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An Introduction to Homogenization and Gamma-convergence

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CONTENTS

This paper contains the abstract of five lectures conceived as an introduction to Γ-convergence methods in the theory of Homogenization, and delivered on September 8–10, 1993 as part of the "School on Homogenization" at the ICTP, Trieste. Its content is strictly linked and complementary to the subject of the courses held at the same School by A. Defranceschi and G. Buttazzo. Prerequisites are some basic knowledge of functional analysis and of Sobolev spaces (as a reference we shall use the books by Adams [3] and Ziemer [29]; see also the Appendix to the Lecture Notes by A. Defranceschi in this volume). A list of notations can be found at the end of this paper.

Lesson One. Gamma-convergence for Integral Functionals

1.1. Introduction

The subject of these lectures is the study of the asymptotic behaviour as ε goes to 0 of integral functionals of the form

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, Du(x))\,dx, \tag{1.1}$$

defined on some (subset of a) Sobolev space $W^{1,p}(\Omega;\mathbb{R}^N)$ (in general, of vector-valued functions), when $f = f(y,\xi)$ is a Borel function, (almost) periodic in the variable y, and satisfying the so-called "natural growth" conditions with respect to the variable ξ. Integrals of this form model various phenomena in Mathematical Physics in the presence of microstructures (the vanishing parameter ε gives the microscopic scale of the media). The function f represents the energy density at this lower scale. As an example we can think of u representing a deformation, and $\mathcal{F}_\varepsilon$ being the stored energy of a cellular elastic material with Ω as a reference configuration. In other applications u is a difference of potential in a condenser composed of periodically distributed material, occupying the region Ω, etc.

The main question we are going to answer is: *does the (medium modeled by the) energy $\mathcal{F}_\varepsilon$ behave as a homogeneous medium in the limit?* (and if so: can we say something about this homogeneous limit?)

First we have to give a precise meaning to this statement. The behaviour of the media described by the integral in (1.1) is given by the behaviour of boundary value problems of the Calculus of Variations of the form

$$\min\Big\{\int_\Omega f(\frac{x}{\varepsilon}, Du(x))\,dx + \int_\Omega gu\,dx \ : u = \phi \text{ on } \gamma_0\Big\}, \tag{1.2}$$

where g is some fixed function, and γ_0 is a portion of $\partial\Omega$ (here we suppose Ω sufficiently smooth). If our media behave as a homogeneous medium when ε tends to 0, we expect that there exists a function $f_{\rm hom}$ (representing the energy density of the latter), which is now "homogeneous", that is, independent on the variable x, such that the minima of the problems in (1.2) converge as $\varepsilon \to 0$ to the minimum of the problem

$$\min\Big\{\int_\Omega f_{\rm hom}(Du(x))\,dx + \int_\Omega gu\,dx\ : u = \phi \text{ on } \gamma_0\Big\}, \tag{1.3}$$

and, what is important, the function $f_{\rm hom}$ does not depend on Ω and on the particular choice we make of g, ϕ and γ_0.

The convergence of these minimum values (and, in some weak sense, also of the minimizing functions in (1.2) to the minimizer of (1.3)) will be obtained as a consequence of the convergence of the functionals $\mathcal{F}_\varepsilon$ to the *homogenized functional*

$$\mathcal{F}_{\rm hom}(u) = \int_\Omega f_{\rm hom}(Du(x))\,dx \tag{1.4}$$

in the variational sense of Γ-convergence, which was introduced by E. De Giorgi in the 70s exactly for dealing with problems of this kind. Special relevance will be given to the illustration of the general method, which can be applied, with the due changes, to the study of other types of functionals, different than those defined on Sobolev spaces of the form (1.1) (for example, with essentially the same proof we can obtain a homogenization result for functionals with volume and surface energies (see [11])). In this spirit, many results have been simplified for expository purposes; more technical and general theorems can be found in the papers cited as references.

1.2. Γ-convergence

The notion of Γ-convergence was introduced in a paper by E. De Giorgi and T. Franzoni in 1975 [18], and was since then much developed especially in connection with applications to problems in the calculus of variation. We refer to the recent book by Dal Maso [15] for a comprehensive introduction to the subject. Here we shall be interested mainly in applications to the asymptotic behaviour of minimum problems for integral functionals defined on Sobolev spaces.

First we shall give an abstract definition of Γ-convergence on a metric space.

Definition 1.1. Let $X = (X, d)$ be a metric space, and for every $h \in \mathbb{N}$ let $F_h : X \to [0, +\infty]$ be a function defined on X. We say that the sequence (F_h) $\Gamma(d)$*-converges* in $x_0 \in X$ to the value $r \in [0, +\infty]$ (and we write $r = \Gamma(d)\text{-}\lim_h F_h(x_0)$) if we have:

(i) for every sequence x_h such that $d(x_h, x_0) \to 0$ we have

$$r \le \liminf_h F_h(u_h); \tag{1.5}$$

(ii) there exists a sequence $\overline{x}_h$ such that $d(\overline{x}_h, x_0) \to 0$, and we have

$$r = \lim_h F_h(u_h) \tag{1.6}$$

(or, equivalently, $r \geq \limsup_h F_h(u_h)$).

If the $\Gamma(d)$-limit $\Gamma(d)\text{-}\lim\limits_h F_h(x)$ exists for all $x \in X$, and the function $F : X \to [0, +\infty]$ verifies $F(x) = \Gamma(d)\text{-}\lim\limits_h F_h(x)$ for all $x \in X$, then we say that the sequence (F_h) $\Gamma(d)$-*converges* to F (on X) and we write $F = \Gamma(d)\text{-}\lim\limits_h F_h$.

Remark 1.2. Note that if $F = \Gamma(d)\text{-}\lim\limits_h F_h$, then F is a *lower semicontinuous* function with respect to the distance d; *i.e.,*

$$\forall x \in X \quad \forall (x_h) : \quad d(x_h, x) \to 0 \quad F(x) \leq \liminf_h F(x_h). \tag{1.7}$$

Remark 1.3. (More remarks on Γ-limits) 1) It can easily be seen, with one-dimensional examples, that the Γ-convergence of a sequence (F_h) is independent from its pointwise convergence. In particular a constant sequence $F_h = F$ $\Gamma(d)$-converges to its constant value F if and only if the function $F : X \to [0, +\infty]$ is *lower semicontinuous* with respect to the distance d.

2) If $F_h = F$ is not lower semicontinuous then we have

$$\Gamma(d)\text{-}\lim_h F_h = \overline{F}, \tag{1.8}$$

where the function $\overline{F}$ is the *d-lower semicontinuous envelope* (or *relaxation*) of F; *i.e.,* the greatest d-lower semicontinuous function not greater than F, whose abstract definition can be expressed as

$$\overline{F}(x) = \inf\Big\{\liminf_h F(x_h) \ : \ d(x_h, x) \to 0\Big\}. \tag{1.9}$$

3) If a sequence Γ-converges, then so does its every subsequence (to the same limit).

4) If $F = \Gamma(d)\text{-}\lim_h F_h$ and G is any d-continuous function then $\Gamma(d)\text{-}\lim_h(F_h + G) = F + G$ (this remark will be extremely useful in applications).

5) The Γ-limit of a sequence of convex functions is convex (here and in the following, we suppose that (X, d) is a topological vector space).

6) The Γ-limit of a sequence of quadratic forms (*i.e.,* $F_h(x+y)+F_h(x-y) = 2F_h(x)+ 2F_h(y)$) is a quadratic form.

7) Let $\alpha > 0$; then the Γ-limit of a sequence of positively α-homogeneous functions (*i.e.*, $F_h(tx) = t^\alpha F_h(x)$ for all $t \geq 0$) is positively α-homogeneous.

We shall easily obtain the property of convergence of minima we are looking for in the case of sequences of equicoercive Γ-converging functionals.

We recall that a subset K of X is *d-compact* if from every sequence (x_h) in K we can extract a subsequence (x_{h_k}) converging to an element $x \in K$.

We say that a function $F : X \to [0, +\infty]$ is *d-coercive* if there exists a d-compact set K such that

$$\inf\{F(x) : \ x \in X\} = \inf\{F(x) : \ x \in K\}. \tag{1.10}$$

Let us also recall here *Weierstrass' Theorem*, which is the fundamental tool of the so-called direct methods of the calculus of variations: *if F is d-coercive and d-lower semicontinuous then there exists a minimizer for F on X.* (*Proof*: by (1.10) there exists a sequence x_h in K such that $F(x_h) \to \inf F$. By the d-compactness of K we can suppose that $x_h \to \overline{x} \in K$. By the d-lower semicontinuity of F we have then $F(\overline{x}) \leq \lim_h F(x_h) = \inf F$; *i.e.*, $\overline{x}$ is a minimizer of F).

We say that a sequence $F_h : X \to [0, +\infty]$ is *d-equicoercive* if there exists a d-compact set K (independent of h) such that

$$\inf\{F_h(x) : \ x \in X\} = \inf\{F_h(x) : \ x \in K\}. \tag{1.11}$$

Theorem 1.4. (The Fundamental Theorem of Γ-convergence) *Let (F_h) be a d-equicoercive sequence $\Gamma(d)$-converging on X to the function F. Then we have the convergence of minima*

$$\min\{F(x) \ : \ x \in X\} = \lim_h \inf\{F_h(x) \ : \ x \in X\}. \tag{1.12}$$

Moreover we have also convergence of minimizers: if $x_h \to x$ and $\lim_h F_h(x_h) = \lim_h \inf F_h$, then x is a minimizer for F.

Proof. Let (h_k) be a sequence of indices such that $\lim_k \inf F_{h_k} = \liminf_h \inf F_h$. Let (x_k) be a sequence in K (K as in (1.11)) verifying

$$\lim_k F_{h_k}(x_k) = \lim_k \inf F_{h_k} = \liminf_h \inf F_h. \tag{1.13}$$

By the d-compactness of K we can suppose (possibly passing to a further subsequence) that $x_k \to x \in K$. We have then by (1.5)

$$F(x) \leq \liminf_k F_{h_k}(x_k) = \liminf_h \inf F_h, \tag{1.14}$$

so that

$$\inf F \le \inf\{F(x) \ : \ x \in K\} \le \liminf_h \inf F_h. \tag{1.15}$$

Since F is d-lower semicontinuous there exists (by Weierstrass' Theorem) a minimum point $\overline{x}$ for F on K. By (1.6) there exists a sequence x_h such that $x_h \to \overline{x}$, and

$$\min\{F(x) \ : \ x \in K\} = F(\overline{x}) = \lim_h F_h(x_h) \ge \limsup_h \inf F_h. \tag{1.16}$$

Hence

$$\min\{F(x) \ : \ x \in K\} = \lim_h \inf F_h. \tag{1.17}$$

In order to prove (1.12) it will be sufficient to show that K satisfies the coercivity property (1.10). Suppose that (1.10) is not verified, then we must have, by (1.17), $\inf F < \lim_h \inf F_h$, so that there exists $x \in X$ such that $F(x) < \lim_h \inf F_h$. This inequality contradicts (1.6), and hence (1.12) is proven.

The convergence of minimizers is a direct consequence of (1.5) and (1.12). □

Note that if F_h is an integral functional with smooth strictly convex integrand, then we obtain from the Γ-convergence of the sequence (F_h) the G-convergence of the corresponding Euler equations. It will be clear in the sequel that no regularity of the integrands is in general necessary for Γ-convergence.

