Chapter 7

Minimizing movements along a sequence of functionals

Gradient flows, and hence minimizing movements, trivially do not commute even with uniform convergence. As a simple example, take $X = \mathbb{R}$ and

$$F_\varepsilon(x) = x^2 - \rho \sin\Big(\frac{x}{\varepsilon}\Big),$$

with $\rho = \rho_\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly converging to $F(x) = x^2$. If also

$$\varepsilon << \rho,$$

then for fixed x_0 the solutions u_ε to the equation

$$\begin{cases} u'_\varepsilon = -2u_\varepsilon + \dfrac{\rho}{\varepsilon} \cos\Big(\dfrac{u_\varepsilon}{\varepsilon}\Big) \\ u_\varepsilon(0) = x_0 \end{cases}$$

converge to the constant function $u_0(t) = x_0$ as $\varepsilon \to 0$, which does not solve

$$\begin{cases} u' = -2u \\ u(0) = x_0 \,. \end{cases}$$

This is easily seen by studying the stationary solutions of

$$-2x + \frac{\rho}{\varepsilon} \cos\Big(\frac{x}{\varepsilon}\Big) = 0 \,.$$

With the remark above in mind, in order to give a meaningful limit for the energy-driven motion along a sequence of functionals it may be useful to vary the definition of minimizing movement as in the following section.

7.1 Minimizing movements along a sequence

Definition 7.1.1 (minimizing movements along a sequence) *Let X be a separable Hilbert space, $F_\varepsilon : X \to [0,+\infty]$ equicoercive and lower semicontinuous and $x_0^\varepsilon \to x_0$ with*

$$F_\varepsilon(x_0^\varepsilon) \le C < +\infty, \tag{7.1}$$

and $\tau_\varepsilon > 0$ converging to 0 as $\varepsilon \to 0$. Fixed $\varepsilon > 0$ we define recursively x_k^ε as a minimizer for the problem

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}\|x - x_{k-1}^\varepsilon\|^2\Big\}, \tag{7.2}$$

and the piecewise-constant trajectory $u^\varepsilon : [0,+\infty) \to X$ given by

$$u^\varepsilon(t) = x_{\lfloor t/\tau_\varepsilon \rfloor}\,. \tag{7.3}$$

A minimizing movement *for F_ε from x_0^ε is any limit of a subsequence u^{ε_j} uniform on compact sets of $[0,+\infty)$.*

After remarking that the Hölder continuity estimates in Proposition 6.1.4 only depend on the bound on $F_\varepsilon(x_0^\varepsilon)$, with the same proof we can show the following result.

Proposition 7.1.2 *For every F_ε and x_0^ε as above there exist minimizing movements for F_ε from x_0^ε in $C^{1/2}([0,+\infty);X)$.*

Remark 7.1.3 (growth conditions) The positiveness of F_ε can be substituted by the requirement that for all $\overline{x}$ the functionals

$$x \mapsto F_\varepsilon(x) + \frac{1}{2\tau}\|x - \overline{x}\|^2$$

be bounded from below; i.e., that there exists $C > 0$ such that

$$x \mapsto F_\varepsilon(x) + C\|x - \overline{x}\|^2$$

be bounded from below.

Example 7.1.4 We give a simple example that shows how the limit minimizing movement may depend on the choice of the mutual behavior of ε and τ. We consider the functions

$$F_\varepsilon(x) = \begin{cases} -x & \text{if } x \le 0 \\ 0 & \text{if } 0 \le x \le \varepsilon \\ \varepsilon - x & \text{if } x \ge \varepsilon, \end{cases}$$

which converge uniformly to $F(x) = -x$. Note the that the energies are not bounded from below, but their analysis falls within the framework in the previous remark. For this example a direct computation is immediately carried on. We consider a fixed initial datum x_0.

If $x_0 > 0$, then for $\varepsilon < x_0$ we have $x_k^\varepsilon = x_{k-1}^\varepsilon + \tau$ for all $k \geq 0$.

If $x_0 \leq 0$ then we have $x_k^\varepsilon = x_{k-1}^\varepsilon + \tau$ if $x_{k-1}^\varepsilon \leq -\tau$. If $0 \geq x_{k-1}^\varepsilon > -\tau$ then $x_k^\varepsilon - x_{k-1}^\varepsilon$ is obtained by minimizing the function

$$f(y) = \begin{cases} -y + \dfrac{1}{2\tau} y^2 & \text{if } 0 \leq y \leq -x_{k-1}^\varepsilon \\ x_{k-1}^\varepsilon + \dfrac{1}{2\tau} y^2 & \text{if } -x_{k-1}^\varepsilon \leq y \leq -x_{k-1}^\varepsilon + \varepsilon \\ \varepsilon - y + \dfrac{1}{2\tau} y^2 & \text{if } y \geq -x_{k-1}^\varepsilon + \varepsilon, \end{cases}$$

whose minimizer is always $y = \tau + x_{k-1}^\varepsilon$ if $\varepsilon - x_{k-1}^\varepsilon > \tau$. In this case $x_k^\varepsilon = 0$. If otherwise $\varepsilon - x_{k-1}^\varepsilon \leq \tau$ the other possible minimizer is $y = \tau$. We then have to compare the values

$$f(-x_{k-1}^\varepsilon) = x_{k-1}^\varepsilon + \frac{1}{2\tau}(x_{k-1}^\varepsilon)^2, \qquad f(\tau) = \varepsilon - \frac{1}{2}\tau.$$

We have three cases:

(a) $\varepsilon - \frac{1}{2}\tau > 0$. In this case we have $x_k^\varepsilon = 0$ (and this holds for all subsequent steps);

(b) $\varepsilon - \frac{1}{2}\tau < 0$. In this case we either have $f(\tau) < f(-x_{k-1}^\varepsilon)$, in which case $x_k^\varepsilon = x_{k-1}^\varepsilon + \tau$ (and this then holds for all subsequent steps); otherwise $x_k^\varepsilon = 0$ and $x_{k+1}^\varepsilon = x_k^\varepsilon + \tau$ (and this holds for all subsequent steps);

(c) $\varepsilon - \frac{1}{2}\tau = 0$. If $x_{k-1}^\varepsilon < 0$ then $x_k^\varepsilon = 0$ (otherwise we already have $x_{k-1}^\varepsilon = 0$). Then, since we have the two solutions $y = 0$ and $y = \tau$, we have $x_j^\varepsilon = 0$ for $k \leq j \leq k_0$ for some $k_0 \in \mathbb{N} \cup +\infty$ and $x_j^\varepsilon = x_{j-1}^\varepsilon + \tau$ for $j > k_0$.

