Textures in discrete systems

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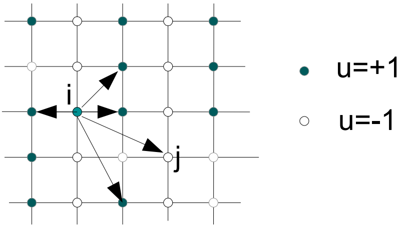
Simple lattice interactions: spin systems

Parameter: $u : \Omega \cap \mathbb{Z}^d \to \{1, -1\}$

Energies: $E(u) = -\sum_{i,j} c_{ij} u_i u_j$, or, up to additive/multiplicative constants

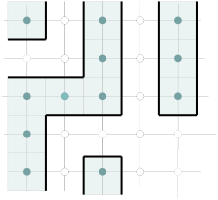
$$E(u) = \sum_{i,j} c_{ij} (u_i - u_j)^2$$



Ferromagnetic interactions

If $c_{ij} \geq 0$ then $u \equiv 1$ or $u \equiv -1$ are *ground states*

For **nearest-neighbour interactions** ($c_{ij} = 0$ for $|i - j| > 1$) E can be viewed as a **surface energy**



Discrete-to-continuous analysis

Scale parameter ε: scaled variable $u : \Omega \cap \varepsilon\mathbb{Z}^d \to \{1, -1\}$

Scaled energies:

$$E_\varepsilon(u) = \sum_{i,j} \varepsilon^{d-1} c^\varepsilon_{ij} (u_i - u_j)^2$$

Asymptotic description: identify u with its *piecewise-constant interpolation* (or with $A = \{u = 1\}$) and compute the **Γ-limit** with respect to the strong $L^1(\Omega)$-convergence.

General Ferromagnetic Homogenization Result

(Braides-Piatnitski, JFA 2012)

Periodicity: $c^\varepsilon_{ij} = C_{\frac{i}{\varepsilon}\frac{j}{\varepsilon}}$ and $C_{(k+K)(l+K)} = C_{kl}$ for $K \in T\mathbb{Z}^d$

Decay: $\sum_{k'\in\mathbb{Z}^d} C_{kk'} < +\infty$ (e.g. $C_{kk'} = 0$ for $|k-k'| > M$)

Then E_ε Γ-converge to an interfacial energy

$$F(u) = \int_{\Omega\cap\partial\{u=1\}} \varphi_{\rm hom}(\nu)d\mathcal{H}^{d-1}$$

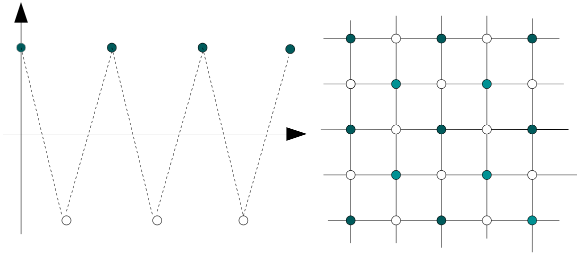
where $\nu =$ normal to the interface $\partial\{u=1\}$ and $\varphi_{\rm hom}$ is given by a discrete least-area homogenization formula.
Periodicity can be substituted by a random dependence (a.s. result).

More complex patterns: antiferromagnetic interactions

Simplest case: nearest-neighbour energies $E(u) = \sum_{NN} u_i u_j$, or, up to additive/multiplicative constants

$$E(u) = \sum_{NN} (u_i + u_j)^2$$

Ground states: alternating states.



Note: can reduce to ferromagnetic interactions introducing the variable $v_i = (-1)^i u_i$ (only for NN systems).