Remark 1.5. The Γ-limit of an arbitrary sequence of functions does not always exist. It will be convenient then to introduce, beside the Γ-limit already studied, also the Γ-limsup and Γ-liminf. Let us define then for $x \in X$

$$\Gamma(d)\text{-}\liminf_h F_h(x) = \inf\{\liminf_h F_h(x_h) \ : \ d(x_h, x) \to 0\}, \tag{1.18}$$

$$\Gamma(d)\text{-}\limsup_h F_h(x) = \inf\{\limsup_h F_h(x_h) \ : \ d(x_h, x) \to 0\}. \tag{1.19}$$

We have $\Gamma(d)\text{-}\liminf_h F_h(x) = \Gamma(d)\text{-}\limsup_h F_h(x) = r$ if and only if there exist the $\Gamma(d)\text{-}\lim_h F_h(x) = r$.

1.3. A Class of Integral Functionals

We have at our disposal now a powerful tool to obtain the desired convergence of minima in (1.2) and (1.3). The next, crucial point now is to understand what the right choice for the space (X, d) is, and how to define the functionals F_h.

At this point, we have to specify the conditions we require on the function f. We shall suppose $p > 1$, and $f : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0, +\infty[$ be a Borel function verifying the so-called "standard growth conditions of order p": there exist constants $c_1 \geq 0$, $C_1 > 0$ such that

$$|\xi|^p - c_1 \leq f(x, \xi) \leq C_1(1 + |\xi|^p), \qquad \text{for all } x \in \mathbb{R}^n, \xi \in \mathrm{M}^{n\times N} \tag{1.20}$$

(here and afterwards $\mathrm{M}^{n\times N}$ will denote the space of $n \times N$ real matrices) so that the functionals $\mathcal{F}_\varepsilon$ in (1.1) are well-defined on $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^N)$ for every Ω open subset of $\mathbb{R}^n$.

Let us face now the choice of the space (X, d); the topology of the metric d should be weak enough to obtain equicoercivity for minimum problems, but strong enough to allow for Γ-convergence. For the sake of simplicity let us suppose that $\phi \equiv 0$, $\gamma_0 = \partial\Omega$, and Ω itself being sufficiently smooth and bounded (some of these hypotheses may be weakened). Let us recall then the following fundamental theorems on Sobolev spaces.

Theorem 1.6. (Poincaré's Inequality) *Let Ω be a bounded open subset of $\mathbb{R}^n$; then there exist a constant $C' > 0$ such that*

$$\int_\Omega |u|^p\, dx \leq C' \int_\Omega |Du|^p\, dx \tag{1.21}$$

for all $u \in \mathrm{W}_0^{1,p}(\Omega; \mathbb{R}^N)$.

Theorem 1.7. (Rellich's Theorem) *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$, and (u_h) be a bounded sequence in $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^N)$. Then there exists a subsequence of u_h converging with respect to the $\mathrm{L}^p(\Omega; \mathbb{R}^N)$ metric.*

Theorem 1.7 can be stated also: "the sets $\{u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^N) \ : \ \|u\|_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)} \leq C\}$ (C any constant) are $\mathrm{L}^p(\Omega; \mathbb{R}^N)$-compact".

By Theorems 1.6 and 1.7 we obtain that the whole family of functionals $(\mathcal{F}_\varepsilon)$ is $\mathrm{L}^p(\Omega; \mathbb{R}^N)$-equicoercive on $\mathrm{W}_0^{1,p}(\Omega; \mathbb{R}^N)$: it is sufficient to set $c_2 = C_1|\Omega| \geq \int_\Omega f(\frac{x}{\varepsilon}, 0)\, dx$, and to notice that the set

$$E = \{u \in \mathrm{W}_0^{1,p}(\Omega; \mathbb{R}^N) \ : \ \mathcal{F}_\varepsilon(u) \leq c_2\}$$

is not empty (the constant 0 belongs to E), and by (1.20) is contained in the set

$$K = \{u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^N) \ : \ \|u\|_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)} \leq (1 + C')^{1/p}(c_1 + c_2)^{1/p}\},$$

which is $L^p(\Omega;\mathbb{R}^N)$-compact (by Theorem 1.7). In fact by (1.20) and Theorems 1.6, if $u\in E$, then

$$\int_\Omega (|u|^p+|Du|^p)\,dx\le(1+C')\int_\Omega |Du|^p\,dx\le(1+C')(\mathcal{F}_\varepsilon(u)+c_1)\le(1+C')(c_1+c_2).$$

With the same kind of computations we obtain that for each fixed $g\in L^{p'}(\Omega;\mathbb{R}^N)$ the family of functionals $\mathcal{F}_\varepsilon(u)+\int_\Omega gu\,dx$ is equicoercive on $W^{1,p}_0(\Omega;\mathbb{R}^N)$.

We are led then to consider $X=W^{1,p}_0(\Omega;\mathbb{R}^N)$, and d the restriction of the $L^p(\Omega;\mathbb{R}^N)$-distance to $W^{1,p}_0(\Omega;\mathbb{R}^N)$.

In order to describe the limit of the problems in (1.2) it is sufficient to consider all limits of problems related to sequences (ε_h) with $\varepsilon_h\to 0$ as $h\to\infty$. Moreover by Remark 1.3(4), since the functionals

$$u\mapsto\int_\Omega ug\,dx \tag{1.22}$$

are continuous (we suppose $g\in L^{p'}(\Omega;\mathbb{R}^N)$), we can neglect this integral. Hence we shall have to study the $\Gamma(L^p(\Omega;\mathbb{R}^N))$-convergence of the functionals

$$F_h(u)=\mathcal{F}^0_{\varepsilon_h}(u)=\begin{cases}\displaystyle\int_\Omega f(\frac{x}{\varepsilon},Du)\,dx & \text{if } u\in W^{1,p}_0(\Omega;\mathbb{R}^N)\\ +\infty & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^N)\setminus W^{1,p}_0(\Omega;\mathbb{R}^N).\end{cases} \tag{1.23}$$

We have preferred to define our functionals by (1.23) on the whole $W^{1,p}(\Omega;\mathbb{R}^N)$, and to deal with the boundary conditions setting the functional to $+\infty$ outside $W^{1,p}_0(\Omega;\mathbb{R}^N)$ since this is a good illustration of a common procedure for dealing with constraints.

The Γ-convergence of $\mathcal{F}^0_{\varepsilon_h}$ will be deduced from the Γ-convergence of the functionals

$$\mathcal{F}_{\varepsilon_h}(u)=\int_\Omega f(\frac{x}{\varepsilon},Du)\,dx \qquad \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^N), \tag{1.24}$$

showing that the boundary condition $u=0$ on $\partial\Omega$ does not affect the form of the Γ-limit (see Lesson Two).

Exercises

Prove the statements 1)–7) of Remark 1.3 by using the definition of Γ-limit.

Lesson Two. A General Compactness Result

2.1. The Localization Method of Γ-convergence. Compactness

The proof of the Γ-convergence of the functionals in (1.1) will follow this line:

(i) prove a *compactness theorem* which allows to obtain from each sequence $(\mathcal{F}_{\varepsilon_h})$ a subsequence Γ-converging to an abstract limit functional;

(ii) prove an *integral representation result*, which allows us to write the limit functional as an integral;

(iii) prove a *representation formula* for the limit integrand which does not depend on the subsequence, showing thus that the limit is well-defined.

The third point is characteristic of homogenization and will be performed in Lesson Three by exploiting the special form of the functionals under examination. Steps (i) and (ii) follow from general theorems in Γ-convergence (see the books by Dal Maso [15] and Buttazzo [13]); here we shall give briefly an idea of the methods involved in the proof (without entering into details, some of which will be given in the course by Buttazzo at this same School).

Let us fix a sequence of Borel functions $f_h : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0,+\infty]$ satisfying the growth condition

$$|\xi|^p - c_1 \leq f_h(x,\xi) \leq C_1(1+|\xi|^p) \tag{2.1}$$

(in our case we will have $f_h(x,\xi) = f(\frac{x}{\varepsilon_h},\xi)$, where (ε_h) is a fixed sequence converging to 0), and let us consider the functionals

$$F_h(u) = \int_\Omega f_h(x,Du)\,dx \tag{2.2}$$

defined for $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. We shall outline the proof of a compactness and integral representation theorem for the sequence (F_h).

Let us first notice that it is easy to obtain, by a diagonal procedure, a compactness theorem for the functionals F_h since the topology of $\mathrm{L}^p(\Omega;\mathbb{R}^N)$ has a countable base (see Dal Maso [15] Theorem 8.5). However, the limit functional thus obtained depends *a priori* heavily on the choice of Ω, and it is not possible to obtain directly an integral representation of it. To overcome this difficulty it was introduced a *localization method* characteristic of Γ-convergence. Instead of considering the functionals in (2.2) for a fixed Ω bounded open subset of $\mathbb{R}^n$, we consider

$$F_h(u,A) = \int_A f_h(x,Du)\,dx \tag{2.3}$$

as a function of the pair (u,A) where $A \in \mathcal{A}_n$ (the family of bounded open subsets of $\mathbb{R}^n$) and $u \in \mathrm{W}^{1,p}(A;\mathbb{R}^N)$ (this is sometimes called a *variational functional*).

We can now fix a countable dense family $\mathcal{Q}$ of $\mathcal{A}_n$[1] (for example all poly-rectangles with rational vertices), and, again using a diagonal procedure, find an increasing sequence of integers (h_k) such that we have the existence of the Γ-limit

$$F(u,A)=\Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\lim_k F_{h_k}(u,A) \tag{2.4}$$

for all $A\in\mathcal{Q}$ and $u\in \mathrm{W}^{1,p}(A;\mathbb{R}^N)$.

Beside this limit we can consider the upper and lower Γ-limits

$$F^+(u,A)=\Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\limsup_k F_{h_k}(u,A) \tag{2.5}$$

$$F^-(u,A)=\Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\liminf_k F_{h_k}(u,A) \tag{2.6}$$

for all $A\in\mathcal{A}_n$ and $u\in \mathrm{W}^{1,p}(A;\mathbb{R}^N)$, so that we have

$$F^+(\cdot,A)=F^-(\cdot,A)=F(\cdot,A) \tag{2.7}$$

for all $A\in\mathcal{Q}$.

The next step (which is rather technical, and relies on the growth conditions (2.1) on f; see Section 2.2) is to notice that the increasing set functions $A\mapsto F^+(u,A)$ and $A\mapsto F^-(u,A)$ are *inner-regular*; that is,

$$F^\pm(u,A)=\sup\Big\{F^\pm(u,A')\ :A'\in\mathcal{A}_n,\ \overline{A'}\subset A\Big\} \tag{2.8}$$

for all $A\in\mathcal{A}_n$ and $u\in \mathrm{W}^{1,p}(A,\mathbb{R}^N)$.

At this point it suffices to notice that the supremum in (2.8) can be taken for $A'\in\mathcal{Q}$, and to recall (2.7), to obtain

$$F^+(u,A)=\sup\Big\{F(u,A')\ :A'\in\mathcal{Q},\ \overline{A'}\subset A\Big\}=F^-(u,A), \tag{2.9}$$

and then the existence of the Γ-limit in (2.4) for all $A\in\mathcal{A}_n$ and $u\in \mathrm{W}^{1,p}(A;\mathbb{R}^N)$.

We have thus obtained a converging subsequence on all $A\in\mathcal{A}_n$.

[1] We say that $\mathcal{Q}$ is dense in $\mathcal{A}_n$ if for every $A,A'\in\mathcal{A}_n$ with $\overline{A'}\subset A$ there exists $Q\in\mathcal{Q}$ such that $A'\subset Q\subset A$

Theorem 2.1. (Compactness) *Let F_h be defined as in* (2.3), *with f_h satisfying* (2.1)*; then there exists an increasing sequence of integers (h_k) such that the limit*

$$F(u,A)=\Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\lim_k F_{h_k}(u,A) \tag{2.10}$$

exists for all $A\in\mathcal{A}_n$ and $u\in \mathrm{W}^{1,p}(A;\mathbb{R}^N)$.

