ESERCIZI TRASFORMATA DI LAPLACE (Tarantello a.a. 2011-12)

alcolare la trasformata di Laplace delle seguenti funzioni:

$) = t \sin^2 t$, $f(t) = t\cos^2 t$, $f(t) = t^2 \sin t$, $f(t) = t^2 \cos t$

$) = t^2 \sin^2 t$, $f(t) = t^2 \cos^2 t$, $f(t) = t^2 \sinh t$, $f(t) = t^2 \cosh t$

$) = \begin{cases} 2+t & 0 \le t < 2 \\ 6-t & t \ge 2 \end{cases}$, $f(t) = \begin{cases} t+1 & t<3 \\ 4 & 3 \le t < 4 \\ t & t \ge 4 \end{cases}$, $f(t) = \begin{cases} 1-t^2 & 0 \le t < 1 \\ 0 & 1 \le t < 3 \\ 2(t-3) & t \ge 3 \end{cases}$

$t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ 0 & t \ge 3 \end{cases}$, $f(t) = \begin{cases} a-t & 0 \le t < a \\ 0 & a \le t < 2a \\ 2a-t & t \ge a \end{cases}$

$a > 0$ arbitrario.

Calcolare l'anti-trasformata di Laplace delle seguenti funzioni:

$(x) = \dfrac{2x+2}{x^2+2x+5}$, $F(x) = \dfrac{x^2+2x}{x^2+4x+5}$, $F(x) = \dfrac{3x}{(x-1)(x^2+4x+5)}$

$(x) = \dfrac{1}{(x^2+2x+5)^2}$, $F(x) = \dfrac{1}{x(x^2+2x+5)^2}$, $F(x) = \dfrac{x^2}{(x+1)(x^2+2x+5)}$

$(x) = \dfrac{2x-3}{x^2-4}$, $F(x) = \dfrac{x}{(x^2-4)^2}$, $F(x) = \dfrac{2x-3}{x^2+4}$,

$(x) = \dfrac{x^3}{(x^2+4)^2}$, $F(x) = \dfrac{8x^2-4x+12}{x(x^2+4)}$, $F(x) = \dfrac{1}{(x^2-4)(x^2+1)^2}$

$(x) = \dfrac{x+3}{x^2+2x+10}$, $F(x) = \dfrac{x^2}{(x^2+2x+10)(x-2)}$, $F(x) = \dfrac{x+1}{(x^2+2x+10)^2}$

$(x) = e^{-cx} F(x)$ per ogni funzione F assegnata sopra.

otazione: In seguito si utilizza la seguente notazione:

$$u_c(t) := H(t-c) = \begin{cases} 1 & t \ge c \\ 0 & t < c. \end{cases}$$

Risolvere i seguenti problemi di Cauchy:

$$\begin{cases} y'' + 2a\, y' + (a^2+1)\, y = e^{-at} + \delta(t-2) \\ y(0) = 1, \quad y'(0) = 1 \end{cases}, \quad \text{con } a \in \mathbb{R}$$

$$\begin{cases} y'' - 2y' + 3y = 2\, u_1(t) + \delta(t-1) \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

$$\begin{cases} y'' + 4y = f(t) \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

: $f(t) = \cos^2 t$, $f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$, $f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$

$f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$, $f(t) = 2\,\delta\left(t - \frac{\pi}{4}\right)$.

$$\begin{cases} y'' + 2y' + y = f(t) \\ y(0) = 1, \quad y'(0) = 1 \end{cases}$$

: $f(t) = u_1(t) + 2t\, u_3(t) - 6\, u_4(t)(t-1)$, $f(t) = \sin(t-\pi)\, u_\pi(t)$,

$f(t) = \begin{cases} 1 & 0 \le t < 3 \\ 4-t & 3 \le t < 4 \\ 0 & t \ge 4 \end{cases}$, $f(t) = \delta(t-1) + u_{2\pi}(t)$.

$$\begin{cases} y'' + 3y' + 2y = f(t) \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

: $f(t) = \begin{cases} 1 & 0 \le t < 2 \\ -1 & t \ge 2 \end{cases}$, $f(t) = \begin{cases} \cos t & 0 \le t < \pi \\ -1 & t \ge \pi \end{cases}$;

$t) = t - u_2(t)$, $f(t) = t\, u_1(t)$, $f(t) = u_\pi(t) - u_{2\pi}(t)$, $f(t) = \delta(t -$

$(t) = \sin t - \delta(t-\pi)$, $f(t) = u_\pi(t) - \delta(t-\pi)$.

