Homogenization of geometric flows on lattices

Andrea Braides (University of Rome *Tor Vergata*)

A Mathematical Tribute to Ennio De Giorgi

Pisa, Centro di Ricerca Matematica Ennio De Giorgi
September 21 2016

My recollections from thirty-five years ago

As a young student of De Giorgi I saw him as an **oracle** dispensing his wisdom (sometimes in misterious statements) for hours in his office at *Scuola Normale* or in the classroom of the *Timpano* building.

Many of his statements are still food for thought

- **positive statements (conjectures)**
- **working methods (lemmas, definitions)**

and (more rarely)

- **negative statements**

I would like to talk about one of these negative statements. . .

(but before I have to recall some of the interests of De Giorgi in his last years)

Minimizing movements

(after Almgren, Taylor and Wang (SIAM JO1993), and many other earlier contributions)

De Giorgi was interested in a notion of gradient flow w/ minimal hypotheses on energies and spaces:

- introduce $\tau > 0$ (time-scale), x_0 (initial data)
- define the discrete orbit x_k^τ by $x_0^\tau = x_0$ and

$$x_{k+1}^\tau \text{ minimizes } x \mapsto F(x) + \frac{1}{2\tau} D(x, x_k^\tau)$$

(F = energy, D= distance squared or dissipation)

- extend x_k^τ to a continuous-time orbit $x^\tau(t) = x^\tau_{\lfloor x/\tau \rfloor}$
- let $\tau \to 0$ and obtain (up to subseq.) $x^\tau(t) \to x(t)$

x is **a minimizing movement** for F with initial datum x_0

Almgren-Taylor-Wang: x = set, F = perimeter, D = L^2-distance of boundaries. Minimizing movement = motion by mean curvature

Note: (up to technicalities) minimizing movements are *curves of maximal slope*; in the smooth (Hilbert) case $x' = -\nabla F(x)$.

(for developments in the last 20 years cf. Ambrosio, Gigli, Savaré)

De Giorgi's negative statement

In June 1995 at the conference "Calculus of Variations and Nonlinear Elasticity" in Cortona we were discussing about minimizing movements.

At that time I had in mind the homogenization of nonlinear functionals and I asked De Giorgi

Would it be possible to use minimizing movements for the homogenization of nonlinear functionals?

After a few seconds his answer was

. . . è un osso duro. . . (it is a hard nut to crack)

I was taken aback by his answer, and unfortunately never had the occasion to ask for more explanations.
. . . but why are oscillating energies hard to "homogenize"?

A simple zero-dimensional example

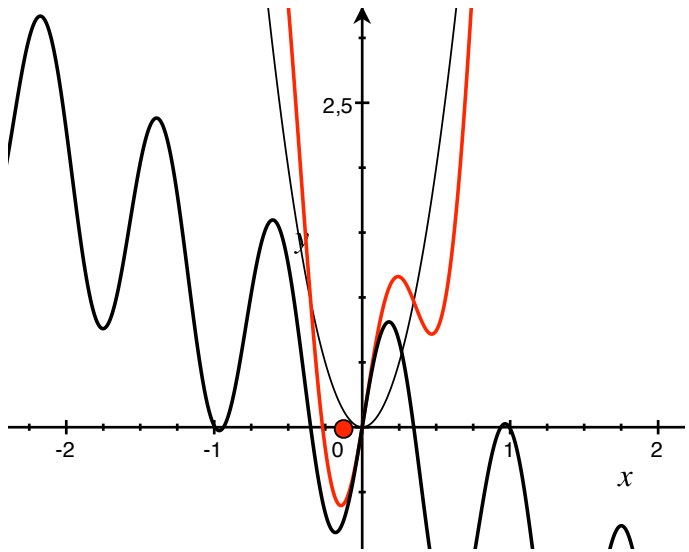
(Ansini-B-Zimmer, 2016)

We now show that even in the simplest example, homogenization is not trivial.

Energy: $F(x) = -x + \varepsilon \sin\left(\frac{x}{\varepsilon}\right)$, Dissipation: $D(x,y) = (x-y)^2$,

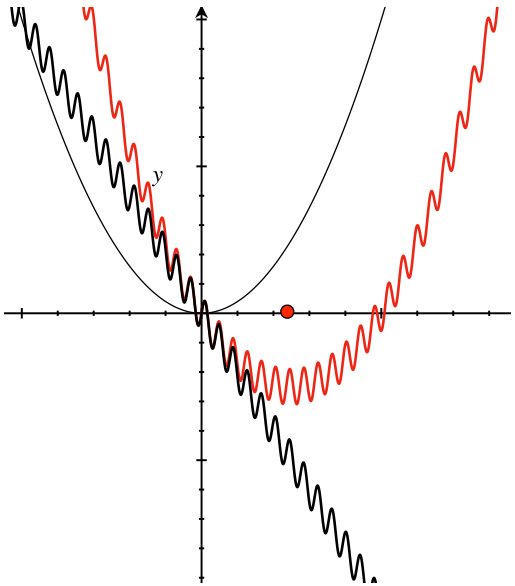
$x_0 = 0$, x_{k+1} minimizing $x \mapsto -x + \varepsilon \sin\left(\frac{x}{\varepsilon}\right) + \frac{1}{2\tau}(x - x_k)^2$

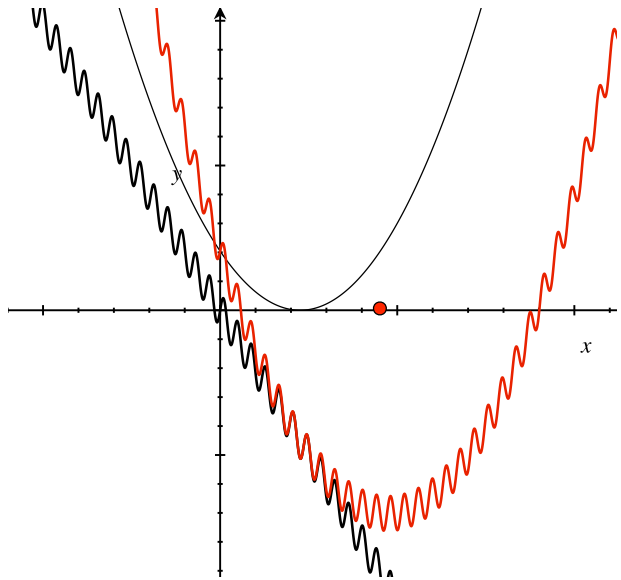
- For $\tau << \varepsilon$ we have the convergence to closest local minimum