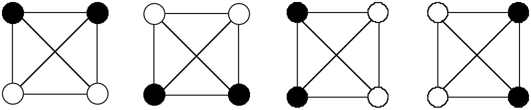
Order parameters from ground states

Alicandro-Braides-Cicalese NHM 2006

Anti-ferromagnetic spin systems in 2D

$$E(u) = c_1 \sum_{NN} u_i u_j + c_2 \sum_{NNN} u_k u_l \qquad u_i \in \{\pm 1\}$$

For suitable positive c_1 and c_2 the ground states are 2-periodic



(representation in the unit cell)

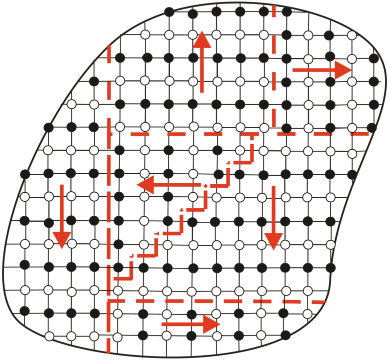
The correct order parameter is the **orientation** $v \in \{\pm e_1, \pm e_2\}$ of the ground state.

Γ-limit of scaled E_ε:

$$F(v) = \int_{S(v)} \psi(v^+ - v^-, \nu)\, d\mathcal{H}^1$$

$S(v)$ = discontinuity lines; ν = normal to $S(v)$
ψ given by an optimal-profile problem

Microscopic picture of a limit state with finite energy



We may give a continuous description also for **variational motions** derived from these energies following an Euler scheme by successive minimization (Almgren-Taylor-Wang, De Giorgi):

• fixed *space-scale* ε and *time-scale* τ define 'discrete motions' u^k by iteration:

$u_0 =$ initial state

u^{k+1} a minimizer of

$$u \mapsto F_\varepsilon(u) + \frac{1}{\tau} D_\varepsilon(u, u^k)$$

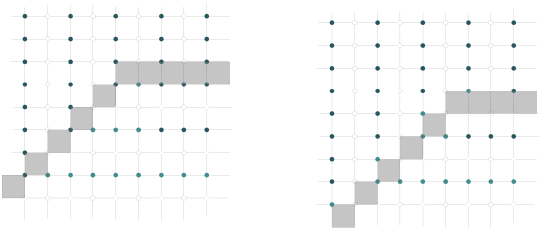
$D_\varepsilon(u, u^k) =$ 'distance' of u from $u^k \approx \varepsilon^d \, \#\{i \in \mathbb{Z}^d : u_i \neq u_i^k\}$

• Define the interpolation

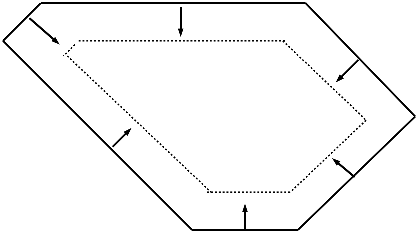
$$u^{\tau,\varepsilon}(t,x) = u^{\lfloor t/\tau \rfloor}_{\lfloor x/\varepsilon \rfloor}$$

• Let $\varepsilon, \tau \to 0$ and prove that $u^{\tau,\varepsilon} \to u$ (the limit may depend on $\frac{\varepsilon}{\tau}$; most interesting regime when $\varepsilon \approx \tau$)

Description of the motion of a two-phase initial datum: discrete picture (e_2-inclusion in an e_1-matrix)



Continuous picture (motion from a local minimizer of the continuous energy)



velocity of each side $v = m\left\lfloor \frac{\alpha}{L} \right\rfloor$

$m =$ mobility (depending on the normal direction)

$\alpha =$ discreteness parameter

Note: a side may be **pinned** (if $L > \alpha$)

Note: this is **not** the variational motion of the Γ-limit

Note: for three phases this method gives a new definition of motion of three-point intersections

Phase-shift energies

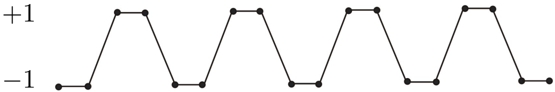
In 1D we may consider

$$F_\varepsilon(u) = \sum_i (\alpha u_i u_{i-1} + u_{i-1}u_{i+1})$$

with dominating NNN antiferromagnetic interactions. More precisely, $|\alpha| < 2$.
In this case (e.g. $\alpha = 0$) ground state are with

$$u_{i-1} = -u_{i+1}$$

e.g., of the form $+, +, -$. This gives that the ground states are these 4-periodic oscillating states