We can summarize the possible minimizing movements with initial datum $x_0 \leq 0$ as follows:

(i) if $\tau < 2\varepsilon$ then the unique minimizing movement is $x(t) = \min\{x_0 + t, 0\}$;

(ii) if $\tau > 2\varepsilon$ then the unique minimizing movement is $x(t) = x_0 + t$;

(iii) if $\tau = 2\varepsilon$ then we have the minimizing movements $x(t) = \max\{\min\{x_0 + t, 0\}, x_1 + t\}$ for $x_1 \leq x_0$.

For $x_0 > 0$ we always have the only minimizing movement $x(t) = x_0 + t$.

7.2 Commutability along 'fast-converging' sequences

We now show that by suitably choosing $\varepsilon = \varepsilon(\tau)$ the minimizing movement along the sequence F_ε from x_ε converges to a minimizing movement for the limit F from x_0.

We consider an equi-coercive sequence F_ε of (non-negative) lower-semicontinuous functionals on a Hilbert space X Γ-converging to F. Note that if $y_\varepsilon \to y_0$ then the solutions of

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}\|x - y_\varepsilon\|^2\Big\} \tag{7.4}$$

converge to solutions of

$$\min\Big\{F(x) + \frac{1}{2\tau}\|x - y_0\|^2\Big\} \tag{7.5}$$

since we have a continuously converging perturbation of a Γ-converging sequence. We want this convergence to hold along the sequences of minimum problems defining minimizing movements.

Let now $x_\varepsilon \to x_0$. Let τ be fixed. We consider the sequence $\{x_k^{\tau,\varepsilon}\}$ defined by iterated minimization of F_ε with initial point x_ε. Since $x_\varepsilon \to x_0$, up to subsequences we have $x_1^{\tau,\varepsilon} \to x_1^{\tau,0}$, which minimizes

$$\min\Big\{F(x) + \frac{1}{2\tau}\|x - x_0\|^2\Big\}. \tag{7.6}$$

The point $x_2^{\tau,\varepsilon}$ converge to $x_2^{\tau,0}$. Since they minimize

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}\|x - x_1^{\tau,\varepsilon}\|^2\Big\} \tag{7.7}$$

and $x_1^{\tau,\varepsilon} \to x_1^{\tau,0}$ their limit is a minimizer of

$$\min\Big\{F(x) + \frac{1}{2\tau}\|x - x_1^{\tau,0}\|^2\Big\}. \tag{7.8}$$

This operation can be repeated iteratively, obtaining (upon subsequences) $x_k^{\tau,\varepsilon} \to x_k^{\tau,0}$, and $\{x_k^{\tau,0}\}$ iteratively minimizes F with initial point x_0.

With fixed $T > 0$, let $K = \lfloor T/\tau \rfloor + 1$. Then, we deduce that there exists $\varepsilon = \varepsilon(\tau)$ such that we have

$$\|x_k^{\tau,\varepsilon} - x_k^{\tau,0}\| \le \tau \text{ for all } k = 1, \dots, K.$$

Upon subsequences of τ these two schemes converge respectively to a minimizing movement along F_ε and a minimizing movement for F. We have then proved the following result.

Theorem 7.2.1 *Let F_ε be a equi-coercive sequence of (non-negative) lower-semicontinuous functionals on a Hilbert space X Γ-converging to F, let $x_\varepsilon \to x_0$. Then there exists a choice of $\varepsilon = \varepsilon(\tau)$ such that every minimizing movement along F_ε with initial data x_ε converge to a minimizing movement for F from x_0 on $[0, T]$ for all T.*

Remark 7.2.2 Note that, given x_ε and F_ε, if F has more than one minimizing movement from x_0 then the approximation gives a choice criterion. As an example take $F(x) = -|x|$, $F_\varepsilon(x) = -|x + \varepsilon|$ and $x_0 = x_\varepsilon = 0$.

Remark 7.2.3 (the convex case) If all F_ε are convex then it can be shown that actually the minimizing movement along the sequence F_ε always coincides with the minimizing movement for their Γ-limit. This (exceptional) case will be dealt with in detail separately.

Example 7.2.4 In dimension one we can take

$$F_\varepsilon(x) = \frac{1}{2}x^2 + \varepsilon\, W\Big(\frac{x}{\varepsilon}\Big),$$

where W is a one-periodic odd Lipschitz function with $\|W'\|_\infty = 1$. Up to addition of a constant is not restrictive to suppose that the average of W is 0. We check that the critical regime for the minimizing movements along F_ε is $\varepsilon \sim \tau$. Indeed, if $\varepsilon << \tau$ then from the estimate

$$\Big|F_\varepsilon(x) - \frac{1}{2}x^2\Big| \leq \frac{\varepsilon}{2}$$

we deduce that

$$\frac{x_k - x_{k-1}}{\tau} = -x_k + O\Big(\frac{\varepsilon}{\tau}\Big),$$

and hence that the limit minimizing movement satisfies $u' = -u$, so that it corresponds to the minimizing movement of the limit $F_0(x) = \frac{1}{2}x^2$.