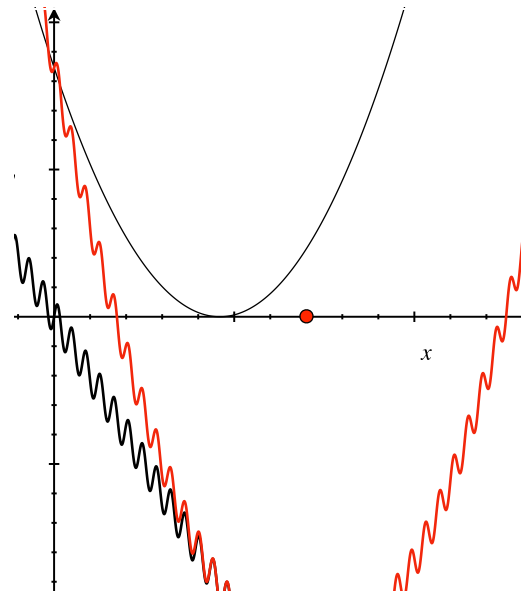


Approximate limit equation $x' = 0$

- For $\tau >> \varepsilon$, x_k approximately sit on local minima of the energy and $x_k = x_{k-1} + \tau$

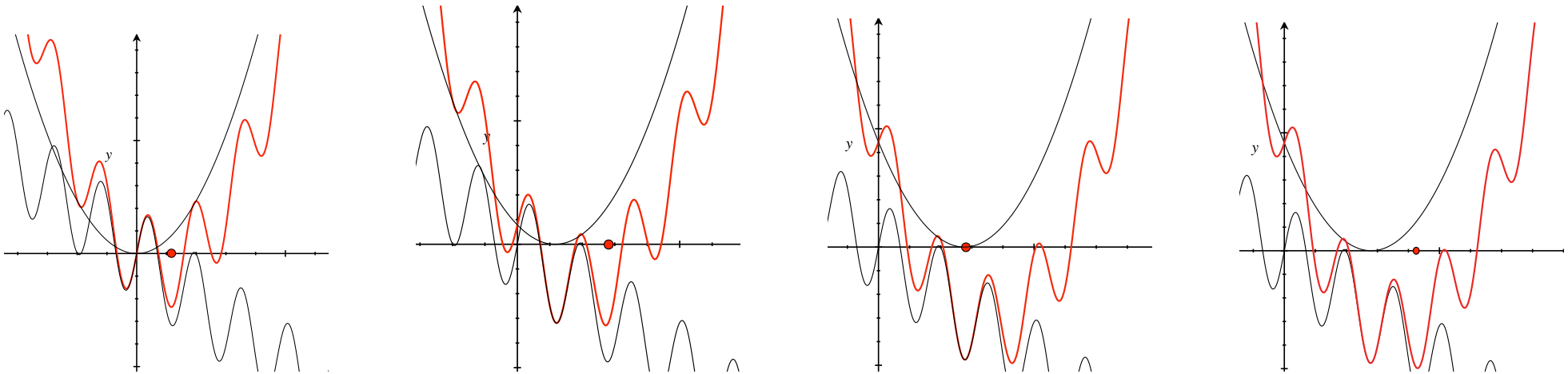






Approximate limit equation $x' = 1$

• For $\tau \sim \varepsilon$, x_k may converge to a local minimum or have almost-periodic oscillations around an averaged uniform motion
$x_{k+1} = x_k + \varepsilon\, C(\frac{\tau}{\varepsilon})$ (independently of x_0)
(If $\frac{\tau}{\varepsilon} \to \gamma$) approximate limit equation $x' = \frac{1}{\gamma} C(\gamma)$



$\Longrightarrow$ behaviour described by the properties of a dynamical system
(Question: what are the properties of $\gamma \mapsto \gamma^{-1} C(\gamma)$?)

$\Longrightarrow$ Homogenized motion depends on the *interplay* between the time and length scales τ and ε

Keywords: pinning threshold, homogenized velocity

Minimizing movements along a sequence F_ε at scale τ

Given F_ε, D_ε (often $D_\varepsilon = D$), $\tau > 0$ (time-scale), x_0 (initial datum)

- define the discrete orbit $x_k^{\tau,\varepsilon}$ by $x_0^{\tau,\varepsilon} = x_0$ and

$$x_{k+1}^{\tau,\varepsilon} \text{ minimizes } x \mapsto F_\varepsilon(x) + \frac{1}{2\tau} D_\varepsilon(x, x_k^{\tau,\varepsilon})$$

- extend $x_k^{\tau,\varepsilon}$ to a continuous-time orbit $x^{\tau,\varepsilon}(t) = x_{\lfloor x/\tau \rfloor}^{\tau,\varepsilon}$
- let $\varepsilon, \tau \to 0$ and obtain (up to subseq.) $x^{\tau,\varepsilon}(t) \to x(t)$

x is **a minimizing movement** along F_ε at scale τ with initial datum x_0 (strictly speaking, with dissipations D_ε)

Minimizing movements and Γ-convergence

$$\Gamma\text{-convergence of } F_\varepsilon \to F$$
$$\Updownarrow$$

For all G_ε continuously converging to G and $F_\varepsilon + G_\varepsilon$ equicoercive we have

$$\inf(F_\varepsilon + G_\varepsilon) \to \min(F + G)$$
$$\text{minimizing sequences } x_\varepsilon \to x \text{ minimizers}$$

As a consequence, if $D_\varepsilon(\cdot, y_k)$ continuously converge to $D(\cdot, y)$ when $y_k \to y$ and F_ε Γ-converges to F then by induction

$$\text{minimizers of } F_\varepsilon(x) + \tfrac{1}{2\tau} D_\varepsilon(x, x_k^{\tau,\varepsilon})$$

converge to $\quad$ minimizers of $F(x) + \frac{1}{2\tau} D(x, x_k)$

Compatibility of Γ-convergence at "slow time scales"
Let $F_\varepsilon + \frac{1}{2\tau} D_\varepsilon$ be equicoercive. If $\varepsilon \to 0$ *"fast enough with respect to τ"*, then any minimizing movement along F_ε at scale τ is a **minimizing movement for the Γ-limit F**

$\Rightarrow$ Minimizing movement for F is a "comparison evolution"

A detour: conditions for commutability

Question: are there conditions that guarantee compatibility of Γ-convergence and minimizing movements at all scales?