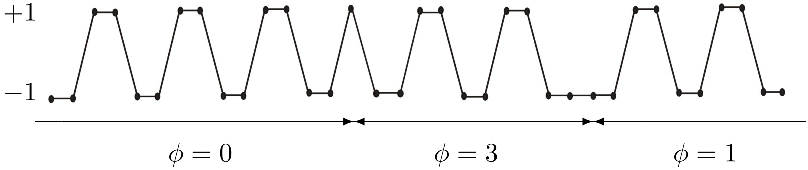


(4 distinct states)

Phase-shift energies
(continued)

The correct order parameter is now the phase $\phi \in \{0, 1, 2, 3\}$ of the ground states

Functions u such that $F_\varepsilon(u) = \min F_\varepsilon(u) + o(1)$ have the form



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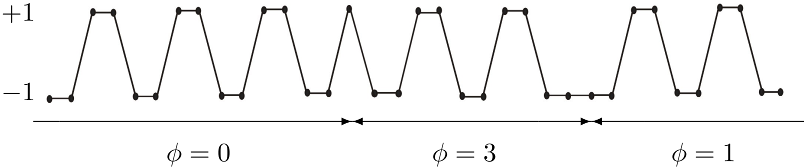

Theorem (Braides-Cicalese 2008 (Lecture Notes))

$$\Gamma\text{-}\lim_{\varepsilon} F_\varepsilon(u) = \sum_{t \in S(\phi)} \psi(\phi^+(t) - \phi^-(t)),$$

where $\phi : I \to \{0, 1, 2, 3\}$ and ψ given by an optimal profile problem.

General Phase-Shift energies

$X \subset \mathbf{R}$ finite space of configurations
For $u : \varepsilon\mathbf{Z}^n \to X$ let $E_\varepsilon(u) = \sum_i \varepsilon^{n-1}\Psi(\{u_{i+j}\}_{j\in\mathbf{Z}^n})$ be such that

H1 (presence of periodic minimizers)
$\exists N, K \in \mathbf{N}$ and $\{v_1, \dots, v_K\}$ Q_N-periodic functions such that
$u \neq v_j$ in $Q_N \Rightarrow E_\varepsilon(u, Q_N) \geq C > 0$
$u = v_j$ in $Q_N \Rightarrow E_\varepsilon(u, Q_N) = 0$

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H2 (incompatibility of minimizers)

$$u = \begin{cases} v_l & \text{in } Q_N \\ v_m & \text{in } Q'_N \end{cases} \quad \Longrightarrow \quad E_\varepsilon(u, Q_N \cup Q'_N) > 0,\ Q_N \cap Q'_N \neq \emptyset$$

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H3 (locality of the energy)
$u = u'$ in $Q_{RN} \Rightarrow |E_\varepsilon(u', Q_N) - E_\varepsilon(u, Q_N)| \leq C_R$ and
$\sum_R C_R R^{n-1} < \infty$

A compactness result

Theorem (Braides-Cicalese)

Compactness:
Let u_ε be such that $E_\varepsilon(u_\varepsilon) \leq C < +\infty$. Then, under $H1, H2$ and $H3$, $\exists A_{1,\varepsilon}, \ldots, A_{K,\varepsilon} \subseteq \mathbf{Z}^N$ (identified with the union of the ε-cubes centered on their points) such that $u_\varepsilon = v_j$ on $A_{j,\varepsilon}$, $A_{j,\varepsilon} \to A_j$ in $L^1_{loc}(\mathbf{R}^n)$ and $A_1, \ldots, A_N$ is a partition of $\mathbf{R}^n$.

