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Two-Scale Convergence and Homogenization of Periodic Structures

School on Homogenization
ICTP, Trieste, September 6–17, 1993

CONTENTS

Introduction.

The topics covered by these lecture notes are the homogenization (or asymptotic analysis) of periodic structures and the two-scale convergence method. These notes correspond roughly to three two-hours courses for graduate students, and thus are a mere introduction to the above subjects. The students are assumed to already have a slight knowledge of homogenization, and of one of its basic techniques : two-scale asymptotic expansions. However, this pre-requisite is by no means essential, since these notes are self-contained. General references for the homogenization of periodic structures are the books [5], [6], and [17] (cf. also the courses of A. Braides [8] and A. Defranceschi [9]). Two-scale convergence is a quite recent method, introduced by G. Nguetseng [16] and the author [2], which is especially well-suited for the problems encountered in the above books.

Section 1 briefly introduces a model problem in periodic homogenization, and recalls the usual method to solve it. Section 2 is devoted to the definition of a new type of convergence, called two-scale convergence. In section 3, it is applied to the homogenization of the model problem of section 1, and it is shown to be both efficient and simple. Section 4 deals with a more involved application of this method : the derivation of Darcy's law for fluid flows in porous media. Finally, section 5 contains a few technical results required in section 4, and concerning mainly an a priori estimate for the pressure.

1. Asymptotic analysis of periodic structures.

The title of this section is taken from the well-known book of A. Bensoussan, J.L. Lions, and G. Papanicolaou [6]. It describes perfectly one of the main applications of the homogenization theory. Indeed, in many fields of science and technology one has to solve boundary value problems in periodic media. Quite often the size of the period is small compared to the size of a sample of the medium, and, denoting by ε their ratio, an asymptotic analysis, as ε goes to zero, is called for. In other words, starting from a microscopic description of a problem, we seek a macroscopic, or effective, description. This process of making an asymptotic analysis and seeking an averaged formulation is called homogenization. Here, we focus on the homogenization of periodic structures, but we recall that homogenization is not restricted to that particular case and can be applied to any kind of disordered media (cf. the Γ-convergence of E. DeGiorgi [9], the G-convergence of S. Spagnolo [18], see also [23], or the H-convergence of L. Tartar [19], [15]).

To fix ideas, we consider the well-known model problem in homogenization : a linear second-order partial differential equation with periodically oscillating coefficients. Such an equation models, for example, the heat conduction in a periodic composite medium. We call Ω the material domain (a bounded open set in $\mathbb{R}^N$), ε the period, and Y the rescaled unit cell (i.e. $Y = [0;1]^N$). Denoting by f the source term (a function of $L^2(\Omega)$), and enforcing a Dirichlet boundary condition for the unknown

u_ε, this equation reads as

$$\begin{cases} -\, div\left[A(\frac{x}{\varepsilon}) \nabla u_\varepsilon \right] = f \quad in \;\; \Omega \\ u_\varepsilon = 0 \quad on \;\; \partial\Omega \end{cases} \tag{1.1}$$

where $A(y)$ is a $L^\infty(Y)$-matrix (the diffusion coefficients), Y-periodic in y, such that there exists two positive constants $0 < \alpha \leq \beta$ satisfying

$$\alpha |\xi|^2 \;\leq\; \sum_{i,j=1}^{N} A_{ij}(x,y)\xi_i\xi_j \;\leq\; \beta |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^N. \tag{1.2}$$

Under assumption (1.2), it is well-known that equation (1.1) admits a unique solution u_ε in $H_0^1(\Omega)$ which satisfies the a priori estimate

$$||u_\varepsilon||_{H_0^1(\Omega)} \;\leq\; C\, ||f\,||_{L^2(\Omega)} \tag{1.3}$$

where C is a positive constant which depends only on Ω and α, and not on ε and f. In view of (1.3), the sequence of solutions u_ε is uniformly bounded in $H_0^1(\Omega)$ as ε goes to zero, and thus there exists a limit u such that, up to a subsequence, u_ε converges weakly to u in $H_0^1(\Omega)$. The homogenization of (1.1) amounts to find a "homogenized" equation which admits the limit u as its unique solution.

Let us briefly recall the classical method for the homogenization of the model problem (1.1). In a first step, the well known two-scale asymptotic expansion method is applied in order to find the precise form of the homogenized equation. The key of that method is to postulate the following ansatz for u_ε

$$u_\varepsilon(x) \;=\; u_0(x,\frac{x}{\varepsilon}) + \varepsilon\, u_1(x,\frac{x}{\varepsilon}) + \varepsilon^2\, u_2(x,\frac{x}{\varepsilon}) + \;\cdots, \tag{1.4}$$

where each term $u_i(x,y)$ is Y-periodic in y. The ansatz (1.4) is inserted in equation (1.1), and a geometric series in ε is obtained by application of the formal rule of differentiation

$$\frac{\partial}{\partial x}\left[u_i(x,\frac{x}{\varepsilon}) \right] \;=\; \frac{\partial u_i}{\partial x}(x,\frac{x}{\varepsilon}) + \varepsilon^{-1}\frac{\partial u_i}{\partial y}(x,\frac{x}{\varepsilon}).$$

Then, identifying the coefficients of this series to zero leads to a cascade of equations. The first one (corresponding to the ε^{-2} term) is

$$\begin{cases} -\, div_y\left[A(y)\nabla_y u_0 \right] = 0 \quad in \;\; Y \\ y \to u_0(x,y) \;\; Y\text{−}periodic. \end{cases}$$

This implies that u_0 doesn't depend on y, namely

$$u_0(x,y) \;=\; u(x).$$

The second one (the ε^{-1} term) is

$$\begin{cases} -\, div_y \left[A(y)[\nabla_y u_1(x,y) + \nabla_x u(x)] \right] = 0 \quad in \ Y \\ y \to u_1(x,y) \quad Y\text{–}periodic. \end{cases} \tag{1.5}$$

From (1.5) we compute u_1 in terms of the gradient of u :

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y),$$

where, for $1 \leq i \leq N$, w_i is the unique solution of the so-called local or cell problem

$$\begin{cases} -\, div_y \left[A(y)[\nabla_y w_i(y) + e_i] \right] = 0 \quad in \ Y \\ y \to w_i(y) \quad Y\text{–}periodic. \end{cases} \tag{1.6}$$

Finally the third one (the ε^0 term) is

$$\begin{cases} -\, div_y \left[A(y)\nabla_y u_2(x,y) \right] = f(x) + div_y \left[A(y)\nabla_x u_1(x,y) \right] \\ \qquad\qquad + \, div_x \left[A(y)[\nabla_y u_1(x,y) + \nabla_x u(x)] \right] \quad in \ Y \\ y \to u_2(x,y) \quad Y\text{–}periodic. \end{cases} \tag{1.7}$$

Applying the Fredholm alternative to (1.7) (the average on Y of the right hand side must be zero), and replacing u_1 by its expression (1.6) leads to the homogenized equation

$$\begin{cases} -\, div \left[A^* \nabla u(x) \right] = f \quad in \ \Omega \\ u = 0 \quad on \ \partial\Omega \end{cases} \tag{1.8}$$

where the entries of the matrix A^* are given by

$$A^*_{ij} = \int_Y A(y)[\nabla_y w_i(y) + e_i].[\nabla_y w_j(y) + e_j] \, dy. \tag{1.9}$$

This method is very simple and powerful, but unfortunately is formal since there is no reason, a priori, for the ansatz (1.4) to hold true. Thus, the two-scale asymptotic expansion method is used only to guess the form of the homogenized equation (1.8), and a second step is needed to prove the convergence of the sequence u_ε to u. To this end, many methods are available (Γ or G-convergence, maximum principle in the scalar case, etc), but the more general and powerful one is the so-called energy method (introduced by L. Tartar [19], [15]). Its name is not really adequate, since its main ingredient is a clever choice of test functions (thus it should have been named "test function method" rather than "energy method", which does not pertain to any kind of energy...). More precisely, the goal of this method is to pass to the limit in the variational formulation of equation (1.1) :

$$\int_\Omega A(\frac{x}{\varepsilon})\nabla u_\varepsilon(x).\nabla\phi(x)\, dx \;=\; \int_\Omega f(x)\phi(x)\, dx \quad \textit{for any } \phi \in H_0^1(\Omega). \tag{1.10}$$