"Classical" answer: F_ε **convex** (cf. Brezis. . . Ambrosio-Gigli,. . .)

Connection with gradient flows for Ginzburg-Landau (Sandier-Serfaty), Mumford-Shah (Gobbino), Lennard-Jones (B-Defranceschi-Vitali), Perona-Malik (Colombo-Gobbino).

A general condition. (B-Colombo-Gobbino-Solci, CRAS, 2016)
Let $D_\varepsilon = D$, and let F_ε Γ-converge to F and be such that
• if $x_\varepsilon \to x$ with $\sup_n \{|F_\varepsilon(x_\varepsilon)| + |\partial F_\varepsilon|(x_\varepsilon)\} < +\infty$, then
$\lim_\varepsilon F_\varepsilon(x_\varepsilon) = F(x)$ and $\liminf_n |\partial F_\varepsilon|(x_\varepsilon) \geq |\partial F|(x)$
• curves of maximal slopes for F are minimiz. movements for F
Then for all $\tau_\varepsilon \to 0$ any minimizing movement along F_ε at scale τ_ε is a minimizing movement for F.

Some DeGiorgian keywords of the proof

Given x minimizing movement along F_ε at scale $\tau = \tau_\varepsilon$

- (use De Giorgi interpolants) $\widetilde{x}^{\tau,\varepsilon}(t) \to x(t)$ for which

$$F_\varepsilon(\widetilde{x}^{\tau,\varepsilon}(s)) - F_\varepsilon(\widetilde{x}^{\tau,\varepsilon}(t)) \geq \frac{1}{2}\int_s^t |(\widetilde{x}^{\tau,\varepsilon})'|^2\,dr + \frac{1}{2}\int_s^t |\partial F_\varepsilon|^2(\widetilde{x}^{\tau,\varepsilon}(r))\,dr + o(1)$$

if $0 \leq s \leq t$

- check that $\sup\{|F_\varepsilon(\widetilde{x}^{\tau,\varepsilon}(r))| + |\partial F_\varepsilon|(\widetilde{x}^{\tau,\varepsilon}(r))\} < +\infty$ at almost all r
- use the liminf inequality for $|\partial F_\varepsilon|$ and that $F_\varepsilon(\widetilde{x}^{\tau,\varepsilon}(s)) \to F_\varepsilon(x(s))$

$$F(x(s)) - F(x(t)) \geq \frac{1}{2}\int_s^t |x'|^2\,dr + \frac{1}{2}\int_s^t |\partial F|^2(x(r))\,dr$$

- the limit is a curve of maximal slope. Hence, by hypothesis it is a minimizing movement for F

Note: in our example $\partial F = 1$ but every point is approximated by local minima of F_ε (for which $\partial F_\varepsilon = 0$); hence the first hypothesis is violated

The passage discrete-to-continuum

A late interest of De Giorgi, fostered by results by Chambolle (approximation of the Mumford-Shah functional by finite-difference energies with highly non-convex (truncated quadratic) potentials)

$\Longrightarrow$ one of De Giorgi's *last conjectures* (Gobbino CPAM 1998)
$\Longrightarrow$ treatment of non-local energies (B-Dal Maso CalcVar 1997)
$\Longrightarrow$ local minimization of Lennard-Jones discrete systems (B-Dal Maso-Garroni ARMA 1999)
$\Longrightarrow$ *discrete-to-continuum analysis* by Γ-convergence (B-Gelli MMS 2002, Alicandro-Cicalese SIMA 2004, . . .)

A *very timely subject*: cf. also Le Bris and Lions, Weinan E, Friesecke and Theil, Ortiz, and collaborators, for progress in the last 20 years. We have missed De Giorgi much in this study

Review papers: Le Bris-Lions. From atoms to crystals. Bull. AMS '05
B.- Variational methods for lattice systems. Proc. 2014 ICM, Seoul (for the lazy. . . the ICM video:www.youtube.com/watch?v=rHGNjjEC5ww)

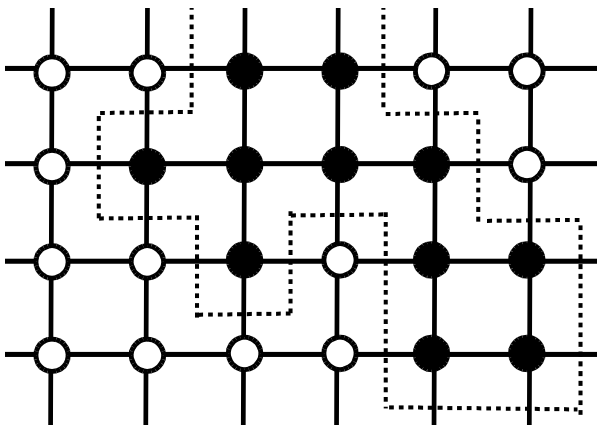
Lattice spin systems

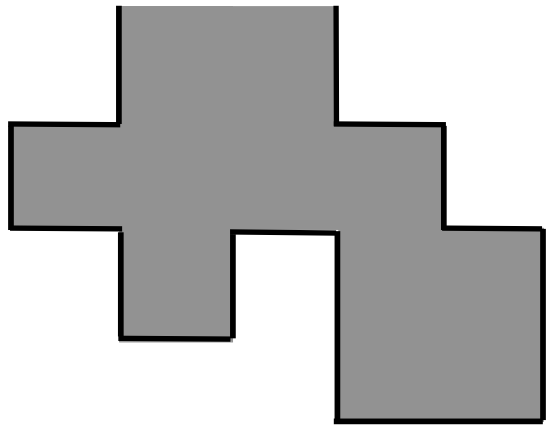
(Caffarelli-de la Llave JSP 2005, Alicandro-B-Cicalese NHM 2006, B-Piatnitsky JFA 2013)
Simplest lattice energy: the **ferromagnetic** nearest-neighbour system in 2D

$$F_\varepsilon(u) = \sum_{\langle i,j\rangle} \varepsilon(u_i - u_j)^2, \qquad u_i \in \{0,1\}$$

