# Chapter 10

# Stability theorems

We now face the problem of determining conditions under which the minimizing-movement scheme commutes with $\Gamma$-convergence. Let $F_\varepsilon$ $\Gamma$-converge to $F$ with initial data $x_\varepsilon$ converging to $x_0$. We have seen in Section 7.2 that by choosing suitably $\varepsilon = \varepsilon(\tau)$ the minimizing movement along the sequence $F_\varepsilon$ from $x_\varepsilon$ converges to a minimizing movement for the limit $F$ from $x_0$. A further issue is whether, by assuming some further properties on $F_\varepsilon$ we may deduce that the same thing happens for any choice of $\varepsilon$. In order to give an answer we will use results from the theory of gradient flows recently elaborated by Ambrosio, Gigli and Savarè, and by Sandier and Serfaty.

## 10.1 Stability for convex energies

We now use the theory of gradient flows to deduce stability results if the functionals satisfy some convexity assumptions. For the sake of simplicity we will assume that $X$ is a Hilbert space and all $F_\varepsilon$ are convex.

### 10.1.1 Convergence estimates

We first recall some results on minimizing movements for a single convex functional $F$.

**Proposition 10.1.1** *Let $F$ be convex, $z \in X$ and let $w$ be a minimizer of*

$$\min\Big\{F_\varepsilon(x) + \frac{1}{2\eta}\|x - z\|^2\Big\}. \tag{10.1}$$

*Then*

$$\|x - w\|^2 - \|x - z\|^2 \le 2\eta(F(x) - F(w)) \tag{10.2}$$

*for all $x \in X$.*

*Proof.* We recall that the inequality

$$\|sx + (1-s)w - z\|^2 \leq s\|x - z\|^2 + (1-s)\|w - z\|^2 - s(1-s)\|x - w\|^2 \tag{10.3}$$

holds for all $x, w, z \in X$ and $s \in [0, 1]$. Using this property and the convexity of $F$, thanks to the minimality of $w$ we have

$$\begin{aligned} F(w) + \frac{1}{2\eta}\|w - z\|^2 &\leq F(sx + (1-s)w) + \frac{1}{2\eta}\|sx + (1-s)w - z\|^2 \\ &\leq sF(x) + (1-s)F(w) \\ &\quad + \frac{1}{2\eta}(s\|x - z\|^2 + (1-s)\|w - z\|^2 - s(1-s)\|x - w\|^2). \end{aligned}$$

After regrouping and dividing by $s$, from this we have

$$\frac{1}{2\eta}(\|w - z\|^2 + (1-s)\|x - w\|^2 - \|x - z\|^2) \leq F(x) - F(w)$$

and then the desired (10.2) after letting $s \to 0$ and dropping the positive term $\|w - z\|^2$. □

**Remark 10.1.2** Let $\{z_k\} = \{z_k^\eta\}$ be a minimizing scheme for $F$ from $z_0$ with time-step $\eta$. Then (10.2) gives

$$\|x - z_{k+1}\|^2 - \|x - z_k\|^2 \leq 2\eta(F(x) - F(z_{k+1})) \tag{10.4}$$

for all $x \in X$.

We now fix $\tau > 0$ and two initial data $x_0$ and $y_0$ and want to compare the resulting $\{x_k\} = \{x_k^\tau\}$ obtained by iterated minimization with time-step $\tau$ and initial datum $x_0$ and $\{y_k\} = \{y_k^{\tau/2}\}$ with time-step $\tau/2$ and initial datum $y_0$. Note that the corresponding continuous-time interpolations are

$$u^\tau(t) := x_{\lfloor t/\tau \rfloor}, \qquad v^{\tau/2}(t) = y_{\lfloor 2t/\tau \rfloor}, \tag{10.5}$$

so that the comparison must be performed between $x_k$ and $y_{2k}$.