For a given test function ϕ one cannot pass to the limit in (1.10), as ε goes to zero, since the left hand side involves the product of two weakly convergent sequences. The main idea is thus to replace the fixed test function ϕ by a carefully chosen sequence ϕ_ε which permits to pass to the limit thanks to some "compensated compactness" phenomenon (see [21] for this notion). The right sequence of test functions is

$$\phi_\varepsilon(x) \;=\; \phi(x) + \varepsilon\sum_{i=1}^N \frac{\partial\phi}{\partial x_i}(x)\tilde{w}_i(\frac{x}{\varepsilon}), \tag{1.11}$$

where ϕ is a smooth function with compact support in Ω, and $\tilde{w}_i$ is the solution of the adjoint cell problem (i.e. equation (1.6) with ${}^tA$ instead of A). Integrating by parts in (1.10) and using the cell equation (1.6) allows us to pass to the limit and to obtain the variational formulation of the homogenized problem (1.8). The convergence of the homogenization process is thus rigorously proved.

Although the asymptotic expansion method leads to both the local and the homogenized problem, the energy method uses only the knowledge of the cell problem to construct the test functions. The homogenized problem is then rederived independently. Clearly the two methods don't cooperate very much, and part of the homogenization process is done twice. On the contrary, we are going to see that the two-scale convergence is efficient because it is self-contained (i.e. it works in a single step). Loosely speaking, it appears as a blend of the two above methods.

2. Two-scale convergence.

Let us begin this section by a few notations : Ω is an open set of $\mathbb{R}^N$ (not necessarily bounded), and $Y = [0;1]^N$ is the closed unit cube. We denote by $C_\#^\infty(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^N$ which are periodic of period Y, and by $C_\#(Y)$ the Banach space of continuous and Y-periodic functions. Eventually, $D[\Omega;C_\#^\infty(Y)]$ denotes the space of infinitely smooth and compactly supported functions in Ω with values in the space $C_\#^\infty(Y)$.

Definition 2.1.

A sequence of functions u_ε in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0(x,y)$ belonging to $L^2(\Omega\times Y)$ if, for any function $\psi(x,y)$ in $D[\Omega;C_\#^\infty(Y)]$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx \;=\; \int_\Omega\int_Y u_0(x,y)\psi(x,y)\, dxdy\ . \tag{2.1}$$

This new notion of "two-scale convergence" makes sense because of the next compactness theorem (see [2] and [16]).

Theorem 2.2.
From each bounded sequence u_ε in $L^2(\Omega)$ one can extract a subsequence, and there exists a limit $u_0(x,y) \in L^2(\Omega\times Y)$ such that this subsequence two-scale converges to u_0.

Before proving Theorem 2.2, we give a few examples of two-scale convergences.

(*) Any sequence u_ε which converges strongly in $L^2(\Omega)$ to a limit $u(x)$, two-scale converges to the same limit $u(x)$.

(**) For any smooth function $a(x,y)$, being Y-periodic in y, the associated sequence $a_\varepsilon(x) = a(x, x/\varepsilon)$ two-scale converges to $a(x,y)$.

(***)For the same smooth and Y-periodic function $a(x,y)$ the other sequence defined by $b_\varepsilon(x) = a(x, \frac{x}{\varepsilon^2})$ has the same two-scale limit and weak-L^2 limit, namely $\int_Y a(x,y)\, dy$ (this is a consequence of the difference of orders in the speed of oscillations for b_ε and the test functions $\psi(x, \frac{x}{\varepsilon})$) Clearly the two-scale limit captures only the oscillations which are in resonance with those of the test functions $\psi(x, \frac{x}{\varepsilon})$.

To establish theorem 2.2, we need the following

Lemma 2.3.
Let $B(\Omega,Y)$ denote the Banach space $L^2[\Omega;C_\#(Y)]$ if Ω is unbounded, or any of the Banach spaces $L^2[\Omega;C_\#(Y)]$, $L^2_\#[Y;C(\overline{\Omega})]$, $C[\overline{\Omega};C_\#(Y)]$, if Ω is bounded. Then, this space $B(\Omega,Y)$ has the following properties :

(i) *$B(\Omega,Y)$ is a separable Banach space (i.e. contains a dense countable family)*

(ii) *$B(\Omega,Y)$ is dense in $L^2(\Omega\times Y)$*

(iii) *for any $\psi(x,y) \in B(\Omega,Y)$, the function $\psi(x, \frac{x}{\varepsilon})$ is measurable and satisfies*

$$\| \psi(x, \frac{x}{\varepsilon}) \|_{L^2(\Omega)} \leq \| \psi(x,y) \|_{B(\Omega,Y)}$$

(iv) *for any $\psi(x,y) \in B(\Omega,Y)$, one has*

$$\lim_{\varepsilon\to 0} \int_\Omega \psi(x, \frac{x}{\varepsilon})^2\, dx = \int_\Omega\int_Y \psi(x,y)^2\, dxdy \ .$$

In the case where Ω is bounded and $B(\Omega,Y)$ is defined as $C[\overline{\Omega};C_\#(Y)]$, lemma 2.3 is easily proved since any function $\psi(x,y)$ in this space is continuous in both variables x and y. In the other cases the delicate point is (iv) which holds true as soon as $\psi(x,y)$ is continuous in one of its arguments (as it is the case when ψ belongs to

$L^2[\Omega;C_\#(Y)]$ or $L^2_\#[Y;C(\overline{\Omega})]$). A complete proof of lemma 2.3 may be found in [2].

Proof of theorem 2.2.

Let u_ε be a bounded sequence in $L^2(\Omega)$: there exists a positive constant C such that

$$\| u_\varepsilon \|_{L^2(\Omega)} \leq C.$$

For any function $\psi(x,y) \in B(\Omega,Y)$, we deduce from (iii) in lemma 2.3 that

$$\left| \int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx \right| \leq C \,\| \psi(x,\frac{x}{\varepsilon}) \|_{L^2(\Omega)} \leq C \,\| \psi(x,y) \|_{B(\Omega,Y)} . \tag{2.2}$$

Thus, for fixed ε, the left hand side of (2.2) turns out to be a bounded linear form on $B(\Omega,Y)$. Let us denote by $B'(\Omega,Y)$ the dual space of $B(\Omega,Y)$. By virtue of the Riesz representation theorem, there exists a unique function $\mu_\varepsilon \in B'(\Omega,Y)$ such that

$$< \mu_\varepsilon,\psi > \; = \int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx \tag{2.3}$$

where the brackets in the left hand side of (2.3) denotes the duality product between $B(\Omega,Y)$ and its dual. Furthermore, in view of (2.2), the sequence μ_ε is bounded in $B'(\Omega,Y)$. Since the space $B(\Omega,Y)$ is separable (see (i) in lemma 2.3), from any bounded sequence of its dual one can extract a subsequence which converges for the weak * topology. Thus, there exists $\mu_0 \in B'(\Omega,Y)$ such that, up to a subsequence, and for any $\psi \in B(\Omega,Y)$

$$< \mu_\varepsilon,\psi > \; \to \; < \mu_0,\psi > . \tag{2.4}$$

By combining (2.3) and (2.4) we obtain, up to a subsequence, and for any $\psi \in B(\Omega,Y)$

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx \; = \; < \mu_0,\psi > . \tag{2.5}$$

By virtue of (iv) in lemma 2.3 we have

$$\lim_{\varepsilon\to 0} \| \psi(x,\frac{x}{\varepsilon}) \|_{L^2(\Omega)} = \| \psi(x,y) \|_{L^2(\Omega\times Y)} . \tag{2.6}$$

Now, passing to the limit in the first two terms of (2.2) with the help of (2.5) and (2.6), we deduce

$$| < \mu_0,\psi > | \leq C \,\| \psi \|_{L^2(\Omega\times Y)} .$$

By density of $B(\Omega,Y)$ in $L^2(\Omega\times Y)$ (see (ii) in lemma 2.3), μ_0 is identified with a function $u_0 \in L^2(\Omega\times Y)$, i.e.

$$< \mu_0,\psi > \; = \int_\Omega\int_Y u_0(x,y)\psi(x,y)\, dxdy \; . \tag{2.7}$$

Equalities (2.5) and (2.7) give the desired result.

