Sympathy for the Devil
(. . . sorry, Mick and Keith . . .)

Andrea Braides (Roma *Tor Vergata*)

Nonconvexity, Nonlocality and Incompatibility:
from Materials to Biology

Conference in honor of Lev Truskinovsky's 60th birthday

Department of Mathematics
University of Pittsburgh
May 5 2017

I know exactly when Lev and I met (. . . *it was almost twenty years ago today*. . .): on the morning of June 23 1997 in Oberwolfach:

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

T a g u n g s b e r i c h t 24/1997

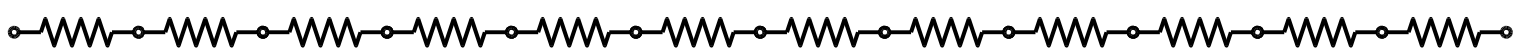
Mathematical Continuum Mechanics

22.06. — 28.06.1997

I was giving my first talk on a discrete-to-continuum limit, and Lev was the chairman.

(2007 reunion)

I was talking about a continuum limit of a chain of Lennard-Jones interactions



Lev had studied *exactly the same problem*, but had obtained a different limit . . .

This could have been the beginning of a fight, but instead was the beginning of a friendship and of a collaboration between Minneapolis (that was my first trip to the States), Trieste, Rome and Paris (we both moved), that carried on while our lives changed . . .

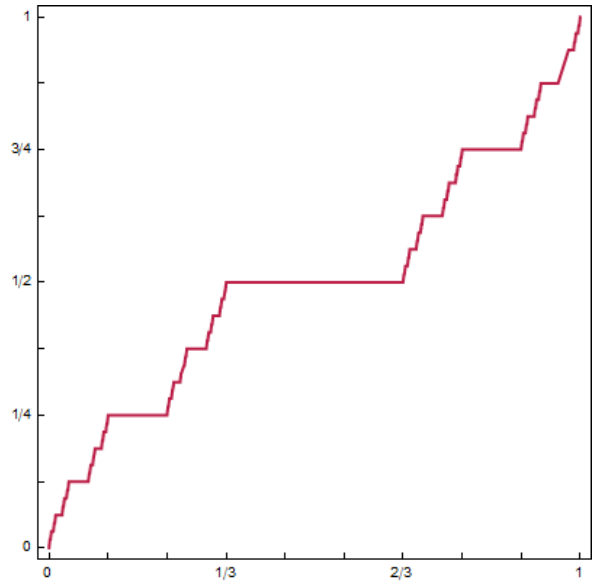
Finally, our *magnum opus* appeared ten years after:

A. Braides and L. Truskinovsky. Asymptotic expansions by Gamma-convergence. *Cont. Mech. Therm.* **20** (2008), 21–62


GUIDO CASTELNUOVO

Why the Devil?

The *Devil's Staircase* (or Cantor's Step Function)



diabolic points = where the function is continuous but not locally constant (a set of measure zero, but where all the derivative concentrates)

This mathematical beast seems to reappear in complex Physics behaviours (a rapid search on the web: ground states of long range 1D lattice gas, chaotic behaviour, fractional quantum Hall effect, . . .)

Some thoughts (with A.Causin, M.Solci and Lev) around two papers:

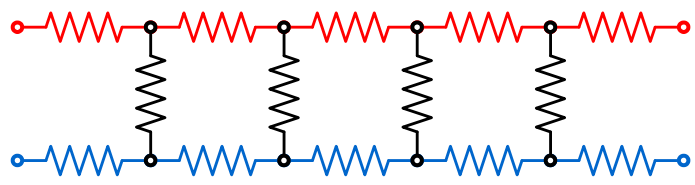
I. Novak and L. Truskinovsky. Segmentation in cohesive systems constrained by elastic environments. Phil. Trans. R. Soc. A *375 (2017)*

I. Novak and L. Truskinovsky. Nonaffine response of skeletal muscles on the descending limb. Math. Mechanics of Solids *20 (2015)*

Environment: double chain of interactions between N sites

$i \to u_i$

$i \to v_i$



Energy: $\sum_i \big(f(v_i - v_{i-1}) + g(u_i - u_{i-1}) + h(u_i - v_i)\big)$

Scaling argument $\varepsilon = 1/N$:

$$\sum_i \varepsilon\Big(f\Big(\frac{v_i - v_{i-1}}{\varepsilon}\Big) + g\Big(\frac{u_i - u_{i-1}}{\varepsilon}\Big) + h\Big(\frac{u_i - v_i}{\varepsilon}\Big)\Big)$$

