Chapter 4

Convergence of local minimizers

In this section we consider a generalization of the fundamental theorem of Γ-convergence when we have strict local minimizers of the Γ-limit.

4.1 Convergence to isolated local minimizers

The following theorem shows that we may extend (part of) the fundamental theorem of Γ-convergence to *isolated local minimizers* of the Γ-limit F; i.e., to points u_0 such that there exists $\delta > 0$ such that

$$F(u_0) < F(u) \quad \text{if} \quad 0 < d(u, u_0) \leq \delta. \tag{4.1}$$

The proof of this theorem essentially consists in remarking that we may at the same time apply Proposition 1.1.2 (more precisely, Remark 1.1.4) to the closed ball of center u_0 and radius δ, and Proposition 1.1.6 to the open ball of center u_0 and radius δ.

Theorem 4.1.1 *Suppose that each F_ε is coercive and lower semicontinuous and the sequence (F_ε) Γ-converge to F and is equicoercive. If u_0 is an isolated local minimizer of F then there exist a sequence (u_ε) converging to u_0 with u_ε a local minimizer of F_ε for ε small enough.*

Proof. Let $\delta > 0$ satisfy (4.1). Note that by the coerciveness and lower semicontinuity of F_ε there exists a minimizer u_ε of F_ε on $\overline{B_\delta(u_0)}$, the closure of $B_\delta(u_0) = \{u : d(u, u_0) \leq \delta\}$. By the equicoerciveness of (F_ε), upon extracting a subsequence, we can suppose that $u_\varepsilon \to \overline{u}$. Since $\overline{u} \in \overline{B_\delta(u_0)}$ we then have

$$\begin{aligned} F(u_0) &\leq F(\overline{u}) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0} \min_{\overline{B_\delta(u_0)}} F_\varepsilon \\ &\leq \limsup_{\varepsilon\to 0} \inf_{B_\delta(u_0)} F_\varepsilon \leq \inf_{B_\delta(u_0)} F = F(u_0), \end{aligned} \tag{4.2}$$

where we have used Proposition 1.1.6 in the last inequality. By (4.1) we have that $\overline{u} = u_0$ and $u_\varepsilon \in B_\delta(u_0)$ for ε small enough, which proves the thesis. □

Remark 4.1.2 In the theorem above it is sufficient to require the coerciveness properties for F_ε only on bounded sets, since they are applied to minimization problems on $\overline{B_\delta(u_0)}$.

Remark 4.1.3 Clearly, the existence of an isolated (local) minimizer in the limit does not imply that the converging (local) minimizers are isolated. It suffices to consider $F_\varepsilon(x) = ((x-\varepsilon) \vee 0)^2$ converging to $F(x) = x^2$.

Remark 4.1.4 In Section 3.4 we have noticed that the limit fracture energy F^λ possesses families of L^1-local minimizers with an arbitrary number of jump points, while the approximating functionals F^λ_ε have local minimizers corresponding to limit functions with only one jump point. This cannot directly be deduced from the result above since those limit local minimizers are not isolated. Anyhow L^1-local minimizers with one jump are strict local minimizers with respect to the distance

$$\begin{aligned} d(u,v) &= \int_0^1 |u-v|\,dx + \sum_{x\in(0,1)} |(u^+-u^-)-(v^+-v^-)| \\ &= \int_0^1 |u-v|\,dx + \sum_{x\in S(u)\cap S(v)} |(u^+-u^-)-(v^+-v^-)| \\ &\qquad + \sum_{x\in S(u)\setminus S(v)} |u^+-u^-| + \sum_{x\in S(v)\setminus S(u)} |v^+-v^-|, \end{aligned}$$

which penalizes (large) jumps of a competitor v outside $S(u)$. Upon suitably defining interpolations of discrete functions in $SBV(0,1)$ (where jumps correspond to difference quotients above the threshold $w_0/\sqrt{\varepsilon}$) it can be shown that the Γ-limit remains unchanged with this convergence, so that we may apply Theorem 4.1.1. Note that for discrete functions the notion of local minimizers is the same as for the L^1-distance since we are in a finite-dimensional space. Note moreover that L^1-local minimizers of F^λ with more than one jump are not strict local minimizers for the distance d above. Indeed, if $u' = 0$ and $S(u) = \{x_1, \ldots, x_N\}$ with $0 \le x_1 < \cdots < x_N$ and $N \ge 2$, then any $u_s = u + s\chi_{(x_1,x_2)}$ is still a local minimizer for F^λ with $F^\lambda(u_s) = F^\lambda(u) = N$ and $d(u,u_s) = s(1+|x_2-x_1|)$.

