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Gamma-convergence and its Applications to Some Problems in the Calculus of Variations

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CONTENTS

Lesson 1. Gamma-convergence: the general framework

We recall the definition of Γ-limits in metric spaces:

$$\Gamma \liminf_{h\to+\infty} F_h(x) = \inf\{\liminf_{h\to+\infty} F_h(x_h) \ : \ x_h \to x\};$$

$$\Gamma \limsup_{h\to+\infty} F_h(x) = \inf\{\limsup_{h\to+\infty} F_h(x_h) \ : \ x_h \to x\};$$

moreover the infima in formulas above are attained (more generally this holds in spaces with first countability axiom). Analogous definitions for families (F_ε) with $\varepsilon \to 0$.

Coerciveness: $F : X \to \overline{\mathbf{R}}$ is said coercive if for every $t \in \mathbf{R}$ there exists a compact subset K_t of X such that

$$\{F \le t\} \subset K_t.$$

Equi-coerciveness: A sequence (F_h) of functionals is said equi-coercive if for every $t \in \mathbf{R}$ there exists a compact subset K_t of X (independent of h) such that

$$\{F_h \le t\} \subset K_t \qquad \forall h \in \mathbf{N}.$$

The main properties of Γ-convergence are (see the book of Dal Maso [Birkhäuser]):

- •) (F_h) equicoercive, $F_h \overset{\Gamma}{\to} F \quad \Rightarrow \quad \min_X F = \lim_h (\inf_X F_h)$;
- •) $F_h \overset{\Gamma}{\to} F$, x_h minimizer of F_h, $x_h \to x \quad \Rightarrow \quad x$ minimizer of F;
- •) $F_h \overset{\Gamma}{\to} F$, x_h minimizer of F_h, (F_h) equicoercive, F has a unique minimum point $x \quad \Rightarrow \quad x_h \to x$ (and $F_h(x_h) \to F(x)$);
- •) $F_h \overset{\Gamma}{\to} F$, G continuous $\quad \Rightarrow \quad F_h + G \overset{\Gamma}{\to} F + G$;
- •) If X is separable the Γ-convergence is a compact convergence, in the sense that from every sequence (F_h) we may extract a subsequence (F_{h_k}) which Γ-converges.

Homogenization. Consider on the Sobolev space $W^{1,p}(\Omega)$ (with $1 < p < +\infty$) the family of functionals

$$F_\varepsilon(u) = \int_\Omega f(x/\varepsilon, Du)\,dx \qquad (\varepsilon \to 0)$$

where $f(x,z)$ satisfies the assumptions:

- •) $f(x,\cdot)$ convex on $\mathbf{R^n}$;
- •) $f(\cdot,z)$ measurable and Y-periodic;
- •) $|z|^p \le f(x,z) \le C(1+|z|^p)$.

Then $F_\varepsilon \overset{\Gamma}{\to} F$ in the weak $W^{1,p}(\Omega)$ convergence, where

$$F(u) = \int_\Omega f_0(Du)\,dx$$

and f_0 is given by the formula

$$f_0(z) = \inf\left\{\frac{1}{|Y|}\int_Y f(x, z + Dw(x))\,dx \ : \ w \in W^{1,p}_{per}\right\}.$$

When $f(x,z)$ is a quadratic form

$$f(x,z) = \sum_{i,j=1}^n a_{ij}(x) z_i z_j$$

then $f_0(z)$ is a quadratic form too

$$f_0(z) = \sum_{i,j} \alpha_{ij} z_i z_j$$

with constant coefficient α_{ij} which can be computed by the formula above.

Other variations on the theme can be made: for instance the Attouch & Buttazzo [Ann. SNS] case of "periodic reinforcement"

$$F_\varepsilon(u) = \int_\Omega |Du|^2\,dx + k\varepsilon \int_{\Omega\cap S_\varepsilon} |D_\tau u|^2\,d\sigma$$

where S_ε is the ε-rescaling of a $(n-1)$ dimensional manifold $S \subset Y$. The Γ-limit F is then

$$F(u) = \int_\Omega f(Du)\,dx$$

with

$$f(z) = \inf\left\{\frac{1}{|Y|}\int_Y |Dw|^2\,dx + \frac{k}{|Y|}\int_S |D_\tau w|^2\,d\sigma \;:\; w - \langle z,\cdot\rangle \in W^{1,2}_{per}\right\}.$$

The homogenization has been widely treated in the other courses of this school. Therefore, the program we intend to follow in these lectures is to show some applications of Γ-convergence different from periodic homogenization. More precisely we shall treat the following topics:
•) limits of periodic Riemannian metrics;
•) limits of singular perturbation problems;
•) a limit problem in phese transitions theory;
•) Γ-convergence and optimal control problems.

Lesson 2. Limits of sequences of Riemannian metrics

We shall study the limit (as $\varepsilon \to 0$) of the functionals

$$F_\varepsilon(u) = \int_0^1 \sum_{i,j=1}^n a_{ij}(\frac{u}{\varepsilon}) u_i' u_j'\,dt$$

where $\{a_{ij}\}$ are the coefficients of a Riemannian metric, or more generally in the so called "Finsler case"

$$F_\varepsilon(u) = \int_0^1 f(\frac{u}{\varepsilon}, u')\,dt$$

where $f : \mathbf{R^n} \times \mathbf{R^n} \to \mathbf{R}$ is a Borel function such that
•) $f(s,\cdot)$ is convex
•) $f(\cdot,z)$ is Y-periodic ($Y = [0,1[^n$)
•) $|z|^p \le f(s,z) \le C(1+|z|^p)$ with $p > 1$.

Theorem. *There exists a convex function φ with*

$$|z|^p \le \varphi(z) \le C(1+|z|^p)$$

such that $F_\varepsilon \xrightarrow{\Gamma} \Phi$ where

$$\Phi(u) = \int_0^1 \varphi(u')\,dt.$$

Moreover, φ is given by

$$\varphi(z)=\lim_{\varepsilon\to 0}\Big[\inf\big\{F_\varepsilon(w)\ :\ w\in W^{1,p}(0,1),\ w(0)=0,\ w(1)=z\big\}\Big]=$$
$$=\lim_{T\to+\infty}\left[\inf\left\{\frac{1}{T}\int_0^T f(w,w')\,dt\ :\ w\in W^{1,p}(0,1),\ w(0)=0,\ w(T)=Tz\right\}\right].$$

We prove the theorem in several steps; for some technical details we refer to the original paper by Acerbi & Buttazzo [JAM]. It will be convenient to localize all functionals by setting for every open subset A of $(0,1)$ (we denote by $\mathcal{A}$ such a class)

$$F_\varepsilon(u,A)=\int_A f(\frac{u}{\varepsilon},u')\,dt$$
$$\Phi(u,A)=\int_A \varphi(u')\,dt.$$

Step 1. There exists a sequence $\varepsilon_h\to 0$ such that for every open set A belonging to a countable base $\mathcal{U}$ of open sets in $(0,1)$ the sequence $F_{\varepsilon_h}(\cdot,A)$ Γ-converges to some Γ-limit we denote by $G(\cdot,A)$.

