \{p_2\} \cup \{p_1\} \times B_2) = (x_1, x_2) = (a_j, b_k) = \theta(B_1 \times \{p_2\} \cup \{p_1\} \times B_2)$. Hence $\widetilde{\theta} = \theta$, so (z_1, z_2) is an upper hypercube decomposition of $[(u_1, u_2), (v_1, v_2)]$.

Conversely, suppose that (z_1, z_2) is an upper hypercube decomposition of $[(u_1, u_2), (v_1, v_2)]$. We will show that z_1 is an upper hypercube decomposition of $[u_1, v_2]$. (A similar argument shows that z_2 is an upper hypercube decomposition of $[u_2, v_2]$.)

Let $x_1, a_1, b_1, y_1 \in [u_1, v_1]$ be such that $x_1 \to a_1 \to y_1$, $x_1 \to b_1 \to y_1$, and $a_1, b_1, y_1 \in [z_1, v_1]$ with $a_1 \neq b_1$. Then we have that $(x_1, z_2) \to (a_1, z_2) \to (y_1, z_2)$, $(x_1, z_2) \to (b_1, z_2) \to (y_1, z_2)$, and $(a_1, z_2), (b_1, z_2), (y_1, z_2) \in [(z_1, z_2), (v_1, v_2)]$, with $(a_1, z_2) \neq (b_1, z_2)$. Hence, since $[(z_1, z_2), (v_1, v_2)]$ is diamond complete, we have that $(x_1, z_2) \in [(z_1, z_2), (v_1, v_2)]$, so $x_1 \in [z_1, v_1]$. Therefore $[z_1, v_1]$ is diamond complete.

Let $p_1 \in [z_1, v_1]$ and A_1 be an antichain in E^{p_1}, where E^{p_1} has the same meaning as in the first part of the proof. Then $(p_1, z_2) \in [(z_1, z_2), (v_1, v_2)]$ and $A_1 \times \{z_2\}$ is an antichain in $E^{(p_1, z_2)}$ (where $E^{(p_1, z_2)}$ has the same meaning as in the first part of the proof, namely $E^{(p_1, z_2)} = \{(x_1, x_2) \in [(u_1, u_2), (v_1, v_2)] : (x_1, x_2) \ngeq (z_1, z_2)$ and $(x_1, x_2) \to (p_1, z_2)\}$). Since (z_1, z_2) is an upper hypercube decomposition, there is a unique embedding $\theta : \mathcal{P}(A_1 \times \{z_2\}) \to B((u_1, u_2), (v_1, v_2))$ of directed graphs such that $\theta(A_1 \times \{z_2\}) = (p_1, z_2)$ and $\theta((A_1 \setminus \{a\}) \times \{z_2\}) = (a, z_2)$ for all $a \in A_1$. We claim that $\theta(B \times \{z_2\}) \subseteq [u_1, v_1] \times \{z_2\}$, for all $B \subseteq A_1$. We prove this by induction on $|A_1 \setminus B|$. Suppose $|B| \leq |A_1| - 2$. Let $b_1, b_2 \in A_1 \setminus B$, $b_1 \neq b_2$. By the induction hypothesis, there are $a_1, a_2 \in [u_1, v_1]$, $a_1 \neq a_2$, such that $\theta((B \cup \{b_1\}) \times \{z_2\}) = (a_1, z_2)$ and $\theta((B \cup \{b_2\}) \times \{z_2\}) = (a_2, z_2)$. Let $(x_1, x_2) = \theta(B)$. Since θ is an embedding of directed graphs, we have that $(x_1, x_2) \to (a_1, z_2)$ and $(x_1, x_2) \to (a_2, z_2)$. By Lemma 5.1, this implies that $x_2 = z_2$, which proves our claim. Therefore, for all $C \subseteq A_1$, there is $\theta_1(C) \in [u_1, v_1]$ such that $\theta(C \times \{z_2\}) = (\theta_1(C), z_2)$. Since θ is an embedding of directed graphs, the map $\theta_1 : \mathcal{P}(A_1) \to B(u_1, v_1)$ is also an embedding of directed graphs, and it satisfies $\theta_1(A_1) = p_1$ and $\theta_1(A_1 \setminus \{a\}) = a$ for all $a \in A_1$. Let $\widetilde{\theta}_1 : \mathcal{P}(A_1) \to B(u_1, v_1)$ be an embedding of directed graphs such that $\widetilde{\theta}_1(A_1) = p_1$ and $\widetilde{\theta}_1(A_1 \setminus \{a\}) = a$ for all $a \in A_1$. Then