4.2 Two examples

We use Theorem 4.1.1 to prove the existence of sequences of converging local minima.

Example 4.2.1 (local minimizers for elliptic homogenization) Consider the functionals in Example 1.4.2. Suppose furthermore that $\overline{g}$ has an isolated local minimum at z_0. We will show that the constant function $u_0(x) = z_0$ is a L^1-local minimizer of $F_{\rm hom} + G$. Thanks to Theorem 4.1.1 we then deduce that there exists a sequence of local minimizers of $F_\varepsilon + G_\varepsilon$ (in particular, if g is differentiable with respect to u, of solutions of the Euler-Lagrange equation (1.15)) converging to u_0.

We only prove the statement in the one-dimensional case, for which $\Omega = (0, L)$. We now consider $\delta > 0$ and u such that

$$\|u - u_0\|_{L^1(0,L)} \le \delta.$$

Since z_0 is an isolated local minimum of $\overline{g}$ there exists $h > 0$ such that $g(z_0) < g(z)$ if $0 < |z - z_0| \le h$. If $\|u - u_0\|_\infty \le h$ then $G(u) \ge G(u_0)$ with equality only if $u = u_0$ a.e., so that the thesis is verified. Suppose otherwise that there exists a set of positive measure A such that $|u - u_0| > h$ on A. We then have

$$h|A| \le \int_A |u - u_0|\, dt \le \delta,$$

so that $|A| \le \delta/h$. We can then estimate

$$G(u) \ge \min \overline{g}|A| + (L - |A|)\overline{g}(z_0) \ge G(u_0) - \frac{\overline{g}(z_0) - \min \overline{g}}{h}\delta\,.$$

On the other hand, there exists a set of positive measure B such that

$$|u(x) - u_0| \le \frac{\delta}{L}$$

(otherwise the L^1 estimate doe not hold). Let $x_1 \in B$ and $x_2 \in A$, we can estimate (we can assume $x_1 < x_2$)

$$F_{\rm hom}(u) \ge \alpha \int_{[x_1,x_2]} |u'|^2\, dt \ge \alpha \frac{(u(x_2) - u(x_1))^2}{x_2 - x_1} \ge \alpha \frac{\left(h - \frac{\delta}{L}\right)^2}{L}$$

(using Jensen's inequality). Summing up we have

$$\begin{aligned} F_{\rm hom}(u) + G(u) &\ge F_{\rm hom}(u_0) + G(u_0) + \alpha \frac{\left(h - \frac{\delta}{L}\right)^2}{L} - \frac{\overline{g}(z_0) - \min \overline{g}}{h}\delta \\ &= F_{\rm hom}(u_0) + G(u_0) + \alpha \frac{h^2}{L} + O(\delta) \\ &> F_{\rm hom}(u_0) + G(u_0) \end{aligned}$$

for δ small as desired.

Example 4.2.2 (Kohn-Sternberg) In order to prove the existence of L^1 local minimizers for the energies F_ε in (1.18) by Theorem 1.4.2 it suffices to prove the existence of isolated local minimizers for the minimal interface problem related to the energy (1.20). In order for this to hold we need some hypothesis on the set Ω (for example, it can be proved that no non-trivial local minimizer exists when Ω is convex).