$\langle i,j\rangle$ = sum on nearest neighbours in $\varepsilon\mathbb{Z}^2$

A spin function $u : \varepsilon\mathbb{Z}^n \to \{0,1\}$ is identified with its piecewise-constant interpolation $\sim$ set $\{u = 1\}$





Continuum limit of ferromagnetic spin systems

By this identification we define the convergence $u_\varepsilon \to A$ as the convergence of (the interpolations of) u_ε to χ_A, and compute the Γ-limit

$$F(A) = \int_{\partial^* A} \|\nu\|_1 d\mathcal{H}^1$$

defined for A *set of finite perimeter*, where $\partial^* A$ = reduced boundary, ν = normal to A and $\|\nu\|_1 = |\nu_1| + |\nu_2|$

This is a **crystalline perimeter**, whose *Wulff shape* is a *square*

Note: more in general we can consider $F_\varepsilon(u) = \sum_{i,j} \varepsilon c_{ij}^\varepsilon (u_i - u_j)^2$ with c_{ij}^ε long-range interaction coefficients, and obtain

$$F(A) = \int_{\partial^* A} \varphi(\nu) d\mathcal{H}^1$$

(analogously in higher dimension).

Minimizing movements for spin systems

(Motivation: motion of interfaces in binary alloys, cf. Cahn, Taylor, . . .)

We take D = (discrete version of the) ATW dissipation

Two general facts

• for $\tau \ll \varepsilon$ such systems are **pinned**
(A set can only decrease by an area $\sim$ perimeter $\times\, O(\varepsilon)$
Dissipation $\geq \frac{\varepsilon}{2\tau} \times$ area $\geq \frac{\varepsilon^2}{\tau} \times$ perimeter $>> \varepsilon \times$ perimeter = decrease in energy
$\Rightarrow$ orbits are constant for small τ)

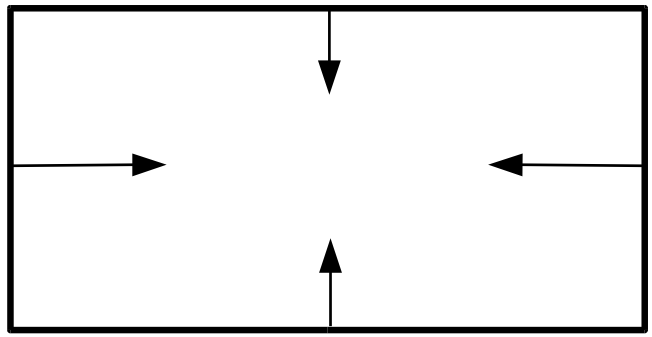
• for $\varepsilon \ll \tau$ the minimizing movement is **flat flow**; i.e., the minimizing movement for the crystalline perimeter
(by "compatibility" of Γ-convergence at slow time scales)

$\Rightarrow$ we have a **critical scaling** $\varepsilon \sim \tau$

Flat flow

Almgren and Taylor showed that the ATW scheme applied with energy F the crystalline perimeter in dimension two (**flat flow**), gives **motion by crystalline curvature**.

A simple test. Motion by crystalline curvature is described by the evolution of *Wulff-like shapes* (in this case, *rectangles*)



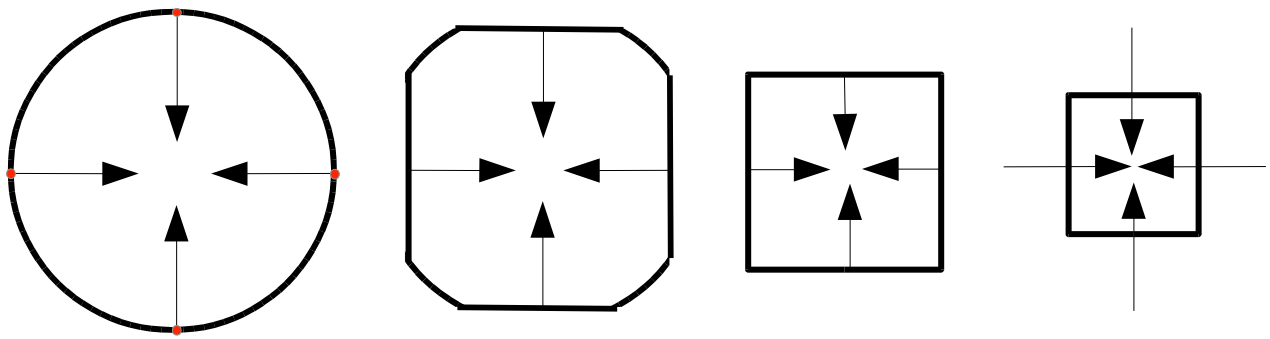
A side of length L moves inwards with velocity $v = \frac{2}{L}$
($\Rightarrow$ homothetical contraction to the centre)

This characterizes evolution of arbitrary sets, by the law

$$v = \kappa \qquad \text{(crystalline curvature)}$$

For simplicity we consider covex sets; in this case

$$\kappa = \begin{cases} \frac{2}{L} & \text{for coordinate sides} \\ 0 & \text{elsewhere} \end{cases}$$



evolution of a circle (with extinction on finite time)