Γ-convergence:

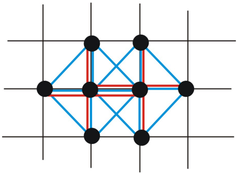
$$\Gamma\text{-}\lim_{\varepsilon} E_\varepsilon(u) = \sum_{i,j} \int_{\partial A_j \cap \partial A_j} \Psi(i, j, \nu)\, d\mathcal{H}^{n-1}$$

A more exotic example

We may consider a $2D$ spin model accounting for NN (nearest neighbors), NNN (next-to-nearest neighbors) and NNNN (next-to-next-to...) interactions

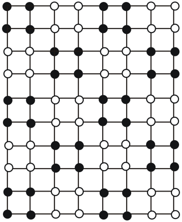
$u : \varepsilon\mathbb{Z}^2 \to \{\pm 1\}$

$$E_\varepsilon(u) = \sum_{NN} \varepsilon u_i u_j + c_1 \sum_{NNN} \varepsilon u_i u_j + c_2 \sum_{NNNN} \varepsilon u_i u_j$$



It is possible to regroup the interactions to study the ground states

For suitable c_1 and c_2, for ε small enough we obtain 4-periodic minimizers as:



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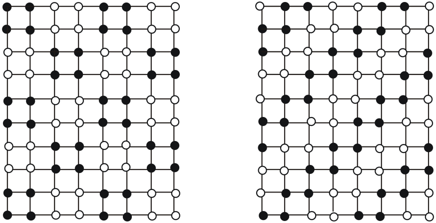

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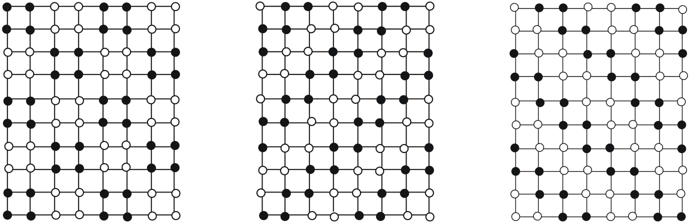

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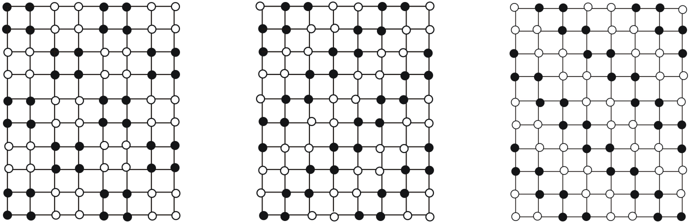
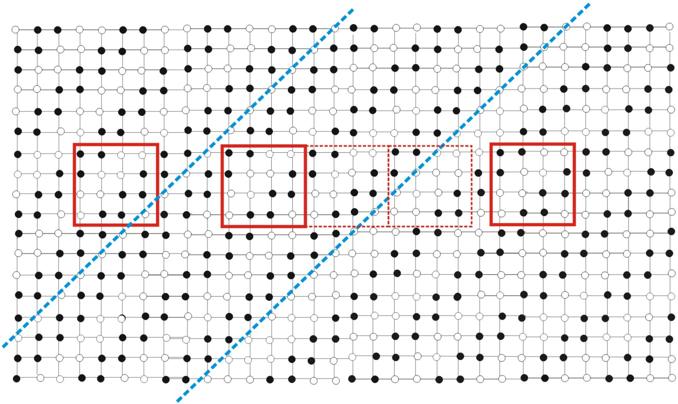


(counting translations they are 16)

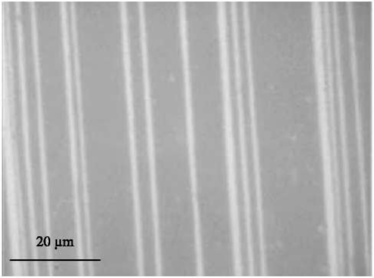


Braides-Cicalese

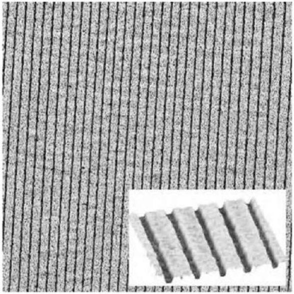
The Γ-limit can be expressed in terms of a **phase variable**.
The limit functional is the energy of the shift transitions in spatially-modulated phases.

Stripe patterns

Formation of stripe patterns during Langmuir-Blodgett condensation



fast process



slow process

Thanks for the attention!