We treat the two-dimensional case only. We suppose that Ω is bounded, regular, and has an "isolated neck"; i.e., it contains a straight segment whose endpoints meet $\partial\Omega$ perpendicularly, and $\partial\Omega$ is strictly concave at those endpoints (see Fig. 4.1). We will show

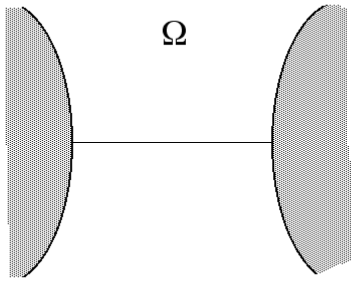

Figure 4.1: a neck in the open set Ω

that the set with boundary that segment (we can suppose that the segment disconnets Ω) is an isolated local minimizer for the perimeter functional.

We can think that the segment is $(0, L) \times \{0\}$. By the strict concavity of $\partial\Omega$ there exist $h > 0$ such that in a rectangular neighbourhood of the form $(a, b) \times (-2h, 2h)$ the lines $x = 0$ and $x = L$ meet $\partial\Omega$ only at $(0, 0)$ and $(0, L)$ respectively. The candidate strict local minimizer is $A_0 = \{(x, y) \in \Omega; x > 0\}$, which we identify with the function $u_0 = -1+2\chi_{A_0}$, taking the value $+1$ in A_0 and -1 in $\Omega \setminus A_0$.

Take another test set A. The L^1 closeness condition for functions translates into

$$|A \triangle A_0| \leq \delta.$$

We may suppose that A is sufficiently regular (some minor extra care must be taken when A is a set of finite perimeter, but the proof may be repeated essentially unchanged).

Consider first the case that A contains a horizontal segment $y = M$ with $M \in [h, 2h]$ and its complement contains a horizontal segment $y = m$ with $m \in [-2h, h]$. Then a portion of the boundary ∂A is contained in the part of Ω in the strip $|y| \leq 2h$, and its length is strictly greater than L, unless it is exactly the minimal segment (see Fig. 4.2).

If the condition above is not satisfied then A must not contain, e.g., any horizontal segment $y = t$ with $t \in [h, 2h]$ (see Fig. 4.3). In particular, the length of the portion of ∂A contained with $h \leq y \leq 2h$ is not less than h. Consider now the one-dimensional set

$$B = \{t \in (0, L) : \partial A \cap (\{t\} \times (-h, h)) = \emptyset\}.$$

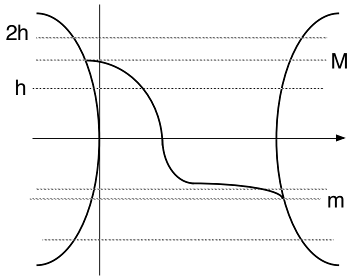

Figure 4.2: comparison with a uniformly close test set

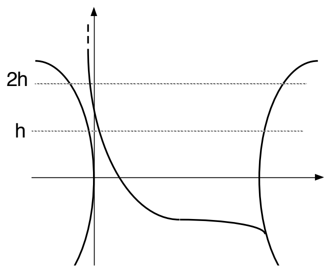

Figure 4.3: comparison with a L^1-close test set

We have

$$\delta \geq |A \triangle A_0| \geq h|B|,$$

so that $|B| \leq \delta/h$, and the portion of ∂A with $h \leq y \leq 2h$ is not less than $L - \delta/h$. Summing up we have

$$\mathcal{H}^1(\partial A) \geq h + L - \frac{\delta}{h} = \mathcal{H}^1(\partial A_0) + h - \frac{\delta}{h},$$

and the desired strict inequality for δ small enough.

4.3 Generalizations

We can give some generalizations of Theorem 4.1.1 in terms of scaled energies.

Proposition 4.3.1 *Let F_ε satisfy the coerciveness and lower-semicontinuity assumptions of Theorem* 4.1.1. *Suppose furthermore that a bounded positive function $f : (0, +\infty) \to (0, +\infty)$ exists and constants m_ε such that the scaled functionals*

$$\widetilde{F}_\varepsilon(u) = \frac{F_\varepsilon(u) - m_\varepsilon}{f(\varepsilon)} \tag{4.3}$$

are equicoercive and Γ-converge on $\overline{B_\delta(u_0)}$ to $\widetilde{F}_0$ given by

$$\widetilde{F}_0(u) = \begin{cases} 0 & \text{if } u = u_0 \\ +\infty & \text{otherwise} \end{cases} \tag{4.4}$$

in $\overline{B_\delta(u_0)}$. Then there exists a sequence (u_ε) converging to u_0 of local minimizers of F_ε.