It can be proven that, as a set function, the limit F behaves in a very nice way. In fact we have:

(a) (measure property) *for every $\Omega\in\mathcal{A}_n$ and $u\in\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ the set function $A\mapsto F(u,A)$ is the restriction to $\mathcal{A}_n(\Omega)$ (the family of all open subsets of Ω) of a regular Borel measure.*

The variational functional F enjoys other properties, which derive from the structure of the Γ-limit:

(b) (semicontinuity) *for every $A\in\mathcal{A}_n$ the functional $F(\cdot,A)$ is $\mathrm{L}^p(A;\mathbb{R}^N)$-lower semicontinuous* (by the lower semicontinuity properties of Γ-limits);

(c) (growth conditions) *we have the inequality*

$$\int_A |Du|^p\,dx - c_1|A|\le F(u,A)\le C_1\Big(|A|+\int_A|Du|^p\,dx\Big)$$

for every $A\in\mathcal{A}_n$ and $u\in\mathrm{W}^{1,p}(A;\mathbb{R}^N)$ (by the growth condition (2.1));

(d) (locality) *if $u=v$ a.e. on $A\in\mathcal{A}_n$, then $F(u,A)=F(v,A)$;*

(e) ("translation invariance") *if $z\in\mathbb{R}^n$ then $F(u,A)=F(u+z,A)$.*

The proofs of the two last statements are trivial since the operation of Γ-limit is local and all functionals F_h are translation invariant.

These properties assure us that it is possible to represent the functional F as an integral.

Theorem 2.2. (Integral Representation Theorem (Buttazzo & Dal Maso; see [13] Chapter 4.3 and [15] Chapter 20)) *If F is a variational integral verifying* (a)–(e)*, then there exists a Carathéodory integrand $\varphi:\mathbb{R}^n\times\mathrm{M}^{n\times N}\to[0,+\infty[$ satisfying*

(*growth condition*) $$|\xi|^p-c_1\le\varphi(x,\xi)\le C_1(1+|\xi|^p)$$

and

(*quasiconvexity*) $$|A|\varphi(x,\xi)\le\int_A\varphi(x,\xi+Du(y))\,dy$$

for all $A\in\mathcal{A}_n$, $x\in\mathbb{R}^n$, $\xi\in\mathrm{M}^{n\times N}$, and $u\in\mathrm{W}^{1,p}_0(A,\mathbb{R}^N)$, such that

$$F(u,A)=\int_A\varphi(x,Du)\,dx \tag{2.11}$$

for all $A\in\mathcal{A}_n$ and $u\in\mathrm{W}^{1,p}(A;\mathbb{R}^N)$.

Remark 2.3. Let us recall that quasiconvexity is a well-known necessary and sufficient condition for the L^p-lower semicontinuity of functionals of the form (2.2) with integrands verifying (2.1) (see Acerbi & Fusco [2]). Convex functions are quasiconvex; the two notions coincide only in the case $n = 1$ or $N = 1$. Examples of quasiconvex non convex functions are *polyconvex* functions: we say that $f : M^{n\times N} \to \mathbb{R}$ is polyconvex if $f(\xi)$ is a convex function of the vector of all minors of the matrix ξ. In the case $n = N = 2$ this means that $f(\xi) = g(\xi, \det\xi)$, with g convex.

Proof of Theorem 2.2. We will just give an idea of the proof. First of all one can obtain a representation for $F(u, A)$ when $u = \xi x$ is linear (or affine, which is the same because of the translation invariance): since $F(\xi x, \cdot)$ is a measure (absolutely continuous with respect to the Lebesgue measure), then, by Riesz Theorem, there exists a function g_ξ such that

$$F(\xi x, A) = \int_A g_\xi(x)\,dx$$

for all $A \in \mathcal{A}_n$.

Let us define then $\varphi(x, \xi) = g_\xi(x)$. If u is piecewise affine then we obtain immediately (2.11) since $F(\xi x, \cdot)$ is a measure. If u is general, then the inequality

$$F(u, A) \le \int_A \varphi(x, Du)\,dx$$

follows by approximating u with piecewise affine functions in the $W^{1,p}$ metric, and then using the lower semicontinuity of F (on the left hand side), and Lebesgue Theorem (on the right hand side).

Fixed u we can define $G(v, A) = F(u + v, A)$. This variational functional still verifies the hypotheses (a)–(e). Hence we can construct as above a function ψ such that $G(v, A) = \int_A \psi(x, Dv)\,dx$ for v piecewise affine, and

$$G(v, A) \le \int_A \psi(x, Du)\,dx$$

for general v. We obtain then (if u_h is a sequence of piecewise affine functions converging strongly in $W^{1,p}(A; \mathbb{R}^N)$ to u)

$$\int_A \psi(x, 0)\,dx = G(0, A) = F(u, A) \le \int_A \varphi(x, Du)\,dx = \lim_h \int_A \varphi(x, Du_h)\,dx$$

$$= \lim_h F(u_h, A) = \lim_h G(u_h - u, A) \le \lim_h \int_A \psi(x, Du_h - Du)\,dx = \int_A \psi(x, 0)\,dx,$$

so that all inequalities are indeed equalities and we get (2.11).

The quasiconvexity of φ follows by the theorem of Acerbi & Fusco. $\square$

We can apply all the machinery above to our functionals. Hence for every fixed sequence (ε_h) there exist a subsequence (ε_{h_k}) and a Carathéodory quasiconvex function φ such that the limit

$$\Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\lim_k \int_A f(\frac{x}{\varepsilon_{h_k}}, Du)\,dx = \int_A \varphi(x, Du)\,dx \tag{2.12}$$

exists for all $A \in \mathcal{A}_n$ and $u \in \mathrm{W}^{1,p}(A, \mathbb{R}^N)$.

2.2. The Fundamental Estimate. Boundary Value Problems

As we have already remarked, the very crucial point in the compactness procedure for integral functionals, described in Section 2.1 is the proof of the properties of the Γ-limit as a set function, namely that it is (the restriction to the family of bounded open sets of) a inner-regular measure. For example, it must be proven the *subadditivity* of $F(u, \cdot)$; that is, for all pairs of sets $A, B \in \mathcal{A}_n$ and $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^N)$ we must have

$$F(u, A \cup B) \le F(u, A) + F(u, B).$$

Recalling the definition of Γ-limit, this means that from the knowledge of the "minimizing sequences" for $F(u, A)$ and $F(u, B)$ we must somehow obtain an estimate for $F(u, A \cup B)$. This is done by elaborating a method for "joining" sequences of functions, without increasing in the limit the value of the corresponding integrals. This procedure is not possible in general for arbitrary integral functionals, and indeed there are examples of Γ-limits which are not measures (as set functions). Anyhow, for functionals whose integrands verify (2.1) the possibility of inexpensive joints was shown by De Giorgi in [17]; his method was later generalized in many papers (see [16], [15] and the references therein), and remains one of the cornerstones of the theory. A general formulation of this property can be found in [15] Definition 18.2.

Lemma 2.4. (Fundamental Estimate) *Let F_h be the functionals in* (2.1), (2.2). *Then, for every $\eta > 0$, and for every $A, A', B \in \mathcal{A}_n$ with $\overline{A'} \subset A$ there exists a constant $M > 0$ with the property: for every $h \in \mathbb{N}$, for every $w \in \mathrm{W}^{1,p}(A; \mathbb{R}^N)$, $v \in \mathrm{W}^{1,p}(B; \mathbb{R}^N)$ there exists a cut-off function[2] ϕ between A' and A such that*

$$F_h(\phi w + (1-\phi)v, A' \cup B) \le (1+\eta)\Big(F_h(w, A) + F_h(v, B)\Big) + M \int_{A\cap B} |w - v|^p\,dx. \tag{2.13}$$

Note that ϕ depends on h, v, and w.

[2] We say that ϕ is a *cut-off function* between A' and A if $\phi \in C_0^\infty(A)$ and $\phi = 1$ on a neighbourhood of $\overline{A'}$.

With this property in mind it is not difficult to prove the inner regularity of $F^{\pm}$, and hence that F is a measure (for useful criteria which give conditions on an increasing set function equivalent to being a measure we refer to De Giorgi and Letta [18]). We are not going to prove these consequences, nor Lemma 2.4 (for a proof see [15] Section 19, and also the paper by Fusco [23] where the vector-valued case is dealt with in detail). Let us remark instead how this property allows us also to deduce the Γ-convergence of functionals defined taking into account (homogeneous) Dirichlet boundary conditions.

Lemma 2.5. (Γ-limits and Boundary Conditions) *Let (F_{h_k}) be the converging subsequence of (F_h) given by Theorem 2.1. If we set*

$$F_h^0(u,A)=\begin{cases}\displaystyle\int_A f_h(x,Du)\,dx & \text{if } u\in \mathrm{W}_0^{1,p}(A;\mathbb{R}^N)\\ +\infty & \text{elsewhere on } \mathrm{W}^{1,p}(A;\mathbb{R}^N),\end{cases}\tag{2.14}$$

then we have for all $A\in\mathcal{A}_n$ and $u\in \mathrm{W}^{1,p}(A,\mathbb{R}^N)$

$$\Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\lim_k F^0_{h_k}(u,A)=F^0(u,A),\tag{2.15}$$

where

$$F^0(u,A)=\begin{cases}\displaystyle\int_A \varphi(x,Du)\,dx & \text{if } u\in \mathrm{W}_0^{1,p}(A;\mathbb{R}^N)\\ +\infty & \text{elsewhere on } \mathrm{W}^{1,p}(A;\mathbb{R}^N),\end{cases}\tag{2.16}$$

and φ is given by Theorem 2.2.

Proof. Let us apply the definition of Γ-convergence. Let us consider a converging sequence $u_k\to u$ in $\mathrm{L}^p(A;\mathbb{R}^N)$. If $u\notin \mathrm{W}_0^{1,p}(A;\mathbb{R}^N)$ then we must have $F_{h_k}(u_k,A)\to +\infty$; otherwise (by the growth conditions (2.1)) (u_k) would be a bounded sequence in $\mathrm{W}_0^{1,p}(A;\mathbb{R}^N)$, so that (a subsequence of it) converges weakly in $\mathrm{W}_0^{1,p}(A;\mathbb{R}^N)$ to u, obtaining thus a contradiction. Hence $F^0(u,A)=+\infty$. If $u\in \mathrm{W}_0^{1,p}(A;\mathbb{R}^N)$ we have trivially

$$F(u,A)\le\liminf_k F_{h_k}(u_k,A)\le\liminf_k F^0_{h_k}(u_k,A)$$

for all $u_k\to u$; that is,

$$\Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\liminf_k F^0_{h_k}(u,A)\ge F^0(u,A).\tag{2.17}$$

Vice versa, let $u_k\to u$ be such that $F(u,A)=\lim_k F_{h_k}(u_k,A)$. Let us fix a compact subset K of A, $A'\subset A$, $\eta>0$, choose in Lemma 2.4 $B=A\setminus K$, $w=u_k$, $v=u$, and define $v_k = \phi u_k + (1-\phi)u \in \mathrm{W}_0^{1,p}(A;\mathrm{I\!R}^N)$, where ϕ is given by Lemma 2.4. We have then $v_k \to u$, and

$$F^0_{h_k}(v_k, A) = F_{h_k}(v_k, A) \leq (1+\eta)\Big(F_{h_k}(u_k, A) + F_{h_k}(u, A \setminus K)\Big) + M \int_{A\setminus K} |u_k - u|^p \, dx.$$

Letting $k \to +\infty$, and recalling (2.1), we obtain

$$\limsup_k F^0_{h_k}(v_k, A) \leq (1+\eta)F(u, A) + (1+\eta)\int_{A\setminus K} C_1(1 + |Du|^p)\, dx,$$

hence by the arbitrariness of K, and letting $\eta \to 0$,

$$\Gamma(\mathrm{L}^p(A;\mathrm{I\!R}^N))\text{-}\limsup_k F^0_{h_k}(u, A) \leq F^0(u, A). \tag{2.18}$$

This inequality completes the proof. $\square$

Exercises

1. State and prove the analog of Lemma 2.5 for the boundary condition $u = \phi$ on γ_0, under appropriate assumptions on the data.
2. Prove (2.8) using (2.13).
3. Prove that the Dirichlet integral $\int_A |Du|^2 \, dx$ verifies the fundamental estimate.