Calcolare l'anti-trasformata di Laplace delle seguenti funzioni

$$F(x) = \frac{3}{x^2+4}, \quad F(x) = \frac{4}{(x-1)^3}, \quad F(x) = \frac{2}{x^2+3x-4}$$

$$F(x) = \frac{3x}{x^2-x-6}, \quad F(x) = \frac{2x-3}{x^2-4}, \quad F(x) = \frac{2x+2}{x^2+2x+5}$$

$$F(x) = \frac{2x+1}{x^2-2x+2}, \quad F(x) = \frac{8x^2-4x+12}{x(x^2+4)}, \quad F(x) = \frac{1-2x}{x^2+4x+5}$$

$$F(x) = \frac{2x-3}{x^2+2x+10}, \quad F(x) = \frac{e^{-2x}}{x^2+x-2}, \quad F(x) = \frac{2e^{-2x}}{x^2-4}$$

$$F(x) = \frac{e^{-x}+e^{-2x}-e^{-3x}-e^{-4x}}{x}, \quad F(x) = \frac{3!}{(x-2)^4},$$

$$F(x) = \frac{2(x-1)e^{-2x}}{x^2-2x+2}, \quad F(x) = \frac{(x-2)e^{-x}}{x^2-4x+3}.$$

Tracciare il grafico e calcolare la trasformata di Laplace delle seguenti funzioni:

(a) $u_1(t) + 2u_3(t) - 6u_4(t)$
(b) $(t-3)u_2(t) - (t-2)u_3(t)$
(c) $f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
(d) $f(t-3)u_3(t)$, where $f(t) = \sin t$
(e) $f(t-1)u_2(t)$, where $f(t) = 2t$
(f) $f(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

2. Find the Laplace transform of each of the following functions.

(a) $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$ (b) $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$

(c) $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$ (d) $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$

(e) $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$ (f) $f(t) = t - u_1(t)(t-1), \quad t \geq 0$

Risolvere i seguenti problemi di Cauchy tramite l'uso della trasformata di Laplace:

11. $y'' - y' - 6y = 0;$ $\quad y(0) = 1, \quad y'(0) = -1$
12. $y'' + 3y' + 2y = 0;$ $\quad y(0) = 1, \quad y'(0) = 0$
13. $y'' - 2y' + 2y = 0;$ $\quad y(0) = 0, \quad y'(0) = 1$
14. $y'' - 4y' + 4y = 0;$ $\quad y(0) = 1, \quad y'(0) = 1$
15. $y'' - 2y' - 2y = 0;$ $\quad y(0) = 2, \quad y'(0) = 0$
16. $y'' + 2y' + 5y = 0;$ $\quad y(0) = 2, \quad y'(0) = -1$
17. $y^{iv} - 4y''' + 6y'' - 4y' + y = 0;$ $\quad y(0) = 0, \; y'(0) = 1, \; y''(0) = 0, \; y'''(0) = 1$
18. $y^{iv} - y = 0;$ $\quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$
19. $y^{iv} - 4y = 0;$ $\quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$
20. $y'' + \omega^2 y = \cos 2t, \quad \omega^2 \neq 4;$ $\quad y(0) = 1, \quad y'(0) = 0$
21. $y'' - 2y' + 2y = \cos t;$ $\quad y(0) = 1, \quad y'(0) = 0$
22. $y'' - 2y' + 2y = e^{-t};$ $\quad y(0) = 0, \quad y'(0) = 1$
23. $y'' + 2y' + y = 4e^{-t};$ $\quad y(0) = 2, \quad y'(0) = -1$

1. $y'' + y = f(t);$ $\quad f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 0, & \pi/2 \le t < \infty \end{cases}$
   $y(0) = 0, \quad y'(0) = 1$
2. $y'' + 2y' + 2y = h(t);$ $\quad h(t) = \begin{cases} 1, & \pi \le t < 2\pi \\ 0, & 0 \le t < \pi \text{ and } t \ge 2\pi \end{cases}$
   $y(0) = 0, \quad y'(0) = 1$
3. $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi);$ $\quad y(0) = 0, \quad y'(0) = 0$
4. $y'' + 4y = \sin t + u_{\pi}(t)\sin(t - \pi);$ $\quad y(0) = 0, \quad y'(0) = 0$
5. $y'' + 2y' + y = f(t);$ $\quad f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$
   $y(0) = 1, \quad y'(0) = 0$
6. $y'' + 3y' + 2y = u_2(t);$ $\quad y(0) = 0, \quad y'(0) = 1$
7. $y'' + y = u_{\pi}(t);$ $\quad y(0) = 1, \quad y'(0) = 0$
8. $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)(t - \pi/2);$ $\quad y(0) = 0, \quad y'(0) = 0$
9. $y'' + y = g(t);$ $\quad g(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$
   $y(0) = 0, \quad y'(0) = 1$
10. $y'' + y' + \frac{5}{4}y = g(t);$ $\quad g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$
    $y(0) = 0, \quad y'(0) = 0$
11. $y'' + 4y = u_{\pi}(t) - u_{2\pi}(t);$ $\quad y(0) = 1, \quad y'(0) = 0$
12. $y^{iv} - y = u_1(t) - u_2(t);$ $\quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$
13. $y^{iv} + 5y'' + 4y = 1 - u_{\pi}(t);$ $\quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$