Remark 4.3.2 (i) First note that the functionals F_ε in Theorem 4.1.1 satisfy the hypotheses of the above proposition, taking, e.g., $f(\varepsilon) = \varepsilon$ and m_ε equal to the minimum of F_ε in $\overline{B_\delta(u_0)}$;

(ii) Note that the hypothesis above is satisfied if there exist constants m_ε such that
(a) $\Gamma\text{-}\limsup_{\varepsilon\to 0}(F_\varepsilon(u_0) - m_\varepsilon) = 0$;
(b) $\Gamma\text{-}\liminf_{\varepsilon\to 0}(F_\varepsilon(u) - m_\varepsilon) > 0$ on $\overline{B_\delta(u_0)} \setminus \{u_0\}$.

Indeed condition (a) implies that we may change the constants m_ε so that the Γ-limit exists, is 0 at u_0, and we have a recovery sequence with $F_\varepsilon(u_\varepsilon) = m_\varepsilon$, while (b) is kept unchanged. At this point is suffices to chose, e.g., $f(\varepsilon) = \varepsilon$.

Proof. The proof follows that of Theorem 4.1.1. Again, let u_ε be a a minimizer of F_ε on $\overline{B_\delta(u_0)}$; we can suppose that $u_\varepsilon \to \overline{u} \in \overline{B_\delta(u_0)}$ we then have

$$\begin{aligned} 0 = \widetilde{F}_0(u_0) &\leq \widetilde{F}_0(\overline{u}) \leq \liminf_{\varepsilon\to 0} \widetilde{F}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0} \min_{\overline{B_\delta(u_0)}} \widetilde{F}_\varepsilon \\ &\leq \limsup_{\varepsilon\to 0} \inf_{\overline{B_\delta(u_0)}} \widetilde{F}_\varepsilon \leq \inf_{B_\delta(u_0)} \widetilde{F}_0 = 0, . \end{aligned} \tag{4.5}$$

so that $\overline{u} = u_0$ and $u_\varepsilon \in B_\delta(u_0)$ for ε small enough, which proves the thesis after remarking that (local) minimization of F_ε and $\widetilde{F}_\varepsilon$ are equivalent up to additive and multiplicative constants. □

Proposition 4.3.3 *Let F_ε satisfy the coerciveness and lower-semicontinuity assumptions of Theorem* 4.1.1. *Suppose furthermore that there exist a bounded positive function $f : (0,+\infty) \to (0,+\infty)$, constants m_ε and ρ_ε with $\rho_\varepsilon > 0$ and $\rho_\varepsilon \to 0$, and $\widetilde{u}_\varepsilon \to u_0$ such that the scaled functionals*

$$\widetilde{F}_\varepsilon(v) = \frac{F_\varepsilon(\widetilde{u}_\varepsilon + \rho_\varepsilon v) - m_\varepsilon}{f(\varepsilon)} \tag{4.6}$$

are equicoercive and Γ-converge on $\overline{B_\delta(v_0)}$ to $\widetilde{F}_0$ with v_0 an isolated local minimum. Then there exists a sequence (u_ε) converging to u_0 of local minimizers of F_ε.