Lesson Three. Homogenization Formulas

3.1. The Asymptotic Homogenization Formula

We have reduced the problem of Γ-convergence of the functionals $\mathcal{F}_\varepsilon$ to the description of the function φ in (2.12). In order to deduce the convergence of the whole sequence it is sufficient now to prove that φ is independent of the sequence (ε_{h_k}). This will be done by proving an asymptotic formula for φ.

We shall make a weaker assumption on f than periodicity, namely a sort of uniform *almost periodicity* (see the book by Besicovitch [5] for a study of different types of almost periodic functions). The motivation for the introduction of this kind of hypothesis lies in its greater flexibility compared to mere periodicity:

(a) sum and product of almost periodic functions are almost periodic (this happens for periodic functions only if they have a common period; think of $\sin x + \sin(\pi x)$);

(b) restriction of an almost periodic function to an affine subspace is still almost periodic (this is not true for periodic functions; think as above of the function $\sin x + \sin y$ restricted to the line $y = \pi x$);

(c) almost periodic functions are "stable under small perturbations" (this concept will be explained and studied later).

Moreover, the techniques are essentially of the same type as in the periodic case, so that we get a stronger result for free.

Let us recall that a continuous function $a : \mathbb{R}^n \to \mathbb{R}$ is *uniformly almost periodic* if the following property holds: for every $\eta > 0$ there exists an *inclusion length* $L_\eta > 0$ and a set $T_\eta \subset \mathbb{R}^n$ (which will be called the set of η-*almost periods* for a) such that $T_\eta + [0, L_\eta]^n = \mathbb{R}^n$, and if $\tau \in T_\eta$ we have

$$|a(x+\tau) - a(x)| \le \eta \qquad \text{for all } x \in \mathbb{R}^n. \tag{3.1}$$

Of course if a is periodic then we can take for all η the lattice of all periods of a as $T = T_\eta$, and $L = L_\eta$ equal to the mesh size of the lattice. Particular uniformly almost periodic functions are *quasiperiodic* functions; that is, functions of the form $a(x) = b(x, \dots, x)$, where b is a continuous periodic function of a higher number of variables. The set of uniformly almost periodic functions can be seen also as the closure of all trigonometric polynomials in the uniform norm.

We can model our hypotheses to fit functionals of the form

$$\int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|Du|^p\,dx, \tag{3.2}$$

with the coefficient a uniformly almost periodic. We say then that a Borel function $f : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0, +\infty]$ is p-*almost periodic* (see [7]) if for every $\eta > 0$ there exists $L_\eta > 0$ and a set $T_\eta \subset \mathbb{R}^n$ such that $T_\eta + [0, L_\eta]^n = \mathbb{R}^n$, and if $\tau \in T_\eta$ we have

$$|f(x+\tau,\xi) - f(x,\xi)| \le \eta(1 + |\xi|^p) \qquad \text{for all } x \in \mathbb{R}^n. \tag{3.3}$$

Notice that we do not require any continuity of f since it will not be necessary in the proofs; hence all Borel functions $f = f(x, \xi)$ periodic in x (with period independent of ξ) satisfy the hypothesis of p-almost periodicity.

The first result we will obtain by exploiting the almost periodicity of f will be the "homogeneity" of the function φ.

Proposition 3.1. *Let us suppose f be p-almost periodic and satisfy the growth condition* (2.1)*. Then the function $\varphi = \varphi(x, \xi)$ in* (2.12) *can be chosen independent of x.*

Proof. (Let us remark that we follow the line of the proof of the corresponding statement in the periodic case by Marcellini [26]; see also [7] Proposition 5.1) Let us fix $x_0, y_0 \in \mathbb{R}^n$, $r > 0$, $K \in \mathbb{N}$, and $\eta > 0$. Let $B = B(x_0, r)$, $B_K = B(x_0, r(1 - 1/K))$, and (τ_k) be a sequence of points of T_η such that $\lim_k \varepsilon_{h_k} \tau_k = y_0 - x_0$. Let (u_k) be a sequence in $\mathrm{W}^{1,p}(B; \mathbb{R}^N)$ with $u_k \to 0$ and

$$\int_B \varphi(x, \xi)\, dx = \lim_k \int_B f(\frac{x}{\varepsilon_{h_k}}, Du_k + \xi)\, dx. \tag{3.4}$$

Let us set $y_k = x_0 + \varepsilon_{h_k} \tau_k$; if k is large enough we have $y_0 + B_K \subset y_k + B$. We have then (using (3.3) and the definition of Γ-limit)

$$\int_B \varphi(x, \xi) \geq \liminf_k \int_B f(\frac{x}{\varepsilon_{h_k}} + \tau_k, Du_k + \xi)\, dx - \eta \limsup_k \int_B (1 + |Du_k + \xi|^p)\, dx$$

$$= \liminf_k \int_{y_k + B} f(\frac{y}{\varepsilon_{h_k}}, Du_k(y - y_k) + \xi)\, dy - \eta c$$

$$\geq \liminf_k \int_{y_0 + B_K} f(\frac{y}{\varepsilon_{h_k}}, Du_k(y - y_k) + \xi)\, dy - \eta c \geq \int_{y_0 + B_K} \varphi(x, \xi)\, dx - \eta c$$

(c a constant depending on (u_k)) . By the arbitrariness of η and K we have

$$\int_B \varphi(x, \xi)\, dx \geq \int_{y_0 + B} \varphi(y, \xi)\, dy = \int_B \varphi(x + y_0, \xi)\, dx, \tag{3.5}$$

and then by symmetry the equality

$$\int_B \varphi(x, \xi) = \int_B \varphi(x + y_0, \xi)\, dx. \tag{3.6}$$

It is easy to see that from this equality we can conclude the proof. □

The independence from the space variable is essential for expressing the value $\varphi(\xi)$ as the solution of a minimum problem. In fact by the quasiconvexity of φ we have

$$|\Omega|\varphi(\xi) = \min\Big\{\int_\Omega \varphi(\xi + Du(y))\,dy \ :\ u \in \mathrm{W}_0^{1,p}(\Omega;\mathbb{R}^N)\Big\}$$

for every $\Omega \in \mathcal{A}_n$; in particular we can choose $\Omega =]0,1[^n$ so that

$$\begin{aligned}\varphi(\xi) &= \min\Big\{\int_{]0,1[^n} \varphi(\xi + Du(y))\,dy \ :\ u \in \mathrm{W}_0^{1,p}(]0,1[^n;\mathbb{R}^N)\Big\}\\ &= \min\Big\{F^0(u + \xi x,]0,1[^n) \ :\ u \in \mathrm{W}^{1,p}(]0,1[^n;\mathbb{R}^N)\Big\}.\end{aligned} \tag{3.7}$$

We can use now the Γ-convergence of $F^0_{h_k}$ to F^0 (Lemma 2.5), the equicoercivity of these functionals (Section 1.3), and the Fundamental Theorem of Γ-convergence (Theorem 1.4), to obtain

$$\varphi(\xi) = \lim_k \inf\Big\{\int_{]0,1[^n} f\big(\frac{y}{\varepsilon_{h_k}}, Du(y) + \xi\big)\,dy :\ u \in \mathrm{W}_0^{1,p}(]0,1[^n;\mathbb{R}^N)\Big\}. \tag{3.8}$$

At this point is is clear that our next step must be the proof of the independence of this limit of the sequence (ε_{h_k}).

Proposition 3.2. (Asymptotic Homogenization Formula) *Let f be as above. Then the limit*

$$f_{\mathrm{hom}}(\xi) = \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{\int_{]0,t[^n} f(x, Du(x) + \xi)\,dx :\ u \in \mathrm{W}_0^{1,p}(]0,t[^n;\mathbb{R}^N)\Big\} \tag{3.9}$$

exists for every $\xi \in \mathrm{M}^{n\times N}$.

Proof. (Let us remark that we follow the line of the proof of the corresponding statement in the periodic case in [6]; see also [7]) Let us fix $\xi \in \mathrm{M}^{n\times N}$ and define for $t > 0$

$$g_t = \frac{1}{t^n} \inf\Big\{\int_{]0,t[^n} f(x, Du(x) + \xi)\,dx :\ u \in \mathrm{W}_0^{1,p}(]0,t[^n;\mathbb{R}^N)\Big\}; \tag{3.10}$$

moreover let $u_t \in \mathrm{W}_0^{1,p}(]0,t[^n;\mathbb{R}^N)$ verify

$$\frac{1}{t^n}\int_{]0,t[^n} f(x, Du_t(x) + \xi)\,dx \le g_t + \frac{1}{t}. \tag{3.11}$$

Let $\eta > 0$. If $s \ge t + L_\eta$ (the inclusion length related to η and f) we can construct $u_s \in \mathrm{W}_0^{1,p}(]0,s[^n;\mathbb{R}^N)$ as follows: for every $\mathbf{i} = (i_1,\dots,i_n) \in \mathbf{Z}^n$ with $0 \le (t + L_\eta)i_j \le s$ for all $j = 1,\dots,n$, we choose $\tau_{\mathbf{i}} \in T_\eta$ with $\tau_{\mathbf{i}} \in (t + L_\eta)\mathbf{i} + [0, L_\eta]^n$, and we define

$$u_s(x) = \begin{cases} u_t(x - \tau_{\mathbf{i}}) & \text{if } x \in \tau_{\mathbf{i}} + [0,t]^n\\ 0 & \text{otherwise.}\end{cases} \tag{3.12}$$

Let us also define $Q_s =]0,s[^n \setminus \bigcup_{\mathbf{i}} (\tau_{\mathbf{i}} + [0,t]^n)$; we have $|Q_s| \le s^n - (s-t-L_\eta)^n \Big(\dfrac{t}{t+L_\eta}\Big)^n$.

We can estimate g_s by using u_s:

(3.13)

$$\begin{aligned}
g_s &\le \frac{1}{s^n} \int_{]0,s[^n} f(x, Du_s(x) + \xi)\,dx \\
&= \frac{1}{s^n} \Big(\sum_{\mathbf{i}} \int_{\tau_{\mathbf{i}}+[0,t]^n} f(x, Du_t(x-\tau_{\mathbf{i}}) + \xi)\,dx + \int_{Q_s} f(x,\xi)\,dx \Big) \\
&\le \frac{1}{s^n} \Big(\sum_{\mathbf{i}} \int_{[0,t]^n} f(y+\tau_{\mathbf{i}}, Du_t + \xi)\,dy + |Q_s| C_1 (1+|\xi|^p) \Big) \\
&\le \frac{1}{s^n} \bigg(\sum_{\mathbf{i}} \int_{[0,t]^n} \Big(f(y, Du_t + \xi) + \eta(1 + |Du_t + \xi|^p) \Big)\,dy + |Q_s| C_1 (1+|\xi|^p) \bigg) \\
&\le (1+\eta) \frac{1}{(t+L_\eta)^n} t^n \Big(g_t + \frac{1}{t} \Big) + \eta + \Big(1 - \Big(\frac{s-t-L_\eta}{s^n} \Big)^n \Big(\frac{t}{t+L_\eta} \Big)^n \Big) C_1 (1+|\xi|^p).
\end{aligned}$$

Taking the limit first in s and then in t we get

$$\limsup_{s\to+\infty} g_s \le (1+\eta) \liminf_{t\to+\infty} g_t + \eta.$$

By the arbitrariness of η we conclude the proof. $\square$

Note that our growth hypotheses guarantee by a density argument that the infima in (3.9) can be computed on smooth functions; hence we can write also

$$f_{\mathrm{hom}}(\xi) = \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{ \int_{]0,t[^n} f(x, Du(x)+\xi)\,dx : \ u \in C_0^\infty(]0,t[^n; \mathbb{R}^N) \Big\}. \tag{3.14}$$

exists for every $\xi \in \mathrm{M}^{n\times N}$.