Conversely, if $\tau << \varepsilon$ then it may be seen that for $|x_0| \leq 1$ the motion is *pinned*; i.e., the resulting minimizing movement is trivial $u(t) = x_0$ for all t. If $W \in C^1$ this is easily seen since in this case the stationary solutions, corresponding to x satisfying

$$x + W'(\frac{x}{\varepsilon}\Big) = 0$$

tend to be dense in the interval $[-1, 1]$ as $\varepsilon \to 0$. Moreover, it is easily seen that in this regime the minimizing movement corresponds to the limit as $\varepsilon \to 0$ to the minimizing movements of F_ε for ε fixed; i.e., solutions u_ε of the gradient flow

$$u_\varepsilon' = -u_\varepsilon - W'\Big(\frac{u_\varepsilon}{\varepsilon}\Big).$$

Integrating between t_1 and t_2 we have

$$\int_{u_\varepsilon(t_1)}^{u_\varepsilon(t_2)} \frac{1}{s + W'(s/\varepsilon)}\, ds = t_1 - t_2.$$

By the uniform convergence $u_\varepsilon \to u$ we can pass to the limit, recalling that the integrand weakly converges to the function $1/g$ defined by

$$\frac{1}{g(s)} = \int_0^1 \frac{1}{s + W'(\sigma)} d\sigma,$$

and obtain the equation

$$u' = -g(u).$$

This equation corresponds to the minimizing movement for the even energy $\widetilde{F}_0$ given for $x \geq 0$

$$\widetilde{F}_0(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \displaystyle\int_1^x g(w)\,dw & \text{if } x \geq 1. \end{cases}$$

The plot of the derivatives of F_ε, F_0 and $\widetilde{F}_0$ is reproduced in Fig. 7.1

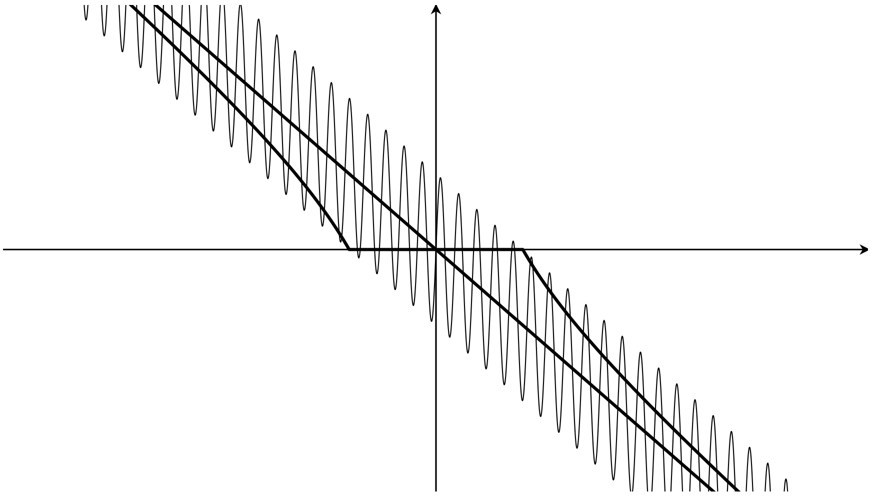

Figure 7.1: derivatives of F_ε, F_0 and $\widetilde{F}_0$

We can explicitly compute the minimizing movement for $\tau << \varepsilon$ in the case $W(x) = |x - \frac{1}{2}| - \frac{1}{4}$ for $0 \leq x \leq 1$. Here, the solutions with initial datum $x_0 > 1$ satisfy the equation

$$u' = \frac{1}{u} - u.$$

Integrating this limit equation we conclude that the minimizing movement along F_ε correspond to that of the effecitve energy

$$\widetilde{F}_0(x) = \Big(\frac{1}{2}x^2 - \log|x| - \frac{1}{2}\Big)^+.$$

7.3 An example: "overdamped dynamics" of Lennard-Jones interactions

We now give an example of a sequence of non-convex energies which commute with the minimizing movement procedure.

Let J be as in Section 3.4 and $\frac{1}{\varepsilon} = N \in \mathbb{N}$. We consider the energies

$$F_\varepsilon(u) = \sum_{i=1}^{N} J\Big(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\Big)$$

with the periodic boundary condition $u_N = u_0$. As proved in Section 3.4, after identification of u with a piecewise-constant function on $[0, 1]$, these energies Γ-converge to the energy

$$F(u) = \int_0^1 |u'|^2\, dt + \#(S(u) \cap [0, 1)), \qquad u^+ > u^-,$$

defined on piecewise-H^1 functions, in this case extended 1-periodically on the whole real line.

In this section we apply the minimizing movements scheme to F_ε as a sequence of functionals in $L^2(0, 1)$. In order to have initial data u_0^ε with equibounded energy, we may suppose that these are the discretization of a single piecewise-H^1 function u_0 (with a slight abuse of notation we will continue to denote all these discrete functions by u_0).

With fixed ε and τ, the time-discretization scheme consists in defining recursively u^k as a minimizer of

$$u \mapsto \sum_{i=1}^{N} J\Big(\frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\Big) + \frac{1}{2\tau}\sum_{i=1}^{N} \varepsilon|u_i - u_i^{k-1}|^2. \tag{7.9}$$

By Proposition 7.1.2, upon extraction of a subsequence, the functions $u^\tau(t) = u_{\lfloor t/\tau \rfloor}$ converge uniformly in L^2 to a function $u \in C^{1/2}([0, +\infty); L^2(0, 1))$. Moreover, since we have $F(u(t)) \leq F(u_0) < +\infty$, $u(t)$ is a piecewise-H^1 function for all t.

We now describe the motion of the limit u. For the sake of simplicity we suppose that u_0 is a piecewise-Lipschitz function and that $S(u_0) \cap \{\varepsilon i : i \in \{1, \ldots, N\}\} = \emptyset$ (so that we do not have any ambiguity in the definition of the interpolations of u_0)..