**Proposition 10.1.3** *For all $j \in \mathbb{N}$ we have*

$$\|x_j - y_{2j}\|^2 - \|x_0 - y_0\|^2 \leq 2\tau F(x_0)\,.$$

*Proof.* We first give an estimate between $x_1$ and $y_2$. We first apply (10.4) with $\eta = \tau$, $z_k = x_0$, $z_{k+1} = y_1$ and $x = y_2$ which gives

$$\|y_2 - x_1\|^2 - \|y_2 - x_0\|^2 \leq 2\tau(F(y_2) - F(x_1))\,. \tag{10.6}$$

If instead we apply (10.4) with $\eta = \tau/2$, $z_k = y_0$, $z_{k+1} = y_1$ and $x = x_0$, or $z_k = y_1$, $z_{k+1} = y_2$ and $x = x_0$ we get, respectively,

$$\begin{aligned}\|x_0 - y_1\|^2 - \|x_0 - y_0\|^2 &\leq \tau(F(x_0) - F(y_1))\\ \|x_0 - y_2\|^2 - \|x_0 - y_1\|^2 &\leq \tau(F(x_0) - F(y_2)),\end{aligned}$$

so that, summing up,

$$\|x_0 - y_2\|^2 - \|x_0 - y_0\|^2 \leq 2\tau F(x_0) - \tau F(y_1) - F(y_2) \leq 2\tau(F(x_0) - F(y_2)), \tag{10.7}$$

where we have used that $F(y_2) \leq F(y_1)$ in the last inequality. Summing up (10.6) and (10.7) we obtain

$$\|x_1 - y_2\|^2 - \|x_0 - y_0\|^2 \leq 2\tau(F(x_0) - F(x_1)). \tag{10.8}$$

We now compare the later indices. We can repeat the same argument with $x_0$ and $y_0$ substituted by $x_1$ and $y_2$, so that by (10.8) we get

$$\|x_2 - y_4\|^2 - \|x_1 - y_2\|^2 \leq 2\tau(F(x_1) - F(x_2)), \tag{10.9}$$

and, summing (10.8),

$$\|x_2 - y_4\|^2 - \|x_0 - y_0\|^2 \leq 2\tau(F(x_0) - F(x_2)). \tag{10.10}$$

Iterating this process we get

$$\|x_j - y_{2j}\|^2 - \|x_0 - y_0\|^2 \leq 2\tau(F(x_0) - F(x_j)) \leq 2\tau F(x_0) \tag{10.11}$$

as desired. □

**Theorem 10.1.4** *Let $F$ be convex and let $F(x_0) < +\infty$. Then there exists a unique minimizing movement $u$ for $F$ from $x_0$ such that, if $u^\tau$ is defined by* (10.5)*, then*

$$\|u^\tau(t) - u(t)\| \leq 6\sqrt{F(x_0)}\sqrt{\tau}$$

*for all $t \geq 0$.*

*Proof.* With fixed $\tau$ we first prove the convergence of $u^{2^{-j}\tau}$ as $j \to +\infty$. By Proposition 10.1.3 applied with $y_0 = x_0$ and $2^{-j}\tau$ in place of $\tau$ we have

$$\|u^{2^{-j}\tau}(t) - u^{2^{-j-1}\tau}(t)\| \leq 2^{-j/2}\sqrt{2\tau}\sqrt{F(x_0)} \tag{10.12}$$

for all $t$. This shows the convergence to a limit $u_\tau(t)$, which in particular satisfies

$$\|u^\tau(t) - u_\tau(t)\| \;\leq\; \sqrt{2}\sum_{j=0}^{\infty} 2^{-j/2}\sqrt{\tau}\sqrt{F(x_0)} \leq 6\sqrt{F(x_0)}\sqrt{\tau}\,. \tag{10.13}$$

The limit $u_\tau$ can be characterized as follows: with fixed $x$, inequality (10.4) applied to $z_k = u^{2^{-j}\tau}((k-1)2^{-j}\tau)$ $(k \geq 1)$ can be seen as describing in the sense of distribution the derivative

$$\frac{d}{dt}\frac{1}{2}\|x - u^{2^{-j}\tau}(t)\|^2 \leq \sum_{k=1}^{\infty}\Big(F(x) - F\Big(u^{2^{-j}\tau}((k-1)2^{-j}\tau)\Big)\Big)2^{-j}\tau\,\delta_{k2^{-j}\tau}\,. \tag{10.14}$$