It is enough to apply the compactness property of Γ-convergence and a diagonal procedure.

Step 2. The sequence $F_{\varepsilon_h}(\cdot,A)$ Γ-converges for all $A\in\mathcal{A}$ to

$$F(u,A)=\sup\big\{G(u,B)\ :\ B\in\mathcal{B},\ B\subset\subset A\big\}.$$

See Acerbi & Buttazzo [JAM].

Step 3. The set function $A\mapsto F(u,A)$ is a measure for all $u\in W^{1,p}(0,1)$.

We prove only the key fact that $F(u,\cdot)$ is a subadditive set function, that is for every $A,B,C\in\mathcal{U}$ with $C\subset\subset A\cup B$ and every $u\in W^{1,p}(A\cup B)$

$$G(u,C)\le G(u,A)+G(u,B).$$

the remaining facts can be found in Acerbi & Buttazzo [JAM].

Let K be a compact subset of A containing $\overline{C}\setminus B$ in its interior, let $\delta=\operatorname{dist}(K,\partial A)$, let $\nu\in\mathbf{N}$ be a fixed integer number, and let for $i=1,\dots,\nu$

$$A_i=\Big\{t\in]0,1[\ :\ \operatorname{dist}(t,K)<i\frac{\delta}{\nu}\Big\}\qquad (A_0=\int K)$$
$$\varphi_i\in C_c^\infty(A_i),\quad 0\le\varphi_i\le 1,\quad \varphi_i=1\text{ on }A_{i-1},\quad |\varphi_i'|\le\frac{2\nu}{\delta}.$$

Moreover let $u_h\to u$ in $L^p(A)$, $v_h\to u$ in $L^p(B)$ be such that

$$G(u,A)=\lim_h F_{\varepsilon_h}(u_h,A)$$
$$G(u,B)=\lim_h F_{\varepsilon_h}(v_h,B).$$

Setting $w_{i,h}=\varphi_i u_h+(1-\varphi_i)v_h$ we have

$$\begin{aligned}F_{\varepsilon_h}(w_{i,h},C)&\le F_{\varepsilon_h}(u_h,A_{i-1})+F_{\varepsilon_h}(v_h,C\setminus A_i)+\\&\quad+C\int_{C\cap(A_i\setminus A_{i-1})}(1+|w_{i,h}'|^p)\,dt\le\\&\le F_{\varepsilon_h}(u_h,A)+F_{\varepsilon_h}(v_h,B)+C(\frac{\nu}{\delta})^p\int_C|u_h-v_h|^p\,dt+\\&\quad+C\int_{C\cap(A_i\setminus A_{i-1})}(1+|u_h'|^p+|v_h'|^p)\,dt.\end{aligned}$$

For every $h \in \mathbf{N}$ choose $i_h \le \nu$ such that

$$\int_{C\cap(A_{i_h}\setminus A_{i_h-1})}(1+|u_h'|^p+|v_h'|^p)\,dt \le \frac{1}{\nu}\int_{C\cap A\cap B}(1+|u_h'|^p+|v_h'|^p)\,dt \le$$
$$\le \frac{C}{\nu}[1+F_{\varepsilon_h}(u_h,A)+F_{\varepsilon_h}(v_h,B)]$$

so that

$$F_{\varepsilon_h}(w_{i_h,h},C) \le (1+\frac{C}{\nu})[F_{\varepsilon_h}(u_h,A)+F_{\varepsilon_h}(v_h,B)]+\frac{C}{\nu}+C(\frac{\nu}{\delta})^p\int_C |u_h-v_h|^p\,dt.$$

It is easy to see that $w_{i_h,h} \to u$ in $L^p(C)$, so that as $h \to +\infty$

$$G(u,C) \le \limsup_h F_{\varepsilon_h}(w_{i_h,h},C) \le$$
$$\le (1+\frac{C}{\nu})[G(u,A)+G(u,B)]+\frac{C}{\nu}$$

and the proof follows by letting now $\nu \to +\infty$.

Step 4. For every $a \in \mathbf{R^n}$ we have

$$F(u+a,A)=F(u,A).$$

Take $a_h \to a$ in $\mathbf{R^n}$ such that $a_h/\varepsilon_h \in \mathbf{Z^n}$ and take $u_h \to u$ such that

$$F(u,A)=\lim_h F_{\varepsilon_h}(u_h,A).$$

Then $u_h + a_h \to u + a$ and so

$$F(u+a,A) \le \liminf_h \int_A f(\frac{u_h}{\varepsilon_h}+\frac{a_h}{\varepsilon_h},u_h')\,dt =$$
$$= \liminf_h \int_A f(\frac{u_h}{\varepsilon_h},u_h')\,dt = F(u,A).$$

The opposite inequality can be proved in a similar way.

Step 5. There exists a convex function φ such that

$$F(u,A)=\int_A \varphi(u')\,dt.$$

This follows from the Buttazzo & Dal Maso [Nonl. An.] and [JMPA] integral representation theorem (valid also in the multiple integrals case):
Let $F : W^{1,p} \times \mathcal{A} \to \mathbf{R}$ be a functional such that
(i) $F(u,\cdot)$ is a measure (proved in Step 3);
(ii) $F(\cdot,A)$ is alower semicontinuous L^p (because it is a Γ-limit);
(iii) $\int_A |u'|^p\,dt \le F(u,A) \le C\int_A(1+|u'|^p)\,dt$ (by the assumptions on f);
(iv) F is local, i.e. $u=v$ a.e. on $B \Rightarrow F(u,B)=F(v,B)$ (we refer to Acerbi & Buttazzo [JAM] for the proof);
(v) $F(u+a,A)=F(u,A)$ for every $a \in \mathbf{R^n}$ (proved in Step 4).
Then there exists a function $\varphi(t,z)$ convex in z such that

$$F(u,A)=\int_A \varphi(t,u')\,dt.$$

The fact that in our case the function φ does not depend on t follows from the *translations invariance* of F (easy to prove):

$$F(u,A) = F(u_\tau, A+\tau) \qquad (u_\tau(t) = u(t-\tau)).$$

Step 6. Setting for every $z \in \mathbf{R^n}$ and $T > 0$

$$M_T(z) = \inf\left\{\frac{1}{T}\int_0^T f(u,u')\,dt \ :\ u \in W^{1,p}(0,T),\ u(0)=0,\ u(T)=Tz\right\}$$

there exists

$$\lim_{T\to+\infty} M_T(z) = M(z).$$

We refer to Acerbi & Buttazzo [JAM] for the proof.