We first write down the Euler-Lagrange equations for u^k, which amount to a simple N-dimensional system of equations obtained by deriving (7.9) with respect to u_i

$$\frac{1}{\sqrt{\varepsilon}}\Big(J'\Big(\frac{u_i^k - u_{i-1}^k}{\sqrt{\varepsilon}}\Big) - J'\Big(\frac{u_{i+1}^k - u_i^k}{\sqrt{\varepsilon}}\Big)\Big) + \frac{\varepsilon}{\tau}(u_i^k - u_i^{k-1}) = 0. \tag{7.10}$$

- With fixed $i \in \{1, \ldots, N\}$ let v_k be defined by

$$v_k = \frac{u_i^k - u_{i-1}^k}{\varepsilon}\,.$$

For simplicity of notation we set

$$J_\varepsilon(w)=\frac{1}{\varepsilon}J(\sqrt{\varepsilon}\,w).$$

By (7.10) and the corresponding equation for $i-1$, which can be rewritten as

$$J'_\varepsilon\Big(\frac{u^k_{i-1}-u^k_{i-2}}{\varepsilon}\Big)-J'_\varepsilon\Big(\frac{u^k_{i}-u^k_{i-1}}{\varepsilon}\Big)+\frac{\varepsilon}{\tau}(u^k_{i-1}-u^{k-1}_{i-1})=0.$$

we have

$$\begin{aligned}\frac{v_k-v_{k-1}}{\tau}&=\frac{1}{\tau}\Big(\frac{u^k_i-u^k_{i-1}}{\varepsilon}-\frac{u^{k-1}_i-u^{k-1}_{i-1}}{\varepsilon}\Big)\\&=\frac{1}{\varepsilon}\Big(\frac{u^k_i-u^{k-1}_{i}}{\tau}-\frac{u^k_{i-1}-u^{k-1}_{i-1}}{\tau}\Big)\\&=\frac{1}{\varepsilon^2}\Bigg(\Big(J'_\varepsilon\Big(\frac{u^k_{i-1}-u^k_{i-2}}{\varepsilon}\Big)-J'_\varepsilon\Big(\frac{u^k_{i}-u^k_{i-1}}{\varepsilon}\Big)\Big)\\&\qquad-\Big(J'_\varepsilon\Big(\frac{u^k_{i}-u^k_{i-1}}{\varepsilon}\Big)-J'_\varepsilon\Big(\frac{u^k_{i+1}-u^k_{i}}{\varepsilon}\Big)\Big)\Bigg),\end{aligned}$$

so that

$$\frac{v_k-v_{k-1}}{\tau}-\frac{2}{\varepsilon^2}J'_\varepsilon(v_k)=-\frac{1}{\varepsilon^2}\Big(J'_\varepsilon\Big(\frac{u^k_{i-1}-u^k_{i-2}}{\varepsilon}\Big)+J'_\varepsilon\Big(\frac{u^k_{i+1}-u^k_{i}}{\varepsilon}\Big)\Big)\geq-\frac{2}{\varepsilon^2}J'_\varepsilon\Big(\frac{w_0}{\sqrt{\varepsilon}}\Big).\quad(7.11)$$

We recall that we denote by w_0 the maximum point of J'.

We can interpret (7.11) as an inequality for the difference system

$$\frac{v_k-v_{k-1}}{\eta}-2J'_\varepsilon(v_k)\geq-2J'_\varepsilon\Big(\frac{w_0}{\sqrt{\varepsilon}}\Big),$$

where $\eta=\tau/\varepsilon^2$ is interpreted as a discretization step. Note that $v_k=w_0/\sqrt{\varepsilon}$ for all k is a stationary solution of the equation

$$\frac{v_k-v_{k-1}}{\eta}-2J'_\varepsilon(v_k)=-2J'_\varepsilon\Big(\frac{w_0}{\sqrt{\varepsilon}}\Big)$$

and that J'_ε are equi-Lipschitz functions on $[0,+\infty)$. If $\eta<<1$ this implies that if $v_{k_0}\leq w_0/\sqrt{\varepsilon}$ for some k_0 then

$$v_k\leq\frac{w_0}{\sqrt{\varepsilon}}\qquad\text{for } k\geq k_0,$$

or, equivalently, that if $\tau<<\varepsilon^2$ the set

$$S^k_\varepsilon=\Big\{i\in\{1,\dots,N\}:\frac{u^k_i-u^k_{i-1}}{\varepsilon}\geq\frac{w_0}{\sqrt{\varepsilon}}\Big\}$$

is decreasing with k. By our assumption on u_0, for ε small enough we then have

$$S^0_\varepsilon = \Big\{ i \in \{1, \dots, N\} : [\varepsilon(i-1), \varepsilon i] \cap S(u_0) \neq \emptyset \Big\},$$

so that, passing to the limit

$$S(u(t)) \subseteq S(u_0) \qquad \text{for all } t \geq 0. \tag{7.12}$$

• Taking into account that we may define

$$u^\tau(t,x) = u^{\lfloor t/\tau \rfloor}_{\lfloor x/\varepsilon \rfloor},$$

we may choose functions $\phi \in C^\infty_0(0,T)$ and $\psi \in C^\infty_0(x_1,x_2)$, with $(x_1,x_2) \cap S(u_0) = \emptyset$, and obtain from (7.10)

$$\begin{aligned} &\int_0^T \int_{x_1}^{x_2} u^\tau(t,x) \Big(\frac{\phi(t) - \phi(t+\tau)}{\tau} \Big) \psi(x)\, dx\, dt \\ = &\; - \int_0^T \int_{x_1}^{x_2} \Big(\frac{1}{\sqrt{\varepsilon}} J' \Big(\sqrt{\varepsilon} \frac{u^\tau(t,x) - u^\tau(t,x-\varepsilon)}{\varepsilon} \Big) \Big) \phi(t) \Big(\frac{\psi(x) - \psi(x+\varepsilon)}{\varepsilon} \Big)\, dx\, dt\,. \end{aligned}$$