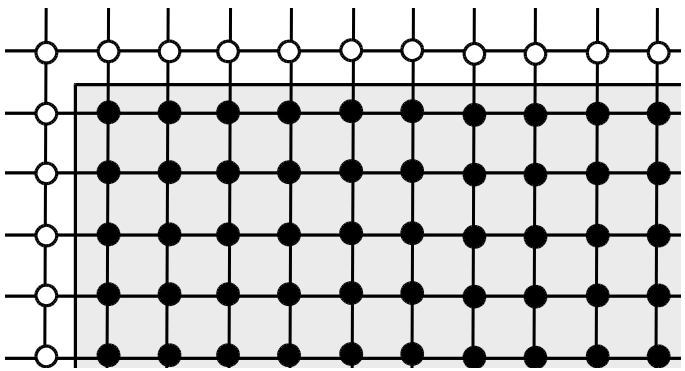
This is a simplified setting (with respect to motion by mean curvature), since it is governed by a system of ODEs

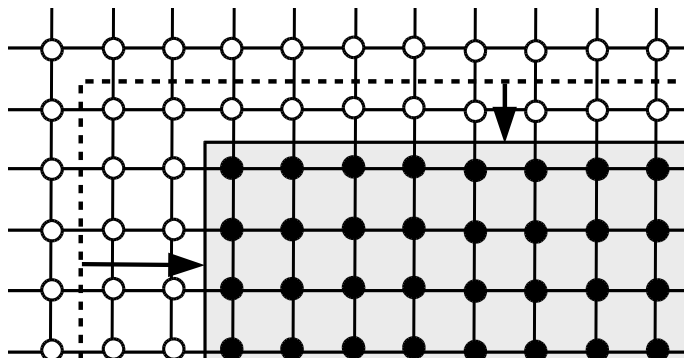
1. Minimizing movements for ferromagnetic spin systems

(B-Gelli-Novaga, ARMA 2010)

At the **critical scaling** $\varepsilon \sim \tau$ we expect some "homogenized" crystalline curvature flow

We test the minimizing-movement scheme on rectangles (this is again sufficient) by picturing what happens close to a corner





- discrete orbit: rectangles evolve in rectangles
- there is motion only if the decrease of the perimeter is sufficiently large with respect to the dissipation
- the velocity is "quantized"
- the limit depends on $\frac{\tau}{\varepsilon}$

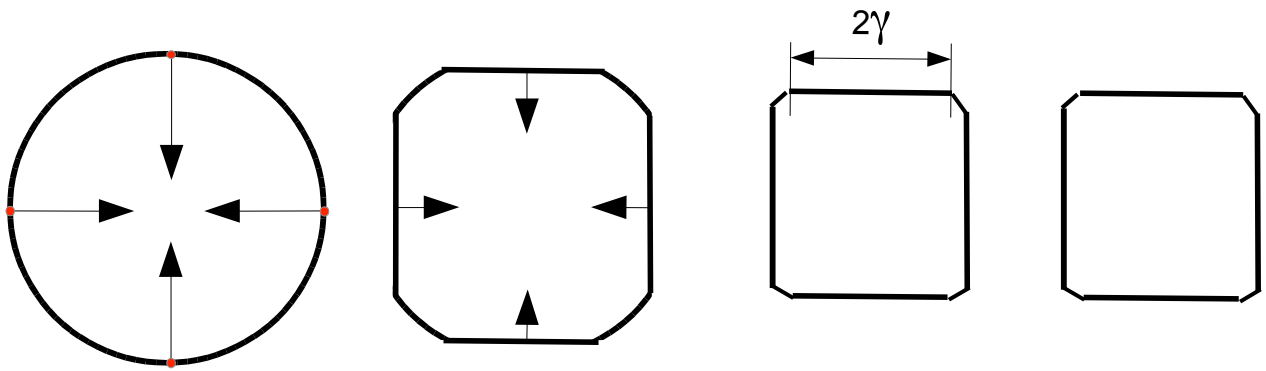
The effective flow

If $\gamma = \lim_{\varepsilon \to 0} \frac{\tau_\varepsilon}{\varepsilon} \in (0, +\infty)$ then the (inward) velocity of the limit motion is given by

$$v = \frac{1}{\gamma} \lfloor \gamma \kappa \rfloor \qquad \lfloor \cdot \rfloor = \text{integer part}$$

Note: this is a system of ODEs with discontinuous right-hand side

- **partial pinning** ($v = 0$ for small κ; i.e., large sides)
- (only) **weak comparison principle**
- only **"generic uniqueness"**
- possibility of **non-trivial motion with eventual pinning**



evolution of a (large) circle (with eventual pinning)

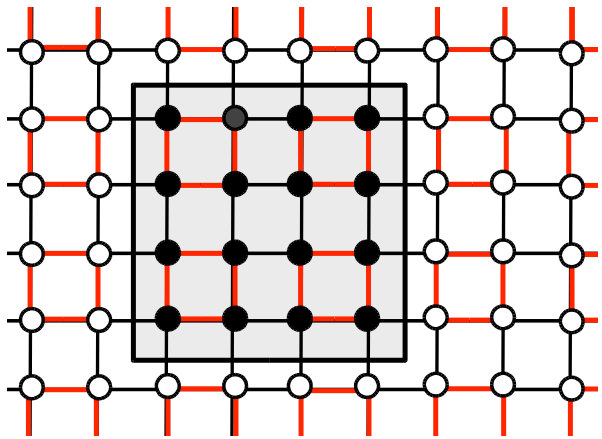
2. Minimizing movements for ferromagnetic spin systems with defects

(B-Scilla, IFB 2013)

We now remark that **minimizing movements do not depend on the Γ-limit only**

We consider $F_\varepsilon(u) = \sum_{\langle i,j \rangle} \varepsilon\, c_{ij} (u_i - u_j)^2$

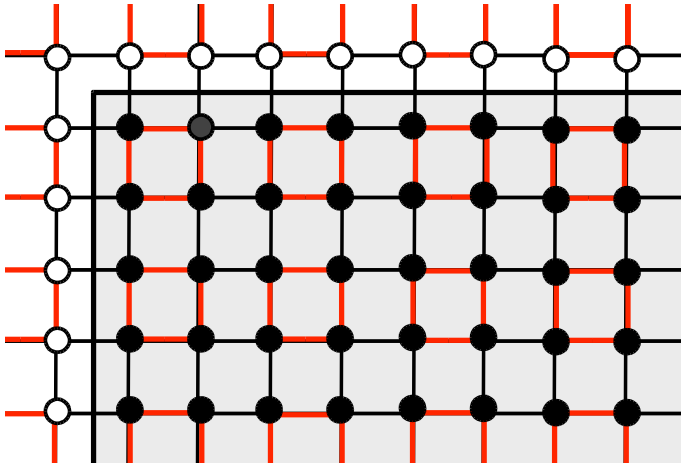
with $c_{ij} > 1$ for some **"defected bonds"** ($c_{ij} = 1$ otherwise)