Remark that the choice of the space $B(\Omega,Y)$ is purely technical and does not affect the final result of theorem 2.2. Remark also that the test function $\psi(x,y)$ in definition 2.1 of the two-scale convergence doesn't need to be very smooth since theorem 2.2 is proved, for example, with $\psi(x,y) \in L^2[\Omega;C_\#(Y)]$.

The next theorem shows that more information is contained in a two-scale limit than in a weak-L^2 limit ; some of the oscillations of a sequence are contained in its two-scale limit. When all of them are captured by the two-scale limit (condition (2.9) below), one can even obtain a strong convergence (a corrector result in the vocabulary of homogenization).

Theorem 2.4.

Let u_ε be a sequence of functions in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega\times Y)$.

(i) *Then u_ε converges also to $u(x) = \int_Y u_0(x,y)\, dy$ in $L^2(\Omega)$ weakly, and we have*

$$\lim_{\varepsilon\to 0} \| u_\varepsilon \|_{L^2(\Omega)} \geq \| u_0 \|_{L^2(\Omega\times Y)} \geq \| u \|_{L^2(\Omega)} . \tag{2.8}$$

(ii) *Assume further that $u_0(x,y)$ is smooth (for example, belongs to $L^2[\Omega;C_\#(Y)]$), and that*

$$\lim_{\varepsilon\to 0} \| u_\varepsilon \|_{L^2(\Omega)} = \| u_0 \|_{L^2(\Omega\times Y)}. \tag{2.9}$$

Then, we have

$$\lim_{\varepsilon\to 0} \| u_\varepsilon(x) - u_0(x,\frac{x}{\varepsilon}) \|_{L^2(\Omega)} = 0. \tag{2.10}$$

Proof.

By taking test functions $\psi(x)$, which depends only on x, in the definition of two-scale convergence, we immediately obtain that u_ε weakly converges to $u(x) = \int_Y u_0(x,y)\, dy$ in $L^2(\Omega)$. To obtain (2.8), we take a smooth and Y-periodic function $\psi(x,y)$ and we compute

$$\int_\Omega [u_\varepsilon(x) - \psi(x,\frac{x}{\varepsilon})]^2\, dx = \int_\Omega u_\varepsilon(x)^2\, dx - 2\int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})\, dx$$
$$+ \int_\Omega \psi(x,\frac{x}{\varepsilon})^2\, dx \geq 0. \tag{2.11}$$

Passing to the limit as ε goes to zero yields

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)^2\, dx \geq 2\int_\Omega\int_Y u_0(x,y)\psi(x,y)\, dxdy - \int_\Omega\int_Y \psi(x,y)^2\, dxdy .$$

Then, using a sequence of smooth functions which converges strongly to u_0 in $L^2(\Omega \times Y)$ leads to

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)^2 \, dx \ \geq \ \int_\Omega \int_Y u_0(x,y)^2 \, dxdy \ .$$

On the other hand, the Cauchy-Schwarz inequality in Y gives the other inequality in (2.8). To obtain (2.10) we use assumption (2.9) when passing to the limit in the right hand side of (2.11). This yields

$$\lim_{\varepsilon \to 0} \int_\Omega [u_\varepsilon(x) - \psi(x, \frac{x}{\varepsilon})]^2 \, dx \ = \ \int_\Omega \int_Y [u_0(x,y) - \psi(x,y)]^2 \, dxdy \ . \tag{2.12}$$

Now, if u_0 is smooth enough as to ensure that $u_0(x, \frac{x}{\varepsilon})$ is measurable and belongs to $L^2(\Omega)$, we can replace ψ by u_0 in (2.12) to obtain (2.10).

We have just seen that the smoothness assumption on u_0 in part (ii) of theorem 2.4 is needed only to achieve the measurability of $u_0(x, \frac{x}{\varepsilon})$ (which otherwise is not guaranteed for a function of $L^2(\Omega \times Y)$). However, one could wonder if all two-scale limits automatically satisfy this property. Unfortunately, this is not true, and it can be shown that any function in $L^2(\Omega \times Y)$ is attained as a two-scale limit (see lemma 1.13 in [2]).

So far we have only considered bounded sequences in $L^2(\Omega)$. The next proposition investigates the case of a bounded sequence in $H^1(\Omega)$.

Proposition 2.5.

Let u_ε be a bounded sequence in $H^1(\Omega)$. Then, there exist $u(x) \in H^1(\Omega)$ and $u_1(x,y) \in L^2[\Omega; H^1_\#(Y)/\mathbb{R}]$ such that, up to a subsequence, u_ε two-scale converges to $u(x)$, and ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$.

Proof.

Since u_ε (resp. ∇u_ε) is bounded in $L^2(\Omega)$ (resp. $[L^2(\Omega)]^N$), up to a subsequence, it two-scale converges to a limit $u_0(x,y) \in L^2(\Omega \times Y)$ (resp. $\chi_0(x,y) \in [L^2(\Omega \times Y)]^N$). Thus for any $\Psi(x,y) \in D[\Omega; C^\infty_\#(Y)]^N$, we have

$$\lim_{\varepsilon \to 0} \int_\Omega \nabla u_\varepsilon(x) . \Psi(x, \frac{x}{\varepsilon}) \, dx \ = \ \int_\Omega \int_Y \chi_0(x,y) . \Psi(x,y) \, dxdy. \tag{2.13}$$

Integrating by parts the left hand side of (2.13) gives

$$\varepsilon \int_\Omega \nabla u_\varepsilon(x) . \Psi(x, \frac{x}{\varepsilon}) \, dx \ = \ - \int_\Omega u_\varepsilon(x) [div_y \Psi(x, \frac{x}{\varepsilon}) + \varepsilon div_x \Psi(x, \frac{x}{\varepsilon})] \, dx \ .$$

Passing to the limit yields

$$0 = -\int_\Omega\int_Y u_0(x,y) div_y \Psi(x,y)\, dxdy \ .$$

This implies that $u_0(x,y)$ does not depend on y. Thus there exists $u(x) \in L^2(\Omega)$, such that $u_0 \equiv u$. Next, in (2.13) we choose a function Ψ such that $div_y \Psi(x,y) = 0$. Integrating by parts we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x) div_x \Psi(x,\frac{x}{\varepsilon})\, dx &= -\int_\Omega\int_Y \chi_0(x,y).\Psi(x,y)\, dxdy \\ &= \int_\Omega\int_Y u(x) div_x \Psi(x,y)\, dxdy. \end{aligned} \tag{2.14}$$

If Ψ does not depend on y, (2.14) proves that $u(x)$ belongs to $H^1(\Omega)$. Furthermore, we deduce from (2.14) that

$$\int_\Omega\int_Y [\chi_0(x,y) - \nabla u(x)].\Psi(x,y)\, dxdy = 0$$

for any function $\Psi(x,y) \in D[\Omega;C_\#^\infty(Y)]^N$ with $div_y \Psi(x,y) = 0$. Recall that the orthogonal of divergence-free functions are exactly the gradients (this well-known result can be very easily proved in the present context by means of Fourier analysis in Y). Thus, there exists a unique function $u_1(x,y)$ in $L^2[\Omega;H_\#^1(Y)/\mathbb{R}]$ such that

$$\chi_0(x,y) = \nabla u(x) + \nabla_y u_1(x,y) \ .$$

For more results about two-scale convergence (including generalizations to the L^p case, to the multi-scale case, or to the non-linear case) the reader is referred to [2].