Note in fact that $x \mapsto \frac{1}{2}\|x - u^{2^{-j}\tau}\|^2$ is a piecewise-constant function with discontinuities in $2^{-j}\tau\mathbb{Z}$, whose size is controlled by (10.4). Since the measures

$$\mu_j = \sum_{k=1}^{\infty} 2^{-j}\tau\,\delta_{k2^{-j}\tau}$$

converge to the Lebesgue measure, and $u^{2^{-j}\tau}(t) \to u_\tau(t)$ for all $t$, so that by the lower semicontinuity of $F$

$$F(u_\tau(t)) \leq \liminf_{j\to+\infty} F\Big(u^{2^{-j}\tau}(t)\Big),$$

we deduce that

$$\frac{d}{dt}\frac{1}{2}\|x - u_\tau(t)\|^2 \leq F(x) - F(u_\tau(t)) \tag{10.15}$$

for all $x$. Equation (10.15) is sufficient to characterize $u_\tau$. We only sketch the argument: suppose otherwise that (10.15) is satisfied by some other $v(t)$. Then we have

$$\langle x - u_\tau, \nabla u_\tau\rangle \leq F(x) - F(u_\tau) \qquad \text{and} \qquad \langle x - v, \nabla v\rangle \leq F(x) - F(v)$$

for all $x$. Inserting $x = v(t)$ and $x = u_\tau(t)$ respectively, and summing the two inequalities we have

$$\frac{d}{dt}\frac{1}{2}\|v(t) - u_\tau(t)\|^2 = \langle v - u_\tau, \nabla v - \nabla u_\tau\rangle \leq 0\,.$$

Since $v(0) = u_\tau(0)$ we then have $v = u_\tau$.

This argument shows that $u = u_\tau$ does not depend on $\tau$. We then have the convergence of the whole sequence, and (10.13) gives the desired estimate of $\|u^\tau - u\|$. □

### 10.1.2 Stability along sequences of convex energies

From the estimates in the previous section, and the convergence argument in Section 7.2 we can deduce the following stability results.

**Theorem 10.1.5** *Let $F_\varepsilon$ be a sequence of lower-semicontinuous coercive positive convex energies $\Gamma$-converging to $F$, and let $x_0^\varepsilon \to x_0$ with $\sup_\varepsilon F_\varepsilon(x_0^\varepsilon) < +\infty$. Then*

(i) *for every choice of $\tau$ and $\varepsilon$ converging to* 0 *the family $u^\varepsilon$ introduced in Definition* 7.1.1 *converges to the unique $u$ given by Theorem* 10.1.4*;*

(ii) *the sequence of minimizing movements $u_\varepsilon$ for $F_\varepsilon$ from $x_0^\varepsilon$ (given by Theorem* 10.1.4 *with $F_\varepsilon$ in place of $F$) also converge to the same minimizing movement $u$.*

*Proof.* We first show (ii). Indeed, by the estimate in Theorem 10.1.4 we have that, after defining $u_\varepsilon^\tau$ following the notation of that theorem,

$$\|u^\tau - u\|_\infty \le M\sqrt{\tau}, \qquad \|u_\varepsilon^\tau - u_\varepsilon\|_\infty \le M\sqrt{\tau},$$

where

$$M = 6\sup_\varepsilon F_\varepsilon(x_0^\varepsilon).$$

In order to show that $u_\varepsilon \to u$ it suffices to show that $u_\varepsilon^\tau \to u^\tau$ for fixed $\tau$. That has already been noticed to hold in Section 7.2.

In order to prove (i) it suffices to use the triangular inequality

$$\|u_\varepsilon^\tau - u\| \le \|u_\varepsilon^\tau - u_\varepsilon\| + \|u_\varepsilon - u\| \le M\sqrt{\tau} + o(1)$$

by Theorem 10.1.4 and (ii). □

**Remark 10.1.6 (compatible topologies)** We may weaken the requirement that $F_\varepsilon$ be equi-coercive with respect to the $X$-convergence. It suffices to require that the $\Gamma$-limit be performed with respect to a topology *compatible* with the $X$-norm; i.e., such that the $\Gamma$-convergence $F_\varepsilon \to F$ ensures that $F_\varepsilon(x) + C\|x - x_0\|^2$ $\Gamma$-converges to $F(x) + C\|x - x_0\|^2$ for fixed $C$ and $x_0$, and with respect to which these energies are equi-coercive. In this way we still have $u_\varepsilon^\tau \to u^\tau$ in the proof above.