We can conclude now the proof of our homogenization result by simply remarking that the limit in (3.8) can be transformed in the form (3.9) by the change of variables $y = \varepsilon_{h_k} x$ (when $t = 1/\varepsilon_{h_k}$), so that $\varphi(\xi) = f_{\mathrm{hom}}(\xi)$ is independent of (ε_{h_k}).

Remark 3.3. By an use of the Fundamental Estimate as in the proof of Lemma 2.5 it is easy to see that an equivalent formula for f_{hom} is the following:

$$f_{\mathrm{hom}}(\xi) = \inf_{k\in\mathbb{N}} \frac{1}{k^n|Q|} \inf\Big\{ \int_{kQ} f(x, Du(x)+\xi)\,dx : \ u \in \mathrm{W}^{1,p}_{\#}(kQ; \mathbb{R}^N) \Big\}, \tag{3.15}$$

where Q is any non-degenerate open parallelogram in $\mathbb{R}^n$, and $W^{1,p}_{\#}(kQ;\mathbb{R}^N)$ denotes the space of functions in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ which are Q-periodic. This formula may be useful in the case of f periodic in x with period Q.

Remark 3.4. We shall see in the next section that a simpler formula, which involves a single minimization problem on the periodicity cell, can be obtained in the *convex and periodic* case. It is important to note that in the (vector-valued) non convex case formula (3.9) *cannot be simplified*, as shown by a counterexample by S.Müller [27]: in a sense homogenization problems in the vector-valued case have an almost periodic nature even if the integrand is periodic.

3.2. The Convex and Periodic Case

In this section we will suppose in addition to the previous hypotheses that for a.e. $x \in \mathbb{R}^n$ the function $f(x,\cdot)$ is *convex* on $M^{n\times N}$. This is no restriction in the scalar case $N = 1$ since it can be seen that in this case an equivalent convex integrand to f (that is, giving the same infima) may be constructed by "convexification" (see Ekeland & Temam [21]). Moreover, we shall suppose that f is 1-*periodic* in the first variable:

$$f(x + e_i, \xi) = f(x, \xi) \qquad \text{for all } x \in \mathbb{R}^n, \xi \in M^{n\times N}, i = 1, \dots, n, \tag{3.16}$$

where $\{e_1, \dots, e_n\}$ denotes the canonical base of $\mathbb{R}^n$ (every periodic function can be reduced to this case by a change of variables).

We can choose $Q =]0,1[^n$ in (3.15) to obtain the formula

$$f_{\mathrm{hom}}(\xi) = \inf_{k\in\mathbb{N}} \frac{1}{k^n} \inf\Big\{\int_{]0,k[^n} f(x, Du(x) + \xi)\, dx : \ u \in W^{1,p}_{\#}(]0,k[^n;\mathbb{R}^N)\Big\}. \tag{3.17}$$

If we define the function $f_{\#} : M^{n\times N} \to [0,+\infty[$ by setting

$$f_{\#}(\xi) = \inf\Big\{\int_{]0,1[^n} f(x, Du(x) + \xi)\, dx : \ u \in W^{1,p}_{\#}(]0,1[^n;\mathbb{R}^N)\Big\} \tag{3.18}$$

we have obviously

$$f_{\mathrm{hom}}(\xi) \le f_{\#}(\xi). \tag{3.19}$$

Thanks to the convexity of f we can reverse this inequality and obtain the following result.

Proposition 3.5. (Convex Homogenization Formula) *Let f be convex and periodic as above. Then we have $f_{\rm hom}(\xi) = f_{\#}(\xi)$ for all $\xi \in {\rm M}^{n\times N}$.*

Proof. Let u_k^ξ be a solution to the minimum problem

$$\frac{1}{k^n}\inf\Big\{\int_{]0,k[^n} f(x, Du(x)+\xi)\,dx : \ u \in {\rm W}^{1,p}_{\#}(]0,k[^n;{\rm I\!R}^N)\Big\} = f^k_{\#}(\xi), \tag{3.20}$$

which exists by the coerciveness and lower semicontinuity of the functional $\mathcal{F}_1$ (see Remark 2.3). Let I_h be the set of $\mathbf{i} = (i_1,\ldots,i_n) \in \mathbf{Z}^n$ with $0 \le i_j < k$, and let us define

$$u^\xi(x) = \frac{1}{k^n}\sum_{\mathbf{i}\in I_h} u_k^\xi(x+\mathbf{i}) \tag{3.21}$$

a convex combination of the translated functions $u_h^\xi(\cdot+\mathbf{i})$. The function u^ξ is 1-periodic, and we have

$$\begin{aligned} f_{\#}(\xi) &\le \int_{]0,1[^n} f(x, Du^\xi(x)+\xi)\,dx = \frac{1}{k^n}\int_{]0,k[^n} f(x, Du^\xi(x)+\xi)\,dx \\ &\le \frac{1}{k^n}\sum_{\mathbf{i}\in I_h}\frac{1}{k^n}\int_{]0,k[^n} f(x, Du_k^\xi(x+\mathbf{i})+\xi)\,dx \qquad (3.22) \\ &= \frac{1}{k^n}\sum_{\mathbf{i}\in I_h}\frac{1}{k^n}\int_{]0,k[^n} f(x, Du_k^\xi(x)+\xi)\,dx = f^k_{\#}(\xi). \end{aligned}$$

Since obviously we have $f_{\#}(\xi) = f^1_{\#}(\xi) \ge f^k_{\#}(\xi)$, by (3.22) and (3.17) we have $f_{\#}(\xi) = \inf_k f^k_{\#}(\xi) = f_{\rm hom}(\xi)$, and we can conclude the proof. □

Remark 3.6. Let us remark that in the convex and periodic case the homogenization formula and the Γ-convergence of the functionals $\mathcal{F}_\varepsilon$ can be proven under the weaker growth hypothesis

$$0 \le f(x,\xi) \le C_1(1+|\xi|^p) \tag{3.23}$$

(see [6] and [15]). Of course, no convergence of minima can be deduced in these hypotheses. The Γ-convergence of the functionals $\mathcal{F}_\varepsilon$ under only the growth hypothesis (3.23) in the general vector valued case is to my knowledge still an open problem.

3.3. Stability of Homogenization

A natural requirement in the study of oscillating media seems to be the stability of the limit under small perturbations. For example we would like our results to remain unchanged if we add to f a function with compact support (we expect the overall properties of a medium to be maintained in the presence of an impurity in a very small and confined region).

Theorem 3.6. (Stability for Homogenization) *Let f be a homogenizable*[3] *quasiconvex Borel function, and let $\psi : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0,+\infty[$ be a quasiconvex Borel function. Let us suppose that both functions verify the growth condition* (2.1)*, and that we have for every $r > 0$*

$$\limsup_{t\to+\infty} \frac{1}{t^n} \int_{]0,t[^n} \sup_{|\xi|\le r} |f(x,\xi) - \psi(x,\xi)|\, dx = 0. \tag{3.24}$$

Then also ψ is homogenizable and $\psi_{\text{hom}} = f_{\text{hom}}$.

Proof. Let us prove that for every $\xi \in \mathrm{M}^{n\times N}$ there exists $\psi_{\text{hom}}(\xi) = f_{\text{hom}}(\xi)$. Let $\varepsilon > 0$, and let us consider a solution u^ξ_ε to the minimum problem (which exists since by the quasiconvexity and growth conditions the integral functional is lower semicontinuous and coercive)

$$\min\Big\{ \int_{]0,1[^n} f(\frac{x}{\varepsilon}, Du(x) + \xi)\, dx : \ u \in \mathrm{W}^{1,p}_0(]0,1[^n;\mathbb{R}^N) \Big\} = f^\varepsilon_{\text{hom}}(\xi). \tag{3.25}$$

Let us recall that $\lim_{\varepsilon\to 0} f^\varepsilon_{\text{hom}}(\xi) = f_{\text{hom}}(\xi)$.

We shall use a partial regularity result which tells us that the solutions to the minimum problems are bounded in some Sobolev space with exponent larger than p (in some sense they behave as if they were Lipschitz continuous).

Theorem 3.7. (Partial Regularity Theorem; Meyers & Elcrat [28]) *There exist $\eta > 0$ and a constant $C > 0$ such that we have*

$$\int_{]0,1[^n} |Du^\xi_\varepsilon + \xi|^{p+\eta}\, dx \le C \tag{3.26}$$

for every $\varepsilon > 0$.

Let us fix $r > 0$ and define

$$E_\varepsilon = \Big\{ x \in]0,1[^n : \ |Du^\xi_\varepsilon + \xi| > r \Big\}.$$

---

[3] We say in general that $f : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0,+\infty[$ is homogenizable if the function f_{hom} gives the integrand of the Γ-limit in (2.12) for all converging sequences. Notice that in this theorem we do not make any hypotheses of periodicity or almost periodicity on f.

Clearly we have

$$|E_\varepsilon| r^p \le \int_{E_\varepsilon} |Du_\varepsilon^\xi + \xi|^p \, dx \le \int_{]0,1[^n} |Du_\varepsilon^\xi + \xi|^p \, dx$$

and by (2.1)

$$\int_{]0,1[^n} |Du_\varepsilon^\xi + \xi|^p \, dx \le C_1(1 + |\xi|^p) = C_\xi,$$

so that

$$|E_\varepsilon| \le r^{-p} C_1(1 + |\xi|^p) = r^{-p} C_\xi. \tag{3.27}$$

By using Hölder's inequality and (3.26) we get also

$$\begin{aligned}\int_{E_\varepsilon} |Du_\varepsilon^\xi + \xi|^p \, dx &\le |E_\varepsilon|^{\eta/(p+\eta)} \Big(\int_{E_\varepsilon} |Du_\varepsilon^\xi + \xi|^{p+\eta} \, dx \Big)^{p/(p+\eta)} \\ &\le r^{-p\eta/(p+\eta)} C_\xi^{\eta/(p+\eta)} C^{p/(p+\eta)} = C'_\xi r^{-p\eta/(p+\eta)}\end{aligned} \tag{3.28}$$

Using u_ε^ξ as a test function in the definition of

$$\min\Big\{ \int_{]0,1[^n} \psi(\frac{x}{\varepsilon}, Du(x) + \xi) \, dx : \ u \in \mathrm{W}_0^{1,p}(]0,1[^n; \mathbb{R}^N) \Big\} = \psi_{\mathrm{hom}}^\varepsilon(\xi) \tag{3.29}$$

we have (using (2.1), (3.25), (3.27) and (3.28))

(3.30)

$$\begin{aligned}\psi_{\mathrm{hom}}^\varepsilon(\xi) &\le \int_{]0,1[^n} \psi(\frac{x}{\varepsilon}, Du_\varepsilon^\xi + \xi) \, dx \\ &= \int_{\{|Du_\varepsilon^\xi + \xi| \le r\}} \psi(\frac{x}{\varepsilon}, Du_\varepsilon^\xi + \xi) \, dx + \int_{E_\varepsilon} \psi(\frac{x}{\varepsilon}, Du_\varepsilon^\xi + \xi) \, dx \\ &\le \int_{\{|Du_\varepsilon^\xi + \xi| \le r\}} (\psi(\frac{x}{\varepsilon}, Du_\varepsilon^\xi + \xi) - f(\frac{x}{\varepsilon}, Du_\varepsilon^\xi + \xi)) \, dx \\ &+ \int_{]0,1[^n} f(\frac{x}{\varepsilon}, Du_\varepsilon^\xi + \xi) \, dx + \int_{E_\varepsilon} C_1(1 + |Du_\varepsilon^\xi + \xi|^p) \, dx \\ &\le \int_{]0,1[^n} \sup_{|z| \le r} |\psi(\frac{x}{\varepsilon}, z) - f(\frac{x}{\varepsilon}, z)| \, dx + f_{\mathrm{hom}}^\varepsilon(\xi) + C_1(r^{-p} C_\xi + r^{-p\eta/(p+\eta)} C'_\xi).\end{aligned}$$