Note: a formal continuum analog

$$\int_0^1 \Big(f(v') + g(u') + h\Big(\frac{u - v}{\varepsilon}\Big)\Big)dx$$

Particular case - I $h(z) = z^2$

$$\int_0^1 \left(f(v') + g(u') + \frac{1}{\varepsilon^2}(u-v)^2 \right) dx$$

or

$$\int_0^1 (\varepsilon^2 f(v') + \varepsilon^2 g(u') + (u-v)^2) dx$$

(singular-perturbation problem)

Particular case - II $h(\xi) = \begin{cases} 0 & \text{if } \xi = 0 \\ +\infty & \text{otherwise,} \end{cases}$ (rigid case: $u = v$)

$f(z) = az^2$, $g(z) = \min\{bz^2, \gamma\}$. Set $\alpha = a,\ \beta = a + b$

$$\int_{\{b|u'|^2 \geq \gamma\}} \beta |u'|^2 dx + \int_{\{b|u'|^2 \leq \gamma\}} (\alpha |u'|^2 + \gamma) dx$$

(damage model with two phases)

A homogenization approach

Energy:

$$E_\varepsilon(u,v) = \sum_i \varepsilon\Big(f\Big(\frac{v_i - v_{i-1}}{\varepsilon}\Big) + g\Big(\frac{u_i - u_{i-1}}{\varepsilon}\Big) + h\Big(\frac{u_i - v_i}{\varepsilon}\Big)\Big)$$

Hypotheses (simplified):
$f(z) \sim z^2 \Longrightarrow$ continuum parameter $v \in H^1$
$h(\xi) \sim \xi^2 \Longrightarrow$ continuum parameter $u = v$
$0 \le g(z) \le cz^2 \Longrightarrow$ limit finite for $u = v \in H^1$

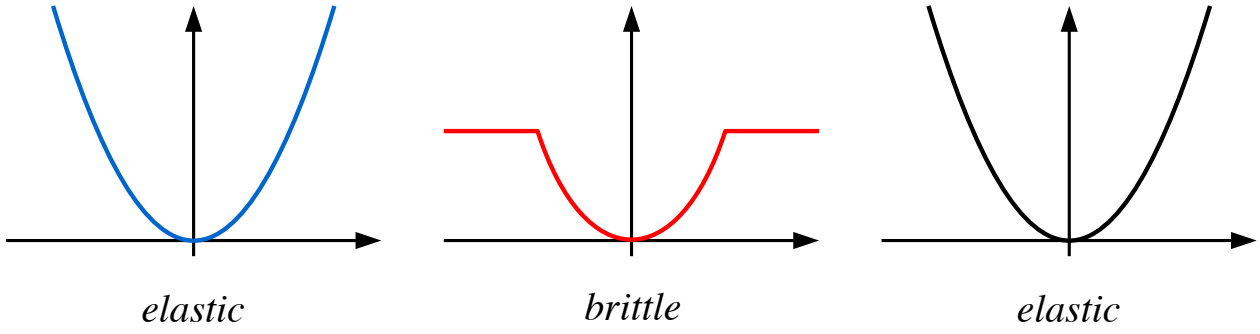
Convergence: $(u_\varepsilon, v_\varepsilon) \to v$ i.e. $v_\varepsilon \to v$ and $u_\varepsilon \to v$ in $L^2(0,1)$

$\Longrightarrow$ **Γ-limit**: $\displaystyle\int_0^1 f^{\mathrm{d}}_{\mathrm{hom}}(v')dx$ for $v \in H^1(0,1)$ (d = discrete)

