Peak algebras, paths in the Bruhat graph and Kazhdan-Lusztig polynomials*

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Abstract

We give a new characterization of the peak subalgebra of the algebra of quasisymmetric functions and use this to construct a new basis for this subalgebra. As an application of these results we obtain a combinatorial formula for the Kazhdan-Lusztig polynomials which holds in complete generality and is simpler and more explicit than any existing one. We point out that, in a certain sense, this formula cannot be simplified.

1. Introduction

In their seminal paper [31] Kazhdan and Lusztig introduced a family of polynomials, indexed by pairs of elements of a Coxeter group W, that are now known as the Kazhdan-Lusztig polynomials of W (see, e.g., [12] or [29]). These polynomials play a fundamental role in several areas of mathematics, including representation theory, the geometry of Schubert varieties, the theory of Verma modules, Macdonald polynomials, canonical bases, immanant inequalities, and the Hodge theory of Soergel bimodules (see, e.g., [1, 5, 11, 17, 23, 24, 26, 27, 28, 32, 41], and the references cited there). Given their importance, it is natural to try to compute these polynomials in full generality (i.e., for all pairs of elements of all Coxeter groups) as explicitly as possible. Initially, the only way to do this would be to use Kazhdan and Lusztig's original existence proof ([31, §2.2], see also [29, §§7.10-7.11] or [8, §5.1]), which involves a fairly complicated recursion on the Bruhat order of W. Starting in 1994 the first author, and various collaborators, have given a series of nonrecursive combinatorial formulas for the polynomials which hold in complete generality ([13, 14, 15, 16, 8]). Essentially, all these formulas express the Kazhdan-Lusztig polynomial of two elements $u, v \in W$ as a sum, where each summand is a product of a number, which depends on u, v and W, and a polynomial, which is independent of u, v and W, defined in terms of lattice paths. A geometric interpretation of one of these formulas has been given (for geometric cases, i.e., for Weyl groups) by Morel in [35].

Quasisymmetric functions were introduced by Gessel in [25] and are related to many topics in algebra, combinatorics, and geometry including descent algebras, Macdonald polynomials, Kazhdan-Lusztig polynomials, enumeration, convex polytopes, noncommutative symmetric functions, Hecke algebras, and Schubert polynomials (see, e.g., [2, 3, 8, 9, 10, 18, 33, 34], and the references cited there). An interesting class of quasisymmetric functions is the class of peak quasisymmetric functions. These quasisymmetric functions were introduced independently by Billey and Haiman in [10] and by Stembridge in [40]. The linear span of these peak quasisymmetric

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functions is an algebra, known as the algebra of peaks (or peak algebra), first defined and studied by Stembridge in [40]. This algebra was later found to be connected to several other topics including the theory of convex polytopes (and, more generally, Eulerian posets [9]), Whitney stratified manifolds ([21]), and Coxeter groups ([8]), often shedding new light on areas that had been studied for many years. In the enumerative theory of convex polytopes a fundamental classical problem is that of characterizing the sequences of integers that are f-vectors of convex polytopes (see, e.g., [42], or [7]). More generally, there is interest in the flag f-vector (which counts flags of faces of prescribed dimensions). In 1985 Bayer and Billera characterized [3] all the linear relations that hold for the flag f-vectors of all convex polytopes. In 1991 Bayer and Klapper [4], elaborating on ideas of Fine, showed that any sequence of numbers satisfying the above relations (now known as the Bayer-Billera relations) can be irredundantly encoded in a polynomial in two noncommuting variables, and that, conversely, any such polynomial encodes a sequence of numbers satisfying the Bayer-Billera relations. They called this polynomial, in the case of polytopes, the *cd*-index of the polytope. Right from the start it was noted that the *cd*-index seemed to possess remarkable nonnegativity properties. Indeed, Fine conjectured ([4, Conj. 5]) and Stanley proved ([39, Cor. 2.2]) that the coefficients of the *cd*-index of any convex polytope are nonnegative. This result was then generalized by Karu in [30], using ideas from intersection cohomology, to all Cohen-Macaulay Eulerian posets, as conjectured by Stanley in [39, Conj. 2.1] (it is interesting to note that the recent work of Elias-Williamson proving the nonnegativity of the coefficients of Kazhdan-Lusztig polynomials [23] also uses ideas from intersection cohomology). In 2000 Bergeron, Mykytiuk, Sottile and van Willigenburg showed [6, Thm 5.4] that there is a close relationship between the peak algebra and the Bayer-Billera relations. This easily implies ([9, Prop. 1.3] see also Theorem 2.1 below) that given any sequence of numbers satisfying the Bayer-Billera relations there is an associated element of the peak algebra, and conversely. In fact, the sequence of numbers associated to a given element of the peak algebra is just the sequence of coefficients obtained by expanding the element as a linear combination of monomial quasisymmetric functions, while the coefficients of the corresponding noncommutative polynomial are (up to a simple explicit factor) the coefficients obtained by expanding the element as a linear combination of the elements of the basis defining the peak algebra ([9, Thm. 2.1]). In [8] Billera and the first author associated a quasisymmetric function to any pair of elements of any Coxeter group. Using results from [15] they showed that this quasisymmetric function is always in the peak algebra. By expanding this quasisymmetric function as a linear combination of the elements of the defining basis of the peak algebra they obtained an analogue of the *cd*-index for any pair of elements of any Coxeter group (called the complete *cd*-index in [8]) and showed that the Kazhdan-Lusztig polynomial of any pair of elements of any Coxeter system can be expressed in a simple way in terms of the coefficients of the complete *cd*-index and explicit combinatorially defined polynomials. The complete *cd*-index seems to possess remarkable nonnegativity properties ([8, Conj. 6.1]) but this is at present unknown.

In this work we give a new characterization of the peak subalgebra of the algebra of quasisymmetric functions and use this to construct a new basis for this subalgebra with certain properties. As an application of these results we obtain a combinatorial formula for the Kazhdan-Lusztig polynomials which holds in complete generality and is simpler and more explicit than any existing one. More precisely, this formula expresses the Kazhdan-Lusztig polynomial of two elements $u, v \in W$ as a sum of at most $f_{\ell(u,v)}$ summands (f_n being the n-th Fibonacci number), each one of which is the product of a number, which depends on u and v, and a polynomial, independent of u, v, and W, and we provide a combinatorial interpretation for both the number and the polynomial. This formula cannot be simplified by means of linear relations if it is to hold in complete generality. We then investigate linear relations between the numbers involved in the formula and show that there are no "homogeneous" relations even for lower intervals of a fixed rank. Our proof uses some new total reflection orderings which may be of independent interest.

The organization of the paper is as follows. In the next section we collect some notation, definitions, and results that are needed in the rest of this work. In §3 we give a new characterization of the peak subalgebra of the algebra of quasisymmetric functions (Theorem 3.1). In §4, using this characterization, we construct a basis of the peak subalgebra of the algebra of quasisymmetric functions with certain properties (Theorem 4.3). In §5, using the results in the previous ones, we obtain a combinatorial formula for the Kazhdan-Lusztig polynomials which holds in complete generality (Theorem 5.1), is simpler and more explicit than any existing one, and cannot be "linearly" simplified. Finally, in §6, we study linear relations between the numbers involved in the formula.

2. Preliminaries

We let $\mathbb{P} \stackrel{\text{def}}{=} \{1,2,3,\dots\}$, $\mathbb{N} \stackrel{\text{def}}{=} \mathbb{P} \cup \{0\}$, $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ be the field of rational numbers, and $\mathbb{R}$ be the field of real numbers; for $a \in \mathbb{N}$ we let $[a] \stackrel{\text{def}}{=} \{1,2,\dots,a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). Given $n, m \in \mathbb{P}$, $n \leqslant m$, we let $[n,m] \stackrel{\text{def}}{=} [m] \setminus [n-1]$, and we define similarly $(n,m]$, (n,m), and $[n,m)$. For $S \subseteq \mathbb{Q}$ we write $S = \{a_1,\dots,a_r\}_<$ to mean that $S = \{a_1,\dots,a_r\}$ and $a_1 < \cdots < a_r$. The cardinality of a set A will be denoted by $|A|$. Given a polynomial $P(q)$, and $i \in \mathbb{Z}$, we denote by $[q^i](P(q))$ the coefficient of q^i in $P(q)$. Given $j \in \mathbb{Z}$ we let $\chi_{\text{odd}}(j) = 1$ if j is odd and $\chi_{\text{odd}}(j) = 0$ if j is even, and $\chi_{\text{even}}(j) = 1 - \chi_{\text{odd}}(j)$. We let f_n be the n-th Fibonacci number defined recursively by $f_0 \stackrel{\text{def}}{=} 0$, $f_1 \stackrel{\text{def}}{=} 1$ and $f_n \stackrel{\text{def}}{=} f_{n-1} + f_{n-2}$ for $n > 1$.

Recall that a *composition* of n ($n \in \mathbb{P}$) is a sequence $(\alpha_1,\dots,\alpha_s)$ (for some $s \in \mathbb{P}$) of positive integers such that $\alpha_1 + \cdots + \alpha_s = n$ (see, e.g., [38, p. 17]). For $n \in \mathbb{P}$ we let C_n be the set of all compositions of n and $C \stackrel{\text{def}}{=} \bigcup_{n \geqslant 1} C_n$. Given $\beta \in C$ we denote by $l(\beta)$ the number of parts of β, by β_i, for $i = 1,\dots,l(\beta)$, the i-th part of β (so that $\beta = (\beta_1,\beta_2,\dots,\beta_{l(\beta)})$), and we let $|\beta| \stackrel{\text{def}}{=} \sum_{i=1}^{l(\beta)} \beta_i$, and $T(\beta) \stackrel{\text{def}}{=} \{\beta_r, \beta_r + \beta_{r-1}, \dots, \beta_r + \cdots + \beta_2\}$ where $r \stackrel{\text{def}}{=} l(\beta)$ (note that this definition is backwards from what would normally be expected, but agrees with [15] and [12]). Given $(\alpha_1,...,\alpha_s), (\beta_1,...,\beta_t) \in C_n$ we say that $(\alpha_1,...,\alpha_s)$ *refines* $(\beta_1,...,\beta_t)$ if there exist $0 < i_1 < i_2 < \cdots < i_{t-1} < s$ such that $\sum_{j=i_{k-1}+1}^{i_k} \alpha_j = \beta_k$ for $k = 1,\dots,t$ (where $i_0 \stackrel{\text{def}}{=} 0$, $i_t \stackrel{\text{def}}{=} s$). We then write $(\alpha_1,...,\alpha_s) \preceq (\beta_1,...,\beta_t)$. It is well known, and easy to see, that the map $\alpha \mapsto T(\alpha)$ is an isomorphism from $(C_n, \preceq)$ to the Boolean algebra B_{n-1} of subsets of $[n-1]$, ordered by reverse inclusion.

We let $\mathbf{2} \stackrel{\text{def}}{=} \{0,1\}$ and for $n \in \mathbb{N}$ we let $\mathbf{2}^n$ be the set of all 0-1 words of length n

$$\mathbf{2}^n = \{E = (E_1 \cdots E_n) : \, E_i \in \mathbf{2}\},$$

$\varepsilon \in \mathbf{2}^0$ be the empty word, and $\mathbf{2}^* \stackrel{\text{def}}{=} \cup_{n \geqslant 0} \mathbf{2}^n$. We consider on $\mathbf{2}^*$ the monoid structure given by concatenation. We say that $E \in \mathbf{2}^*$ is *sparse* if either $E = \varepsilon$ or E belongs to the submonoid generated by 0 and 01 and we let $\mathbf{2}^*_s$ be the monoid of sparse sequences. We also let $1\mathbf{2}^* \stackrel{\text{def}}{=} \{1E : E \in \mathbf{2}^*\}$ and $\overline{1\mathbf{2}^*} \stackrel{\text{def}}{=} 1\mathbf{2}^* \cup \{\epsilon\}$, and we similarly define $\mathbf{2}^*1$ and $\overline{\mathbf{2}^*1}$. If $E \in \mathbf{2}^n$ we let $\check{E} \stackrel{\text{def}}{=} (E_1 \cdots E_{n-1}(1-E_n))$ if $n \geqslant 1$, and $\check{\varepsilon} = \varepsilon$, $\overline{E}$ be the complementary string (so the i-th element of $\overline{E}$ is 1 if and only if the i-th element of E is 0, for $i \in [n]$), and E^{op} be the opposite string of E (so the i-th element of E^{op} is 1 if and only if the $(n+1-i)$-th element of E is 1, for $i \in [n]$). For notational convenience, for $1 \leqslant i \leqslant j$ we also let $E_{i,j} \stackrel{\rm def}{=} 0^{i-1}10^{j-i}$. Finally, we let $S(E) \stackrel{\rm def}{=} \{i \in [n] : E_i = 1\}$, $m_i(E) \stackrel{\rm def}{=} |\{j \in [n] : E_j = i\}|$, for $i \in \mathbf{2}$, and $\ell(E) \stackrel{\rm def}{=} n$. We consider on $\mathbf{2}^n$ the natural partial order $\leqslant$ defined by $E' \leqslant E$ if and only if $S(E') \subseteq S(E)$. If $E, E' \in \mathbf{2}^n$ we let $E \vee E'$ be the only element of $\mathbf{2}^n$ such that $S(E \vee E') = S(E) \cup S(E')$.

A formal power series $F \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is a *quasisymmetric function* if it is of bounded degree and for all $\alpha_1, \alpha_2, \ldots \in \mathbb{N}$, the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ in F equals the coefficient of $x_{\sigma(1)}^{\alpha_1} x_{\sigma(2)}^{\alpha_2} \cdots$ in F for all $\sigma : \mathbb{P} \to \mathbb{P}$ which are strictly increasing. Clearly, a symmetric function is quasisymmetric, but not conversely. The set $\mathcal{Q}$ of quasisymmetric functions is a $\mathbb{Q}$-algebra, graded by the usual degree. We denote by $\mathcal{Q}_i$ the ith homogeneous part of $\mathcal{Q}$, so $\mathcal{Q} = \mathcal{Q}_0 \oplus \mathcal{Q}_1 \oplus \cdots$.

There are (at least) two important bases of $\mathcal{Q}_n$ both indexed by C_n. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in C_n$ let

$$M_\alpha \stackrel{\rm def}{=} \sum_{1 \leqslant i_1 < \cdots < i_r} x_{i_1}^{\alpha_1} \ldots x_{i_r}^{\alpha_r}$$

and

$$L_\alpha \stackrel{\rm def}{=} \sum_{\{\beta \in C_n :\, \beta \preceq \alpha\}} M_\beta .$$

These are called the *monomial* and *fundamental* bases (respectively) of $\mathcal{Q}_n$.

If $E \in \mathbf{2}^{n-1}$ and $S(E) = \{s_1, \ldots, s_t\}_<$ we let $oc(E) = (n - s_t, s_t - s_{t-1}, \cdots, s_2 - s_1, s_1)$; $oc(E)$ is a composition of n and we denote by M_E the monomial quasisymmetric function $M_{oc(E)}$, and by L_E the fundamental quasisymmetric function $L_{oc(E)}$. So, for example, $M_\varepsilon = L_\varepsilon = \sum_{i \geqslant 1} x_i$, $M_{001010} = \sum_{1 \leqslant i_1 < i_2 < i_3} x_{i_1}^2 x_{i_2}^2 x_{i_3}^3$ and, for $E \in \mathbf{2}^{n-1}$, $L_E = \sum_{E' \geqslant E} M_{E'}$. We emphasize that what we denote L_E is denoted by $L_{oc(E)}$ in [37, §7.19]. Note that the degree of M_E and L_E is $\ell(E)+1$.

An element $E = (E_1 \cdots E_n) \in \mathbf{2}^n$ is said to be *peak* if $E = \varepsilon$ or if $E_1 = E_n = 0$ and $E_i = 1$ implies $E_{i-1} = E_{i+1} = 0$ for all $i = 2, \ldots, n-1$. So, for $n \geqslant 1$, E is peak if and only if $(E_1 \cdots E_{n-1})$ is sparse and $E_n = 0$. Given such a peak string $E \in \mathbf{2}^n$ we let

$$K_E \stackrel{\rm def}{=} \sum_{\{A \in \mathbf{2}^{n-1} :\, E \leqslant A0 \vee 0A\}} 2^{m_1(A)+1} M_A,$$

and $K_\varepsilon \stackrel{\rm def}{=} 1$. The *peak algebra* Π of $\mathcal{Q}$ is defined to be the subspace spanned by the quasisymmetric functions K_E as E ranges over all peak strings, for all $n \geqslant 0$. It is known ([40, Thm. 3.1]) that Π is indeed an algebra.

The following result is known (see [9, Prop. 1.3] and also [6, Thm. 5.4]), and can also be taken as a definition of Π.

Theorem 2.1. *Let $F = c + \sum_{E \in \mathbf{2}^*} c_E M_E \in \mathcal{Q}$. Then the following are equivalent:*

i) *$F \in \Pi$;*

ii) *for all $A \in \overline{\mathbf{2}^* 1}$, $B \in \overline{1\, \mathbf{2}^*}$, and $j \geqslant 1$*

$$\sum_{i=1}^{j} (-1)^{i-1} c_{A E_{i,j} B} = 2 \chi_{odd}(j) c_{A\, 0^j B}. \qquad (1)$$

The relations in part ii) of the above result are known as the *Bayer-Billera* (or *generalized Dehn-Sommerville*) relations (see, e.g., [3]).

Let $\mathscr{V}_n$ be the $\mathbb{Q}$-vector space of functions on $\mathbf{2}^n$ taking values in $\mathbb{Q}$. In particular, $\dim_{\mathbb{Q}}(\mathscr{V}_n) = 2^n$. If $\alpha \in \mathscr{V}_n$ and $E \in \mathbf{2}^n$ we let $\alpha_E \stackrel{\text{def}}{=} \alpha(E)$.

