A handbook of Γ-convergence*

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Preface

The notion of Γ-convergence has become, over the more than thirty years after its introduction by Ennio De Giorgi, the commonly-recognized notion of convergence for variational problems, and it would be difficult nowadays to think of any other 'limit' than a Γ-limit when talking about asymptotic analysis in a general variational setting (even though special convergences may fit better specific problems, as Mosco-convergence, two-scale convergence, G- and H-convergence, etc.). This short presentation is meant as an introduction to the many applications of this theory to problems in Partial Differential Equations, both as an effective method for solving asymptotic and approximation issues and as a means of expressing results that are derived by other techniques. A complete introduction to the general theory of Γ-convergence is the by-now-classical book by Gianni Dal Maso [85], while a user-friendly introduction can be found in my book 'for beginners' [46], where also simplified one-dimensional versions of many of the problems in this article are treated.

These notes are addressed to an audience of experienced mathematicians, with some background and interest in Partial Differential Equations, and are meant to direct the reader to what I regard as the most interesting features of this theory. The style of the exposition is how I would present the subject to a colleague in a neighbouring field or to an interested PhD student: the issues that I think will likely emerge again and link a particular question to others are presented with more detail, while I refer to the main monographs or recent articles in the literature for in-depth knowledge of the single issues. Necessarily, many of the proofs are sketchy, and some expert in the field of the Calculus of Variations might shudder at the liberties I will take in order to highlight the main points without entering in details that can be dealt with only in a more ample and dedicated context.

The choice of the issues presented in these notes has been motivated by their closeness to general questions of Partial Differential Equations. Many interesting applications of Γ-convergence that are a little further from that field, and would need a wider presentation of their motivations are only briefly mentioned (for example, the derivation of low-dimensional theories in Continuum Mechanics [108], functionals on BV and SBV [20, 55], the application of Γ-convergence to modeling problems in Mechanics [76, 32], etc.), or not even touched at all (for example, non-convex energies defined on measures [42, 16], stochastic Γ-convergence in a continuous or discrete setting [87, 64], applications of Γ-convergence to Statistical Mechanics [7, 40] and to finite-element methods [73], etc.). The reference to those applications listed here are just meant to be a first suggestion to the interested

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reader.

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1 Introduction

Γ-convergence is designed to express the convergence of minimum problems: it may be convenient in many situations to study the asymptotic behaviour of a family of problems

$$m_\varepsilon = \min\{F_\varepsilon(x) : x \in X_\varepsilon\} \tag{1.1}$$

not through the study of the properties of the solutions x_ε, but by defining a limit energy F_0 such that, as $\varepsilon \to 0$, the problem

$$m_0 = \min\{F_0(x) : x \in X_0\} \tag{1.2}$$

is a 'good approximation' of the previous one; i.e. $m_\varepsilon \to m_0$ and $x_\varepsilon \to x_0$, where x_0 is itself a solution of m_0. This latter requirement must be thought upon extraction of a subsequence if the 'target' minimum problem admits more than a solution. Note that the convergence problem above can also be stated in the reverse direction: given F_0 for which solutions are difficult to characterize, find approximate F_ε whose solution are more at hand. Of course, in order for this procedure to make sense we must require a *equi-coerciveness* property for the energies F_ε; i.e., that we may find a pre-compact *minimizing sequence* (that is, $F_\varepsilon(x_\varepsilon) \leq \inf F_\varepsilon + o(1)$) such that the convergence $x_\varepsilon \to x_0$ can take place.

The existence of such an F_0, the Γ-*limit* of F_ε, is a consequence of the two following conditions:

(i) *liminf inequality*: for every $x \in X_0$ and for every $x_\varepsilon \to x$ we have

$$F_0(x) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon). \tag{1.3}$$

In other words, F_0 is a *lower bound* for the sequence F_ε, in the sense that $F_0(x) \leq F_\varepsilon(x_\varepsilon) + o(1)$ whenever $x_\varepsilon \to x$. If the family F_ε is equi-coercive, then this condition immediately implies one inequality for the minimum problems: if (x_ε) is a minimizing sequence and (upon subsequences) $x_\varepsilon \to x_0$ then

$$\inf F_0 \leq F_0(x_0) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \liminf_{\varepsilon\to 0} \inf F_\varepsilon \tag{1.4}$$

(to be precise, in this argument we take care to start from x_{ε_j} such that $\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \lim_j F_{\varepsilon_j}(x_{\varepsilon_j})$);

(ii) *limsup inequality* or *existence of a recovery sequence*: for every $x \in X_0$ we can find a sequence $\overline{x}_\varepsilon \to x$ such that

$$F_0(x) \geq \limsup_{\varepsilon\to 0} F_\varepsilon(\overline{x}_\varepsilon). \tag{1.5}$$

Note that if (i) holds then in fact $F_0(x) = \lim_{\varepsilon\to 0} F_\varepsilon(\overline{x}_\varepsilon)$, so that the lower bound is sharp. From (1.5) we get in particular that $F_0(x) \geq \limsup_{\varepsilon\to 0} \inf F_\varepsilon$, and since this holds for all x we conclude that

$$\inf F_0 \geq \limsup_{\varepsilon\to 0} \inf F_\varepsilon. \tag{1.6}$$

An F_0 satisfying (1.5) is an *upper bound* for the sequence (F_ε) and its computation is usually related to an *ansatz* leading to the construction of the sequence $\overline{x}_\varepsilon$.

From the two inequalities (1.4) and (1.6) we obtain the convergence of the infima m_ε in (1.1) to the minimum m_0 in (1.2). Not only: we also obtain that every cluster point of a minimizing sequence is a minimum point for F_0. This is the *fundamental theorem of* Γ*-convergence*, that is summarized by the implication

$$\Gamma\text{-convergence} + \text{equi-coerciveness} \implies \text{convergence of minimum problems.}$$

A hidden element in the procedure of the computation of a Γ-limit is the *choice of the right notion of convergence* $x_\varepsilon \to x$. This is actually one of the main issues in the problem: a convergence is not given beforehand and should be chosen in such a way that it implies the equi-coerciveness of the family F_ε. The choice of a weaker convergence, with many converging sequences, makes this requirement easier to fulfill, but at the same time makes the liminf inequality more difficult to hold. In the following we will not insist on the motivation of the choice of the convergence, that in most cases will be a strong L^p-convergence (the choice of a separable metric convergence makes life easier). The reader is anyhow advised that this is one of the main points of the Γ-convergence approach. Another related issue is that of the *correct energy scaling*. In fact, in many cases the given functionals F_ε will not give rise to an equi-coercive family with respect to a meaningful convergence, but the right scaled functionals, e.g., $\varepsilon^{-\alpha}F_\varepsilon$, will turn out to better describe the behaviour of minimum problems. The correct scaling is again usually part of the problem.

Applications of Γ-convergence to Partial Differential Equations can be generally related to the behaviour of the Euler-Lagrange equations of some integral energy. The prototype of such problems can be written as

$$m_\varepsilon = \inf\Big\{\int_\Omega f_\varepsilon(x, Du)\,dx - \int_\Omega \langle g, u\rangle\,dx : u = \varphi \text{ on } \partial\Omega\Big\}. \tag{1.7}$$

In these notes Ω will always stand for an open bounded (sufficiently smooth) subset of $\mathbb{R}^n$, unless otherwise specified. Note that the possibility of defining a Γ-limit related to these problems will not be linked to the properties (or even the existence) of the solutions of the related Euler-Lagrange equations.

It must be noted that the functionals related to (1.7) of which we want to compute the Γ-limit are usually defined on some Sobolev space $W^{1,p}(\Omega;\mathbb{R}^m)$ and can be written as

$$F_\varepsilon^{\varphi,g}(u) = \begin{cases} \displaystyle\int_\Omega f_\varepsilon(x,Du)\,dx - \int_\Omega \langle g,u\rangle\,dx & \text{if } u-\varphi\in W_0^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise,}\end{cases} \tag{1.8}$$

or, equivalently, we can think of $F_\varepsilon^{\varphi,g}$ as defined on $L^p(\Omega;\mathbb{R}^m)$ extended to $+\infty$ outside $W^{1,p}(\Omega;\mathbb{R}^m)$. We will often use this extension to $+\infty$ in the paper, leaving it as understood in most cases.

As written above, the functional $F_\varepsilon^{\varphi,g}$ depends both on the forcing term g and on the boundary datum φ. Suppose now that the growth conditions on f_ε ensure strong pre-compactness of minimizing sequences in L^p (this is usually obtained by Poincaré's inequality and Rellich's Embedding Theorem), and that $g\in L^{p'}(\Omega)$. A first important property, following directly from the definition of Γ-convergence is the *stability of* Γ*-convergence with respect to continuous perturbations*: if G is a continuous function and $F_0=\Gamma\text{-}\lim_{\varepsilon\to0}F_\varepsilon$, then $\Gamma\text{-}\lim_{\varepsilon\to0}(F_\varepsilon+G)=F_0+G$. This implies that we can neglect the forcing term in the computation of the Γ-limit of $F_\varepsilon^{\varphi,g}$ and add it *a posteriori*.

Another, more particular, property is the *compatibility* of boundary conditions, which says that also the boundary condition $u=\varphi$ can be added after computing the Γ-limit of the 'free' energy

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f_\varepsilon(x,Du)\,dx & \text{if } u\in W^{1,p}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise.}\end{cases} \tag{1.9}$$

This property of compatibility of boundary conditions is not always true, but holds for a large class of integrals (see Section 4.2.1). Other compatibility properties are available for other types of functionals, as that of volume constraints for Cahn-Hilliard functionals (see Section 7.2.2).

We have hence seen that to characterize the Γ-limit of $F_\varepsilon^{\varphi,g}$ it will be sufficient to compute the Γ-limit of F_ε. We will see that growth conditions on f_ε ensure that the limit always exists (up to subsequences) and can be represented again through an integral functional $F_0(u)=\int_\Omega f_0(x,Du)\,dx$, independently of the regularity and convexity properties of f_ε. This can be done through a general *localization and compactness procedure* due to De Giorgi [91] (see Sections 3.3 and 4.2). As a consequence we obtain a limit problem

$$m_0=\inf\Big\{\int_\Omega f_0(x,Du)\,dx-\int_\Omega\langle g,u\rangle\,dx : u=\varphi \text{ on } \partial\Omega\Big\}, \tag{1.10}$$

with f_0 *independent of the data* g and φ, and also of Ω.

We will see other classes of energies than the integrals as above for which a general approach is possible showing compactness with respect to Γ-convergence and representation results for the Γ-limit. We now briefly outline the description of some specific problems. Many more examples are included in the text.

A 'classical' example of Γ-limit is for functionals of the type (1.9) when $f_\varepsilon(x,\xi)=f(x/\varepsilon,\xi)$, and f is a fixed function that is 1-periodic in the first variable (i.e., $f(x+e_i,\xi)=f(x,\xi)$ for the elements e_i of the standard basis of $\mathbb{R}^n$). The Γ-limit is also called the *homogenized functional* of the F_ε (see Section 5).

It is interesting to note how some general issues arise in the study of these functionals. First, one notes that the limit energy f_0, that always exists up to subsequences by what remarked above, is *homogeneous*; i.e., $f_0(x,\xi) = f_0(\xi)$ by the vanishing periodicity of f_ε. At the same time, a general property is the *lower-semicontinuity of* Γ*-limits*, that in this case implies that f_0 is *quasiconvex*, or, what is more important, that the value $f_0(\xi)$ at a fixed $\xi \in \mathbb{M}^{m\times n}$ can be expressed as a minimum problem;

$$f_0(\xi) = \min\Big\{\int_{(0,1)^n} f_0(\xi + D\varphi)\,dx : u \in W_0^{1,p}((0,1)^n;\mathbb{R}^m)\Big\}. \tag{1.11}$$

Now the property of convergence of minima provides an *ansatz* for the function f_0, as the value given by the *asymptotic homogenization formula*

$$\begin{aligned} f_{\rm hom}(\xi) &= \lim_{\varepsilon\to 0}\min\Big\{\int_{(0,1)^n} f\Big(\frac{x}{\varepsilon},\xi + D\varphi\Big)\,dx : u \in W_0^{1,p}((0,1)^n;\mathbb{R}^m)\Big\} \\ &= \lim_{T\to+\infty}\frac{1}{T^n}\min\Big\{\int_{(0,T)^n} f\Big(y,\xi + D\varphi\Big)\,dy : u \in W_0^{1,p}((0,T)^n;\mathbb{R}^m)\Big\}. \end{aligned}$$

Note that we have made use both of the compatibility of boundary conditions, and of the fact that we have Γ-convergence on all open sets (and in particular on $(0,1)^n$). The problem is thus reduced to showing that this last limit exists, giving a form for the limit independent of a particular subsequence. The derivation of suitable formulas is one of the recurrent issues in the characterization of Γ-limits.

The problem above is made easier in the convex and scalar case. In particular we can apply it to the study of the behaviour of linear equations

$$\begin{cases} -\displaystyle\sum_{i,j=1}^n D_i\big(a_{ij}\big(\frac{x}{\varepsilon}\big)D_j u\big) = g & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{1.12}$$

with a_{ij} periodic (given the usual boundedness and uniform ellipticity), that are the Euler equations of the minimum problem

$$m_\varepsilon = \inf\Big\{\int_\Omega \sum_{i,j=1}^n a_{ij}\Big(\frac{x}{\varepsilon}\Big)D_j u D_i u\,dx - \int_\Omega gu\,dx : u = \varphi \text{ on } \partial\Omega\Big\}. \tag{1.13}$$

In this case an additional property of Γ-limits can be used: that Γ-limits of quadratic forms are still quadratic forms, so that we obtain $f_{\rm hom}(\xi) = \sum_{i,j} q_{ij}\xi_i\xi_j$ with q_{ij} constant coefficients. Since in this case the limit problem has a unique solution we deduce that the solutions u_ε weakly converge in $H^1(\Omega)$ to the solution of the simpler problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^n q_{ij}D_iD_j u = g & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

Note that the coefficients q_{ij} depend in a non-trivial way on all the coefficients of the matrix a_{ij}, and in particular differ from their averages $\overline{a}_{ij}$, which give the pointwise limit $\int_\Omega \sum_{ij}\overline{a}_{ij}D_iD_j u\,dx$ of $F_\varepsilon(u)$. Even in the simple case $a_{ij} \in \{\alpha\delta_{ij},\beta\delta_{ij}\}$ the characterization of the homogenized matrices is not a trivial task (see Section 5.4). More interesting effects of the form of the Γ-limit are obtained by introducing Dirichlet or Neumann boundary conditions on varying domains (see Section 6).

A feature of Γ-convergence is that it is not linked to a particular assumption of the form of solutions, but relies instead on energetic approaches, tracing the behaviour of energies. In this way we could end up with problems of a different nature than those we started with. One of the first examples of this fact, included in an early paper by Modica and Mortola [124], is the study of the asymptotic behaviour of minimizers of

$$m_\varepsilon = \min\Big\{\int_\Omega (1-|u|)^2\,dx + \varepsilon^2\int_\Omega |Du|^2\,dx : u\in H^1(\Omega),\ \int_\Omega u\,dx = C\Big\}, \tag{1.15}$$

where $|C| < |\Omega|$. This is a problem connected to the Cahn-Hilliard theory of liquid-liquid phase transitions. It is also known as the 'scalar Ginzburg-Landau' energy. In this case it is easily seen by energy considerations that minimizers are weakly pre-compact in $L^2(\Omega)$ and that in that topology the Γ-limit is simply $\int_\Omega W^{**}(u)\,dx$ (W^{**} denotes the convex hull of $W(u) = (1-|u|)^2$). Since $W^{**}(u) = 0$ for $|u|\le 1$ we conclude that minimizers converge weakly in $L^2(\Omega)$ to functions with $|u|\le 1$, and that all such functions arise as limit of minimizing sequences. This is clearly not a satisfactory description, and a more meaningful scaling must be performed, considering the functionals

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega (1-|u|)^2\,dx + \varepsilon\int_\Omega |Du|^2\,dx. \tag{1.16}$$

These functionals are equi-coercive with respect to the strong convergence in $L^1(\Omega)$ and from the first term we may deduce that the limit u of a sequence u_ε equi-bounded in energy satisfies $|u| = 1$ a.e. in Ω. By using compactness arguments for sets of finite perimeter, we actually may deduce that $E = \{u = 1\}$ is a set of finite perimeter. By the invariance properties and representation theorems we may deduce that $F_0(u) = \sigma\mathcal{H}^{n-1}(\Omega\cap\partial E)$. Once the Γ-limit is computed we may show the *compatibility* of the integral constraint thus expressing the limit of u_ε in terms of a set E that minimizes

$$m_0 = \min\{\mathcal{H}^{n-1}(\Omega\cap\partial E) : 2|E| - |\Omega| = C\}. \tag{1.17}$$

This form of the limit problem is common to many phase-transition energies (see Section 7).

The computation of this Γ-limit shows some remarkable features, such as the one-dimensional nature of minimizers (i.e., their value essentially depends only on the distance to ∂E) that allows for a *slicing procedure* (see Section 3.4), the necessity of finding a 'correct scaling' for the energies, the use of Γ-convergence as a selection criterion when we have many solutions to a variational problem (in this case problem (1.15) with $\varepsilon = 0$), and, not least, the 'change of type' in the limit energy that turns from a 'bulk' energy into a 'surface' energy.

Other types of limits present an even more dramatic change of type, such as the (complex) Ginzburg-Landau energies (note the different scaling with respect to the 'scalar' ones)

$$F_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon^2|\log\varepsilon|}\int_\Omega (1-|u|)^2\,dx + \frac{1}{|\log\varepsilon|}\int_\Omega |Du|^2\,dx \tag{1.18}$$

where $\Omega\subset\mathbb{R}^2$ and $u:\Omega\to\mathbb{R}^2$. In this problem the relevant objects for a sequence with $F_\varepsilon(u_\varepsilon)$ equi-bounded for which we have a compactness property are the distributional Jacobians $J(u_\varepsilon)$ that converge (upon subsequences) to measures $\mu = \pi\sum_i d_i\delta_{x_i}$, with $x_i\in\Omega$ and d_i integers, describing the formation of 'vortices'. The limit energy is then defined as $F_0(\{(x_i,d_i)\}_i) = 2\pi\sum_i |d_i|$ (see Section 8.1).

Another type of problems with concentration can be dealt with, as the *Bernoulli free-boundary problem*: look for an open set A and a function u which is the weak solution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus A \\ u = 1 & \text{on } \partial A \\ u = 0 & \text{on } \partial \Omega \\ \dfrac{\partial u}{\partial \nu} = q & \text{on } \partial A, \end{cases}$$

and describe the asymptotic behaviour of such A and u when $q \to +\infty$. These can be considered as the Euler-Lagrange equations of the following family of variational problems depending on a small parameter $\varepsilon > 0$

$$\begin{aligned} S^V_\varepsilon(\Omega) &= \max\Big\{ |\{u \geq 1\}| : u \in H^1_0(\Omega)\,, \quad \int_\Omega |\nabla u|^2 dx \leq \varepsilon^2 \Big\} \\ &= \max\Big\{ |A| : \mathrm{Cap}(A, \Omega) \leq \varepsilon^2 \Big\} \end{aligned} \tag{1.19}$$

(where $\mathrm{Cap}(A, \Omega)$ denotes the capacity of A with respect to Ω) with $q = q_\varepsilon$ a Lagrange multiplier. Again, maximizers show concentration phenomena since maximal sets A will shrink to a point, and can be treated by Γ-convergence (actually, since we have a *maximum* problem we must use the symmetric notion fit for maxima). This asymptotic description can be adapted to treat more general concentration problems (see Section 8.2).

Beside the study of asymptotic properties of minimum problems, we will describe other uses of Γ-convergence. One is the construction of suitable Γ-converging functionals F_ε to a given F_0. This is the case for example of functionals in Computer Vision, such as the Mumford-Shah functional, that are difficult to treat numerically. Their approximation by elliptic functionals (such as the Ambrosio-Tortorelli approximation) provides approximate solutions and numerical schemes, but also many other types of approximating functionals are available (see Section 10). Γ-convergence is particularly suited to such issues, not being linked to a particular form or domain for the energies to be constructed. Moreover, Γ-convergence is also used in the 'justification' of physical theories through a limit procedure. One example is the derivation of low-dimensional theories from three-dimensional elasticity (see Section 9), another one is the deduction of properties in Continuum Mechanics from atomistic potentials (see Section 11).

Notation. We will use standard notation and results for Lebesgue, Sobolev and BV spaces (see [97, 146, 20]). For functionals defined on Sobolev spaces we will identify the distributional derivative Du with its density, so that the same symbol will denote the corresponding L^1-function. We will abandon this identification when we will deal with functions of bounded variation, where the approximate gradient will be denoted by ∇u (Sections 10 and 11).

In the proofs we will often use the letters c and C to indicate unspecified strictly positive constants. A family of objects, even if parameterized by a continuous parameter, will also be often called a 'sequence' not to overburden notation.

2 General theory of Γ-convergence

This first chapter is devoted to an introduction to the main properties of Γ-convergence, in particular to those that are useful in the actual computation of Γ-limits. The reader is referred to [46] for a proof of these results, many of which are nevertheless simple applications of the definitions, and to [85] for a more refined introduction (see also [27]). The original papers on the subject by De Giorgi and collaborators are colected in [92].

In what follows we will usually compute the Γ-limit of a *family* F_ε of functionals indexed by the positive parameter ε. Within the proofs of the lower bounds it is generally useful to invoke some compactness argument, and consider the problem of computing the Γ-limit of a *sequence* $G_j := F_{\varepsilon_j}$. We will give the definitions for the whole family F_ε. The reader can easily rewrite every definition for functionals depending on a discrete parameter j.

2.1 The definitions of Γ-convergence

We have seen in the introduction how a definition of Γ-convergence can be given in terms of properties of the functions along converging sequences. That one will be the definition we will normally use. For the sake of completeness we now consider the most general case of a family $F_\varepsilon : X \to [-\infty, +\infty]$ defined on a *topological space* X. In that case we say that F_ε Γ*-converges* to $F : X \to [-\infty, +\infty]$ at $x \in X$ as $\varepsilon \to 0$ if we have

$$\begin{aligned} F(x) &= \sup_{U\in\mathcal{N}(x)} \liminf_{\varepsilon\to 0} \inf_{y\in U} F_\varepsilon(y) \Big(= \sup_{U\in\mathcal{N}(x)} \sup_{0<\rho} \inf_{\varepsilon<\rho} \inf_{y\in U} F_\varepsilon(y)\Big) \\ &= \sup_{U\in\mathcal{N}(x)} \limsup_{\varepsilon\to 0} \inf_{y\in U} F_\varepsilon(y) \Big(= \sup_{U\in\mathcal{N}(x)} \inf_{0<\rho} \sup_{\varepsilon<\rho} \inf_{y\in U} F_\varepsilon(y)\Big), \end{aligned} \tag{2.1}$$

where $\mathcal{N}(x)$ denotes the family of all neighbourhoods of x in X. In this case we say that $F(x)$ is the Γ-limit of F_ε at x and we write

$$F(x) = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(x). \tag{2.2}$$

If (2.2) holds for all $x \in X$ then we say that F_ε Γ*-converges* to F (on the whole X).

Note that we sometime will consider families of functionals $F_\varepsilon : X_\varepsilon \to [-\infty, +\infty]$, where the domain may depend on ε. In this case it is understood that we identify such functionals with

$$\widetilde{F}_\varepsilon(x) = \begin{cases} F_\varepsilon(x) & \text{if } x \in X_\varepsilon \\ +\infty & \text{if } X \setminus X_\varepsilon, \end{cases}$$

where X is a space containing all X_ε where the convergence takes place.

The definition above makes sense in any topological space and is rather suggestive as it shows as these limits are constructed from the elementary operations of 'sup' and 'inf'. This definition is sometimes handy to prove general properties as compactness or lower semicontinuity; however it is less suited to most direct computations. In applications we will usually deal with metric spaces (as L^p spaces) or metrizable spaces (as bounded subsets of Sobolev spaces or of spaces of measures, equipped with the weak topology), that in addition are also separable. For such spaces the definitions above are simplified as follows.