Step 7. $M(z) = \varphi(z)$ for every $z \in \mathbf{R^n}$.
We refer to Acerbi & Buttazzo [JAM] for the proof.

We can conclude now the proof of the main Γ-convergence theorem because for every $\varepsilon_h \to 0$ we may extract (ε_{h_k}) such that $F_{\varepsilon_{h_k}}$ Γ-converges to some $\int_A \varphi(u')\,dt$ with φ possibly depending on the subsequence choosen. By Step 7 the function *varphi* is identified in a way which is independent of the subsequence choosen; therefore the entire (F_ε) Γ-converges to $\int_A \varphi(u')\,dt$.

Example. Let $n = 2$ and consider the chessboard structure with $f(s,z) = a(s)|z|^2$ (a is considered extended periodically). Therefore $f_\varepsilon(s,z) = a(s/\varepsilon)|z|^2$ corresponds to the Riemannian metric with coefficients $a(s/\varepsilon)\delta_{ij}$.

The rescaled coefficient $a(s)$

We know that as $\varepsilon \to 0$ the limit functional is of the form

$$\int_0^1 \varphi(u')\,dt$$

with φ convex. In the theorem above it is easy to prove that $f(s,z)$ is positively p-homogeneous in z so is $\varphi(z)$; then in our case $\varphi(z)$ is positively 2-homogeneous with

$$\alpha|z|^2 \le \varphi(z) \le \beta|z|^2.$$

The following fact hold.

- •) If $\alpha \neq \beta$ then $\varphi(z)$ is not a quadratic form (see Acerbi & Buttazzo [JAM]); therefore the variational limit of a sequence of Riemannian metrics may be not Riemannian but only a Finsler metric; the class of Finsler metrics on the contrary is closed under Γ-convergence.
- •) If β/α is large enough then the function φ depends only on α and has the form

$$\varphi(z) = \alpha\big((\sqrt{2}-1)|z_1| \wedge |z_2| + |z_1| \vee |z_2|\big)^2.$$

Lesson 3. Gamma-convergence for a class of singular perturbation problems

We want to study the asymtotic behaviour (in terms of Γ-convergence) of problems of the form

$$F_\varepsilon(u) = \int_\Omega f(x, u, \varepsilon Du, \varepsilon^2 D^2 u, \dots, \varepsilon^m D^m u)\, dx.$$

For instance the optimal control problem (u is the state, v is the control)

$$\min \int_\Omega \big(k|v|^2 + |u - u_0(x)|^p\big)\, dx$$

with state equation

$$\begin{cases} \varepsilon^2 \Delta u + g(u) = v \\ u \in H_0^1(\Omega) \end{cases}$$

reduces to the functional

$$\int_\Omega k|\varepsilon^2 \Delta u + g(u)|^2 + |u - u_0(x)|^p\, dx.$$

The first difficulty to overcome is the lack of equi-coerciveness in the Sobolev spaces; therefore we study the Γ-limit in the weak L^p topology.

We make the following assumptions on $f(x,s,z)$ where s represents u and z represents $(Du, D^2u, \dots, D^m u)$:

(i) there exist $a \in L^1,\ c \geq 1,\ p > 1,\ 1 \leq r \leq p$ such that

$$-a(x) + |s|^p \leq f(x,s,z) \leq a(x) + C[|s|^p + |z|^r];$$

(ii) there exist continuity moduli w and σ such that

$$|f(x,s,z) - f(y,t,w)| \leq w(x, |y-x|) + \sigma(|y-x| + |t-s| + |w-z|)(a(x) + f(x,s,z));$$

(iii) $f(x,s,z) + |s|^p + a(x) \geq \gamma(s,z)$ where γ is such that

$$\int_A \sum_{|\alpha| \leq m} |D^\alpha u|^r\, dx \leq \lambda(A, A') \int_{A'} \gamma(u, Du, \dots, D^m u)\, dx \qquad \forall A \subset\subset A'$$

where $\lambda(A, A')$ is such that

$$\lim_{t \to +\infty} \lambda(tA, tA') < +\infty.$$

For instance, if

$$F_\varepsilon(u) = \int_\Omega k|\varepsilon^2 \Delta u + g(u)|^2 + |u - u_0(x)|^p\, dx$$

the assumptions above are fulfilled with $(m = r = 2)$

$$f(x,s,z) = k|\sum_{i=1}^{n} z_{ii} + g(s)|^2 + |s - u_0(x)|^p$$
$$\gamma(s,z) = C_1 \left[\left|\sum_{i=1}^{n} z_{ii}\right|^2 + |s|^2\right]$$
$$\lambda(A,A') = C_2 \max\{1, \operatorname{dist}^{-4}(A, \partial A')\}$$

provided g is such that

$$|g(s)| \leq C(1 + |s|^{p/2})$$
$$|g(s) - g(t)| \leq \omega(t - s|)(1 + |s|^{p/2}).$$

Theorem. *There exists a function $\psi(x,s)$ convex in s such that*

$$F_\varepsilon(u,A) \overset{\Gamma}{\to} \int_A \psi(x,u)\,dx \qquad \textit{(weakly in } L^p(A))$$

for every $A \in \mathcal{A}$. Moreover

$$f_{s,z}^{**}(x,s,0) \leq \psi(x,s) \leq f_s^{**}(x,s,0)$$

where $f_{s,z}^{**}$ and f_s^{**} represent the convexification of f in (s,z) and in z respectively. A representation formula for ψ is $(Y =]0,1[^n)$

$$\psi(x,s) = \lim_\varepsilon \left[\inf\left\{F_\varepsilon(x,u) \;:\; \int_Y u\,dy = s\right\}\right] =$$
$$= \inf\left\{F_\varepsilon(x,u) \;:\; \varepsilon > 0, \; \int_Y u\,dy = s\right\}$$

where

$$F_\varepsilon(x,u) = \int_Y f(x, u(y), \varepsilon Du(y), \ldots, \varepsilon^m D^m u(y))\,dy.$$