$\widetilde{\theta} : \mathcal{P}(A_1 \times \{z_2\}) \to B((u_1, u_2), (v_1, v_2))$ defined by $\widetilde{\theta}(C \times \{z_2\}) = (\widetilde{\theta}_1(C), z_2)$ for all $C \subseteq A_1$ is also an embedding of directed graphs, $\widetilde{\theta}(A_1 \times \{z_2\}) = (p_1, z_2)$, and $\widetilde{\theta}((A_1 \setminus \{a\}) \times \{z_2\}) = (a, z_2)$ for all $a \in A_1$. Since (z_1, z_2) is an upper hypercube decomposition, we have that $\widetilde{\theta} = \theta$, so $\widetilde{\theta}_1 = \theta_1$. Hence, z_1 is an upper hypercube decomposition of $[u_1, v_1]$. □

Our next result shows that shortcuts also behave well under direct products.

Proposition 5.3. *Let (W_1, S_1), (W_2, S_2), and (W, S) be three Coxeter systems and $u_1, v_1 \in W_1$, $u_2, v_2 \in W_2$, and $u, v \in W$ be such that $[u, v] \simeq [u_1, v_1] \times [u_2, v_2]$ (isomorphic as posets). Let $x \mapsto (x_1, x_2)$ be an isomorphism, $z \in [u, v]$, and $p \in [z, v]$. Then the following conditions are equivalent:*

i) $p \in W^z_{[u,v]}$;

ii) $p_i \in (W_i)^{z_i}_{[u_i,v_i]}$, *for* $i = 1, 2$.

Proof. As in the proof of Theorem 5.2, we may suppose that $W = W_1 \times W_2$ and $[u, v] = [(u_1, u_2), (v_1, v_2)]$.

From Lemma 5.1, we have that every Bruhat path from u to p in $B(u, v)$ is the shuffle of a Bruhat path from u_1 to p_1 in $B(u_1, v_1)$ and one from u_2 to p_2 in $B(u_2, v_2)$. In particular, $d(u, p) = d(u_1, p_1) + d(u_2, p_2)$.

Assume first that p_1 and p_2 are both shortcuts. Let $(u_1, u_2) = (a_0, b_0) \to (a_1, b_1) \to \cdots \to (a_{k-1}, b_{k-1}) \to (a_k, b_k) = (p_1, p_2)$ be a path of minimal length in $B(u, v)$. Then, by Lemma 5.1, either $a_{k-1} = p_1$ and $b_{k-1} \to p_2$, or $b_{k-1} = p_2$ and $a_{k-1} \to p_1$. In the first case, by what was just observed, $b_{k-1} \to p_2$ is the last edge of a path of minimal length from u_2 to p_2 in $B(u_2, v_2)$, so $b_{k-1} \notin [z_2, v_2]$ and hence $(a_{k-1}, b_{k-1}) \notin [(z_1, z_2), (v_1, v_2)]$. Similarly in the second case. Hence (p_1, p_2) is a shortcut of $[u, v]$ with respect to z.