Theorem 2.1 (equivalent definitions of Γ-convergence) *Let X be a metric space and let $F_\varepsilon, F : X \to [-\infty, +\infty]$. Then the Γ-convergence of F_ε to F at x is equivalent to any of the following conditions*

(a) *we have*

$$F(x)=\inf\Big\{\liminf_{\varepsilon\to 0}F_\varepsilon(x_\varepsilon): x_\varepsilon\to x\Big\}=\inf\Big\{\limsup_{\varepsilon\to 0}F_\varepsilon(x_\varepsilon): x_\varepsilon\to x\Big\};\tag{2.3}$$

(b) *we have*

$$F(x)=\min\Big\{\liminf_{\varepsilon\to 0}F_\varepsilon(x_\varepsilon): x_\varepsilon\to x\Big\}=\min\Big\{\limsup_{\varepsilon\to 0}F_\varepsilon(x_\varepsilon): x_\varepsilon\to x\Big\};\tag{2.4}$$

(c) (sequential Γ-convergence) *we have*

(i) **(liminf inequality)** *for every sequence* (x_ε) *converging to* x

$$F(x)\le\liminf_{\varepsilon\to 0}F_\varepsilon(x_\varepsilon);\tag{2.5}$$

(ii) **(limsup inequality)** *there exists a sequence* (x_ε) *converging to* x *such that*

$$F(x)\ge\limsup_{\varepsilon\to 0}F_\varepsilon(x_\varepsilon);\tag{2.6}$$

(d) *the liminf inequality* (c)(i) *holds and*

(ii)' **(existence of a recovery sequence)** *there exists a sequence* (x_ε) *converging to* x *such that*

$$F(x)=\lim_{\varepsilon\to 0}F_\varepsilon(x_\varepsilon).\tag{2.7}$$

(e) *the liminf inequality* (c)(i) *holds and*

(ii)" **(approximate limsup inequality)** *for all* $\eta>0$ *there exists a sequence* (x_ε) *converging to* x *such that*

$$F(x)\ge\limsup_{\varepsilon\to 0}F_\varepsilon(x_\varepsilon)-\eta.\tag{2.8}$$

Moreover, the Γ*-convergence of* F_ε *to* F *on the whole* X *is equivalent to*

(f) **(limits of minimum problems)** *inequality*

$$\inf_U F\ge\limsup_{\varepsilon\to 0}\inf_U F_\varepsilon\tag{2.9}$$

holds for all open sets U *and inequality*

$$\inf_K F\le\sup\{\liminf_{\varepsilon\to 0}\inf_U F_\varepsilon:\ U\supset K,\ U\ \textit{open}\}\tag{2.10}$$

holds for all compact sets K.

Finally, if d *denotes a distance on* X *and we have a uniform lower bound* $F_\varepsilon(x)\ge -c(1+d(x,x_0)^p)$ *for some* $p>0$ *and* $x_0\in X$*, then the* Γ*-convergence of* F_ε *to* F *on the whole* X *is equivalent to*

(g) **(convergence of Moreau-Yosida transforms)** *we have*

$$\begin{aligned}F(x)&=\sup_{\lambda\ge 0}\liminf_{\varepsilon\to 0}\inf_{y\in X}\{F_\varepsilon(y)+\lambda d(x,y)^p\}\\&=\sup_{\lambda\ge 0}\limsup_{\varepsilon\to 0}\inf_{y\in X}\{F_\varepsilon(y)+\lambda d(x,y)^p\}.\end{aligned}\tag{2.11}$$

Proof The equivalence between (a)–(e) is easily proved, as their equivalence with definition (2.1). In particular note that (2.7) is equivalent to (2.6) since (2.5) holds, and (2.8) is equivalent to (2.7) by the arbitrariness of η and a diagonal argument. The proofs of (f) and (g) are only a little more delicate (see [46] Section 1.4 for details). □

The two inequalities in (c) are usually taken as the definition of Γ-convergence (for first-countable spaces). Note the asymmetry between inequalities with upper and lower limits, due to Γ-convergence being 'unbalanced' towards minimum problems.

Remark 2.2 From the definitions above we can make some immediate but interesting observation:

1) *Stability under continuous perturbations*: if (F_ε) Γ-converges to F and $G : X \to [-\infty, +\infty]$ is a d-continuous function then $(F_\varepsilon + G)$ Γ-converges to $F + G$. This is an immediate consequence of the definition (e.g. from condition (d));

2) Γ*-limit of a constant sequence*: Γ-convergence does not enjoy the property that a constant family $F_\varepsilon = F$ converges to F. In fact if this were true, then from the liminf inequality we would have $F(x) \le \liminf_{\varepsilon\to 0} F(x_\varepsilon)$ for all $x_\varepsilon \to x$; i.e., F would be *lower semicontinuous* (which is not always the case);

3) *Comparison with uniform and pointwise limits*. The previous observation in particular shows that we cannot deduce the existence of the Γ-limit from pointwise convergence. If F_ε converge to G pointwise and $F = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon$ then $F \le G$. However, if F_ε converge uniformly to a *continuous* F on an open set U then we easily see that F_ε Γ-converge to F;

4) *Dependence on the metric*. Note that the value and the existence of the Γ-limit depend on the metric d. If we want to highlight the role of the metric, we can add the dependence on the distance d, and write $\Gamma(d)$-limit, $\Gamma(d)$-convergence, and so on. When two distances d and d' are comparable then from (2.3) we get an inequality between the two Γ-limits (if they exist); in general, note that the existence of the Γ-limit in one metric does not imply the existence of the Γ-limit in the second.

2.1.1 Upper and lower Γ-limits

As for usual limits, it is convenient to define quantities that always exist (as upper and lower limits) and rephrase the existence of a Γ-limit as an equality between those two quantities. Theorem 2.1(a) suggests the following definition of *lower and upper* Γ*-limits*:

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(x) = \inf\Big\{\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) : \ x_\varepsilon \to x\Big\}, \tag{2.12}$$

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(x) = \inf\Big\{\limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) : \ x_\varepsilon \to x\Big\}, \tag{2.13}$$

respectively. In this way the existence of the $\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(x) = F(x)$ is stated as

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(x) = \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(x) = F(x). \tag{2.14}$$

Remark 2.3 If (F_{ε_k}) is a subsequence of (F_ε) then

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon \le \Gamma\text{-}\liminf_{k} F_{\varepsilon_k}, \qquad \Gamma\text{-}\limsup_{k} F_{\varepsilon_k} \le \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon.$$

In particular, if $F = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon$ exists then for every infinitesimal sequence (ε_k) $F = \Gamma\text{-}\lim_k F_{\varepsilon_k}$.

2.2 Γ-convergence and lower semicontinuity

As remarked above the Γ-limit of a constant family $F_\varepsilon = F$ does not converge to F. This is true, however, if F is d-lower semicontinuous. More, the class of lower-semicontinuous functions provides a 'stable class' for Γ-convergence. This is summarized in the following propositions.

Proposition 2.4 (lower semicontinuity of Γ-limits) *The Γ-upper and lower limits of a family F_ε are d-lower semicontinuous functions.*

Proposition 2.5 (Γ-limits and lower-semicontinuous envelopes – relaxation)
1) *The Γ-limit of a constant sequence $F_\varepsilon = F$ is equal to*

$$\overline{F}(x) = \liminf_{y\to x} F(y); \tag{2.15}$$

that is, the lower-semicontinuous envelope *of F, defined as the largest lower-semicontinuous function not greater than F. This operation is also called* relaxation.

2) *The Γ-limit is stable by substituting F_ε by its lower-semicontinuous envelope $\overline{F}_\varepsilon$; i.e., we have*

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon = \Gamma\text{-}\liminf_{\varepsilon\to 0} \overline{F}_\varepsilon, \qquad \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon = \Gamma\text{-}\limsup_{\varepsilon\to 0} \overline{F}_\varepsilon. \tag{2.16}$$

Remark 2.6 If $F_\varepsilon \to F$ pointwise then $\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon \leq F$, and hence, taking both lower-semicontinuous envelopes, also $\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon \leq \overline{F}$.

Remark 2.7 The following observations are often useful in computations:

(i) the supremum of a family (non necessarily finite or countable) of lower-semicontinous functions is itself lower semicontinuous;

(ii) if $f : X \to \overline{\mathbb{R}}$ is bounded from below and $p > 0$; then for all $x \in X$

$$\overline{f}(x) = \sup_{\lambda\geq 0} \inf_{y\in X} \{f(y) + \lambda d(x,y)^p\}.$$

In particular, every lower-semicontinuous function bounded from below is the supremum of an increasing family of Lipschitz functions.

2.3 Computation of Γ-limits

For some classes of functionals, a common compactness procedure has been formalized (see Section 3 below), but in general the computation of the Γ-limit of a family (F_ε) is usually divided into the computation of a separate lower and an upper bound. A *lower bound* is a functional G such that $G \leq \Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon$; i.e., such that

$$G(x) \leq \liminf_j F_{\varepsilon_j}(x_j) \text{ for all } \varepsilon_j \to 0 \text{ and } x_j \to x. \tag{2.17}$$

The lower semicontinuity of Γ-limit allows us to limit our search for lower bounds to the class of lower-semicontinuos G. If we can characterize a large enough family $\mathcal{G}$ of G satisfying (2.17) then the optimal lower bound is obtained as $\overline{G}(x) := \sup\{G(x) : G \in \mathcal{G}\}$. Note that this function is lower semicontinuous, being the supremum of a family of lower semicontinuous functions.

The optimization of the lower bound usually suggests an *ansatz* (or more) to approximate a target element $x \in X$ by a family $\overline{x}_\varepsilon \to x$, thus defining $H(x) := \lim_{\varepsilon\to 0} F_\varepsilon(\overline{x}_\varepsilon)$. By definition

$H \geq \Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon$, so that H is an *upper bound* for the Γ-limit. If we use more *ansätze*, we obtain a family $\mathcal{H}$ and then a candidate optimal upper bound as $\overline{H}(x) = \inf\{H(x) : H \in \mathcal{H}\}$. The existence (and computation) of the Γ-limit is then expressed in the equality $\overline{G} = \overline{H}$.

Remark 2.8 (a density argument) The lower semicontinuity of the Γ-limsup can be used to reduce its computation to a dense class. Let d' be a distance on X inducing a topology which is not weaker than that induced by d; i.e., $d'(x_\varepsilon, x) \to 0$ implies $d(x_\varepsilon, x) \to 0$, and suppose that

(i) $\mathcal{D}$ is a dense subset of X for d';

(ii) we have $\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(x) \leq F(x)$ on $\mathcal{D}$, where F is a function which is continuous with respect to d;

then we have $\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon \leq F$ on X.

To check this, it suffices to note that if $d'(x_k, x) \to 0$ and $x_k \in \mathcal{D}$ then

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(x) &\leq \liminf_k \Big(\Gamma\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(x_k)\Big) \\ &\leq \liminf_k F(x_k) = F(x).\end{aligned}$$

This method will be used very often in the following, without repeating these arguments. It will be applied for example for integral functionals with d the L^p-topology and d' the strong $W^{1,p}$-topology.

2.4 Properties of Γ-convergence

Definition 2.9 *We will say that a sequence* $F_\varepsilon : X \to \overline{\mathbb{R}}$ *is* equi-coercive *if for all* $t \in \mathbb{R}$ *there exists a compact set* K_t *such that* $\{F_\varepsilon \leq t\} \subset K_t$.

If needed, we will also use the more general definition: for all $\varepsilon_j \to 0$ *and* x_j *such that* $F_{\varepsilon_j}(x_j) \leq t$ *there exist a subsequence of* j *(not relabeled) and a converging sequence* x'_j *such that* $F_{\varepsilon_j}(x'_j) \leq F_{\varepsilon_j}(x_j) + o(1)$.

We can state the main convergence result of Γ-convergence, whose proof has been shown in the Introduction.

Theorem 2.10 (fundamental theorem of Γ-convergence) *Let* (X, d) *be a metric space, let* (F_ε) *be a equi-coercive sequence of functions on* X*, and let* $F = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon$*; then*

$$\exists \min_X F = \lim_{\varepsilon\to 0} \inf_X F_\varepsilon. \tag{2.18}$$

Moreover, if (x_ε) *is a precompact sequence such that* $\lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0} \inf_X F_\varepsilon$*, then every limit of a subsequence of* (x_ε) *is a minimum point for* F.

Remark 2.11 (Γ-convergence as a choice criterion) If in the theorem above all functions F_ε admit a minimizer x_ε then, up to subsequences, x_ε converge to a minimum point of F. The converse is clearly not true: we may have minimizers of F which are not limits of minimizers of F_ε. A trivial example is $F_\varepsilon(t) = \varepsilon t^2$ on the real line. This situation is not exceptional; on the contrary: we may often view some functional as a Γ-limit of some particular perturbations, and single out from its minima those chosen as limits of minimizers.

Γ-limits of monotone sequences

We can state some simple but important cases when the Γ-limit does exist, and it is easily computed.

Remark 2.12 (i) If $F_{j+1} \le F_j$ for all $j \in \mathbb{N}$, then

$$\Gamma\text{-}\lim_j F_j = \overline{(\inf_j F_j)} = \overline{(\lim_j F_j)}. \tag{2.19}$$

In fact as $F_j \to \inf_k F_k$ pointwise, by Remark 2.6 we have $\Gamma\text{-}\limsup_j F_j \le \overline{(\inf_k F_k)}$, while the other inequality comes trivially from the inequality $\overline{(\inf_k F_k)} \le \inf_k F_k \le F_j$;

(ii) if $F_j \le F_{j+1}$ for all $j \in \mathbb{N}$, then

$$\Gamma\text{-}\lim_j F_j = \sup_j \overline{F}_j = \lim_j \overline{F}_j; \tag{2.20}$$

in particular if F_j is l.s.c. for every $j \in \mathbb{N}$, then

$$\Gamma\text{-}\lim_j F_j = \lim_j F_j. \tag{2.21}$$

In fact, since $\overline{F}_j \to \sup_k \overline{F}_k$ pointwise,

$$\Gamma\text{-}\limsup_j F_j = \Gamma\text{-}\limsup_j \overline{F}_j \le \sup_k \overline{F}_k$$

by Remark 2.6. On the other hand $\overline{F}_k \le F_j$ for all $j \ge k$ so that the converse inequality easily follows.

Γ-limits and pointwise properties

Proposition 2.13 *If each element of the family (F_ε) is positively homogeneous of degree d (respectively, convex, a quadratic form) then their Γ-limit is F_0 is positively homogeneous of degree d (respectively, convex, a quadratic form).*

Proof The proof follows directly from the definition. The only care is to note that $F : X \to [0, +\infty]$ is a quadratic form if and only if $F(0) = 0$, $F(x + x') + F(x - x') \le 2(F(x) + F(x'))$ and $F(tx) \le t^2 F(x)$ for all $x, x' \in X$ and $t > 0$. □

2.4.1 Topological properties of Γ-convergence

Proposition 2.14 (Compactness) *Let (X, d) be a separable metric space, and for all $j \in \mathbb{N}$ let $F_j : X \to \overline{\mathbb{R}}$ be a function. Then there is an increasing sequence of integers (j_k) such that the Γ-$\lim_k F_{j_k}(x)$ exists for all $x \in X$.*

Proof The proof follows easily from the topological definition (2.1) by extracting a subsequence such that $\inf_{y \in U} F_{j_k}(y)$ converges in a countable basis of open sets U. □

Proposition 2.15 (Urysohn property) *We have Γ-$\lim_j F_j = F$ if and only if for every subsequence (f_{j_k}) there exists a further subsequence which Γ-converges to F.*

Remark 2.16 (Metrizability) Γ-convergence on spaces of lower-semicontinuous functions satisfying some uniform equi-coerciveness properties is metrizable (see [85] Chapter 10 for more detailed statements). This property is often useful, for example in the definition of 'diagonal' Γ-converging sequences.

2.5 Development by Γ-convergence

In many cases a first Γ-limit provides a functional with a lot of minimizers. In this case a further 'Γ-limit of higher order', with a different scaling, may bring more information, as formalized in this result by Anzellotti and Baldo [25] (see also [26]).

Theorem 2.17 (development by Γ-convergence) *Let $F_\varepsilon : X \to \overline{\mathbb{R}}$ be a family of d-equi-coercive functions and let $F^0 = \Gamma(d)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon$. Let $m_\varepsilon = \inf F_\varepsilon$ and $m^0 = \min F^0$. Suppose that for some $\delta_\varepsilon > 0$ with $\delta_\varepsilon \to 0$ there exists the Γ-limit*

$$F^1 = \Gamma(d')\text{-}\lim_{\varepsilon\to 0} \frac{F_\varepsilon - m^0}{\delta_\varepsilon}, \tag{2.22}$$

and that the sequence $F^1_\varepsilon = (F_\varepsilon - m^0)/\delta_\varepsilon$ is d'-equi-coercive for a metric d' which is not weaker than d. Define $m^1 = \min F^1$ and suppose that $m^1 \neq +\infty$; then we have that

$$m_\varepsilon = m^0 + \delta_\varepsilon m^1 + o(\delta_\varepsilon) \tag{2.23}$$

and from all sequences (x_ε) such that $F_\varepsilon(x_\varepsilon) - m_\varepsilon = o(\delta_\varepsilon)$ (in particular this holds for minimizers, if any) there exists a subsequence converging in (X, d') to a point x which minimizes both F^0 and F^1.

3 Localization methods

The abstract compactness properties of Γ-convergence (Proposition 2.14) always ensure the existence of a Γ-limit, upon passing to a subsequence, but in general the limit function defined in this way remains an abstract object, that needs more information to be satisfactorily identified. In applications to minimum problems of the Calculus of Variations, we often encounter functionals as volume or surface integrals depending on the 'local' behaviour of some function u; e.g., *integral functionals* of the form

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(x, Du(x))\,dx \qquad u \in W^{1,p}(\Omega;\mathbb{R}^m) \tag{3.1}$$

(see Section 4) or *free-discontinuity energies*

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(x, Du(x))\,dx + \int_{S(u)} \varphi(x, u^+ - u^-, \nu_u) d\mathcal{H}^{n-1} \tag{3.2}$$

for $u \in SBV(\Omega;\mathbb{R}^m)$ (see Section 10). These energies can be *localized* on open subsets A of Ω; i.e., we may define

$$F_\varepsilon(u, A) = \int_A f_\varepsilon(x, Du(x))\,dx \qquad u \in W^{1,p}(\Omega;\mathbb{R}^m) \tag{3.3}$$

and

$$F_\varepsilon(u, A) = \int_A f_\varepsilon(x, \nabla u(x))\,dx + \int_{S(u)\cap A} \varphi_\varepsilon(x, u^+ - u^-, \nu_u) d\mathcal{H}^{n-1} \tag{3.4}$$

($u \in SBV(\Omega;\mathbb{R}^m)$), respectively.

The essential property defining *local functionals* $F(u,A)$ is that

$$F(u,A) = F(v,A) \qquad \text{if } u = v \text{ a.e. on } A. \tag{3.5}$$

For the rest of the section, with the examples above in mind we suppose to have a sequence of functionals $F_j(u)$ (i.e., we may fix a subsequence (F_{ε_j}) of some (F_ε)) that may be 'localized' by defining $F_j(u,A)$ for all open subsets of some open set Ω.

3.1 Supremum of measures

The localization methods can be used to simplify the computation of lower bounds. A simple but useful observation is that if F_j are local, then, for fixed u, the set function $A \mapsto F'(u,A) := \Gamma$-$\liminf_j F_j(u,A)$ is a super-additive set function on open sets with disjoint compact closures; i.e.,

$$F'(u, A \cup B) \geq F'(u,A) + F'(u,B)$$

if $\overline{A} \cap \overline{B} = \emptyset$, $\overline{A} \cup \overline{B} \subset\subset \Omega$. This inequality directly derives from the definition of Γ-liminf since test functions for $F'(u, A \cup B)$ can be used as test functions for both $F'(u,A)$ and $F'(u,B)$.

If we have a family of lower bounds of the form

$$F'(u,A) \geq G_i(u,A) =: \int_A \psi_i \, d\lambda,$$

where λ is a positive measure and ψ_i are positive Borel functions, then we can apply to $\mu(A) = F'(u,A)$ the following general lemma (see, e.g., [46] Lemma 15.2).

Lemma 3.1 (supremum of a family of measures) *Let μ be a function defined on the family of open subsets of Ω which is super-additive on open sets with disjoint compact closures, let λ be a positive measure on Ω, let ψ_i be positive Borel functions such that $\mu(A) \geq \int_A \psi_i \, d\lambda$ for all open sets A and let $\psi(x) = \sup_i \psi_i(x)$. Then $\mu(A) \geq \int_A \psi \, d\lambda$ for all open sets A.*

3.2 The blow-up technique

The procedure described above highlights that for local functionals the liminf inequality can be itself localized on open subsets. Another type of localization argument is by the 'blow-up' technique introduced by Fonseca and Müller [104] (see also [43]). It applies to the lower estimate along a sequence $F_j(u_j)$ with $u_j \to u$, for energies that for fixed j can be written as measures; i.e., $F_j(u_j, A) = \mu_j(A)$. For the functionals in (3.3), (3.4) we have $\mu_j = f_{\varepsilon_j}(x, Du_j)\mathcal{L}^n$ and $\mu_j = f_{\varepsilon_j}(x, \nabla u_j)\mathcal{L}^n + \varphi_{\varepsilon_j}(x, u_j^+ - u_j^-, \nu_{u_j})\mathcal{H}^{n-1} \llcorner S(u_j)$, respectively.

Step 1*: definition of a limit measure.* If $\liminf_j F_j(u_j)$ is finite (which is the non-trivial case) then we deduce that the family of measures (μ_j) is finite and hence, up to subsequences, we may suppose it converges weakly* to some measure μ. We fix some measure λ (whose choice is driven by the target function u) and consider the decomposition $\mu = (d\mu/d\lambda)\lambda + \mu^s$ in a part absolutely continuous with respect to λ and a singular part. In the case of Sobolev functionals and $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ we expect the limit to be again an integral of the same type and we choose $\lambda = \mathcal{L}^n$; for free-discontinuity energies instead we expect the limit to have an additional term concentrated on $S(u)$, and we may choose $\lambda = \mathcal{L}^n$ or $\lambda = \mathcal{H}^{n-1} \llcorner S(u)$;

Step 2: local analysis. We fix a 'meaningful' x_0; i.e., such that: x_0 is a Lebesgue point for μ with respect to λ; i.e.,

$$\frac{d\mu}{d\lambda}(x_0) = \lim_{\rho\to 0}\frac{\mu(x_0+\rho D)}{\lambda(x_0+\rho D)},$$

where D is a suitable open set properly chosen for the problem. In the case when $\lambda = \mathcal{L}^n$, for example, we may choose $D = (-1/2, 1/2)^n$ so that $\lambda(x_0 + \rho D) = \rho^n$. Note that for all ρ except for a countable set, we have $\mu(\partial(x_0 + \rho D)) = 0$, and hence $\mu(x_0 + \rho D) = \lim_j \mu_j(x_0 + \rho D) = \lim_j F_j(u_j, x_0 + \rho D)$; for $\lambda = \mathcal{H}^{n-1} \llcorner S(u)$ we choose D a cube with one face orthogonal to the normal vector $\nu_u(x_0)$ to $S(u)$ at x_0, so that $\lambda(x_0 + \rho D) = \rho^{n-1} + o(\rho^{n-1})$;

Step 3: blow up. We choose $\rho_j \to 0$ such that we still have

$$\frac{d\mu}{d\lambda}(x_0) = \lim_{j}\frac{F_j(u_j, x_0+\rho_j D)}{\lambda(x_0+\rho_j D)},$$

and change our variables obtaining functionals

$$G_j(v_j, D) = \lambda(x_0+\rho_j D)^{-1} F_j(u_j, x_0+\rho_j D).$$

Up to a proper choice of scaling we may suppose that v_j converges to a meaningful v_0. In the case $\lambda = \mathcal{L}^n$ we have $v_j(x) := \frac{1}{\rho_j}(u_j(x_0 + \rho_j x) - u_j(x_0)) \to \langle\nabla u(x_0), x - x_0\rangle =: v_0(x)$; in the case $\lambda = \mathcal{H}^{n-1} \llcorner S(u)$ we choose $v_j(x) = u_j(x_0 + \rho_j x)$, converging to the function taking the only two values $u^\pm(x_0)$ jumping across the linear hyperplane orthogonal to $\nu_u(x_0)$;

Step 4: local estimates. At this point we only have to estimate the limit of the scaled energies $G_j(v_j, D)$ when v_j converges to a simple target v_0. This is done in different ways depending on the type of energies, obtaining then an inequality $\frac{d\mu}{d\lambda}(x_0) \geq \varphi^\lambda(x_0)$, and some formulas linking $\varphi^\lambda(x_0)$ with the local behaviour of u at x_0. In the case $\lambda = \mathcal{L}^n$, $\varphi^\lambda(x_0) = f_0(x, \nabla u(x_0))$; if $\lambda = \mathcal{H}^{n-1} \llcorner S(u)$ then $\varphi^\lambda(x_0) = \varphi_0(x, u^+(x_0) - u^-(x_0), \nu_u(x_0))$;

Step 5: global estimates. We integrate the local estimates above. For integral functionals, for example, we then conclude that

$$F_0(u,\Omega) = \mu(\Omega) \geq \int_\Omega \frac{d\mu}{d\lambda}\,d\lambda = \int_\Omega f_0(x, \nabla u(x_0))\,dx. \tag{3.6}$$

3.3 A general compactness procedure

If the functionals F_ε Γ-converge to some F, we may look at the behaviour of the localized limit functionals $F(u, A)$ both with respect to u and A, and deduce enough information to give a description of F (e.g., that it is itself an integral in the cases above). The great advantage of this piece of information is that it reduces the computation of a particular Γ-limit within that class to the pointwise characterization of its energy densities. We briefly describe a procedure introduced by De Giorgi [91] that leads e.g. to compactness results for classes of integrals as above (see [85, 54] for more details).

Step 1: compactness The first step is to apply the compactness result (not only on Ω, but also) on a dense countable family $\mathcal{V}$ of open subsets of Ω. For example we can choose as $\mathcal{V}$ the family of all unions of open polyrectangles with rational vertices. Since $\mathcal{V}$ is countable, by a diagonal argument, upon extracting a subsequence we can suppose that all $F_j(\cdot, A)$ Γ-converge for $A \in \mathcal{V}$. We denote by $F_0(\cdot, A)$ the Γ-limit, whose form may *a priori* depend on A.