Proof. We can apply Theorem 4.1.1 to the functionals $\widetilde{F}_\varepsilon(v)$ concluding that there exist local minimizers v_ε of $\widetilde{F}_\varepsilon$ converging to v_0. The corresponding $u_\varepsilon = \widetilde{u}_\varepsilon + \rho_\varepsilon v_\varepsilon$ are local minimizers for F_ε converging to u_0. □

Example 4.3.4 We illustrate the proposition with the simple example

$$F_\varepsilon(x) = \sin\Big(\frac{x}{\varepsilon}\Big) + x,$$

whose Γ-limit $F(x) = x - 1$ has no local (or global) minimizers. Take any $x_0 \in \mathbb{R}$, $x_\varepsilon \to x_0$ any sequence with $\sin(x_\varepsilon/\varepsilon) = -1$, $m_\varepsilon = x_\varepsilon - 1$, $\rho_\varepsilon = \varepsilon^\beta$ with $\beta \geq 1$, and $f(\varepsilon) = \varepsilon^\alpha$ with $\alpha \geq 0$, so that

$$\begin{aligned} \widetilde{F}_\varepsilon(t) &= \frac{\sin\Big(\frac{x_\varepsilon + \varepsilon^\beta t}{\varepsilon}\Big) + 1}{\varepsilon^\alpha} + \varepsilon^{\beta-\alpha} t \\ &= \frac{\sin\Big(\varepsilon^{\beta-1} t - \frac{\pi}{2}\Big) + 1}{\varepsilon^\alpha} + \varepsilon^{\beta-\alpha} t = \frac{1 - \cos(\varepsilon^{\beta-1} t)}{\varepsilon^\alpha} + \varepsilon^{\beta-\alpha} t. \end{aligned}$$

In this case the Γ-limit $\widetilde{F}$ coincides with the pointwise limit of $\widetilde{F}_\varepsilon$. If $\beta = 1$ and $0 \leq \alpha \leq 1$ then we have (local) minimizers of $\widetilde{F}$ at all points of $2\pi\mathbb{Z}$; indeed if $\alpha = 0$ then the sequence converges to $\widetilde{F}(x) = 1 - \cos x$, if $0 < \alpha < 1$ we have

$$\widetilde{F}(x) = \begin{cases} 0 & \text{if } x \in 2\pi\mathbb{Z} \\ +\infty & \text{otherwise,} \end{cases}$$

and if $\alpha = 1$

$$\widetilde{F}(x) = \begin{cases} x & \text{if } x \in 2\pi\mathbb{Z} \\ +\infty & \text{otherwise.} \end{cases}$$

In the case $2 > \beta > 1$ we have two possibilities: if $\alpha = 2\beta - 2$ then $\widetilde{F}(x) = \frac{1}{2}t^2$; if $\beta \geq \alpha > 2\beta - 2$ then

$$\widetilde{F}(x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

If $\alpha = \beta = 2$ then $\widetilde{F}(x) = \frac{1}{2}t^2 + t$. In all these cases we have isolated local minimizers in the limit.

Note that in this computation x_ε are not themselves local minimizers of F_ε.

We now consider an infinite-dimensional example in the same spirit as the one above.

Example 4.3.5 (existence of infinitely many local minima for oscillating metrics) Let the 1-periodic coefficient $a : \mathbb{R}^2 \to \{1, 2\}$ be defined on $[0,1]^2$ as

$$a(v_1, v_2) = \begin{cases} 1 & \text{if either } (v_1 - v_2)(v_1 + v_2 - 1) = 0 \text{ or} \\ 4 & \text{otherwise.} \end{cases} \tag{4.7}$$

Let

$$F^0_\varepsilon(u) = \int_0^1 a\Big(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}\Big)(1 + |u'|^2)\, dx$$

defined on

$$X = \{u \in W^{1,\infty}((0,1);\mathbb{R}^2), u(0) = 0,\ u(1) = 1\}$$

equipped with the L^2-convergence. It may be useful to remark that F^0_ε can be rewritten in terms of the curve $\gamma(x) = (x, u(x))$ as the energy

$$\int_0^1 a\Big(\frac{\gamma}{\varepsilon}\Big)|\gamma'(x)|^2\,dx,$$

of an inhomogeneous Riemannian metric which favors curves lying on the network where $a = 1$ (we will call that the 1-network), which is a sort of *opus reticolatum* as pictured in Fig. 4.4.