3. Homogenization of a second order elliptic equation.

We go back to the model problem introduced in the first section :

$$\begin{cases} -div\left(A(\frac{x}{\varepsilon})\nabla u_\varepsilon\right) = f & in \ \Omega \\ u_\varepsilon = 0 & on \ \partial\Omega \end{cases} \tag{3.1}$$

where $A(y)$ is a Y-periodic matrix satisfying the coercivity hypothesis (1.2). We recall that equation (3.1) admits a unique solution u_ε in $H_0^1(\Omega)$ which satisfies the a priori estimate

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \le C \|f\|_{L^2(\Omega)} \tag{3.2}$$

where C is a positive constant which does not depend on ε.

We now describe what we call the "two-scale convergence method" for homogenizing problem (3.1). In a **first step,** we deduce from the a priori estimate (3.2) the precise form of the two-scale limit of the sequence u_ε. Applying proposition 2.5, we know that there exists two functions, $u(x) \in H_0^1(\Omega)$ and $u_1(x,y) \in L^2[\Omega;H_\#^1(Y)/\mathbb{R}]$, such that, up to a subsequence, u_ε two-scale converges to $u(x)$, and ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$. In view of these limits, u_ε is expected to behave as $u(x) + \varepsilon u_1(x,x/\varepsilon)$.

Thus, in a **second step,** we multiply equation (3.1) by a test function similar to the limit of u_ε, namely $\phi(x) + \varepsilon\phi_1(x,x/\varepsilon)$, where $\phi(x) \in D(\Omega)$ and $\phi_1(x,y) \in D[\Omega;C_\#^\infty(Y)]$. This yields

$$\int_\Omega A(\frac{x}{\varepsilon})\nabla u_\varepsilon . \left[\nabla\phi(x)+\nabla_y\phi_1(x,\frac{x}{\varepsilon})+\varepsilon\nabla_x\phi_1(x,\frac{x}{\varepsilon})\right]dx = \int_\Omega f(x)[\phi(x)+\varepsilon\phi_1(x,\frac{x}{\varepsilon})]dx. \quad (3.3)$$

Regarding ${}^tA(x/\varepsilon)[\nabla\phi(x) + \nabla_y\phi_1(x,x/\varepsilon)]$ as a test function for the two-scale convergence (cf. definition 2.1), we pass to the two-scale limit in (3.3) for the sequence ∇u_ε. (Although this test function is not necessarily very smooth, it belongs at least to $L_\#^2[Y;C(\overline{\Omega})]$ which is enough for the two-scale convergence theorem 2.2 to hold.) Thus, the two-scale limit of (3.3) is

$$\int_\Omega\int_Y A(y)[\nabla u(x) + \nabla_y u_1(x,y)].[\nabla\phi(x) + \nabla_y\phi_1(x,y)]\, dxdy = \int_\Omega f(x)\phi(x)\, dx. \quad (3.4)$$

In a **third step,** we read off a variational formulation for (u,u_1) in (3.4). By density, (3.4) holds true for any (ϕ,ϕ_1) in the Hilbert space $H_0^1(\Omega)\times L^2[\Omega;H_\#^1(Y)/\mathbb{R}]$. Endowing this space with the norm $\|\nabla u(x)\|_{L^2(\Omega)} + \|\nabla_y u_1(x,y)\|_{L^2(\Omega\times Y)}$, we check the conditions of the Lax-Milgram lemma for (3.4). Let us focus on the coercivity of the bilinear form defined by the left hand side of (3.4)

$$\int_\Omega\int_Y A(y)[\nabla\phi(x) + \nabla_y\phi_1(x,y)].[\nabla\phi(x) + \nabla_y\phi_1(x,y)]\, dxdy \geq$$

$$\alpha\int_\Omega\int_Y|\nabla\phi(x) + \nabla_y\phi_1(x,y)|^2\, dxdy = \alpha\int_\Omega|\nabla\phi(x)|^2\, dx + \alpha\int_\Omega\int_Y|\nabla_y\phi_1(x,y)|^2\, dxdy\ .$$

Thus, by application of the Lax-Milgram lemma, there exists a unique solution (u,u_1) of the variational formulation (3.4) in $H_0^1(\Omega)\times L^2[\Omega;H_\#^1(Y)/\mathbb{R}]$. Consequently, the entire sequences u_ε and ∇u_ε converge to $u(x)$ and $\nabla u(x)+\nabla_y u_1(x,y)$. An easy integration by parts shows that (3.4) is a variational formulation associated to the following system of equations that we call the "two-scale homogenized problem"

$$\begin{cases} -\,div_y\left[A(y)[\nabla u(x) + \nabla_y u_1(x,y)]\right] = 0 \quad in\ \Omega\times Y \\ -\,div_x\left[\int_Y A(y)[\nabla u(x) + \nabla_y u_1(x,y)]\, dy\right] = f \quad in\ \Omega \\ u(x) = 0 \quad on\ \partial\Omega \\ y \to u_1(x,y) \quad Y\text{-}periodic. \end{cases} \quad (3.5)$$

It is easily seen that (3.5) is equivalent to the usual homogenized and cell equations

(1.6)-(1.8) through the relation

$$u_1(x,y) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(y).$$

At this point, the homogenization process could be considered as achieved since the entire sequence of solutions u_ε converges to the solution of a well-posed limit problem, namely the two-scale homogenized problem (3.5). However, it is usually preferable, from a physical or numerical point of view, to eliminate the microscopic variable y (one doesn't want to solve the small scale structure).

Thus, in a **fourth (and optional) step,** we can eliminate from (3.5) the y variable and the u_1 unknown. This is an easy algebra exercise (left to the reader) to derive from (3.5) the usual homogenized and cell equations (1.6)-(1.8). Due to the simple form of our model problem the two equations of (3.5) can be decoupled in a macroscopic and microscopic equations, but we emphasize that it is not always possible, and sometimes it leads to very complicate forms of the homogenized equation, including integro-differential operators and non-explicit equations. Thus, the homogenized equation does not always belong to a class for which an existence and uniqueness theory is easily available, on the contrary of the two-scale homogenized system, which is, in most cases, of the same type as the original problem, but with twice more variables (x and y) and unknowns (u and u_1). The supplementary, microscopic, variable and unknown play the role of "hidden" variables in the vocabulary of mechanics. Although their presence doubles the size of the limit problem, it greatly simplifies its structure (which could be useful for numerical purposes too), while eliminating them introduces "strange" effects (like memory or non-local effects) in the usual homogenized problem. In short, both formulations ("usual" or two-scale) of the homogenized problem have their pros and cons, and none should be eliminated without second thoughts. Particularly striking examples of the above discussion may be found in [2], [3], [4].

Corrector results are easily obtained with the two-scale convergence method. By application of theorem 2.4, we are going to prove that

$$\left[u_\varepsilon(x) - u(x) - \varepsilon\, u_1(x, \frac{x}{\varepsilon}) \right] \to 0 \quad \textit{in } H^1(\Omega) \textit{ strongly.} \tag{3.6}$$

This rigorously justifies the two first term in the usual asymptotic expansion (1.4) of the solution u_ε. Let us first remark that, by standard regularity results for the solutions $w_i(y)$ of the cell problem (1.6), the term $u_1(x, x/\varepsilon) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x) w_i(x/\varepsilon)$ does actually belong to $L^2(\Omega)$ and can be seen as a test function for the two-scale convergence. Bearing this in mind, we write

$$\int_\Omega A(\frac{x}{\varepsilon})[\nabla u_\varepsilon(x) - \nabla u(x) - \nabla_y u_1(x, \frac{x}{\varepsilon})]^2\, dx = \int_\Omega f(x) u_\varepsilon(x)\, dx$$

$$+\int_\Omega A(\frac{x}{\varepsilon})[\nabla u(x)+\nabla_y u_1(x,\frac{x}{\varepsilon})]^2\,dx\;-2\int_\Omega A(\frac{x}{\varepsilon})\nabla u_\varepsilon(x).[\nabla u(x)+\nabla_y u_1(x,\frac{x}{\varepsilon})]\,dx.$$

Using the coercivity condition for A, and passing to the two-scale limit yields

$$\alpha \lim_{\varepsilon\to 0} \|\nabla u_\varepsilon(x)-\nabla u(x)-\nabla_y u_1(x,\frac{x}{\varepsilon})\|^2_{L^2(\Omega)} \le \int_\Omega f(x)u(x)dx$$
$$-\int_\Omega\int_Y A(y)[\nabla u(x)+\nabla_y u_1(x,y)]^2 dxdy. \tag{3.7}$$

In view of (3.5), the right hand side of (3.7) is equal to zero, which gives the desired result (3.6).