Step 2: inner regularization The next idea is to consider $F_0(u,\cdot)$ as a set function and prove some properties that lead to some (integral) representation. The first property is 'inner regularity' (see below). In general there may be exceptional sets where this property is not valid; hence, in place of F_0, we define the set function $\overline{F}_0(u,\cdot)$, the *inner-regular envelope* of F_0, on all open subsets of Ω by setting

$$\overline{F}_0(u,A) = \sup\{F_0(u,A') : A' \in \mathcal{V}, A' \subset\subset A\}.$$

In this way, $\overline{F}_0(u,\cdot)$ is automatically *inner regular*; i.e., $\overline{F}_0(u,A) = \sup\{\overline{F}_0(u,A') : A' \subset\subset A\}$. An alternative approach is directly proving that $F_0(u,\cdot)$ can be extended to an inner-regular set function (which is not always the case).

Step 3: subadditivity A crucial property (see Step 4 below) of $\overline{F}_0$ is subadditivity; i.e., that

$$\overline{F}_0(u,A\cup B) \le \overline{F}_0(u,A) + \overline{F}_0(u,B)$$

(which is enjoyed for example by non-negative integral functionals). This is usually the most technical part to prove that may involve a complex analysis of the behaviour of the functionals F_j.

Step 4: measure property The next step is to prove that $\overline{F}_0(u,\cdot)$ is the restriction of a finite Borel measure to the open sets of Ω. To this end it is customary to use the *De Giorgi Letta Measure Criterion* (see below).

Step 5: integral representation Since $\overline{F}_0(u,\cdot)$ is (the restriction of) a measure we may write it as an integral. For example, in the case of Sobolev spaces if $\overline{F}_0(u,\cdot)$ is absolutely continuous with respect to the Lebesgue measure, then it can be written as

$$\overline{F}_0(u,A) = \int_A f_u(x)\,dx;$$

subsequently, by combining the properties of $\overline{F}_0$ as a set function with those with respect to u we deduce that indeed we may write $f_u(x) = f(x, Du(x))$.

This step is usually summarized in separate integral representation theorems that state that a local functional $F_0(u,A)$ satisfying suitable growth conditions, that is lower-semicontinuous in u and that is a measure in A, can be written as an integral functional (see Section 4.1.1 for the case of functionals on Sobolev spaces).

Step 6: recovery of the Γ*-limit* The final step is to check that, taking $A = \Omega$, indeed $\overline{F}_0(u,\Omega) = F_0(u,\Omega)$ so that the representation we have found holds for the Γ-limit (and not for its 'inner regularization'). This last step is an inner regularity result on Ω and for some classes of problems is sometime directly proved in Step 2.

Remark 3.2 (fundamental estimate) The subadditivity property in Step 3 is often derived by showing that the sequence F_j satisfies the so-called *fundamental estimate.* In the case of Γ-limits with respect to the L^p-convergence this is stated as follows: for all A, A', B open subsets of Ω with $A' \subset\subset A$, and for all $\sigma > 0$, there exists $M > 0$ such that for all u, v in the domain of F_j one may find a function w such that $w = u$ in A', $w = v$ on $B \setminus A$ such that

$$\begin{aligned} F_j(w, A'\cup B) \quad &\le \quad (1+\sigma)(F_j(u,A) + F_j(v,B)) \\ &\qquad + M\int_{(A\cap B)\setminus A'} |u-v|^p\,dx + \sigma\,, \end{aligned} \tag{3.7}$$

and

$$\|u - w\|_{L^p} + \|v - w\|_{L^p} \le C\|u - v\|_{L^p}.$$

In the case of functionals on Sobolev spaces w is usually of the form $\varphi u + (1 - \varphi)v$ with $\varphi \in C_0^\infty(A; [0,1])$, $\varphi = 1$ in A', but can also be constructed differently (e.g., solving some auxiliary minimum problem in $A \setminus A'$ with data 0 on ∂A and 1 on $\partial A'$).

The subadditivity of $\overline{F}_0$ in Step 3 above is easily proved from this property by directly using the definition of Γ-convergence.

The proof of the following characterization of measures can be found in [95] (see also [54])

Lemma 3.3 (De Giorgi Letta Measure Criterion) *If a set function α defined on all open subsets of a set Ω satisfies*

(i) $\alpha(A) \le \alpha(B)$ is $A \subset B$ (α is increasing);

(ii) $\alpha(A) = \sup\{\alpha(B) : B \subset\subset A\}$ (α is inner regular);

(iii) $\alpha(A \cup B) \le \alpha(A) + \alpha(B)$ (α is subadditive);

(iv) $\alpha(A \cup B) \ge \alpha(A) + \alpha(B)$ if $A \cap B = \emptyset$ (α is superadditive),

then α is the restriction to all open sets of Ω of a regular Borel measure.

3.4 The 'slicing' method

In this section we describe a fruitful method to recover the liminf inequality for Γ-limits through the study of one-dimensional problems by a 'sectioning' argument. An example of application of this procedure will be given by the proof of Theorem 7.3.

The main idea of this method is the following. Let F_ε be a sequence of functionals defined on a space of functions with domain a fixed open set $\Omega \subset \mathbb{R}^n$. Then we may examine the behaviour of F_ε on one-dimensional sections as follows: for each $\xi \in S^{n-1}$ we consider the hyperplane

$$\Pi_\xi := \{z \in \mathbb{R}^n : \langle z, \xi\rangle = 0\} \tag{3.8}$$

passing through 0 and orthogonal to ξ. For each $y \in \Pi_\xi$ we then obtain the one-dimensional set

$$\Omega_{\xi,y} := \{t \in \mathbb{R} : y + t\xi \in \Omega\}, \tag{3.9}$$

and for all u defined on Ω we define the one-dimensional function

$$u_{\xi,y}(t) = u(y + t\xi) \tag{3.10}$$

defined on $\Omega_{\xi,y}$. We may then give a lower bound for the Γ-liminf of F_ε by looking at the limit of some functionals 'induced by F_ε' on the one-dimensional sections.

Step 1. *We 'localize' the functional F_ε highlighting its dependence on the set of integration.* This is done by defining functionals $F_\varepsilon(\cdot, A)$ for all open subsets $A \subset \Omega$ as in (3.3) and (3.4).

Step 2. *For all $\xi \in S^{n-1}$ and for all $y \in \Pi_\xi$, we find functionals $F_\varepsilon^{\xi,y}(v, I)$, defined for $I \subset \mathbb{R}$ and $v \in L^1(I)$, such that setting*

$$F_\varepsilon^\xi(u, A) = \int_{\Pi_\xi} F_\varepsilon^{\xi,y}(u_{\xi,y}, A_{\xi,y})\, d\mathcal{H}^{n-1}(y) \tag{3.11}$$

we have $F_\varepsilon(u, A) \ge F_\varepsilon^\xi(u, A)$. This is usually an application of Fubini's Theorem.

Step 3. *We compute the* $\Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon^{\xi,y}(v,I) = F^{\xi,y}(v,I)$ *and define*

$$F^\xi(u,A) = \int_{\Pi_\xi} F^{\xi,y}(u_{\xi,y}, A_{\xi,y})\, d\mathcal{H}^{n-1}(y)\,. \tag{3.12}$$

Step 4. *Apply Fatou's Lemma.* If $u_\varepsilon \to u$, we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, A) &\geq \liminf_{\varepsilon\to 0^+} F_\varepsilon^\xi(u_\varepsilon, A) = \liminf_{\varepsilon\to 0^+} \int_{\Pi_\xi} F_\varepsilon^{\xi,y}((u_\varepsilon)_{\xi,y}, A_{\xi,y})\, d\mathcal{H}^{n-1}(y)\\
&\geq \int_{\Pi_\xi} \liminf_{\varepsilon\to 0^+} F_\varepsilon^{\xi,y}((u_\varepsilon)_{\xi,y}, A_{\xi,y})\, d\mathcal{H}^{n-1}(y)\\
&\geq \int_{\Pi_\xi} F^{\xi,y}(u_{\xi,y}, A_{\xi,y})\, d\mathcal{H}^{n-1}(y) = F^\xi(u,A)\,.
\end{aligned}$$

Hence, we deduce that $\Gamma\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u,A) \geq F^\xi(u,A)$ for all $\xi \in S^{n-1}$.

Step 5. *Describe the domain of the limit.* From the estimates above and some directional characterization of function spaces we deduce the domain of the Γ-lim inf. For example, if $F^\xi(u,A) \geq \int_A |\langle Du, \xi\rangle|^p\, dx$ for $p > 1$ for ξ in a basis of $\mathbb{R}^n$ we deduce that the limit is a $W^{1,p}$-function.

Step 6. *Optimize the lower estimate.* We finally deduce that $F'(u,A) \geq \sup\{F^\xi(u,A) : \xi \in S^{n-1}\}$. If the latter supremum is obtained also by restricting to a countable family $(\xi_i)_i$ of directions, if possible use Lemma 3.1 to get an explicit form of a lower bound.

4 Local integral functionals on Sobolev spaces

The most common functionals encountered in the treatment of Partial Differential Equations are integral functionals defined on some subset of a Sobolev space; i.e., of the form (we limit to first derivatives)

$$F_\varepsilon(u) = \int_\Omega g_\varepsilon(x, u, Du)\, dx.$$

It must be previously noted that if we can isolate the explicit dependence on u; i.e., if we can write

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(x, Du)\, dx + \int_\Omega h(x,u)\, dx,$$

and h is a Carathéodory function satisfying a growth condition of the form $|h(x,u)| \leq c(\alpha(x) + |u|^p)$ with $\alpha \in L^1(\Omega)$, then the second integral is a continuous functional on L^p, and can therefore be dropped when computing the Γ-limit in the L^p topology. Note that this observation in particular applies to the continuous linear perturbations

$$u \mapsto \int_\Omega \langle \psi, u\rangle\, dx,$$

where $\psi \in L^{p'}(\Omega; \mathbb{R}^m)$.

4.1 A prototypical compactness theorem

Integral functionals of the form

$$F(u) = \int_\Omega f(x, Du)\, dx \qquad u \in W^{1,p}(\Omega; \mathbb{R}^m) \tag{4.1}$$

$$c_1(|\xi|^p - 1) \le f(x, \xi) \le c_2(|\xi|^p + 1) \tag{4.2}$$

($f : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ Borel function) with Ω a bounded open subset of $\mathbb{R}^n$, $p > 1$ and $c_i > 0$, represent a 'classical' class of energies for which the general compactness procedure in Section 3.3 can be applied. For minimum problems of the form

$$\min_{u=\varphi \text{ on } \partial\Omega} F(u)\,, \qquad \min\Big\{F(u) - \int_\Omega \langle \psi, u\rangle\, dx : u \in W^{1,p}(\Omega; \mathbb{R}^m)\Big\} \tag{4.3}$$

(with the usual conditions of Ω connected and $\int_\Omega \psi\, dx = 0$ in the second case) we easily infer that minimizers satisfy some $W^{1,p}$ bound depending only on c_i, φ or ψ. Hence the correct convergence to use in the computation of the Γ-limit is the weak convergence in $W^{1,p}$ or, equivalently by Rellich's Theorem, the strong L^p convergence.

Remark 4.1 (growth conditions) The growth conditions (4.2) can be relaxed to cover more general cases. The case $p = 1$ can be dealt with in the similar way, but in this case the natural domain for these energies is the space of functions with bounded variation $BV(\Omega; \mathbb{R}^m)$ on which the relaxation of F take a more complex form (see [20]), whose description goes beyond the scopes of this presentation. Moreover, we may also deal with conditions of the form

$$c_1(|\xi|^p - 1) \le f(x, \xi) \le c_2(|\xi|^q + 1) \tag{4.4}$$

if the gap between p and q is not too wide. The methods in this chapter work exactly the same if $q < p^*$ by the Sobolev embedding theorem, but wider gaps can also be treated (see the book by Fonseca and Leoni [103]). Outside the convex context, a long-standing conjecture that I learned from De Giorgi is that it should be sufficient to deal with the class of f that satisfy a condition

$$c_1(\psi(\xi) - 1) \le f(x, \xi) \le c_2(\psi(\xi) + 1) \tag{4.5}$$

where ψ is such that $\Psi(u) = \int_\Omega \psi(Du)\, dx$ is lower semicontinuous. This is a completely open problem, in particular because in general we do not have a characterization of good dense sets in the domain of Ψ.

Theorem 4.2 (Compactness of local integral energies) *Let $p > 1$ and let f_j be a sequence of Borel functions uniformly satisfying the growth condition* (4.2). *Then there exist a subsequence of f_j (not relabeled) and a Carathéodory function f_0 satisfying the same condition* (4.2) *such that, if we set*

$$F_0(u, A) = \int_A f_0(x, Du)\, dx \qquad u \in W^{1,p}(A; \mathbb{R}^m), \tag{4.6}$$

and the localized functionals defined by

$$F_j(u, A) = \int_A f_j(x, Du)\, dx \qquad u \in W^{1,p}(A; \mathbb{R}^m), \tag{4.7}$$

then $F_j(\cdot, A)$ converges to $F_0(\cdot, A)$ with respect to the $L^p(A; \mathbb{R}^m)$ convergence for all A open subsets of Ω.

Proof To prove the theorem above the steps in Section 3.3 can be followed. In this case the Γ-limit can be proved to be inner regular (this can be done by using the argument in Section 4.2.1 below, which also proves the subadditivity property) and we may use the integral representation result in the next section. □

Remark 4.3 (convergence of minimum problems with Neumann boundary conditions) In the class above we immediately obtain the convergence of minimum problems as the second one in (4.3). The only thing to check is equi-coerciveness, which follows from the Poincaré-Wirtinger inequality. Note in fact that we may reduce to the case $\int u\,dx = 0$ upon a translation of a constant vector.

4.1.1 An integral representation result

The prototype of the integral representation results is the following classical theorem in Sobolev spaces due to Buttazzo and Dal Maso (see [66, 54]).

Theorem 4.4 (Sobolev integral representation theorem) *If $F = F(u, A)$ is a functional defined for $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and A open subset of Ω satisfying*

(i) (lower semicontinuity) $F(\cdot, A)$ *is lower semicontinuous with respect to the L^p convergence;*

(ii) (growth estimate) $0 \le F(u, A) \le C \int_A (1 + |Du|^p)\,dx$*;*

(iii) (measure property) $F(u, \cdot)$ *is the restriction of a regular Borel measure;*

(iv) (locality) F *is local:* $F(u, A) = F(v, A)$ *if* $u = v$ *a.e. on* A*,*

then there exists a Borel function f such that

$$F(u, A) = \int_A f(x, Du)\,dx.$$

Remark 4.5 Note that $f(x, \xi)$ can be obtained by derivation for all $\xi \in \mathbb{M}^{m\times n}$ and a.a. $x \in \Omega$ as

$$f(x, \xi) = \lim_{\rho\to 0^+} \frac{F(u_\xi, B_\rho(x))}{|B_\rho(x)|}, \tag{4.8}$$

where $u_\xi(y) = \xi y$.

A remark by Dal Maso and Modica shows that this formula can be used to give an indirect description of f_0 from f_j in Theorem 4.2, by introducing the functions

$$M_j(x, \xi, \rho) := \min_{w\in W_0^{1,p}(B_\rho(x);\mathbb{R}^m)} \int_{B_\rho(x)} f_j(y, \xi + \nabla w)\,dy\,.$$

Then F_j Γ-converges to F_0 if and only if

$$f_0(x, \xi) = \liminf_{\rho\to 0} \liminf_j \frac{M_j(x, \xi, \rho)}{|B_\rho(x)|} = \limsup_{\rho\to 0} \limsup_j \frac{M_j(x, \xi, \rho)}{|B_\rho(x)|}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{M}^{m\times n}$.

This fact can be proved by a blow-up argument, upon using the argument in Section 4.2.1 to match the boundary conditions.

Remark 4.6 (Other classes of integral functionals) The compactness procedure above can also be applied to energies of the form

$$F_\varepsilon(u) = \int_\Omega f_\varepsilon(x, u, Du)\, dx, \tag{4.9}$$

with f_ε satisfying analogous growth conditions. It must be remarked that more complex integral-representation results must be used, for which we refer to the book by Fonseca and Leoni [103].

4.1.2 Convexity conditions

It is useful to note that the Borel function f_0 in the compactness theorem enjoys some convexity properties in the gradient variable due to the fact that the Γ-limit is a lower semicontinuous functional (more precisely sequentially lower semicontinuous with respect to the weak convergence in $W^{1,p}$). The following theorem by Acerbi and Fusco [4] shows that the function $f_0(x,\cdot)$ satisfies Morrey's *quasiconvexity* condition [127].

Theorem 4.7 (Quasiconvexity and lower semicontinuity) *Let f be a Borel function satisfying* (4.2) *and let $F(u) = \int_\Omega f(x, Du)\, dx$ be defined on $W^{1,p}(\Omega;\mathbb{R}^m)$. Then F is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ if and only if $f(x,\cdot)$ is* quasiconvex*; i.e., we have*

$$|D| f(x,\xi) = \min\Big\{ \int_D f(x, \xi + D\varphi(y))\, dy : \varphi \in W^{1,p}_0(D;\mathbb{R}^m) \Big\}, \tag{4.10}$$

where D is any open subset of $\mathbb{R}^n$.

Remark 4.8 1) In the scalar case $m = 1$ or for curves $n = 1$ quasiconvexity reduces to the usual convexity and (4.10) to Jensen's inequality; in all other cases convexity is a restrictive condition. A family of non-convex quasiconvex functions is that of *polyconvex* functions (i.e., convex functions of the minors of ξ; for example, $\det \xi$ if $m = n$) (see [29, 82] and[54] Chapter 5).

2) Quasiconvexity implies convexity on rank-1 lines, in particular in the coordinate directions. This, together with the growth condition (4.2) implies a locally uniform Lipschitz condition on f_0 with a Lipschitz constant growing like $|\xi|^{p-1}$ (see [54] Remark 5.15).

3) Condition (4.10) can be equivalently stated with the more general condition of φ periodic (and D a periodicity set for φ).

4) The integrand of the lower-semicontinuous envelope of a functional as in (4.1) is given by the *quasiconvex envelope* Qf of f

$$Qf(x,\xi) = \min\Big\{ |D|^{-1} \int_D f(x, \xi + D\varphi(y))\, dy : \varphi \in W^{1,p}_0(D;\mathbb{R}^m) \Big\} \tag{4.11}$$

(see e.g. [4, 82, 54]). Note that here x acts as a parameter.

4.2 Useful technical results

4.2.1 The De Giorgi method for matching boundary values

From the Γ-convergence of functionals F_j to F_0 as in Theorem 4.2 we do not immediately deduce the convergence of minimum problems with Dirichlet boundary conditions. In fact, to do so we must prove the *compatibility* of the condition $u=\varphi$ on $\partial\Omega$; i.e., that the functionals

$$F_j^{\varphi}(u)=\begin{cases}F_j(u)=\displaystyle\int_\Omega f_j(x,Du)\,dx & \text{if } u=\varphi \text{ on } \partial\Omega\\ +\infty & \text{otherwise}\end{cases}$$

Γ-converge to F_0^{φ} analogously defined. The liminf inequality is an immediate consequence of the stability of the boundary condition under weak $W^{1,p}$-convergence. It remains to prove an approximate limsup inequality; i.e., that for all $u\in\varphi+W_0^{1,p}$ and fixed $\eta>0$ there exists a sequence $u_j\in\varphi+W_0^{1,p}$ such that $\limsup_j F_j(u_j)\le F_0(u)+\eta$.

The proof of this fact will use a method introduced by De Giorgi [91]. From the Γ-convergence of the F_j we know that there exists a sequence $v_j\to u$ such that $F_0(u)=\lim_j F_j(v_j)$; in particular the $W^{1,p}$-norms of v_j are equi-bounded. We wish to modify v_j close to the boundary. For simplicity suppose that the Lebesgue measure of $\partial\Omega$ is zero and all f_j are positive. We consider functions $w_j=\phi v_j+(1-\phi)u$, where $\phi=0$ on $\partial\Omega$ and $\phi(x)=1$ if $\operatorname{dist}(x,\partial\Omega)>\eta$. Such functions tend to u and satisfy the desired boundary condition. However, we may obtain only the estimate

$$F_j(w_j)\le F_j(v_j)+C\int_{\Omega^\eta}\Big(|Du|^p+|Dv_j|^p+\frac{1}{\eta^p}|u-v_j|^p\Big)\,dx, \tag{4.12}$$

(where $\Omega^\eta=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)<\eta\}$), and passing to the limit we get

$$\limsup_j F_j(w_j)\le F_0(u)+C\int_{\Omega^\eta}(|Du|^p+|Dv_j|^p)\,dx, \tag{4.13}$$

which is not sufficient to conclude since we do not know if we can choose $\int_{\Omega^\eta}|Dv_j|^p\,dx$ arbitrarily small. Note that this would be the case if $(|Dv_j|^p)$ were an equi-integrable sequence (see Section 4.2.2 below).

This choice of w_j can be improved by fixing an integer N and considering ϕ_k (for $k=1,\dots,N$) such that $\phi_k=0$ on $\Omega^{(N+k-1)\eta/N}$, $\phi_k=1$ on $\Omega\setminus\Omega^{(N+k)\eta/N}$ and $|D\phi_k|\le N/\eta$. With this ϕ_k in place of ϕ we define w_j^k and obtain for each j the estimate

$$F_j(w_j^k)\le F_j(v_j,\Omega\setminus\Omega^\eta)+CF_j(u,\Omega^{2\eta})+C\int_{D_\eta^k}\Big(|Dv_j|^p+\frac{N^p}{\eta^p}|u-v_j|^p\Big)\,dx, \tag{4.14}$$

where $D_\eta^k=\Omega^{(N+k)\eta/N}\setminus\Omega^{(N+k-1)\eta/N}$. Now, summing up for $k=1,\dots,N$, we obtain

$$\sum_{k=1}^N F_j(w_j^k)\le NF_j(v_j,\Omega\setminus\Omega^\eta)+NCF_j(u,\Omega^{2\eta})+C\int_{\Omega^{2\eta}\setminus\Omega^\eta}(|Dv_j|^p+\frac{N^p}{\eta^p}|u-v_j|^p)\,dx, \tag{4.15}$$

and we may choose $k=k_j$ such that

$$F_j(w_j^k)\le F_j(v_j,\Omega\setminus\Omega^\eta)+C\,F_j(u,\Omega^\eta)+\frac{C}{N}\int_{\Omega^{2\eta}\setminus\Omega^\eta}\Big(|Dv_j|^p+\frac{N^p}{\eta^p}|u-v_j|^p\Big)\,dx. \tag{4.16}$$

If we define $u_j=w_j^{k_j}$ we then obtain

$$\limsup_j F_j(u_j)\le F_0(u)+C\int_{\Omega^{2\eta}}(1+|Du|^p)\,dx+\frac{C}{N}\sup_j\int_\Omega|Dv_j|^p\,dx, \tag{4.17}$$

which proves the approximate limsup inequality.

Remark 4.9 1) The idea of the method above consists in finding suitable 'annuli', where the energy corresponding to $|Du_j|^p$ does not concentrate, and then taking *cut-off* functions with gradient supported in those annuli to 'join' u_j and u through a convex combination. An alternative way to construct such annuli would be to consider sets that are not charged by the weak* limit of the measures $\mu_j = |Dv_j|^p \mathcal{L}^n$ (this method is for instance used in the book by Evans [97]).

2) (*Proof of the inner regularity and fundamental estimate*) We can use the method above to prove the inner regularity in Step 2 of Section 3.3. Note that we have $F_0(u, B) \le c_2 \int_B (1+|Du|^p)\,dx$ for all B. Fix an open set A, $\eta > 0$ and set $A' = A \setminus A^\eta$ in the notation above. Then from inequality (4.16) with A in the place of Ω and (u_j) an optimal sequence for $F_0(u, A')$ we have

$$F_0(u, A) \le \limsup_j F_j(u_j, A) \le F_0(u, A') + C \int_{A^{2\eta}} (1 + |Du|^p)\,dx + \frac{C}{N} \tag{4.18}$$

so that we obtain the inner regularity of F_0 by the arbitrariness of η and N.

In a similar fashion we may use the same argument to 'join' recovery sequences on sets A' and B and prove the fundamental estimate (Remark 3.2), and hence the subadditivity property of Step 3 of Section 3.3.

3) The method described above is very general and can be extended also to varying domains (see, e.g., Lemma 6.1). In the scalar case $m = 1$ and with a fixed Ω a simpler truncation argument can be used (see [46] Section 2.7).

Remark 4.10 (convergence of minimum problems with Dirichlet boundary conditions) The result above immediately implies the convergence of problems with Dirichlet boundary conditions from the Γ-convergence of the energies, as remarked in the Introduction.

4.2.2 An equi-integrability lemma

As remarked in the proof exhibited in the previous section, sequences of functions with $|Du_j|^p$ equi-integrable are often easier to handle. A method introduced by Acerbi and Fusco [4] shows that this is essentially always the case, as stated by the following theorem due to Fonseca, Müller and Pedregal [105].

Theorem 4.11 (equivalent sequences with equi-integrability properties) *Let (u_j) be a sequence weakly converging to u in $W^{1,p}(\Omega;\mathbb{R}^m)$; then there exist a subsequence of (u_j), not relabeled, and a sequence (w_j) with $(|Dw_j|^p)$ equi-integrable such that*

$$\lim_j |\{x \in \Omega : u_j(x) \ne w_j(x)\}| = 0$$

and still converging to u.