We prove only the key fact that the $\Gamma - \limsup$ is subadditive, by referring for all other details to Buttazzo and Dal Maso [CRAS], [Ann. SNS].
Setting

$$F^+(u,A) = \inf\{\limsup_h F_{\varepsilon_h}(u_h, A) \;:\; u_h \to u \; wL^p\}$$

we have to prove for every $u \in L^p(A \cup B)$ and $C \subset\subset A \cup B$

$$F^+(u,C) \leq F^+(u,A) + F^+(u,B).$$

Fix $K = \overline{C} \setminus B$ and A_0, B_0 with $K \subset A_0 \subset\subset B_0 \subset\subset A$. Fix an integer ν and let $(A_i)_{1 \leq i \leq \nu}$ be such that $A_0 \subset\subset A_1 \subset\subset \ldots \subset\subset A_\nu \subset\subset B_0$. Denote by $S = C \cap (B_0 \setminus A_0)$ and by $S_i = C \cap (A_i \setminus \overline{A}_{i-1})$ and let $\varphi_i \in C_c^\infty(A_i)$ be such that $0 \leq \varphi_i 1$ and $\varphi_i = 1$ on A_{i-1}. We have

$$F^+(u,A) = \limsup_h F_{\varepsilon_h}(u_h, A)$$
$$F^+(u,B) = \limsup_h F_{\varepsilon_h}(v_h, B)$$

for suitable sequences (u_h) and (v_h) converging to u. Setting

$$w_{i,h} = \varphi_i u_h + (1-\varphi_i) v_h$$

we have

$$\begin{aligned}
F_{\varepsilon_h}(w_{i,h}, C) \le\; & F_{\varepsilon_h}(u_h, C\cap A_{i-1}) + F_{\varepsilon_h}(v_h, C\setminus \overline{A}_i) + \\
& + C\int_{S_i}\left[a(x) + |w_{i,h}|^p + \sum_{k=1}^m |\varepsilon_h^k D^k w_{i,h}|^r\right] dx \\
\le\; & F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) + \\
& + C\int_{S_i}\left[a(x) + |u_h|^p + |v_h|^p + \sum_{k=1}^m \varepsilon_h^{kr}(|D^k u_h|^r + |D^k v_h|^r)\right] dx \\
& + C_\nu \int_{S_i}\sum_{k=1}^m \varepsilon_h^{kr}\sum_{j=0}^{k-1}(|D^j u_h|^r + |D^j v_h|^r)\, dx
\end{aligned}$$

where C_ν depends on $\|\varphi_i\|_{C^m}$ for $i = 1, \ldots, \nu$. Let i_h be an index such that

$$\int_{S_{i_h}}\left[a(x) + |u_h|^p + |v_h|^p + \sum_{k=1}^m \varepsilon_h^{kr}(|D^k u_h|^r + |D^k v_h|^r)\right] dx \le$$
$$\frac{1}{\nu}\int_S \left[a(x) + |u_h|^p + |v_h|^p + \sum_{k=1}^m \varepsilon_h^{kr}(|D^k u_h|^r + |D^k v_h|^r)\right] dx$$

and set $w_h = w_{i_h,h}$. Then $w_h \to u$ and

$$\begin{aligned}
F_{\varepsilon_h}(w_h, C) \le\; & F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) + \\
& + \frac{C}{\nu}\int_S \left[a(x) + |u_h|^p + |v_h|^p + \sum_{k=1}^m \varepsilon_h^{kr}(|D^k u_h|^r + |D^k v_h|^r)\right] dx + \\
& + C_\nu \int_S \sum_{k=1}^m \varepsilon_h^{kr}\sum_{j=0}^{k-1}(|D^j u_h|^r + |D^j v_h|^r)\, dx.
\end{aligned}$$

As $h \to +\infty$

$$\begin{aligned}
F^+(u, C) \le \limsup_h F_{\varepsilon_h}(w_h, C) \le\; & F^+(u, A) + F^+(u, B) + \\
& + \frac{C}{\nu} + \frac{C}{\nu}\limsup_h \int_S \sum_{k=1}^m \varepsilon_h^{kr}(|D^k u_h|^r + |D^k v_h|^r)\, dx + \\
& + C_\nu \limsup_h \int_S \sum_{k=1}^m \varepsilon_h^{kr}\sum_{j=0}^{k-1}(|D^j u_h|^r + |D^j v_h|^r)\, dx.
\end{aligned}$$

Set now $U_h(x) = u_h(\varepsilon_h x)$ and $V_h(x) = v_h(\varepsilon_h x)$; then, for every $S' \subset\subset A\cap B$

$$\begin{aligned}
& \int_{S'}\sum_{k=1}^m \varepsilon_h^{kr}(|D^k u_h|^r + |D^k v_h|^r)\, dx = \varepsilon_h^m \int_{S'/\varepsilon_h}\sum_{k=1}^m (|D^k U_h|^r + |D^k V_h|^r)\, dx \le \\
& \le \varepsilon_h^n \lambda(S'/\varepsilon_h, \frac{A\cap B}{\varepsilon_h})\int_{(A\cap B)/\varepsilon_h}[\gamma(U_h, \ldots, D^m U_h) + \gamma(V_h, \ldots, D^m V_h)]\, dx = \\
& = \lambda(S'/\varepsilon_h, \frac{A\cap B}{\varepsilon_h})\int_{A\cap B}[\gamma(u_h, \ldots, \varepsilon_h^m D^m u_h) + \gamma(v_h, \ldots, \varepsilon_h^m D^m v_h)]\, dx \le \\
& \le \lambda(S'/\varepsilon_h, \frac{A\cap B}{\varepsilon_h})\int_{A\cap B}[2a(x) + |u_h|^p + |v_h|^p + f(x, u_h, \ldots, \varepsilon_h^m D^m u_h) + \\
& \quad + f(x, v_h, \ldots, \varepsilon_h^m D^m v_h)]\, dx \le \\
& \le \lambda(S'/\varepsilon_h, \frac{A\cap B}{\varepsilon_h})\int_{A\cap B}[C + F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B)] \le C.
\end{aligned}$$