The Γ-limit of F^0_ε is of the form

$$F^0_{\rm hom}(u) = \int_0^1 \varphi(u')\,dx\,.$$

with domain X. It can be shown that $\varphi(z) = \sqrt{2}$ if $|z| \le 1$, and that for functions with $|u'| \le 1$ recovery sequences for $F^0_{\rm hom}(u)$ are functions with $a(x/\varepsilon, u_\varepsilon(x)/\varepsilon) = 1$ a.e. (i.e., that follow the lines of the 1-network). This will also follow from the computations below.

We consider the functionals

$$F_\varepsilon(u) = F^0_\varepsilon(u) + G(u), \qquad \text{where} \qquad G(u) = \int_0^1 |u|^2\,dx$$

(perturbation more general than G can be added). Since G is a continuous perturbation the Γ-limit of F_ε is simply $F = F^0_{\rm hom} + G$. Since G is strictly convex, then F is also strictly convex, and hence admits no local minimizers other than the absolute minimizer $u = 0$. We will show that F_ε admit infinitely many local minimizers. To that end we make some simplifying hypotheses: we suppose that ε are of the form 2^{-k}. In this way both $(0,0)$ and $(1,0)$ (corresponding to the boundary conditions) belong to the 1-network for all ε, and 1-networks are decreasing (in the sense of inclusion) with ε. We consider any function $u_0 \in X$ such that $a(x2^{k_0}, u_0(x)2^{k_0}) = 1$ a.e. for some k_0, and hence for all $k \ge k_0$; i.e., a function following the lines of the 1-network for all ε sufficiently small. We will prove that every such u_0 is a local minimum for F_ε if ε is small enough.

We consider the scaled functionals

$$\widetilde{F}_\varepsilon(v) = \frac{F_\varepsilon(u_0 + \varepsilon^2 v) - F_\varepsilon(u_0)}{\varepsilon^2}\,.$$

We note that the term deriving from G still gives a continuously converging term, and can be dealt with separately, since

$$\frac{G(u_0 + \varepsilon^2 v) - G(u_0)}{\varepsilon^2} = 2\int_0^1 u_0 v\,dx + \varepsilon^2\int_0^1 |v|^2\,dx.$$

We concentrate our analysis on the term of $\widetilde{F}_\varepsilon$ coming from F^0_ε: let $\widetilde{v}_\varepsilon$ be such that

$$\|\widetilde{v}_\varepsilon - u_0\|_{L^2} \le \varepsilon^2\delta; \tag{4.8}$$

i.e., that $\widetilde{v}_\varepsilon = u_0 + \varepsilon^2 v_\varepsilon$ with $\|v_\varepsilon\|_{L^2} \le \delta$, and $\widetilde{F}_\varepsilon(\widetilde{v}_\varepsilon) \le C_1 < +\infty$.

We denote $\widetilde{\gamma}_\varepsilon(x) = (x, \widetilde{v}_\varepsilon(x))$ and $\gamma_0(x) = (x, u_0(x))$. Note that if we set $e^1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $e^2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ then $x \mapsto \langle \gamma_0(x), e^1\rangle$ and $x \mapsto \langle \gamma_0(x), e^2\rangle$ are both non decreasing. We may then suppose that the same holds for $\widetilde{\gamma}_\varepsilon$. We also denote

$$\|z\|_1 = |\langle z, e^1\rangle| + |\langle z, e^2\rangle|.$$

For each ε fixed we consider points $0 = x_0 < x_1 < \ldots < x_N = 1$ such that

$$a\Big(\frac{\widetilde{\gamma}_\varepsilon}{\varepsilon}\Big) = 1 \qquad \text{a.e. or} \qquad a\Big(\frac{\widetilde{\gamma}_\varepsilon}{\varepsilon}\Big) = 4 \qquad \text{a.e. alternately on } [x_{k-1}, x_k]\,;$$

we can suppose that the first case occurs for e.g. k odd and the second one for k even. In the first case, by convexity and taking into account that the image of $\widetilde{\gamma}_\varepsilon$ is contained in the 1-network, we have

$$\int_{x_{k-1}}^{x_k} a\Big(\frac{\widetilde{\gamma}_\varepsilon}{\varepsilon}\Big)|\widetilde{\gamma}_\varepsilon|^2\,dx \ge (x_k - x_{k-1})\Big\|\frac{\widetilde{\gamma}_\varepsilon(x_k) - \widetilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}}\Big\|_1^2$$