Thanks to this result we can limit our analysis to such (w_j) with $(|Dw_j|^p)$ equi-integrable. In fact, set $\Omega_j = \{u_j = w_j\}$ and note that $Du_j = Dw_j$ a.e. on Ω_j and $\lim_j \int_{\Omega\setminus\Omega_j} |Dw_j|^p\,dx = 0$ by the equi-integrability property. Then we have (for simplicity assume $f_j \ge 0$)

$$\begin{aligned}
\liminf_j \int_\Omega f_j(x, Dw_j)\,dx &\le \liminf_j \int_{\Omega_j} f_j(x, Dw_j)\,dx + \lim_j \int_{\Omega\setminus\Omega_j} c_2(1 + |Dw_j|^p)\,dx \\
&= \liminf_j \int_{\Omega_j} f_j(x, Dw_j)\,dx = \liminf_j \int_{\Omega_j} f_j(x, Du_j)\,dx \\
&\le \liminf_j \int_\Omega f_j(x, Du_j)\,dx,
\end{aligned}$$

with c_2 given by (4.2), so that if we give a lower bound for such (w_j) we obtain a lower bound also for a general (u_j).

4.2.3 Higher-integrability results

When using the characterization of Γ-limits of integral functionals via Moreau-Yosida transforms with respect to the $L^p(\Omega;\mathbb{R}^m)$ convergence it is often useful to resort to some regularity properties of solutions of variational problems stated as follows.

Theorem 4.12 (Meyers regularity theorem) *Let f be as in* (4.2)*, let A be a bounded open set with smooth boundary and $\overline{u} \in C^\infty(\overline{A};\mathbb{R}^m)$. Then there exists $\eta = \eta(c_1, c_2, A, \overline{u}) > 0$ such that for all $\lambda > 0$ any solution u_λ of*

$$\min\Big\{\int_A f(x, Du + D\overline{u})\,dx + \lambda\int_A |u|^p\,dx : u \in W^{1,p}(A;\mathbb{R}^m)\Big\} \tag{4.19}$$

belongs to $W^{1,p+\eta}(A;\mathbb{R}^m)$, and there exists $C = C(\lambda, c_1, c_2, \Omega, \overline{u})$ such that

$$\|u_\lambda\|_{W^{1,p+\eta}(A;\mathbb{R}^m)} \leq C. \tag{4.20}$$

This theorem shows that for fixed λ minimizers of the Moreau-Yosida transforms related to a family F_ε as in (4.1) (see (2.11)) satisfy a uniform bound (4.20) independent of ε.

4.3 Convergence of quadratic forms

From the stability property of quadratic forms (Proposition 2.13) we have the following particular case of the compactness Theorem 4.2 (for simplicity we treat the scalar case $n = 1$ only).

Theorem 4.13 (compactness of quadratic forms) *Let $A_j : \Omega \to \mathbb{M}^{m\times n}$ be a sequence of symmetric matrix-valued measurable functions, and suppose that $\alpha, \beta > 0$ exist such that $\alpha\,\mathrm{Id} \leq A_j \leq \beta\,\mathrm{Id}$ for all j. Then there exist a subsequence of A_j, not relabeled, and a matrix-valued function A satisfying the same conditions, such that*

$$\int_\Omega \langle A(x)Du, Du\rangle\,dx = \Gamma\text{-}\lim_j \int_\Omega \langle A_j(x)Du, Du\rangle\,dx \tag{4.21}$$

with respect to the $L^2(\Omega)$-convergence, for all $u \in H^1(\Omega)$.

As a consequence of this theorem we have a result of convergence for the related Euler equations. Note that all functionals are strictly convex, so that the solutions to minimum problems with Dirichlet boundary conditions are unique.

Corollary 4.14 (G-convergence) *If A_j, A are as above then for all $\varphi \in H^1(\Omega)$ and $f \in L^2(\Omega)$ the solutions u_j of*

$$\begin{cases} -\mathrm{div}(A_j Du_j) = f & \textit{in } \Omega \\ u_j - \varphi \in H^1_0(\Omega) \end{cases}$$

weakly converge in $H^1(\Omega)$ to the solution u of

$$\begin{cases} -\mathrm{div}(ADu) = f & \textit{in } \Omega \\ u - \varphi \in H^1_0(\Omega). \end{cases}$$

*This is usually referred to as the G-*convergence *of the differential operators $G_j(v) = -\mathrm{div}(A_j Dv)$.*

4.4 Degenerate limits

If the growth conditions of order p are not uniformly satisfied, then the limit of a family of integral functionals may take a different form, and in particular lose the locality property. In this section we give two examples of such a case.

4.4.1 Functionals of the sup norm

A simple example of a family of functionals not satisfying uniformly a p-growth condition is the following

$$F_\varepsilon(u) = \varepsilon \int_\Omega |a(x)Du|^{1/\varepsilon}\, dx, \qquad u \in W^{1,1/\varepsilon}(\Omega), \tag{4.22}$$

where $a \in L^\infty(\Omega)$ and $\inf a > 0$. These functionals can be thought to be defined on $W^{1,1}(\Omega)$, and each F_ε satisfies a $1/\varepsilon$-growth condition. Limits of problems involving these functionals are described by the following result by Garroni, Nesi and Ponsiglione [114].

Theorem 4.15 (i) *The functionals F_ε Γ-converge with respect to the L^1-convergence to the functional F_0 given by*

$$F_0(u) = \begin{cases} 0 & \text{if } \|aDu\|_\infty \le 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{4.23}$$

(ii) *The functionals G_ε given by $G_\varepsilon(u) = (F_\varepsilon(u))^\varepsilon$ Γ-converge with respect to the L^1-convergence to the functional G_0 given by*

$$G_0(u) = \|aDu\|_\infty. \tag{4.24}$$

Proof (i) the liminf inequality follows by noticing that if $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ and $u_\varepsilon \to u$ in L^1, then actually $u_\varepsilon \rightharpoonup u$ in $W^{1,q}(\Omega)$ for each $q > 1$, so that

$$|\{|aDu| > t\}|t^q \le \int_\Omega |aDu|^q\, dx \le \liminf_{\varepsilon\to 0} \int_\Omega |aDu_\varepsilon|^q\, dx \le \liminf_{\varepsilon\to 0} \Big(\frac{1}{\varepsilon} F_\varepsilon(u)\Big)^{\varepsilon q} = 1,$$

and we get $|\{|aDu| > t\}| = 0$ for all $t > 1$. A recovery sequence is trivially $u_\varepsilon = u$.

(ii) follows from the increasing convergence of $\varepsilon^{-\varepsilon} G_\varepsilon$ to G_0. □

We can apply the result above to minimum problems of the form

$$m_\varepsilon = \min\Big\{\int_\Omega |a(x)Du|^{1/\varepsilon} dx : u = \varphi \text{ on } \partial\Omega\Big\}; \tag{4.25}$$

note however that the limit minimum problem (of the scaled functionals)

$$\min\Big\{\|a(x)Du\|_\infty : u = \varphi \text{ on } \partial\Omega\Big\} \tag{4.26}$$

possesses many solutions. Hence, the limit of the (unique) solutions of m_ε should be characterized otherwise (see, e.g., [30, 81]).

For generalizations of this result we refer e.g. to the paper by Champion, De Pascale and Prinari [74].

4.4.2 The closure of quadratic forms

We consider now the problem of characterizing the closure of all quadratic forms when the coefficients do not satisfy uniform bounds from above and below as in Theorem 4.13. To this end we have to introduce some definitions (for details see the book by Fukushima [110]).

Definition 4.16 (Dirichlet form) *A quadratic form F on $L^2(\Omega)$ is called a* Dirichlet form *if:*

(i) *it is* closed*; i.e., its domain* Dom(F) *(where $F(u) = B(u,u) < +\infty$, B a bilinear form) endowed with the scalar product $(u,v)_F = B(u,v) + \int_\Omega uv\,dx$ is a Hilbert space;*

(ii) *it is* Markovian *(or* decreasing by truncature*); i.e., $F((u \vee 0) \wedge 1) \leq F(u)$ for all $u \in L^2(\Omega)$.*

The following remark helps to get an intuition of a general Dirichlet form.

Remark 4.17 (Deny-Beurling integral representation) A *regular Dirichlet form F* is such that $\mathrm{Dom}(F) \cap C^0(\Omega)$ is both dense in $C^0(\Omega)$ with respect to the uniform norm and in $\mathrm{Dom}(F)$. Such F admits the representation

$$F(u) = \sum_{i,j} \int_\Omega D_i u D_j u \, d\mu_{ij} + \int_\Omega |u|^2 \, d\nu + \int_{\Omega\times\Omega} (u(x) - u(y))^2 d\mu \tag{4.27}$$

for $u \in \mathrm{Dom}(F) \cap C^1_0(\Omega)$, where μ_{ij}, ν and μ are Radon measures such that $\mu(\{(x,x) : x \in \Omega\}) = 0$ and $\sum_{i,j} z_i z_j \mu_{ij}(K) \geq 0$ for all compact subsets K of Ω and $z \in \mathbb{R}^n$.

For the use of Dirichlet form for the study of asymptotic problems we refer to Mosco [129]. The following theorem is due to Camar-Eddine and Seppecher [69].

Theorem 4.18 (closure of quadratic forms) *Let $n \geq 3$. The closure with respect to the $L^2(\Omega)$-convergence of isotropic quadratic forms of* diffusion type *, i.e. of the form*

$$F_\alpha(u) = \int_\Omega \alpha(x)|Du|^2 \, dx, \qquad u \in H^1(\Omega) \tag{4.28}$$

where $0 < \inf \alpha \leq \sup \alpha < +\infty$ (but not equi-bounded) is the set of all Dirichlet form that are objective*; i.e., $F(u + c) = F(u)$ for all constants c.*

We do not give a proof of this result, referring to the paper [69]. We only remark that the density of isotropic quadratic forms in all (coercive) quadratic forms can be obtained by local homogenization (see Remark 5.8 below). The prototype of a non-local term is $\mu = \delta_{x_0,y_0}$ in the Deny-Beurling formula; this can be reached by taking $\alpha \to +\infty$ on a set composed of two balls centred on x_0 and y_0 and a tubular neighbourhood of the segment joining the two points with suitable (vanishing) radius. Note that the use of this construction is not possible in dimension two.

5 Homogenization of integral functionals

An important case of limits of integral functionals is that of energies within the theory of *homogenization*; i.e., when we want to take into account fast-oscillating inhomogeneities. The simpler way to model such a behaviour is to consider a function $f : \mathbb{R}^n \times \mathbb{M}^{m\times n} \to \mathbb{R}$ *periodic* in the first variable (up to a change of basis, we may suppose it is T-periodic (if not otherwise specified $T = 1$); i.e.,

$$f(x + Te_i, \xi) = f(x, \xi) \qquad \text{for all } x, \xi$$

for all vectors e_i of the standard basis of $\mathbb{R}^n$), and examine the asymptotic behaviour of energies

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, Du\Big)\,dx \qquad u \in W^{1,p}(\Omega;\mathbb{R}^m). \tag{5.1}$$

Note that we may apply the general compactness Theorem 4.2 to any sequence (f_j), where $f_j(x,\xi) = f(x/\varepsilon_j, \xi)$, thus obtaining the existence of Γ-converging subsequences. The main issues here are:

(i) prove that the whole family (F_ε) Γ-converges;

(ii) give a description of the energy density of the Γ-limit in terms of the properties of f.

In this section we give a simple account of the main features of this problem referring to the book [54] for more details.

The natural *ansatz* for the Γ-limit of (F_ε) is that it is "homogeneous"; i.e., its energy density does not depend on x, so that it takes the form

$$F_{\rm hom}(u) = \int_\Omega f_{\rm hom}(Du)\,dx \qquad u \in W^{1,p}(\Omega;\mathbb{R}^m). \tag{5.2}$$

If such a Γ-limit exists it will be called the *homogenized* functional of F_ε. As a consequence of the theorem on the convergence of minimum problems, we obtain that families of problems with solutions with highly oscillating gradients are approximated by solutions of simpler problems with $F_{\rm hom}$ in place of F_ε, where oscillations are 'averaged out'.

5.1 The asymptotic homogenization formula

From the localization methods we can easily derive an *ansatz* for a formula describing $f_{\rm hom}$. As a first remark, recall that $f_{\rm hom}$ is quasiconvex, so that it can be expressed as a minimum problem; e.g., choosing $D = (0,1)^n$ in (4.10), for all $\xi \in \mathbb{M}^{m\times n}$ we may write

$$f_{\rm hom}(\xi) = \min\Big\{\int_{(0,1)^n} f_{\rm hom}(\xi + D\varphi)\,dy : \varphi \in W^{1,p}_0((0,1)^n;\mathbb{R}^m)\Big\}. \tag{5.3}$$

Now, from the convergence of minima and the compatibility of addition of boundary conditions, we obtain

$$f_{\rm hom}(\xi) = \lim_{\varepsilon\to 0}\inf\Big\{\int_{(0,1)^n} f\Big(\frac{y}{\varepsilon}, \xi + D\varphi\Big)\,dy : \varphi \in W^{1,p}_0((0,1)^n;\mathbb{R}^m)\Big\}. \tag{5.4}$$

The final *asymptotic homogenization formula* is obtained from this by the change of variables $y = \varepsilon x$ that isolates the dependence on ε in a scaling argument (here we set $T = 1/\varepsilon$)

$$f_{\rm hom}(\xi) = \lim_{T\to+\infty}\frac{1}{T^n}\inf\Big\{\int_{(0,T)^n} f(x, \xi + D\varphi)\,dx : \varphi \in W^{1,p}_0((0,T)^n;\mathbb{R}^m)\Big\}. \tag{5.5}$$

To make this *ansatz* into a theorem we need just to prove that the candidate $f_{\rm hom}$ is indeed homogeneous, and that the limit on the right-hand side exists. In this way we can use the compactness theorem, and prove that the limit is independent of the sequence (ε_j), being characterized by formula (5.4).

Theorem 5.1 (homogenization theorem) *Let f be as above. If F_ε are defined as in* (5.1)*, then Γ-$\lim_{\varepsilon\to 0} F_\varepsilon = F_{hom}$, given by* (5.2) *and* (5.5) *with respect to the $L^2(\Omega;\mathbb{R}^m)$ convergence.*

Proof The 'homogeneity' of an $f_{\rm hom} = f_{\rm hom}(x,\xi)$ given by the compactness theorem can be easily obtained thanks to the ε-periodicity of F_ε, that ensures that $F_\varepsilon(u,A) = F_\varepsilon(u_y, y+A)$ for all open sets A and $y \in \varepsilon\mathbb{Z}^n$, where $u_y(x) = u(x-y)$. This implies, by a translation and approximation argument, that $F_{\rm hom}(u,A) = F_{\rm hom}(u_y, y+A)$ for all open sets A and $y \in \mathbb{R}^n$, so that $f_{\rm hom}(x,\xi) = f_{\rm hom}(x+y,\xi)$ by derivation (see (4.8)). The existence of the limit in (5.5) can be derived from the scaling argument in Remark 5.3 below. □

Proposition 5.2 (asymptotic behaviour of subadditive functions) *Let g be a function defined on finite unions of cubes of $\mathbb{R}^n$ which is subadditive (i.e., $g(A\cup B) \le g(A)+g(B)$ if $|A\cap B| = 0$) such that $g(z+A) = g(A)$ for all $z \in \mathbb{Z}^n$ and $g(A) \le c|A|$. Then there exists the limit*

$$\lim_{T\to+\infty} \frac{g((0,T)^n)}{T^n}. \tag{5.6}$$

Proof It suffices to check that if $S > T$ then we have $g((0,S)^n) \le (S/T)^n g((0,T)^n) + C(T,S)$, with $\lim_{T\to+\infty} \lim_{S\to+\infty} C(T,S) = 0$, and then take the limsup in S first and eventually the liminf in T. □

Remark 5.3 To prove the existence of the limit in (5.5) it suffices to apply the previous proposition to

$$g(A) = \inf\Big\{\int_A f(x,\xi + D\varphi)\,dx : \varphi \in W^{1,p}_0(A;\mathbb{R}^m)\Big\}. \tag{5.7}$$

5.1.1 A periodic formula

We can easily derive alternative formulas for $f_{\rm hom}$, for instance taking periodic minimum problems. The following *asymptotic periodic formula* is due to Müller [130] (we suppose that f is 1-periodic)

$$f_{\rm hom}(\xi) = \inf_{k\in\mathbb{N}} \frac{1}{k^n} \inf\Big\{\int_{(0,k)^n} f(x,\xi + D\varphi)\,dx : \varphi \in W^{1,p}_\#((0,k)^n;\mathbb{R}^m)\Big\}, \tag{5.8}$$

where $W^{1,p}_\#((0,k)^n;\mathbb{R}^m)$ denotes the space of k-periodic functions in $W^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)$. Note that since $W^{1,p}_\#((0,k)^n;\mathbb{R}^m) \subset W^{1,p}_\#((0,lk)^n;\mathbb{R}^m)$ for all $l \in \mathbb{N}$ with $l \ge 1$, the $\inf_k$ is actually a $\lim_k$. Moreover, since $W^{1,p}_0((0,k)^n;\mathbb{R}^m) \subset W^{1,p}_\#((0,k)^n;\mathbb{R}^m)$ the right-hand side of formula (5.8) is not greater than the value for $f_{\rm hom}(\xi)$ given by (5.5). It remains to prove the opposite inequality. To this end we will make use of the following lemma, which is a fundamental tool for dealing with oscillating energies.

Lemma 5.4 (Riemann-Lebesgue lemma) *Let g be an $L^1_{\rm loc}$ periodic function of period Y, let $g_\varepsilon(x) = g\big(\frac{x}{\varepsilon}\big)$ and let $\overline{g} = |Y|^{-1}\int_Y g\,dy$. Then $g_\varepsilon \rightharpoonup \overline{g}$. In particular $\int_D g_\varepsilon\,dx \to \overline{g}|D|$ for all bounded open subsets D.*

To conclude the proof of formula (5.8) it suffices to fix k and φ a test function for the corresponding minimum problem. Define $u_\varepsilon(x) = \xi x + \varepsilon\varphi\big(\frac{x}{\varepsilon}\big)$, so that $u_\varepsilon \to \xi x$. We can use these functions in the liminf inequality to get

$$\begin{aligned}
|\Omega| f_{\rm hom}(\xi) &= F_{\rm hom}(\xi x,\Omega) \le \liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon,\Omega)\\
&= \lim_{\varepsilon\to0}\int_\Omega f\Big(\frac{x}{\varepsilon},\xi + D\varphi\big(\frac{x}{\varepsilon}\big)\Big)\,dx = |\Omega|\frac{1}{k^n}\int_{(0,k)^n} f(y,\xi+D\varphi)\,dy,
\end{aligned}$$

where we have used Lemma 5.4 with $Y = (0,k)^n$ and $g(y) = f(y, \xi + D\varphi(y))$. By the arbitrariness of φ and k the desired inequality is proved.

5.2 The convex case: the cell-problem formula

In the convex case (i.e., $f(x,\cdot)$ convex for a.a. x) the formula for $f_{\rm hom}$ is further simplified. In fact, in this case we have a single *cell-problem formula*: (we suppose that f is 1-periodic)

$$f_{\rm hom}(\xi) = \inf\Big\{\int_{(0,1)^n} f(x, \xi + D\varphi)\,dx : \varphi \in W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m)\Big\}. \tag{5.9}$$

To check this, by (5.8) it is sufficient to prove that

$$\begin{aligned} &\frac{1}{k^n}\inf\Big\{\int_{(0,k)^n} f(x, \xi + D\varphi)\,dx : \varphi \in W^{1,p}_{\#}((0,k)^n;\mathbb{R}^m)\Big\} \\ \geq\ \ &\inf\Big\{\int_{(0,1)^n} f(x, \xi + D\varphi)\,dx : \varphi \in W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m)\Big\}\end{aligned}$$

for all k, the converse inequality being trivial since $W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m) \subset W^{1,p}_{\#}((0,k)^n;\mathbb{R}^m)$. Now, take φ a k-periodic test function and define

$$\widetilde{\varphi}(x) = \frac{1}{k^n}\sum_{j\in\{1,\dots,k\}^n} \varphi(x+j).$$

Then $\widetilde{\varphi}$ is 1-periodic and it is a convex combination of periodic translations of φ. By the convexity of f then

$$\begin{aligned} &\int_{(0,1)^n} f(x, \xi + D\widetilde{\varphi})\,dx = \frac{1}{k^n}\int_{(0,k)^n} f(x, \xi + D\widetilde{\varphi})\,dx \\ \leq\ \ &\frac{1}{k^n}\sum_j \frac{1}{k^n}\int_{j+(0,k)^n} f(x, \xi + D\varphi)\,dx = \frac{1}{k^n}\int_{(0,k)^n} f(x, \xi + D\varphi)\,dx,\end{aligned}$$

that proves the inequality.

Remark 5.5 (homogenization and convexity conditions) Note that convexity is preserved by Γ-convergence also in the vectorial case; i.e., $f_{\rm hom}$ is convex if $f(x,\cdot)$ is convex for a.a. x. On the contrary it can be seen that the same fails for the condition that $f(x,\cdot)$ be polyconvex (see [46]).

5.2.1 Müller's counterexample

The convex formula above proves the *ansatz* that recovery sequences for convex homogenization problems can be though locally periodic of minimal period (ε in the case above). Note that in the case $n = 1$ or $m = 1$ convexity is not a restrictive hypothesis, since we may consider the lower-semicontinuous envelope of F_ε in its place, whose integrand is convex.

The local-periodicity *ansatz* is false if the problem is vectorial, as shown by a counterexample by Müller [130, 54]. We do not enter in the detail of the example, but try to give an interpretation of the physical idea behind the construction: the function f is defined in the periodicity cell $(0,1)^3$ as

$$f(x,\xi)=\begin{cases} f_1(\xi) & \text{if } x\in B_{1/4}(\frac{1}{2},\frac{1}{2})\times(0,1)\\ f_0(\xi) & \text{otherwise,}\end{cases}$$

where f_1 is a (suitable) polyconvex function and f_0 is a 'weak' convex energy; e.g., $f_0(\xi)=\delta|\xi|^p$ with δ small enough (we may think f_0 being 0, even though that case is not covered by our results). We may interpret the energy F_ε as describing a periodic array or thin vertical bars. For suitable polyconvex f_1 we will have buckling instabilities and the array of thin bars will sustain much less vertical compression than the single bar in the periodicity cell. This corresponds to the inequality

$$f_{\rm hom}(\xi)<\inf\Big\{\int_{(0,1)^n} f(x,\xi+D\varphi)\,dx:\varphi\in W^{1,p}_{\#}((0,1)^n;\mathbb{R}^m)\Big\}\tag{5.10}$$

for $\xi=-e_3\otimes e_3$.

5.3 Homogenization of quadratic forms

As remarked in Theorem 4.13, quadratic forms are closed under Γ-convergence. In the case of homogenization we can give the following characterization (for simplicity we deal with the scalar case $m=1$ only)

Theorem 5.6 (homogenization of quadratic forms) *Let $A:\mathbb{R}^n\to\mathbb{M}^{m\times n}$ be a 1-periodic symmetric matrix-valued measurable function, and suppose that $\alpha,\beta>0$ exist such that $\alpha\,{\rm Id}\le A\le\beta\,{\rm Id}$ for all j. Then we have*

$$\int_\Omega\langle A_{\rm hom}Du,Du\rangle\,dx=\Gamma\text{-}\lim_{\varepsilon\to0}\int_\Omega\Big\langle A\Big(\frac{x}{\varepsilon}\Big)Du,Du\Big\rangle\,dx\tag{5.11}$$

with respect to the $L^2(\Omega)$-convergence, for all $u\in H^1(\Omega)$, where the constant matrix $A_{\rm hom}$ is given by

$$\langle A_{\rm hom}\xi,\xi\rangle=\inf\Big\{\int_{(0,1)^n}\langle A(x)(\xi+D\varphi),\xi+D\varphi\rangle\,dx:\varphi\in H^1_{\#}((0,1)^n)\Big\}.\tag{5.12}$$

Remark 5.7 (one-dimensional homogenization) In the one-dimensional case, when we simply have $\langle A(x)\xi,\xi\rangle=a(x)\xi^2$ with $a:\mathbb{R}\to[\alpha,\beta]$ 1-periodic, the limit energy density is of the simple form $a_{\rm hom}\xi^2$. The coefficient $a_{\rm hom}$ is easily computed and is the *harmonic mean* of a

$$a_{\rm hom}=\underline{a}:=\Big(\int_0^1\frac{1}{a(y)}\,dy\Big)^{-1}.\tag{5.13}$$

Remark 5.8 (laminates) We can easily compute the homogenized matrix for an A of the form $a(\langle x,\nu\rangle){\rm Id}$

$$a(s)=\begin{cases}\alpha & \text{if } 0<s-[s]<t\\ \beta & \text{if } t<s-[s]<1\end{cases}\qquad\text{([}s\text{] is the \textit{integer part} of }s).\tag{5.14}$$

The corresponding energy density is called a *lamination* of the two energies densities $\alpha|\xi|^2$ and $\beta|\xi|^2$ in direction ν with *volume fractions* t and $1-t$ respectively.