Conversely, suppose that p is a shortcut. Let $u_1 = a_0 \to a_1 \to \cdots \to a_k = p_1$ and $u_2 = b_0 \to b_1 \to \cdots \to b_h = p_2$ be two Bruhat paths of minimal length from u_1 to p_1 and from u_2 to p_2, respectively. Then $(u_1, u_2) = (a_0, u_2) \to (a_1, u_2) \to \cdots \to (a_{k-1}, u_2) \to (p_1, u_2) \to (p_1, b_1) \to \cdots \to (p_1, b_{h-1}) \to (p_1, p_2)$ is a Bruhat path of minimal length from (u_1, u_2) to (p_1, p_2). Since p is a shortcut, $(p_1, b_{h-1}) \notin [(z_1, z_2), (v_1, v_2)]$, so $b_{h-1} \notin [z_2, v_2]$. Hence p_2 is a shortcut. Similarly for p_1. □

We now introduce the second new combinatorial concept of this work.

Definition 5.4. Let W be a Coxeter group and $u, v \in W$. Let z be an upper hypercube decomposition of $[u, v]$. We say that the interval $[z, v]$ (or just z, for short) is a *join upper hypercube decomposition* of $[u, v]$ provided that, for all w in $[u, v]$, the intersection $[w, v] \cap [z, v]$ has a unique minimal element (i.e., w and z have a join).

Similarly, we have the concept of a *meet lower hypercube decomposition*.

Remark 5.5. As already noted before, the intersection of any interval with any coset of a standard parabolic subgroup is itself an interval ([27]). Hence, the canonical hypercube decompositions (Remark 4.2) are join (or meet) hypercube decompositions. Note that not all upper hypercube decompositions are join upper hypercube decompositions. For example, if $u = 1234$, $v = 3412$, and $z = 2143$, then z is an upper hypercube decomposition of $[u, v]$ but not a join upper hypercube decomposition. This is, so far, the only known combinatorially defined class of elements that is strictly contained in the class of hypercube decompositions but still includes the canonical decompositions.

The following simple result shows that elements that have a join with every other element in a poset also behave well under direct products.

Proposition 5.6. *Let P, P_1 and P_2 be posets such that $P \simeq P_1 \times P_2$. Let $x \mapsto (x_1, x_2)$ be an isomorphism, and $z \in P$. Then the following are equivalent:*

i) z has a join with every other element of P;

ii) z_i has a join with every other element of P_i, for $i = 1, 2$.

Proof. Suppose that i) holds. Then (z_1, z_2) has a join with every other element of $P_1 \times P_2$. Let $(x_1, x_2) \in P_1 \times P_2$ and $(y_1, y_2) := (z_1, z_2) \vee (x_1, x_2)$. Then, as it is easy to show, $y_i = z_i \vee x_i$ for $i = 1, 2$. Similarly if ii) holds. □

We can now prove the main result of this section.

Corollary 5.7. *Let (W, S), (W_1, S_1), and (W_2, S_2) be three Coxeter systems and $u, v \in W$, $u_i, v_i \in W_i$ with $u_i \neq v_i$, for $i = 1, 2$, be such that $[u, v] \simeq [u_1, v_1] \times [u_2, v_2]$ (isomorphic as posets). If every join upper hypercube decomposition of any Bruhat interval of rank less than $\ell(u, v)$ is an R-element, then every join upper hypercube decomposition of $[u, v]$ is an R-element.*

Proof. Let $x \mapsto (x_1, x_2)$ be an isomorphism from $[u,v]$ to $[u_1,v_1] \times [u_2,v_2]$. Let $z \in [u,v]$ be a join upper hypercube decomposition of $[u,v]$, and $p \in W^z_{[u,v]}$. By Theorem 5.2 and Proposition 5.6 z_1 and z_2 are join upper hypercube decompositions of $[u_1,v_1]$ and $[u_2,v_2]$, respectively. Further, by Proposition 5.3, $p_1 \in (W_1)^{z_1}_{[u_1,v_1]}$ and $p_2 \in (W_2)^{z_2}_{[u_2,v_2]}$.