We can pass to the limit first as $\varepsilon \to 0$, and then as $r \to +\infty$, recalling (3.24), obtaining

$$\limsup_{\varepsilon\to 0} \psi^{\varepsilon}_{\text{hom}}(\xi) \leq f_{\text{hom}}(\xi);$$

since f and ψ play symmetric roles, we can interchange $\psi^{\varepsilon}_{\text{hom}}(\xi)$ and $f^{\varepsilon}_{\text{hom}}(\xi)$ in (3.30) so that we obtain

$$\liminf_{\varepsilon\to 0} \psi^{\varepsilon}_{\text{hom}}(\xi) \geq f_{\text{hom}}(\xi).$$

This proves the existence of $\psi_{\text{hom}}(\xi) = \lim_{\varepsilon\to 0} \psi^{\varepsilon}_{\text{hom}}(\xi) = f_{\text{hom}}(\xi)$. The rest of the proof of Theorem 3.8 follows easily by using a compactness argument and showing that all converging subsequences can be represented by means of $\psi_{\text{hom}}(\xi)$ (the only delicate point is the proof of the homogeneity of the limit integrand, that can be obtained by a similar argument as above; for details see [8] Section 3). $\square$

Remark 3.8. (Stability by Compact Support Perturbation) If for every $r > 0$ there exists $T_r > 0$ such that $f(x,\xi) = \psi(x,\xi)$ for $|x| > T_r$ and $|\xi| \leq r$ then 3.16 is verified; hence in this sense the homogenization is stable under compact support perturbations.

Remark 3.9. The hypothesis that ψ verifies (2.1) is essential. In [8] Section 3 it can be found an example of a functions ψ not homogenizable (the Γ-liminf different from the Γ-limsup) which verifies (3.24) with $f(x,\xi) = |\xi|^2$.

Remark 3.10. (A Stronger Homogenization Theorem) With the same type of arguments as in Theorem 3.6 we can prove a Closure Theorem for the Homogenization: *let f_h be a sequence of homogenizable Borel functions and let ψ be a Borel function. Let us suppose all these functions be quasiconvex, verify* (2.1), *and*

$$\lim_h \limsup_{t\to+\infty} \frac{1}{t^n} \int_{]0,t[^n} \sup_{|\xi|\leq r} |f_h(x,\xi) - \psi(x,\xi)|\, dx = 0 \tag{3.31}$$

for all $r > 0$. Then also ψ is homogenizable and $\psi_{\text{hom}} = \lim_h f_{\text{hom}}$.

Using this result and a suitable approximation procedure we can prove a stronger homogenization theorem under the only hypothesis of f verifying (2.1) and $f(\cdot,\xi)$ being Besicovitch-almost periodic[4] (details in [8] Sections 3 and 4). The class $\mathcal{F}$ of these functions is stable under perturbations as in (3.24); that is, if $f \in \mathcal{F}$ and ψ verifies (3.24), then $\psi \in \mathcal{F}$.

Exercises

Rewrite the proofs of Propositions 3.1 and 3.2 in the case of f periodic in x, using its periods instead of its almost periods.

[4] We say that f is *Besicovitch-almost periodic* if there exists a sequence of trigonometric polynomials (P_h) such that $\lim_h \limsup_{t\to+\infty} t^{-n} \int_{]0,t[^n} |f(x) - P_h(x)|\, dx = 0$ (*i.e.*, $P_h \to f$ *in the mean*).

Lesson Four. Examples: Homogenization without Standard Growth Conditions

We have now a quite general homogenization theorem; but, what is more important, we are in possession of a general and flexible procedure that can be also applied to face different kinds of problems[5]. We shall here describe some situations in which the same machinery can be applied. In this lesson we shall deal with functions f which fail to satisfy the standard growth conditions.

4.1. Condenser with Conducting Inclusions

Let us consider a condenser with small well-separated and uniformly distributed impurities. We can model this situation, introducing a proper periodic energy functional. Let us consider a compact set $K \subset]0,1[^n$, and let us define the 1-periodic energy density $f : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty]$ (this is a scalar model: $N = 1$) by setting

$$f(x,\xi) = \begin{cases} |\xi|^2 & \text{if } x \in [0,1]^n \setminus K \\ 0 & \text{if } x \in K \text{ and } \xi = 0 \\ +\infty & \text{if } x \in K \text{ and } \xi \neq 0. \end{cases} \tag{4.1}$$

on $[0,1]^n \times \mathbb{R}^n$, and extended by periodicity to $\mathbb{R}^n \times \mathbb{R}^n$. The set K represents the region occupied by the perfect conductor, where the potential must be constant, hence Du must be 0 in K. This constraint is included in the energy density f by the position $f(x,\xi) = +\infty$ if $x \in K$ and $\xi \neq 0$.

The function f does not satisfy the hypotheses of our homogenization theorem. However, the region where the growth conditions (2.1) are violated is composed of "well isolated" domains. This fact gives us hope that the homogenization process may be carried over all the same.

Let us remark that the compactness argument of Lesson Two applies to completely abstract functionals, once we prove the inner regularity of the localized Γ-liminf and Γ-limsup. These properties, in their turn, can be derived from the Fundamental Estimate (2.13). It is easy to see that the Dirichlet integral verifies the Fundamental Estimate. Hence fixed A, A', B, η, v, w as in Lemma 2.4 we can find a cut-off function ϕ between A and A' such that

$$\int_{A'\cup B} |D(\phi w + (1-\phi)v)|^2\,dx \leq (1+\eta)\int_A |Dw|^2\,dx + \int_B |Dv|^2 + M\int_{A\cap B} |v-w|^2. \tag{4.2}$$

[5] This is an appreciable feature, since by Murphy's Law however complete is a theory every time we try to apply it we find an exception.

Fixed ε we can modify the cut-off function ϕ as to obtain another cut-off function $\widetilde{\phi}$ between A and A' such that $D\widetilde{\phi} = 0$ on $\varepsilon\mathbf{Z}^n + \varepsilon K$, $\widetilde{\phi} = \phi$ outside a neighbourhood of $\varepsilon\mathbf{Z}^n + \varepsilon K$, giving

$$\mathcal{F}_\varepsilon(\widetilde{\phi}w + (1-\widetilde{\phi})v, A' \cup B) \leq (1+\eta)(\mathcal{F}_\varepsilon(w,A) + \mathcal{F}_\varepsilon(v,B)) + M(\zeta)\int_{A\cap B} |v-w|^2\,dx \tag{4.3}$$

where $\zeta > 0$ is small enough so that the ζ-neighbourhood of K is still compactly contained in $]0,1[^n$ (the proof of this statement, which goes beyond the scope of the course, relies on the construction of ϕ). Hence our functionals still verify the Fundamental Estimate, and we can infer the existence of the limit in (2.10). In the same way we prove the "measure property" (a). The properties (b), (d) and (e) still hold trivially, as does the growth estimate from below:

$$\int_A |Du|^2\,dx \leq F(u,A). \tag{4.4}$$

Let us prove that we have also a growth inequality from above. By the lower semicontinuity of F it suffices to prove the estimate for piecewise constant u; by the measure property of F it suffices to prove it for u affine; by the translation invariance it is enough to prove it for $u = \xi x$ linear. Let us consider a function $u_\xi \in \mathrm{W}^{1,2}_0(]0,1[^n)$ such that $u_\xi(x) = -\xi x$ on K and $\int_{]0,1[^n} |Du|^2\,dx \leq C|\xi|^2$ (C independent of ξ), and let us define

$$u_k(x) = \xi x + \varepsilon_{h_k} u_\xi(\frac{x}{\varepsilon_{h_k}}), \tag{4.5}$$

so that

$$u_k \to \xi x, \qquad Du_k = 0 \text{ on } \varepsilon_{h_k}(\mathbf{Z}^n + K) \qquad Du_k = \xi + Du_\xi(\frac{x}{\varepsilon_{h_k}}). \tag{4.6}$$

We have then

$$\begin{aligned} F(\xi x, A) &\leq \lim_k \int_A f(\frac{x}{\varepsilon_{h_k}}, Du_k(x))\,dx = |A| \int_{]0,1[^n} f(y, \xi + Du_\xi(y))\,dy \\ &= |A| \int_{]0,1[^n} |\xi + Du_\xi(y)|^2\,dy \leq 2C|A||\xi|^2. \end{aligned} \tag{4.7}$$

Recalling (4.4) we obtain for the functional F also the growth conditions (c), and we can apply the integral representation Theorem 2.2, obtaining (2.11). At this point the use of the Fundamental Estimate allows us to deal with boundary value problems, so that the proof of the Asymptotic Homogenization Formula can be repeated without changing a word (the formula itself must be slightly modified to take into account the constraint $Du = 0$ on $\mathbf{Z} + K$). Finally we use again the Fundamental Estimate to obtain the Convex Homogenization Formula, that can be rewritten as

$$f_\#(\xi) = \min\Big\{\int_{]0,1[^n} |Du(y) + \xi|^2\,dy \ : \ u \in \mathrm{W}^{1,2}_\#(]0,1[^n),\ Du = -\xi \text{ on } K\Big\}. \tag{4.8}$$

Remark 4.1. We can take, instead of $f(x,\xi) = |\xi|^2$ on $[0,1]^n \setminus K$, any function f satisfying the growth conditions (2.1). The same procedure works in the vector-valued case, except of course the proof of the Convex Homogenization Formula if $N > 1$ and f is not convex.

4.2. Homogenization of Connected Media

Let us consider now the case of an elastic body occupying a region Ω with a microscopically periodic structure (as an example we can think of a sponge). If $W : \mathrm{M}^{n\times N} \to [0,+\infty]$ represents the elastic stored energy density of the material, we have then to study functionals of the form

$$\mathcal{F}_\varepsilon(u) = \int_{\Omega\cap\varepsilon E} W(Du)\,dx, \tag{4.9}$$

where E is a periodic set describing the microscopical structure of the medium.

Let us suppose that the function W satisfies a standard growth condition of order p. We can try to apply our methods to the function

$$f(x,\xi) = \begin{cases} W(\xi) & \text{if } x \in E \\ 0 & \text{otherwise,} \end{cases} \tag{4.10}$$

which globally satisfies a growth condition of the form

$$0 \le f(x,\xi) \le C_1(1+|\xi|^p). \tag{4.11}$$

Even though we do not have a homogenization theorem for functions satisfying only (4.11), in this case it is possible to prove, for the functionals $\mathcal{F}_\varepsilon$, the fundamental estimate (2.13) thanks to the special form of f (that is identically 0 where it does not satisfy a growth condition of order p), and then to carry over the proof of Theorems 2.1 and 2.2 (notice that in Theorem 2.2 we do not need the growth condition from below). The proof of Proposition 3.2 needs no change, so that we obtain a homogenized integrand by the asymptotic homogenization formula

$$f_{\mathrm{hom}}(\xi) = \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{ \int_{]0,t[^n\cap E} W(Du(x)+\xi)\,dx : \ u \in \mathrm{W}_0^{1,p}(]0,t[^n;\mathbb{R}^N) \Big\}, \tag{4.12}$$

with f_{hom} verifying

$$0 \le f_{\mathrm{hom}}(\xi) \le C_1(1+|\xi|^p). \tag{4.13}$$

It is interesting to understand under what conditions the limit function verifies also a growth condition from below, that is when it maintains the elastic properties of the material described by W. If E is not connected it is easy to see in general that we may have $f_{\mathrm{hom}} \equiv 0$. If E is connected and has Lipschitz boundary (more in general if E contains a connected subset with Lipschitz boundary) then f_{hom} does satisfy a growth condition from below. This fact can be deduced from the following extension lemma, which assures that the functionals $(\mathcal{F}_\varepsilon)$ are equicoercive even though the function f does not satisfy (2.1) pointwise.