We conclude this short presentation of the two-scale convergence method by saying that it is a very general method which can handle all possible difficulties in periodic homogenization, as perforated domains, non-linear (monotone) equations, memory or non-local effects, highly heterogeneous coefficients, etc.

4) Application to fluid flow in porous media.

In this section, two-scale convergence is applied to the homogenization of a more complicated problem. We consider the steady Stokes equations in a porous medium Ω_ε with a Dirichlet boundary condition. We denote by u_ε and p_ε the velocity and pressure of the fluid, and f the density of forces acting on the fluid (u_ε and f are vector-valued functions, while p_ε is scalar). We assume that the density of the fluid is equal to 1, and we scale its viscosity to ε^2 (where ε is the period). The system of equations is

$$\begin{cases} \nabla p_\varepsilon - \varepsilon^2\,\Delta u_\varepsilon = f & in\ \Omega_\varepsilon \\ div\ u_\varepsilon = 0 & in\ \Omega_\varepsilon \\ u_\varepsilon = 0 & on\ \partial\Omega_\varepsilon. \end{cases} \tag{4.1}$$

Remark that the scaling of the viscosity is perfectly legitimate since by linearity of the equations one can always replace u_ε by $\varepsilon^2 u_\varepsilon$. We will see in Remark 4.2 below the precise reason of this scaling, which simplifies the exposition. The originality of system (4.1) compared to (3.1) is that the periodic oscillations are not in the coefficients of the operator but in the geometry of the porous medium Ω_ε. Roughly speaking, Ω_ε is a periodically perforated domain, i.e. it has many small holes of size ε, which represents solid obstacles that the fluid cannot penetrate.

Let us describe this domain Ω_ε in more details. As usual, a periodic porous medium is defined by a domain Ω and an associated microstructure, or periodic cell $Y=[0;1]^N$, which is made of two complementary parts : the fluid part Y_f, and the solid part Y_s ($Y_f \cup Y_s = Y$ and $Y_f \cap Y_s = \emptyset$). More precisely, we assume that Ω is a smooth, bounded, connected set in $\mathbb{R}^N$, and that Y_s is a smooth and connected set strictly included in Y (i.e. Y_s does not touch the faces of Y). The microscale of a porous medium is a (small) positive number ε. The domain Ω is covered by a regular mesh of size ε : each cell Y_i^ε is of the type $[0;\varepsilon]^N$, and is divided in a fluid part $Y_{f_i}^\varepsilon$ and a solid part $Y_{s_i}^\varepsilon$, i.e. is similar to the unit cell Y rescaled to size ε. The fluid part Ω_ε of a porous medium is defined by

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} Y_{s_i}^\varepsilon = \Omega \cap \bigcup_{i=1}^{N(\varepsilon)} Y_{f_i}^\varepsilon \tag{4.2}$$

where the number of cells is $N(\varepsilon) = |\Omega| \varepsilon^{-N} [1+o(1)]$. Throughout this section, we assume that Ω_ε is itself a smooth, connected set in $\mathbb{R}^N$. This last assumption on Ω_ε and that on Y_s are of no fundamental importance for the result, but it makes things simpler in the proofs (see [1] for some generalizations).

To obtain an existence and uniqueness result for (4.1), the forcing term is assumed to have the usual regularity : $f(x) \in L^2(\Omega)^N$. Then, as well-known (see [22] for details), the Stokes equations (4.1) admits a unique solution

$$u_\varepsilon \in H_0^1(\Omega_\varepsilon)^N \ , \quad p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}. \tag{4.3}$$

The next step is to obtain a priori estimates of the solution $(u_\varepsilon, p_\varepsilon)$, which are *independent* of ε. These estimates will be used to extract weakly convergent subsequences ; but to do so, the sequence $(u_\varepsilon, p_\varepsilon)$ needs to be defined in a fixed Sobolev space, independent of ε. Unfortunately, it is not the case in view of (4.3), and thus a new difficulty arises, which is to extend the solution $(u_\varepsilon, p_\varepsilon)$ to the whole domain Ω. It is easy to extend the velocity by zero in $\Omega \setminus \Omega_\varepsilon$ (this is compatible with its Dirichlet boundary condition on $\partial\Omega_\varepsilon$) to obtain a function $\tilde{u}_\varepsilon$

$$\begin{cases} \tilde{u}_\varepsilon = u_\varepsilon & in \ \Omega_\varepsilon \\ \tilde{u}_\varepsilon = 0 & in \ \Omega \setminus \Omega_\varepsilon \end{cases} \tag{4.4}$$

which belongs to $H_0^1(\Omega)^N$. The definition of the proposed extension $\tilde{p}_\varepsilon$ of the pressure is slightly more complicated

$$\tilde{p}_\varepsilon = p_\varepsilon \quad in \ \Omega_\varepsilon \ , \ \text{ and } \ \tilde{p}_\varepsilon = \frac{1}{|Y_{f_i}^\varepsilon|} \int_{Y_{f_i}^\varepsilon} p_\varepsilon \quad in \ each \ Y_{s_i}^\varepsilon \tag{4.5}$$

but it turns out to be convenient to obtain an a priori estimate for the pressure.

Proposition 4.1.

The extensions $\tilde{u}_\varepsilon$ and $\tilde{p}_\varepsilon$ of the solution $(u_\varepsilon, p_\varepsilon)$, defined in (4.4), (4.5) satisfy the a priori estimates

$$\| \tilde{u}_\varepsilon \|_{L^2(\Omega)^N} + \varepsilon \| \nabla \tilde{u}_\varepsilon \|_{L^2(\Omega)^{N \times N}} \ \leq \ C \tag{4.6}$$

and

$$\|\tilde{p}_\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C, \tag{4.7}$$

where the constant C does not depend on ε.

Remark 4.2.

In view of the a priori estimates of Proposition 4.1, the scaling ε^2 of the viscosity in the Stokes equations (4.1) can now be well understood. It is exactly chosen in order for the velocity u_ε to have a bounded and non-zero limit. In other words, the very small viscosity ε^2 balances exactly the friction on the solid parts of the porous medium due to the no-slip (Dirichlet) boundary condition.

The proof of Proposition 4.1 is a little technical, and it does not use any arguments from two-scale convergence. Thus, we prefer to postpone it until section 5, and to proceed to the homogenization of system (4.1). According to the two-scale convergence method described in section 3, we now look for the precise form of the two-scale limit of the sequence of solutions $(u_\varepsilon, p_\varepsilon)$.

Lemma 4.3.

There exists two-scale limits $u_0(x,y) \in L^2[\Omega\,;H^1_\#(Y)^N]$ and $p(x) \in L^2(\Omega)/\mathbb{R}$ such that, up to a subsequence, the sequences $\tilde{u}_\varepsilon$, $\varepsilon\nabla\tilde{u}_\varepsilon$, and $\tilde{p}_\varepsilon$ two-scale converge to u_0, $\nabla_y u_0$, and $p(x)$ respectively. Furthermore, u_0 satisfies

$$\left\{\begin{array}{l} div_y\, u_0(x,y) = 0 \quad in \;\; \Omega\times Y, \text{ and } \quad div_x\,[\int_Y u_0(x,y)\;dy] = 0 \quad in \;\; \Omega \\ \\ u_0(x,y) = 0 \quad in \;\; \Omega\times Y_s, \text{ and } \quad [\int_Y u_0(x,y)\;dy].n = 0 \quad on \;\; \partial\Omega. \end{array}\right. \tag{4.8}$$

Proof.