It is not restrictive, upon a rotation, to consider $\nu = e_1$. By a symmetry argument, we see that $A_{\rm hom}$ is diagonal. By (5.12) the coefficient a_{kk} of $A_{\rm hom}$ is computed by considering the minimum problem

$$a_{kk} = \inf\Big\{\int_{(0,1)^n} a(x_1)|e_k + D\varphi|^2\,dx : \varphi \in H^1_\#((0,1)^n)\Big\}. \tag{5.15}$$

For $k = 1$ the solution is $\varphi(x) = \varphi_1(x_1)$, where φ_1 is the solution of the one-dimensional problem with coefficient a; hence $a_{11} = \underline{a}$ as defined in (5.13). For $k > 1$ we may easily see that the solution is $\varphi(x) = 0$, so that, in conclusion,

$$a_{11} = \frac{\alpha\beta}{t\beta + (1-t)\alpha}, \qquad a_{kk} = t\alpha + (1-t)\beta \text{ for } k > 1.$$

Note that we have obtained a non-isotropic matrix by homogenization of isotropic ones. Of course, by varying ν we obtain all symmetric matrices with the same eigenvalues.

The same computation can be performed for $A(x) = \Pi_{k=1}^n a_i(x_i)|\xi|^2$, with $\alpha \le a_i(y) \le \beta$. Note that if we vary α and β and a_i then we may obtain all symmetric matrices as homogenization of isotropic ones.

5.4 Bounds on composites

The example of lamination above shows that mixtures of two simple energies can give rise to more complex ones. A general question is to describe all possible mixtures of a certain number of 'elementary' energies. This is a complex task giving rise to numerous types of questions, most of which still open (see [122]). Here we want to highlight a few connections with the theory of homogenization as presented above, by considering only the case of mixtures of two isotropic energies $\alpha|\xi|^2$ and $\beta|\xi|^2$.

A 'mixture' will be given by a choice of measurable sets $E_j \subset \Omega$. We will consider energies of the form

$$F_j(u) = \alpha\int_{E_j} |Du|^2\,dx + \beta\int_{\Omega\setminus E_j} |Du|^2\,dx. \tag{5.16}$$

Note that we can rewrite $F_j(u) = \int_\Omega a_j(x)|Du|^2\,dx$ and apply Theorem 4.13, thus obtaining, upon subsequences, a Γ-limit of the form

$$F_0 = \int_\Omega \langle A_0(x)Du, Du\rangle\,dx. \tag{5.17}$$

The problem is to give the best possible description of the possible reachable A_0.

The limit local (statistical) description of the behaviour of (E_j) is given by the weak*-limit of χ_{E_j}, which will be denoted by θ and called the *local volume fraction* of the energy α. Note that by Remark 5.8 the knowledge of θ is not sufficient to describe the limit of F_j (since we can take laminates in two different directions with the same $\theta = t$ but different limit energies).

5.4.1 The localization principle

With fixed $\overline{\theta}$ we can consider the set of all *matrices obtained by homogenization* of energies α and β with volume fraction $\overline{\theta}$ of α, corresponding by (5.12) to matrices A satisfying

$$\langle A\xi,\xi\rangle = \inf\Big\{\alpha\int_E |\xi + D\varphi|^2\,dx + \beta\int_{(0,1)^n\setminus E} |\xi + D\varphi|^2\,dx : \varphi \in H^1_\#((0,1)^n)\Big\}. \tag{5.18}$$

for some measurable $E \subset (0,1)^n$ with $|E| = \overline{\theta}$. We will denote by $\mathcal{H}(\overline{\theta})$ the closure of the set of all such matrices; E is called an *underlying microgeometry* of such A.

The matrices A_0 in (5.17) are characterized by a *localization principle* ([143], [136]).

Proposition 5.9 (localization principle) *$A_0(x) \in \mathcal{H}(\theta(x))$ for almost all $x \in \Omega$.*

Proof We only sketch the main points of the proof. Let $\overline{x}$ be a Lebesgue point for $\theta(x)$. Upon a translation argument we can suppose that $\overline{x} = 0$. For all open sets U the functional defined by $\int_U \langle A_0(0)Du, Du\rangle\, dx$ is the Γ-limit of $\int_U \langle A_0(\rho x)Du, Du\rangle\, dx$ as $\rho \to 0$ since $A_0(\rho x)$ converges to $A_0(0)$ in L^1 on U. Let $Q_\rho(x)$ denote the coordinate cube centered at x and with side length ρ. We can then infer that, for any fixed ξ

$$\begin{aligned}\langle A_0(\overline{x})\xi,\xi\rangle &= \min\Big\{\int_{Q_1(0)} \langle A_0(\rho x)(\xi + D\varphi),(\xi+D\varphi)\rangle : \varphi \text{ 1-periodic}\Big\} + o(1)\\ &= \rho^{-n}\min\{F_0(\xi + D\varphi, Q_\rho(0)) : \varphi\ \rho\text{-periodic}\} + o(1)\\ &= \rho^{-n}\min\{F_j(\xi + D\varphi, Q_\rho(0)) : \varphi\ \rho\text{-periodic}\} + o(1)\end{aligned}$$

as $\rho \to 0$ and $j \to +\infty$. Upon scaling, the formula in the last limit is of type (5.18) for some θ^j_ρ tending to $\theta(\overline{x})$ as $\rho \to 0$ and $j \to +\infty$, and the proposition is proved, upon remarking that the limit of matrices in $\mathcal{H}(\theta^j_\rho)$ belongs to $\mathcal{H}(\theta(\overline{x}))$. □

The previous proposition reduces the problem of characterizing all $A_0(x)$ to that of studying the sets $\mathcal{H}(\theta)$ for fixed $\theta \in [0,1]$.

Remark 5.10 (set of all reachable matrices) From the trivial one-dimensional estimates we have

$$\frac{\alpha\beta}{\theta\beta + (1-\theta)\alpha} \le \lambda_i \le \theta\alpha + (1-\theta)\beta, \tag{5.19}$$

where λ_i denote the eigenvalues of the matrices in $\mathcal{H}(\theta)$.

In the two-dimensional case we deduce that all such matrices have eigenvalues satisfying

$$\frac{\alpha\beta}{\alpha+\beta-\lambda_1} \le \lambda_2 \le \alpha + \beta - \frac{\alpha\beta}{\lambda_1}, \tag{5.20}$$

and actually all matrices satisfying (5.20) belong to some $\mathcal{H}(\theta)$.

5.4.2 Optimal bounds

The computation of $\mathcal{H}(\theta)$ is obtained by exhibiting ‘optimal bounds’; it is due to Murat and Tartar (see [143]; see also the derivation of Cherkaev and Lurie in the two–dimensional case [120]). It is not based on Γ-convergence arguments, so for completeness we only include the (two–dimensional) optimal bounds, which only constrain the eigenvalues λ_1, λ_2 of the macroscopic conductivity tensor A_0. The formula is

$$\begin{cases} \dfrac{1}{\lambda_1 - \alpha} + \dfrac{1}{\lambda_2-\alpha} \le \dfrac{1}{\overline{a}(\theta)-\alpha} + \dfrac{1}{\underline{a}(\theta)-\alpha} \\[2ex] \dfrac{1}{\beta-\lambda_1} + \dfrac{1}{\beta-\lambda_2} \le \dfrac{1}{\beta-\overline{a}(\theta)} + \dfrac{1}{\beta-\underline{a}(\theta)}, \end{cases} \tag{5.21}$$

where $\underline{a}(\theta)$ and $\overline{a}(\theta)$ are the harmonic and arithmetic means of α and β with proportion θ:

$$\underline{a}(\theta) = \frac{\alpha\beta}{\theta\beta + (1-\theta)\alpha}, \qquad \overline{a}(\theta) = \theta\alpha + (1-\theta)\beta.$$

Note that the two ‘extremal’ geometries are given by laminates.

5.5 Homogenization of metrics

We conclude this chapter with some observation regarding another type of homogenization, that of functionals of the type

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{u}{\varepsilon}, Du\Big)\,dx, \tag{5.22}$$

with f periodic in the first variable and satisfying the usual growth conditions. In this case, by Remark 4.6 we can carry over the compactness procedure and also represent the limit as an integral of the usual form

$$f_{\rm hom}(\xi) = \lim_{T\to+\infty}\frac{1}{T^n}\inf\Big\{\int_{(0,T)^n} f(u+\xi y, Du+\xi)\,dy : \ u \in W^{1,p}_0((0,T)^n;\mathbb{R}^m)\Big\} \tag{5.23}$$

We do not treat the general case (for details see [54] Chapter 15), but briefly outline two applications.

5.5.1 The closure of Riemannian metrics

We consider the one-dimensional integrals, related to distances on a periodic isotropic Riemannian manifold

$$F_\varepsilon(u) = \int_0^1 a\Big(\frac{u}{\varepsilon}\Big)|u'|^2\,dt, \qquad u \in H^1((0,1);\mathbb{R}^m). \tag{5.24}$$

The following result states that the limit of such energies corresponds to a *homogeneous Finsler metric* (see Acerbi and Buttazzo [1]) and the converse is also true; i.e., every homogeneous Finsler metric can be approximated by homogenization of (isotropic) Riemennian metrics (see Braides, Buttazzo and Fragalà [47]).

Theorem 5.11 (closure of Riemannian metrics by homogenization) (i) *Let a be a 1-periodic function satisfying $0 < \alpha \le a \le \beta < +\infty$; then the Γ-limit of F_ε is*

$$F_{\rm hom}(u) = \int_0^1 f_{\rm hom}(u')\,dt, \tag{5.25}$$

where

$$f_{\rm hom}(z) = \lim_{T\to+\infty}\frac{1}{T}\inf\Big\{\int_0^1 a(v)|v'|^2\,dt : v(0)=0,\ v(T)=Tz\Big\}; \tag{5.26}$$

(ii) *for all $\psi : \mathbb{R}^m \to [0,+\infty)$ even, convex, positively homogeneous of degree two and such that $\alpha|z|^2 \le \psi(z) \le \beta|z|^2$, and for all $\eta > 0$ there exists f_{hom} as above such that $|f_{\rm hom}(z) - \psi(z)| \le \eta|z|^2$* for all $z \in \mathbb{R}^m$.

Proof (i) can be achieved as outlined above. The formula follows from the representation of $f_{\rm hom}(z)$ as a minimum problem;

(ii) let (ν_i) be a sequence of rational directions (i.e., such that for all i there exists $T_i\in\mathbb{R}$ such that $T_j\nu_j\in\mathbb{Z}^m$) dense in S^{m-1}, fix M and define a^M as follows:

$$a^M(s)=\begin{cases}\psi(\nu_i) & \text{if } s\in(\mathbb{Z}^m+\nu_i\mathbb{R})\setminus\bigcup_{j\neq i,1\le j\le M}(\mathbb{Z}^m+\nu_j\mathbb{R}),\ 1\le i\le M\\ \beta & \text{otherwise.}\end{cases}\tag{5.27}$$

The coefficient a^M is β except on a $\mathbb{Z}^m$ periodic set of lines in the directions ν_i. Then from formula (5.26) for the corresponding $f^M_{\rm hom}$ we easily get that $f^M_{\rm hom}(\nu_i)=\psi(\nu_i)$ on all ν_i for $i\le M$. Note in fact that for all v we have $\int_0^1 a^M(v)|v'|^2\,dt\ge\int_0^1\psi(v')\,dt$ and equality holds on the functions $v(t)=t\nu_i$. Since ψ is positively homogeneous of degree two and convex this implies that $f^M_{\rm hom}\to\psi$ uniformly on S^{m-1} as $M\to+\infty$, as desired. □

This result has been generalized to the approximation of arbitrary (non-homogeneous) Finsler metrics by Davini [90].

5.5.2 Homogenization of Hamilton-Jacobi equations

The solution u_ε of a Hamilton-Jacobi equation of the form

$$\begin{cases}\dfrac{\partial u_\varepsilon}{\partial t}+H\Big(\dfrac{x}{\varepsilon},Du_\varepsilon(x,t)\Big)=0 & \text{in } \mathbb{R}^m\times[0,+\infty)\\ u_\varepsilon(x,0)=\varphi(x) & \text{in } \mathbb{R}^m,\end{cases}\tag{5.28}$$

where H is a quadratic *Hamiltonian* and φ is a smooth bounded initial datum, is given by the *Lax formula*

$$\begin{aligned}u_\varepsilon(x)&=\inf\{\varphi(y)+S_\varepsilon(x,t;y,s):y\in\mathbb{R}^m,\ 0\le s<t\},\\ S_\varepsilon(x,t;y,s)&=\inf\Big\{\int_s^t L\Big(\frac{u}{\varepsilon},u'\Big)\,d\tau:u(s)=y,u(t)=x\Big\},\\ L(x,z)&=\sup\{\langle z,z'\rangle-H(x,z'):z'\in\mathbb{R}^m\}\end{aligned}$$

(the *Legendre transform* of H). By the Γ-convergence of the integrals above, the pointwise limit of S_ε is given by

$$S_{\rm hom}(x,t;y,s)=\inf\Big\{\int_s^t L_{\rm hom}(u')\,d\tau:u(s)=y,u(t)=x\Big\}=(t-s)\,L_{\rm hom}\Big(\frac{x-y}{t-s}\Big),$$

where $L_{\rm hom}$ is obtained through formula (5.23), and u_ε converge uniformly on compact sets to the corresponding u. As a conclusion we may prove that u satisfies the *homogenized Hamilton-Jacobi equation*

$$\begin{cases}\dfrac{\partial u}{\partial t}+H_{\rm hom}(Du(x,t))=0 & \text{in } \mathbb{R}^m\times[0,+\infty)\\ u(x,0)=\varphi(x) & \text{in } \mathbb{R}^m,\end{cases}\tag{5.29}$$

where $H_{\rm hom}$ is given by

$$H_{\rm hom}(x,z)=\sup\{\langle z,z'\rangle-L_{\rm hom}(x,z'):z'\in\mathbb{R}^m\}.$$

Details can be found in [46] Section 3.4 (see also [99]).

6 Perforated domains and relaxed Dirichlet problems

A class of problems that cannot be directly framed within the class of integral functionals considered above are those defined on varying domains. The prototype of these domains are *perforated domains*; i.e., obtained from a fixed Ω by removing some periodic set, the simplest of which is a periodic array of closed sets:

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i\in\mathbb{Z}^n} (\varepsilon i + \delta_\varepsilon K). \tag{6.1}$$

On the set K, we suppose that it is a bounded closed set. On the boundary of Ω_ε (or on the boundary of Ω_ε interior to Ω) we can consider various types of conditions. We will examine Dirichlet and Neumann boundary conditions, leading to different relevant scales for δ_ε and technical issues.

6.1 Dirichlet boundary conditions: a direct approach

We first treat the model case of Ω_ε as in (6.1) and $u = 0$ on $\partial\Omega_\varepsilon$, with in mind minimum problems of the form

$$\min\Big\{ \int_\Omega |Du|^2\, dx - 2\int_\Omega gu\, dx : u = 0 \text{ on } \partial\Omega_\varepsilon \Big\}. \tag{6.2}$$

The results of this section can be extended to vector u, to different boundary conditions on $\partial\Omega$ (provided we introduce a 'safe zone' close to $\partial\Omega$ vanishing with ε where the perforation is absent in order not to make the boundary conditions interact) and to general integrands satisfying the growth conditions of Section 4.

The observation that for suitable δ_ε the solutions u_ε of the equations

$$\begin{cases} -\Delta u_\varepsilon = g & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{6.3}$$

extended to 0 inside the perforation, may converge to a function u satisfying

$$\begin{cases} -\Delta u + Cu = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.4}$$

for some $C > 0$, goes back to Marchenko and Khruslov [121], and was subsequently recast in a variational framework by Cioranescu and Murat [75]. We want to re-read this phenomenon on problems in (6.2), of which (6.3) is the Euler equation.

We will extend our functions to the whole Ω by setting $u = 0$ on the perforation. As usual, we will neglect the continuous part $2\int_\Omega gu\, dx$ since it commutes with the Γ-limit, and consider the functionals

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega |Du|^2\, dx & \text{if } u \in H^1_0(\Omega) \text{ and } u = 0 \text{ on } \Omega\setminus\Omega_\varepsilon \\ +\infty & \text{otherwise.} \end{cases} \tag{6.5}$$

It remains to understand the role of δ_ε. We have to understand the meaningful scalings of the energy and possibly an optimal minimum formula describing the limit. To this end, consider a sequence $u_\varepsilon \to u$. We make the assumption that the energy 'far from the perforation' gives a term of Dirichlet form that can be dealt with separately, and focus on the contribution 'close to the perforation'. We also assume that the energy due to each set $\varepsilon i + \delta_\varepsilon K$ can be dealt with separately.

Suppose for the time being that u is continuous; since $u_\varepsilon \to u$ close to $\varepsilon i + \delta_\varepsilon K$ the function u_ε will be close to the limit value $u(\varepsilon i)$. Assume that this is true (and then that we may directly suppose $u_\varepsilon = u(\varepsilon i)$) on the boundary of some ball $B_{\varepsilon R}(\varepsilon i)$ containing $\varepsilon i + \delta_\varepsilon K$. We have

$$
\begin{aligned}
& \int_{B_{\varepsilon R}(\varepsilon i)} |Du_\varepsilon|^2\,dx \\
\geq\ & \min\Big\{\int_{B_{\varepsilon R}(0)} |Dv|^2\,dx : v = 0 \text{ on } \delta_\varepsilon K,\ v = u(\varepsilon i) \text{ on } \partial B_{\varepsilon R}(0)\Big\} \\
=\ & \delta_\varepsilon^{n-2} \min\Big\{\int_{B_{\varepsilon R/\delta_\varepsilon}(0)} |Dv|^2\,dy : v = 0 \text{ on } K,\ v = u(\varepsilon i) \text{ on } \partial B_{\varepsilon R/\delta_\varepsilon}(0)\Big\} \\
\geq\ & \delta_\varepsilon^{n-2} \inf_T \min\Big\{\int_{B_T(0)} |Dv|^2\,dy : v = 0 \text{ on } K,\ v = u(\varepsilon i) \text{ on } \partial B_T(0)\Big\} \\
=\ & \delta_\varepsilon^{n-2} |u(\varepsilon i)|^2 \inf_T \min\Big\{\int_{B_T(0)} |Dv|^2\,dy : v = 0 \text{ on } K,\ v = 1 \text{ on } \partial B_T(0)\Big\}. \qquad (6.6)
\end{aligned}
$$

For the sake of simplicity we suppose that $n > 2$; in that case the last minimum problem is the *capacity* of the set K (with respect to $\mathbb{R}^n$), that we will denote by $\mathrm{Cap}(K)$. We then have a lower estimate on the contribution 'close to the perforation' of the form

$$
\sum_i \int_{B_{\varepsilon R}(\varepsilon i)} |Du_\varepsilon|^2\,dx \geq \mathrm{Cap}(K) \sum_i \delta_\varepsilon^{n-2} |u(\varepsilon i)|^2 \qquad (6.7)
$$

The last is a Riemann sum provided that $\delta_\varepsilon^{n-2} = M\varepsilon^n + o(\varepsilon)$. This given a guess for the correct meaningful scaling for which the limit is influenced by the perforation (we may suppose $M = 1$ upon scaling K)

$$
\delta_\varepsilon = \varepsilon^{\frac{n}{n-2}}. \qquad (6.8)
$$

We will see that all other scaling can be reduced to this one by a comparison argument.

The argument above needs some refinement if $n = 2$, due to the scaling-invariance properties of the Dirichlet integral. In that case, the minimum problem on B_T in (6.6) scales as $(\log T)^{-1}$, so that taking the infimum in T would give a trivial lower bound. Instead, to obtain an inequality as in (6.7) from the first inequality in (6.6), we choose δ_ε so that $(\log T)^{-1} = \varepsilon^2$ ($T = \varepsilon R/\delta_\varepsilon$). This choice gives the correct scaling $\delta_\varepsilon = e^{-c/\varepsilon^2}$. Note that the dependence on K in the limit disappears. In this section we will always assume that $n \geq 3$.

6.1.1 A joining lemma on perforated domains

In the argument in (6.6) we have supposed that it is not restrictive to vary the value of a sequence u_ε on some sets surrounding the perforation. This can be obtained easily if the family $(|Du_\varepsilon|^2)$ is equi-integrable. Unfortunately, Theorem 4.11 cannot be directly used since the modified sequence might violate the constraint $u_\varepsilon = 0$ on the perforation. Nevertheless, we can modify De Giorgi's method to match boundary conditions and obtain the following technical lemma proved by Ansini and Braides. We suppose that $K \subset B_1(0)$ for simplicity.

Lemma 6.1 *Let (u_ε) converge weakly to u in $H^1(\Omega)$. Let $k \in \mathbb{N}$ be fixed and $R < 1/2$. Let Z_ε be the set of all $i \in \mathbb{Z}^n$ with $\mathrm{dist}(\varepsilon i, \partial\Omega) > n\varepsilon$. For each such i there exists $k_i \in \{0, \dots, k-1\}$ such* *that, having set*

$$C_i^\varepsilon = \Big\{x \in \Omega : \ 2^{-k_i-1}R\varepsilon < |x - \varepsilon i| < 2^{-k_i}R\varepsilon\Big\}, \tag{6.9}$$

$$u_\varepsilon^i = \frac{1}{|C_i^\varepsilon|}\int_{C_i^\varepsilon} u_\varepsilon\,dx \qquad \textit{and} \qquad \rho_\varepsilon^i = \frac{3}{4}2^{-k_i}R\varepsilon \tag{6.10}$$

(the mean value *of u_ε on C_i^ε and the* middle radius *of C_i^ε, respectively), there exists a sequence (w_ε), with $w_\varepsilon \rightharpoonup u$ in $H^1(\Omega)$ such that*

$$w_\varepsilon = u_\varepsilon \ \textit{on}\ \Omega \setminus \bigcup_{i\in Z_\varepsilon} C_i^\varepsilon, \qquad w_\varepsilon(x) = u_\varepsilon^i \ \textit{if}\ |x - \varepsilon i| = \rho_\varepsilon^i \tag{6.11}$$

and

$$\int_\Omega \Big||Dw_\varepsilon|^2 - |Du_\varepsilon|^2\Big|\,dx \le c\frac{1}{k}. \tag{6.12}$$

Proof The proof of the lemma follows the idea of the De Giorgi method for matching boundary values. In this case the value to match is u_ε^i, and the choice where to operate the cut-off procedure is between the annuli C_i^ε, $i \in \{1, \dots, N\}$. The proof is a little more complex since we have to use Poincaré's inequality on C_i^ε to estimate the excess of energy due to this process (note that the annuli are all homothetic in order to control the Poincaré constant by the scaling ratio). We refer to [22] for the details of the proof. □

With this lemma, it is relatively easy to describe the Γ-limit of F_ε.

Theorem 6.2 *Let $n > 2$ and let F_ε be given by* (6.5) *and $\delta_\varepsilon = \varepsilon^{\frac{n}{n-2}}$. Then the Γ-limit of F_ε with respect to the $L^2(\Omega)$ convergence is given by*

$$F_0(u) = \int_\Omega |Du|^2\,dx + \mathrm{Cap}(K)\int_\Omega |u|^2\,dx \tag{6.13}$$

on $H_0^1(\Omega)$.

Proof By Lemma 6.1 we can use the argument in (6.6) with u_ε^i in place of $u(\varepsilon i)$ to give a lower bound on the contribution close to the perforation with $\mathrm{Cap}(K)\sum_{i\in Z_\varepsilon}|u_\varepsilon^i|^2$, which converges to $\mathrm{Cap}(K)\int_\Omega |u|^2\,dx$. As for the contribution away from the perforation, we can write it as $\int_\Omega |Dz_\varepsilon|^2\,dx$, where z_ε is the $H^1(\Omega)$-extension of w_ε which is constant on each ball $B_{\rho_\varepsilon^i}(\varepsilon i)$. The limit of z_ε is still u so that we have the inequality $\liminf_{\varepsilon\to 0}\int_\Omega |Dz_\varepsilon|^2\,dx \ge \int_\Omega |Du|^2\,dx$, completing the lower bound.