Theorem 4.2. (Extension Lemma (Acerbi, Chiadò Piat, Dal Maso & Percivale [1])) *If $\Omega \in \mathcal{A}_n$ and $\varepsilon > 0$, then there exists a linear and continuous operator $T_\varepsilon : \mathrm{W}^{1,p}(\Omega \cap \varepsilon E) \to \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$, and three constants $k_0, k_1, k_2 > O$, independent of Ω and ε, such that $T_\varepsilon u = u$ in $\Omega \cap \varepsilon E$, and*

(4.14)

$$\int_{\Omega(\varepsilon k_0)} |T_\varepsilon u|^p \, dx \leq k_1 \int_{\Omega \cap \varepsilon E} |u|^p \, dx, \qquad \int_{\Omega(\varepsilon k_0)} |D(T_\varepsilon u)|^p \, dx \leq k_2 \int_{\Omega \cap \varepsilon E} |Du|^p \, dx,$$

where $\Omega(\varepsilon k_0) = \{x \in \Omega \ : \ \mathrm{dist}\,(x, \partial\Omega) > \varepsilon k_0\}$.

With the help of Theorem 4.2 it is possible to prove that

$$\frac{1}{k_2}|\xi| - c_1 \leq f_{\mathrm{hom}}(\xi),$$

where k_2 is the constant in (4.14) (for further details see [1] and [12]).

4.3. Homogenization with Non-Standard Growth Conditions

Our model problem will be the study of the fine mixture of two materials whose energy densities have different (but not *too* different) growths at infinity. For the sake of simplicity let us assume that the energy density of the first material is exactly $|\xi|^p$, while the energy density of the second material is $|\xi|^q$, with $q > p$. We have to study then the functional

$$\mathcal{F}_\varepsilon(u) = \int_{\Omega \cap \varepsilon E_1} |Du|^p \, dx + \int_{\Omega \cap \varepsilon E_2} |Du|^q \, dx, \tag{4.15}$$

where E_1, E_2 are two disjoint measurable 1-periodic sets with $E_1 \cup E_2 = \mathbb{R}^n$. This functional can be rewritten in the form (1.1) by considering the function f defined by

$$f(x, \xi) = \begin{cases} |Du|^p & \text{if } x \in E_1 \\ |Du|^q & \text{if } x \in E_2. \end{cases} \tag{4.16}$$

This function globally satisfies a growth condition of the form

$$|\xi|^p - c_1 \leq f(x, \xi) \leq C_1(1 + |\xi|^q). \tag{4.17}$$

In this case it can be seen that the functionals $\mathcal{F}_\varepsilon$ satisfy a weaker fundamental estimate, obtaining in (2.13)

$$\mathcal{F}_\varepsilon(\phi w + (1-\phi)v, A' \cup B) \leq (1+\eta)\Big(\mathcal{F}_\varepsilon(w, A) + \mathcal{F}_\varepsilon(v, B)\Big) + M \int_{A \cap B} |w - v|^q \, dx. \tag{4.18}$$

By the Rellich-Kondrachov compactness theorem (see [29] Section 2.5) this estimate is sufficient to carry over the proof of the compactness Theorem 2.1 in the case $q < \frac{np}{n-p}$ (any q if $p \geq n$).

In this case however, the limit functional can be finite on a different space than $\mathrm{W}^{1,p}(\Omega; \mathbb{R}^N)$, as shown by the following examples.

Example 4.3. (see [12]) In general, if we make no assumptions on the connectedness of the set E_2, the function f_{hom} may have different growth with respect to different directions. For instance, let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty[$ be of the form $f(x, \xi) = |\xi|^{P(x_1)}$, where $x = (x_1, x_2)$, and P is the periodic function given by

$$P(t) = \begin{cases} p & \text{if } 0 \le t \le \frac{1}{2} \\ q & \text{if } \frac{1}{2} < t < 1. \end{cases}$$

In this case, since $f(x, \cdot)$ is convex, $f_{\text{hom}} = f_\#$. Now, as f is independent of x_2, for every $\xi = (\xi_1, 0)$ the formula for $f_\#$ reduces to a one dimensional minimum problem, namely

$$f_\#(\xi) = \inf_v \Big\{ \int_0^1 |v'(t)|^{P(t)} dt : v(0) = 0, v(1) = \xi_1 \Big\}.$$

Taking $v(t) = 2t\xi_1$ if $0 \le t \le \frac{1}{2}$, and $v(t) = \xi_1$ if $\frac{1}{2} < t \le 1$, we obtain $f_\#(\xi) \le 2^{q-1}|\xi|^q$ when $\xi = (\xi_1, 0)$. Conversely, if $\xi = (0, \xi_2)$, we have

$$f_\#(\xi) = \inf_v \Big\{ \int_0^1 \int_0^1 |v'(t) + \xi_2|^{P(s)}\, dt\, ds : v(0) = v(1) = 0 \Big\} = \int_0^1 |\xi_2|^{P(s)} ds = \frac{1}{2}|\xi|^p + \frac{1}{2}|\xi|^q,$$

hence f_0 has a growth of order p at infinity along the direction $e_2 = (0, 1)$. In conclusion the domain of the homogenized functional is a “non-isotropic Sobolev space”.

When E_2 is Lipschitz, connected and periodic then by the arguments of Section 4.2 we obtain that $f_\#$ still satisfies a growth condition of order p.

Example 4.4. In [24] an example is given of a function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of the form $f(x, \xi) = |\xi|^{P(x)}$ such that we have

$$c_2|\xi|^p \log(e + |\xi|) \le f_\#(\xi) \le C_2(1 + |\xi|^p \log(e + |\xi|));$$

hence the domain of the homogenized functional may be in general a Orlicz-Sobolev space.

Exercises

1. find a function $P : [0, 1] \to \{q, p\}$ such that, if we define $f(x, \xi) = |\xi|^{P(x)}$, then $f_\#$ satisfies a growth condition of order p.
2. find a function $P : [0, 1] \to \{q, p\}$ such that, if we define $f(x, \xi) = |\xi|^{P(x)}$, then $f_\#$ satisfies a growth condition of order q.

Lesson Five. Examples: Other Homogenization Formulas

In this lesson we shall deal with homogenization problems which give rise to different asymptotic formulas.

5.1. Singular Perturbation and Homogenization

In some theories of non linear elasticity higher order gradients have been introduced to explain the formation of the so-called shear bands under severe loadings. Francfort and Müller have analyzed in [22] the effect of such perturbations at a microscopical scale by introducing functionals of the form

$$\mathcal{F}_\varepsilon^\gamma(u) = \int_\Omega f(\frac{x}{\varepsilon}, Du)\,dx + \varepsilon^\gamma \int_\Omega |\triangle u|^2\,dx, \qquad u \in \mathrm{W}^{2,2}(\Omega;\mathbb{R}^N) \cap \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N) \tag{5.1}$$

where $\gamma > 0$ is a parameter relating the microscopical scale ε and the strength of the perturbation. We shall suppose that the function $f : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0,+\infty[$ satisfies the conditions of Lessons Two and Three, so that it is homogenizable with homogenized function $f_{\rm hom}$.

Let us consider the case $\gamma = 2$. In this case let us define the function $f^2_{\rm hom} : \mathrm{M}^{n\times N} \to [0,+\infty[$ by setting

$$\begin{aligned} f^2_{\rm hom}(\xi) &= \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{\int_{]0,t[^n} f(x, Du(x)+\xi)\,dx + \int_{]0,t[^n} |\triangle u|^2\,dx \\ &\qquad : \ u \in \mathrm{W}^{2,2}(]0,t[^n;\mathbb{R}^N) \cap \mathrm{W}^{1,p}_0(]0,t[^n;\mathbb{R}^N)\Big\} \\ &= \lim_{\varepsilon\to 0} \inf\Big\{\int_{]0,1[^n} f(\frac{x}{\varepsilon}, Du(x)+\xi)\,dx + \varepsilon^2 \int_{]0,1[^n} |\triangle u|^2\,dx \\ &\qquad : \ u \in \mathrm{W}^{2,2}(]0,1[^n;\mathbb{R}^N) \cap \mathrm{W}^{1,p}_0(]0,1[^n;\mathbb{R}^N)\Big\} \end{aligned} \tag{5.2}$$

(the existence of this limit can be proven by following the proof of Proposition 3.2). It is not difficult to follow the proof of the compactness and integral representation results of Section 2, and to realize that the proof fits also these functionals, as well as Proposition 3.1 does. By (5.2) we have then that the whole family $\mathcal{F}^2_\varepsilon$ Γ-converges to the functional

$$\mathcal{F}^2(u) = \int_\Omega f^2_{\rm hom}(Du)\,dx \qquad u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N), \tag{5.3}$$

and the singular perturbation contributes to the definition of $f^2_{\rm hom}$.

In the other cases it is still possible to obtain by Theorems 2.1 and 2.2 and by Proposition 3.1 the integral representation

$$\mathcal{F}^\gamma(u) = \int_\Omega \varphi^\gamma(Du)\,dx \qquad u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N), \tag{5.4}$$

for Γ-converging subsequences of $(\mathcal{F}^\gamma_\varepsilon)$. Francfort and Müller have proven that we have the following two cases (different from $\gamma = 2$):

$\gamma > 2$: the singular perturbation turns out to be irrelevant in the limit, and we have $\varphi^\gamma = f_{\rm hom}$ for all γ;

$\gamma < 2$: (*i.e.*, the length scale ε is small compared with the singular perturbation) the singular perturbation has the dominant role, and forbids large oscillations in minimizing sequences. In this case we have $\varphi^\gamma = \overline{f}$ for all γ, where $\overline{f}$ is the largest quasiconvex function not greater than

$$\widetilde{f}(\xi) = \lim_{t\to+\infty} \frac{1}{t^n}\int_{]0,t[^n} f(x,\xi)\,dx \tag{5.5}$$

(for a proof in the periodic case see [22]).

5.2. Reiterated Homogenization

Let us consider a medium with two different scales of microstructures. The overall behavior in such a case can be modeled by the asymptotic behaviour of functionals of the form

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Du)\,dx \tag{5.5}$$

where ε^2 represents the finer microstructure. Again, if f verifies a growth condition of order p we can repeat word by word the proofs of Theorems 2.1 and 2.2 and Proposition 3.1, to obtain the integral representation

$$\mathcal{F}(u) = \int_\Omega \varphi(Du)\,dx \tag{5.6}$$

for Γ-converging subsequences of $(\mathcal{F}_\varepsilon)$.