Asymptotic homogenization formula

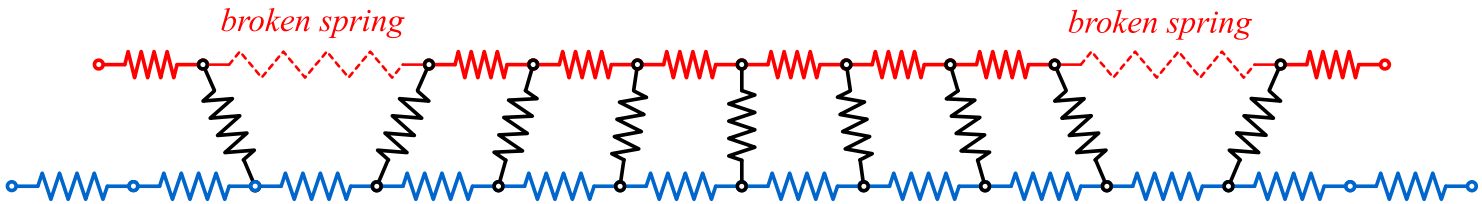
$$f^{\mathrm{d}}_{\mathrm{hom}}(z) = \lim_{M\to+\infty} \frac{1}{M} \min\Big\{ \sum_{i=1}^{M} \big(f(v_i - v_{i-1}) + g(u_i - u_{i-1}) + h(u_i - v_i)\big) \\ u_0 = v_0 = 0,\ u_M = v_M = zM \Big\}$$

Analysis of the effect of brittle interactions (NT 2017)

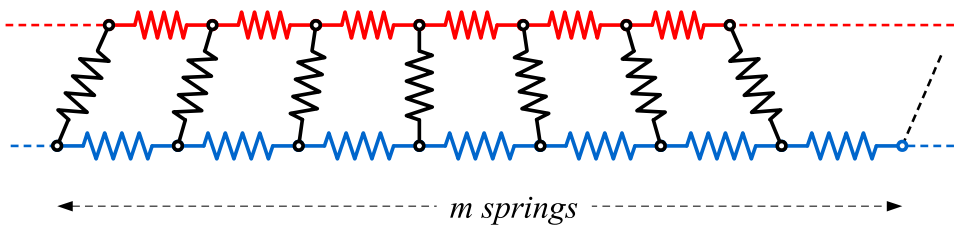


$$f(z) = \tfrac{1}{2}az^2 \qquad g(z) = \tfrac{1}{2}\min\{z^2, \eta\} \qquad h(\xi) = \tfrac{1}{2}\xi^2$$

Interpretation of $f^{\mathrm{d}}_{\mathrm{hom}}$: a minimization on the location of broken springs



or on the optimization of the location and length of 'unbroken islands'



Averaged elastic energy of a single island: $\frac{1}{2}c_m z^2 + \frac{\eta}{2}\frac{1}{m}$

$$c_m = \frac{m(a+1)a}{ma+\tanh(m\theta)\coth(\theta)}, \qquad \theta = \sinh^{-1}\left(\tfrac{1}{2}\sqrt{\tfrac{b(a+1)}{a}}\right)$$

Effect of *microstructure*: $f^{\mathrm{d}}_{\mathrm{hom}}(z) = \left(\frac{1}{2}\inf_m\left\{c_m z^2 + \frac{\eta}{m}\right\}\right)^{**}$

(may set: $c_\infty = a+1$ (all unbroken springs) so that $\inf = \min$)

(mixture of *infinitely many* damage phases corresponding to different m)

Properties: ($z \geq 0$)

- for $z \leq z_*$ $f^{\mathrm{d}}_{\mathrm{hom}}(z) = \dfrac{a+1}{2}z^2$ (no fracture)
- for $z \geq z^*$ $f^{\mathrm{d}}_{\mathrm{hom}}(z) = \dfrac{a}{2}z^2 + \dfrac{\eta}{2}$ (complete fracture)

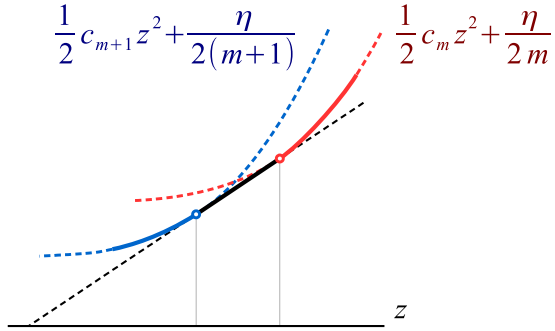
$$z_* = \sqrt{\frac{a\eta}{(a+1)\coth\theta}} \qquad\qquad z^* = \sqrt{\frac{\eta(2a+b(a+1))}{ab}}$$