We claim that $\widetilde{R}_{p,v} = \widetilde{R}_{p_1,v_1}\,\widetilde{R}_{p_2,v_2}$. Indeed, consider the direct product $W_1 \times W_2$ of the two Coxeter systems and the interval $[(p_1,p_2),(v_1,v_2)]$ in this direct product. The Subword property (see, e.g., [4, Theorem 2.2.2]) implies that $[(p_1,p_2),(v_1,v_2)] = [p_1,v_1] \times [p_2,v_2]$ as posets. Therefore, $[p,v] \simeq [(p_1,p_2),(v_1,v_2)]$ as posets. Since every join upper hypercube decomposition of any Bruhat interval of rank $< \ell(u,v)$ is an R-element the CIC holds for $[p,v] \simeq [(p_1,p_2),(v_1,v_2)]$. Hence $\widetilde{R}_{p,v} = \widetilde{R}_{(p_1,p_2),(v_1,v_2)}$. But, by [5, Proposition 1.7], $\widetilde{R}_{(p_1,p_2),(v_1,v_2)} = \widetilde{R}_{p_1,v_1}\,\widetilde{R}_{p_2,v_2}$, so our claim follows.

Therefore,

$$\begin{aligned}
\widetilde{R}^z_{u,v} &= \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \widetilde{R}_{p,v}(q) \\
&= \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u_1,p_1)} q^{\mathrm{d}(u_2,p_2)} \widetilde{R}_{p_1,v_1}(q) \widetilde{R}_{p_2,v_2}(q) \\
&= \sum_{p_1 \in W^{z_1}_{[u_1,v_1]}} q^{\mathrm{d}(u_1,p_1)} \widetilde{R}_{p_1,v_1}(q) \sum_{p_2 \in W^{z_2}_{[u_2,v_2]}} q^{\mathrm{d}(u_2,p_2)} \widetilde{R}_{p_2,v_2}(q) = \widetilde{R}^{z_1}_{u_1,v_1}\,\widetilde{R}^{z_2}_{u_2,v_2},
\end{aligned}$$

where the second equality follows from our claim, and the third one from Proposition 5.3. Since all join upper hypercube decompositions of $[u_1,v_1]$ and $[u_2,v_2]$ are R-elements, we have that $\widetilde{R}^{z_1}_{u_1,v_1} = \widetilde{R}_{u_1,v_1}$ and $\widetilde{R}^{z_2}_{u_2,v_2} = \widetilde{R}_{u_2,v_2}$. Therefore $\widetilde{R}^z_{u,v}$ does not depend on z and hence $\widetilde{R}^z_{u,v} = \widetilde{R}_{u,v}$, so z is an R-element. □

6 Open problems

In this section we discuss some conjectures and open problems arising from our work, and the evidence that we have about them.

It is natural to wonder whether all upper hypercube decompositions are R-elements. This is not true. For example, if $u = 432156$, $v = 645231$, and $z = 543216$, then z is an upper hypercube decomposition of $[u,v]$ but not an R-element: indeed, $\widetilde{R}^z_{u,v} = t^7 + 3t^5$ while $\widetilde{R}_{u,v} = t^7 + 2t^5$. However, we feel that the following holds.

Conjecture 6.1. *Let W be a Coxeter group of type A, and $u, v \in W$ with $u \leq v$. Then every join upper hypercube decomposition z of $[u, v]$ is an R-element, i.e.*

$$\widetilde{R}_{u,v}(q) = \sum_{p \in W^z_{[u,v]}} q^{\mathrm{d}(u,p)} \widetilde{R}_{p,v}(q). \tag{6}$$

Note that Conjecture 6.1 implies the Combinatorial Invariance Conjecture for the symmetric groups. We have checked Conjecture 6.1 for all intervals in S_n, for $n \leq 6$. By Corollary 3.10, we have that Conjecture 6.1 holds for the canonical upper hypercube decompositions. Furthermore, Proposition 3.4 implies that Conjecture 6.1 holds for all intervals such that every edge in the Bruhat graph has length 1. Finally, our results in Section 5 show that, if Conjecture 6.1 holds for all Bruhat intervals of rank $< d$ for some $d \in \mathbb{N}$, and $[u, v]$ is a Bruhat interval of rank d that is the direct product of two smaller Bruhat intervals, then Conjecture 6.1 holds for $[u, v]$.