In the second case, again by convexity and by the inequality $\|z\|_1 \le \sqrt{2}|z|$,

$$\begin{aligned}\int_{x_{k-1}}^{x_k} a\Big(\frac{\widetilde{\gamma}_\varepsilon}{\varepsilon}\Big)|\widetilde{\gamma}_\varepsilon'|^2\,dx &\ge 4(x_k - x_{k-1})\Big|\frac{\widetilde{\gamma}_\varepsilon(x_k) - \widetilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}}\Big|^2 \\ &\ge 2(x_k - x_{k-1})\Big\|\frac{\widetilde{\gamma}_\varepsilon(x_k) - \widetilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}}\Big\|_1^2.\end{aligned}$$

As a first consequence, we deduce that

$$\begin{aligned}F^0_\varepsilon(\widetilde{\gamma}_\varepsilon) &\ge \sum_{k=1}^N (x_k - x_{k-1})\Big\|\frac{\widetilde{\gamma}_\varepsilon(x_k) - \widetilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}}\Big\|_1^2 + \sum_{k \text{ even}} (x_k - x_{k-1})\Big\|\frac{\widetilde{\gamma}_\varepsilon(x_k) - \widetilde{\gamma}_\varepsilon(x_{k-1})}{x_k - x_{k-1}}\Big\|_1^2 \\ &\ge \Big\|\sum_{k=1}^N \widetilde{\gamma}_\varepsilon(x_k) - \widetilde{\gamma}_\varepsilon(x_{k-1})\Big\|_1^2 + \sum_{k \text{ even}} \frac{1}{x_k - x_{k-1}}\|\widetilde{\gamma}_\varepsilon(x_k) - \widetilde{\gamma}_\varepsilon(x_{k-1})\|_1^2 \\ &= \|\widetilde{\gamma}_\varepsilon(1) - \widetilde{\gamma}_\varepsilon(0)\|_1^2 + \sum_{k \text{ even}} \frac{1}{x_k - x_{k-1}}\|\widetilde{\gamma}_\varepsilon(x_k) - \widetilde{\gamma}_\varepsilon(x_{k-1})\|_1^2 \\ &= F^0_\varepsilon(u_0) + \sum_{k \text{ even}} \frac{1}{x_k - x_{k-1}}\|\widetilde{\gamma}_\varepsilon(x_k) - \widetilde{\gamma}_\varepsilon(x_{k-1})\|_1^2.\end{aligned}$$

From the energy bound we then deduce that for each k even

$$(x_k - x_{k-1})\Big(1 + \Big(\frac{\widetilde{v}_\varepsilon(x_k) - \widetilde{v}_\varepsilon(x_{k-1})}{x_k - x_{k-1}}\Big)\Big) \le C\varepsilon^2 .$$

so that both $(x_k - x_{k-1}) \le C\varepsilon^2$ and $|\widetilde{v}_\varepsilon(x_k) - \widetilde{v}_\varepsilon(x_{k-1})| \le C\varepsilon$. This implies that $\widetilde{\gamma}_\varepsilon$ can be deformed with a perturbation with $o(\varepsilon^2)$ L^2-norm to follow the 1-network between x_{k-1} and x_k. Hence, possible competitors essentially follow the 1-network (see Fig. 4.4). If δ is