In the case of $f = f(x,y,\xi)$ 1-periodic in x and in y, convex in ξ, and piecewise uniformly continuous in x, it is possible to prove a representation formula that permits to conclude the homogenization procedure. As in Remark 3.3 we have to describe the limit

$$\lim_k \frac{1}{k^n}\inf\Big\{\int_{]0,k[^n} f(x,kx,Du(x)+\xi)\,dx :\ u \in \mathrm{W}^{1,p}_\#(]0,k[^n;\mathbb{R}^N)\Big\}, \tag{5.7}$$

In this case, proceeding as in Proposition 3.5 we see that

$$\inf\Big\{\int_{]0,k[^n} f(x,kx,Du(x)+\xi)\,dx:\ u\in W^{1,p}_{\#}(]0,k[^n;\mathbb{R}^N)\Big\}$$

(5.8)

$$= \inf\Big\{\int_{]0,1[^n} f(x,kx,Du(x)+\xi)\,dx:\ u\in W^{1,p}_{\#}(]0,1[^n;\mathbb{R}^N)\Big\}.$$

In the second formula of (5.8) the first x acts as a parameter (see [6]), so that we have the asymptotic formula

$$\lim_k \inf\Big\{\int_{]0,1[^n} f(x,kx,Du(x)+\xi)\,dx:\ u\in W^{1,p}_{\#}(]0,1[^n;\mathbb{R}^N)\Big\}$$

(5.9)

$$= \inf\Big\{\int_{]0,1[^n} f_{\rm hom}(x,Du(x)+\xi)\,dx:\ u\in W^{1,p}_{\#}(]0,1[^n;\mathbb{R}^N)\Big\},$$

where

$$f_{\rm hom}(x,\xi)=\inf\Big\{\int_{]0,1[^n} f(x,y,Du(y)+\xi)\,dy:\ u\in W^{1,p}_{\#}(]0,1[^n;\mathbb{R}^N)\Big\} \tag{5.10}$$

In conclusion, in order to obtain the homogenized integrand we have to "iterate" the convex homogenization formula. For further details we refer to [4], [6] and [9].

Exercises

1. Recalling that if $n=1$ and $f(x,\xi)=a(x)\xi^2$, with

$$a(x)=\begin{cases}\alpha & \text{if } 0\le x<1/2\\ \beta & \text{if } 1/2\le x<1\end{cases}$$

($\alpha,\beta>0$), then $f_{\rm hom}(\xi)=\dfrac{2\alpha\beta}{\alpha+\beta}\,\xi^2$, compute the homogenized functionals of

1) $$\mathcal{F}_\varepsilon(u)=\int_{]0,1[} a(\frac{x}{\varepsilon})\,a(\frac{x}{\varepsilon^2})(u'(x))^2\,dx$$

2) $$\mathcal{F}_\varepsilon(u)=\int_{]0,1[} \Big(a(\frac{x}{\varepsilon})+a(\frac{x}{\varepsilon^2})\Big)(u'(x))^2\,dx$$

3) $$\mathcal{F}_\varepsilon(u)=\int_{\Omega} a(\frac{x}{\varepsilon})\,a(\frac{y}{\varepsilon^2})\Big((\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2\Big)\,dx\,dy$$

4) $$\mathcal{F}_\varepsilon(u)=\int_{\Omega} \Big(a(\frac{x}{\varepsilon})+a(\frac{y}{\varepsilon^2})\Big)\Big((\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2\Big)\,dx\,dy$$

2. State the homogenization theorem for a medium with n different scales of microstructures.

5.3. Homogenization of Hamilton-Jacobi Equations

Let us consider a *Hamiltonian* $H = H(t, x, \xi) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$ verifying a growth condition of order p', 1-periodic in the first two variables and convex in the last variable.

We shall study the limiting behaviour of the (viscosity) solutions of the Cauchy problem

$$(5.11) \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + H(\dfrac{t}{\varepsilon}, \dfrac{x}{\varepsilon}, Du_\varepsilon) = 0 & \text{in } \mathbb{R}^n \times [0, +\infty[\\ u_\varepsilon(x, 0) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where φ is a given bounded and uniformly continuous function in $\mathbb{R}^n$ (see [20], [10]).

Let us define the *Legendre transform* of H:

$$L(t, u, \xi) = \sup_{\xi' \in \mathbb{R}^n} \{(\xi, \xi') - H(t, u, \xi')\},$$

for every (t, u, ξ). Let us remark that L verifies a growth condition of order p.

Following P.L.Lions [25] Theorem 11.1 we can define for $x, y \in \mathbb{R}^n$ and $0 \le s < t$

$$S_\varepsilon(x, t; y, s) = \inf \left\{ \int_s^t L(\frac{\tau}{\varepsilon}, \frac{u(\tau)}{\varepsilon}, u'(\tau))d\tau \ : \ u(s) = y, \ u(t) = x, \ u \in \mathrm{W}^{1,\infty}((s, t); \mathbb{R}^n) \right\}$$

$$= \inf \Big\{ \int_s^t L(\frac{\tau}{\varepsilon}, \frac{u(\tau)}{\varepsilon}, u'(\tau))d\tau \ : \ u(\tau) - \big(\frac{y - x}{s - t}(\tau - s) + y\big) \in \mathrm{W}_0^{1,p}((s, t); \mathbb{R}^n) \Big\}.$$

Then the unique *viscosity solution* to problem (5.11) is given by the Lax formula:

$$u_\varepsilon(x, t) = \inf\{\varphi(y) + S_\varepsilon(x, t; y, s) \ : \ y \in \mathbb{R}^n, 0 \le s < t\}.$$

In order to study the asymptotic behaviour as $\varepsilon \to 0$ of the functions u_ε we have to compute the limits of $S_\varepsilon(x, t; y, s)$, and hence to study the Γ-convergence of the functionals

$$(5.12) \qquad \mathcal{F}_\varepsilon(u) = \int_s^t L(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}, Du)\, dx.$$

As in the previous Sections, since L verifies a growth condition of order p we can repeat almost word by word the proofs of Theorems 2.1 and 2.2 and Proposition 3.1, to obtain the integral representation

$$(5.13) \qquad \mathcal{F}(u) = \int_\Omega \varphi(Du)\, dx$$

for Γ-converging subsequences of $(\mathcal{F}_\varepsilon^\gamma)$. The function φ is identified by the following proposition.

Proposition 5.1. *The limit*

$$\overline{L}(\xi) = \lim_{T\to\infty} \frac{1}{T} \inf\left\{ \int_0^T L(\tau, u(\tau)+\xi\tau, u'(\tau)+\xi)d\tau \ : \ u \in \mathrm{W}_0^{1,p}((0,T);\mathbb{R}^n) \right\} \tag{5.14}$$

exists for every $\xi \in \mathbb{R}^n$.

Proof. The proposition is analogous to the Asymptotic Homogenization Formula, and the proof follows the same line. Note however that while in Proposition 3.2 we consider perturbations of the function $f(x,\xi)$, here we have to deal with the function $L(x,\xi x,\xi)$, which in general is not periodic; hence we have to consider the function L as uniformly almost periodic, and use the fact that restrictions of uniformly almost periodic functions to linear subspaces are still uniformly almost periodic (see [5]). For more details see [10]. □

At this point we can infer as in Section 3 that $\varphi = \overline{L}$. By the fundamental theorem of the Γ-convergence we have then that for every $x, y \in \mathbb{R}^n$ and $0 \le s < t$

$$\begin{aligned} S_\varepsilon(x,t;y,s) \to \min &\left\{ \int_s^t \overline{L}(u'(\tau))d\tau \ : \ u(\tau) - \Big(\frac{y-x}{s-t}(\tau - s) + y\Big) \in \mathrm{W}_0^{1,q}((s,t);\mathbb{R}^n) \right\} \\ =&(t-s)\,\overline{L}\Big(\frac{y-x}{s-t}\Big), \end{aligned}$$

the last equality following by the convexity of $\overline{L}$ and Jensen's inequality. By the growth hypothesis on L we obtain that the functions $S_\varepsilon(x,t;\cdot,\cdot)$ are equicontinuous in $\{y \in \mathbb{R}^n, 0 \le s \le t-\eta\}$, and then

$$u_\varepsilon(x,t) \to u(x,t)$$

pointwise, where

$$u(x,t) = \inf\{\varphi(y) + (t-s)\overline{L}\Big(\frac{y-x}{s-t}\Big) \ : \ y \in \mathbb{R}^n, 0 \le s < t\}.$$

Since the functions u_ε are equicontinuous on compact sets, the convergence is uniform on bounded sets. Again by the Lax formula in [25] Theorem 11.1, u is the unique viscosity solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times [0,+\infty[\\ u(x,0) = \varphi(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{5.15}$$

where the *effective Hamiltonian* $\overline{H}$ is defined by

$$\overline{H}(\xi) = \sup_{\xi' \in \mathbb{R}^n} \{(\xi,\xi') - \overline{L}(\xi')\}. \tag{5.16}$$

We have then the following convergence result.

Theorem 5.2. *Let φ be a given bounded and uniformly continuous function in* $\mathbb{R}^n$*, and let u_ε be the unique viscosity solution of* (5.11)*; then u_ε converges uniformly on compact sets as $\varepsilon \to 0$ to the unique viscosity solution of the Cauchy problem* (5.15)*, with $\overline{H}$ given by* (5.14)*,* (5.16)*.*

Example 5.3. Let $n = 1$ and $H(x,\xi) = |\xi|^2 - V(x)$, with V uniformly almost periodic and $\inf V = 0$; then we can give an alternative definition of $\overline{H}$: for every $\xi \in \mathbb{R}^n$, $\overline{H}$ is the unique constant such that the stationary problem

$$H(x, \xi + Du(x)) = |\xi + Du(x)|^2 - V(x) = \overline{H}(\xi) \tag{5.16}$$

has a uniformly almost periodic solution u with u' continuous. When $\overline{H}(\xi) > 0$, from equation (5.16) we have

$$|u'(x) + \xi|^2 = V(x) + \overline{H}(\xi) > 0\,,$$

hence, by the requirement that u' be continuous,

$$u'(x) = -\xi + \sqrt{V(x) + \overline{H}(\xi)} \quad \text{or} \quad u'(x) = -\xi - \sqrt{V(x) + \overline{H}(\xi)}\,.$$

The function u is then uniformly almost periodic if and only if the mean value of u' is zero; *i.e.*,

$$|\xi| = \lim_{t\to+\infty} \frac{1}{2t} \int_{[-t,t]} \sqrt{V(x) + \overline{H}(\xi)}\, dx := \fint \sqrt{V(x) + \overline{H}(\xi)}\, dx.$$

Since $\overline{H}$ is positive and convex, we obtain the formula

$$\overline{H}(\xi) = \begin{cases} 0 & \text{if } |\xi| \le \fint \sqrt{V(x)}\ dx \\[2ex] \alpha & \text{if } |\xi| = \fint \sqrt{V(x) + \alpha}\ dx. \end{cases}$$

The flat piece in the graph of $\overline{H}$ corresponds to the lack of differentiability of $\overline{L}$ in 0, as already observed by Buttazzo & Dal Maso [14] Section 4a.

Notation

$B(x,r)$ open ball of center x and radius r;

$\{e_1,\dots,e_n\}$ canonical base of $\mathbb{R}^n$: $e_1=(1,0,\dots,0)$, $e_2=(0,1,0,\dots,0),\dots$;

$|E|$ Lebesgue measure of the set E;

$\mathcal{A}_n$ family of all bounded open subsets of $\mathbb{R}^n$;

$\mathcal{A}_n(\Omega)$ family of all bounded open subsets of $\Omega\subset\mathbb{R}^n$;

$\mathrm{W}^{k,p}(\Omega;\mathbb{R}^N)$ Sobolev space of $\mathbb{R}^N$-vaued functions on Ω with p-summable weak derivatives up to the order k (if $N=1$ we write $\mathrm{W}^{k,p}(\Omega)$); $\mathrm{L}^p(\Omega;\mathbb{R}^N)=\mathrm{W}^{0,p}(\Omega;\mathbb{R}^N)$;

$\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)=\mathrm{H}^{1,p}_0(\Omega;\mathbb{R}^N)$ closure in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ of compactly supported smooth functions;

p' conjugate exponent of p, *i.e.*, $\dfrac{1}{p}+\dfrac{1}{p'}=1$;

$\triangle u$ Laplacian of u.

References

We just list the papers directly referred to in the lessons. For a complete reference list see [15].

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