Let P be an Eulerian partially ordered set of rank $n+1$ with minimum $\hat{0}$ and maximum $\hat{1}$; we always assume that a chain $\mathcal{C} = (x_1, \ldots, x_k)$ in P does not contain $\hat{0}$ and $\hat{1}$. Given such a chain we define $E(\mathcal{C}) \in \mathbf{2}^n$ by

$$E(\mathcal{C})_i = 1 \Leftrightarrow \exists j \in [k] : \, \rho(x_j) = i,$$

where ρ is the rank function of P. The *flag f-vector* of P is the element $f(P) \in \mathscr{V}_n$ given by

$$f(P)_E \stackrel{\text{def}}{=} |\{\text{chains } \mathcal{C} \text{ in } P : \, E(\mathcal{C}) = E\}|$$

for all $E \in \mathbf{2}^n$.

Let $\mathscr{A}_n$ be the subspace of $\mathscr{V}_n$ generated by the flag f-vectors $f(P)$ of all Eulerian posets of rank $n+1$. The following result is then well known (see [3]).

Theorem 2.2. *The vector space $\mathscr{A}_n$ has dimension f_{n+1} and is determined by the following linear relations: given $\alpha \in \mathscr{V}_n$ we have $\alpha \in \mathscr{A}_n$ if and only if for all $A \in \overline{\mathbf{2}^*1}$, $B \in \overline{1\mathbf{2}^*}$ and $j \geqslant 1$ such that $A0^jB \in \mathbf{2}^n$, we have*

$$\sum_{i=1}^{j} (-1)^{i-1} \alpha_{A\,E_{i,j}\,B} = 2\chi_{odd}(j)\alpha_{A\,0^j\,B}.$$

Note that these relations are exactly the relations that appear in the characterization of the peak algebra in Theorem 2.1.

If P is a graded poset of rank $n+1$, the function $h(P) \in \mathscr{V}_n$ which is uniquely determined by

$$f(P)_E = \sum_{E' \leqslant E} h(P)_{E'},$$

is called the *flag h-vector* of P. The definition is clearly equivalent to

$$h(P)_E = \sum_{E' \leqslant E} (-1)^{m_1(E) - m_1(E')} f(P)_{E'}$$

by the Principle of Inclusion-Exclusion.

We follow [12] for general Coxeter groups notation and terminology. In particular, given a Coxeter system (W,S) and $u \in W$ we denote by $\ell(u)$ the length of u in W, with respect to S, by e the identity of W, and we let $T \stackrel{\text{def}}{=} \{usu^{-1} : u \in W, \; s \in S\}$ be the set of *reflections* of W. We always assume that W is partially ordered by *Bruhat order*. Recall (see, e.g., [12, §2.1]) that this means that $x \leqslant y$ if and only if there exist $r \in \mathbb{N}$ and $t_1, \ldots, t_r \in T$ such that $t_r \cdots t_1\, x = y$ and $\ell(t_i \cdots t_1\, x) > \ell(t_{i-1} \cdots t_1 x)$ for $i = 1, \ldots, r$. Given $u, v \in W$ we let $[u,v] \stackrel{\text{def}}{=} \{x \in W : u \leqslant x \leqslant v\}$. We consider $[u,v]$ as a poset with the partial ordering induced by W. It is well known (see, e.g., [12, Cor. 2.7.11]) that intervals of W are Eulerian posets. Recall (see, e.g., [12, §2.1]) that the *Bruhat graph* of a Coxeter system (W,S) is the directed graph $B(W,S)$ obtained by taking W as vertex set and putting a directed edge from x to tx for all $x \in W$ and $t \in T$ such that $\ell(x) < \ell(tx)$.

We denote by $\mathcal{H}(W)$ the *Hecke algebra* associated to W. Recall (see, e.g., [29, Chap. 7]) that this is the free $\mathbb{Z}[q,q^{-1}]$-module having the set $\{T_w : \, w \in W\}$ as a basis and multiplication such that

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w, & \text{if } \ell(ws) < \ell(w), \end{cases} \tag{2}$$

for all $w \in W$ and $s \in S$. It is well known that this is an associative algebra having T_e as unity and that each basis element is invertible in $\mathcal{H}(W)$. More precisely, we have the following result (see [29, Prop. 7.4]).

Proposition 2.3. *Let $v \in W$. Then*

$$(T_{v^{-1}})^{-1} = q^{-\ell(v)} \sum_{u \leqslant v} (-1)^{\ell(v)-\ell(u)} R_{u,v}(q)\, T_u \,,$$

where $R_{u,v}(q) \in \mathbb{Z}[q]$.

The polynomials $R_{u,v}(q)$ defined by the previous proposition are called the *R-polynomials* of W. It is easy to see that $\deg(R_{u,v}(q)) = \ell(v) - \ell(u)$, and that $R_{u,u}(q) = 1$, for all $u, v \in W$, $u \leqslant v$. It is customary to let $R_{u,v}(q) \stackrel{\text{def}}{=} 0$ if $u \not\leqslant v$.

The R-polynomials can be used to define the Kazhdan-Lusztig polynomials. The following result is not hard to prove and a proof can be found, e.g., in [29, §§7.9-11], or [31, §2.2].

Theorem 2.4. *There is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbb{Z}[q]$, such that, for all $u, v \in W$:*

- **i)** $P_{u,v}(q) = 0$ *if* $u \not\leqslant v$*;*
- **ii)** $P_{u,u}(q) = 1$*;*
- **iii)** *deg*$(P_{u,v}(q)) < \frac{1}{2}(\ell(v) - \ell(u))$*, if* $u < v$*;*
- **iv)**

$$q^{\ell(v)-\ell(u)} P_{u,v}\left(\frac{1}{q}\right) = \sum_{z \in [u,v]} R_{u,z}(q)\, P_{z,v}(q) \,, \tag{3}$$

if $u \leqslant v$*.*

The polynomials $P_{u,v}(q)$ defined by the preceding theorem are called the *Kazhdan-Lusztig polynomials* of W.

We denote by Φ^+ the set of positive roots of (W, S) (see, e.g., [12, §4.4]). Recall (see, e.g., [12, §5.2]) that a total ordering $\prec$ on Φ^+ is a *reflection ordering* if whenever $\alpha, \beta, c_1\alpha + c_2\beta \in \Phi^+$ for some $c_1, c_2 \in \mathbb{R}_{>0}$ and $\alpha \prec \beta$ then $\alpha \prec c_1\alpha + c_2\beta \prec \beta$. The existence of reflection orderings (and many of their properties) is proved in [20, §2] (see also [12, §5.2]). By means of the canonical bijection between Φ^+ and T (see, e.g., [12, §4.4]) we transfer the reflection ordering also on T.

Let $\prec$ be a reflection ordering of T. Given a path $\Delta = (a_0, a_1, \ldots, a_r)$ in $B(W, S)$ from a_0 to a_r, we define its *length* to be $l(\Delta) \stackrel{\text{def}}{=} r$, and its *descent string* with respect to $\prec$ to be the sequence $E_{\prec}(\Delta) \in \mathbf{2}^{r-1}$ given by

$$E_{\prec}(\Delta)_{r-i} = 1 \Leftrightarrow a_i(a_{i-1})^{-1} \succ a_{i+1}(a_i)^{-1}.$$

Given $u, v \in W$, and $k \in \mathbb{N}$, we denote by $B_k(u, v)$ the set of all the directed paths in $B(W, S)$ from u to v of length k, and we let $B(u, v) \stackrel{\text{def}}{=} \bigcup_{k \geqslant 0} B_k(u, v)$. For $u, v \in W$, and $E \in \mathbf{2}^{n-1}$, we let, following [14],

$$c(u, v)_E \stackrel{\text{def}}{=} |\{\Delta \in B_n(u, v) :\ E_{\prec}(\Delta) \leqslant E\}|, \tag{4}$$

and

$$b(u, v)_E \stackrel{\text{def}}{=} |\{\Delta \in B_n(u, v) :\ E_{\prec}(\Delta) = E\}|. \tag{5}$$

Note that these definitions imply that

$$c(u,v)_E = \sum_{\{E' \in \mathbf{2}^{n-1} :\, E' \leqslant E\}} b(u,v)_{E'} \tag{6}$$

for all $u, v \in W$ and $E \in \mathbf{2}^{n-1}$. It follows immediately from Proposition 4.4 of [14] that $c(u,v)_E$ (and hence $b(u,v)_E$) are independent of the reflection ordering $\prec$ used to define them.

Let $[u,v]$ be a Bruhat interval of rank $r+1$ in a Coxeter group, and $\Delta = (x_0, x_1, \ldots, x_{n+1})$ a path in the Bruhat graph from u to v. So $x_0 = u$, $x_{n+1} = v$, and for all $i \in [n+1]$ we have $x_{i-1} < x_i$ and the element t_i given by $x_i = x_{i-1} t_i$ is a reflection. We then sometimes denote such a path by

$$\Delta = (x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{n+1}} x_{n+1}).$$

If $\Delta \in B_{n+1}(u,v)$, $\prec$ is a reflection ordering and $E = E_{\prec}(\Delta)$ we let $m_{\prec}(\Delta) \stackrel{\text{def}}{=} \mu_{E_n} \cdots \mu_{E_1} \in \mathbb{Z}\langle a, b\rangle$, where $\mu_0 = a$ and $\mu_1 = b$. In other words, if $\Delta = (x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{n+1}} x_{n+1})$, then $m_{\prec}(\Delta)$ is the product of n factors, the i-th factor being a if $t_i \prec t_{i+1}$ and b otherwise. We will usually drop the subscript $\prec$ from the notation $m_{\prec}(\Delta)$ when it is clear from the context.

If $[u,v]$ is a Bruhat interval of rank $r+1$ the *cd-index* of $[u,v]$ is the polynomial

$$\Psi_{[u,v]} \stackrel{\text{def}}{=} \sum_{E \in \mathbf{2}^r} h([u,v])_E \, \mu_E,$$

where $h([u,v])$ is the flag h-vector of $[u,v]$ and $\mu_E \stackrel{\text{def}}{=} \mu_{E_1} \cdots \mu_{E_r}$. It is known that $\Psi_{[u,v]}$ is a polynomial in $c = a+b$ and $d = ab+ba$, as $[u,v]$ is a Eulerian poset. It is also known that if $[u,v]$ has rank $r+1$ then there exists a unique path $\Delta \in B_{r+1}(u,v)$ such that $E_{\prec}(\Delta) = 0^r \in \mathbf{2}^r$. This implies (see [38, Thm. 3.13.2]) that for all $E \in \mathbf{2}^r$ we have that $b([u,v])_{E^{op}} = h([u,v])_E$. Therefore the *cd*-index of a Bruhat interval $[u,v]$ of length $r+1$ can be expressed as

$$\Psi_{[u,v]} = \sum_{\Delta \in B_{r+1}(u,v)} m_{\prec}(\Delta),$$

where $\prec$ is any reflection ordering. We consider the natural extension of this polynomial to all paths in the Bruhat graph

$$\tilde{\Psi}_{[u,v]}(a,b) \stackrel{\text{def}}{=} \sum_{\Delta \in B(u,v)} m_{\prec}(\Delta).$$

The polynomial $\tilde{\Psi}_{[u,v]}$ has been introduced by Billera and the first author in [8] and can also be expressed as a polynomial in the variables $c = a+b$ and $d = ab+ba$ and therefore it is called the *complete cd-index* of the interval $[u,v]$. We will use the simpler notation $\tilde{\Psi}_{u,v}$ instead of $\tilde{\Psi}_{[u,v]}$ to denote the complete *cd*-index of the Bruhat interval $[u,v]$.

Let $\mathcal{A} = \mathbb{Z}\langle a, b\rangle$. Following [22], we define a coproduct $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ on $\mathcal{A}$ as the unique linear map such that for all $n \in \mathbb{N}$ and all $v_1, \ldots, v_n \in \{a, b\}$,

$$\delta(v_1 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n.$$

One can observe that the algebra $\mathcal{A}$ endowed with the coproduct δ has also a Newtonian coalgebra structure, though this is not needed in the sequel.

Now let $\mathcal{P}$ be the k-vector space consisting of formal finite linear combinations of Bruhat intervals. We define also on $\mathcal{P}$ a coproduct $\delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ in the following way. We let

$$\delta([u,v]) = \sum_{x \in (u,v)} [u,x] \otimes [x,v].$$

The following result is proved in [8, Prop. 2.11]

Proposition 2.5. *The complete cd-index $\tilde{\Psi} : \mathcal{P} \to \mathcal{A}$ is a coalgebra map, i.e.*

$$\sum_{x \in (u,v)} \tilde{\Psi}_{u,x} \otimes \tilde{\Psi}_{x,v} = \delta(\tilde{\Psi}_{u,v}).$$

For all $x \in \mathcal{A}$ write $\delta(x) = \sum_i x_i(1) \otimes x_i(2)$ where $x_i(1), x_i(2) \in \mathcal{A}$. Then for any $y \in \mathcal{A}$ we can consider the following map

$$D_y(x) = \sum_i x_i(1) \cdot y \cdot x_i(2).$$

One can easily verify that this is a well-defined linear map, and that it is a derivation, i.e. it satisfies the Leibniz rule on products, for all $y \in \mathcal{A}$. The following is then an immediate consequence of Proposition 2.5.

Corollary 2.6. *Let $[u,v]$ be any Bruhat interval. Then*

$$D_y(\tilde{\Psi}_{u,v}) = \sum_{x \in (u,v)} \tilde{\Psi}_{u,x} \cdot y \cdot \tilde{\Psi}_{x,v}.$$

Given $u, v \in W$, $u \leqslant v$, we let, following [8],

$$\widetilde{F}(u,v) \stackrel{\text{def}}{=} \sum_{E \in \mathbf{2}^*} b(u,v)_E L_E,$$

if $u < v$ and $\widetilde{F}(u,u) \stackrel{\text{def}}{=} 1$. This definition is different from the one given in [8] but equivalent to it by Theorem 2.2 of [8]. The following is proved in [8, Theorem 2.2] (see also [15, Theorem 8.4]).

Theorem 2.7. *Let (W,S) be a Coxeter system and $u, v \in W$, $u < v$. Then $\widetilde{F}(u,v) \in \Pi$.*

Let $n \in \mathbb{N}$. A *lattice path* of length n is a function $\Gamma : [0,n] \to \mathbb{Z}$ such that $\Gamma(0) = 0$ and $|\Gamma(i) - \Gamma(i-1)| = 1$ for all $i \in [n]$ and we denote by $\mathcal{L}(n)$ the set of all the lattice paths of length n. Given $\Gamma \in \mathcal{L}(n)$ we let $N(\Gamma) \in \mathbf{2}^{n-1}$ be given by

$$N(\Gamma)_i = 1 \Longleftrightarrow \Gamma(i) < 0,$$

and $d_+(\Gamma) \stackrel{\text{def}}{=} |\{i \in [n] : \Gamma(i) - \Gamma(i-1) = 1\}|$. Note that

$$d_+(\Gamma) = \frac{\Gamma(n) + n}{2}. \tag{7}$$

For $E \in \mathbf{2}^{n-1}$ we define, following [12, §5.4], a polynomial $\Upsilon_E(q) \in \mathbb{Z}[q]$ by

$$\Upsilon_E(q) = (-1)^{m_0(E)} \sum_{\Gamma \in \mathcal{L}(E)} (-q)^{d_+(\Gamma)}, \tag{8}$$

where $\mathcal{L}(E) \stackrel{\text{def}}{=} \{\Gamma \in \mathcal{L}(n) : N(\Gamma) = E\}$. Note that what we denote Υ_E is denoted by $\Upsilon_{oc(E)}$ in [12, §5.4]. For example, $\Upsilon_{001010}(q) = q^4 - q^3$.

Following [8] we define a linear map $\mathcal{K} : QSym \to \mathbb{Z}[q^{1/2}, q^{-1/2}]$ by $\mathcal{K}(L_E) \stackrel{\text{def}}{=} q^{-\frac{\ell(E)+1}{2}} \Upsilon_E$, for all $E \in \mathbf{2}^*$. We then have the following result (see [8, Prop. 3.1], and [12, Thm. 5.5.7]).

THEOREM 2.8. *Let $u, v \in W$, $u < v$. Then*

$$\mathcal{K}(\widetilde{F}(u,v)) = q^{\frac{-\ell(u,v)}{2}} P_{u,v}(q) - q^{\frac{\ell(u,v)}{2}} P_{u,v}\left(\frac{1}{q}\right). \tag{9}$$

Given $E \in \mathbf{2}^*$ we let the *exponent composition* of E be the unique composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ such that

$$E = \begin{cases} \underbrace{1\ldots1}_{\alpha_1}\,\underbrace{0\ldots0}_{\alpha_2}\,\underbrace{1\ldots1}_{\alpha_3}\ldots, & \text{if } E_1 = 1, \\ \underbrace{0\ldots0}_{\alpha_1}\,\underbrace{1\ldots1}_{\alpha_2}\,\underbrace{0\ldots0}_{\alpha_3}\ldots, & \text{if } E_1 = 0. \end{cases}$$

So, for example, the exponent composition of 00110 is $(2,2,1)$. The following is a restatement of Corollary 6.7 of [15]. Note that $\Upsilon_E \neq 0$ if the exponent composition of E has only one part.

COROLLARY 2.9. *Let $E \in \mathbf{2}^*$ be such that $\ell(\alpha) \geqslant 2$, where α is the exponent composition of E. Then $\Upsilon_E \neq 0$ if and only if $\alpha_2 \equiv \alpha_3 \equiv \cdots \equiv \alpha_{\ell(\alpha)-1} \equiv 1 \pmod 2$ and $\alpha_1 \equiv E_1 \pmod 2$.* □

3. A characterization of the peak algebra

Our purpose in this section is to give a new characterization of the peak subalgebra of the algebra of quasisymmetric functions. More precisely, we give necessary and sufficient conditions on the coefficients of a quasisymmetric function F, when expressed as a linear combination of fundamental quasisymmetric functions, for F to be in the peak subalgebra. Our result is the following.

THEOREM 3.1. *Let $F = \sum_{E \in \mathbf{2}^*} \beta_E L_E \in \mathcal{Q}$. Then the following are equivalent:*

i) *$F \in \Pi$;*

ii) *for all $A, B \in \mathbf{2}^*$*

$$\beta_{AB} + \beta_{\check{A}B} = \beta_{A\bar{B}} + \beta_{\check{A}\bar{B}}.$$

The rest of this section is devoted to the proof of “ii) implies i)”, while the proof of “i) implies ii)” will be given as a consequence of the main result in §4.