A diabolic behaviour at a point

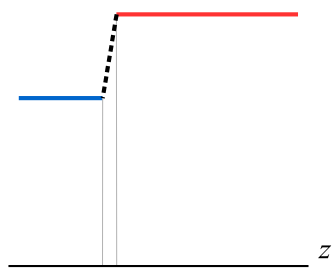
- between z_* and z^* we have alternating:

parabolic pieces corresponding to regularly alternating island of length m (with the density of broken springs $1/m$)

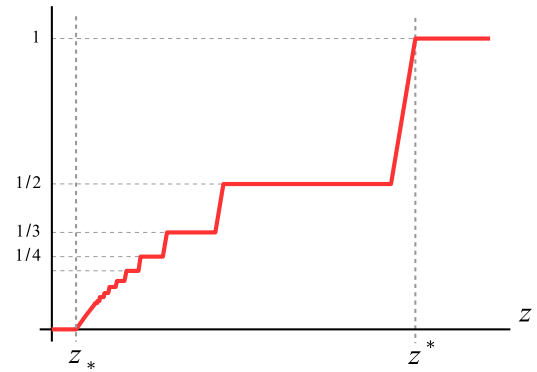
affine pieces = mixtures of islands of length m and $m+1$ (with the density of broken springs interpolating between $1/(m+1)$ and $1/m$)



energy



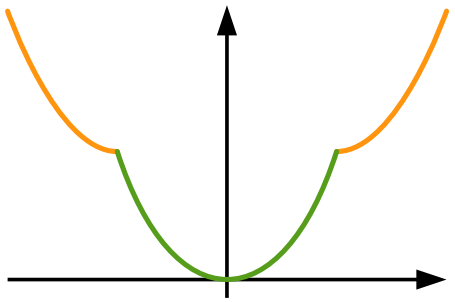
density of broken springs



diabolic behaviour = concentration of infinitely many phases at z_*

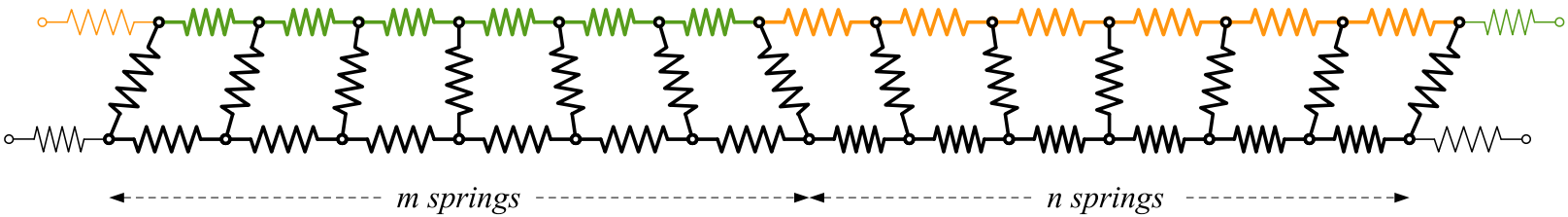
More diabolic points

In NT 2015 the case of bistable springs is considered: $f(z) = \min\{z^2, 1 + (|z| - 1)^2\}$

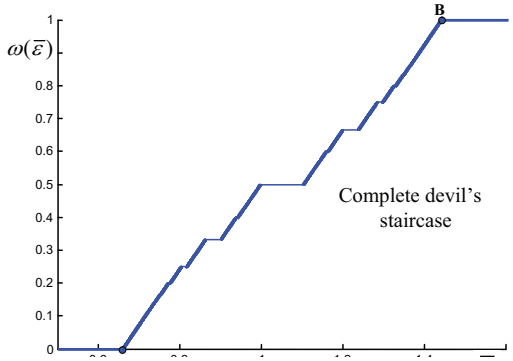


bistable

$f^{\mathrm{c}}_{\mathrm{hom}}$ is obtained by optimization on mixtures of pairs of 'islands' of springs with u in the two phases



(minimization on m and n instead of m; i.e., on n/m instead of $1/m$)



. . . still have to grasp the details (and *the Devil is in the details*!. . .)

. . . non-commensurability effect $\Longrightarrow$ nontrivial Γ-development?