**Example 10.1.7 (parabolic homogenization)** We can consider $X = L^2(0, T)$,

$$F_\varepsilon(u) = \int_0^T a\Big(\frac{x}{\varepsilon}\Big)|u'|^2\,dx, \qquad F(u) = \underline{a}\int_0^T |u'|^2\,dx$$

with the notation of Section 1.4. We take as initial datum $u_0$ independent of $\varepsilon$. Since all functionals are convex, lower semicontinuous and coercive, from Theorem 10.1.5 we deduce the converge of the corresponding minimizing movements. From this we deduce the convergence of the solutions of the parabolic problem with oscillating coefficients

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \dfrac{\partial}{\partial x}\Big(a\Big(\dfrac{x}{\varepsilon}\Big)\dfrac{\partial u_\varepsilon}{\partial x}\Big) \\ u_\varepsilon(x, 0) = u_0(x) \end{cases}$$

to the solution of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \underline{a}\,\dfrac{\partial^2 u}{\partial x^2} \\ u_\varepsilon(x, 0) = u_0(x)\,. \end{cases}$$

**Example 10.1.8 (high-contrast media)** We consider a discrete system parameterized on $\{0, \dots, N\}$ with $N$ even. We set $\varepsilon = 1/N$ and consider the energies

$$F_\varepsilon(u) = \frac{1}{2}\sum_{l=1}^{N/2} \varepsilon \Big|\frac{u_{2l} - u_{2l-2}}{\varepsilon}\Big|^2 + \frac{c_\varepsilon}{2}\sum_{j=1}^{N} \varepsilon \Big|\frac{u_j - u_{j-1}}{\varepsilon}\Big|^2$$

with a *periodic boundary condition* $u_N = u_0$.

This is a simple model where two elliptic energies interact possibly on different scales. The critical scale is when

$$c_\varepsilon = \varepsilon^2,$$

condition that will be assumed in the rest of the example. The first sum is a strong next-to-nearest-neighbor interaction between even points, and the second one is a weak nearest-neighbor interaction between all points.

Upon identifying $u_i$ with the piecewise-constant function $u \in L^2(0,1)$ with $u(x) = u_{\lfloor x/\varepsilon \rfloor}$ we may regard $F_\varepsilon$ as defined on $X = L^2(0,1)$ and consider the minimizing movement of $F_\varepsilon$ with respect to the $L^2$-norm, which we can write

$$\|u\|^2 = \sum_{j=1}^{N} \varepsilon |u_i|^2$$

on the domain of $F_\varepsilon$, so that the iterated minimum problem giving $u^k$ reads

$$\min\Big\{\frac{1}{2}\sum_{l=1}^{N/2} \varepsilon \Big|\frac{u_{2l} - u_{2l-2}}{\varepsilon}\Big|^2 + \frac{1}{2}\sum_{j=1}^{N} \varepsilon^3 \Big|\frac{u_j - u_{j-1}}{\varepsilon}\Big|^2 + \frac{1}{2\tau}\sum_{j=1}^{N} \varepsilon (u_j - u_j^{k-1})^2\Big\}.$$

We consider as initial datum (the sampling on $\varepsilon\mathbb{Z} \cap [0,1]$ of) a smooth 1-periodic datum $u^0$ (for simplicity independent of $\varepsilon$).

Since all $F_\varepsilon$ are convex, we may describe their minimizing movement through the gradient flow of their $\Gamma$-limit. Since $F_\varepsilon$ is not equi-coercive with respect to the $L^2$ norm, we have to choose a different topology for the $\Gamma$-limit.

Among the different choices we may consider the following two.

**(1)** We choose the strong $L^2$-convergence of the even piecewise-constant interpolations only; i.e.,

$$\|u - v\|^2_{even} = \sum_{j=1}^{N} \varepsilon |u_{2j} - v_{2j}|^2.$$

Note that $F_\varepsilon$ are equi-coercive and their $\Gamma$-limit is simply

$$F^s(u) = \int_0^1 |u'|^2 \, dx.$$

To check this it suffices to remark that, if we consider the even piecewise-affine interpolation $\widetilde{u}$ of $u_i$, then we have

$$\sum_{l=1}^{N/2}\varepsilon\Big|\frac{u_{2l}-u_{2l-2}}{\varepsilon}\Big|^2=2\sum_{l=1}^{N/2}2\varepsilon\Big|\frac{u_{2l}-u_{2l-2}}{2\varepsilon}\Big|^2=2\int_0^1|\widetilde{u}'|^2\,dx,$$

so that $F^s$ is a lower bound, while a recovery sequence is simply obtained by taking $u_\varepsilon$ the interpolation of $u$, for which

$$F_\varepsilon(u_\varepsilon)=\int_0^1|u'|^2\,dx+\frac{\varepsilon^2}{2}\int_0^1|u'|^2\,dx+o(1).$$