The upper bound can be achieved by a direct construction. We first show it for $u = 1$ constant (even though this does not satisfy the boundary condition $u \in H_0^1(\Omega)$). In this case we simply choose $T > 0$ and v_T minimizing the last minimum problem in (6.6), and define

$$u_\varepsilon(x) = \begin{cases} v_T(\varepsilon^{\frac{n}{2-n}}(x - \varepsilon i)) & \text{on } \varepsilon^{\frac{n}{n-2}}B_T(\varepsilon i) \\ 1 & \text{otherwise.} \end{cases} \tag{6.14}$$

Then $u_\varepsilon \to u$ and $\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = |\Omega|\int_{B_T(0)} |Dv_T|^2\,dx$, which proves the approximate limsup inequality. For $u \in C_0^\infty(\Omega)$ we can use the recovery sequence $\widetilde{u}_\varepsilon = u_\varepsilon u$ with u_ε as in (6.14), and for $u \in H_0^1(\Omega)$ use a density argument. □

Remark 6.3 (other limits) (i) As a first remark, note that we may consider perforations with locally varying size. For example, we can fix a smooth bounded function $g:\mathbb{R}^n\to[0,+\infty)$ and take

$$\Omega_\varepsilon=\Omega\setminus\bigcup_{i\in\mathbb{Z}^n}(\varepsilon i+\varepsilon^{\frac{n}{n-2}}g(\varepsilon i)K). \tag{6.15}$$

We can follow word for word the proof above, noting that a term $(g(x))^{n-2}$ appears in (6.6), and obtain the limit functional

$$F_0(u)=\int_\Omega|Du|^2\,dx+\int_\Omega a(x)|u|^2\,dx \tag{6.16}$$

on $H^1_0(\Omega)$, where $a(x)=\mathrm{Cap}(K)(g(x))^{n-2}$. By approximation, in this way we may obtain any $a\in L^1_{\rm loc}(\mathbb{R}^n)$ in the limit functional;

(ii) the 'non-critical cases' can be easily dealt with by comparison. If we take a perforation as in (6.1) with

$$\lim_{\varepsilon\to0}\frac{\delta_\varepsilon}{\varepsilon^{\frac{n}{n-2}}}=0 \tag{6.17}$$

then we have an upper bound for the Γ-limit by any functional of the form (6.16) with a any fixed constant, so that we may take $a=0$ and obtain that the contribution of the perforation disappears leaving only the Dirichlet integral (the lower bound is trivial in this case by the lower semicontinuity of the Dirichlet integral). Conversely, if

$$\lim_{\varepsilon\to0}\frac{\delta_\varepsilon}{\varepsilon^{\frac{n}{n-2}}}=+\infty, \tag{6.18}$$

then the functionals (6.16) with a any fixed constant are a lower bound and in the limit we obtain that the functional is finite (and its value is 0) only on the constant 0 (for which the upper bound is trivial);

(iii) we can extend the method outlined above to cover other cases. For example we can require the unilateral condition $u\ge0$ on the perforation. In this case the proof above shows a limit of the form

$$F_0(u)=\int_\Omega|Du|^2\,dx+\mathrm{Cap}(K)\int_\Omega|u^-|^2\,dx, \tag{6.19}$$

where the sole *negative part* of u contributes to the extra term. It is sufficient to note that

$$\begin{aligned}&\min\Big\{\int_{B_T(0)}|Dv|^2\,dy: v\ge0 \text{ on } K,\ v=u(\varepsilon i)\text{ on } B_T(0)\Big\}\\ =\ &|u^-(\varepsilon i)|^2\min\Big\{\int_{B_T(0)}|Dv|^2\,dy: v=0\text{ on } K,\ v=1\text{ on } B_T(0)\Big\}\end{aligned}$$

in the last equality of (6.6).

Remark 6.4 (perforated domains as degenerate quadratic forms) We note that for $n\ge3$ at fixed ε the functional in (6.5) can be seen as the Γ-limit of a family of usual quadratic integral functionals on $H^1_0(\Omega)$. We can easily check this by a double-limit procedure. We first fix $\rho>0$ and consider the set $\mathcal{G}_\rho=\{x:\mathrm{dist}(x,\mathcal{G})\le\rho\}$, where $\mathcal{G}$ is the 'integer grid' $\mathcal{G}=\{x\in\mathbb{R}^n:\#\{i: x_i\notin\mathbb{Z}\}\le1\}$. It is not restrictive to suppose that $0\in K$ and K connected, so that the set

$C_\rho = \mathcal{G}_\rho \cup \bigcup_i (\varepsilon i + \delta_\varepsilon K)$ is periodic and connected (and in particular connected with $\partial\Omega$). We can then define

$$a_n^\rho(x) = \begin{cases} n & \text{if } x \in C_\rho \\ 1 & \text{otherwise,} \end{cases} \qquad F_n^\rho(u) = \int_\Omega a_n^\rho |Du|^2\,dx, \quad u \in H_0^1(\Omega). \tag{6.20}$$

As $n \to +\infty$ F_n^ρ converge increasingly to the functionals F^ρ defined by the Dirichlet integral with zero boundary conditions on $\partial\Omega \cup C_\rho$ (note that we may use Remark 2.12(ii) to deduce that F^ρ is also the Γ-limit of F_n^ρ). We now let $\rho \to 0$ so that F^ρ converge decreasingly to F_ε as defined in (6.5) and use Remark 2.12(i) to deduce their Γ-convergence. Note that here we use that $\mathcal{G}$ has zero capacity (this fact is not true for $n = 2$). A diagonal sequence (that we may construct thanks to Remark 2.16) does the job.

6.2 Relaxed Dirichlet problems

The problem of the computation of the Γ-limit for an arbitrary family of perforations needs a general setting including both the original constraint $u = 0$ on some E, and the limit case obtained in the previous section with the 'extra term' $\int |u|^2\,dx$. To this end, note that both energies can be written as

$$F(u) = \int_\Omega |Du|^2\,dx + \int_\Omega |u|^2 d\mu,$$

where the Borel measure μ is defined either as $\mu = \mathrm{Cap}(K)\mathcal{L}^n$ or as $\mu = \infty_E$, where

$$\infty_E(B) = \begin{cases} 0 & \text{if } \mathrm{Cap}(B \setminus E) = 0 \\ +\infty & \text{otherwise,} \end{cases} \tag{6.21}$$

corresponding to the zero condition on the set E. The notion of capacity is naturally linked to H^1-functions, that are defined up to sets of zero capacity, so that in the second case the condition $F(u) < +\infty$ can be equivalently read as $u \in H^1(\Omega)$ and $u = 0$ (up to a set of zero capacity) on E.

Definition 6.5 (relaxed Dirichlet problems) *We denote by $\mathcal{M}_0$ the set of all (possibly non-finite) non-negative Borel measures μ on $\mathbb{R}^n$ such that $\mu(B) = 0$ for every Borel set $B \subset \mathbb{R}^n$ of zero capacity.*

A relaxed Dirichlet problem *is a minimum problem of the form*

$$\min\Big\{ \int_\Omega |Du|^2\,dx + \int_\Omega |u|^2\,d\mu - 2\int_\Omega gu\,dx : u \in H_0^1(\Omega) \Big\},$$

where $\mu \in \mathcal{M}_0$ and $g \in L^2(\Omega)$; the solution to this minimum problem solves the problem

$$\begin{cases} -\Delta u + u\mu = g \\ u \in H_0^1(\Omega). \end{cases}$$

*We define the γ-*convergence *of μ_j to μ as the Γ-convergence of the functionals defined on $H_0^1(A)$ by*

$$F_{\mu_j}(u, A) = \int_A |Du|^2\,dx + \int_A |u|^2\,d\mu_j$$

to the corresponding

$$F_\mu(u,A)=\int_A |Du|^2\,dx+\int_A |u|^2\,d\mu$$

for all A bounded open subset of $\mathbb{R}^n$.

For the class $\mathcal{M}_0$ we have a compactness and density result as follows.

Theorem 6.6 (closure of relaxed Dirichlet problems) (i) *For every sequence (μ_j) in $\mathcal{M}_0$ there exist a subsequence, not relabeled, and μ in $\mathcal{M}_0$ such that μ_j γ-converge to μ.*

(ii) *For every $\mu\in\mathcal{M}_0$ there exists a sequence (K_j) of compact subsets of $\mathbb{R}^n$ such that the measures $\mu_j=\infty_{K_j}$ (defined as in* (6.21) *with $E=K_j$) γ-converge to μ.*

Proof For a complete proof we refer to the paper by Dal Maso and Mosco [88] Section 4. Here we want to highlight that for the proof of (i) (a variation of) the compactness method in Section 3.3 can be applied. In this case the limit F_0 of the functionals F_{μ_j}, obtained by a compactness and localization argument, can be written as $\int_A |Du|^2\,dx+G(u,A)$, and a suitable representation theorem for G by Dal Maso (see [83]) shows that $G(u,A)=\int_A g(x,u)\,d\mu$. Eventually, as F_0 is a quadratic form, we deduce that we may take $g(x,u)=|u|^2$, so that $F_0=F_\mu$.

As for (ii) note that the case $\mu=a(x)\mathcal{L}^n$ with $a\in L^\infty$ is taken care in Remark 6.3(i). In the general case, one proceeds by approximation. □

Remark 6.7 (computation of the limit of perforated domains) The construction in the previous section shows that in the case of functionals $F_{\infty_{K_\varepsilon}}$, where $K_\varepsilon=\bigcup_i \varepsilon i+\varepsilon^{\frac{n}{n-2}}K$ the measure μ in limit energy F_μ can be computed as the weak* limit of the measures $\sum_i \varepsilon^n \mathrm{Cap}(K)\delta_{\varepsilon i}$ (here $\delta_{\varepsilon i}$ stands for the Dirac mass at εi), and the effect of the capacity of the set K_ε can be decomposed as the sum of the capacities of each $\varepsilon i+\varepsilon^{\frac{n}{n-2}}K$. This is not the case in general, since the capacity is not an additive set function. Nevertheless, a formula for the limit of a family F_{μ_j} can be proved (in particular we may have $\mu_j=\infty_{K_j}$, with K_j an arbitrary perforation): the limit is F_μ if a Radon measure ν exists and $f:\mathbb{R}^n\to[0,+\infty]$ is such that $f(x)<+\infty$ up to a set of zero capacity and

$$\begin{aligned} f(x) &= \liminf_{\rho\to0}\liminf_j \frac{\mathrm{Cap}_{\mu_j}(B_\rho(x),B_{2\rho}(x))}{\nu(B_\rho(x))}\\ &= \liminf_{\rho\to0}\limsup_j \frac{\mathrm{Cap}_{\mu_j}(B_\rho(x),B_{2\rho}(x))}{\nu(B_\rho(x))}, \end{aligned}$$

where the μ_j*-capacity* is defined as

$$\mathrm{Cap}_{\mu_j}(E,A)=\min\Big\{\int_A |Du|^2\,dx+\int_E u^2\,d\mu_j : u-1\in H^1_0(A)\Big\} \tag{6.22}$$

then we have $\mu=f\,\nu$ (see Buttazzo, Dal Maso and Mosco [68] Theorem 5.2).

In particular, if $\mu_j=\infty_{K_j}$, then we have to compute the behaviour of

$$\begin{aligned} \mathrm{Cap}_{\mu_j}(E,A) &= \mathrm{Cap}(E\cap K_j,A)\\ &= \min\Big\{\int_A |Du|^2\,dx : u-1\in H^1_0(A),\ u=1 \text{ on } K_j\cap E\Big\}. \end{aligned}$$

This formula describes the local behaviour of the energies due to a perforation in terms of the μ_j-capacities.

Another way to express the measure μ is as the least superadditive set function satisfying

$$\mu(A) \geq \inf_{\substack{U \text{ open} \\ A \subseteq U}} \sup_{\substack{B \text{ compact} \\ B \subseteq U}} \limsup_j \mathrm{Cap}(K_j \cap B, \Omega)$$

for every Borel subset $A \subseteq \Omega$ (see Dal Maso [84]).

Remark 6.8 (limits of obstacle problems) As noted in Remark 6.3(iii) problems on perforated domains can be extended to problems with (unilateral or bilateral) obstacles. In particular the condition $u = 0$ on the perforation can be seen as a particular case of bilateral obstacle. We refer to the paper by Dal Maso [83] for the treatment of limits of such problems, and in particular for their integral representation, which is used to represent limits of relaxed Dirichlet problems.

Remark 6.9 (closure of quadratic forms with Dirichlet boundary conditions) Theorem 4.18 shows that the closure of quadratic forms of *diffusion type* are all objective Dirichlet forms. On the other hand, Remark 6.4 has shown that functionals on perforated domains, and hence also all functionals of relaxed Dirichlet problems by the result above, can be obtained as limits of quadratic forms of diffusion type on $H^1_0(\Omega)$. Note that relaxed Dirichlet problems possess the missing non-objective part in the Deny-Beurling formula. In fact, a result by Camar-Eddine and Seppecher shows that *the closure of quadratic forms of diffusion type are all Dirichlet forms.* We refer to [69] for details.

6.3 Neumann boundary conditions: an extension lemma

The issues in the treatment of Neumann boundary conditions are different; the first one being which convergence to use in the definition of Γ-limit; the second one being the most general hypothesis under which a limit exists and defines a non-degenerate functional. The fundamental tool to answer these questions is an *extension lemma* by Acerbi, Chiadò Piat, Dal Maso and Percivale [3] (see also [54] Appendix).

Lemma 6.10 *Let E be a periodic, connected, open subset of $\mathbb{R}^n$, with Lipschitz boundary. Given a bounded open set $\Omega \subset \mathbb{R}^n$, and a real number $\varepsilon > 0$, there exist a linear and continuous extension operator $T_\varepsilon : W^{1,p}(\Omega \cap \varepsilon E) \to W^{1,p}_{\mathrm{loc}}(\Omega)$ and three constants $k_0, k_1, k_2 > 0$, such that*

$$T_\varepsilon u = u \quad \text{a.e. in } \Omega \cap \varepsilon E, \tag{6.23}$$

$$\int_{\Omega(\varepsilon k_0)} |T_\varepsilon u|^p dx \leq k_1 \int_{\Omega \cap \varepsilon E} |u|^p dx, \tag{6.24}$$

$$\int_{\Omega(\varepsilon k_0)} |D(T_\varepsilon u)|^p dx \leq k_2 \int_{\Omega \cap \varepsilon E} |Du|^p dx, \tag{6.25}$$

where we use the notation $A(\lambda)$ for the retracted set $\{x \in A : \mathrm{dist}\,(x, \partial A) > \lambda\}$, *for every $u \in W^{1,p}(\Omega \cap \varepsilon E)$. The constants k_0, k_1, k_2 depend on E, n, p, but are independent of ε and Ω.*

With this result in mind we can look for the behaviour of solutions to problems of the form

$$m_\varepsilon = \min\Big\{\int_{\Omega\cap\varepsilon E} f\Big(\frac{x}{\varepsilon}, Du\Big)\,dx - \int_{\Omega\cap\varepsilon E} gu\,dx : u = \varphi \text{ on } \partial\Omega\Big\}. \tag{6.26}$$

In fact, if u_ε is a solution to m_ε, then we can consider $T_\varepsilon u_\varepsilon$ as defined above (componentwise if $u : \Omega\cap\varepsilon E \to \mathbb{R}^m$ with $m > 1$). If f satisfies the growth condition of Theorem 4.2 then we infer that $(T_\varepsilon u_\varepsilon)$ is locally bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$ so that a limit $u \in W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^m)$ exists up to subsequences. Actually, it is easily seen that we obtain a uniform bound on $|Du|^p$ on each $\Omega(\lambda)$ independent of λ, so that $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. The notion of Γ-convergence in this case must follow this compactness result.

Theorem 6.11 (homogenization of perforated domains) *Let f satisfy the hypotheses of Theorem* 5.1*, let E be a periodic, connected, open subset of $\mathbb{R}^n$, with Lipschitz boundary and let*

$$F_\varepsilon(u) = \int_{\Omega\cap\varepsilon E} f\Big(\frac{x}{\varepsilon}, Du\Big)\,dx \qquad u \in W^{1,p}(\Omega\cap\varepsilon E;\mathbb{R}^m). \tag{6.27}$$

Then F_ε Γ-converge with respect to the weak convergence in $W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^m)$ to the functional defined on $W^{1,p}(\Omega;\mathbb{R}^m)$ by $F_{\rm hom}(u) = \int_\Omega f_{\rm hom}(Du)\,dx$, with $f_{\rm hom}$ still satisfying a growth condition as f, given by

$$f_{\rm hom}(\xi) = \lim_{T\to+\infty}\frac{1}{T^n}\inf\Big\{\int_{(0,T)^n\cap E} f(x,\xi+D\varphi)\,dx : \varphi \in W^{1,p}_0((0,T)^n\cap E;\mathbb{R}^m)\Big\}. \tag{6.28}$$

This formula can be simplified to a cell-problem formula if $f(y,\cdot)$ is convex. Furthermore, if Ω has a Lipschitz boundary then problems m_ε converge to

$$m_{\rm hom} = \min\Big\{\int_\Omega f_{\rm hom}(Du)\,dx - C\int_\Omega gu\,dx : u = \varphi \text{ on } \partial\Omega\Big\}, \tag{6.29}$$

where $C = |E\cap(0,1)^n|$.

It must be noted that the result of Γ-convergence still holds if we only suppose that E is connected and contains a periodic connected set with Lipschitz boundary (for example we can take E as the complement of a periodic array of 'cracks'; i.e., of $n-1$-dimensional closed sets). Of course, in this case in general the solutions to m_ε cannot be extended to $W^{1,p}_{\rm loc}$-functions in Ω. For details we refer to [54] Chapter 20.

6.4 Double-porosity homogenization

The homogenization of perforated media presents an interesting variant when the 'holes' are not 'empty', but the energy density therein has a different scaling. The prototype of such problems is of the form

$$m_\varepsilon = \min\Big\{\int_{\Omega\cap\varepsilon E} |Du|^2\,dx + \varepsilon^2\int_{\Omega\setminus\varepsilon E} |Du|^2\,dx - \int_\Omega gu\,dx : u = \varphi \text{ on } \partial\Omega\Big\}, \tag{6.30}$$

where E is a periodic open subset of $\mathbb{R}^n$, with Lipschitz boundary, not necessarily connected. Contrary to (6.26) on the part $\Omega\setminus\varepsilon E$ we consider a 'weak' energy scaling as ε^2 (other scalings as usual can be considered giving less interesting results). Note moreover that now the forcing term $\int_\Omega gu\,dx$ is considered on the whole Ω.

A first observation is that if E is also connected then the Γ-limit with respect to the weak convergence in $W^{1,p}_{\rm loc}(\Omega;\mathbb{R}^m)$ of $\int_{\Omega\cap\varepsilon E}|Du|^2\,dx+\varepsilon^2\int_{\Omega\setminus\varepsilon E}|Du|^2\,dx$ is the same as that of $\int_{\Omega\cap\varepsilon E}|Du|^2\,dx$ by Theorem 6.11 and a simple comparison argument. However, it must be noted that we cannot derive from this result the convergence of problems m_ε, as we cannot obtain a bound on the L^2 norms of the gradients of solutions (u_ε).

A way to overcome this lack of compactness is by considering only the part of Ω where we can apply the compactness argument in the previous section: we define a limit u considering only the limit of $T_\varepsilon u_\varepsilon$. In this way the contribution of $\varepsilon^2\int_{\Omega\setminus\varepsilon E}|Du_\varepsilon|^2\,dx-\int_{\Omega\setminus E}gu_\varepsilon\,dx$ can be considered as a perturbation. In order to understand its effect, suppose that $K=(0,1)^n\setminus E$ is compactly contained in $(0,1)^n$, and g is continuous.

We focus on the energy contained on a set $\varepsilon i+\varepsilon K$. Since $u_\varepsilon\to u$, we may suppose that $u_\varepsilon=u(\varepsilon i)$ on $\partial(\varepsilon i+\varepsilon K)$, so that we may estimate the contribution

$$\begin{aligned}
&\varepsilon^2\int_{\varepsilon i+\varepsilon K}|Du_\varepsilon|^2\,dx-\int_{\varepsilon i+\varepsilon K}gu_\varepsilon\,dx\\
\geq\ &\inf\Bigl\{\varepsilon^2\int_{\varepsilon K}|Dv|^2\,dx-\int_{\varepsilon K}g(\varepsilon i+x)v\,dx : v=u(\varepsilon i)\text{ on }\partial\varepsilon K\Bigr\}\\
=\ &\varepsilon^n\inf\Bigl\{\int_K|Dv|^2\,dx-\int_K g(\varepsilon i+\varepsilon x)v\,dx : v=u(\varepsilon i)\text{ on }\partial K\Bigr\}.
\end{aligned}$$

If we set

$$\phi(x,u)=\inf\Bigl\{\int_K|Dv|^2\,dx-g(x)\int_K v\,dx : v=u\text{ on }\partial K\Bigr\}$$

then we deduce a lower estimate of the limit of the contributions on $\Omega\setminus\varepsilon E$ by $\int_\Omega\phi(x,u(x))\,dx$.

This argument can be carried over rigorously and also removing the assumption that E consists of a single connected component, as stated in the next section.

6.4.1 Multi-phase limits

Let $E=\bigcup_{i=1}^N E_i$ where E_i are periodic connected open subsets of $\mathbb{R}^n$ with Lipschitz boundary and such that $\overline{E}_i\cap\overline{E}_j=\emptyset$ for $i\neq j$. We also set $E_0=\mathbb{R}^n\setminus E$. Let the extension operators T^j_ε corresponding to $\Omega\cap E_j$ be defined as in Theorem 6.10. We define the convergence on $H^1(\Omega)$ as the $L^2_{\rm loc}$ convergence of these extensions. Namely, we will write that $u_\varepsilon\to(u_1,\dots,u_N)$ if $T^j_\varepsilon u_\varepsilon\to u_j$ for all $j=1,\dots,N$, or, equivalently, if

$$\lim_{\varepsilon\to0}\sum_{i=1}^N\int_{\Omega\cap\varepsilon E_i}|u_\varepsilon-u_i|^2\,dx=0. \tag{6.31}$$

We define the energies

$$F_\varepsilon(u)=\int_{\Omega\cap\varepsilon E}|Du|^2\,dx+\varepsilon^2\int_{\Omega\cap\varepsilon E_0}|Du|^2\,dx+\int_\Omega|u|^2\,dx \tag{6.32}$$

for $u\in H^1(\Omega)$. For simplicity we only consider the quadratic perturbation $\int_\Omega|u|^2\,dx$. By Theorem 6.10 F_ε are equicoercive with respect to the convergence above.

Note that by the closure of quadratic forms there exist $A^j_{\rm hom}$ constant matrices such that

$$\int_\Omega \langle A^j_{\rm hom} Du, Du\rangle\, dx = \Gamma\text{-}\lim_{\varepsilon\to 0}\int_{\Omega\cap\varepsilon E_j} |Du|^2 dx, \tag{6.33}$$

in the sense of Theorem 6.11. Moreover we define

$$\phi(z_1,\dots,z_N) = \min\Big\{\int_{E_0\cap(0,1)^n} (|Dv|^2 + |v|^2)\, dy : \\ v\in H^1_\#((0,1)^n),\ v = z_j \text{ on } E_j,\ j=1,\dots,N\Big\}. \tag{6.34}$$

The following theorem is a particular case of a result by Braides, Chiadò Piat and Piatnitski [50], where general integrands in the vector case are considered.

Theorem 6.12 *If $|\partial\Omega| = 0$ then the functionals F_ε defined by* (6.32) *Γ-converge with respect to the convergence* (6.31) *to the functional $F_{\rm hom}$ with domain $H^1(\Omega;\mathbb{R}^N)$ defined by*

$$F_{\rm hom}(u_1,\dots,u_N) = \sum_{j=1}^N \int_\Omega (\langle A^j_{\rm hom} Du_j, Du_j\rangle + C_j|u_j|^2)\, dx + \int_\Omega \phi(u_1,\dots,u_N)\, dx, \tag{6.35}$$

where $C_j = |E_j\cap(0,1)^n|$ and $A^j_{\rm hom}$ and ϕ are given by (6.33) *and* (6.34)*, respectively.*

7 Phase-transition problems

In the previous chapters we have examined sequences of functionals defined on Sobolev spaces, whose minimizers satisfy some weak compactness properties, so that the limit is automatically defined on a Sobolev space, even though the actual form of the limit takes into account oscillations and compactness effects. In this section we will consider families of functionals whose minimizers tend to generate sharp interfaces between zones where they are approximately constant. In the limit we expect the relevant properties of such minimizers to be described by energies, whose domain are partitions of the domain Ω into sets (the *phase domains*).

7.1 Interfacial energies

The types of energies we have in mind are functionals defined on partitions of a reference set Ω into sets, which take into account some measure of the interface between those sets. The simplest of such functionals is the 'perimeter functional', suitably defined to suit problems in the Calculus of Variations.

7.1.1 Sets of finite perimeter

The simplest way to have a definition of *perimeter* which is lower semicontinuous by the L^1-convergence of the sets is by *lower-semicontiuity*: if $E\subset\mathbb{R}^n$ is of class C^1 define the perimeter $\mathcal{P}(E,\Omega)$ of the set E inside the open set Ω in a classical way, and then for an arbitrary set, define

$$\mathcal{P}(E,\Omega) = \inf\{\liminf_j \mathcal{P}(E_j,\Omega) : \chi_{E_j}\to\chi_E \text{ in } L^1(\Omega),\ E_j \text{ of class } C^1\}.$$

Another choice leading to the same definition is to start with E_j of polyhedral type.

If $\mathcal{P}(E,\Omega) < +\infty$, then we say that E is a *set of finite perimeter* or *Caccioppoli set* in Ω. For such sets it is possible to define a notion of measure-theoretical boundary, where a normal is defined, so that we may heuristically picture those sets as having a smooth boundary. In order to make these concepts more precise we recall the definition of the *k-dimensional Hausdorff measure* (in this context we will limit ourselves to $k \in \mathbb{N}$). If E is a Borel set in $\mathbb{R}^n$, then we define

$$\mathcal{H}^k(E) = \sup_{\delta>0} \frac{\omega_k}{2^k} \inf\Big\{\sum_{i\in\mathbb{N}} (\operatorname{diam} E_i)^k : \ \operatorname{diam} E_i \le \delta,\ E \subseteq \bigcup_{i\in\mathbb{N}} E_i\Big\},$$

where ω_k is the Lebesgue measure of the unit ball in $\mathbb{R}^k$.