Thanks to the bounds of Proposition 4.1, by application of Theorem 2.2, there exists three functions $u_0(x,y)$, $\xi_0(x,y)$, and $p_0(x,y)$ in $L^2(\Omega\times Y)$ such that

$$\left\{\begin{array}{l} \lim\limits_{\varepsilon\to 0} \int_\Omega \tilde{u}_\varepsilon(x).\psi(x,\frac{x}{\varepsilon})\;dx \;\; = \;\; \int_\Omega\int_Y u_0(x,y).\psi(x,y)\;dxdy \\ \\ \lim\limits_{\varepsilon\to 0} \int_\Omega \varepsilon\nabla\tilde{u}_\varepsilon(x).\Xi(x,\frac{x}{\varepsilon})\;dx \;\; = \;\; \int_\Omega\int_Y \xi_0(x,y).\Xi(x,y)\;dxdy \\ \\ \lim\limits_{\varepsilon\to 0} \int_\Omega \tilde{p}_\varepsilon(x)\phi(x,\frac{x}{\varepsilon})\;dx \;\; = \;\; \int_\Omega\int_Y p_0(x,y)\phi(x,y)\;dxdy \end{array}\right. \tag{4.9}$$

for any ψ, Ξ, and ϕ in $D[\Omega;C^\infty_\#(Y)]^N$. Integrating by parts and passing to the two-scale limit in the second lines of (4.9) yields

$$\lim_{\varepsilon\to 0} \int_\Omega \tilde{u}_\varepsilon.div_y\,\Xi(x,\frac{x}{\varepsilon})\;dx = -\int_\Omega\int_Y \xi_0.\Xi(x,y)\;dxdy = \int_\Omega\int_Y u_0.div_y\,\Xi(x,y)\;dxdy.$$

Desintegrating by parts shows that $\xi_0 = \nabla_y u_0$. On the other hand, multiplying the first equation in (4.1) by $\varepsilon\psi(x, x/\varepsilon)$ and integrating by parts, leads to

$$\lim_{\varepsilon\to 0} \int_\Omega \tilde{p}_\varepsilon div_y \psi(x, \frac{x}{\varepsilon})\, dx = 0. \tag{4.10}$$

Combining the last line of (4.9) and (4.10) shows that $p_0(x, y)$ does not depend on y. Thus, there exists $p(x) \in L^2(\Omega)/\mathbb{R}$ such that $p_0(x, y) = p(x)$. To obtain the incompressibility conditions (4.8), the same type of arguments is used : multiply the equation $div\ u_\varepsilon = 0$ by a test function $\psi(x, x/\varepsilon)$, integrate by parts, and pass to the two-scale limit.

The next step in the two-scale convergence method is to multiply system (4.1) by a test function having the form of the two-scale limit u_0 (as explicited in Lemma 4.3), and to read off a variational formulation for the limit. This is the focus of the following theorem.

Theorem 4.4.

The extension $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ of the solution of (4.1) two-scale converges to the unique solution $(\ u_0(x, y)\ ,\ p(x)\ )$ of the two-scale homogenized problem

$$\begin{cases} \nabla_y p_1(x, y) + \nabla_x p(x) - \Delta_{yy} u_0(x, y) = f(x) \quad in\ \Omega\times Y_f \\ div_y u_0(x, y) = 0 \quad in\ \Omega\times Y_f \quad \text{and} \quad div_x \left[\int_Y u_0(x, y)\, dy \right] = 0 \quad in\ \Omega \\ u_0(x, y) = 0 \quad in\ \Omega\times Y_s \quad \text{and} \quad \left[\int_Y u_0(x, y)\, dy \right].n = 0 \quad on\ \partial\Omega \\ y \to u_0,\ p_1\ \ Y\text{-}periodic. \end{cases} \tag{4.11}$$

Proof.

We choose a test function $\psi(x, y) \in D[\Omega; C_\#^\infty(Y)]^N$ with $\psi(x, y) \equiv 0$ in $\Omega\times Y_s$ (thus, $\psi(x, \frac{x}{\varepsilon}) \in [H_0^1(\Omega_\varepsilon)]^N$). Furthermore, we assume that ψ satisfies the incompressibility conditions (4.8), i.e. $div_y \psi(x, y) = 0$ and $div_x [\int_Y u_0(x, y)\, dy] = 0$. Multiplying equation (4.1) by $\psi(x, \frac{x}{\varepsilon})$, and integrating by parts yields

$$-\int_{\Omega_\varepsilon} p_\varepsilon(x) div_x \psi(x, \frac{x}{\varepsilon})\, dx + \int_{\Omega_\varepsilon} \varepsilon\nabla u_\varepsilon(x).\nabla_y \psi(x, \frac{x}{\varepsilon})\, dx = \int_{\Omega_\varepsilon} f(x).\psi(x, \frac{x}{\varepsilon})\, dx + O(\varepsilon) \tag{4.12}$$

where $O(\varepsilon)$ stands for the the remaining terms of order ε. In (4.12), the domain of integration Ω_ε can be replaced by Ω since the test function is zero in $\Omega\setminus\Omega_\varepsilon$. Thus, we can use the two-scale convergences (4.9). When passing to the two-scale limit, the first term in (4.12) contributes nothing because the two-scale limit of $\tilde{p}_\varepsilon$ does not depend on y and ψ satisfies $div_x [\int_Y u_0(x,y)\,dy] = 0$. Finally, (4.12) gives

$$\int_\Omega \int_{Y_f} \nabla_y u_0(x,y).\nabla_y \psi(x,y)\,dxdy = \int_\Omega \int_Y f(x).\psi(x,y)\,dxdy. \tag{4.13}$$

By density (4.13) holds for any function ψ in the Hilbert space V defined by

$$V = \left\{ \begin{array}{l} \psi(x,y) \in L^2[\Omega;H^1_\#(Y)]^N, \textit{ such that} \\ div_y \psi(x,y) = 0 \ \ in \ \ \Omega\times Y, \text{ and } \ \ div_x [\int_Y \psi(x,y)\,dy] = 0 \ \ in \ \ \Omega \\ \psi(x,y) = 0 \ \ in \ \ \Omega\times Y_s, \text{ and } \ \ [\int_Y \psi(x,y)\,dy].n = 0 \ \ on \ \ \partial\Omega \end{array} \right\}. \tag{4.14}$$

It is not difficult to check that the hypothesis of the Lax-Milgram lemma holds for the variational formulation (4.13) in the Hilbert space V, which, by consequence, admits a unique solution u_0 in V. Furthermore, by Lemma 4.5 below, the orthogonal of V with respect to the usual scalar product in $L^2(\Omega\times Y)$ is made of gradients of the form $\nabla_x q(x) + \nabla_y q_1(x,y)$ with $q(x) \in L^2(\Omega)/I\!R$ and $q_1(x,y) \in L^2[\Omega;L^2_\#(Y_f)]$. Thus, by integration by parts, the variational formulation (4.13) is equivalent to the two-scale homogenized system (4.11). (There is a subtle point here ; one must check that the pressure $p(x)$ arising as a Lagrange multiplier of the incompressibility constraint $div_x [\int_Y u_0(x,y)\,dy] = 0$ is the same as the two-scale limit of the pressure $\tilde{p}_\varepsilon$. This can easily be done by multiplying equation (4.1) by a test function ψ which is divergence free only in y, and identifying limits.) Since (4.11) admits a unique solution, then the entire sequence $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ converges to its unique solution $(u_0(x,y), p(x))$. This completes the proof of Theorem 4.4.

Lemma 4.5.

The orthogonal $V^\perp$ of the Hilbert space V, defined in (4.14), has the following characterization

$$V^\perp = \left\{ v(x,y) = \nabla_x \phi(x) + \nabla_y \phi_1(x,y) \ \text{ with } \phi \in H^1(\Omega), \ \text{ and } \phi_1 \in L^2(\Omega;L^2_\#(Y_f)) \right\}.$$

Proof.