By using inequalities as

$$\int_S |D^k w|^r \, dx \le \sigma \int_{S'} |D^m w|^r \, dx + C_\nu \int_{S'} |w|^r \, dx$$

for every $1 \le k \le m-1$ and every $\sigma > 0$, where $S \subset\subset S'$, we have

$$\begin{aligned} C_\nu \sum_{k=1}^m \varepsilon_h^{kr} \sum_{j=0}^{k-1} \int_S |D^j u_h|^r \, dx \le \\ \le C_\nu \sum_{k=1}^m \varepsilon_h^{kr} \int_{S'} [\sigma |D^k u_h|^r + C_\sigma |u_h|^r] \, dx \le \\ \le \sigma C_\nu \int_{S'} \sum_{k=1}^m \varepsilon_h^{kr} |D^k u_h|^r \, dx + \varepsilon_h C_\nu C_\sigma \end{aligned}$$

and analogously for v_h. Therefore, arguing as before we obtain

$$C_\nu \limsup_h \int_S \sum_{k=1}^m \varepsilon^{kr} \sum_{j=0}^{k-1} (|D^j u_h)^r + |D^j v_h|^r) \, dx \le \sigma C_\nu$$

so that

$$F^+(u, C) \le F^+(u, A) + F^+(u, B) + \frac{C}{\nu} + \sigma C_\nu$$

and the subadditivity follows by taking first the limit as $\sigma \to 0$ and then the limit as $\nu \to +\infty$.

Once the subadditivity is proved, standard methods prove that the Γ-limit $F(u, \cdot)$ is a measure, and by the Buttazzo and Dal Maso [RM] integral representation theorem

$$F(u, A) = \int_A \psi(x, u) \, dx$$

for a suitable $\psi(x, s)$ convex in s. The inequality $f_{s,z}^{**}(x, s, 0) \le \psi(x, s)$ is trivial because if $u_h \to u$ weakly L^p

$$\begin{aligned} \int_A f_{s,z}^{**}(x, u, 0) \, dx \le \liminf_h \int_A f_{s,z}^{**}(x, u_h, \varepsilon_h D u_h, \ldots, \varepsilon_h^m D^m u_h) \, dx \le \\ \le \liminf_h \int_A f(x, u_h, \varepsilon_h D u_h, \ldots, \varepsilon_h^m D^m u_h) \, dx \end{aligned}$$

and so

$$\int_A f_{s,z}^{**}(x, u, 0) \, dx \le \int_A \psi(x, u) \, dx.$$

For the inequality $\psi(x, s) \le f_s^{**}(x, s, 0)$ it is enough to show that

$$\int_A \psi(x, u) \, dx \le \int_A f(x, u, 0) \, dx$$

for every u. This can be proved by taking $u_h = \rho_h * u$ where $\rho_h(x) = \varepsilon_h^{-n\theta} \rho(\varepsilon_h^{-\theta} x)$; if θ is small enough ($\theta < 1/n+1$) we have $u_h \to u$ strongly in L^p and $\varepsilon_h^k D^k u_h \to 0$ strongly inL^p, so that

$$\int_A \psi(x, u) \, dx \le \liminf_h \int_A f(x, u_h, \ldots, \varepsilon_h^m D^m u_h) \, dx = \int_A f(x, u, 0) \, dx.$$

In the case $f(x, s, z) = k|\sum_{i=1}^n z_{ii} + g(s)|^2 + |s - u_0(x)|^p$ it is possible to prove (see Buttazzo & Dal Maso [Ann. SNS])

- •) g affine $\Rightarrow$ $\psi(x,s) = f_s^{**}(x,s,0) = k|g(s)|^2 + |s - u_0(x)|^p$;
- •) g decreasing $\Rightarrow$ $\psi(x,s) = f_s^{**}(x,s,0) = k|g(s)|^2 + |s - u_0(x)|^p$;
- •) the equality $\psi(x,s) = f_s^{**}(x,s,0)$ is not true in general;
- •) $g \geq 0$ convex $\Rightarrow$ $\psi(x,s) = k|g(s)|^2 + |s - u_0(x)|^p$.

Note that in the case

$$F_\varepsilon(u) = \int_\Omega [\varepsilon^2 |Du|^2 + W(u)]\, dx$$

we have $f(s,z) = |z|^2 + W(s)$ so that

$$f_{s,z}^{**}(s,z) = f_s^{**}(s,z) = |z|^2 + W^{**}(s).$$

Hence $\psi(x,s) = W^{**}(s)$.

Lesson 4. A limit problem in phase transitions theory

Let $W : \mathbf{R} \to \mathbf{R}$ be a positive continuous function with only two zeros (say at -1 and at 1); consider the functionals

$$F_\varepsilon(u) = \int_\Omega \left[\varepsilon |Du|^2 + \frac{1}{\varepsilon} W(u)\right] dx$$

where Ω is a bounded open Lipschitz subset of $\mathbf{R^n}$. We shall prove that the Γ-limit (as $\varepsilon \to 0$) in the topology $L^1(\Omega)$ is

$$F(u) = \begin{cases} C_0 \displaystyle\int_\Omega |Du| & \text{if } |u(x)| = 1 \text{ for a.e. } x \in \Omega \\ +\infty & \text{otherwise} \end{cases}$$

defined for all $u \in BV(\Omega)$, where $C_0 = 2\int_{-1}^1 \sqrt{W(s)}\, ds$.

It is convenient to introduce the function

$$\phi(t) = \int_0^t \sqrt{W(s)}\, ds$$

and to write $F(u)$ for $|u| \equiv 1$ as

$$F(u) = 2 \int_\Omega |D(\phi \circ u)|.$$

Then the inequality

$$F(u) \leq \Gamma - \liminf_\varepsilon F_\varepsilon(u)$$

is rather easy to prove. Indeed, when $|u| \not\equiv 1$ we have

$$\liminf_\varepsilon F_\varepsilon(u_\varepsilon)) \geq \liminf_\varepsilon \frac{1}{\varepsilon} \int_\Omega W(u_\varepsilon)\, dx = +\infty$$

whereas if $|u| \equiv 1$

$$\liminf_\varepsilon F_\varepsilon(u_\varepsilon) \geq \liminf_\varepsilon \int_\Omega 2|Du_\varepsilon| \sqrt{W(u_\varepsilon)}\, dx =$$
$$= \liminf_\varepsilon \int_\Omega 2|D(\phi \circ u_\varepsilon)|\, dx \geq 2 \int_\Omega |D(\phi \circ u)|$$

where the first inequality follows from the standard $a^2 + b^2 \geq 2ab$ and the last one from the lower semicontinuity of the total variation functional.