Let $\mathscr{B}_n$ be the vector subspace of $\mathscr{V}_n$ generated by the flag h-vectors of all Eulerian posets of rank $n+1$. Then $\mathscr{B}_n$ has clearly dimension f_{n+1} by Theorem 2.2, and Theorem 3.1 is clearly equivalent to the following one.

THEOREM 3.2. *Let $\alpha, \beta \in \mathscr{V}_n$ be such that $\alpha_E = \sum_{E' \leqslant E} \beta_{E'}$ for all $E \in \mathbf{2}^n$. Then the following are equivalent*

- *α satisfies Bayer-Billera relations (i.e. $\alpha \in \mathscr{A}_n$);*
- *for all $A, B \in \mathbf{2}^*$ such that $AB \in \mathbf{2}^n$*

$$\beta_{AB} + \beta_{\check{A}B} = \beta_{A\bar{B}} + \beta_{\check{A}\bar{B}}. \tag{10}$$

We refer to the relations appearing in (10) as the *dual Bayer Billera-relations*. Note that the relations $\beta_E = \beta_{\bar{E}}$ appear as a special case of (10) by letting $A = \varepsilon$.

Let $\mathscr{B}'_n$ be the subspace of $\mathscr{V}_n$ defined by the relations (10). Theorem 3.2 can therefore be simply stated by saying that $\mathscr{B}_n = \mathscr{B}'_n$ for all $n \in \mathbb{N}$.

We begin our study with a technical result.

Lemma 3.3. *If $\beta \in \mathscr{B}'_n$ then for all $A, B \in \mathbf{2}^*$ and $j > 0$ such that $A1^jB \in \mathbf{2}^n$ we have*

$$\beta_{A1^jB} = \sum_{i=1}^{j-1}(-1)^{i-1}\beta_{AE_{i,j}\bar{B}} + (-1)^{j-1}\beta_{AE_{j,j}B} + \chi_{\text{even}}(j)\beta_{A0^j\bar{B}}.$$

Proof. We proceed by induction on j. If $j = 1$ the relation is a trivial identity and so we assume $j \geqslant 2$. We then have, using Eq. (10) and our induction hypothesis, that

$$\begin{aligned}\beta_{A1^jB} &= \beta_{A10^{j-1}\bar{B}} + \beta_{A0^j\bar{B}} - \beta_{A01^{j-1}B} \\ &= \beta_{AE_{1,j}\bar{B}} + \beta_{A0^j\bar{B}} - \sum_{i=1}^{j-2}(-1)^{i-1}\beta_{A0E_{i,j-1}\bar{B}} + (-1)^{j-1}\beta_{A0E_{j-1,j-1}B} \\ &\quad - \chi_{\text{even}}(j-1)\beta_{A0^j\bar{B}} \\ &= \sum_{i=1}^{j-1}(-1)^{i-1}\beta_{AE_{i,j}\bar{B}} + (-1)^{j-1}\beta_{AE_{j,j}B} + \chi_{\text{even}}(j)\beta_{A0^j\bar{B}}.\end{aligned}$$

□

Lemma 3.4. *If $\beta \in \mathscr{B}'_n$ then for all $A \in \mathbf{2}^*$ and $B \in \overline{\mathbf{12}^*}$ such that $AB \in \mathbf{2}^n$, we have*

$$\sum_{B' \leqslant B} \beta_{AB'} = \sum_{B' \leqslant B} \beta_{A\overline{B'}}.$$

Proof. If $B = \varepsilon$ the result is trivial, so we can assume $B = 1C$ for some $C \in \mathbf{2}^*$. Then

$$\sum_{B' \leqslant B} \beta_{AB'} = \sum_{C' \leqslant C} (\beta_{A0C'} + \beta_{A1C'})$$

while

$$\sum_{B' \leqslant B} \beta_{A\overline{B'}} = \sum_{C' \leqslant C} (\beta_{A\overline{0C'}} + \beta_{A\overline{1C'}}) = \sum_{C' \leqslant C} (\beta_{A1\overline{C'}} + \beta_{A0\overline{C'}}),$$

and the result follows from (10). □

Using the relations $\beta_E = \beta_{\bar{E}}$, one can obtain in an analogous way the following symmetric version of Lemma 3.4: for all $\beta \in \mathscr{B}'_n$, $A \in \overline{\mathbf{2}^*1}$ and $B \in \mathbf{2}^*$ such that $AB \in \mathbf{2}^n$ we have

$$\sum_{A' \leqslant A} \beta_{A'B} = \sum_{A' \leqslant A} \beta_{\overline{A'}B}.$$

The next result completes the proof that $\mathscr{B}'_n \subseteq \mathscr{B}_n$.

Proposition 3.5. *Let $\beta \in \mathscr{B}'_n$ and $\alpha \in V_n$ be given by*

$$\alpha_E = \sum_{E' \leqslant E} \beta_{E'}$$

for all $E \in \mathbf{2}^n$. Then $\alpha \in \mathscr{A}_n$.

Proof. Let $j \geqslant 1$, $A \in \overline{\mathbf{2}^*1}$, $B \in \overline{\mathbf{12}^*}$ be such that $A0^jB \in \mathbf{2}^n$. We have to show that

$$2\chi_{\text{odd}}(j)\alpha_{A0^jB} = \sum_{i=1}^{j}(-1)^{i-1}\alpha_{AE_{i,j}B}. \tag{11}$$

By the definition of α, Eq. (11) is equivalent to

$$2\chi_{\text{odd}}(j)\sum_{A'\leqslant A,\,B'\leqslant B}\beta_{A'0^jB'}=\sum_{A'\leqslant A,\,B'\leqslant B}\sum_{i=1}^{j}(-1)^{i-1}(\beta_{A'E_{i,j}B'}+\beta_{A'0^jB'}).$$

Now we observe that clearly $\sum_{i=1}^{j}(-1)^{i-1}\beta_{A'0^jB'}=\chi_{\text{odd}}(j)\beta_{A'0^jB'}$ and therefore the preceding equation simplifies to

$$\chi_{\text{odd}}(j)\sum_{A'\leqslant A,\,B'\leqslant B}\beta_{A'0^jB'}=\sum_{A'\leqslant A,\,B'\leqslant B}\sum_{i=1}^{j}(-1)^{i-1}\beta_{A'E_{i,j}B'}.$$

By Lemma 3.3 this reduces to

$$\chi_{\text{odd}}(j)\sum_{A'\leqslant A,\,B'\leqslant B}\beta_{A'0^jB'}=$$
$$\sum_{A'\leqslant A,\,B'\leqslant B}\Big(\beta_{A'1^j\overline{B'}}+(-1)^{j-1}\beta_{A'E_{j,j}B'}+(-1)^{j}\beta_{A'E_{j,j}\overline{B'}}-\chi_{\text{even}}(j)\beta_{A'0^jB'}\Big)$$

and using the relations $\beta_{A'1^j\overline{B'}}=\beta_{\overline{A'}0^jB'}$ and $\beta_{A'E_{j,j}B'}+\beta_{A'0^jB'}=\beta_{A'E_{j,j}\overline{B'}}+\beta_{A'0^j\overline{B'}}$ together with the simple observation $(-1)^j-\chi_{\text{even}}(j)=-\chi_{\text{odd}}(j)$, to conclude the proof we only have to verify that

$$2\chi_{\text{odd}}(j)\sum_{A'\leqslant A,\,B'\leqslant B}\beta_{A'0^jB'}=\sum_{A'\leqslant A,\,B'\leqslant B}(\beta_{\overline{A'}0^jB'}+(-1)^{j-1}\beta_{A'0^j\overline{B'}});$$

but this is an immediate consequence of Lemma 3.4 and of its symmetric version. □

4. A basis for the peak algebra

In this section we define a family of quasisymmetric functions and show, using the results in the previous one, that the ones that are nonzero are a basis for the peak subalgebra of the algebra of quasisymmetric functions. We also show how to expand any peak quasisymmetric function as a linear combination of elements of this basis. As a consequence of this fact, this basis can also be characterized as the image of the fundamental quasisymmetric functions indexed by sparse sequences under the unique projection of $\mathcal{Q}$ on Π whose kernel is spanned by the fundamental quasisymmetric functions indexed by non sparse sequences. These results are used in the next section in the proof of our main result.

For $E\in\mathbf{2}^{n-1}$ let

$$\partial(E)\stackrel{\text{def}}{=}\{i\in[n-2]:E_i\neq E_{i+1}\}\cup\{n-1\}.$$

Note that $\partial(E)=\{x_1,\dots,x_r\}_<$ if and only if the exponent composition of E is $(x_1,x_2-x_1,x_3-x_2,\dots,x_r-x_{r-1})$. Let $T\in\mathbf{2}^{n-1}$, $S(T)=\{s_1,\dots,s_t\}_<$, $s_0\stackrel{\text{def}}{=}0$, $s_{t+1}\stackrel{\text{def}}{=}n$, and

$$I_j\stackrel{\text{def}}{=}(s_j,s_{j+1})=\{s_j+1,s_j+2,\dots,s_{j+1}-1\},$$

for all $j\in[0,t]$. We let $\mathcal{G}(T)$ be the set of all $E=\mathbf{2}^{n-1}$ such that

i) $\partial(E)\cap I_j\neq\emptyset$ for all $j\in[0,t-1]$;

ii) if $x,y\in\partial(E)\cap I_j$ then $x\equiv y\pmod 2$ for all $j\in[0,t]$;

Given such an E we define

$$sgn(E,T)\stackrel{\text{def}}{=}(-1)^{\sum_{j=1}^{t}(s_j-x_j-1)},$$

where x_j is any element of $\partial(E) \cap I_{j-1}$ for $j \in [t]$, and we let

$$D_T \stackrel{\text{def}}{=} \sum_{E \in \mathcal{G}(T)} sgn(E,T)\, L_E \in \mathcal{Q}.$$

So, for example, if $T = 00100$ then $\mathcal{G}(T) = \{01111, 01100, 00111, 00100, 10000, 10011, 11000, 11011\}$ and $D_T = -L_{01111} - L_{01100} + L_{00100} - L_{10000} - L_{10011} + L_{00111} + L_{11000} + L_{11011}$. Note that D_T is homogeneous of degree $\ell(T)+1$ and that $\mathcal{G}(T) = \emptyset$, and hence $D_T = 0$, if T is not sparse. Given $E, T \in \mathbf{2}^{n-1}$ we let

$$h_{E,T} \stackrel{\text{def}}{=} [L_E](D_T),$$

so, by our definitions,

$$h_{E,T} = \begin{cases} sgn(E,T), & \text{if } E \in \mathcal{G}(T), \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Note that, since $h_{E,T}$ depends only on $\partial(E) \setminus T$, given $S \subseteq [n-1]$ we will sometimes write $h_{S,T}$ rather than $h_{E,T}$ if $\partial(E) = S$.

The next property is crucial in the proof of the main result of this section.

Proposition 4.1. *Let $E, T \in \mathbf{2}^{n-1}$, and $i \in [2, n-2]$ be such that $i-1, i \notin \partial(E)$. Then*

$$h_{\partial(E),T} + h_{\partial(E)\cup\{i-1,i\},T} = h_{\partial(E)\cup\{i\},T} + h_{\partial(E)\cup\{i-1\},T}. \tag{13}$$

Proof. We may clearly assume that T is sparse. If $i \in T$ then $\partial(E) \setminus T = (\partial(E) \cup \{i\}) \setminus T$ and $(\partial(E) \cup \{i-1\}) \setminus T = (\partial(E) \cup \{i-1, i\}) \setminus T$ so (13) clearly holds. Similarly if $i-1 \in T$. We may therefore assume that $i, i-1 \notin T$.

Suppose first that $E \in \mathcal{G}(T)$. Then $\partial(E) \cup \{i-1, i\} \notin \mathcal{G}(T)$ while exactly one of $\partial(E) \cup \{i-1\}$, $\partial(E) \cup \{i\}$ is in $\mathcal{G}(T)$, and it is easy to see that it has the same sign as E, so (13) holds.

Suppose now that $E \notin \mathcal{G}(T)$. Then either there is $j \in [0, t-1]$ such that $\partial(E) \cap I_j = \emptyset$ or there exists $j \in [0, t]$ such that $\partial(E) \cap I_j = \{x_1, \ldots, x_p\}_<$ and there exists $r \in [2, p]$ such that $x_r - x_{r-1} \equiv 1 \pmod 2$. If either $i < s_j$ or $s_{j+1} < i-1$ then $\partial(E) \cup \{i-1\}, \partial(E) \cup \{i\}, \partial(E) \cup \{i-1, i\} \notin \mathcal{G}(T)$ so (13) holds. So assume $s_j < i-1 < i < s_{j+1}$.

Suppose first that $\partial(E) \cap I_j = \emptyset$ for some $j \in [0, t-1]$. Then $\partial(E) \cup \{i-1, i\} \notin \mathcal{G}(T)$ while either both or none of $\partial(E) \cup \{i-1\}$, $\partial(E) \cup \{i\}$ are in $\mathcal{G}(T)$ and in the first case $sgn(\partial(E) \cup \{i-1\}, T) = -sgn(\partial(E) \cup \{i\}, T)$ so (13) holds.

Suppose now that $\partial(E) \cap I_j = \{x_1, \ldots, x_p\}_<$ for some $j \in [0, t]$ and there exists $r \in [2, p]$ such that $x_r - x_{r-1} \equiv 1 \pmod 2$. Then $\partial(E) \cup \{i-1, i\}, \partial(E) \cup \{i-1\}, \partial(E) \cup \{i\} \notin \mathcal{G}(T)$ and (13) holds. □

We can now prove the first main result of this section, namely that the quasisymmetric functions D_T are in the peak subalgebra of the algebra of quasisymmetric functions.

Theorem 4.2. *Let $T \in \mathbf{2}^{n-1}$. Then $D_T \in \Pi_n$.*

Proof. Note first that, since $h_{E,T}$ depends only on $\partial(E) \setminus T$, $h_{E,T} = h_{\bar{E},T}$ for all $E \in \mathbf{2}^{n-1}$. Now let $i \in [2, n-2]$, $A \in \mathbf{2}^{i-1}$, and $B \in \mathbf{2}^{n-1-i}$. We claim that then

$$h_{A0B,T} + h_{A1B,T} = h_{A0\bar{B},T} + h_{A1\bar{B},T}. \tag{14}$$

In fact, we may clearly assume that $B_1 = 0$. Let $\{x_1, \ldots, x_r\}_< \stackrel{\text{def}}{=} \partial(A)$ and $\{y_1, \ldots, y_k\}_< \stackrel{\text{def}}{=} \partial(B)$ (so $x_r = i-1$ and $y_k = n-1-i$). If $A_{i-1} = 0$, then $\partial(A0B) = \{x_1, \ldots, x_{r-1}, y_1 + i, \ldots, y_k + i\}$,

$\partial(A1B) = \{x_1,\ldots,x_r,i,y_1+i,\ldots,y_k+i\}$, $\partial(A0\bar{B}) = \{x_1,\ldots,x_{r-1},i,y_1+i,\ldots,y_k+i\}$, and $\partial(A1\bar{B}) = \{x_1,\ldots,x_r,y_1+i,\ldots y_k+i\}$ so (14) follows from Proposition 4.1. Similarly, if $A_{i-1}=1$ then we have that $\partial(A1\bar{B}) = \{x_1,\ldots,x_{r-1},y_1+i,\ldots,y_k+i\}$, $\partial(A0\bar{B}) = \partial(A1\bar{B})\cup\{x_r,i\}$, $\partial(A0B) = \partial(A1\bar{B})\cup\{x_r\}$, and $\partial(A1B) = \partial(A1\bar{B})\cup\{i\}$ and (14) again follows from Proposition 4.1. This shows that the function β given by $\beta_E = h_{E,T}$ belongs to $\mathscr{B}_{n-1}$ by Proposition 3.5. This implies the result, by Theorem 2.1 and the proven part of Theorem 3.2. □

Let $E \in \mathbf{2}^{n-1}$, $\{y_1,\ldots y_k\}_< \stackrel{\text{def}}{=} \partial(E)$. Note that E is sparse if and only if $E_1 = 0$ and

$$y_{2i} = y_{2i-1}+1 \tag{15}$$

for all $1 \leqslant i \leqslant \left\lfloor \frac{k}{2} \right\rfloor$.

The following is the main result of this section.

Theorem 4.3. *The linear map $\pi : \mathcal{Q} \to \Pi$ given by $\pi(1) = 1$ and*

$$\pi(L_T) = \begin{cases} D_T, & \text{if } T \in \mathbf{2}^*_s, \\ 0, & \text{otherwise} \end{cases}$$

is a projection on Π. In other words the set $\{D_T : T \in \mathbf{2}^*_s\}\cup\{1\}$ is a basis of Π and if $F = \sum_{E\in\mathbf{2}^*} h_E L_E \in \Pi$, then $F = \sum_{T\in\mathbf{2}^*_s} h_T D_T$.

Proof. Let $T \in \mathbf{2}^{n-1}_s$, $T = \{s_1,\ldots,s_t\}_<$, $s_0 \stackrel{\text{def}}{=} 0$, $s_{t+1} \stackrel{\text{def}}{=} n$. We claim that

$$h_{E,T} = \delta_{E,T}. \tag{16}$$

for all $E \in \mathbf{2}^{n-1}_s$. Note first that $\partial(T)\cap I_j = \{s_{j+1}-1\}$ for all $j \in [0,t-1]$, while $|\partial(T)\cap I_t| \leqslant 1$. Hence $T \in \mathcal{G}(T)$ and $sgn(T,T) = 1$ so $h_{T,T} = 1$.

Suppose now that $E \in \mathcal{G}(T)$, E sparse, $\{y_1,\ldots,y_k\}_< \stackrel{\text{def}}{=} \partial(E)$. We claim that

$$\partial(E)\cap I_{j-1} = \{s_j - 1\},$$

and that

$$s_j = y_{2j}$$

for all $j \in [t]$.