Remark that $V = V_1 \cap V_2$ with

$$V_1 = \left\{ v(x,y) \in L^2(\Omega;H^1_\#(Y)^N) \ / \ div_y v = 0 \ in \ \Omega\times Y, \ v = 0 \ in \ \Omega\times Y_s \right\}$$

$$V_2 = \left\{ v(x,y) \in L^2(\Omega;H^1_\#(Y)^N) \ / \ div_x [\int_{Y_f} v dy] = 0 \ in \ \Omega, \ [\int_{Y_f} v dy].n_x = 0 \ on \ \partial\Omega \right\}.$$

It is a well-known result (see, e.g., [15], [16]) that

$$V_1^{\perp} = \left\{ \nabla_y \phi_1(x,y) \ / \ \phi_1 \in L^2(\Omega;L^2_\#(Y_f)) \right\}, \text{ and } V_2^{\perp} = \left\{ \nabla_x \phi(x) \ / \ \phi \in H^1(\Omega) \right\}.$$

Since V_1 and V_2 are two closed subspaces, it is equivalent to say $(V_1 \cap V_2)^{\perp} = V_1^{\perp} + V_2^{\perp}$ or $V_1 + V_2 = \overline{V_1 + V_2}$. Indeed, we are going to prove that $V_1 + V_2$ is equal to $L^2(\Omega;H^1_\#(Y_f)^N)$, which establishes that $V_1 + V_2$ is closed, and thus completes the proof of this lemma.

Introducing the divergence-free solutions $[w_i(y)]_{1\le i\le N}$ of the local Stokes problem (4.17) defined below, for any given $v(x,y) \in L^2(\Omega;H^1_\#(Y_f)^N)$, we define a unique solution $q(x)$ in $H^1(\Omega)/\mathbb{R}$ of the Neuman problem

$$\begin{cases} div_x \left[A \nabla q(x) - \int_{Y_f} v(x,y)dy \right] = 0 \quad in \ \Omega \\ \left[A \nabla q(x) - \int_{Y_f} v(x,y)dy \right].n = 0 \quad on \ \partial\Omega, \end{cases}$$

where the positive definite matrix A defined in (4.16) satisfies $Ae_i = \int_{Y_f} w_i(y)\, dy$ ($(e_i)_{1\le i\le N}$ being the orthonormal basis). Then, decomposing v as

$$v(x,y) = \sum_{i=1}^{N} w_i(y) \frac{\partial q}{\partial x_i}(x) + \left[v(x,y) - \sum_{i=1}^{N} w_i(y) \frac{\partial q}{\partial x_i}(x) \right],$$

it is easily seen that the first term of this decomposition belongs to V_1, while the second one belongs to V_2.

We now arrive to the last and optional step of the two-scale convergence method which amounts to eliminate, if possible, the microscopic variable y in the homogenized system. This is the focus of the next theorem.

Theorem 4.6.

The extension $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ of the solution of (4.1) converges, weakly in $[L^2(\Omega)]^N \times [L^2(\Omega)/\mathbb{R}]$, to the unique solution (u,p) of the homogenized problem

$$\begin{cases} u(x) = A[f(x) - \nabla p(x)] \quad in \ \Omega \\ div \ u(x) = 0 \quad in \ \Omega \\ u(x).n = 0 \quad on \ \partial\Omega \end{cases} \tag{4.15}$$

where the limit velocity u is the average of u_0 ($u(x) = \int_{Y_f} u_0(x,y)\, dy$), and A is a symmetric, positive definite, tensor defined by its entries

$$A_{ij} = \int_{Y_f} \nabla w_i(y).\nabla w_j(y)\, dy \tag{4.16}$$

where, for $1 \le i \le N$, $w_i(y)$ *denotes the unique solution in* $[H^1_\#(Y_f)]^N$ *of the local, or unit cell, Stokes problem*

$$\begin{cases} \nabla q_i - \Delta w_i = e_i \ , \ \ div \ w_i = 0 \ \ in \ Y_f \\ w_i = 0 \ \ in \ \partial Y_s \ , \ \ q_i, w_i \ \ Y\text{-periodic.} \end{cases} \tag{4.17}$$

Furthermore, the two-scale homogenized problem (4.11) is equivalent to (4.15)-(4.17) through the relation

$$u_0(x,y) \ = \ \sum_{i=1}^{N} \left(f_i(x) - \frac{\partial p}{\partial x_i}(x) \right) w_i(y).$$

Proof.

The derivation of (4.15) from the two-scale homogenized problem (4.11) is an easy algebra exercise (left to the reader). Let us point out that (4.15) is a well-posed problem since it is simply a second order elliptic equation for the pressure p (with Neumann boundary condition). As is well-known, the local problem is also well-posed with periodic boundary condition, and it is easily checked, by integration by parts, that

$$A_{ij} \ = \ \int_{Y_f} \nabla w_i(y).\nabla w_j(y) \ dy \ = \ \int_{Y_f} w_i(y).e_j \ dy ,$$

which implies that A is symmetric and positive definite.

Remark 4.7.

The two-scale homogenized problem is also called a two-pressure Stokes system. The homogenized problem (4.15) is called Darcy's law (i.e. the flow rate u is proportional to the balance of forces including the pressure). The matrix A is called the permeability tensor of the porous media (it depends only on the microstructure Y_f). The homogenization results of this section are a rigorous justification of the well-known physical principle which says that Darcy's law is the asymptotic behavior of Stokes equations in porous media. Quite early, many papers have been devoted to this topic (see for example [11], [12], [17]). The first rigorous proof (including the difficult estimate (4.7) for the pressure) appeared in [20]. Further extensions are to be found in [1], [13], and [14]. A good reference for physical aspects of this problem (as well as mathematical ones) is the book [10]. Finally, as in section 3 one can prove corrector results (see [3]).

5) Estimate of the pressure in a porous medium.

This section is devoted to the proof of Proposition 4.1 which constructs extensions and establishes uniform estimates for the velocity and pressure of a Stokes fluid in a porous medium. This proof is rather technical and does not appeal to any notion of two-scale convergence. Consequently, readers who are willing to accept this proof can safely skip this section, which is provided for the sake of completeness of these lecture notes.

Basically, we reproduce the original proof of L. Tartar [20] which has been further generalized in [1] and [13]. We begin by two technical lemmas on Poincaré inequality in Ω_ε and a restriction operator from $H_0^1(\Omega)^N$ into $H_0^1(\Omega_\varepsilon)^N$ preserving divergence-free vectors.

Lemma 5.1.

There exists a constant C independent of ε, such that, for any function $v \in H_0^1(\Omega_\varepsilon)$, one has

$$\| v \|_{L^2(\Omega_\varepsilon)} \leq C \| \nabla v \|_{L^2(\Omega_\varepsilon)}. \tag{5.1}$$

Proof.

For any function $w(y) \in H^1(Y_f)$ such that $w = 0$ on ∂Y_s, the Poincaré inequality in Y_f states that

$$\| w \|^2_{L^2(Y_f)} \leq C \| \nabla w \|^2_{L^2(Y_f)}, \tag{5.2}$$

where the constant C depends only on Y_f. By a change of variable $x = \varepsilon y$, we rescale (5.2) from Y_f to $Y^e_{f_i}$. This yields that, for any function $w(x) \in H^1(Y^e_{f_i})$ such that $w = 0$ on $\partial Y^e_{s_i}$, one has

$$\| w \|^2_{L^2(Y^e_{f_i})} \leq C \varepsilon^2 \| \nabla w \|^2_{L^2(Y^e_{f_i})}, \tag{5.3}$$

with the same constant C as in (5.2). Summing the inequalities (5.3) arising from all the fluid cells $Y^e_{f_i}$, which cover the domain Ω_ε, gives the desired result (5.1).

Lemma 5.2.

There exists a linear continuous operator R_ε acting from $H_0^1(\Omega)^N$ into $H_0^1(\Omega_\varepsilon)^N$ such that

$$R_\varepsilon v = v \quad \text{in } \Omega_\varepsilon , \ \text{ if } v \in H_0^1(\Omega_\varepsilon)^N \tag{5.4}$$

$$\operatorname{div} (R_\varepsilon v) = 0 \quad \text{in } \Omega_\varepsilon , \ \text{ if } \operatorname{div} v = 0 \quad \text{in } \Omega \tag{5.5}$$

$$\| R_\varepsilon v \|_{L^2(\Omega_\varepsilon)} + \varepsilon \| \nabla (R_\varepsilon v) \|_{L^2(\Omega_\varepsilon)} \leq C \left[\| v \|_{L^2(\Omega)} + \varepsilon \| \nabla v \|_{L^2(\Omega)} \right], \tag{5.6}$$

for any $v \in H_0^1(\Omega)^N$ (the constant C is independent of v and ε).