The approximating sequence

We prove now the opposite inequality

$$F(u) \geq \Gamma \limsup_{\varepsilon} F_\varepsilon(u)$$

only for functions u of the form $-1_A + 1_{\Omega \setminus A}$ where A is an open set with a smooth boundary Σ transversal to $\partial\Omega$. We refer to the original papers of Modica & Mortola [BUMI], [BUMI] for the proof that from this particular case we can deduce, by a density argument, the general case.

We want to construct an approximating sequence u_ε as in the picture, where the thickness of the transition layer and the transition itself have to be suitably choosen.

Set for every $t \in \mathbf{R}$

$$\psi_\varepsilon(t) = \int_{-1}^{t} \frac{\varepsilon}{\sqrt{\varepsilon + W(s)}}\, ds$$

$$\varphi_\varepsilon(t) = \begin{cases} -1 & \text{if } t \leq 0 \\ \psi_\varepsilon^{-1}(t) & \text{if } 0 \leq t \leq \psi_\varepsilon(1) \\ 1 & \text{if } t \geq \psi_\varepsilon(1) \end{cases}$$

and, if $d(x) = dist(x, A)$

$$u_\varepsilon(x) = \varphi_\varepsilon(d(x)).$$

We have $u_\varepsilon \to u$ in $L^1(\Omega)$ and, if $\Sigma_\varepsilon = \{x \in \Omega \;:\; 0 < d(x) < \psi_\varepsilon(1)\}$

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_\Omega \left[\varepsilon |\varphi_\varepsilon^1(d(x))|^2 + \frac{1}{\varepsilon} W(\varphi_\varepsilon(d(x))) \right] dx = \\ &= \int_{\Sigma_\varepsilon} \left[\varepsilon |\varphi_\varepsilon^1(d)|^2 + \frac{1}{\varepsilon} W(\varphi_\varepsilon(d)) \right] dx = \\ &= |\Sigma| \int_0^{\psi_\varepsilon(1)} \left[\varepsilon |\varphi_\varepsilon^1(t)|^2 + \frac{1}{\varepsilon} W(\varphi_\varepsilon(t)) \right] dt. \end{aligned}$$

Since

$$\varphi_\varepsilon' = \frac{1}{\psi_\varepsilon'(\psi_\varepsilon^{-1})} = \frac{\sqrt{\varepsilon + W(\psi_\varepsilon^{-1})}}{\varepsilon} = \frac{1}{\varepsilon}\sqrt{\varepsilon + W(\varphi_\varepsilon)}$$

we get

$$F_\varepsilon(u_\varepsilon) = |\Sigma| \int_0^{\psi_\varepsilon(1)} \left[\frac{\varepsilon + W(\varphi_\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon} W(\varphi_\varepsilon) \right] dt \le$$
$$\le \frac{2|\Sigma|}{\varepsilon} \int_0^{\psi_\varepsilon(1)} [\varepsilon + W(\varphi_\varepsilon)] \frac{dt}{d\varphi_\varepsilon} d\varphi_\varepsilon$$
$$= 2|\Sigma| \int_{-1}^{1} \sqrt{\varepsilon + W(s)}\, ds.$$

Therefore

$$\limsup_\varepsilon F_\varepsilon(u_\varepsilon) \le C_0 |\Sigma|.$$

Other cases have been considered in the Modica and Mortola paper; for instance if W is periodic and $t_\varepsilon \to +\infty$

$$F_\varepsilon(u) = \int_\Omega \left[\varepsilon |Du|^2 + \frac{1}{\varepsilon} W(t_\varepsilon u) \right] dx$$

Γ-converge to

$$F(u) = C_0 \int_\Omega |Du| \qquad (\forall u \in BV(\Omega))$$

where (T is the period of W)

$$C_0 = \frac{2}{T} \int_0^T \sqrt{W(s)}\, ds.$$

Lesson 5. Gamma-convergence in optimal control theory

The abstract framework is the following:
-) Y space of states;
-) U space of controls;
-) $J(u, y)$ cost functional;
-) $E \subset U \times Y$ admissible set given by the state equation.

The optimal control problem is then

$$\min\{J(u, y) \ : \ (u, y) \in E\}$$

or equivalently

$$\min\{F(u, y) \ : \ (u, y) \in U \times Y\} \qquad \text{where } F = J + \chi_E.$$

When we deal with sequences of problems

$$\min\{F_\varepsilon(u, y) \ : \ (u, y) \in U \times Y\} \qquad \text{where } F_\varepsilon = J_\varepsilon + \chi_{E_\varepsilon}$$

we have to study the Γ-convergence of F_ε in the product space $U \times Y$.

The typical case is:
-) $U = L^p(0, T; \mathbf{R^m})$ topology $w - L^p$;
-) $Y = W^{1,1}(0, T; \mathbf{R^m})$ topology strong L^∞;
-) $J_\varepsilon(u, y) = \int_0^T f_\varepsilon(t, y, u)\, dt$;
-) $E_\varepsilon = \{y' = a_\varepsilon(t, y) + b_\varepsilon(t, y)u, \ y(0) = \xi_\varepsilon\}$.

We would like to study the Γ-limits of J_ε and of χ_{E_ε} separatly, but the equality

$$\Gamma \lim F_\varepsilon = \Gamma \lim J_\varepsilon + \Gamma \lim \chi_{E_\varepsilon}$$

is false in general. To bypass this difficulty we introduce the multiple Γ-limits for functions on a product space.

$$\Gamma(U^-,Y^-)\liminf_\varepsilon F_\varepsilon(u,y)=\inf_{u_\varepsilon\to u}\inf_{y_\varepsilon\to y}\liminf_\varepsilon F_\varepsilon(u_\varepsilon,y_\varepsilon)$$
$$\Gamma(U^-,Y^+)\liminf_\varepsilon F_\varepsilon(u,y)=\inf_{u_\varepsilon\to u}\sup_{y_\varepsilon\to y}\liminf_\varepsilon F_\varepsilon(u_\varepsilon,y_\varepsilon)$$
$$\Gamma(U^+,Y^-)\liminf_\varepsilon F_\varepsilon(u,y)=\sup_{u_\varepsilon\to u}\inf_{y_\varepsilon\to y}\liminf_\varepsilon F_\varepsilon(u_\varepsilon,y_\varepsilon)$$
$$\Gamma(U^+,Y^+)\liminf_\varepsilon F_\varepsilon(u,y)=\sup_{u_\varepsilon\to u}\sup_{y_\varepsilon\to y}\liminf_\varepsilon F_\varepsilon(u_\varepsilon,y_\varepsilon)$$

and analogously for the Γ-limits with lim sup. When two of them coincide we use notations as

$$\Gamma(U,Y^-)\liminf_\varepsilon F_\varepsilon,\quad \Gamma(U,Y)\limsup_\varepsilon F_\varepsilon,\quad \Gamma(U^-,Y)\lim_\varepsilon F_\varepsilon.$$