Taking into account that

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} J'(\sqrt{\varepsilon} w) = 2w,$$

we can pass to the limit and obtain that

$$-\int_0^T \int_{x_1}^{x_2} u(t,x) \phi'(t) \psi(x)\, dx\, dt = \int_0^T \int_{x_1}^{x_2} 2 \frac{\partial u}{\partial x} \phi(t) \psi'(x)\, dx\, dt\,;$$

i.e., that

$$\frac{\partial u}{\partial t} = -2 \frac{\partial^2 u}{\partial x^2} \tag{7.13}$$

in the sense of distributions (and hence also classically) in $(0,T) \times (x_1,x_2)$. By the arbitrariness of the interval (x_1,x_2) we have that equation (7.13) is satisfied for x in $(0,1) \setminus S(u_0)$.

• We now derive boundary conditions on $S(u(t))$. Let $i_0 + 1$ belong to S^0_ε, and suppose that $u^+(t,x) - u^-(t,x) \geq c > 0$. Then we have

$$\lim_{\tau \to 0} \frac{1}{\sqrt{\varepsilon}} J' \left(\frac{u^{\lfloor t/\tau \rfloor}_{i_0} - u^{\lfloor t/\tau \rfloor}_{i_0 - 1}}{\sqrt{\varepsilon}} \right) = 0\,.$$

If $i < i_0$, from (7.10) it follows, after summing up the indices from i to i_0, that

$$\sum_{j=i}^{i_0} \frac{\varepsilon}{\tau} (u^k_j - u^{k-1}_j) = -\frac{1}{\sqrt{\varepsilon}} J' \Big(\frac{u^k_i - u^k_{i-1}}{\sqrt{\varepsilon}} \Big)\,. \tag{7.14}$$

We may choose $i = i_\varepsilon$ such that $\varepsilon i_\varepsilon \to \overline{x}$ and we may deduce from (7.14) that

$$\int_{\overline{x}}^{x_0} \frac{\partial u}{\partial t}\, dx = -2\frac{\partial u}{\partial x}(\overline{x}),$$

where $x_0 \in S(u(t))$ is the limit of εi_0. Letting $\overline{x} \to x_0^-$ we obtain

$$\frac{\partial u}{\partial x}(x_0^-) = 0.$$

Similarly we obtain the homogeneous Neumann condition at x_0^+.

Summarizing, the minimizing movement of the scaled Lennard-Jones energies F_ε from a piecewise-H^1 function consists in a piecewise-H^1 motion, following the heat equation on $(0,1)\setminus S(u_0)$, with homogeneous Neumann boundary conditions on $S(u_0)$ (as long as $u(t)$ has a discontinuity at the corresponding point of $S(u_0)$).

Note that for $\varepsilon \to 0$ sufficiently fast Theorem 7.2.1 directly ensures that the minimizing movement along F_ε coincides with the minimizing movement for the functional F. The computation above shows that this holds also for $\tau << \varepsilon^2$ (i.e., $\varepsilon \to 0$ “sufficiently slow”), which then must be regarded as a technical condition.

7.4 Homogenization of minimizing movements

We now examine minimizing movements along oscillating sequences (with many local minima), treating two model cases in the real line.

7.4.1 Minimizing movements for piecewise-constant energies

We apply the minimizing-movement scheme to the functions

$$F_\varepsilon(x) = -\left\lfloor \frac{x}{\varepsilon} \right\rfloor \varepsilon$$

converging to $F(x) = -x$ (see Fig. 7.2). This is a prototype of a function with many local minimizers (actually, in this case all points are local minimizers) converging to a function with few local minimizers (actually, none).

Note that, with fixed ε, for any initial datum x_0 the minimizing movement for F_ε is trivial: $u(t) = x_0$, since all points are local minimizers. Conversely the corresponding minimizing movement for the limit is $u(t) = x_0 + t$.

We now fix an initial datum x_0, the space scale ε and the time scale τ, and examine the successive-minimization scheme from x_0. Note that it is not restrictive to suppose that $0 \le x_0 < 1$ up to a translation in $\varepsilon\mathbb{Z}$.

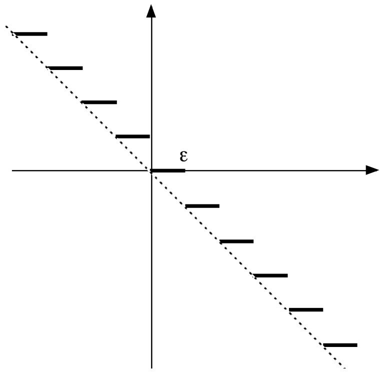

Figure 7.2: the function F_ε

The first minimization, giving x_1 is

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}(x - x_0)^2\Big\}. \tag{7.15}$$

The function to minimize is pictured in Figure 7.3 in normalized coordinates ($\varepsilon = 1$); note that it equals $-x + \frac{1}{2\tau}(x - x_0)^2$ if $x \in \varepsilon\mathbb{Z}$.