We say that $x \in E$ is a *point of density* $t \in [0,1]$ if the limit $\lim_{\rho\to 0+}(\omega_n)^{-1}\rho^{-n}|E \cap B_\rho(x)| = t$ exists. The set of all points of density t will be denoted by E_t. If E is a set of finite perimeter in Ω then the De Giorgi's *essential boundary* of E, denoted by $\partial^* E$, is defined as the set of points $x \in \Omega$ with density $1/2$.

Theorem 7.1 (De Giorgi's Rectifiability Theorem) *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter in Ω. Then $\partial^* E$ is* rectifiable*; i.e., there exists a countable family (Γ_i) of graphs of C^1 functions of $(n-1)$ variables such that $\mathcal{H}^{n-1}(\partial^* E \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$. Moreover the perimeter of E in $\Omega' \subseteq \Omega$ is given by*

$$\mathcal{P}(E,\Omega') = \mathcal{H}^{n-1}(\partial^* E \cap \Omega').$$

By the previous theorem and the Implicit Function Theorem a *internal normal* $\nu = \nu_E(x)$ to E is defined at $\mathcal{H}^{n-1}$-almost all points x of $\partial^* E$ as the normal of the corresponding Γ_i. A *generalized Gauss-Green formula* holds, which states that the distributional derivative of χ_E is a vector measure given by

$$D\chi_E(B) = \int_B \nu_E \, d\mathcal{H}^{n-1}.$$

In particular, we have $\mathcal{P}(E,\Omega) = |D\chi_E|(\Omega)$, the total variation of the measure $D\chi_E$ on Ω, so that χ_E is a *function with bounded variation*.

A finite *Caccioppoli partition*; i.e., a partition of Ω into sets of finite perimeter $E_1,\dots,E_M$ can be identified with an element $u \in BV(\Omega;T)$, where $\#T = M$. In this case we will also use the notation $S(u)$ for $\bigcup_i \partial^* E_i$, which is the *jump set* of u. This notation also holds if $u = \chi_E$.

7.1.2 Convexity and subadditivity conditions

From the characterization above we easily see that the characteristic functions of a sequence of sets with equi-bounded perimeter are bounded in BV, so that we may extract a converging subsequence in L^1, and $\mathcal{P}(E,\Omega) = |D\chi_E|(\Omega) \le \liminf_j |D\chi_{E_j}|(\Omega) = \liminf_j \mathcal{P}(E_j,\Omega)$ by the lower semicontinuity of the total variation, so that $\mathcal{P}(\cdot,\Omega)$ is a lower semicontinuous functional.

Actually, it may be easily seen that functionals of the form

$$F(E) = \int_{\Omega\cap\partial^* E} \varphi(\nu)\, d\mathcal{H}^{n-1} \tag{7.1}$$

are L^1-lower semicontinuous if and only if the positively homogeneous extension of degree 1 of φ to $\mathbb{R}^n$ is convex.

7.1.3 Integral representation

The application of the localization methods often necessitates the representation of functionals defined on sets of finite perimeter or on (finite) Caccioppoli partitions. An analogue of the representation theorem for integral functionals is the following, of which a simple proof can be obtained from that in the paper by Braides and Chiadò Piat [49] Section 3, where it is directly proved for infinite Caccioppoli partitions.

Theorem 7.2 (integral representation on Caccioppoli partitions) *Let T be a finite set and $F : BV(\Omega;T)\times\mathcal{B}(\Omega)\to[0,+\infty)$ be a function defined on pairs Caccioppoli partition/Borel subset of Ω, satisfying*

(i) *$F(u,\cdot)$ is a measure for every $u\in BV(\Omega;T)$*

(ii) *F is* local *on open sets; i.e.,$F(u,A)=F(v,A)$ whenever $u=v$ a.e. in A*

(iii) *$F(\cdot,A)$ is L^1-lower semicontinuous for all open sets A*

(iv) *there exist constants $c_1,c_2>0$ such that $c_1\mathcal{H}^{n-1}(B\cap S(u))\le F(u,B)\le c_2\mathcal{H}^{n-1}(B\cap S(u))$.*

Then there exist Borel functions $\varphi_{ij}:\Omega\times S^{n-1}\to[0,+\infty)$ such that

$$F(u,B)=\sum_{i\neq j}\int_{B\cap\partial^*E_i\cap\partial^*E_j}\varphi_{ij}(x,\nu_j)d\mathcal{H}^{n-1} \tag{7.2}$$

for all $u\in BV(\Omega;T)$ identified with the partition $E_1,\dots,E_M$ with inner normal ν_j to E_j, and every Borel subset B of Ω.

For Caccioppoli partitions lower-semicontinuity conditions are more complex than the simple convexity: for homogeneous functionals F as in (7.2) of the form

$$F(E_1,\dots,E_M)=\sum_{i<j}\int_{\Omega\cap\partial^*E_i\cap\partial^*E_j}\varphi_{ij}(\nu_i)\,d\mathcal{H}^{n-1}, \tag{7.3}$$

where ν_i is the interior normal to E_i, necessary conditions are the convexity of each φ_{ij}, and their *subadditivity*: $\varphi_{ij}(\nu)\le\varphi_{ik}(\nu)+\varphi_{kj}(\nu)$ for all ν. These two combined conditions are not sufficient, and a more complex condition called *BV-ellipticity*, that mirrors the notion of quasiconvexity, turns out to be necessary and sufficient [18].

7.1.4 Energies depending on curvature terms

In the literature other types of energies defined on boundaries of sets have been introduced, especially for Computer Vision models. One type of energy (in a two dimensional setting) is the *elastica functional* (see Mumford [132])

$$F(E)=\int_{\partial E}(1+\kappa^2)d\mathcal{H}^1, \tag{7.4}$$

defined on sets with $W^{2,2}$ boundary, where κ denotes the curvature of ∂E. Note that on one hand the $W^{2,2}$ bounds ensure easier compactness properties for sets with equi-bounded energy, while on the other hand parts of the boundary of such sets may 'cancel' in the limit, giving rise to sets with cusp singularities. As a consequence the functional is not lower semicontinuous and its relaxation exhibits complex non-local effects that have been studied by Bellettini, Dal Maso and Paolini [33] and more recently by Bellettini and Mugnai [34].

7.2 Gradient theory of phase transitions

It is well known that the minimization of a non-convex energy often leads to minimizing sequences with oscillations, highlighted by a relaxation of the energy. This is not the case if we add a singular perturbation with a gradient term. We will be looking at the behaviour of minimum problems

$$\min\Big\{\int_\Omega W(u)\,dx+\varepsilon^2\int_\Omega |Du|^2\,dx:\int_\Omega u\,dx=C\Big\},\tag{7.5}$$

where $u:\Omega\to\mathbb{R}$, and W is a non-convex energy. Upon an affine translation of u, that does not change the minimizers of problem (7.5), is not restrictive to suppose that

$$W\geq 0 \text{ and } W(u)=0 \text{ only if } u=0,1\tag{7.6}$$

(or two other points). W is called a *double-well energy* and the energies above are related to the Cahn-Hilliard theory of liquid-liquid phase transitions.

Note that under the hypotheses above, if $0<C<|\Omega|$ then the minimum of $\int_\Omega W(u)\,dx$ is 0, and is achieved on any $u=\chi_E$ with $|E|=C$. The gradient term however forbids such configurations, and we expect the creation of interfaces to be penalized by the second integral. A heuristic scaling argument can be performed in dimension one to understand the scale of this penalization: if the transition of u is on an interval I of size δ, where the gradient is of the order $1/\delta$, we have

$$\int_I W(u)\,dx+\varepsilon^2\int_I |u'|^2\,dx\approx\delta+\frac{\varepsilon^2}{\delta}.\tag{7.7}$$

The minimization in δ gives $\delta=\varepsilon$ and a contribution of order ε. This argument suggests a scaling of the problem and to consider

$$m_\varepsilon=\min\Big\{\frac{1}{\varepsilon}\int_\Omega W(u)\,dx+\varepsilon\int_\Omega |Du|^2\,dx:\int_\Omega u\,dx=C\Big\},\tag{7.8}$$

whose minimizers are clearly the same as the problem above.

7.2.1 The Modica-Mortola result

The Γ-limit of the energy above is one of the first examples in the literature and is due to Modica and Mortola [124] (see also [123, 142, 41, 45, 5]). In this section we will examine the behaviour of the energies

$$F_\varepsilon(u)=\frac{1}{\varepsilon}\int_\Omega W(u)\,dx+\varepsilon\int_\Omega |Du|^2\,dx,\qquad u\in H^1(\Omega),\tag{7.9}$$

with W as above and such that $W(u)\geq c(|u|^2-1)$.

Note that the *volume constraint* $\int_\Omega u\,dx=C$ is not continuous, so that it cannot be simply added to the Γ-limit, so that a separate argument must, and will, be used.

Theorem 7.3 (Modica-Mortola's theorem) *The functionals above Γ-converge with respect to the $L^1(\Omega)$ convergence to the functional*

$$F_0(u)=\begin{cases}c_W\mathcal{P}(\{u=1\},\Omega)=c_W\mathcal{H}^{n-1}(\partial^*\{u=1\}\cap\Omega) & \text{if } u\in\{0,1\} \text{ a.e.}\\ +\infty & \text{otherwise,}\end{cases}\tag{7.10}$$

where $c_W=2\int_0^1\sqrt{W(s)}\,ds$.

Proof *The one-dimensional case.* Suppose that $\lim_j F_{\varepsilon_j}(u_j) < +\infty$. Fix some $\eta > 0$ and consider an interval I such that u_j takes the values η and $1-\eta$ at the endpoints of the interval. We can use the following *Modica-Mortola trick* to estimate

$$\int_I \Big(\frac{1}{\varepsilon_j} W(u_j) + \varepsilon_j |u'|^2\Big)\, dt \geq 2 \int_I \sqrt{W(u_j)}|u_j'|\, dt \geq 2 \int_\eta^{1-\eta} \sqrt{W(s)}\, ds =: C_\eta, \tag{7.11}$$

(we have simply used the algebraic inequality $a^2 + b^2 \geq 2ab$ and the change of variables $s = u_j(t)$). From this inequality we easily deduce that the number of transitions between η and $1-\eta$ is equibounded. Since $\int_\Omega W(u_j)\, dt \leq \varepsilon C$ we also deduce that $u_j \to \{0,1\}$ in measure, so that we have (up to subsequences) $u_j \to u$, where u is a piecewise-constant function taking values in $\{0,1\}$. If we denote by $S(u)$ the set of discontinuity points of u the inequality above yields

$$\liminf_j F_{\varepsilon_j}(u_j) \geq C_\eta\, \#(S(u)), \tag{7.12}$$

and then the lower bound is achieved by the arbitrariness of η.

To prove the limsup inequality, take v the solution of

$$v'(s) = \sqrt{W(v)} \qquad v(0) = \frac{1}{2} \tag{7.13}$$

(suppose for simplicity that we have a global solution to this problem), and define $v_\varepsilon(t) = v(t/\varepsilon)$. Note that v_ε tends to $H = \chi_{[0,+\infty)}$ (the Heaviside function with jump in 0) and it optimizes the inequality in (7.11): $\frac{1}{\varepsilon} W(v_\varepsilon) + \varepsilon |v_\varepsilon'|^2 = 2\sqrt{W(v_\varepsilon)}|v_\varepsilon'|$ so that it gives a recovery sequence for $u(t) = H(t)$. For a general $u \in \{0,1\}$ we easily construct a recovery sequence by suitably gluing the functions $v_\varepsilon((\overline{t} \pm t)/\varepsilon)$, where $\overline{t} \in S(u)$.

The n-dimensional case. In order to to apply the 'slicing procedure' we will need a result characterizing sets of finite perimeter through their sections. For a piecewise-constant function u on an open set of $\mathbb{R}$ we use the notation $S(u)$ for its set of discontinuity points (if u is thought as an L^1 function we mean its essential discontinuity points). We use the notation for one-dimensional sections introduced in Chapter 3.4.

Theorem 7.4 (sections of sets of finite perimeter)

(a) *Let E be a set of finite perimeter in a smooth open set $\Omega \subset \mathbb{R}^n$ and let $u = \chi_E$. Then for all $\xi \in S^{n-1}$ and for $\mathcal{H}^{n-1}$-a.a. $y \in \Pi_\xi$ the function $u_{\xi,y}$ is piecewise constant on $\Omega_{\xi,y}$. Moreover, for such y we have $S(u_{\xi,y}) = \{t \in \mathbb{R} :\ y + t\xi \in \Omega \cap \partial^* E\}$, and for all Borel functions g*

$$\int_{\Pi_\xi} \sum_{t \in S(u_{\xi,y})} g(t)\, d\mathcal{H}^{n-1}(y) = \int_{\Omega \cap \partial^* E} g(x) |\langle \nu_E, \xi\rangle| d\mathcal{H}^{n-1}\,. \tag{7.14}$$

(b) *Conversely, if $E \subset \Omega$ and for all $\xi \in \{e_1, \dots, e_n\}$ and for $\mathcal{H}^{n-1}$-a.a. $y \in \Pi_\xi$ the function $u_{\xi,y}$ is piecewise constant in each interval of $\Omega_{\xi,y}$ and*

$$\int_{\Pi_\xi} \#(S(u_{\xi,y}))\, d\mathcal{H}^{n-1}(y) < +\infty\,, \tag{7.15}$$

then E is a set of finite perimeter in Ω.

We follow the steps outlined in Section 3.4.
Step 1. The localized functionals are

$$F_\varepsilon(u,A)=\frac{1}{\varepsilon}\int_A W(u)\,dx+\varepsilon\int_A|Du|^2\,dx. \tag{7.16}$$

Step 2. We choose

$$F_\varepsilon^{\xi,y}(v,I)=\frac{1}{\varepsilon}\int_I W(v)\,dt+\varepsilon\int_I|v'|^2\,dt \tag{7.17}$$

(in this case $F_\varepsilon^{\xi,y}$ is independent of y). We then have, by Fubini's Theorem,

$$F_\varepsilon^{\xi}(u,A)=\frac{1}{\varepsilon}\int_A W(u)\,dx+\varepsilon\int_A|\langle\xi,Du\rangle|^2\,dx \tag{7.18}$$

Note that $F_\varepsilon^\xi\leq F_\varepsilon$.
Step 3. By the one-dimensional proof the Γ-limit

$$F^{\xi,y}(v,I):=\Gamma\text{-}\lim_{\varepsilon\to 0}F_\varepsilon^{\xi,y}(v,I)=c_W\#(S(v))\text{ if }v\in\{0,1\}\text{ a.e. on }I. \tag{7.19}$$

($+\infty$ otherwise). We define F^ξ as in (3.12). Note that $F^\xi(u,A)$ is finite if and only if $u\in\{0,1\}$ a.e. in Ω, $u_{\xi,y}$ is piecewise constant on $A_{\xi,y}$ for $\mathcal{H}^{n-1}$-a.a. $y\in\Pi_\xi$, and (7.15) holds.
Step 4. From Fatou's Lemma we deduce that

$$\Gamma\text{-}\liminf_{\varepsilon\to0^+}F_\varepsilon(u,A)\geq\Gamma\text{-}\liminf_{\varepsilon\to0^+}F_\varepsilon^\xi(u,A)\geq F^\xi(u,A)\,,$$

for all $\xi\in S^{n-1}$.
Step 5. By Step 3 and Theorem 7.4(b) we deduce that the Γ-lower limit $F'(u,A)=\Gamma\text{-}\liminf_{\varepsilon\to0^+}F_\varepsilon(u,A)$ is finite only if $u=\chi_E$ for some set E of finite perimeter in A. Moreover, $C>0$ exists such that $F'(u,A)\geq C\mathcal{H}^{n-1}(A\cap\partial^*E)$.
Step 6. If $u=\chi_E$ for some set E of finite perimeter, from Theorem 7.4(a) we have

$$F^\xi(u,A)=c_W\int_{A\cap\partial^*E}|\langle\xi,\nu_u\rangle|d\mathcal{H}^{n-1}(y)\,. \tag{7.20}$$

Hence

$$F'(u,A)\geq c_W\int_{A\cap\partial^*E}|\langle\xi,\nu_u\rangle|d\mathcal{H}^{n-1}(y)\,. \tag{7.21}$$

Step 7. Since all F_ε are local, then if $u=\chi_E$ for some set E of finite perimeter the set function $\mu(A)=F'(u,A)$ is superadditive on disjoint open sets. From Theorem 3.1 applied with $\lambda=\mathcal{H}^{n-1}\llcorner\partial^*E$, and $\psi_i(x)=\chi_{\partial^*E}|\langle\xi_i,\nu_u\rangle|$, where (ξ_i) is a dense sequence in S^{n-1}, we conclude that

$$F'(u,A)\geq c_W\int_{S(u)\cap\partial^*E}\sup_i\{|\langle\xi_i,\nu\rangle|\}d\mathcal{H}^{n-1}. \tag{7.22}$$

The liminf inequality follows noticing that $\sup_i\{|\langle\xi_i,\nu\rangle|\}=1$.

The liminf inequality is sharp for functions with a 'unidimensional' profile; i.e., that on lines orthogonal to ∂^*E follow the one-dimensional recovery sequences. This argument can be easily carried over if ∂^*E is smooth; the general case will then be achieved by approximation via the following result (whose proof easily follows from Sard's theorem by using the coarea formula).

Proposition 7.5 (density of smooth sets) *If Ω is a Lipschitz set and E is a set of finite perimeter in Ω, then there exists a sequence (E_j) of sets of finite perimeter in Ω, such that $\lim_j |E\Delta E_j| = 0$, $\lim_j \mathcal{P}(E_j,\Omega) = \mathcal{P}(E,\Omega)$, and for every open set Ω' with $\Omega \subset\subset \Omega'$ there exist sets E'_j of class C^∞ in Ω' and such that $E'_j \cap \Omega = E_j$.*

It remains to exhibit a recovery sequence when ∂E is smooth. In that case it suffices to take

$$u_\varepsilon(x) = v\Big(\frac{d(x)}{\varepsilon}\Big) \tag{7.23}$$

where $d(x) = \mathrm{dist}(x,\Omega\setminus E) - \mathrm{dist}(x,E)$ is the *signed distance function* to ∂E. A simple computation using the coarea formula in the form

$$\int_\Omega f(x)|Dd|\,dx = \int_{-\infty}^{+\infty}\int_{\{d=t\}\cap A} f(y)d\mathcal{H}^{n-1}(y)\,dt \tag{7.24}$$

valid if f is a Borel function (recalling that $|Dd| = 1$ a.e.) gives the desired estimate. □

Remark 7.6 (optimal profile problem) We may rewrite the constant c_W as the minimum problem

$$c_W = \min\Big\{\int_{-\infty}^{+\infty}(W(v) + |v'|^2)\,dt : u(-\infty) = 0,\ u(+\infty) = 1\Big\}. \tag{7.25}$$

By the proof above we get that the function v defined in (7.13) is a solution to this minimum problem (*optimal profile* problem). The proof of the limsup inequality shows that a recovery sequence is obtained by scaling an optimal profile.

Remark 7.7 (generalizations) By some easy convexity arguments we can adapt the proof above to the case when we substitute $|Du|^2$ by a more general $\varphi^2(Du)$ with φ convex and positively homogeneous of degree one (see [45] Section 4.1.2), in which case the limit is given by

$$F_0(u) = c_W \int_{\Omega\cap\partial^* E} \varphi(\nu)d\mathcal{H}^{n-1} \tag{7.26}$$

if $u = \chi_E$. As a particular case, we may take $\varphi(\nu) = \sqrt{\langle A\nu,\nu\rangle}$.

7.2.2 Addition of volume constraints

As for boundary conditions for integral functionals, to apply the theorem above to the convergence of the minimum problems m_ε we have to prove that the volume constraint $\int_\Omega u\,dx = C$ is *compatible* with the Γ-limit. Clearly, the constraint is closed under $L^1(\Omega)$-convergence. By taking the density argument into account, the compatibility then amounts to proving the following.

Proposition 7.8 (compatibility of volume constraints) *Let E be a set with smooth boundary; then there exist $\overline{u}_\varepsilon \to \chi_E$ such that $\int_\Omega \overline{u}_\varepsilon\,dx = |E\cap\Omega|$ and $\lim\limits_{\varepsilon\to+\infty} F_\varepsilon(\overline{u}_\varepsilon) = c_W\mathcal{H}^{n-1}(\partial E\cap\Omega)$.*

Proof The proof of this proposition can be easily achieved by adding to the recovery sequence constructed above u_ε a suitable perturbation. For example, if W is smooth in 0 and 1, we can choose a ball B contained in E (or $\Omega\setminus E$; it is not restrictive to suppose that such B exists by approximation) and $\phi_\varepsilon \in C_0^\infty(B)$ with $0 \le \phi_\varepsilon \le 1$, $\phi_\varepsilon \to 1$ and $\int_B(|\phi_\varepsilon|^2 + \varepsilon^2|D\phi_\varepsilon|^2)\,dx = O(\varepsilon)$, and consider $\overline{u}_\varepsilon = u_\varepsilon + c_\varepsilon\phi_\varepsilon$ where $c_\varepsilon \to 0$ are such that $\int_\Omega \overline{u}_\varepsilon\,dx = |E\cap\Omega|$. □

With this proposition the proof of the Γ-limit of the functionals in (7.8) is complete. Note moreover that the sequence is equi-coercive by the proof of Theorem 7.3. We then obtain the convergence result as follows.

Corollary 7.9 (convergence to minimal sharp interfaces) *Let u_ε be a minimizer for problem m_ε as defined in* (7.8)*. Then, we have $m_\varepsilon \to m$, and, up to subsequences, $u_\varepsilon \to u$, where $u = \chi_E$, and E is a minimizer of the problem*

$$m = \min\Big\{c_W \mathcal{P}(E;\Omega) : |E| = C\Big\}. \tag{7.27}$$

7.2.3 A selection criterion: minimal interfaces

The convergence above can also be read as a result on the convergence of the original minimum problems in (7.5). Note that we have

$$\int_\Omega W^{**}(u)\,dx = \Gamma\text{-}\lim_{\varepsilon\to 0}\int_\Omega (W(u) + \varepsilon^2|Du|^2)\,dx \tag{7.28}$$

with respect to the weak-$L^2(\Omega)$ convergence, so that the limit of (7.5) can be expressed as

$$\min\Big\{\int_\Omega W^{**}(u)\,dx : \int_\Omega u\,dx = C\Big\}, \tag{7.29}$$

where W^{**} is the convex envelope of W. In our case $W^{**} = 0$ on $[0,1]$ so that the first minimum is 0 and is achieved on all test functions with $0 \le u \le 1$.

In other words, sequences (u_ε) with $\int_\Omega (W(u_\varepsilon) + \varepsilon^2|Du_\varepsilon|^2)\,dx = o(1)$ may converge weakly in $L^2(\Omega)$ to any $0 \le u \le 1$. On the contrary if $\int_\Omega (W(u_\varepsilon) + \varepsilon^2|Du_\varepsilon|^2)\,dx = o(\varepsilon)$, then $u_\varepsilon \to \chi_E$, where E is a set with minimal perimeter. Since the minimum in (7.29) coincides with

$$\min\Big\{\int_\Omega W(u)\,dx : \int_\Omega u\,dx = C\Big\} \tag{7.30}$$

the addition of the singular perturbation represents a choice criterion between all minimizers of this non-convex variational problem.

7.2.4 Addition of boundary values

It is interesting to note that boundary values are trivially *not compatible* with this Γ-limit, as the limit energy is defined only on characteristic functions. Nevertheless the limit of an energy of the form

$$F_\varepsilon^\varphi(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega W(u)\,dx + \varepsilon\int_\Omega |Du|^2\,dx & \text{if } u = \varphi \text{ on } \partial\Omega \\ +\infty & \text{otherwise,}\end{cases} \tag{7.31}$$

can be easily computed.

To check this, we may first consider the one-dimensional case, with $\Omega = (0,1)$ and the boundary condition $u(0) = u_0$. The same line of proof as above shows that the Γ-limit is again finite only if $u \in \{0,1\}$ and is piecewise-constant, but we have the additional boundary term $\phi(u_0, u(0+))$, where

$$\phi(s,t) = 2\Big|\int_s^t \sqrt{W(\tau)}\,d\tau\Big|, \tag{7.32}$$

and $u(0+)$ is the (approximate) right-hand side limit of u at 0. This term accounts for the boundary mismatch of the two wells from the boundary condition, and is derived again using the Modica-Mortola trick. The same applies with a boundary condition $u(1) = u_1$.

In the general n-dimensional case, we may obtain results of the following form.

Theorem 7.10 (relaxed boundary data) *Let Ω be a set with smooth boundary and φ be a continuous function. Then the Γ-limit of the functionals $F^{\varphi}_{\varepsilon}$ in (7.31) is given by*

$$F^{\varphi}_0(u) = c_W \mathcal{P}(\{u = 1\}; \Omega) + \int_{\partial\Omega} \phi(\varphi(y), u(y)) d\mathcal{H}^{n-1}(y), \tag{7.33}$$

where $u(y)$ for $y \in \partial\Omega$ is understood as the inner trace of u at y.

7.3 A compactness result

As for integral functionals, the gradient theory of phase transitions above can be framed in a more abstract framework by proving a compactness theorem via the localization methods. This has been done for example by Ansini, Braides and Chiadò Piat [23], to get a result as follows.