Proof.

As in Lemma 5.1, we proceed by rescaling of a similar operator R acting from $H^1(Y)^N$ into $H^1(Y_f)^N$. For any function $u \in H^1(Y)^N$, there exists a unique solution, denoted Ru, in $H^1(Y_f)^N$ of the following Stokes problem

$$\begin{cases} \nabla q - \Delta Ru = -\Delta u & in\ Y_f \\ div\ Ru = div\ u + \dfrac{1}{|Y_f|}\displaystyle\int_{Y_s} div\ u & in\ Y_f \\ Ru = 0 & on\ \partial Y_s \\ Ru = u & on\ \partial Y. \end{cases} \tag{5.7}$$

Remark that since Y_s is strictly included in Y, the boundary of Y_f is made of two disjoint parts, ∂Y_s and ∂Y. Note also that the compatibility condition for (5.7) is satisfied, namely one checks that the identity

$$\int_{Y_f} div\ Ru = \int_{\partial Y_f} (Ru).n$$

is compatible with the right hand side of (5.7). Furthermore, standard estimates for non-homogeneous Stokes system yields

$$||Ru||_{H^1(Y_f)^N} \leq C\, ||u||_{H^1(Y)^N}$$

where the constant C depends only on Y_f. Thus, R is a linear continuous operator. Now, rescaling R from Y to any cell Y_i^ε, we obtain an operator R_ε acting from $H_0^1(\Omega)^N$ into $H_0^1(\Omega_\varepsilon)^N$ defined in each cell Y_i^ε by

$$\begin{cases} \nabla q_\varepsilon - \Delta R_\varepsilon u = -\Delta u & in\ Y_{f_i}^\varepsilon \\ div\ R_\varepsilon u = div\ u + \dfrac{1}{|Y_{f_i}^\varepsilon|}\displaystyle\int_{Y_{s_i}^\varepsilon} div\ u & in\ Y_{f_i}^\varepsilon \\ R_\varepsilon u = 0 & on\ \partial Y_{s_i}^\varepsilon \\ R_\varepsilon u = u & on\ \partial Y_i^\varepsilon, \end{cases} \tag{5.8}$$

and, by summation over i, satisfying the rescaled estimate

$$||R_\varepsilon u||^2_{L^2(\Omega_\varepsilon)} + \varepsilon^2 ||\nabla(R_\varepsilon u)||^2_{L^2(\Omega_\varepsilon)} \leq C\left[||u||^2_{L^2(\Omega)} + \varepsilon^2 ||\nabla u||^2_{L^2(\Omega)}\right].$$

Finally, the reader can easily check properties (5.4) and (5.5) for this operator R_ε.

We now have the main tools to complete the

Proof of Proposition 4.1.

We begin with the estimate of the velocity. Multiplying equation (4.1) by u_ε and integrating by parts gives

$$\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{\Omega_\varepsilon} f.u_\varepsilon\ . \tag{5.9}$$

Using Poincaré inequality (Lemma 5.1) in (5.9) leads to

$$\varepsilon^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C\,\varepsilon \|f\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}.$$

Thus

$$\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \|f\|_{L^2(\Omega)},$$

and using again Poincaré inequality

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \|f\|_{L^2(\Omega)}.$$

We turn to the case of the pressure. Let us explain briefly why things are more delicate in this case. From equation (4.1), we easily obtain that ∇p_ε is uniformly bounded in $H^{-1}(\Omega_\varepsilon)^N$. Then, a well-known theorem of functional analysis (see, e.g. Proposition 1.2, Chapter I, [22]) states that p_ε belongs to $L^2(\Omega_\varepsilon)$ with the following estimate

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C(\Omega_\varepsilon) \|\nabla p_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)^N}. \tag{5.10}$$

Unfortunately, the above estimate is useless since the constant depends on the domain Ω_ε and thus may be not uniformly bounded when ε goes to zero. Consequently, another argument is required, which turns out to be an extension of the pressure to the whole domain Ω.

Since R_ε is a linear operator from $H_0^1(\Omega)^N$ into $H_0^1(\Omega_\varepsilon)^N$, we can define a function $F_\varepsilon \in H^{-1}(\Omega)^N$ by the following formula

$$< F_\varepsilon, v >_{H^{-1},H_0^1(\Omega)} = < \nabla p_\varepsilon, R_\varepsilon v >_{H^{-1},H_0^1(\Omega_\varepsilon)} \quad \text{for any } v \in H_0^1(\Omega)^N. \tag{5.11}$$

Replacing ∇p_ε by $f - \varepsilon^2 \Delta u_\varepsilon$, integrating by parts in (5.11), and using the estimates on u_ε and R_ε shows that F_ε is uniformly (i.e. independently of ε) bounded in $H^{-1}(\Omega)^N$. By property (5.5), we see that $<F_\varepsilon, v> = 0$ if the function v satisfies $div\ v = 0$. Thus, F_ε, being orthogonal to divergence-free functions, is the gradient of some function P_ε in $L^2(\Omega)$ (see, e.g. Proposition 1.1, Chapter I, [22]). By property (5.4), ∇P_ε and ∇p_ε coincide on $H^{-1}(\Omega_\varepsilon)^N$, implying, by virtue of inequality (5.10), that P_ε and p_ε are equal in Ω_ε up to a constant. (This constant does not matter since a pressure is always defined up to a constant.) It remains to prove that P_ε is identical to the extension $\tilde{p}_\varepsilon$ introduced in (4.5), i.e. that

$$P_\varepsilon = \frac{1}{|Y^\varepsilon_{f_i}|} \int_{Y^\varepsilon_{f_i}} p_\varepsilon \quad in\ each\ Y^\varepsilon_{s_i}.$$

This is done in two steps. First, we introduce in definition (5.11) a smooth function v_ε, with compact support in one of the solid parts $Y^\varepsilon_{s_i}$. For such a function, $R_\varepsilon v_\varepsilon$ is zero in $Y^\varepsilon_{f_i}$, and thus

$$< \nabla P_\varepsilon, v >_{H^{-1},H_0^1(Y^\varepsilon_{s_i})} = 0,$$

which implies that P_ε is constant in $Y^\varepsilon_{s_i}$. In a second step, we choose a test function v_ε, with compact support in the entire cell Y_i^ε. Integrating by parts in (5.11) leads to

$$\int_{Y_i^\varepsilon} P_\varepsilon \, div \; v_\varepsilon = \int_{Y_{f_i}^\varepsilon} p_\varepsilon \, div \; (R_\varepsilon v_\varepsilon). \tag{5.12}$$

Using definition (5.8) of $div(R_\varepsilon v_\varepsilon)$ and the properties of P_ε (constant in $Y_{s_i}^\varepsilon$, equal to p_ε in $Y_{f_i}^\varepsilon$), (5.12) becomes

$$\int_{Y_{f_i}^\varepsilon} p_\varepsilon \, div \; v_\varepsilon + P_\varepsilon(Y_{s_i}^\varepsilon) \int_{Y_{s_i}^\varepsilon} div \; v_\varepsilon = \int_{Y_{f_i}^\varepsilon} p_\varepsilon \, div \; v_\varepsilon + \frac{1}{|Y_{f_i}^\varepsilon|} \left(\int_{Y_{s_i}^\varepsilon} div \; v_\varepsilon \right) \left(\int_{Y_{f_i}^\varepsilon} p_\varepsilon \right),$$

which gives the desired value

$$P_\varepsilon(Y_{s_i}^\varepsilon) = \frac{1}{|Y_{f_i}^\varepsilon|} \left(\int_{Y_{f_i}^\varepsilon} p_\varepsilon \right).$$

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