In fact, since, by (15), $y_2 - y_1 = 1$, this is clear if $j = 1$. Suppose that it is true for some $j \in [t-1]$. Then $s_j = y_{2j}$. Furthermore, $|\partial(E)\cap I_j| = 1$ (for if $|\partial(E)\cap I_j| \geqslant 2$ then $y_{2j+1}, y_{2j+2} \in \partial(E)\cap I_j$ which, by (15), contradicts the fact that $E \in \mathcal{G}(T)$). Hence $s_j < y_{2j+1} < s_{j+1} \leqslant y_{2j+2}$ which, by (15), implies that $y_{2j+1} = s_{j+1}-1 = y_{2j+2}-1$. If $\partial(E)\cap I_t = \emptyset$ then $s_t = n-1$, so $k = 2t$ and $\partial(E) = \{s_1-1, s_1,\ldots,s_t-1,s_t\}$ so $E = T$. If $\partial(E)\cap I_t \neq \emptyset$ then, since $s_t = y_{2t}$, $y_{2t+1} \in \partial(T)\cap I_t$. But, by (15), this implies that $y_{2t+1} = n-1$. Hence $k = 2t+1$ and $\partial(E) = \{s_1-1,s_1,\ldots,s_t-1,s_t,n-1\}$ which again implies that $E = T$. This shows that $\mathcal{G}(T)\cap\mathbf{2}^*_s = \{T\}$ and hence proves (16). Therefore $\{D_T : T \in \mathbf{2}^{n-1}_s\}$ is a linearly independent set and this, since $\dim(\Pi_n) = |\mathbf{2}^{n-1}_s|$, proves the result. □

The flag f-vector $f(P)$ of a polytope P of rank $n+1$ is uniquely determined by the values it takes on sparse sequences: this was already observed by Bayer and Billera in [3] as a consequence of Theorem 2.2. Therefore one can express any entry of the flag f-vector as a linear combination of the entries indexed by sparse sequences. In other words for every $n \in \mathbb{N}$, $E \in \mathbf{2}^n$ and $T \in \mathbf{2}^n_s$

there exists $a_{E,T} \in \mathbb{Q}$ such that

$$f(P)_E = \sum_{T \in \mathbf{2}_s^n} a_{E,T} f(P)_T,$$

for every polytope P of rank $n+1$. This fact can also be reformulated in terms of peak functions and quasisymmetric functions in the following way. For $T \in \mathbf{2}_s^n$ let

$$D'_T = \sum_{E \in \mathbf{2}^n} a_{E,T} M_E.$$

Then the set $\{D'_T : T \in \mathbf{2}_s^*\} \cup \{1\}$ is a basis of Π. Moreover, the (unique) projection $\tau : \mathcal{Q} \to \Pi$ such that the kernel of τ is spanned by the monomial quasi symmetric functions indexed by non sparse sequences is given by $\tau(1) = 1$ and

$$\tau(M_T) = \begin{cases} D'_T, & \text{if } T \in \mathbf{2}_s^*, \\ 0, & \text{otherwise.} \end{cases}$$

The projection τ is called the Eulerian projection and studied in [9] (where it is denoted by π) but the problem of describing it explicitly is still open. Theorem 4.3 therefore can also be seen as a solution to the analogous problem involving the fundamental basis instead of the monomial basis.

Theorem 4.3 also allows us to complete the proof of Theorem 3.1 (and of the equivalent Theorem 3.2).

Proof of Theorem 3.1. Recall the definitions of $\mathscr{B}_n$ and $\mathscr{B}'_n$ from §3 and that the statement that we have to prove is equivalent to $\mathscr{B}'_n = \mathscr{B}_n$. Also recall that we have already proved that $\mathscr{B}'_n \subseteq \mathscr{B}_n$ in Proposition 3.5.

Consider the linear map $\phi : \mathscr{V}_n \to \mathcal{Q}_{n+1}$ given by

$$\phi(\alpha) = \sum_{E \in \mathbf{2}^n} \alpha_E L_E,$$

for all $\alpha \in \mathscr{V}_n$. By Theorems 2.1 and 2.2 we have $\phi(\mathscr{B}_n) = \Pi_{n+1}$. Furthermore, the proof of Theorem 4.2 shows that $D_T \in \phi(\mathscr{B}'_n)$ for all $T \in \mathbf{2}_s^n$, and since these functions span Π_{n+1} by Theorem 4.3 we conclude that

$$\phi(\mathscr{B}'_n) = \Pi_{n+1}.$$

As $\dim \Pi_{n+1} = \dim \mathscr{B}_n$ we conclude that necessarily $\mathscr{B}'_n = \mathscr{B}_n$. □

As a corollary of Theorem 3.1 we also have the following characterization of $\mathbb{Q}\langle a+b, ab+ba\rangle$ as a subspace of $\mathbb{Q}\langle a, b\rangle$.

Corollary 4.4. *Let $\beta \in \mathscr{V}_n$. Then the polynomial*

$$P(a,b) = \sum_{E \in \mathbf{2}^n} \beta_E \mu_E \in \mathbb{Z}\langle a, b\rangle,$$

can be expressed as a polynomial in $a+b, ab+ba$ if and only if β satisfies Eq. (10).

5. Kazhdan-Lusztig polynomials

In this section, using the results in the two previous ones, we prove a nonrecursive combinatorial formula for the Kazhdan-Lusztig polynomials which holds in complete generality, and which is simpler and more explicit than any existing one.

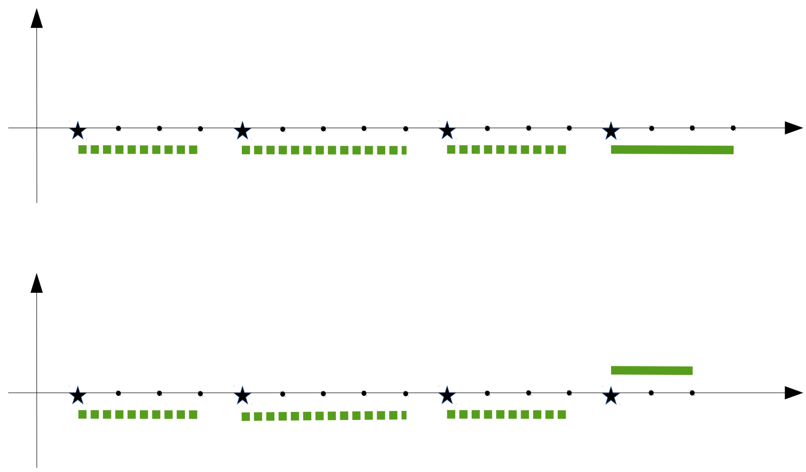

FIGURE 1. The constraints of a slalom path in $\mathcal{SL}(T)$, where $T = 0001000010001000$ (top) and $T = 000100001000100$ (bottom).

Let $n \in \mathbb{P}$, $T \in \mathbf{2}_s^{n-1}$, $S(T) \stackrel{\text{def}}{=} \{s_1, \dots, s_t\}_<$, $s_0 \stackrel{\text{def}}{=} 0$, $s_{t+1} \stackrel{\text{def}}{=} n$. We say that a lattice path Γ is a T-*slalom* (the reader may want to consult Figure 1 (top) for an illustration where n is odd, and Figure 1 (bottom) for an illustration where n is even) if and only if

- $\ell(\Gamma) = n$;
- $\Gamma(s_i + 1) \neq 0$ for all $i \in [t]$ (i.e. Γ does not passes through the "stars" in the examples in Figure 1);
- Γ crosses the segment $\{y = -\frac{1}{2}, x \in [s_{i-1} + 1, s_i]\}$ (the dotted segments in Figure 1) exactly once for all $i \in [t]$;
- $\Gamma(x) \geqslant \chi_{\text{even}}(n)$ for all $x > s_t + 1$ (i.e. the path Γ remains above the solid segment in Figure 1).

We denote by $\mathcal{SL}(T)$ the set of T-slaloms. For $T \in \mathbf{2}_s^{n-1}$ we let

$$\Omega_T(q) \stackrel{\text{def}}{=} (-1)^{s_1 + \cdots + s_t + t} \sum_{\Gamma \in \mathcal{SL}(T)} (-q)^{d_-(\Gamma)},$$

where $d_-(\Gamma) = n - d_+(\Gamma)$ is the number of down-steps of Γ. For example, if $T = 00100$ there are exactly three paths in $\mathcal{SL}(T)$ (see Figure 2) and $\Omega_{00100}(q) = -q + 2q^2$.

We can now state the main result of this section.

THEOREM 5.1. *Let (W, S) be a Coxeter system, $u, v \in W$, $u \leqslant v$ and $\ell = \ell(v) - \ell(u)$. Then*

$$P_{u,v}(q) = \sum_{T \in \mathbf{2}_s^*} b(u,v)_T \, q^{\frac{\ell - \ell(T) - 1}{2}} \, \Omega_T(q).$$

The rest of this section is devoted to the proof of Theorem 5.1.

Let $T \in \mathbf{2}^{n-1}$, $S(T) \stackrel{\text{def}}{=} \{s_1, \dots, s_t\}_<$, $s_0 \stackrel{\text{def}}{=} 0$, $s_{t+1} \stackrel{\text{def}}{=} n$, and $I_j = (s_j, s_{j+1})$ for all $j \in [0, t]$. We define $\mathcal{J}(T)$ to be the set of all $E \in \mathbf{2}^{n-1}$ such that:

i) $|\partial(E)\cap I_j|=1$ for all $j\in[0,t-1]$;
ii) $|\partial(E)\cap I_t|\leqslant 2$;
iii) if $\partial(E)\cap I_t=\{x,n-1\}$ then $x\equiv n-1 \pmod 2$.

Given such an E we define $sgn(E,T)\stackrel{\text{def}}{=}(-1)^{\sum_{j=1}^{t}(s_j-x_j-1)}$ where $\{x_j\}\stackrel{\text{def}}{=}\partial(E)\cap I_{j-1}$ for $j\in[t]$, and let

$$\tilde{\Omega}_T(q)\stackrel{\text{def}}{=}\sum_{E\in\mathcal{J}(T)} sgn(E,T)\,\Upsilon_E(q).$$

We also set $\mathcal{J}(\varepsilon)\stackrel{\text{def}}{=}\{\varepsilon\}$ and $\tilde{\Omega}_\varepsilon(q)\stackrel{\text{def}}{=}\Upsilon_\varepsilon(q)$.

For the reader convenience we recall that

$$\Upsilon_E(q)=(-1)^{m_0(E)}\sum_{\Gamma\in\mathcal{L}(E)}(-q)^{d_+(\Gamma)},$$

where $\mathcal{L}(E)\stackrel{\text{def}}{=}\{\Gamma\in\mathcal{L}(n):N(\Gamma)=E\}$, and we make the following observation: if Γ is a lattice path such that $N(\Gamma)=E$ then ∂E is given by the points where Γ crosses the line $y=-\frac{1}{2}$.

Example 5.2. If $T=00100$, then $\mathcal{J}(T)=\{01111,01100,00111,00100,10000,10011,11000,11011\}$ and $\tilde{\Omega}_T(q)=-\Upsilon_{01111}-\Upsilon_{01100}+\Upsilon_{00111}+\Upsilon_{00100}-\Upsilon_{10000}-\Upsilon_{10011}+\Upsilon_{11000}+\Upsilon_{11011}=\Upsilon_{00111}+\Upsilon_{00100}-\Upsilon_{10000}=q^5-2q^4+2q^2-q$.

Note that $\mathcal{J}(T)=\emptyset$ if T is not sparse, and that $\mathcal{J}(T)\subseteq\mathcal{G}(T)$.

Proposition 5.3. *Let (W,S) be a Coxeter system and $u,v\in W$, $u<v$. Then*

$$P_{u,v}(q)-q^{\ell(u,v)}P_{u,v}\left(\frac{1}{q}\right)=\sum_{T\in\mathbf{2}^*}q^{\frac{\ell(u,v)-\ell(T)-1}{2}}\,b(u,v)_T\,\tilde{\Omega}_T(q).$$

Proof. Note first that, since $u<v$, $\widetilde{F}(u,v)$ has no constant term. Hence from Theorems 2.7 and 4.3 we have that

$$\widetilde{F}(u,v)=\sum_{T\in\mathbf{2}^*_s}b(u,v)_T D_T.$$

Applying the linear map $\mathcal{K}$ to this equality we get, by Theorem 2.8, that

$$q^{-\frac{\ell(u,v)}{2}}P_{u,v}(q)-q^{\frac{\ell(u,v)}{2}}P_{u,v}\left(\frac{1}{q}\right)=\sum_{T\in\mathbf{2}^*_s}b(u,v)_T\mathcal{K}(D_T).$$

But, by our definitions, we have that

$$\mathcal{K}(D_T)=\sum_{E\in\mathcal{G}(T)}sgn(E,T)\,\mathcal{K}(L_E)=\sum_{E\in\mathcal{G}(T)}sgn(E,T)\,q^{-\frac{\ell(E)+1}{2}}\Upsilon_E(q). \tag{17}$$

Recall that $\mathcal{J}(T)\subseteq\mathcal{G}(T)$. Let $E\in\mathcal{G}(T)\setminus\mathcal{J}(T)$, $\{y_1,\ldots,y_k\}_<\stackrel{\text{def}}{=}\partial(E)$. Then either $|\partial(E)\cap I_t|\geqslant 3$ or $|\partial(E)\cap I_j|\geqslant 2$ for some $j\in[0,t-1]$. But if either of these conditions hold then $k\geqslant 3$ and there exists $j\in[k-2]$ such that $y_{j+1}\equiv y_j \pmod 2$ and this, by Corollary 2.9, implies that $\Upsilon_E=0$. Hence we conclude from (17) that

$$\mathcal{K}(D_T)=\sum_{E\in\mathcal{J}(T)}sgn(E,T)\,q^{-\frac{\ell(E)+1}{2}}\,\Upsilon_E(q)=q^{-\frac{\ell(T)+1}{2}}\tilde{\Omega}_T(q),$$

and the result follows. □

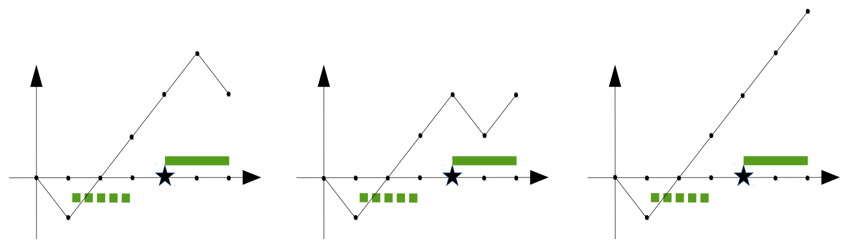

FIGURE 2. Slalom paths associated to $T = 00100$

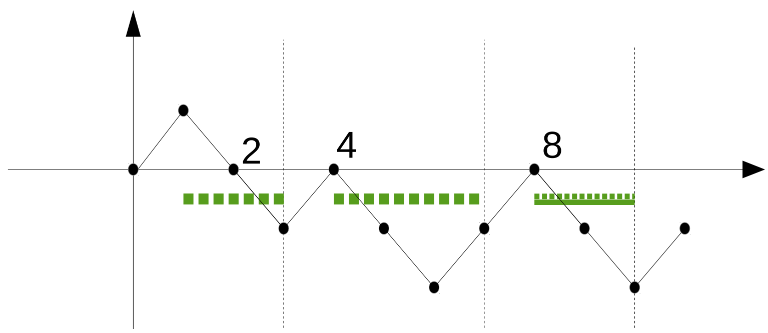

FIGURE 3. A path in $\mathcal{L}(0010001000)$.

Let $T \in \mathbf{2}_s^{n-1}$ be a sparse sequence of length $n-1$, $s_0 = 0$ and $S(T) = \{s_1, \ldots, s_t\}_<$, and $I_j = (s_j, s_{j+1})$ for all $j \in [0,t]$. We let $\mathcal{L}(T)$ be the set of all lattice paths Γ of length n such that $N(\Gamma) \in \mathcal{J}(T)$. For $\Gamma \in \mathcal{L}(T)$ and $j \in [t]$ we let $x_j(\Gamma)$ be the unique element in $\partial(N(\Gamma)) \cap I_{j-1}$ and

$$\varepsilon_T(\Gamma) \stackrel{\text{def}}{=} \sum_{j=1}^{t} (s_j - x_j(\Gamma) - 1).$$

We also let

$$\eta(\Gamma) \stackrel{\text{def}}{=} m_0(N(\Gamma)) = |\{a\ in[n-1] : \Gamma(a) \geqslant 0\}|.$$

We will usually write $\varepsilon(\Gamma)$ instead of $\varepsilon_T(\Gamma)$ when the sparse sequence T is clear from the context.

EXAMPLE 5.4. Let $T = 0010001000$, so $t = 2$, $n-1 = 10$, $S(T) = \{s_1, s_2\}$ with $s_1 = 3$ and $s_2 = 7$. We also have $I_0 = \{1,2\}$, $I_1 = \{4,5,6\}$ and $I_2 = \{8,9,10\}$. The definition of $\mathcal{L}(T)$ implies that a lattice path belongs to $\mathcal{L}(T)$ if and only if it has length $n = 11$, it crosses the two dotted segments in Figure 3 exactly once, and crosses the solid-dotted segment at most once, but only form NW to SE. The path Γ depicted in Figure 3 therefore belongs to $\mathcal{L}(T)$. In this case $N(\Gamma) = 0010111011$ and one can easily check that $x_1(\Gamma) = 2$ and $x_2(\Gamma) = 4$. Finally, we can observe that $\partial(N(\Gamma)) \cap I_t = \{8, 10\}$. Hence we have $\varepsilon(\Gamma) = (s_1 - x_1(\Gamma) - 1) + (s_2 - x_2(\Gamma) - 1) = 0 + 2 = 2$. Moreover, we have $\eta(\Gamma) = 4$ and $d_+(\Gamma) = 5$.

The following result is a direct consequence of the definitions of the polynomials $\tilde{\Omega}_T$ and Υ_E

and so we omit its proof.