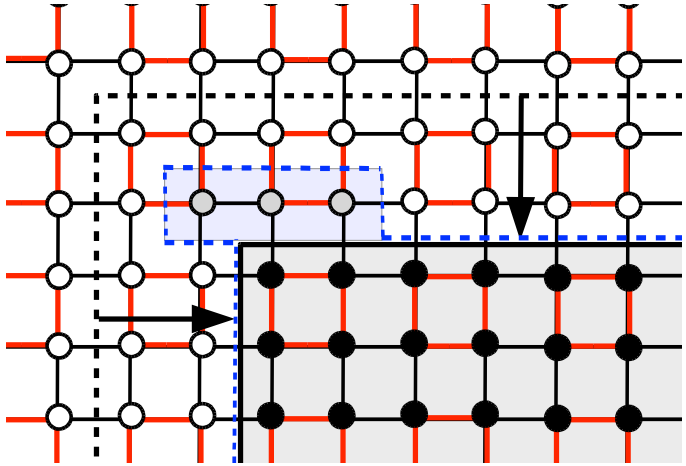


The Γ-limit is still the same **crystalline perimeter**
(optimal sets avoid defects)

Homogenized motion

The motion depends on a *local optimization* argument





- rectangle evolve in rectangles **avoiding the defects**
- the local behaviour of a side is described (at given L) by the orbit of an **auxiliary dynamical system**

The velocity law is

$$v = \frac{1}{\gamma} f_{\mathrm{hom}}(\gamma\kappa)$$

$f_{\mathrm{hom}}(z)$ = "Poincaré rotation number" of a discrete (in space and time) dynamical system

Again $f_{\mathrm{hom}}(z) = 0$ for z below a **pinning threshold** and $\lim_{\gamma\to+\infty} \frac{1}{\gamma} f_{\mathrm{hom}}(\gamma\kappa) = \kappa$ (compatibility with Γ-convergence)

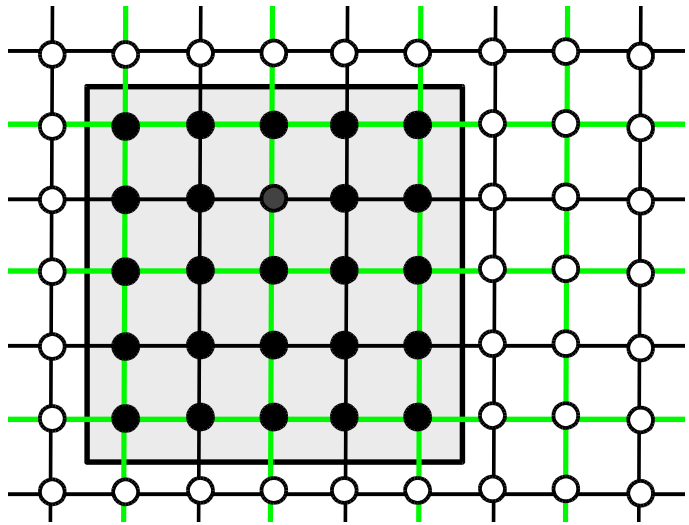
3. Ferromagnetic Systems with Bulk Microstructure

(B-Solci, JNLS 2016)

The previous examples did not develop **spatial microstructure**, which is characteristic of oscillating problems. We now introduce inhomogeneities which favour microstructure.

We consider $F_\varepsilon(u) = \sum_{\langle i,j \rangle} \varepsilon c_{ij}^{\varepsilon} (u_i - u_j)^2$

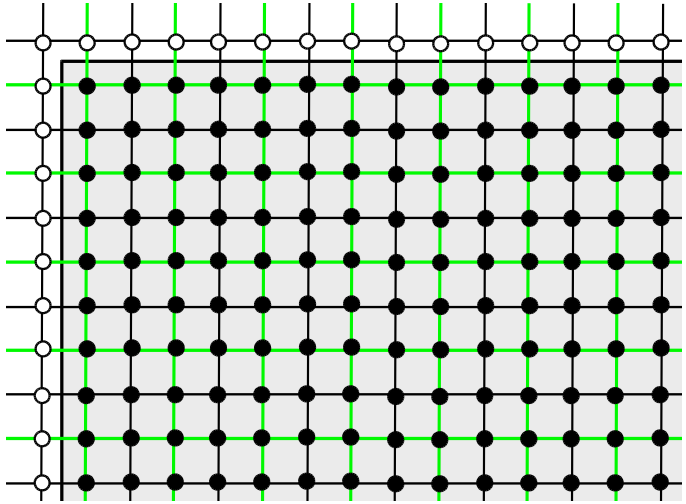
with $c_{ij}^{\varepsilon} = \varepsilon$ for some **weak bonds** ($c_{ij}^{\varepsilon} = 2$ otherwise)

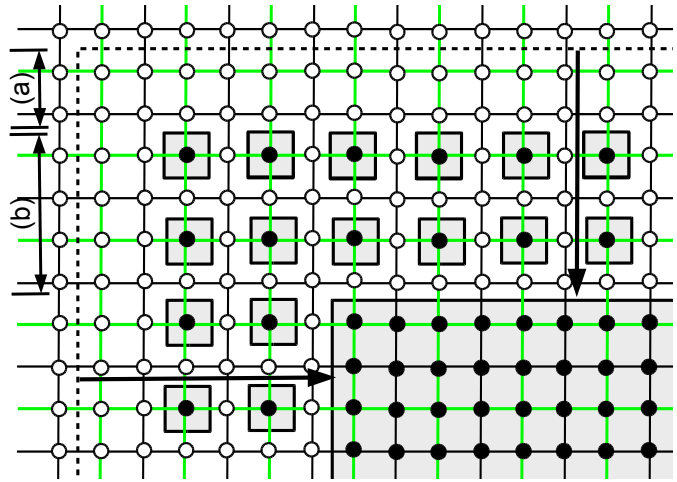


The Γ-limit is still the same **crystalline perimeter**
(optimal sets use the maximum number of weak bonds)