1. $y'' + 2y' + 2y = \delta(t - \pi);$ $\quad y(0) = 1, \quad y'(0) = 0$
2. $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi);$ $\quad y(0) = 0, \quad y'(0) = 0$
3. $y'' + 2y' + y = \delta(t) + u_{2\pi}(t);$ $\quad y(0) = 0, \quad y'(0) = 1$
4. $y'' - y = 2\delta(t - 1);$ $\quad y(0) = 1, \quad y'(0) = 0$
5. $y'' + 2y' + 3y = \sin t + \delta(t - \pi);$ $\quad y(0) = 0, \quad y'(0) = 1$
6. $y'' + \omega^2 y = \delta(t - \pi/\omega);$ $\quad y(0) = 1, \quad y'(0) = 0$
7. $y'' + y = \delta(t - \pi)\cos t;$ $\quad y(0) = 0, \quad y'(0) = 1$
8. $y'' + 4y = 2\delta(t - \pi/4);$ $\quad y(0) = 0, \quad y'(0) = 0$
9. $y'' + y = u_{\pi/2}(t) + \delta(t - \pi) - u_{3\pi/2}(t);$ $\quad y(0) = 0, \quad y'(0) = 0$
10. $y'' + 4y = 4\delta(t - \pi/6)\sin t;$ $\quad y(0) = 0, \quad y'(0) = 0$
11. $y'' + 2y' + 2y = \cos t + \delta(t - \pi/2);$ $\quad y(0) = 0, \quad y'(0) = 0$
12. $y^{iv} - y = \delta(t - 1);$ $\quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$

Determinare l'equazione differenziale soddisfatta di $Y(x) = \mathcal{L}(y(t))$ dove $y = y(t)$ è soluzione del seguente problema di Cauchy:

$$\begin{cases} y'' - t\,y = 0 \\ y(0) = 1, \; y'(0) = 0 \end{cases} \quad \text{(equazione di Airy)}$$

$$\begin{cases} (1-t^2)\,y'' - 2t\,y' + \alpha(\alpha+1)\,y = 0 \\ y(0) = 0, \; y'(0) = 1 \end{cases} \quad \text{(equazione di Legendre)}$$

Trovare le trasformate di Laplace per le seguenti funzioni:

$f(t) = \int_0^t (t-\tau)^2 \cos 2\tau \, d\tau$, (b) $f(t) = \int_0^t e^{-(t-\tau)} \sin\tau \, d\tau$

$f(t) = \int_0^t (t-\tau)\, e^{\tau} \, d\tau$, (d) $f(t) = \int_0^t \sin(t-\tau) \cos\tau \, d\tau$

Nei problemi di Cauchy seguenti esprimere la soluzione in termini di un integrale di convoluzione:

$$\begin{cases} y'' + \omega^2 y = g(t) \\ y(0) = 0, \; y'(0) = 1 \end{cases} \qquad \begin{cases} y'' + 2y' + 2y = \sin\alpha t \\ y(0) = 0, \; y'(0) = 0 \end{cases} \qquad \begin{cases} 4y'' + 4y' + 17y = g(t) \\ y(0) = 0, \; y'(0) = 0 \end{cases}$$

$$\begin{cases} y'' + y' + \frac{5}{4} y = 1 - u_\pi(t) \\ y(0) = 1, \; y'(0) = -1 \end{cases} \qquad \begin{cases} y'' + 4y' + 4y = g(t) \\ y(0) = 2, \; y'(0) = -3 \end{cases} \qquad \begin{cases} y'' + 3y' + 2y = \cos\alpha t \\ y(0) = 1, \; y'(0) = 0 \end{cases}$$

$$\begin{cases} y^{(4)} - y = g(t) \\ y(0) = y'(0) = y''(0) = y'''(0) = 0 \end{cases} \qquad \begin{cases} y^{(4)} + 5y'' + 4y = g(t) \\ y(0) = 1, \; y'(0) = y''(0) = y'''(0) = 0 \end{cases}$$