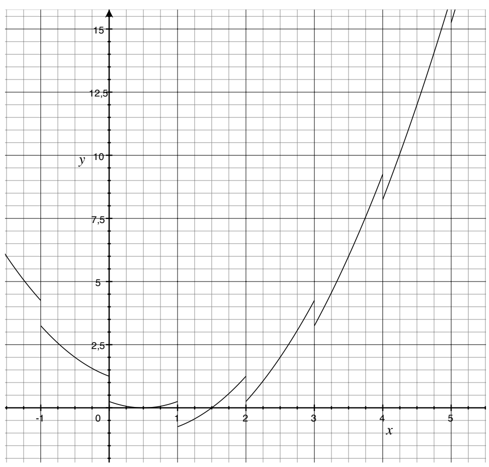

Figure 7.3: the function in the minimization problem (7.15)

Except for some exceptional cases that we deal separately below, we have two possibilities:

(i) if $\frac{\tau}{\varepsilon} < \frac{1}{2}$ then the motion is trivial. If we set $x_0 = s\varepsilon$ with $0 \le s < 1$. Then we have two sub-cases:

(a) the minimizer x_1 belongs to $[0, \varepsilon)$. This occurs exactly if $F_\varepsilon(\varepsilon) + \frac{1}{2\tau}(\varepsilon - x_0)^2 > 0$;

i.e.,

$$\tau < \frac{(s-1)^2\varepsilon}{2}. \tag{7.16}$$

In this case the only minimizer is still x_0. This implies that we have $x_k = x_0$ for all k. Otherwise, we have that $x_1 = \varepsilon$. This implies that, up to a translation we are in the case $x_0 = 0$ with $s = 0$, and (7.16) holds since $\tau < \frac{\varepsilon}{2}$. Hence, $x_k = x_1$ for all $k \geq 1$;

(ii) if $\frac{\tau}{\varepsilon} > \frac{1}{2}$ then for ε small the minimum is taken on $\varepsilon\mathbb{Z}$. So that again we may suppose that $x_0 = 0$.

Note that we are leaving out for now the case when $x_0 = 0$ and $\frac{\tau}{\varepsilon} = \frac{1}{2}$. In this case we have a double choice for the minimizer; such situations will be examined separately.

If $x_0 = 0$ then x_1 is computed by solving

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\tau}x^2 : x \in \varepsilon\mathbb{Z}\Big\}, \tag{7.17}$$

and is characterized by

$$x_1 - \frac{1}{2}\varepsilon \leq \tau \leq x_1 + \frac{1}{2}\varepsilon.$$

We then have

$$x_1 = \Big\lfloor \frac{\tau}{\varepsilon} + \frac{1}{2}\Big\rfloor \varepsilon \qquad \text{if } \frac{\tau}{\varepsilon} + \frac{1}{2} \notin \mathbb{Z}$$

(note again that we have two solutions for $\frac{\tau}{\varepsilon} + \frac{1}{2} \in \mathbb{Z}$, and we examine this case separately). The same computation is repeated at each k giving

$$\frac{x_k - x_{k-1}}{\tau} = \Big\lfloor \frac{\tau}{\varepsilon} + \frac{1}{2}\Big\rfloor \frac{\varepsilon}{\tau}.$$

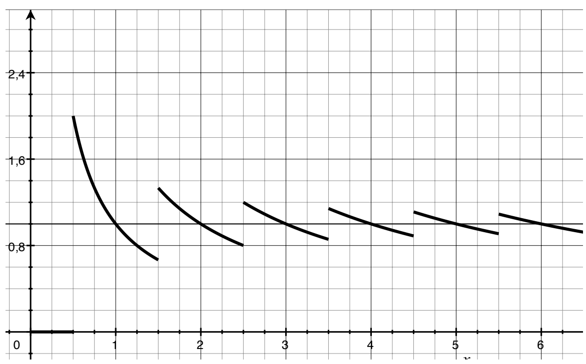

Figure 7.4: the velocity v in terms of w

We can now choose τ and ε tending to 0 simultaneously and pass to the limit. The behaviour of the limit minimizing movements is governed by the quantity

$$w = \lim_{\varepsilon\to 0}\frac{\tau}{\varepsilon}, \tag{7.18}$$

which we may suppose exists up to subsequences. If $w+\frac{1}{2}\notin\mathbb{Z}$ then the minimizing movement along F_ε from x_0 is uniquely defined by

$$u(t) = x_0 + vt, \ \text{with } v = \Big\lfloor w+\frac{1}{2}\Big\rfloor\frac{1}{w}, \tag{7.19}$$

so that the whole sequence converges if the limit in (7.18) exists. Note that

• **(pinning)** we have $v=0$ exactly when $\frac{\tau}{\varepsilon}<\frac{1}{2}$ for ε small. In particular this holds for $\tau << \varepsilon$ (i.e., for $w=0$);

• **(limit motion for slow times)** if $\varepsilon << \tau$ then the motion coincides with the gradient flow of the limit, with velocity 1;

• **(discontinuous dependence of the velocity)** the velocity is a discontinuous function of w at points of $\frac{1}{2}+\mathbb{Z}$. Note moreover that it may be actually greater than the limit velocity 1. The graph of v is pictured in Figure 7.4

• **(non-uniqueness at $w\in\frac{1}{2}+\mathbb{Z}$)** in these exceptional cases we may have either of the two velocities $1+\frac{1}{2w}$ or $1-\frac{1}{2w}$ in the cases $\frac{\varepsilon}{\tau}+\frac{1}{2}>w$ or $\frac{\varepsilon}{\tau}+\frac{1}{2}<w$ for all ε small respectively, but we may also have any $u(t)$ with

$$1-\frac{1}{2w}\le u'(t)\le 1+\frac{1}{2w}$$

if we have precisely $\frac{\varepsilon}{\tau}+\frac{1}{2}=w$ for all ε small, since in this case at every time step we may choose any of the two minimizers giving the extremal velocities. Note therefore that in this case the limit is not determined only by w, and in particular it may depend on the subsequence even if the limit (7.18) exists.

We remark that the functions F_ε above can be substituted by functions with isolated local minimizers; e.g. by taking ($\alpha>0$)

$$F_\varepsilon(x) = -\Big\lfloor\frac{x}{\varepsilon}\Big\rfloor\varepsilon + \alpha\Big(x-\Big\lfloor\frac{x}{\varepsilon}\Big\rfloor\varepsilon\Big),$$

with isolated local minimizers at $\varepsilon\mathbb{Z}$ (for which the computations run exactly as above), or

$$F_\varepsilon(x) = -x + (1+\alpha)\varepsilon\sin\Big(\frac{x}{\varepsilon}\Big).$$

Note that the presence of an energy barrier between local minimizers does not influence the velocity of the final minimizing movement, that can always be larger than 1 (the velocity as $\varepsilon << \tau$).