The rectangle test: mushy layers

- Rectangles develop a (temporary) “mushy layer” (cf. fluid dynamics)





(a) = dissipation energetically convenient
(b) = dissipation **not** energetically convenient

- at the next step the mushy layer disappears (thanks to the dissipation), and the process is repeated inside the new rectangle
- define $A(t)$ the limit of the “rectangular evolution”

A more general limit motion

The limit velocity is

$$v = \frac{2}{\gamma}\left\lfloor \max\left\{\frac{2}{3}\gamma\kappa - \frac{2}{3}\gamma + \frac{1}{6}, \frac{1}{2}\gamma k + \frac{1}{4}\right\}\right\rfloor$$

Note: ("Compatibity" is violated!) if $\frac{\varepsilon}{\tau} >> 1$ then

$$v = \max\left\{\frac{4}{3}\kappa - \frac{4}{3}, k\right\}$$

so that $v = \frac{4}{3}\kappa - \frac{4}{3} \neq \kappa$ if $\kappa > 4$

Reason: lack of equicoerciveness $\Rightarrow$ the Γ-limit should take into account also a new variable = limit of the mushy region.
In this new parameter D is not a continuous perturbation.

General Issue: Interaction between energy and dissipation
(cf B. *Local Minimization, Variational Evolution and* Γ*-convergence*. Springer LNM 2014)

4. Antiferromagnetic systems with surface microstructure

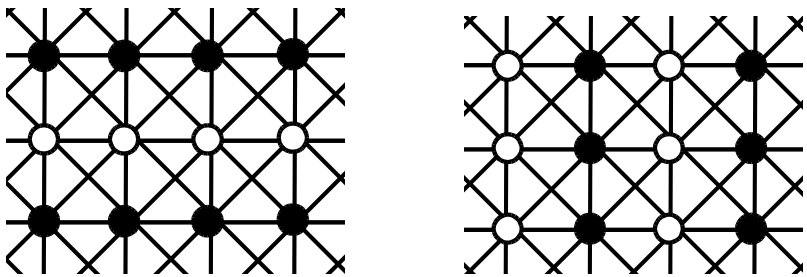
(B-Cicalese-Yip, JSP 2016)

We consider a system of next-to-nearest antiferromagnetic interactions in $\varepsilon\mathbb{Z}^2$

$$F_\varepsilon(u) = -\sum_{\langle\langle i,j\rangle\rangle} \varepsilon\, c_{ij}(u_i - u_j)^2$$

with $c_{ij} = 1$ (NN) and $c_{ij} = 2$ (NNN)

• ground states are **striped patterns** (Alicandro-B-Cicalese NHM 2016)



• we may parameterize the limit as a partition into four sets of finite perimeter (Ambrosio-B JMPA 1990)

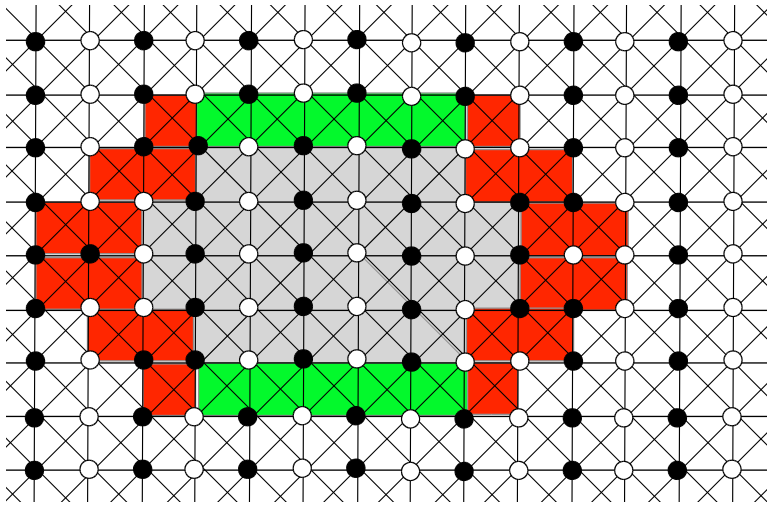
• in general the limit motion will be described as an evolution of networks (cf Kinderlehrer-Liu, Mantegazza-Novaga-Tortorelli, . . .)

Evolution of a “single crystal”

In the case when in the limit we have only (parameters corresponding to the two “variants” of) vertical stripes, we have a crystalline energy

$$F(A) = \int_{\partial^* A} \varphi(\nu) d\mathcal{H}^1$$

where the *Wulff shape* is an irregular hexagon



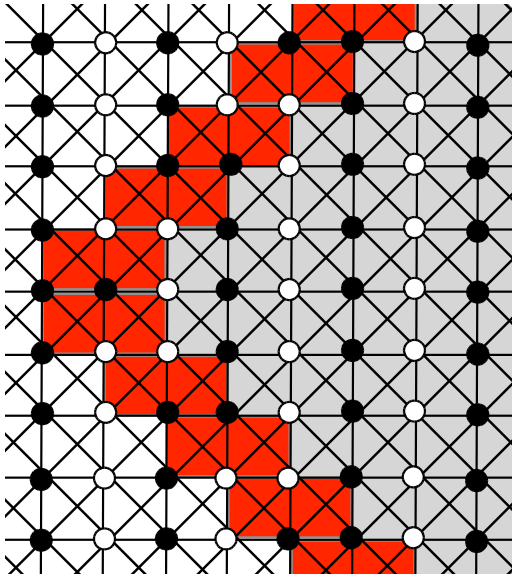
(a discrete Wulff shape with “microstructure” of boundary interfaces)

We may apply the Algren-Taylor scheme to F and obtain the crystalline flow $v = \kappa$ (κ takes into account a “mobility” factor)