**(2)** We choose the strong $L^2$-convergence of the even piecewise-constant interpolations and the weak $L^2$-convergence of the odd piecewise-constant interpolations. A function $u$ is then identified with a pair $(u_e,u_o)$ (even and odd piecewise-constant interpolations), so that

$$F_\varepsilon(u)=F^w(u_e,u_o):=\int_0^1|u_e'|^2\,dx+\frac{1}{2}\int_0^1|u_e-u_o|^2\,dx.$$

The functional $F^w$ thus defined is the $\Gamma$-limit in this topology, which is compatible with the $L^2$-distance (interpreted as the sum of the $L^2$-distances of the even/odd piecewise-constant interpolations).

We can apply Theorem 10.1.5, together with Remark 10.1.6, and deduce that the minimizing movement for $F_\varepsilon$ is given by the solution $(u_e,u_o)=(u_e(x,t),u_o(x,t))$ of the gradient flow for $F^w$, which is

$$\begin{cases}\dfrac{\partial u_e}{\partial t}=2\dfrac{\partial^2u_e}{\partial x^2}-u_e+u_o\\[2mm]\dfrac{\partial u_o}{\partial t}=u_o-u_e\\[2mm]u_o(x,0)=u_e(x,0)=u^0(x)\end{cases},$$

with periodic boundary conditions for $u_e$.

Note that $F^s$ is not compatible with the $L^2$-norm since it does not contain the odd interpolations, and its gradient flow is simply a heat equation. Note however that we may use $u_e$ as a single parameter with respect to which to describe the minimizing movement of $F_\varepsilon$, as suggested by the choice of $F^s$ as $\Gamma$-limit. Indeed, we may integrate the second equation of the system above expressing $u_o$ in terms of $u_e$. Plugging its expression in the first equation we obtain the integro-differential problem satisfied by $u_e$

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=2\dfrac{\partial^2u(x,t)}{\partial x^2}-u(x,t)+u^0(x)e^{-t}+\displaystyle\int_0^t e^{s-t}u(x,s)\,ds\\[2mm]u(x,0)=u^0(x)\end{cases}$$

with periodic boundary conditions.

## 10.2 Sandier-Serfaty theory

We have already remarked that for some non-convex problems minimizing movements commute with Γ-convergence, as for approximations of the Mumford-Shah functional. We conclude this section by giving a brief (and simplified) account of another very fruitful approach to gradient flows that allows to prove the stability of certain solutions with respect to Γ-convergence related to non-convex energies.

We consider a family of Hilbert spaces $X_\varepsilon$ and functionals $F_\varepsilon : X_\varepsilon \to (-\infty, +\infty]$, which are $C^1$ on their domain. We denote by $\nabla_{X_\varepsilon} F_\varepsilon$ the gradient of $F_\varepsilon$ in $X_\varepsilon$.

**Definition 10.2.1** *Let $T > 0$; we say that $u_\varepsilon \in H^1([0,T); X_\varepsilon)$ is a* a.e. solution for the gradient flow of $F_\varepsilon$ *if*

$$\frac{\partial u_\varepsilon}{\partial t} = -\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon)$$

*almost everywhere on $(0,T)$. Such solution for the a gradient flow is* conservative *if*

$$F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(s)) = \int_0^s \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|^2_{X_\varepsilon} dt$$

*for all $\tau \in (0,T)$.*

We suppose that there exists a Hilbert space $X$ and a notion of metrizable convergence $x_\varepsilon \to x$ of families of elements of $X_\varepsilon$ to an element of $X$. With respect to that convergence, we suppose that $F_\varepsilon$ Γ-converge to a functional $F$, which is also $C^1$ on its domain.