In this way it is possible to sum with the Γ-limits. More precisely we have:

$$\Gamma(U^-,Y^-)\lim_\varepsilon(F_\varepsilon+G_\varepsilon)=\Gamma(U^-,Y)\lim_\varepsilon F_\varepsilon+\Gamma(U,Y^-)\lim_\varepsilon G_\varepsilon$$

(see Buttazzo and Dal Maso [JOTA]). Since the Γ-limits which we want to study is the

$$\Gamma(U^-,Y^-)\lim_\varepsilon(J_\varepsilon+\chi_{E_\varepsilon})$$

we have to identify the limits

$$\Gamma(U^-,Y)\lim_\varepsilon J_\varepsilon$$
$$\Gamma(U,Y^-)\lim_\varepsilon \chi_{E_\varepsilon}.$$

We restrict our analysis to the case (for other cases see Buttazzo and Dal Maso [JOTA])

$$J_\varepsilon(u,y)=\int_0^T f_\varepsilon(t,y,u)\,dt$$
$$E_\varepsilon=\{y'=a_\varepsilon(t,y)+b_\varepsilon(t,y)u,\ y(0)=\xi_\varepsilon\}.$$

Case when b_ε is strongly convergent.

Assumptions on $f_\varepsilon :]0,T[\times\mathbf{R^n}\times\mathbf{R^m}\to\overline{\mathbf{R}}$ Borel functions:

(i) $f_\varepsilon(t,s,\cdot)$ is convex and l.s.c. on $\mathbf{R^m}$;
(ii) $f_\varepsilon(t,s,z)\ge|z|^p \qquad (p>1)$;
(iii) for every $R>0$ there exists a continuity modulus ω_R such that

$$|f_\varepsilon(t,s_1,z)-f_\varepsilon(t,s_2,z)|\le\omega_R(|s_1-s_2|)(1+f_\varepsilon(t,s,z))$$

for every $t\in]0,T[$, $z\in\mathbf{R^m}$, $s_1,s_2\in\mathbf{R^n}$ with $|s_1|,|s_2|\le R$;
(iv) there exists $u_\varepsilon\in L^p$ such that $f_\varepsilon(t,0,u_\varepsilon(t))$ is weakly compact in L^1.

Then the $\Gamma(U^-,Y)\lim_\varepsilon J_\varepsilon$ can be computed in the following way (see Marcellini and Sbordone [Ric. Mat. 1977]: for every $s\in\mathbf{R^n}$ and $z^*\in\mathbf{R^m}$

$$\varphi(\cdot,s,z^*)=w-L^1\lim_\varepsilon f_\varepsilon^*(\cdot,s,z^*)$$
$$f(t,s,z)=\varphi^*(t,s,z)$$
$$\Gamma(U^-,Y)\lim_\varepsilon J_\varepsilon(u,y)=\int_0^T f(t,y,u)\,dt.$$

For instance if $f_\varepsilon(t,s,z)=a_\varepsilon(t)|z|^2+|s-y_0(t)|^2$ we have $f(t,s,z)=a(t)|z|^2+|s-y_0(t)|^2$ where

$$\frac{1}{a_\varepsilon}\to\frac{1}{a}\quad\text{weakly in}\quad L^1(0,T).$$

Concerning the differential state equations we assume:
(i) $|a_\varepsilon(t,s_1)-a_\varepsilon(t,s_2)|\le\alpha_\varepsilon(t)|s_1-s_2|$ with $\sup_\varepsilon\int_0^T\alpha_\varepsilon\,dt<+\infty$;
(ii) $|b_\varepsilon(t,s_1)-b_\varepsilon(t,s_2)|\le\beta_\varepsilon(t)|s_1-s_2|$ with $\sup_\varepsilon\int_0^T\beta_\varepsilon^{p'}\,dt<+\infty$;
(iii) $\sup_\varepsilon\int_0^T|a_\varepsilon(t,0)|dt<+\infty$;
(iv) $\sup_\varepsilon\int_0^T|b_\varepsilon(t,0)|^{p'}dt<+\infty$;
(v) $a_\varepsilon(\cdot,s)\to a(\cdot,s)$ weakly in L^1 $\quad\forall s\in\mathbf{R^n}$;
(vi) $b_\varepsilon(\cdot,s)\to b(\cdot,s)$ strongly in $L^{p'}$ $\quad\forall s\in\mathbf{R^n}$;
(vii) $\xi_\varepsilon\to\xi$ in $\mathbf{R^n}$.
Then $\Gamma(U,Y^-)\lim_\varepsilon\chi_{E_\varepsilon}=\chi_R$ where

$$E=\{y'=a(t,y)+b(t,y)u,\ y(0)=\xi\}.$$

Therefore the limit control problems is

$$\min\left\{\int_0^T f(t,y,u)\,dt\ :\ y'=a(t,y)+b(t,y)u,\ y(0)=\xi\right\}.$$

Case when b_ε is only weakly convergent.

Assume for the sake of simplicity that $b_\varepsilon=b_\varepsilon(t)$ and that (vi) is substituted by
(vi') $b_\varepsilon\to b$ weakly in $L^{p'}$.
The simplest situation is when $|b_\varepsilon|^{p'}$ is equi-uniformly integrable (we shall remove later this assumption). In this case it is convenient to introduce an auxiliary variable $v\in V=L^1(0,T)$ and rewrite the control problems in the form

$$\min\left\{\int_0^T[f_\varepsilon(t,y,u)+\chi_{v=b_\varepsilon(t)u}]\,dt\ :\ y'=\frac{a}{\varepsilon}(t,y)+v,\ y(0)=\xi_\varepsilon\right\}.$$