Minimizing movements at the discrete level

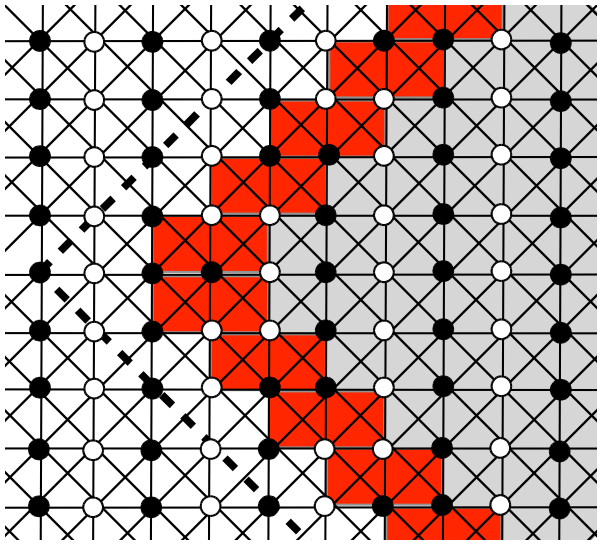
- It is sufficient to examine *Wulff-like sets*
- Wulff-like sets evolve into Wulff-like sets (not trivial!)
- $(\frac{\tau}{\varepsilon} \to \gamma)$ velocity of the horizontal sides is again $v = \frac{1}{\gamma}\lfloor \gamma\kappa \rfloor$
- motion of the "bisectric sides" are governed by **surface microstructure**

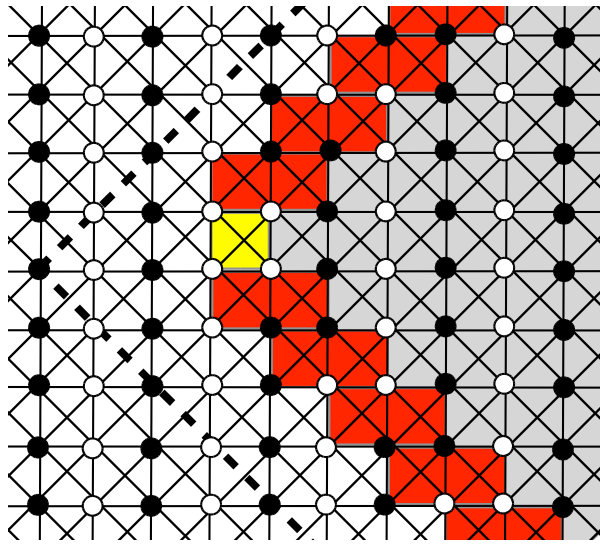
We examine the behaviour close to a corner with optimal microstructure



$\Rightarrow$ introduction of "defects" may iteract with the dissipation

Limit motion of the bisectric sides





We may have

• evolution on non-defected interfaces (after an initial step, possibly)

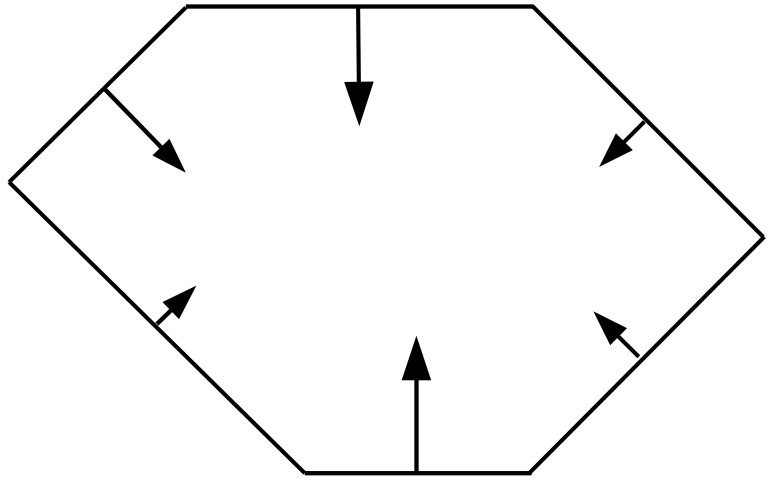
• "toggling" between defected/non-defected interfaces

$\Rightarrow$ "non-local" crystalline motion of bisectric sides with velocity

$$v = \frac{1}{\gamma} f(\gamma\kappa, \gamma\kappa')$$

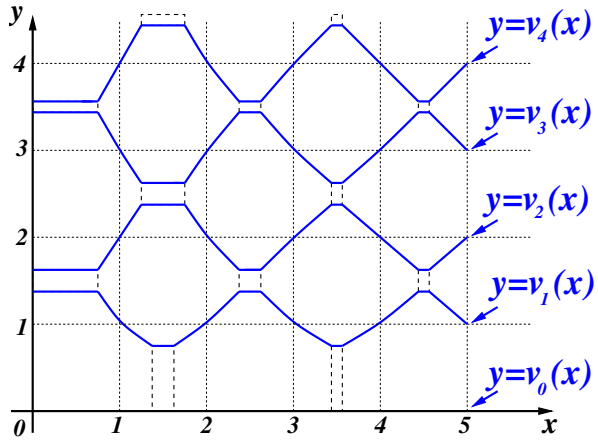
(k, k' = crystalline curvatures of neighbouring bisectric sides)

The final picture



Velocity of vertical sides: $v = \frac{1}{\gamma}\lfloor \gamma\kappa \rfloor$

Velocity of bisectric sides $v = \frac{1}{\gamma} f(\gamma\kappa, \gamma\kappa')$



$f(x, y) = n$ if $v_n(x) \leq y < v_{n+1}(x)$

Conclusions

The application of the minimizing-movement scheme to simple lattice energies gives rise to possibly non-local crystalline motion with pinning, non-uniqueness, homogenization of the velocity non-commutativity effects, that depart from the description given by the simple minimizing movement of the Γ-limit in a unforeseen way.

After we discovered these effects my thoughts naturally went back to De Giorgi's remarks.

Is this what De Giorgi had in mind?
Probably not, but it was nice to go back with the memory to those years and see once more that his teachings linger on.

Thank you for your attention !