There is in the literature one other conjecture that gives a combinatorial recipe for computing the $\widetilde{R}$-polynomials of the symmetric group (which also uses hypercube decompositions), namely Conjecture 1.2 of [1]. These conjectures were obtained independently. We here point out the similarities and differences between them. To do so, we find it convenient to recall the conjecture in [1] (stating the upper version of it). Let $z \in [u, v]$ be an upper hypercube decomposition and keep notation as in Section 2. Then z is called a *strong* hypercube decomposition ([1, Def. 3.6]) if for each $p \in [z, v]$ and all $E_1, E_2 \subseteq E^p$ such that $|E_1| = |E_2| = |E_1 \cap E_2| + 1$, the sources of the edges of E_i are an antichain for $i = 1, 2$ (say, for short, that E_1 and E_2 are *edge antichains*), and there is a $w \in [u, v]$ such that $w \to \theta_{E_1}(\emptyset) \to \theta_{E_1 \cap E_2}(\emptyset)$ and $w \to \theta_{E_2}(\emptyset) \to \theta_{E_1 \cap E_2}(\emptyset)$, then $E_1 \cup E_2$ is also an edge antichain and $w = \theta_{E_1 \cup E_2}(\emptyset)$. The following is [1, Conj. 1.2].

Conjecture 6.2. *Let W be a Coxeter group of type A, $u, v \in W$, $u \leq v$, and $z \in [u, v]$ be a strong upper hypercube decomposition of $[u, v]$. Then*

$$\widetilde{R}_{u,v}(t) \leq \sum_{p \in [z,v]} \sum_{E \subseteq E^p} t^{|E|} \widetilde{R}_{p,v}(t) \tag{7}$$

(coefficientwise) where E runs over all subsets of E^p that are edge antichains, and $\theta_E(\emptyset) = u$.

The main difference between the two conjectures, of course, is that Conjecture 6.2 is an inequality while Conjecture 6.1 is an equality. Also, the two conjectures concern different kinds of hypercube decompositions (strong and join). While, as remarked previously, Conjecture 6.1 fails for general hypercube decompositions, Conjecture 6.2 could be true for all hypercube decompositions since it is an open question (see [1, Rem. 3.7]) whether all hypercube decompositions are strong. It is not known if the polynomials on the right hand sides of (6) and (7) coincide for all join hypercube decompositions.

There is also, in the literature, a conjecture that gives a combinatorial recipe to compute the Kazhdan–Lusztig polynomials of the symmetric group, namely Conjecture 3.8 in [7]. No logical implications between Conjecture 6.2 (i.e., [1, Conj. 1.2]), Conjecture 6.1, and [7, Conj. 3.8] are known.

In the same spirit as [1, Conj. 1.2], the following problem might be worth investigating.

Problem 6.3. *Let W be a Coxeter group of type A. For which intervals $[u, v]$ does the inequality $\widetilde{R}^z_{u,v} \geq \widetilde{R}_{u,v}$ hold for every upper hypercube decomposition z?*

We have checked that the inequality in Problem 6.3 holds for every upper hypercube decomposition in all intervals in S_n, for $n \leq 6$.

Regarding general hypercube decompositions, we feel that, in view of Theorem 4.5, the following problem is natural, should shed light also on the relationships between Conjecture 6.1 and [1, Conj. 1.2], and could therefore be worth investigating. We keep notation as in Theorem 4.5.

Problem 6.4. *Let W be a Coxeter group of type A, and $u, v \in W$, with $u \leq v$. Let z be an upper hypercube decomposition of $[u, v]$, and $p \in [z, v]$. Is it true that $p \in W^z_{u,v}$ if and only if $\theta_p(\emptyset) = u$?*

By Theorem 4.5, the answer to the previous problem is positive for the canonical upper hypercube decompositions.

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