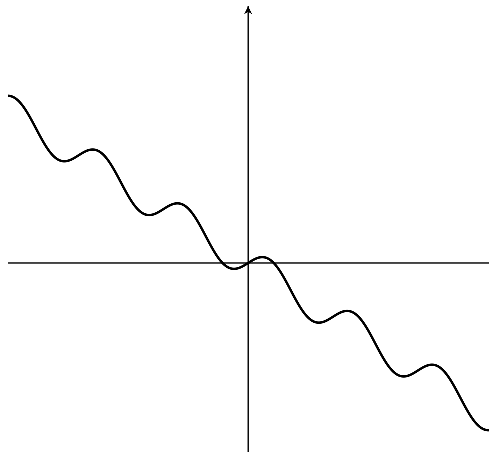

Figure 7.5: other potentials giving the same homogenization pattern

We also remark that the same result can be obtained by a "discretization" of F; i.e., taking

$$F_\varepsilon(x) = \begin{cases} -x & \text{if } x \in \varepsilon\mathbb{Z} \\ +\infty & \text{otherwise.} \end{cases}$$

7.4.2 A heterogeneous case

We briefly examine a variation of the previous example introducing a heterogeneity parameter $1 \le \lambda \le 2$ and defining

$$F^\lambda(x) = \begin{cases} -2\left\lfloor \dfrac{x}{2} \right\rfloor & \text{if } 2\left\lfloor \dfrac{x}{2} \right\rfloor \le x < 2\left\lfloor \dfrac{x}{2} \right\rfloor + \lambda \\ -2\left\lfloor \dfrac{x}{2} \right\rfloor - \lambda & \text{if } 2\left\lfloor \dfrac{x}{2} \right\rfloor + \lambda \le x < 2\left\lfloor \dfrac{x}{2} \right\rfloor + 1. \end{cases} \tag{7.20}$$

If $\lambda = 1$ we are in the previous situation; for general λ the function F^λ is pictured in Fig. 7.6.

We apply the minimizing-movement scheme to the functions

$$F_\varepsilon(x) = F_\varepsilon^\lambda(x) = \varepsilon F^\lambda\Big(\frac{x}{\varepsilon}\Big).$$

Arguing as above, we can reduce to the two cases

$$\text{(a) } x_k \in 2\varepsilon\mathbb{Z}, \qquad \text{or} \qquad \text{(b) } x_k \in 2\varepsilon\mathbb{Z} + \varepsilon\lambda.$$

Taking into account that x_{k+1} is determined as the point in $2\varepsilon\mathbb{Z} \cup (2\varepsilon\mathbb{Z} + \varepsilon\lambda)$ closer to τ (as above, we only consider the cases when we have a unique solution to the minimum problems in the iterated procedure), we can characterize it as follows.

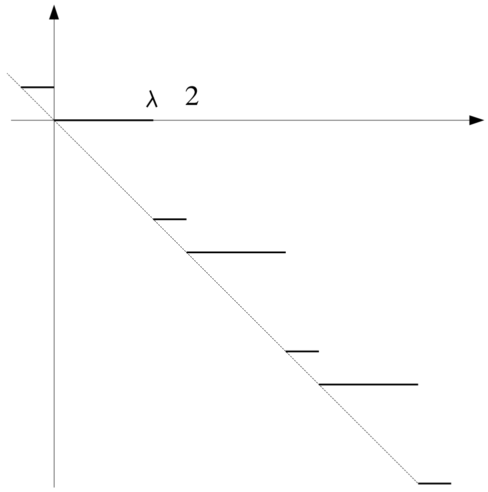

Figure 7.6: the function F^λ

In case (a) we have the two sub cases:

(a$_1$) if we have
$$2n < \frac{\tau}{\varepsilon} - \frac{\lambda}{2} < 2n+1$$
for some $n \in \mathbb{N}$ then
$$x_{k+1} = x_k + (2n+\lambda)\varepsilon.$$
In particular $x_{k+1} \in 2\varepsilon\mathbb{Z} + \varepsilon\lambda$;

(a$_2$) if we have
$$2n-1 < \frac{\tau}{\varepsilon} - \frac{\lambda}{2} < 2n$$
for some $n \in \mathbb{N}$ then
$$x_{k+1} = x_k + 2n\varepsilon.$$
In particular $x_{k+1} \in 2\varepsilon\mathbb{Z}$. Note that $x_{k+1} = x_k$ (pinning) if $\dfrac{\tau}{\varepsilon} < \dfrac{\lambda}{2}$.

In case (b) we have the two sub cases:

(b$_1$) if we have
$$2n < \frac{\tau}{\varepsilon} + \frac{\lambda}{2} < 2n+1$$
for some $n \in \mathbb{N}$ then
$$x_{k+1} = x_k + 2n\varepsilon.$$
In particular $x_{k+1} \in 2\varepsilon\mathbb{Z} + \varepsilon\lambda$. Note that $x_{k+1} = x_k$ (pinning) if $\dfrac{\tau}{\varepsilon} < 1 - \dfrac{\lambda}{2}$, which is implied by the pinning condition in (a$_2$);

(b_2) if we have

$$2n-1<\frac{\tau}{\varepsilon}+\frac{\lambda}{2}<2n$$

for some $n\in\mathbb{N}$ then

$$x_{k+1}=x_k+2n\varepsilon-\varepsilon\lambda.$$

In particular $x_{k+1}\in 2\varepsilon\mathbb{Z}$.