An equivalent (?) continuum model

From Braides-Truskinovsky we have that $\sum_i \varepsilon \frac{1}{2} \min\left\{ \left(\frac{u_i - u_{i-1}}{\varepsilon} \right)^2, \eta \right\}$ is uniformly equivalent (if a finite number of jumps are considered) to

$$\frac{1}{2} \int_0^1 |u'|^2 + \varepsilon \frac{\eta}{2} \#(S(u)) \qquad \text{Griffith fracture}$$

(u piecewise-H^1, $S(u)$ = set of discontinuity/fracture points of u)

A continuum energy with discontinuous u

$$F_\varepsilon(u,v) = \frac{a}{2} \int_0^1 |v'|^2 dx + \frac{1}{2} \int_0^1 |u'|^2 dx + \frac{b}{2\varepsilon^2} \int_0^1 (u-v)^2 dx + \varepsilon \frac{\eta}{2} \#(S(u))$$

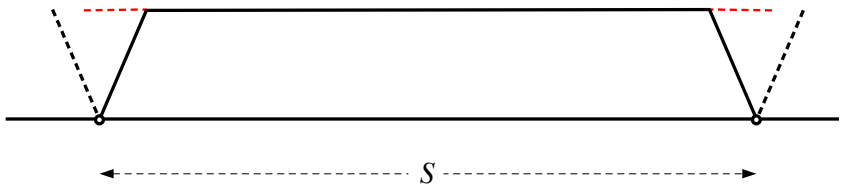
The Γ**-limit** is again local: $\int_0^1 f^{\mathrm{c}}_{\mathrm{hom}}(v') dx$ (c = continuum)

(+ corresponding homogenization formula)

Comparison with the discrete case

- $f^{\mathrm{c}}_{\mathrm{hom}}(z) = \frac{1}{2}\inf_{s>0}\left(C_s z^2 + \frac{\eta}{2s}\right)$ with $C_s = \frac{(a+1)\omega\frac{s}{2}}{\omega\frac{s}{2}+\frac{1}{a}\tanh\omega\frac{s}{2}}$, $\omega = 2\sinh\theta$

obtained by minimizing with boundary conditions $v(s) - v(0) = s$



$$C_\infty = (a+1)$$

- $f^{\mathrm{c}}_{\mathrm{hom}}$ is strictly convex (no convexification = no mixtures involved)

- (no fracture) $f^{\mathrm{c}}_{\mathrm{hom}}(z) = \frac{a+1}{2}z^2$ if $z \le z^c_*$, where $z^c_* = z_*\sqrt{\cosh\theta} > z_*$

Consequence: in order for the discrete and continuous energies to be equivalent up to z_* we need to 'correct' the continuum facture energy to

$$\frac{1}{2}\int_0^1 |u'|^2 + \varepsilon\frac{\eta}{2\cosh\theta}\#(S(u))$$

(effective fracture toughness)

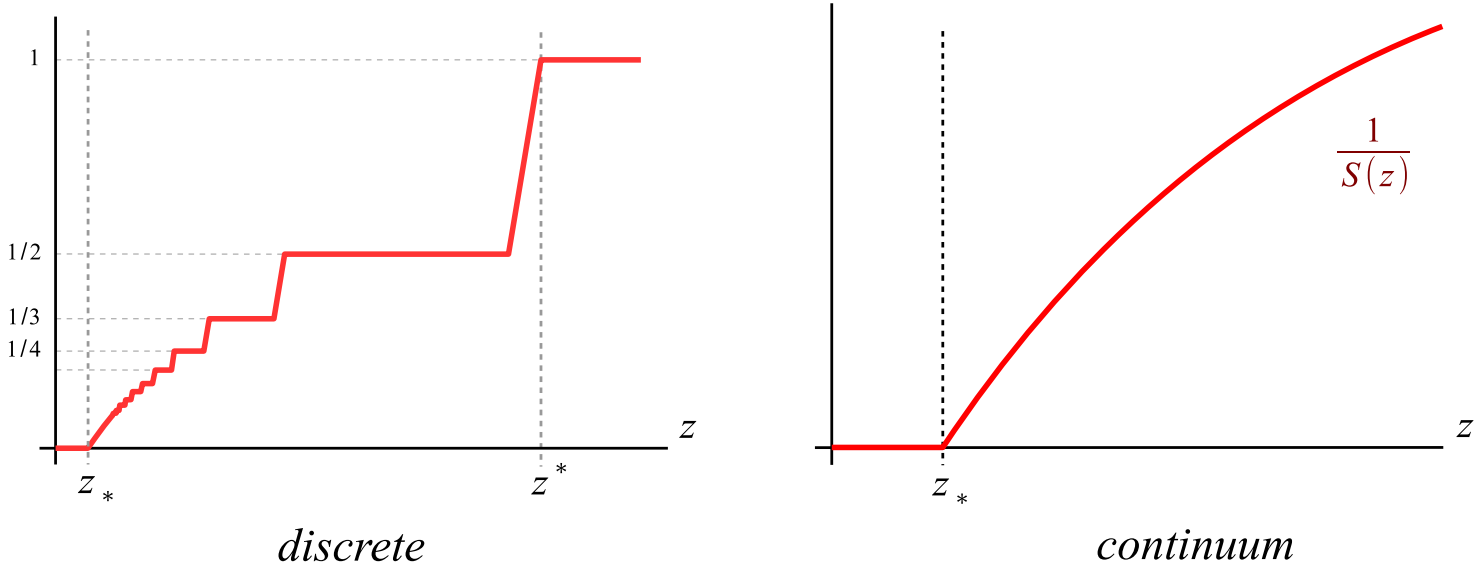
- $f^{c}_{\mathrm{hom}}(z)$ is asymptotic to $\frac{a}{2}z^2 + cz^{2/3}$ for $|z| \to \infty$,