Proposition 5.5. *Let $n \in \mathbb{P}$ and $T \in \mathbf{2}_s^{n-1}$. Then*

$$\tilde{\Omega}_T(q) = \sum_{\Gamma \in \mathcal{L}(T)} (-1)^{\varepsilon(\Gamma)+\eta(\Gamma)+d_+(\Gamma)} q^{d_+(\Gamma)}.$$

Our next target is to simplify the sum in Proposition 5.5. For this we introduce the following notation: if $T \in \mathbf{2}_s^{n-1}$, $s_0 = 0$ and $S(T) = \{s_1, \ldots, s_t\}_<$, we let $r_j \stackrel{\text{def}}{=} s_j + 1$ for $j \in [0,t]$ and let

$$\mathcal{L}_0(T) \stackrel{\text{def}}{=} \{\Gamma \in \mathcal{L}(T) : \Gamma(r_j) = 0 \text{ for some } j \in [t]\}.$$

For example the path Γ depicted in Figure 3 belongs to $\mathcal{L}_0(0010001000)$ as $\Gamma(4) = 0$.

Proposition 5.6. *Let $n \in \mathbb{P}$ and $T \in \mathbf{2}_s^{n-1}$. Then*

$$\sum_{\Gamma \in \mathcal{L}_0(T)} (-1)^{\varepsilon(\Gamma)+\eta(\Gamma)+d_+(\Gamma)} q^{d_+(\Gamma)} = 0.$$

Proof. For $j \in [t]$ let $\mathcal{L}_0^{(j)}(T) = \{\Gamma \in \mathcal{L}_0(T) : \min\{i \in [t] : \Gamma(r_i) = 0\} = j\}$. The result follows if we can find an involution

$$\phi : \mathcal{L}_0^{(j)}(T) \to \mathcal{L}_0^{(j)}(T)$$

such that

- $d_+(\Gamma) = d_+(\phi(\Gamma))$,
- $\varepsilon(\Gamma) + \eta(\Gamma) \equiv \varepsilon(\phi(\Gamma)) + \eta(\phi(\Gamma)) + 1 \pmod 2$,

for all $\Gamma \in \mathcal{L}_0^{(j)}(T)$. The bijection ϕ is defined as follows. Fix an arbitrary path $\Gamma \in \mathcal{L}_0^{(j)}(T)$. Let i be the maximum index smaller than j such that $\Gamma(s_i) = 0$. In the interval $[r_h, s_{h+1}]$, where $h \in [i, j-1]$, the path $\phi(\Gamma)$ is defined as follows (see Figure 4 for an illustration)

$$\phi(\Gamma)(x) = \begin{cases} \Gamma(x), & \begin{array}{l}\text{if there exist } a, b \in \mathbb{N} \text{ such that} \\ r_h < a < x < b < s_{h+1} \text{ and } \Gamma(a) = \Gamma(b) = 0,\end{array} \\ -\Gamma(x), & \text{otherwise,} \end{cases}$$

for all $x \in [r_h, s_{h+1}]$. Finally we let $\phi(\Gamma)(x) = \Gamma(x)$ if $x \notin [r_i, s_j]$. Note that, since $\Gamma(s_i) = \Gamma(s_j + 1) = 0$, we have that $|\Gamma(r_i)| = |\Gamma(s_j)| = 1$ so $\phi(\Gamma)$ is still a lattice path.

Since $\Gamma(r_h) \neq 0$ and $\Gamma(s_{h+1}) \neq 0$ for all $h \in [i, j-1]$ by construction, one can easily see that ϕ is an involution on $\mathcal{L}_0^{(j)}(T)$. It is also clear that $d_+(\Gamma) = d_+(\phi(\Gamma))$ as $\Gamma(n) = \phi(\Gamma)(n)$.

Now we observe that a lattice path crosses the line $\{y = -\frac{1}{2}\}$ from NW to SE always at an even index, and from SW to NE always at an odd index. It then follows from our construction that $x_h(\Gamma) \equiv x_h(\phi(\Gamma)) + 1 \pmod 2$ for all $h \in [i+1, j]$ (see also Figure 4), and that clearly $x_h(\Gamma) = x_h(\phi(\Gamma))$ if $h \notin [i+1, j]$. Therefore

$$\varepsilon(\Gamma) + \varepsilon(\phi(\Gamma)) \equiv j - i \pmod 2. \tag{18}$$

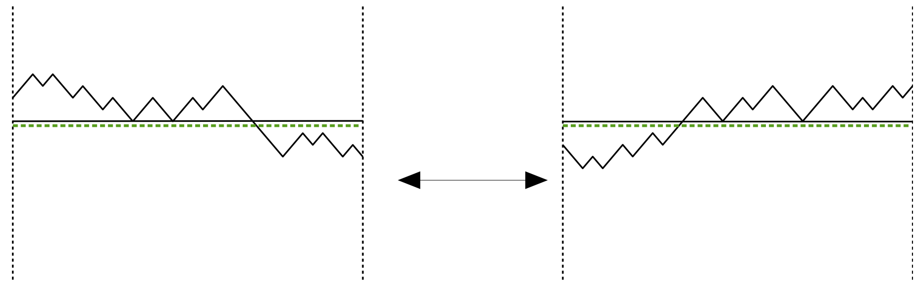

Figure 4. The bijection ϕ in $[r_h, s_{h+1}]$.

Finally, we have that (see also Figure 4)

$$\begin{aligned}\eta(\Gamma)+\eta(\phi(\Gamma)) &\equiv \sum_{h=i}^{j-1}(x_{h+1}(\phi(\Gamma))-s_h)+\sum_{h=i}^{j-1}(s_{h+1}-x_{h+1}(\Gamma))\\ &\equiv \sum_{h=i}^{j-1}(x_{h+1}(\phi(\Gamma))-x_{h+1}(\Gamma)+s_{h+1}-s_h)\\ &\equiv j-i+s_j-s_i\\ &\equiv j-i+1 \pmod 2 \qquad (19)\end{aligned}$$

since $\Gamma(s_i)=\Gamma(s_j+1)=0$.

The result then follows from (18) and (19). □

If n is even other cancellations may occur in Proposition 5.5 and to describe this, for $T \in \mathbf{2}_s^{n-1}$ we also let

$$\mathcal{L}_0'(T)=\{\Gamma \in \mathcal{L}(T) : \Gamma(x)=0 \text{ for some } x \in [r_t, n]\}.$$

Proposition 5.7. *Let $n \in \mathbb{P}$ be even and $T \in \mathbf{2}_s^{n-1}$. Then*

$$\sum_{\Gamma \in \mathcal{L}_0'(T)\setminus \mathcal{L}_0(T)} (-1)^{\varepsilon(\Gamma)+\eta(\Gamma)+d_+(\Gamma)} q^{d_+(\Gamma)} = 0.$$

Proof. This proof is similar to that of Proposition 5.6. We show that there exists an involution ψ on $\mathcal{L}_0'(T)\setminus\mathcal{L}_0(T)$ such that $d_+(\Gamma)=d_+(\psi(\Gamma))$ and $\varepsilon(\Gamma)+\eta(\Gamma)\equiv\varepsilon(\psi(\Gamma))+\eta(\psi(\Gamma))+1 \pmod 2$, for all $\Gamma \in \mathcal{L}_0'(T)\setminus\mathcal{L}_0(T)$.

If $\Gamma \in \mathcal{L}_0'(T)\setminus\mathcal{L}_0(T)$ let $a_0=\min\{a \in [r_t,n] : \Gamma(a)=0\}$ and $i=\max\{j\in[0,t] : \Gamma(s_i)=0\}$. The path $\psi(\Gamma)$ is defined in the following way: if $r_i \leqslant x \leqslant s_t$ we let

$$\psi(\Gamma)(x)=\begin{cases}\Gamma(x), & \text{if there exist } h\in[t] \text{ and } a,b\in\mathbb{N} \text{ such that}\\ & r_{h-1}<a<x<b<s_h \text{ and } \Gamma(a)=\Gamma(b)=0,\\ -\Gamma(x), & \text{otherwise.}\end{cases}$$

Observe that in this situation $\Gamma(r_{h-1})\neq 0$ since $\Gamma\notin\mathcal{L}_0(T)$ and $\Gamma(s_h)\neq 0$ by the maximality of the index i. Finally, we let $\psi(\Gamma)(x)=-\Gamma(x)$ if $x\in[r_t,a_0]$ and $\psi(\Gamma)(x)=\Gamma(x)$ if $x>a_0$ or $x\leqslant s_i$.

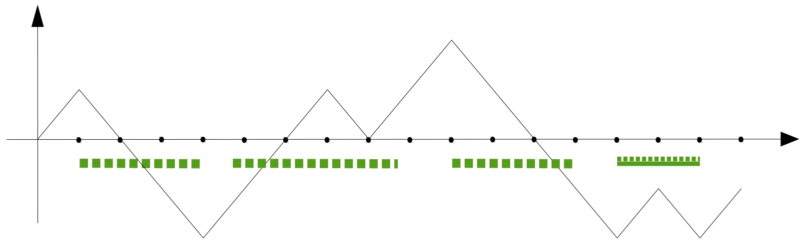

Figure 5. A path in $\tilde{\mathcal{L}}(0001000010001000)$.

Note that, since $n-1$ is odd we have $\Gamma(x) \geqslant 0$ for all $x \geqslant a_0$. By reasoning as in the proof of Proposition 5.6 one obtains that $\varepsilon(\Gamma)+\varepsilon(\psi(\Gamma)) \equiv t-i \pmod 2$ and $\eta(\Gamma)+\eta(\psi(\Gamma)) \equiv t-i-1 \pmod 2$ (since $\Gamma(s_i)=\Gamma(a_0)=0$), thereby completing the proof. $\square$

Propositions 5.6 and 5.7 lead us to consider the set of paths $\tilde{\mathcal{L}}(T)$ given by $\tilde{\mathcal{L}}(T) \stackrel{\text{def}}{=} \mathcal{L}(T) \setminus (\mathcal{L}_0(T) \cup \mathcal{L}'_0(T))$ if n is even and $\tilde{\mathcal{L}}(T) \stackrel{\text{def}}{=} \mathcal{L}(T) \setminus \mathcal{L}_0(T)$ if n is odd associated to $T \in \mathbf{2}_s^{n-1}$. Note that $\Gamma(n) \neq 0$ if $\Gamma \in \tilde{\mathcal{L}}(T)$. Figure 5 shows an example of a path in $\tilde{\mathcal{L}}(T)$.

Now we want to show that there are no further cancellations in the sum appearing in Proposition 5.5. More precisely, if we let $\chi_{\Gamma(n)>0} = 1$ if $\Gamma(n) > 0$ and $\chi_{\Gamma(n)>0} = 0$ otherwise, we have the following result.

Theorem 5.8. *Let $n \in \mathbb{P}$, $T \in \mathbf{2}_s^{n-1}$ and $\Gamma \in \tilde{\mathcal{L}}(T)$. Then*

$$\varepsilon(\Gamma)+\eta(\Gamma) \equiv \chi_{\Gamma(n)>0}\chi_{even}(n) + \sum_{j=1}^{t} r_j \pmod 2.$$

Proof. Suppose $i,k \in [0,t]$, $i<k$ are such that $\Gamma(s_i)=\Gamma(s_k)=0$ and $\Gamma(s_h) \neq 0$ for all $h \in (i,k)$. We consider $\eta_{i,k}(\Gamma) \stackrel{\text{def}}{=} |\{a \in [r_i,s_k] : \Gamma(a) \geqslant 0\}|$ and $\zeta_{i,k}(\Gamma) \stackrel{\text{def}}{=} \sum_{h=i+1}^{k} x_h(\Gamma)$ and we claim that

$$\eta_{i,k}(\Gamma)+\zeta_{i,k}(\Gamma) \equiv 0 \pmod 2. \tag{20}$$

As $\Gamma(s_k)=0$ we have that Γ crosses the segment $\{y=-\frac{1}{2}, x \in [r_{k-1},s_k]\}$ from SW to NE and therefore $x_k(\Gamma) \equiv 1 \pmod 2$. Consequently Γ crosses the segment $\{y=-\frac{1}{2}, x \in [r_{k-2},s_{k-1}]\}$ from NW to SE and hence $x_{k-1}(\Gamma) \equiv 0 \pmod 2$. Iterating this observation we have that $x_k(\Gamma) \equiv x_{k-2}(\Gamma) \equiv \cdots \equiv 1 \pmod 2$ and $x_{k-1}(\Gamma) \equiv x_{k-3}(\Gamma) \equiv \cdots \equiv 0 \pmod 2$. Therefore

$$\zeta_{i,k}(\Gamma) = \sum_{h=i+1}^{k} x_h(\Gamma) \equiv \lfloor \frac{k-i+1}{2} \rfloor \pmod 2.$$

If $k-i$ is odd we have that $\Gamma(r_i) < 0$ and the number of maximal intervals contained in $[r_i,s_k]$ where Γ takes nonnegative values is $\frac{k-i+1}{2}$ and all such intervals contain an odd number of elements. If $k-i$ is even then $\Gamma(r_i) > 0$ and there are $\frac{k-i}{2}$ such intervals with an odd number of elements and one interval with an even number of elements (i.e. the one containing r_i). In both cases we deduce that $\eta_{i,k}(\Gamma) \equiv \lfloor \frac{k-i+1}{2} \rfloor$. Therefore

$$\eta_{i,k}(\Gamma)+\zeta_{i,k}(\Gamma) \equiv \lfloor \frac{k-i+1}{2} \rfloor + \lfloor \frac{k-i+1}{2} \rfloor \equiv 0 \pmod 2$$

Now let i be the maximum index such that $\Gamma(s_i)=0$. We let in this case $\eta_{i,t+1}(\Gamma) \stackrel{\text{def}}{=} |\{a \in [r_i, n-1] : \Gamma(a) \geqslant 0\}|$ and $\zeta_{i,t+1}(\Gamma) \stackrel{\text{def}}{=} \sum_{h=i+1}^{t} x_h(\Gamma)$. We leave to the reader to verify that if $\Gamma(n)>0$ and n is even then $\eta_{i,t+1}(\Gamma) \equiv \lfloor \frac{t-i-1}{2} \rfloor$ and $\zeta_{i,t+1}(\Gamma) \equiv \lfloor \frac{t-i+1}{2} \rfloor$ and therefore

$$\eta_{i,t+1}(\Gamma)+\zeta_{i,t+1}(\Gamma) \equiv 1.$$

In all the other cases we have $\eta_{i,t+1}(\Gamma)+\zeta_{i,t+1}(\Gamma) \equiv 0$; in fact, with an argument similar to the one used in the previous case one can show that:

- if n is even and $\Gamma(n)<0$ we have $\eta_{i,t+1}(\Gamma) \equiv \zeta_{i,t+1}(\Gamma) \equiv \lfloor \frac{t-i}{2} \rfloor$;
- if n is odd and $\Gamma(n)>0$ we have $\eta_{i,t+1}(\Gamma) \equiv \zeta_{i,t+1}(\Gamma) \equiv \lfloor \frac{t-i+1}{2} \rfloor$;
- if n is odd, $\Gamma(n)<0$ and $\Gamma(r_t)<0$ we have $\eta_{i,t+1}(\Gamma) \equiv \zeta_{i,t+1}(\Gamma) \equiv \lfloor \frac{t-i}{2} \rfloor$;
- if n is odd, $\Gamma(n)<0$ and $\Gamma(r_t)>0$ we have $\eta_{i,t+1}(\Gamma) \equiv \zeta_{i,t+1}(\Gamma) \equiv \lfloor \frac{t-i-1}{2} \rfloor$.

Now we can conclude the proof. Let $\Gamma \in \tilde{\mathcal{L}}(T)$ and let $\{i_1,\ldots,i_z\}_< = \{h \in [0,t] : \Gamma(s_h) = 0\} \cup \{t+1\}$. We have

$$\begin{aligned}\eta(\Gamma)+\varepsilon(\Gamma) &\equiv \sum_{v=1}^{z-1} \left(\eta_{i_v,i_{v+1}}(\Gamma)+\zeta_{i_v,i_{v+1}}(\Gamma)\right) + r_1+\cdots+r_t \pmod 2\\ &\equiv \chi_{\Gamma(n)>0}\chi_{\text{even}}(n)+r_1+\cdots+r_t \pmod 2.\end{aligned}$$

□

Corollary 5.9. *Let $n \in \mathbb{P}$ and $T \in \mathbf{2}_s^{n-1}$. Then*

$$[q^i]\tilde{\Omega}_T = (-1)^{i+r_1+\cdots+r_t+\chi_{2i>n}\chi_{\text{even}}(n)} |\{\Gamma \in \tilde{\mathcal{L}}(T) : \Gamma(n) = 2i-n\}|,$$

where $\chi_{2i>n}=1$ if $2i>n$ and $\chi_{2i>n}=0$ otherwise.

Proof. This follows immediately from Propositions 5.5, 5.6 and 5.7, and Theorem 5.8, together with (7). □

Corollary 5.10. *Let $n \in \mathbb{P}$, and $T \in \mathbf{2}_s^{n-1}$. Then*

$$q^n \tilde{\Omega}_T\left(\frac{1}{q}\right) = -\tilde{\Omega}_T(q).$$

Proof. A bijection ϕ constructed as in the proof of Proposition 5.6 shows that for all $i \in [0,n]$

$$|\{\Gamma \in \tilde{\mathcal{L}}(T) : \Gamma(n) = n-2i\}| = |\{\Gamma \in \tilde{\mathcal{L}}(T) : \Gamma(n) = 2i-n\}|.$$

and the result follows from Corollary 5.9. □

Let $T \in \mathbf{2}_s^{n-1}$. Note that a lattice path Γ is a T-slalom if and only if $\Gamma \in \tilde{\mathcal{L}}(T)$ and $\Gamma(n)>0$. We can now complete the proof of the main result of this work.