We can now apply the previous analysis with

$$\begin{aligned}&\widetilde{Y}=Y\\&\widetilde{U}=U\times V\\&\widetilde{f}_\varepsilon(t,s,z,w)=f_\varepsilon(t,s,z)+\chi_{w=b_\varepsilon(t)z}\\&\widetilde{a}_\varepsilon(t,s)=a_\varepsilon(t,s)\\&\widetilde{b}_\varepsilon(t,s)\cdot(z,w)=w\end{aligned}$$

obtaining as a limit problem

$$\min\left\{\int_0^T\widetilde{f}(t,y,u,v)\,dt\ :\ y'=a(t,y)+v,\ y(0)=\xi\right\}$$

being

$$\widetilde{f}(t,s,z,w)=(w-L^1\lim_\varepsilon(f_\varepsilon(t,s,z)+\chi_{w=b_\varepsilon(t)z})^*)^*$$

where the $*$ operator is now made with respect to the variables (z, w). Finally we eliminate the variable v by solving $v = y' - a(t, y)$ and plugging into the cost functional

$$\min\left\{\int_0^T \widetilde{f}(t, y, u, y' - a(t, y))\, dt \ :\ y(0) = \xi\right\}.$$

Note that

$$\left(f_\varepsilon(t, s, z) + \chi_{w=b_\varepsilon(t)z}\right)^* (t, s, z^*, w^*) = f_\varepsilon^*(t, s, z^* + b_\varepsilon(t)w^*)$$

and in some cases the function $\widetilde{f}$ is finite everywhere, that is the state equation may disappear in the limit problem. Consider for instance the case

$$f_\varepsilon(t, s, z) = |z|^2 + |s - y_0(t)|^2 \qquad \text{(for every } \varepsilon\text{)}$$

and

$$\begin{cases} y' = a_\varepsilon(t, y) + b_\varepsilon(t)u \\ y(0) = \xi_\varepsilon \end{cases}$$

with $a_\varepsilon(\cdot, s)$ weakly L^1 convergent to $a(\cdot, s)$ and $b_\varepsilon \to b$ weakly L^2 with $b_\varepsilon^2 \to \beta^2$ weakly L^1. Then some easy computations give

$$\widetilde{f}(t, s, z, w) = |z|^2 + \frac{(w - b(t)z)^2}{\beta^2(t) - b^2(t)}$$

so that the limit problem is

$$\min\left\{\int_0^T \left[|u|^2 + |y - y_0(t)|^2 + \frac{|y' - a(t, y) - b(t)u|^2}{\beta^2(t) - b^2(t)}\right] dt \ :\ y(0) = \xi\right\}$$

and the relaxed form of the limit state equation is now in a penalization term.

For instance $b_\varepsilon(t) = \sin(t/\varepsilon)$ gives $b \equiv 0$, $\beta^2 \equiv 1/2$ so that the limit problem becomes

$$\min\left\{\int_0^T [|u|^2 + |y - y_0|^2 + 2|y' - a(t, y)|^2]\, dt \ :\ y(0) = \xi\right\}.$$

We want now to drop the assumption that $|b_\varepsilon|^{p'}$ is equi-uniformly integrable. In this case we may only obtain (up to extracting subsequences) that $|b_\varepsilon|^{p'}$ converges to a suitable measure μ in the weak* convergence of measures. Assume for simplicity that the cost integrands are of the form

$$f_\varepsilon(t, s, z) = \varphi_\varepsilon(t, z) + \psi(t, s).$$

In this case the limit problem is expressed by means of the measure μ in the following way (see Buttazzo and Freddi [AMSA]). As before consider the auxiliary variable $v = b_\varepsilon(t)u$ and the polar integrand (with respect to (z, w))

$$\left(\varphi_\varepsilon(t, z) + \chi_{w=b_\varepsilon(t)z}\right)^* (t, z^*, w^*).$$

It is possible to show that (up to subsequences) this integrand converges weakly* in $\mathcal{M}(\overline{\Omega})$ to a measure of the form

$$g(t, z^*, w^*) \cdot \nu \qquad (\text{with } \nu = dt + \mu_s)$$

where $g(t, \cdot, \cdot)$ is convex. Then the limit problem is with cost

$$\int_{\overline{\Omega}} g^*(t, u, \frac{dv}{d\nu})\, d\nu + \int_\Omega \psi(t, y)\, dt + \chi_{\{v << \nu\}}$$

and state equation

$$\begin{cases} y' = a(t,y) + v & \text{(in the sense of } \mathcal{M}(\overline{\Omega})) \\ y(0) = \xi. \end{cases}$$

Eliminating the variable v (which varies in the space of measures) we obtain again that the limit differential state equation may disappear becoming a penalization:

$$\int_\Omega g^*(t,u,y'_r - a(t,y))\,dt + \int_\Omega g^*(t,0,\frac{dy'_s}{d\mu_s})\,d\mu_s +$$
$$+ g^*(0,0,\frac{y(0^+)-\xi}{\mu(\{0\})})\mu(\{0\}) + \int_\Omega \psi(t,y)\,dt + \chi_{\{y'_s << \mu_s\}}$$

where $y' = y'_r \cdot dt + y'_s$ is the decomposition of the measure y' into absolutely continuous and singular parts with respect to the Lebesgue measure dt, and the last term is the constraint that y'_s must be absolutely continuous with respect to μ_s.

In the previous example

$$f_\varepsilon(t,s,z) = |z|^2 + |s - y_0(t)|^2 \qquad \begin{cases} y' = a_\varepsilon(t,y) + b_\varepsilon(t)u \\ y(0) = \xi_\varepsilon \end{cases}$$

with $a_\varepsilon(\cdot,s) \to a(\cdot,s)$ weakly L^1, $b_\varepsilon \to b$ weakly L^2 but now $b_\varepsilon^2 \to \mu$ weakly* in the sense of measures, we get at the limit

$$\int_0^T \left[|u|^2 + |y - y_0(t)|^2 + \frac{|y'_r - a(t,y) - b(t)u|^2}{\mu_r(t) - b^2(t)}\right] dt +$$
$$+ \int_{]0,T[} \left|\frac{dy'_s}{d\mu_s}\right|^2 + \frac{|y(0^+) - \xi|^2}{\mu(\{0\})} + \chi_{\{y'_s << \mu_s\}}.$$

For instance, if $b_\varepsilon(t) = sin(t/\varepsilon) + \frac{1}{\sqrt{\varepsilon}} 1_{]0,\varepsilon[}(t)$ we have $b \equiv 0$ and $\mu = \frac{1}{2}dt + \delta_0$ so that the limit problem is

$$\min_{u \in L^2,\ y \in W^{1,1}} \left\{ \int_0^T [|u|^2 + |y - y_0(t)|^2 + 2|y' - a(t,y)|^2]\,dt + |y(0^+) - \xi|^2 \right\}.$$

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