where $c = c(a, b, \eta) = \sqrt[3]{\frac{4a}{3(a+1)\eta\omega}}$

The optimal distribution of fracture at given strain is for $s = s(z) \sim z^{-2/3}$

(cf. Müller, Alberti-Müller, etc.)

- $\frac{1}{s(z)}$ replaces the percentage of broken springs



- The result can be exported to dimension d with

$$\begin{aligned} F_\varepsilon(u,v) &= \frac{a}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx \\ &\quad + \frac{b}{2\varepsilon^2}\int_\Omega (u-v)^2 dx + \varepsilon\frac{\eta}{2}\mathcal{H}^{d-1}(S(u)) \end{aligned}$$

with Γ-limit $\displaystyle\int_\Omega f^{\mathrm{c}}_{\mathrm{hom}}(|\nabla v|)dx$

($1d$ geometry, with fracture sites perpendicular to ∇u)

The Γ-convergence result with asymptotic formulas holds for general f, g, h and (possibly cohesive) fracture energy

$$\int_{S(u)} \varphi(|u^+ - u^-|)d\mathcal{H}^{d-1}$$

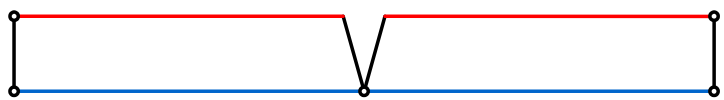
($u^\pm$ traces of u on $S(u)$)

Recovery of a diabolic behaviour

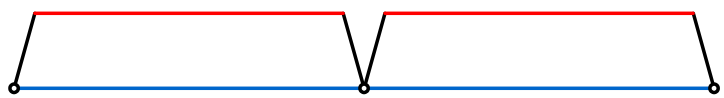
Boundary-displacement-parameterized evolution with increasing fracture site

At given ε increase z and minimize with conditions $u(0) = v(0) = 0$, $u(1) = v(1) = z$ (i.e., $u(x) - zx$ and $v(x) - zx$ 1-periodic) (Hard device) subjected to $S(u^\varepsilon(z)) \subset S(u^\varepsilon(z'))$ if $z \leq z'$