Proof of Theorem 5.1. First a notation. If $P(q)$ is a polynomial and $x \in \mathbb{Q}$ we let

$$D_x(P)(q) = \sum_{i=0}^{\lfloor x \rfloor} [q^i](P(q))\, q^i,$$

the polynomial obtained by deleting the homogeneous components of P of degree greater than x. By Proposition 5.3 we have that

$$P_{u,v}(q) - q^{\ell(u,v)} P_{u,v}(1/q) = \sum_{T \in \mathbf{2}_s^*} q^{\frac{\ell(u,v)-\ell(T)-1}{2}} b(u,v)_T \tilde{\Omega}_T(q).$$

Since $\deg P_{u,v} \leqslant \frac{\ell(u,v)-1}{2}$ we have that

$$P_{u,v}(q) = \sum_{T \in \mathbf{2}_s^*} q^{\frac{\ell(u,v)-\ell(T)-1}{2}} b(u,v)_T D_{\frac{\ell(T)}{2}}(\tilde{\Omega}_T)(q).$$

Now we observe that if $\ell(T) = n-1$ then $i \leqslant \frac{\ell(T)}{2}$ if and only if $2i - n < 0$ and so in this case, by Corollaries 5.9 and 5.10 we have

$$\begin{aligned}
[q^i]\tilde{\Omega}_T(q) &= (-1)^{i+r_1+\cdots+r_t} |\{\Gamma \in \tilde{\mathcal{L}}(T) : \Gamma(n) = 2i - n\}| \\
&= (-1)^{i+r_1+\cdots+r_t} |\{\Gamma \in \tilde{\mathcal{L}}(T) : \Gamma(n) = n - 2i\}| \\
&= (-1)^{i+r_1+\cdots+r_t} |\{\Gamma \in \mathcal{SL}(T) : d_-(\Gamma) = i\}| \\
&= [q^i]\Omega_T(q).
\end{aligned}$$

As $\deg \Omega_T \leqslant \frac{\ell(T)}{2}$ the proof is complete $\square$.

We illustrate the preceding theorem with some examples. If $\ell(u,v) = 1$ then we have from Theorem 5.1 and our definitions that

$$P_{u,v}(q) = q^{-\frac{\ell(\varepsilon)}{2}} b_\varepsilon \, \Omega_\varepsilon(q) = 1.$$

Similarly we obtain

$$P_{u,v}(q) = b_0 \, \Omega_0(q) = 1$$

if $\ell(u,v) = 2$ (where we have used the fact that $b_0(u,v) = 1$ if $\ell(u,v) = 2$) and

$$\begin{aligned}
P_{u,v}(q) &= b_{00} \, \Omega_{00}(q) + b_{01} \, \Omega_{01}(q) + q \, b_\varepsilon \, \Omega_\varepsilon(q) \\
&= b_{00}(1 - 2q) + b_{01}(q) + q \, b_\varepsilon \\
&= 1 + q(-2 + b_{01} + b_\varepsilon),
\end{aligned}$$

if $\ell(u,v) = 3$.

We feel that the formula obtained in Theorem 5.1 is the simplest and most explicit nonrecursive combinatorial formula known for the Kazhdan-Lusztig polynomials that holds in complete generality since this formula, as the one in [8, Corollary 3.2], expresses the Kazhdan-Lusztig polynomial of $u, v \in W$ as a sum of at most $f_{\ell(u,v)}$ summands, as opposed to $2^{\ell(u,v)} + 2^{\ell(u,v)-2} + \cdots$ for the one obtained in [15, Theorem 7.2], each one of which is the product of a number, which depends on u, v, and W, with a polynomial, that is independent of u, v, and W. However, this formula is more explicit than the one obtained in [8, Corollary 3.2] since in the formula obtained in [8] the polynomials have a combinatorial interpretation, but no combinatorial interpretation is known for the numbers, while in the formula obtained in Theorem 5.1 both the numbers and the polynomials have a combinatorial interpretation.

We conclude this section by observing that the explicit formula appearing in Theorem 5.1 cannot be further simplified by means of linear relations in a sense that we are going to make precise. The main point here is the following result of Reading [36].

Theorem 5.11. *For all $n \in \mathbb{N}$ the vector subspace of $\mathbb{Q}\langle a, b\rangle$ spanned by the cd-indices of all Bruhat intervals of rank $n+1$ equals the space of cd-polynomials of degree n and has therefore dimension f_{n+1}.*

Consider the vector space $\mathscr{V}$ consisting of functions $\alpha : \mathbf{2}^* \to \mathbb{Q}$ such that $\alpha_E \neq 0$ for at most a finite number of sequences E. Also let $\mathscr{W}$ be the subspace of $\mathscr{V}$ given by the functions that satisfy the dual Bayer-Billera relations. Note that we have $b(u,v) \in \mathscr{W}$ by Corollary 4.4.

Corollary 5.12. *The vector space $\mathscr{W}$ is spanned by the functions $b(u,v)$ as $[u,v]$ varies among all possible Bruhat intervals.*

Proof. The vector space $\mathscr{W}$ has a natural filtration given by

$$\mathscr{W} = \bigcup_{n=1}^{\infty} \mathscr{W}_n,$$

where

$$\mathscr{W}_n = \{\beta \in \mathscr{W} : \beta_E = 0 \text{ whenever } \ell(E) > n\}.$$

It is therefore enough to show that $\mathscr{W}_n$ is spanned by the functions $b(u,v)$ it contains and for this we proceed by induction on n. If $n = 0$ we let $[u,v]$ be any Bruhat interval of length 1 and in this case we have $b(u,v)_\varepsilon = 1$ by definition.

So let $n > 0$ and observe that $\mathscr{W}_n/\mathscr{W}_{n-1}$ has dimension f_{n+1} and that if $\ell(u,v) = n+1$ then $b(u,v) \in \mathscr{W}_n$. Moreover we have that the values $b(u,v)_E$ as E ranges in $\mathbf{2}^n$ are precisely the coefficients of the cd-index of $[u,v]$. Therefore, by Theorem 5.11, if $\beta \in \mathscr{W}_n$ then there exist Bruhat intervals $[u_1,v_1],\ldots,[u_{f_{n+1}},v_{f_{n+1}}]$ and coefficients $c_1,\ldots,c_{f_{n+1}} \in \mathbb{Q}$ such that

$$\beta - \sum_{i=1}^{f_{n+1}} c_i b(u_i,v_i) \in \mathscr{W}_{n-1},$$

and the result follows by induction. $\square$

The following is now a straightforward consequence that makes precise the sense in which Theorem 5.1 can not be linearly simplified.

Corollary 5.13. *Let $a_T \in \mathbb{Q}$, $T \in \mathbf{2}_s^*$ be such that*

$$\sum_{T \in \mathbf{2}_s^*} a_T b(u,v)_T = 0$$

for all Bruhat intervals $[u,v]$. Then $a_T = 0$ for all $T \in \mathbf{2}_s^*$.

6. Linear relations for Bruhat paths

In this section we study and explicitly describe the complete cd-index of some families of Bruhat intervals. In this study we give a new general construction of reflection orderings in arbitrary Coxeter groups, and we provide a new descent criterion in some infinite Coxeter groups; both of these result may be of independent interest.

We also show that that for any $m,n \in \mathbb{N}$, $m \leqslant n$ and $m \equiv n \pmod 2$ the homogeneous components of degree m of the complete cd-indices of all possible Bruhat intervals of rank $n+1$ span a vector space of dimension f_{k+1}. This generalizes Theorem 5.11 that can be reinterpreted as the particular case $m = n$. We conclude this work by stating a conjecture that would further generalize this result.

6.1 Construction of reflection orderings

We start with a general construction of reflection orderings. Let (W,S) be a Coxeter system, Π be the associated set of simple roots, and $\Phi^+ = \Phi^+(W)$ the associated set of positive roots. A *weight* function on Φ^+ is a map $p : \Phi^+ \to \mathbb{R}_{\geqslant 0}$ which is linear, in the sense that if $\beta = c_1\beta_1 + c_2\beta_2$, with $\beta, \beta_1, \beta_2 \in \Phi^+$ and $c_1, c_2 \in \mathbb{N}$, then $p(\beta) = c_1 p(\beta_1) + c_2 p(\beta_2)$. It is clear that a weight function p is uniquely determined by its images on Π and that the set $\Phi_0^+(p) = \{\beta \in \Phi^+ : p(\beta) = 0\}$ is the set of positive roots of a parabolic subgroup of W. Let $I = (\alpha_1, \ldots, \alpha_l)$ be an indexing (total ordering) of the elements in Π. Then the associated lexicographic order on the root space $\mathbb{R}\alpha_1 \oplus \cdots \oplus \mathbb{R}\alpha_l$ is given by $\sum c_i \alpha_i < \sum d_i \alpha_i$ if $(c_1, \ldots, c_l)$ is smaller than $(d_1, \ldots, d_l)$ lexicographically.

Let W be a Coxeter group, p be a weight function on $\Phi^+(W)$, and W' the parabolic subgroup of W given by $\Phi^+(W') = \Phi_0^+(p)$. Let $\prec$ be a reflection ordering on $\Phi^+(W')$ and I an indexing of Π. Then we define a total ordering $\ll$ on Φ^+ depending on $p, \prec, I$ in the following way: for $\beta, \beta' \in \Phi^+$ we let $\beta \ll \beta'$ if one of the following conditions apply:

- $p(\beta) = p(\beta') = 0$ and $\beta \prec \beta'$;
- $p(\beta) \neq 0$ and $p(\beta') = 0$;
- $p(\beta), p(\beta') \neq 0$ and $\frac{\beta}{p(\beta)} < \frac{\beta'}{p(\beta')}$ in the lexicographic order associated to I.

It is clear that $\ll$ is a total ordering on $\Phi^+(W)$.

PROPOSITION 6.1. *The total ordering $\ll$ on $\Phi^+(W)$ constructed above is a reflection ordering.*

Proof. We have to show that if $\beta = c_1\beta_1 + c_2\beta_2$, with $\beta, \beta_1, \beta_2 \in \Phi^+$, $c_1, c_2 \in \mathbb{R}_{>0}$, and $\beta_1 \ll \beta_2$ then $\beta_1 \ll \beta \ll \beta_2$.

- If $p(\beta_1) = p(\beta_2) = 0$ then $\beta_1, \beta_2 \in \Phi^+(W')$ and hence also $\beta \in \Phi^+(W')$; the result follows since $\prec$ is a reflection ordering on $\Phi^+(W')$;
- if $p(\beta_1) \neq 0$ and $p(\beta_2) = 0$ then $p(\beta) = c_1 p(\beta_1) > 0$ and in particular $\beta \ll \beta_2$. Moreover, if we denote by $x_i(\beta)$ the i-th coordinate of β with respect to the chosen indexing I on Π (so $\beta = \sum_{i=1}^{l} x_i(\beta)\alpha_i$) we have
$$\frac{x_i(\beta)}{p(\beta)} = \frac{x_i(c_1\beta_1 + c_2\beta_2)}{c_1 p(\beta_1)} = \frac{c_1 x_i(\beta_1) + c_2 x_i(\beta_2)}{c_1 p(\beta_1)} \geqslant \frac{x_i(\beta_1)}{p(\beta_1)}.$$
- if $p(\beta_1), p(\beta_2) \neq 0$ then
$$\frac{x_i(\beta)}{p(\beta)} = \frac{x_i(c_1\beta_1 + c_2\beta_2)}{c_1 p(\beta_1) + c_2 p(\beta_2)} = \frac{c_1 p(\beta_1)\frac{x_i(\beta_1)}{p(\beta_1)} + c_2 p(\beta_2)\frac{x_i(\beta_2)}{p(\beta_2)}}{c_1 p(\beta_1) + c_2 p(\beta_2)}$$
which shows that $\frac{x_i(\beta)}{p(\beta)}$ is a convex linear combination of $\frac{x_i(\beta_1)}{p(\beta_1)}$ and $\frac{x_i(\beta_2)}{p(\beta_2)}$, completing the proof.

□

Note that Proposition 6.1 vastly generalizes Proposition 5.2.1 of [12].

COROLLARY 6.2. *Let (W, S) be a Coxeter system and P be the maximal parabolic subgroup generated by $S \setminus \{s\}$, for some $s \in S$. Then there exists a reflection ordering $\ll$ on Φ^+ such that*

- *$t \ll s$ for every reflection t in W;*
- *if t is a reflection in P then $t \ll sts$;*
- *if t and t' are reflections in P then $t \ll t'$ if and only if $sts \ll st's$.*

Proof. Consider an indexing I of $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ with $\alpha_l = \alpha_s$ and the weight function given by $p(\alpha_i) = 1$ if $i < l$ and $p(\alpha_l) = 0$. Let $\ll$ be the reflection ordering constructed above with respect to p and I (there is no choice for $\prec$ in this case). It is clear that α_s is the maximal element. If t is a reflection in P we have that

$$\alpha_{sts} = s(\alpha_t) = \alpha_t + c\alpha_l,$$

for some nonnegative integer c. In particular $p(\alpha_{sts}) = p(\alpha_t)$ and, by Proposition 6.1, $\alpha_t \ll \alpha_{sts} \ll \alpha_s$. Now let t, t' be reflections in P. We clearly have $p(t), p(t') \neq 0$. Since $p(\alpha_{sts}) = p(\alpha_t)$ and all the coordinates but the last one of α_t and α_{sts} coincide (and similarly for t') we deduce that $t \ll t'$ if and only if $sts \ll st's$.

□

6.2 The pyramid over a Bruhat interval

Let (W, S) be a Coxeter system and $[u, v]$ be an interval in W. We say that an interval $[u, vs]$ is a *pyramid* over $[u, v]$ if $s \in S$ and $s \not\leq v$. The name pyramid comes from the fact that if $[u, v]$ is isomorphic as a poset to the face lattice of a polytope P then $[u, vs]$ is isomorphic to the face lattice of a pyramid over P.

The following result states that the complete *cd*-index of a pyramid over a Bruhat interval does not depend on s, generalizes [22] and expresses the complete *cd*-index of the pyramid $[u, vs]$ in terms of the complete *cd*-index of $[u, v]$ and of smaller intervals.

Proposition 6.3. *Let $[u, v]$ be a Bruhat interval and $[u, vs]$ be a pyramid over $[u, v]$. Then*

$$\tilde{\Psi}_{u,vs} = \frac{1}{2}\Big(\tilde{\Psi}_{u,v}c + c\tilde{\Psi}_{u,v} + \sum_{x\in(u,v)} \tilde{\Psi}_{u,x} d \tilde{\Psi}_{x,v}\Big).$$

In particular $\tilde{\Psi}_{u,vs}$ does not depend on s.

Proof. We start with an observation. If $x < v$ then any path Δ in the Bruhat graph $\Delta = (x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{r+1}} x_{r+1})$ from x to v corresponds to a path $\Delta' = (x_0 s \xrightarrow{st_1 s} x_1 s \xrightarrow{st_2 s} \cdots \xrightarrow{st_{r+1} s} x_{r+1} s)$ from xs to vs. This correspondence is a bijection between paths from x to v and paths from xs to vs; moreover, if we consider the reflection ordering $\ll$ defined in Corollary 6.2, we have that if Δ corresponds to Δ' in this correspondence then $m_{\ll}(\Delta) = m_{\ll}(\Delta')$. We also observe that if we consider the *lower s-conjugate* $\ll_s$ of $\ll$ (see [8, Proposition 5.2.3]) given in this case by $r \ll_s r'$ if and only if either $r = s$ or $srs \ll sr's$, we still obtain $m_{\ll_s}(\Delta) = m_{\ll_s}(\Delta')$.

Given a path $\Delta = (u_0 \xrightarrow{t_1} u_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{r+1}} u_{r+1})$ from u to vs there exists a unique $x \in [u, v]$ such that $u_i = x$ and $u_{i+1} = xs$. We denote it by $x(\Delta)$ and in the computation of the complete *cd*-index

$$\tilde{\Psi}_{u,vs} = \sum_{\Delta\in B(u,vs)} m(\Delta)$$

we split the sum on the right-hand side according to $x(\Delta)$. We first consider the reflection ordering $\ll$. In this case we have

$$\sum_{\{\Delta\in B(u,vs):\, x(\Delta)=u\}} m_{\ll}(\Delta) = b \cdot \sum_{\Delta'\in B(us,vs)} m_{\ll}(\Delta') = b \cdot \sum_{\Delta\in B(u,v)} m_{\ll}(\Delta) = b \cdot \tilde{\Psi}_{u,v}.$$

and

$$\sum_{\{\Delta\in B(u,vs):\, x(\Delta)=v\}} m_{\ll}(\Delta) = \sum_{\Delta\in B(u,v)} m_{\ll}(\Delta) \cdot a = \tilde{\Psi}_{u,v} \cdot a,$$

where we have used the fact that s is the maximal reflection in the ordering $\ll$ and the observation at the beginning of the present proof. If $x \in (u,v)$ we have

$$\sum_{\{\Delta \in B(u,vs):\, x(\Delta)=x\}} m_{\ll}(\Delta) = \sum_{\Delta \in B(u,x)} m_{\ll}(\Delta) \cdot ab \cdot \sum_{\Delta' \in B(xs,vs)} m_{\ll}(\Delta') = \tilde{\Psi}_{u,x} ab \tilde{\Psi}_{x,v}$$

and we conclude that

$$\tilde{\Psi}_{u,vs} = b \cdot \tilde{\Psi}_{u,v} + \tilde{\Psi}_{u,v} \cdot a + \sum_{x \in (u,v)} \tilde{\Psi}_{u,x} ab \tilde{\Psi}_{x,v}.$$

By reasoning in a similar way with the ordering $\ll_s$ we can obtain the analogous formula

$$\tilde{\Psi}_{u,vs} = a \cdot \tilde{\Psi}_{u,v} + \tilde{\Psi}_{u,v} \cdot b + \sum_{x \in (u,v)} \tilde{\Psi}_{u,x} ba \tilde{\Psi}_{x,v}$$

and the result follows by "averaging" these two expressions for $\tilde{\Psi}_{u,vs}$. □

Consider the derivation D_d on $\mathcal{A}$. One easily checks that D_d restricts to a derivation on the space of cd-polynomials as $\delta(c) = \delta(a+b) = 2(1 \otimes 1)$ and so $D_d(c) = 2d$ and $\delta(d) = \delta(ab+ba) = a \otimes 1 + 1 \otimes b + b \otimes 1 + 1 \otimes a$ and so $D_d(d) = ad + db + bd + da = dc + cd$. Corollary 2.6 and Proposition 6.3 therefore allow us to write

$$\tilde{\Psi}_{u,vs} = \frac{1}{2}\Big(\tilde{\Psi}_{u,v} c + c\tilde{\Psi}_{u,v} + D_d(\tilde{\Psi}_{u,v})\Big)$$

which shows that $\tilde{\Psi}_{u,vs}$ depends on $\tilde{\Psi}_{u,v}$ only. Let G' be the derivation on $\mathcal{A}$ given by $G'(a) = ab$ and $G'(b) = ba$ so that $G'(c) = d$ and $G'(d) = dc$. The next result is then a consequence of [22, Lemma 5.1 and Theorem 5.2].