**Theorem 10.2.2 (Sandier-Serfaty Theorem)** *Let $F_\varepsilon$ and $F$ be as above with $F_\varepsilon$ Γ-converging to $F$, let $u_\varepsilon$ be a family of conservative solutions for the gradient flow of $F_\varepsilon$ with initial data $u_\varepsilon(0) = u^\varepsilon$ converging to $u^0$. Suppose furthermore that*

- (well-preparedness of initial data) *$u^\varepsilon$ is a recovery sequence for $F(u^0)$;*
- (lower bound) *upon subsequences $u_\varepsilon$ converges to some $u \in H^1((0,T); X)$ and*

$$\liminf_{\varepsilon\to 0} \int_0^s \left\| \frac{\partial u_\varepsilon}{\partial t} \right\|^2_{X_\varepsilon} dt \geq \int_0^s \left\| \frac{\partial u}{\partial t} \right\|^2_{X} dt \tag{10.16}$$

$$\liminf_{\varepsilon\to 0} \|\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(s))\|^2_{X_\varepsilon} \geq \|\nabla_X F(u(s))\|^2_X \tag{10.17}$$

*for all $s \in (0,T)$.*

*Then $u$ is a solution for the gradient flow of $F$ with initial datum $u^0$, $u_\varepsilon(t)$ is a recovery sequence for $F(u(t)$ for all $t$ and the inequalities in* (10.16) *and* (10.17) *are equalities.*

*Proof.* Using the fact that $u_\varepsilon$ is conservative and that for all $t$

$$\Big\|\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon) + \frac{\partial u_\varepsilon}{\partial t}\Big\|^2_{X_\varepsilon} = 0,$$

and hence

$$-\Big\langle \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(t)), \frac{\partial u_\varepsilon}{\partial t}\Big\rangle = \frac{1}{2}\Big(\|\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon(t))\|^2_{X_\varepsilon} + \Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|^2_{X_\varepsilon}\Big).$$

We then get

$$\begin{aligned} F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t)) &= \int_0^t \Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|^2_{X_\varepsilon} ds \\ &= -\int_0^t \Big\langle \nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon), \frac{\partial u_\varepsilon}{\partial t}\Big\rangle_{X_\varepsilon} ds \\ &= \frac{1}{2}\int_0^t \Big(\|\nabla_{X_\varepsilon} F_\varepsilon(u_\varepsilon)\|^2_{X_\varepsilon} + \Big\|\frac{\partial u_\varepsilon}{\partial t}\Big\|^2_{X_\varepsilon}\Big) ds \end{aligned}$$

By the lower-bound assumption then we have

$$\begin{aligned} \liminf_{\varepsilon\to 0}(F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t))) &\geq \frac{1}{2}\int_0^t \Big(\|\nabla_X F(u)\|^2_X + \Big\|\frac{\partial u}{\partial t}\Big\|^2_X\Big) ds \\ &\geq -\int_0^t \Big\langle \nabla_X F(u), \frac{\partial u}{\partial t}\Big\rangle_X ds. \end{aligned} \tag{10.18}$$

The last term equals

$$-\int_0^t \frac{d}{dt}F(u)\, ds = F(u(0)) - F(u(t)),$$

so that we have

$$\liminf_{\varepsilon\to 0}(F_\varepsilon(u_\varepsilon(0)) - F_\varepsilon(u_\varepsilon(t))) \geq F(u(0)) - F(u(t)).$$

Since $u_\varepsilon(0)$ is a recovery sequence for $F(u(0))$ we then have

$$F(u(t)) \geq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon(t)), \tag{10.19}$$

so that $u_\varepsilon(t)$ is a recovery sequence for $u(t)$ and indeed we have equality in (10.19) and hence both inequalities in (10.18) are equalities. The second one of those shows that

$$\Big\|\nabla_X F(u) + \frac{\partial u}{\partial t}\Big\|^2_X = 0,$$

for all $t$, and hence the thesis. □

**Example 10.2.3 (Ginzburg-Landau vortices)** The theory outlined above has been successfully applied by Sandier and Serfaty to obtain the motion of vortices as the limit of the gradient flows of Ginzburg-Landau energies. We give a short account of their setting without entering into details.