Eventually, we have the two cases:

(1) when

$$\Big|\frac{\tau}{\varepsilon}-2n\Big|<\frac{\lambda}{2}$$

for some $n\in\mathbb{N}$ then, after possibly one iteration, we are either in the case (a_2) or (b_1). Hence, either $x_k\in 2\varepsilon\mathbb{Z}$ or $x_k\in 2\varepsilon\mathbb{Z}+\varepsilon\lambda$ for all k. The velocity in this case is then

$$\frac{x_{k+1}-x_k}{\tau}=2n\frac{\varepsilon}{\tau};$$

(2) when

$$\Big|\frac{\tau}{\varepsilon}-(2n+1)\Big|<1-\frac{\lambda}{2}$$

for some $n\in\mathbb{N}$ then we are alternately in case (a_1) or (b_2). In this case we have an

• **averaged velocity**: the speed of the orbit $\{x_k\}$ oscillates between two values with a mean speed given by

$$\frac{x_{k+2}-x_k}{2\tau}=\frac{2n\varepsilon+\lambda\varepsilon}{2\tau}+\frac{2(n+1)\varepsilon-\lambda\varepsilon}{2\tau}=(2n+1)\frac{\varepsilon}{\tau}.$$

This is an additional feature with respect to the previous example.

Summarizing, if we define w as in (7.18) then (taking into account only the cases with a unique limit) the minimizing movement along the sequence F_ε with initial datum x_0 is given by $x(t)=x_0+vt$ with $v=f(w)\frac{1}{w}$, and f is given by

$$f(w)=\begin{cases}2n & \text{if } |w-2n|\le\dfrac{\lambda}{2},\ n\in\mathbb{N}\\[2ex] 2n+1 & \text{if } |w-(2n+1)|<1-\dfrac{\lambda}{2},\ n\in\mathbb{N}\end{cases}$$

(see Fig. 7.7). Note that the pinning threshold is now $\lambda/2$.

We can compare this minimizing movement with the one given in (7.19) by examining the graph of $w\mapsto\lfloor w+1/2\rfloor-f(w)$ in Fig. 7.8. For $2n+1/2<w<2n+\lambda/2$ the new minimizing movement is slower, while for $2n+2-\lambda/2<w<2n+2-1/2$ it is faster.

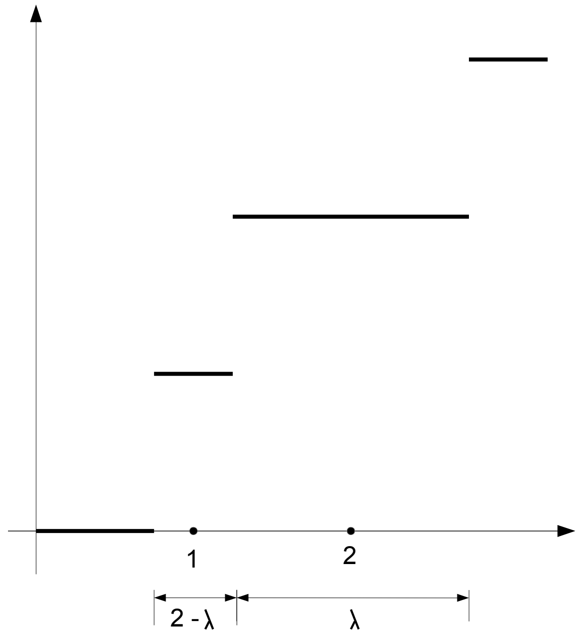

Figure 7.7: The function f describing the effective velocity

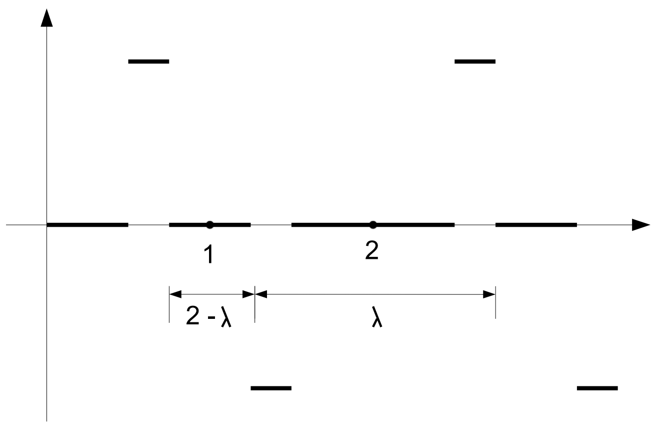

Figure 7.8: comparison with the homogeneous case

7.5 References to Chapter 7

The definition of minimizing movement along a sequence of functionals formalizes a natural extension to the notion of minimizing movement, and follows the definition given in the paper

A. Braides, M.S. Gelli, and M. Novaga. Motion and pinning of discrete interfaces. Arch. Ration. Mech. Anal. 95 (2010), 469–498.

The energies in Example 7.2.4 have been taken as a prototype to model plastic phenomena in

G. Puglisi and L. Truskinovsky. Thermodynamics of rate-independent plasticity. Journal of Mechanics and Physics of solids 53 (2005) 655–679.

More recently this example has been recast in the framework of quasi static motion in the papers

A. Mielke and L.Truskinovsky. From discrete visco-elasticity to continuum rate-independent plasticity: rigorous results. Arch. Rational Mech. Anal. 203 (2012), 577–619

A. Mielke. Emergence of rate-independent dissipation from viscous systems with wiggly energies. Continuum Mech. Thermodyn. 24 (2012), 591–606

The example of the minimizing movement for Lennard Jones interactions is original, and is part of ongoing work with A. Defranceschi and E. Vitali. It is close in spirit to a semi-discrete approach (i.e., the study of the limit of the gradient flows for the discrete energies) in

M. Gobbino. Gradient flow for the one-dimensional Mumford-Shah functional. Ann. Scuola Norm. Sup. Pisa (IV) 27 (1998), 145–193.