Corollary 6.4. *Let (W,S) be a Coxeter system, $u, v \in W$, $u < v$, and $s \in S$ be such that $s \not\leq v$. Then*

$$\tilde{\Psi}_{u,vs} = c\tilde{\Psi}_{u,v} + G'(\tilde{\Psi}_{u,v}).$$

Similarly, one can prove the following "left version" of Corollary 6.4.

Corollary 6.5. *Let (W,S) be a Coxeter system, $u, v \in W$, $u < v$, and $s \in S$ be such that $s \not\leq v$. Then*

$$\tilde{\Psi}_{u,sv} = c\tilde{\Psi}_{u,v} + G'(\tilde{\Psi}_{u,v}).$$

6.3 3-complete Coxeter systems

Let (W,S) be the Coxeter system of rank l such that $m(s,s') = 3$ for all $s, s' \in S$, $s \neq s'$. We call this the 3-complete Coxeter system (or group) of rank l.

Our first result can be interpreted as a concrete criterion to determine the set of (left) descents of a generic element in a 3-complete Coxeter group: it is used in the sequel in the construction of reflection orderings, but is interesting in its own right.

Let (W,S) be a 3-complete Coxeter system of rank l, $S = \{s_1, \ldots, s_l\}$ and let W' be the parabolic subgroup of W generated by $S \setminus \{s_1\}$ (note that W' is a 3-complete Coxeter group of rank $l-1$). We also let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ where α_i is the simple root corresponding to s_i for all $i \in [l]$. We observe that

$$s_i(\alpha_j) = \begin{cases} \alpha_i + \alpha_j & \text{if } i \neq j; \\ -\alpha_i & \text{if } i = j. \end{cases}$$

We consider the W'-orbit of the simple root α_1 to study the sets of left descents of elements in W'. We adopt the following notation: for all $w \in W'$ we let $c_i(w), d_i(w) \in \mathbb{Z}$, $i \in [l]$, be given by

$$w(\alpha_1) = \sum_{i=1}^{l} c_i(w)\alpha_i$$

and $d_i(w) \stackrel{\text{def}}{=} 2c_i(w) - \sum_{k\neq i} c_k(w)$. We note that in the notation of [29, Section 5.3] we have $d_i(w) = B(w(\alpha_1), \alpha_i)$, where B denote the inner product on the root space of an arbitrary Coxeter group. Before proving the main result about the coefficients $d_i(w)$ we need the following preliminary result.

Lemma 6.6. *Let $u \in W'$. Then for all $i, j \in [2, l]$, $i \neq j$, we have*
- (a) $d_i(s_i u) = -d_i(u)$;
- (b) $d_j(s_i u) = d_i(u) + d_j(u)$.

Proof. We first observe that for all $u \in W'$ we have

$$c_i(s_j u) = \begin{cases} c_i(u), & \text{if } i \neq j; \\ \sum_{k\neq i} c_k(u) - c_i(u), & \text{if } i = j. \end{cases} \tag{21}$$

(a) We have

$$d_i(s_i u) = 2c_i(s_i u) - \sum_{k\neq i} c_k(s_i u) = 2\sum_{k\neq i} c_k(u) - 2c_i(u) - \sum_{k\neq i} c_k(u) = -d_i(u).$$

(b) We have

$$\begin{aligned} d_j(s_i u) &= 2c_j(s_i u) - \sum_{k\neq i,j} c_k(s_i u) - c_i(s_i u) = 2c_j(u) - \sum_{k\neq i,j} c_k(u) - \sum_{k\neq i} c_k(u) + c_i(u) \\ &= 2c_j(u) - \sum_{k\neq j} c_k(u) + 2c_i(u) - \sum_{k\neq i} c_k(u) = d_i(u) + d_j(u). \end{aligned}$$

$\square$

For $w \in W$ we let $\mathrm{Des}_L(w) \stackrel{\text{def}}{=} \{i \in [l] : s_i w < w\}$.

Proposition 6.7. *Let $w \in W'$ and $i \in [2, l]$. Then $d_i(w) \neq 0$ and*

$$d_i(w) > 0 \Leftrightarrow i \in \mathrm{Des}_L(w).$$

Proof. We proceed by induction on $\ell(w)$. If $\ell(w) = 0$ then $d_i(w) = -1$ and $i \notin \mathrm{Des}_L(w)$, and the statement is true. So let $\ell(w) \geqslant 1$.

If $i \in \mathrm{Des}_L(w)$ let $w = s_i u$, with $i \notin \mathrm{Des}_L(u)$. By the induction hypothesis we have $d_i(u) < 0$ and so, by Lemma 6.6, we have $d_i(w) = -d_i(u) > 0$.

If $i \notin \mathrm{Des}_L(w)$ let j be such that $j \in \mathrm{Des}_L(w)$ and $w = s_j u$, with $j \notin \mathrm{Des}_L(u)$. Now two cases occur: if $i \notin \mathrm{Des}_L(u)$ we have by induction that $d_i(u), d_j(u) < 0$ and so, by Lemma 6.6 we conclude that $d_i(w) = d_i(u) + d_j(u) < 0$. If $i \in \mathrm{Des}_L(u)$ we let $\tilde{u}$ be such that $w = s_j s_i \tilde{u}$, with

$\ell(w) = \ell(\tilde{u}) + 2$. We have that $j \notin \mathrm{Des}_L(\tilde{u})$ since otherwise we could obtain a reduced expression for w starting with s_i. Therefore, by induction we have $d_j(\tilde{u}) < 0$. Using Lemma 6.6 we then conclude that $d_i(w) = d_i(s_i\tilde{u}) + d_j(s_i\tilde{u}) = -d_i(\tilde{u}) + d_j(\tilde{u}) + d_i(\tilde{u}) = d_j(\tilde{u}) < 0$. □

Corollary 6.8. *For all $w \in W'$ we have $\mathrm{ht}(w(\alpha_1)) \geqslant \ell(w) + 1$, where ht denotes the height function defined by $\mathrm{ht}(\sum c_i\alpha_i) = \sum c_i$.*

Proof. We proceed by induction on $\ell(w)$, the result being trivial if $\ell(w) = 0$. So let $\ell(w) > 0$, $i \in \mathrm{Des}_L(w)$ and $w = s_i u$. Using Eq. (21) we easily have that $c_i(w) = c_i(u) - d_i(u)$. Therefore we have

$$\mathrm{ht}(w(\alpha_1)) = \mathrm{ht}(u(\alpha_1)) - d_i(u) \geqslant \ell(u) + 1 - d_i(u) = \ell(w) - d_i(u)$$

and the result follows since $d_i(u) < 0$ by Proposition 6.7.

□

We now show the existence of reflection orderings in a 3-complete Coxeter group satisfying some particular properties.

Lemma 6.9. *Let (W, S) be a 3-complete Coxeter system, $s \in S$ and P be the parabolic subgroup of W generated by $S \setminus \{s\}$. Then there exists a reflection ordering $\ll$ such that for any reflection $t \in P$ and any element $z \in P$, $\ell(z) \geqslant 2$, we have*

$$t \ll szsz^{-1}s \ll sts \ll s.$$

Proof. We consider the reflection ordering $\ll$ constructed as in Proposition 6.1, where the weight $p = \mathrm{ht}$ is the height function, and the indexing $I = (\alpha_1, \ldots, \alpha_l)$ is such that $\alpha_1 = \alpha_s$ (there is no choice for $\prec$ here, since $\Phi_0^+(\mathrm{ht}) = \emptyset$), so $s \in S$ is the simple reflection corresponding to α_1. It is enough to show that

$$\frac{x_1(\alpha_t)}{\mathrm{ht}(\alpha_t)} < \frac{x_1(sz(\alpha_s))}{\mathrm{ht}(sz(\alpha_s))} < \frac{x_1(s(\alpha_t))}{\mathrm{ht}(s(\alpha_t))} < \frac{x_1(\alpha_s)}{\mathrm{ht}(\alpha_s)}.$$

Since $x_1(\alpha_t) = 0$ and $\frac{x_1(\alpha_s)}{\mathrm{ht}(\alpha_s)} = 1$ we have to show that

$$0 < \frac{x_1(sz(\alpha_s))}{\mathrm{ht}(sz(\alpha_s))} < \frac{x_1(s(\alpha_t))}{\mathrm{ht}(s(\alpha_t))} < 1.$$

Recall that we have $r(\alpha_{r'}) = \alpha_r + \alpha_{r'}$ for all $r, r' \in S$, $r \neq r'$. In particular we have, since $t \in P$

$$s(\alpha_t) = \alpha_t + \mathrm{ht}(\alpha_t)\alpha_s.$$

It follows that $\frac{x_1(s(\alpha_t))}{\mathrm{ht}(s(\alpha_t))} = \frac{1}{2}$ and so to conclude the proof we only have to show that

$$0 < \frac{x_1(sz(\alpha_s))}{\mathrm{ht}(sz(\alpha_s))} < \frac{1}{2}$$

for all $z \in P$, $\ell(z) \geqslant 2$. So let $z(\alpha_s) = \alpha_s + \sum_{i \geqslant 2} c_i\alpha_i$. By Corollary 6.8 we have $\mathrm{ht}(z(\alpha_s)) = 1 + \sum_{i \geqslant 2} c_i \geqslant \ell(z) + 1$ and in particular we have $c \stackrel{\mathrm{def}}{=} \sum_{i \geqslant 2} c_i \geqslant 2$. Therefore $sz(\alpha_s) = (c-1)\alpha_s + \sum_{i \geqslant 2} c_i\alpha_i$ and so $x_1(sz(\alpha_s)) = c - 1$ and $\mathrm{ht}(sz(\alpha_s)) = 2c - 1$. The result follows.

□

Proposition 6.10. *Let (W, S) be a 3-complete Coxeter system. Let $r, s \in S$, $r \neq s$. Let P be the parabolic subgroup of W generated by $S \setminus \{r, s\}$. Then there exists a reflection ordering $\ll$ such that for every $t, z, w \in P$, t a reflection, $\ell(z) \geqslant 2$, we have*

$$t \ll szsz^{-1}s \ll sts \ll s$$

and

$$t \ll swrw^{-1}s \ll sts.$$

Moreover $t \ll wsw^{-1}$, $t \ll wrw^{-1}$ *and* $r \ll s$.

Proof. We consider the weight p given by $p(\alpha) = 1$ for all $\alpha \in \Pi \setminus \{\alpha_r\}$ and $p(\alpha_r) = 2$. We also consider an indexing I of $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ such that $\alpha_1 = \alpha_s$ and $\alpha_2 = \alpha_r$ and we let $\ll$ be the reflection ordering associated to p and I (again there is no choice for $\prec$ as $\Phi_0^+(p) = \emptyset$). As the restriction of $\ll$ to the parabolic subgroup generated by $S \setminus \{r\}$ is the reflection ordering considered in Lemma 6.9 the first part of the statement follows.

For the second part we first observe that $w(\alpha_r) = \alpha_r + \sum_{i \geqslant 3} c_i \alpha_i$. Let $c \overset{\text{def}}{=} \sum_{i \geqslant 3} c_i \geqslant 0$. Then $sw(\alpha_r) = (c+1)\alpha_s + \alpha_r + \sum_{i \geqslant 3} c_i \alpha_i$ and $x_1(sw(\alpha_r)) = c + 1 \geqslant 1$, which implies $t \ll swrw^{-1}s$. Moreover we have

$$\frac{x_1(sw(\alpha_r))}{p(sw(\alpha_r))} = \frac{c+1}{2c+3} < \frac{1}{2} = \frac{x_1(s(\alpha_t))}{p(s(\alpha_t))}$$

implying $swrw^{-1}s \ll sts$. The relations $t \ll wsw^{-1}$, $t \ll wrw^{-1}$ and $r \ll s$ are all clear from the definition. $\square$

We can now prove the second main result of this section.

Theorem 6.11. *Let (W, S) be a 3-complete Coxeter system. Let $r, s \in S$, $r \neq s$, and P be the parabolic subgroup generated by $S \setminus \{s, r\}$. Then for all $v \in P$, $v \neq e$, we have*

$$\tilde{\Psi}_{e,svs} + d \cdot \tilde{\Psi}_{e,v} = \tilde{\Psi}_{e,rvs} + \tilde{\Psi}_{e,v}.$$

Proof. We establish the result by means of an explicit bijection. In particular we exhibit a bijection σ between $B(e, svs) \cup B(e, v) \cup \overline{B(e, v)}$ and $B(e, rvs) \cup B(e, v)$, where $\overline{B(e, v)}$ is just a copy of $B(e, v)$, which is well-behaved with respect to the contributions of these paths to the corresponding complete *cd*-indices in the following sense. If $\Delta \in B(e, svs)$ or $\Delta \in B(e, rvs)$ we consider the monomial $m(\Delta) = m_{\ll}(\Delta)$ with respect to the reflection ordering $\ll$ studied in Proposition 6.10. If $\Delta \in B(e, v)$ (or $\Delta \in \overline{B(e, v)}$) we consider the monomial $m_{\ll_s}(\Delta)$ with respect to the lower s-conjugate $\ll_s$ of $\ll$. With this convention we will show that the bijection σ has the following properties:

(i) if $\Delta \in B(e, svs)$ then $m(\sigma(\Delta)) = m(\Delta)$;

(ii) if $\Delta \in B(e, v)$ then $m(\sigma(\Delta)) = ab \cdot m(\Delta)$;

(iii) if $\Delta \in \overline{B(e, v)}$ then $m(\sigma(\Delta)) = ba \cdot m(\Delta)$.

Consider the Bruhat graph of $[e, svs]$: the vertices of this graph can be visualized as in Figure 6, where the four shaded regions correspond respectively from left to right to: (1) elements of the form sxs, for some $x \leqslant v$; (2) elements of the form sx for some $x \leqslant v$; (3) elements of the form xs for some $x \leqslant v$; (4) elements smaller than or equal to v. The bijection σ is defined as follows. Let $\Delta \in B(e, svs)$. If the smallest element in the path Δ which is strictly greater than s is of the form sxs for some $x \leqslant v$ then by the Exchange Condition (see, e.g., [12, Theorem 1.4.3]), $sxs \in T$ and Δ is necessarily of the form $\Delta = (x_0 \xrightarrow{st_1s} sx_1s \xrightarrow{st_2s} \cdots \xrightarrow{st_rs} sx_rs)$, with $t_i \in P$ and $x_i \leqslant v$ for all $i \in [r]$ (see Figure 6, left), and we define $\sigma(\Delta) = (x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_r} x_r) \in B(e, v)$; since $st_is \ll st_{i+1}s$ if and only if $t_i \ll_s t_{i+1}$ we clearly have $m_{\ll_s}(\sigma(\Delta)) = m_{\ll}(\Delta)$.

Suppose now that the smallest element in the path Δ which is strictly greater than s is of the form xs for some $x \leqslant v$ (see Figure 6, right).

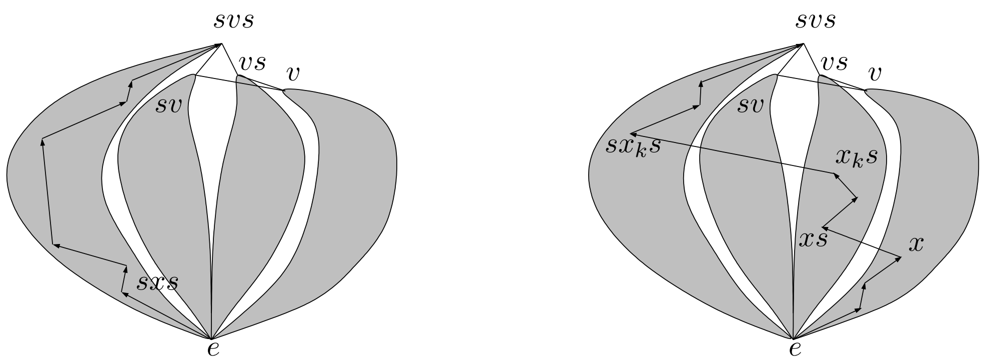

Figure 6. Paths in the Bruhat graph of $[e, svs]$

Then Δ is of the form

$$\begin{aligned}\Delta = (x_0 &\xrightarrow{t_1} \cdots \xrightarrow{t_{i-1}} x_{i-1} \xrightarrow{s} x_{i-1}s \\ &\xrightarrow{st_is} x_is \cdots \xrightarrow{st_ks} x_ks \xrightarrow{sx_k^{-1}sx_ks} sx_ks \xrightarrow{st_{k+1}s} \cdots \xrightarrow{st_rs} sx_rs)\end{aligned}$$

for some integers i, k such that $r \geqslant k \geqslant i-1 \geqslant 0$, $k \geqslant 1$, where $t_1, \ldots, t_r \in P$, $x_1, \ldots, x_r \leqslant v$. In this case we define $\sigma(\Delta) \in B(e, rvs)$ essentially by replacing the letter s "on the left" by r. More precisely we let

$$\begin{aligned}\sigma(\Delta) =(x_0 &\xrightarrow{t_1} \cdots \xrightarrow{t_{i-1}} x_{i-1} \xrightarrow{s} x_{i-1}s \\ &\xrightarrow{st_is} x_is \cdots \xrightarrow{st_ks} x_ks \xrightarrow{sx_k^{-1}rx_ks} rx_ks \xrightarrow{st_{k+1}s} \cdots \xrightarrow{st_rs} rx_rs)\end{aligned}$$

and it follows from Proposition 6.10 that $m_{\ll}(\sigma(\Delta)) = m_{\ll}(\Delta)$ (we observe here that if $\ell(x_k) = 1$ then $sx_k^{-1}sx_ks = x_k$ and in particular we still have $sx_k^{-1}sx_ks \ll sts$ for all reflections $t \in P$).