Let $\Omega$ be a bounded regular open subset of $\mathbb{R}^2$ and $N \in \mathbb{N}$; the Hilbert spaces $X_\varepsilon$ and $X$ are chosen as

$$X_\varepsilon = L^2(\Omega; \mathbb{R}^2), \qquad X = \mathbb{R}^{2N}$$

with scalar products

$$\langle u, v\rangle_{X_\varepsilon} = \frac{1}{|\log \varepsilon|} \int_\Omega \langle u(x), v(x)\rangle_{\mathbb{R}^2}\, dx, \qquad \langle x, y\rangle_X = \frac{1}{\pi}\langle x, y\rangle_{\mathbb{R}^{2N}},$$

respectively.

The energies $F_\varepsilon : H^1(\Omega; \mathbb{R}^2) \to \mathbb{R}$ are defined as

$$F_\varepsilon(u) = \frac{1}{2} \int_\Omega \Big( |\nabla u|^2 + \frac{1}{\varepsilon^2}(1 - |u|^2)^2 \Big)\, dx.$$

The convergence of $u_\varepsilon$ is defined as follows: if we write in polar coordinates

$$u_\varepsilon(x) = \rho_\varepsilon(x) e^{i\varphi_\varepsilon(x)}$$

then $u_\varepsilon \to (x^1, \ldots, x^N)$ if we have

$$\lim_{\varepsilon\to 0} \operatorname{curl}(\rho_\varepsilon^2 \nabla\varphi_\varepsilon) = 2\pi \sum_{j=1}^N d_j \delta_{x^j}$$

weak* in the sense of measures for some integers $d_j$, where $\operatorname{curl}(A_1, A_2) = \frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1}$. This convergence describes the location of *vortices* at the points $x^j$ with a *degree* $d_j$. For $u_\varepsilon(x) \to x/|x|$ we have $N = 1$, $x^1 = 0$ and $d_1 = 1$.

It can be proved that there exists a function $W = W_{(d_1,\ldots,d_N)}$ such that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \Big( F_\varepsilon(u) - \pi N |\log \varepsilon| \Big) = W(x^1, \ldots, x^N).$$

The function $W$ can be characterized in terms of the Green function of $\Omega$. Its precise definition is not relevant to this example.

The well-preparedness condition for the initial data amounts to requiring that

$$u_\varepsilon^0 \to (x_0^1, \ldots, x_0^N) \qquad \text{and } d_j \in \{-1, 1\}.$$

Under these conditions we may apply Theorem 10.2.2 to the scaled energies $F_\varepsilon - \pi N|\log\varepsilon|$. This yields solutions $u_\varepsilon = u_\varepsilon(x,t)$ to the equation

$$\begin{cases} \dfrac{1}{|\log\varepsilon|}\dfrac{\partial u_\varepsilon}{\partial t} = \Delta u + \dfrac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2) & \text{in } \Omega \\ \dfrac{\partial u_\varepsilon}{\partial n} = 0 \qquad \text{on } \partial\Omega \\ u_\varepsilon(x,0) = u^0_\varepsilon(x) \end{cases}$$

converging to $x(t) = (x^1(t),\dots,x^N(t)) = (x_1(t),\dots,x_{2N}(t))$. The limit vortices move following the system of ODE

$$\frac{dx_i}{dt} = -\frac{1}{\pi}\frac{\partial W(x)}{\partial x_i}.$$

This description is valid until the first *collision time* $T^*$ when $x_j(T^*) = x_k(T^*)$ for some $j$ and $k$ with $j \neq k$.

## 10.3 References to Chapter 10

The results in Section 10.1.1 and part (ii) of Theorem 10.1.4 are a simplified version of the analogous results for geodesic-convex energies in metric spaces that can be found in the notes

L. Ambrosio and N. Gigli. A user's guide to optimal transport, in *Modelling and Optimisation of Flows on Networks* (B. Piccoli and M. Rascle eds.) Lecture Notes in Mathematics. Springer, Berlin, 2013, pp 1–155.

The result by Sandier and Serfaty (with weaker hypotheses) is contained in the seminal paper

E. Sandier and S. Serfaty, Gamma-Convergence of Gradient Flows and Application to Ginzburg-Landau, Comm. Pure Appl. Math. 57 (2004), 1627–1672.