$$\frac{1}{2} c_\infty z^2$$



$$\frac{1}{2} c_{\frac{1}{\varepsilon}} z^2 + \varepsilon \frac{\eta}{2}$$



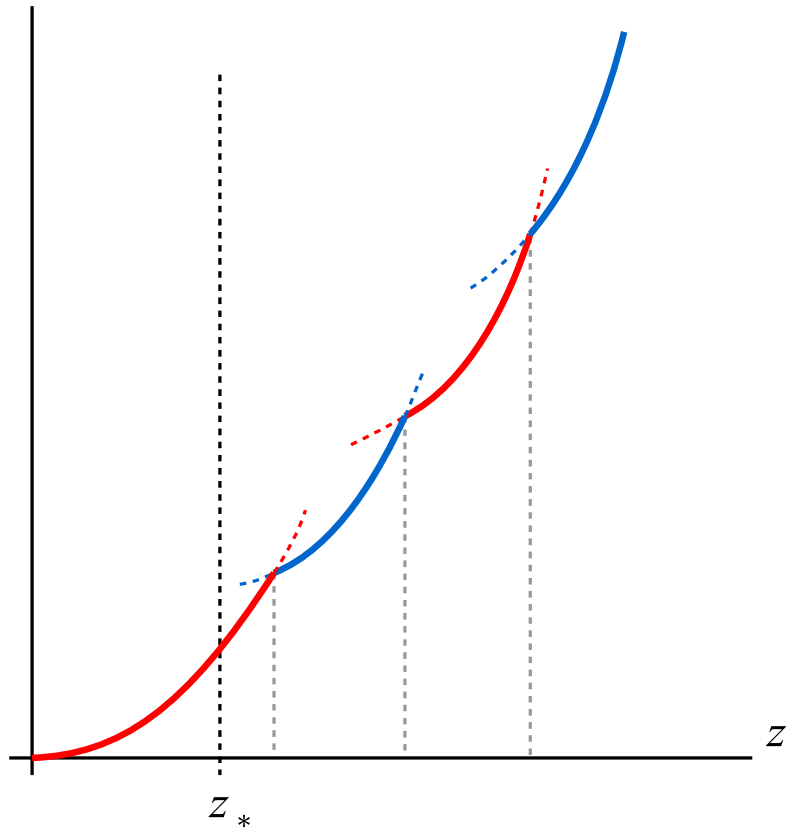
$$\frac{1}{2} c_{\frac{1}{2\varepsilon}} z^2 + 2\varepsilon \frac{\eta}{2}$$

$\ldots$

$$\frac{1}{2} c_{\frac{1}{2^n \varepsilon}} z^2 + 2^n \varepsilon \frac{\eta}{2}$$

If we plot the energy in function of z the minimum value is

$$m^{\varepsilon}(z) = \frac{1}{2}\Big(C_{\infty} z^2 \wedge \min\{C_{\frac{1}{\varepsilon 2^j}} z^2 + \varepsilon 2^j \eta : j \geq 0\}\Big)$$



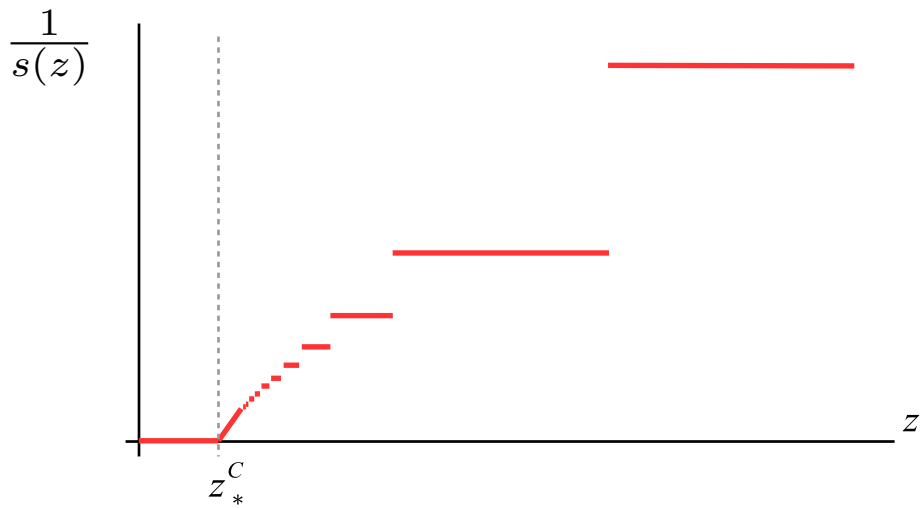
If we let $\varepsilon \to 0$, up to subsequences, the limit "evolution" is described by an envelope of infinitely many parabolas "accumulating in z_*^c." This is an example when Γ-limit and "quasistatic evolution" *do not commute*.

Example: take $\varepsilon = 2^{-k}$ then (we set $m_k = m^{2^{-k}}$ for short)

$$m_k(z) = \frac{1}{2}\Big(C_\infty z^2 \wedge \min\{C_{2^{k-j}} z^2 + 2^{k-j}\eta : j \geq 0\}\Big)$$

tend to

$$m(z) = \frac{1}{2}\Big(C_\infty z^2 \wedge \min\{C_{2^{-n}} z^2 + 2^n \eta : n \in \mathbb{Z}\}\Big)$$



(it's *the Devil in disguise*!...)

Conclusions

Conclusions

Happy birthday, Lev!!