Finally, if the smallest element strictly greater than s in the path Δ is of the form sx for some $x \leqslant v$, then Δ is of the form

$$\begin{aligned}\Delta = (x_0 &\xrightarrow{t_1} \cdots \xrightarrow{t_{i-1}} x_{i-1} \xrightarrow{x_{i-1}^{-1}sx_{i-1}} sx_{i-1} \\ &\xrightarrow{t_i} sx_i \cdots \xrightarrow{t_k} sx_k \xrightarrow{s} sx_ks \xrightarrow{st_{k+1}s} \cdots \xrightarrow{st_rs} sx_rs)\end{aligned}$$

for some integers i, k such that $r \geqslant k \geqslant i-1 \geqslant 0$, $k \geqslant 1$, where $t_1, \ldots, t_r \in P$, $x_1, \ldots, x_r \leqslant v$, and we let $\sigma(\Delta) \in B(e, rvs)$ be defined by

$$\begin{aligned}\sigma(\Delta) = (x_0 &\xrightarrow{t_1} \cdots \xrightarrow{t_{i-1}} x_{i-1} \xrightarrow{x_{i-1}^{-1}rx_{i-1}} rx_{i-1} \\ &\xrightarrow{t_i} rx_i \cdots \xrightarrow{t_k} rx_k \xrightarrow{s} rx_ks \xrightarrow{st_{k+1}s} \cdots \xrightarrow{st_rs} rx_rs).\end{aligned}$$

Also in this case it follows from Proposition 6.10 that $m_{\ll}(\sigma(\Delta)) = m_{\ll}(\Delta)$. We have considered in this way all paths in $B(e, svs)$ and we have obtained all paths in $B(e, v)$ and all paths in $B(e, rvs)$ except those passing through rs.

If $\Delta \in B(e, v)$, with

$$\Delta = (x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{r+1}} x_{r+1})$$

then we let

$$\sigma(\Delta) = (x_0 \xrightarrow{r} r \xrightarrow{s} rs \xrightarrow{st_1s} rx_1s \xrightarrow{st_2s} \cdots \xrightarrow{st_{r+1}s} rx_{r+1}s)$$

and if the same path Δ is considered in $\overline{B(e,v)}$ we let

$$\sigma(\Delta) = (x_0 \xrightarrow{s} s \xrightarrow{srs} rs \xrightarrow{st_1s} rx_1s \xrightarrow{st_2s} \cdots \xrightarrow{st_{r+1}s} rx_{r+1}s).$$

In the first case we have $m_{\ll}(\sigma(\Delta)) = ab \cdot m_{\ll_s}(\Delta)$ by Proposition 6.10. In the second case we similarly have $m_{\ll}(\sigma(\Delta)) = ba \cdot m_{\ll_s}(\Delta)$ as $srs \ll st_1s$ by Proposition 6.10 used with $w = e$. $\square$

6.4 Homogeneous components and linear relations

Following [36] we consider a set W_n of elements in the 3-complete Coxeter group W of rank $n+1$ generated by $s_1, \ldots, s_{n+1}$ constructed recursively in the following way: we let $W_0 = \{s_1\}$, $W_1 = \{s_1 s_2\}$ and, for $n \geqslant 2$,

$$W_n = \{ws_{n+1} : w \in W_{n-1}\} \cup \{s_{n+1} w s_{n+1} : w \in W_{n-2}\}.$$

We now consider the following space of *cd*-polynomials

$$V_n = Span\{\tilde{\Psi}_{e,v} : v \in W_n\}.$$

Since $\ell(v) = n+1$ for all $v \in W_n$ we deduce that V_n is contained in the space of *cd*-polynomials of degree bounded by n. A set of generators for V_n can also be described in the following way. Let $A_0 = \{1\}$, $A_1 = \{c\}$ and

$$A_n = \{c \cdot P + P' : P \in A_{n-1}\} \cup \{(d-1) \cdot P : P \in A_{n-2}\},$$

where for all $P \in \mathcal{A}$ we let $P' \stackrel{\text{def}}{=} G'(P)$. We claim that A_n is a spanning set for V_n and we proceed by induction on n, the result being clear if $n = 0, 1$. So assume that $n \geqslant 2$ and let $v \in W_n$. If $v = ws_{n+1}$ for some $w \in W_{n-1}$ we have $\tilde{\Psi}_{e,v} = c\tilde{\Psi}_{e,w} + \tilde{\Psi}'_{e,w}$ by Corollary 6.4 and the result follows by induction. If $v = s_{n+1} w s_{n+1}$ for some $w \in W_{n-2}$ we have, by Theorem 6.11 and Corollary 6.5 $\tilde{\Psi}_{e,s_{n+1}ws_{n+1}} = c\tilde{\Psi}_{e,vs_{n+1}} + \Psi'_{e,vs_{n+1}} + (1-d)\tilde{\Psi}_{e,v}$ and the result follows again by induction as $\tilde{\Psi}_{e,vs_n} = \tilde{\Psi}_{e,vs_{n+1}}$ by Corollary 6.4.

We observe that $|A_n| = f_{n+1}$ and we denote its elements by $P_{n,1}, \ldots, P_{n,f_{n+1}}$ in the following way. We let $P_{0,1} = 1$, $P_{1,1} = c$ and

$$P_{n,j} = \begin{cases} cP_{n-1,j} + P'_{n-1,j} & \text{if } 1 \leqslant j \leqslant f_n \\ (d-1)P_{n-2,j-f_n} & \text{if } f_n < j \leqslant f_n + f_{n-1} \end{cases}$$

The next result follows immediately from the above recursion.

Lemma 6.12. *Let $P_{n,j} = \sum_M a_M M$, the sum being over all monomials of degree at most n (and of the same parity as n). If M is a monomial of degree $n-2i$ ($i \geqslant 0$) then $a_M(-1)^i \geqslant 0$.*

We consider the lexicographic order $\prec$ on the set of *cd*-monomials of degree n for all $n \in \mathbb{N}$, where we let $c \prec d$. So for example, if $n = 4$ we have $c^4 \prec c^2 d \prec cdc \prec dc^2 \prec d^2$. The proof of the following result is a simple verification, and is left to the reader.

Lemma 6.13. *Let M, I be cd-monomials of the same degree such that $I \preceq M$. Then the cd-polynomial M' is a sum of monomials which are all $\succeq cI$.*

If P is a *cd*-polynomial with non-zero homogeneous component of degree n, we call the minimum monomial of degree n appearing in P with non-zero coefficient the n-th initial term of P. We denote by $M_{n,j}$ the n-th initial term of $P_{n,j}$.

Lemma 6.14. *For all $n \in \mathbb{N}$ we have $M_{n,1} \prec M_{n,2} \prec \cdots \prec M_{n,f_{n+1}}$.*

Proof. This follows from Lemma 6.13 and the observation that if P is a polynomial of degree n and M is the n-th initial term of P, then cM is the $n+1$-st initial term of cP and dM is the $n+2$-nd initial term of $(d-1)P$. □

Lemma 6.15. *Let $P_{n,j} = \sum_M a_M M$, the sum being over all monomials of degree at most n (and of the same parity as n). If M_0 is a monomial of degree $n-2i$, with $i>0$, and $a_{M_0} \neq 0$ then there exists a monomial $\tilde{M}_0$ of degree $n-2i+2$ with $a_{\tilde{M}_0} \neq 0$ such that M_0 is obtained from $\tilde{M}_0$ be deleting a letter d.*

Proof. We proceed by induction on n, the cases $n = 0, 1$ being empty. We consider the two cases:

(i) if $P_{n,j} = cQ + Q'$ for some $Q \in A_{n-1}$ we let $Q = \sum b_m m$, the sum being over monomials m of degree bounded by $n-1$ and of the same parity as $n-1$. The monomial M_0 will appear as a summand in $cm_0 + m_0'$ for some m_0 such that $\deg(m_0) = n-1-2i$ and $b_{m_0} \neq 0$. By induction there exists $\tilde{m}_0$ of degree $n+1-2i$ such that $b_{\tilde{m}_0} \neq 0$ and such that m_0 is obtained from $\tilde{m}_0$ by deleting a letter d. Then it is not hard to see that in $c\tilde{m}_0 + \tilde{m}_0'$ there is a monomial obtained by inserting a letter d in M_0. Since, by Lemma 6.12, all monomials of the same degree appearing in Q have coefficients with the same sign, there cannot be cancellations when expanding $cQ + Q'$ and therefore we necessarily have $a_{\tilde{M}_0} \neq 0$.

(ii) $P = (d-1)Q$ for some $Q \in A_{n-2}$. This is similar and simpler and is left to the reader.

□

We can now prove the main result of this section.

Theorem 6.16. *Let $k, n \in \mathbb{N}$. Then the homogeneous parts of degree n of the polynomials $(d-1)^k P_{n,j}$, for $j \in [f_{n+1}]$, are linearly independent.*

Proof. By Lemma 6.14, the result will follow if we show that the initial term of the homogeneous part of degree n of $(d-1)^k P_{n,j}$ equals the initial term $M_{n,j}$ of the homogeneous part of degree n of $P_{n,j}$.

We need the following notation: if M is a monomial of degree n we let $i(M) = \max\{i \in \mathbb{N} : M = d^i \cdot m \text{ for some monomial } m\}$ and for all $j \leqslant i(M)$ we let $M^{(j)}$ be the monomial obtained from M be deleting its first j factors so $M = d^j M^{(j)}$. For example, if $M = d^2cd$ then $i(M) = 2$, $M^{(0)} = M$, $M^{(1)} = dcd$ and $M^{(2)} = cd$.

Let $P_{n,j} = \sum_M a_M M$ and $(d-1)^k P_{n,j} = \sum_M b_M M$. Then, for every monomial M, $deg(M) \leqslant n+2k$, we have

$$b_M = \sum_{j=0}^{\min(i(M),k)} (-1)^{k-j} \binom{k}{j} a_{M^{(j)}}. \tag{22}$$

If M has degree n we have that $(-1)^j a_{M^{(j)}} \geqslant 0$ for all $j \geqslant 0$ by Lemma 6.12 and in particular we have that $b_M \neq 0$ if $a_M \neq 0$. In particular $b_{M_{n,j}} \neq 0$. Now we have to show that if M_0 is a monomial of degree n such that $b_{M_0} \neq 0$ then $M_{n,j} \prec M_0$. It follows from (22) that $a_{M_0^{(j)}} \neq 0$ for some $0 \leqslant j \leqslant \min(i(M), k)$. Repeated applications of Lemma 6.14 imply that there exists a monomial $\tilde{M}$ of degree n, with $a_{\tilde{M}} \neq 0$ such that $M_0^{(j)}$ can be obtained by deleting j factors d from $\tilde{M}$. Therefore M_0 can be obtained from $\tilde{M}$ by moving some factors d to the left and so $\tilde{M} \prec M_0$; finally, $a_{\tilde{M}} \neq 0$ implies $M_{n,j} \prec \tilde{M}$, completing the proof. □

The following consequence of Theorem 6.16 is the main motivation for the results in this section.

Corollary 6.17. *Let $n, k \in \mathbb{N}$. Let $a_T \in \mathbb{Q}$, $T \in \mathbf{2}^n_s$ be such that*

$$\sum_{T \in \mathbf{2}^n_s} a_T b(e,v)_T = 0$$

for all Coxeter groups W and all $v \in W$ such that $\ell(v) = n + 2k + 1$. Then $a_T = 0$ for all $T \in \mathbf{2}^n_s$.

Proof. If P is a *cd*-polynomial let $P^{(n)}$ be the homogeneous component of degree n of P. By Theorem 6.16 and our definitions we have that the *cd*-polynomials $\tilde{\Psi}^{(n)}_{e,v}$, as v ranges in W_{n+2k+1}, span the whole space of homogeneous *cd*-polynomials of degree n, which has dimension f_{n+1}. But by definition of the complete *cd*-index we have

$$\tilde{\Psi}^{(n)}_{e,v} = \sum_{E \in \mathbf{2}^n} b(e,v)_E \mu_{E^{op}},$$

and so the result now follows since $b(e,v) \in \mathscr{B}_n$.

□

6.5 Another family of complete *cd*-indices and a conjecture

We conclude this work by giving a recursive way to compute the complete *cd*-index of another family Bruhat intervals. This result will allow us to state a general conjecture about possible relations among all the coefficients of the complete cd-index of any Bruhat interval.

Theorem 6.18. *Let (W,S) be a 3-complete Coxeter system, $e \neq v \in W$ and $s \in S$ be such that $s \not\leqslant v$. Then*

$$\tilde{\Psi}_{s,svs} = \tilde{\Psi}_{e,v} \cdot c + \sum_{x \in (e,v)} \tilde{\Psi}_{e,x} \cdot d \cdot \tilde{\Psi}_{x,v}.$$

Proof. Consider a path $\Delta \in B(s,svs)$. Then two cases occur: either Δ is of the form $\Delta = (s \to sx_1 \cdots)$ or $\Delta = (s \to x_1 s \cdots)$ for some $e \neq x_1 \leqslant v$. Call $B_1(s,svs)$ the family of paths of the first kind and $B_2(s,svs)$ the family of paths of the second kind. We claim that there is a bijection $B_1(s,svs) \longleftrightarrow \bigcup_{x \in (e,v]} B(e,x) \times B(x,v)$. Furthermore, if we consider on paths in $B_1(s,svs)$ and in $B(e,x)$ the order $\ll$ described in Lemma 6.9 and on paths in $B(x,v)$ the lower s-conjugate $\ll_s$ of $\ll$ we claim that if $\Delta \in B_1(s,svs)$ corresponds to $(\Delta',\Delta'') \in B(e,x) \times B(x,v)$ with $x \neq v$ then $m_{\ll}(\Delta) = m_{\ll}(\Delta') \cdot ab \cdot m_{\ll_s}(\Delta'')$ and if $(\Delta',\Delta'') \in B(e,v) \times B(v,v)$ then $m_{\ll}(\Delta) = m_{\ll}(\Delta') \cdot a$. The bijection is defined as follows: if $\Delta \in B_1(s,svs)$ then it is necessarily of the form

$$\Delta = (s \xrightarrow{t_1} sx_1 \xrightarrow{t_2} \cdots \xrightarrow{t_i} sx_i \xrightarrow{s} sx_i s \xrightarrow{st_{i+1}s} \cdots \xrightarrow{st_r s} sx_r s).$$

for some $1 \leqslant i \leqslant r$. Then we define $\Delta' = (e \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_i} x_i)$ and $\Delta'' = (x_i \xrightarrow{t_{i+1}} \cdots \xrightarrow{t_r} x_r)$. The fact that this is a bijection is clear and that the monomial $m_{\ll}(\Delta)$ satisfies the stated properties follows from Lemma 6.9 and the definition of $\ll_s$. We deduce that

$$\begin{aligned}\sum_{\Delta \in B_1(s,svs)} m_{\ll}(\Delta) &= \sum_{\Delta' \in B(e,v)} m_{\ll}(\Delta') \cdot a + \sum_{x \in (e,v)} \sum_{\substack{\Delta' \in B(e,x)\\ \Delta'' \in B(x,v)}} m_{\ll}(\Delta') \cdot ab \cdot m_{\ll_s}(\Delta'') \\ &= \tilde{\Psi}_{e,v} \cdot a + \sum_{x \in (e,v)} \tilde{\Psi}_{e,x} \cdot ab \cdot \tilde{\Psi}_{x,v}.\end{aligned}$$

We also claim that there is a bijection $B_2(s,svs) \longleftrightarrow \bigcup_{x \in (e,v]} B(e,x) \times B(x,v)$ such that if Δ corresponds to $(\Delta',\Delta'') \in B(e,x) \times B(x,v)$ with $x \neq v$ then $m_{\ll}(\Delta) = m_{\ll_s}(\Delta') \cdot ba \cdot m_{\ll_s}(\Delta'')$

and if $(\Delta', \Delta'') \in B(e,v) \times B(v,v)$ then $m_{\ll}(\Delta) = m_{\ll_s}(\Delta') \cdot b$. In this case, if $\Delta \in B_2(s, svs)$, then Δ is of the form

$$\Delta = (s \xrightarrow{st_1 s} x_1 s \xrightarrow{st_2 s} \cdots \xrightarrow{st_i s} x_i s \xrightarrow{sx_i^{-1} s x_i s} sx_i s \xrightarrow{st_{i+1} s} \cdots \xrightarrow{st_r s} sx_r s),$$

and we define $\Delta' = (e \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_i} x_i)$ and $\Delta'' = (x_i \xrightarrow{t_{i+1}} \cdots \xrightarrow{t_r} x_r)$. It follows that

$$\sum_{\Delta \in B_2(s,svs)} m_{\ll}(\Delta) = \tilde{\Psi}_{e,v} \cdot b + \sum_{x \in (e,v)} \tilde{\Psi}_{e,x} \cdot ba \cdot \tilde{\Psi}_{x,v},$$

and the result follows. □

If W_n is the subset of elements of the 3-complete Coxeter group constructed in the previous subsection, this result allows us to easily compute all the complete *cd*-indices of the Bruhat intervals $[s_{n+1}, s_{n+1} v s_{n+1}]$ as v ranges in W_{n-1}. This has allowed us to verify the following conjecture for $n \leqslant 17$.

Conjecture 6.19. *For all $n > 0$ the complete cd-indices of all Bruhat intervals of rank $n+1$ span the whole space of cd-polynomials of degree bounded by n whose nonzero homogeneous components have degree of the same parity as n.*

This conjecture implies the following one, which in turn would imply that the formula obtained in Theorem 5.1 cannot be "linearly" simplified, even if we content ourselves with a formula that only holds for all Bruhat intervals of a fixed rank.

Conjecture 6.20. *Let $n > 0$. Then there are no nontrivial relations of the form*

$$\sum_{i \in \{n, n-2, \dots\}} \sum_{T \in \mathbf{2}_s^i} a_T b(u,v)_T = 0,$$

valid for all Coxeter groups W and all $u, v \in W$ such that $\ell(v) - \ell(u) = n$.

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