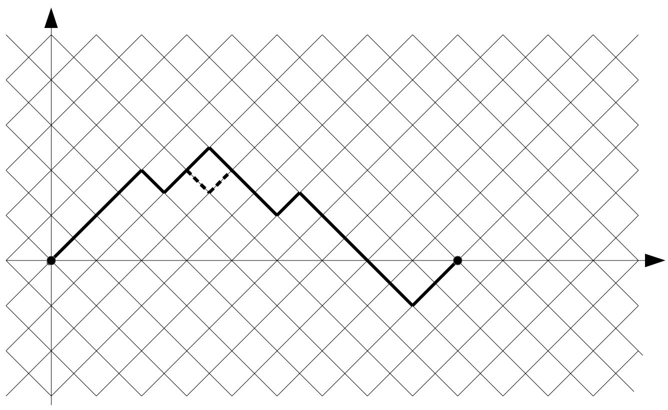

Figure 4.4: a local minimizer and a competitor

small enough then in order that (4.8) hold we must have $v_\varepsilon \to 0$. This shows that the limit of $\widetilde{F}_\varepsilon$ is finite only at $v = 0$ on $B_\delta(0)$ as desired.

As a consequence of the computation above we deduce that for all $u \in X$ with $\|u'\|\infty \le 1$ we have a sequence $\{u_\varepsilon\}$ of local minimizers of F_ε converging to u.

Example 4.3.6 (density of local minima for oscillating distances) We may consider a similar example to the one above for oscillating distances; i.e., length functionals defined on curves. Let the 1-periodic coefficient $a : \mathbb{R}^2 \to \{1, 2\}$ be defined as

$$a(v_1, v_2) = \begin{cases} 1 & \text{if either } v_1 \text{ or } v_2 \in \mathbb{Z} \\ 4 & \text{otherwise.} \end{cases} \tag{4.9}$$

This is the same type of coefficient as in the previous example up to a rotation and a scaling factor. Let

$$F_\varepsilon(u) = \int_0^1 a\Big(\frac{u}{\varepsilon}\Big)|u'|\,dx$$

be defined on
$$X = \{u \in W^{1,1}((0,1);\mathbb{R}^2), u(0) = v_0,\ u(1) = v_1\}$$
equipped with the L^1-convergence.

The Γ-limit of F_ε is
$$F(u) = \int_0^1 \|u'\|_1 dx,$$
where
$$\|z\|_1 = |z_1| + |z_2|.$$
This is easily checked after remarking that recovery sequences (u_ε) are such that $a(u_\varepsilon(t)/\varepsilon) = 1$ a.e. (except possibly close to 0 and 1 if $a(v_0/\varepsilon) \neq 1$ or $a(v_1/\varepsilon) \neq 1$) and then that $|u'_\varepsilon| = |(u_\varepsilon)'_1| + |(u_\varepsilon)'_2|$. For example, if both components of (u_ε) are monotone, then
$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_0^1 a\Big(\frac{u_\varepsilon}{\varepsilon}\Big)|u'|\,dx = \int_0^1 |u'_\varepsilon|\,dx + o(1) \\ &= \int_0^1 (|(u_\varepsilon)'_1| + |(u_\varepsilon)'_2|)\,dx + o(1) \\ &= \Big|(v_1)_1 - (v_0)_1\Big| + \Big|(v_1)_2 - (v_0)_2\Big| + o(1) \\ &= \int_0^1 (|u'_1| + |u'_2|)\,dx + o(1) = F(u) + o(1). \end{aligned}$$

For all these energies there are no strict local minimizers since energies are invariant with respect to reparameterization. Anyhow, if we consider equivalence classes with respect to reparameterization (e.g., by taking only functions in
$$X^1 = \{u \in X:\ \|u'\|_1 \text{ constant a.e.}\})$$
then an argument similar to the one in the previous example shows that local minimizers are L^1 dense, in the sense that for all $u \in X^1$ there exists a sequence of local minimizers of F_ε (restricted to X^1) converging to u.

As a technical remark, we note that in order to have coercivity the limit F should be extended to the space of curves with bounded variations. Anyhow, since functionals are invariant by reparameterization, it suffices to consider bounded sequences in $W^{1,\infty}$ after a change of variables.

4.4 References to Chapter 4

The use of Theorem 4.1.1 for proving the existence of local minimizers, together with Example 4.2.2 are due to

R.V. Kohn and P. Sternberg, Local minimizers and singular perturbations. Proc. Roy. Soc. Edinburgh A, 111 (1989), 69–84