Let $W : \mathbb{R} \to [0, +\infty)$ be a continuous function satisfying the conditions above, let $V_\varepsilon : \mathbb{R}^n \times \mathbb{R} \to [0, +\infty)$ be functions satisfying $c_1 W(u) \le V_\varepsilon(x, u) \le c_2 W(u)$ and let f_ε be a sequence of integrands as in Theorem 4.2. We will consider the functionals $G_\varepsilon : L^1_{\rm loc}(\mathbb{R}^n) \times \mathcal{A} \to [0, +\infty]$ defined by

$$G_\varepsilon(u, A) = \int_A \Big(\frac{V_\varepsilon(x, u)}{\varepsilon} + \varepsilon f_\varepsilon(x, Du)\Big)\, dx, \qquad u \in H^1(A) \tag{7.34}$$

(extended to $+\infty$ elsewhere), where $\mathcal{A}$ denotes the family of bounded open subsets of $\mathbb{R}^n$.

Theorem 7.11 (compactness by Γ-convergence) *For every sequence (ε_j) converging to 0, there exist a subsequence (not relabeled) and a functional $G : L^1_{loc}(\mathbb{R}^n) \times \mathcal{A} \to [0, +\infty]$, such that (G_{ε_j}) Γ-converges to G for every A bounded Lipschitz open set, and for every $u \in L^1_{loc}(\mathbb{R}^n)$ such that $u = \chi_E$ with E a set of finite perimeter, with respect to the strong topology of $L^1(A)$. Moreover, there exists a Borel function $\varphi : \mathbb{R}^n \times S^{n-1} \to [0, +\infty)$ such that*

$$G(u, A) = \int_{\partial^* E \cap A} \varphi(x, \nu)\, d\mathcal{H}^{n-1} \tag{7.35}$$

for every open set A.

Proof We may follow the localization method in Section 3.3. Note that we may follow the same line as in Section 4.2.1 to prove the fundamental estimate, with some finer technical changes. Moreover, by a simple comparison argument we obtain $c_1 c_W \mathcal{P}(E, A) \le G(u, A) \le c_2 c_W \mathcal{P}(E, A)$, so that we may apply the representation theorem in Section 7.1.3. Details are found in [23]. □

Remark 7.12 (a formula for the interfacial energy density) Note that a simple derivation formula for φ as in Theorem 4.4 does not hold; in particular φ is not determined by the behaviour of $G(u, A)$ when $S(u)$ is an hyperplane (this would be the analogue of an affine function in a Sobolev setting). To see this it is sufficient to take $\varphi(x, \nu) = 2 - \chi_{\partial B_1(0)}(x)$, which gives a lower-semicontinuous G; in this case, $G(u, A) = 2\mathcal{H}^{n-1}(A \cap S(u))$ if $S(u)$ is an hyperplane, but for example

$G(\chi_{B_1(0)},A)=\mathcal{H}^{n-1}(A\cap\partial B_1(0))$. Nevertheless, if G is translation invariant then $\varphi=\varphi(\nu)$ and hence by convexity

$$\varphi(\nu)=\min\Big\{G(u,\overline{Q_\nu}):u=\chi_E\ \nu^\perp\text{-periodic},\ u=1\text{ on }Q_\nu^+,\ u=0\text{ on }Q_\nu^-\Big\},\qquad(7.36)$$

where Q_ν is a cube with centre 0, side length 1 and $Q_\nu^\pm=\partial Q_\nu\cap\{\langle x,\nu\rangle=\pm 1/2\}$ are the two faces of Q_ν orthogonal to ν. The $\nu^\perp$-periodicity of u must be understood as periodicity in the $n-1$ directions given by the edges of the cube Q_ν other than ν, the values on $Q_\nu^\pm$ are taken in the sense of traces, and the functional G is extended to a measure on all Borel sets.

The formula above is useful to derive a characterization of φ in terms of the approximating G_ε simply by applying the theorem on convergence of minimum problems after Γ-convergence, in the same spirit of the derivation of the homogenization formula in Section 5.1.

7.4 Other functionals generating phase-transitions

In this section we present some other types of functionals whose Γ-limit is a phase-transition energy, briefly highlighting differences and analogies with the gradient theory outlined above.

7.4.1 A non-local model

Another class of energies giving rise to phase transitions, and linked to some models deriving from Ising systems, have been studied by Alberti and Bellettini [6] (see also [7]). They have the form

$$F_\varepsilon(u):=\frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx+\frac{\varepsilon}{4}\int_{\Omega\times\Omega}J_\varepsilon(x'-x)\Big(\frac{u(x')-u(x)}{\varepsilon}\Big)^2dx\,dx'$$

where $J_\varepsilon(y):=\frac{1}{\varepsilon^N}J(\frac{y}{\varepsilon})$, and J is an even positive L^1 kernel with $\int_{\mathbb{R}^n}J(h)|h|\,dh<+\infty$. Note that F_ε can be obtained from the functional studied by Modica and Mortola (Section 7.2.1) by replacing the term $|Du(x)|$ in the second integral in (7.9) with the average of the finite differences $\frac{1}{\varepsilon}|u(x+\varepsilon h)-u(x)|$ with respect to the measure $J_\varepsilon\mathcal{L}^n$. An equi-coerciveness property for the functionals F_ε in $L^2(\Omega)$ can be proved. The Γ-limit is finite only on characteristic functions of sets of finite perimeter; its characterization is the following.

Theorem 7.13 *The Γ-limit of F_ε in the $L^2(\Omega)$ topology is given by*

$$F(u)=\int_{\Omega\cap\partial^*E}g(\nu)\,d\mathcal{H}^{n-1}\qquad(7.37)$$

on $u=\chi_E$ characteristic functions of sets of finite perimeter, where the anisotropic phase-transition energy density g is defined as

$$g(\nu)=\lim_{T\to+\infty}\frac{1}{T^{n-1}}\inf\Big\{\int_{TQ_\nu}W(u)\,dx+\int_{TQ_\nu\times\mathbb{R}^n}J(h)(u(x+h)-u(x))^2\,dx\,dh:$$
$$u\ T\nu^\perp\text{-periodic},\ u(x)=1\text{ if }\langle x,\nu\rangle\geq\frac{T}{2},\ u=0\text{ if }\langle x,\nu\rangle\leq-\frac{T}{2}\Big\}$$

(we use the notation of Remark 7.12*).*

Proof Even though the functions are non-local, the limit contribution to the surface energy can be computed by using the arguments of Remark 7.12 (which explains the form of g). A particular care must be used to deal with the boundary data. Details can be found in [6]. □

7.4.2 A two-parameter model

An intermediate model between the local Cahn-Hilliard model and the non-local above deriving from Ising systems, can be obtained by considering energies depending on one more parameter v, of the form

$$F_\varepsilon(u,v)=\frac{1}{\varepsilon}\int_\Omega W(u)\,dx+\frac{\alpha}{\varepsilon}\int_\Omega (u-v)^2\,dx+\varepsilon\int_\Omega |Dv|^2\,dx \tag{7.38}$$

for $u\in L^2(\Omega)$ and $v\in H^1(\Omega)$. These functionals arise independently in the study of thin bars, with the additional variable taking into account the deviation from one-dimensional deformations (see Rogers and Truskinovsky [138]), and their Γ-limit has been studied by Solci and Vitali [141]. In the case $\alpha=+\infty$ we recover the Modica-Mortola functionals.

Note that the second term in (7.38) forces $u=v$ as $\varepsilon\to 0$ and the first one gives $u\in\{0,1\}$. Note however that the variable u may be discontinuous at fixed $\varepsilon>0$. An equi-coerciveness theorem can be proved for the family F_ε with respect to the L^2-convergence, as well as that the Γ-limit is finite only on characteristic functions of sets of finite perimeter. The characterization of the Γ-limit (which is finite for $u=v=\chi_E$) is the following.

Theorem 7.14 *Let F_ε be as above. Then Γ-$\lim_{\varepsilon\to0}F_\varepsilon(u,v)=F^\alpha(u)$, where*

$$F^\alpha(u)=c_W^\alpha\mathcal{H}^{n-1}(\Omega\cap\partial^*E), \tag{7.39}$$

if $u=\chi_E$, and c_W^α is defined as

$$\begin{aligned}c_W^\alpha \;=\; \sqrt{\alpha}\inf\Big\{&\int_{\mathbb{R}} W(\varphi)\,dx+\frac{\alpha^2}{4}\int_{\mathbb{R}^2}e^{-\alpha|x-y|}(\varphi(x)-\varphi(y))^2\,dx\,dy:\\ &\varphi(-\infty)=0,\ \varphi(+\infty)=1\Big\}.\end{aligned}$$

Furthermore, we have $\lim\limits_{\alpha\to+\infty}c_W^\alpha=2\int_0^1\sqrt{W(s))}\,ds$.

Proof The idea of the proof is to reduce to the one-dimensional case by slicing, and then minimize the effect of v for fixed u. In this way we recover a non-local one-dimensional functional as in Theorem 7.13. Details can be found in [141]. □

Note that we may recover anisotropic functionals by considering terms of the form $g^2(Dv)$ in the place of $|Dv|^2$, with g a norm.

7.4.3 A perturbation with the $H^{1/2}$ norm

Energies similar to those in Section 7.4.1 are the following ones studied by Alberti, Bouchitté and Seppecher [9]

$$G_\varepsilon(u)=\int_\Omega W(u)\,dt+\varepsilon^2\int_{\Omega\times\Omega}\Big|\frac{u(t)-u(s)}{t-s}\Big|^2\,dt\,ds, \tag{7.40}$$

on a one-dimensional set $\Omega=(a,b)$. In this case, a different scaling is needed. Adapting the argument in Section 7.2 we can argue that the first term forces $u\in\{0,1\}$, and we look at a transition from 0 to 1 taking place on an interval $[t,t+\delta]$. We then have

$$G_\varepsilon(u)\geq C\delta+2\varepsilon^2\int_{(a,t)\times(t+\delta,b)}\Big|\frac{1}{t-s}\Big|^2\,dt\,ds\geq C\delta-2\varepsilon^2(\log\delta+C).$$

By optimizing the last expression we get $\delta = 2\varepsilon^2/C$ and hence $G_\varepsilon(u) \geq 4\varepsilon^2|\log\varepsilon| + O(\varepsilon^2)$. We are then led to the scaled energies

$$F_\varepsilon(u) = \frac{1}{\varepsilon^2|\log\varepsilon|}\int_\Omega W(u)\,dt + \frac{1}{|\log\varepsilon|}\int_{\Omega\times\Omega}\Big|\frac{u(t)-u(s)}{t-s}\Big|^2\,dt\,ds, \tag{7.41}$$

for which we have the following Γ-convergence result (see [9], to which we also refer for the proof of their equi-coerciveness).

Theorem 7.15 (phase transitions generated by a $H^{1/2}$-singular perturbation) *The Γ-limit F_0 of F_ε with respect to the L^1-convergence is finite only on piecewise-constant functions, for which $F_0(u) = 4\,\#(S(u))$.*

Proof The crucial point is a compactness and rearrangement argument that allows to reduce to the case where u is close to 0 or 1 except for a finite number of intervals, to which the computation above can be applied. A recovery sequence is obtained by taking $u_\varepsilon = u$ except on intervals of length ε^2 around $S(u)$. □

It must be noted that contrary to the energies considered until now, here we do not have an 'equi-partition' of the energy in the two terms of F_ε, but the whole lower bound is due to the double integral. As a consequence we do not obtain an optimal profile problem by scaling the energy. The loss of such a property makes the problem more difficult and will be found again for Ginzburg-Landau energies (see Section 8.1).

Remark 7.16 By renaming ε the scaling factor $\frac{1}{|\log\varepsilon|}$ we obtain the Γ-convergence of the energies

$$H_\varepsilon(u) = \lambda_\varepsilon\int_\Omega W(u)\,dt + \varepsilon\int_{\Omega\times\Omega}\Big|\frac{u(t)-u(s)}{t-s}\Big|^2\,dt\,ds, \tag{7.42}$$

to $2K\#(S(u))$ whenever $\varepsilon\log\lambda_\varepsilon \to K$.

Remark 7.17 (An application: the line-tension effect) The result above has been applied by Alberti, Bouchitté and Seppecher [10] to the study of energies defined on $\Omega\subset\mathbb{R}^3$ with smooth boundary by

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega W_1(u)\,dx + \lambda_\varepsilon\int_{\partial\Omega} W_2(u)d\mathcal{H}^2 + \varepsilon\int_\Omega |Du|^2\,dx, \tag{7.43}$$

where W_i are two double-well potentials.

We can give a heuristic derivation of the limit of such energies, and for the sake of simplicity we suppose that $\Omega\subset\mathbb{R}^2$ (the case treated in [10] uses further blow-up and slicing arguments). If $\mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$, then $u_\varepsilon \to u$ in Ω and, if v_ε is the trace of u_ε on $\partial\Omega$, then (up to subsequences) $v_\varepsilon \to v$ as $\varepsilon\to 0$. Note that we can rewrite $\mathcal{F}_\varepsilon$ as $\mathcal{F}^1_\varepsilon + \mathcal{F}^2_\varepsilon$, where

$$\mathcal{F}^1_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega W_1(u)\,dx + \delta\varepsilon\int_\Omega |Du|^2\,dx, \tag{7.44}$$

$$\mathcal{F}^2_\varepsilon(u) = \lambda_\varepsilon\int_{\partial\Omega} W_2(u)d\mathcal{H}^1 + (1-\delta)\varepsilon\int_\Omega |Du|^2\,dx. \tag{7.45}$$

This latter term will give the contribution due to v. Upon a blow-up close to each essential discontinuity points of v in $\partial\Omega$ and change of variables argument we can reduce to treat the case

$\{x_1^2+x_2^2<r, x_2>0\}$ a half-disk and $(0,0)$ is a discontinuity point of v on $(-r,r)$, for which a lower bound for this latter functional is

$$\mathcal{G}^2_\varepsilon(v_\varepsilon)=\lambda_\varepsilon\int_{-r}^{r}W_2(v_\varepsilon)dx_1+\frac{1}{2\pi}(1-\delta)\varepsilon\int_{(-r,r)^2}\Big|\frac{v_\varepsilon(t)-v_\varepsilon(s)}{t-s}\Big|^2\,dt\,ds \tag{7.46}$$

(the last term obtained by minimization at fixed v_ε).

If $\varepsilon\log\lambda_\varepsilon\to K$, the use of the previous result (see Remark 7.16) adapted to $\mathcal{G}^2_\varepsilon$ gives $v\in\{0,1\}$ and provides a term in the limit energy of the form $(1-\delta)K\#(S(v))/\pi$, where $S(v)$ denotes the essential discontinuity points of v in $\partial\Omega$. Since $u_\varepsilon\to u$ in Ω with limit relaxed boundary condition v (see Section 7.2.4), from the limit of $\mathcal{F}^1_\varepsilon$ we get that $u\in\{0,1\}$ and a term in the limit energy of the form

$$\sqrt{\delta}\Big(c_{W_1}\mathcal{H}^1(S(u)\cap\Omega)+2\int_{\partial\Omega}\Big|\int_u^v\sqrt{W_1(s)}\,ds\Big|d\mathcal{H}^1\Big). \tag{7.47}$$

Note that this last term can also be written as $c_{W_1}\mathcal{H}^1(\partial\Omega\cap\{u\neq v\})$ taking into account that both u and v may only take the value 0 and 1, but gives a general form if the wells of W_i differ. We can use Lemma 3.1 to optimize the role of $\sqrt{\delta}$ and $(1-\delta)$ separately. We refer to [10] for the construction of a recovery sequence which optimizes these lower bounds.

In the three-dimensional case and for W_2 with wells α and β possibly different from 0 and 1 we have the Γ-limit of the form

$$\mathcal{F}_0(u,v)=K\frac{(\beta-\alpha)^2}{\pi}\mathcal{H}^1(S(v))+c_{W_1}\mathcal{H}^2(S(u)\cap\Omega)+c_{W_1}2\int_{\partial\Omega}\Big|\int_u^v\sqrt{W_1(s)}\,ds\Big|d\mathcal{H}^2. \tag{7.48}$$

Note that to get a more formally correct statement we should identify $\mathcal{F}_\varepsilon$ with the functional defined on $H^1(\Omega)\times H^{1/2}(\partial\Omega)$ by

$$\mathcal{F}_\varepsilon(u,v)=\begin{cases}\mathcal{F}_\varepsilon(u) & \text{if } v \text{ is the trace of } u\\ +\infty & \text{otherwise,}\end{cases} \tag{7.49}$$

where now $S(v)$ denotes the essential boundary of $\{v=\alpha\}$ on $\partial\Omega$.

These functionals have applications in the study of capillarity phenomena. Similar functionals arise in the study of dislocations, where additional difficulties are related to the presence of an infinite-wells potential (see Garroni and Müller [113]).

7.4.4 A phase transition with a Gibbs' phenomenon

A one-dimensional perturbation problem deriving from a non-linear model of a shell-membrane transition has been studied by Ansini, Braides and Valente [24]. The energies F_ε take the form

$$F_\varepsilon(u)=\frac{1}{\varepsilon^3}\int_0^1\Big(\int_0^t u(u-1)\,ds\Big)^2dt+\varepsilon\int_0^1|u'|^2\,dt,\qquad u\in H^1(0,1). \tag{7.50}$$

Note that this energy can be compared with the corresponding Modica-Mortola functional

$$\frac{1}{\varepsilon}\int_0^1(u(u-1))^2\,dt+\varepsilon\int_0^1|u'|^2\,dt,\qquad u\in H^1(0,1) \tag{7.51}$$

with $W(s)=s^2(s-1)^2$, which shows a different scaling in ε. Another feature of these energies is that it is not true that $F_\varepsilon((0\vee u)\wedge 1)\le F_\varepsilon(u)$ so that truncation arguments are not applicable (see Remark 7.19 below).

Theorem 7.18 *The functionals F_ε Γ-converge to F with respect to the L^1-convergence, whose domain are piecewise-constant functions taking only the values 0 and 1, and for such functions $F(u) = C\#(S(u))$, where*

$$
\begin{aligned}
C \quad = \quad & \inf_{T>0} \min\Big\{ \int_{-T}^{T} \Big(\int_{-T}^{t} \varphi(\varphi - 1)\, ds \Big)^2 dt + \int_{-T}^{T} |\varphi|^2\, dt : \\
& \varphi \in H^1(-T,T),\ \varphi(-T) = 0,\ \varphi(T) = 1,\ \int_{-T}^{T} \varphi(\varphi-1)\, ds = 0 \Big\},
\end{aligned} \tag{7.52}
$$

and $S(u)$ denotes the set of essential discontinuity points of u.

Proof The proof of the equi-coerciveness, and hence of the liminf inequality of the functionals is particularly tricky, since the function $u(u-1)$ in the double integral may change sign. As a consequence, the truncation arguments that make computations easier in the Γ-limits considered above do not hold, and in particular recovery sequences do not satisfy $0 \le u_\varepsilon \le 1$ (and hence are not monotone). This is a kind of Gibbs' phenomenon (see the remark below). Note that the conditions for φ in (7.52) make it easy to construct a recovery sequence. It is a much more technical issue to prove that we may always reduce to sequences (u_ε) such that $\int_{t-\varepsilon T_\varepsilon}^{t+\varepsilon T_\varepsilon} u_\varepsilon(u_\varepsilon - 1)\, ds = 0$ and $u_\varepsilon(t \pm \varepsilon T_\varepsilon) \in \{0,1\}$ for some T_ε, from which derives the possibility of localization of the computation of the Γ-limit on $S(u)$. For details we refer to [24]. □

Remark 7.19 (Gibbs' phenomenon as a scaling effect) The condition $\int_{-T}^{T} \varphi(\varphi-1)\, ds = 0$ in (7.52) cannot be satisfied if $0 \le \varphi \le 1$; this implies that the same observation is valid for recovery sequences u_ε. More precisely, if $u_\varepsilon \to u$, $0 \le u_\varepsilon \le 1$ and u is not constant, then we have

$$
\liminf_{\varepsilon \to 0} \varepsilon^{1/6} F_\varepsilon(u_\varepsilon) > 0.
$$

This shows that the addition of the constraint $0 \le u_\varepsilon \le 1$ not only is not compatible with the construction of recovery sequences, but even gives a different scaling of the energy.

7.5 Some extensions

7.5.1 The vector case: multiple wells

The vector case of the Modica-Mortola functional when $u : \Omega \to \mathbb{R}^m$ and $W : \mathbb{R}^m \to [0, +\infty)$ possesses a finite number of wells (or a discrete set of zeros) can be dealt with similarly. Note that such a setting is necessary when dealing with phases parameterizing mixtures of more than two fluids. In this case, we may suppose that W is continuous, with superlinear growth at infinity and $\{W = 0\} = \{\alpha_1, \dots, \alpha_M\}$. The L^1-limit u of a sequence with equi-bounded energy can therefore be identified with a partition (E_i) with $E_i = \{u = \alpha_i\}$, and the Γ-limit is described by the following theorem (see Baldo [28]).

Theorem 7.20 (multiple phase transitions) *The Γ-limit of the energies*

$$
F_\varepsilon(u) = \frac{1}{\varepsilon} \int_\Omega W(u)\, dx + \varepsilon \int_\Omega |Du|^2\, dx, \qquad u \in H^1(\Omega; \mathbb{R}^m) \tag{7.53}
$$

is described by the functional

$$F(E_1,\dots,E_M)=\sum_{i>j}c_{ij}\mathcal{H}^{n-1}(\Omega\cap\partial^*E_i\cap\partial^*E_j)$$

$$c_{ij}=\inf\Big\{\int_{\mathbb{R}}(W(u)+|u'|^2)\,dt: u(-\infty)=\alpha_i,\ u(+\infty)=\alpha_j\Big\}. \tag{7.54}$$

Proof We may follow the line of the proof of the scalar case through the localization procedure. The Γ-limit can be represented as in (7.3). By (7.36) we obtain that

$$\begin{aligned}\varphi_{ij}(\nu) &= \lim_{\varepsilon\to 0}\min\Big\{\frac{1}{\varepsilon}\int_\Omega W(u)\,dx+\varepsilon\int_\Omega |Du|^2\,dx:\\ &\qquad\qquad u\ \nu^\perp\text{-periodic},\ u=\alpha_i \text{ on } Q_\nu^+,\ u=\alpha_j \text{ on } Q_\nu^-\Big\}\\ &\ge \lim_{\varepsilon\to 0}\min\Big\{\int_{-1/2\varepsilon}^{1/2\varepsilon}(W(\phi)+|\phi'|^2)\,dt:\phi(-\frac{1}{2\varepsilon})=\alpha_i,\ \phi(\frac{1}{2\varepsilon})=\alpha_j\Big\},\end{aligned}$$

the last inequality obtained by testing on $u(x)=\phi(\langle x,\nu\rangle/\varepsilon)$. Taking the limit as $\varepsilon\to 0$ we obtain the inequality $\varphi_{ij}(\nu)\ge c_{ij}$. The converse inequality is obtained by a direct construction with one-dimensional scalings of optimal profiles. □

Note that the constants c_{ij} automatically satisfy the *wetting condition* $c_{ij}\le c_{ik}+c_{kj}$ corresponding to the necessary subadditivity constraint.

7.5.2 Solid-solid phase transitions

The Cahn-Hilliard theory accounts for liquid-liquid phase transitions. The inclusion of functionals of elastic problems into this framework would translate into the Γ-limit of energies of the form

$$F_\varepsilon(u)=\frac{1}{\varepsilon}\int_\Omega W(Du)\,dx+\varepsilon\int_\Omega |D^2u|^2\,dx,\qquad u\in H^2(\Omega;\mathbb{R}^n) \tag{7.55}$$

where $u:\Omega\to\mathbb{R}^n$ represents a deformation, and W is an energy density possessing at least two minimizers A and B in $\mathbb{M}^{m\times n}$.

If W represents a hyperelastic free energy, it must be remarked that the physical assumption of *frame-indifference* would actually force W to vanish on the set $SO(n)A\cup SO(n)B$, where $SO(n)$ is the set of *rotations* in $\mathbb{R}^n$. Non-affine weak solutions for the limiting problem may exist if the two wells are *rank-one connected* (*Hadamard's compatibility condition*); i.e., there exist $R,R'\in SO(n)$ and vectors a, ν such that $RA-R'B=a\otimes\nu$.

We state the Γ-convergence result only in a simplified version obtained by neglecting the frame-indifference constraint, as in the following theorem by Conti, Fonseca and Leoni [77].

Theorem 7.21 (solid-solid phase transitions) *Let Ω be a convex set of $\mathbb{R}^n$, let A and B be $n\times n$ matrices and suppose that vectors a, ν exist such that $A-B=a\otimes\nu$. We suppose that W is continuous, positive, growing more than linearly at infinity and vanishing exactly on $\{A,B\}$. Then the Γ-limit F of F_ε is finite only on continuous piecewise-affine functions u such that $Du\in\{A,B\}$ almost everywhere. If $S(Du)$ denotes the set of discontinuity points for Du then $S(Du)$ is the union of parallel hyperplanes orthogonal to ν, and*

$$F(u)=c_{A,B}\mathcal{H}^{n-1}(S(Du)), \tag{7.56}$$

where

$$c_{A,B} = \inf\Big\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, Q_\nu) : u_\varepsilon \to u_{A,B}\Big\} \tag{7.57}$$

and

$$u_{A,B}(x) = \begin{cases} Ax & \textit{if } \langle x,\nu\rangle \geq 0 \\ Bx & \textit{if } \langle x,\nu\rangle < 0 \end{cases} \tag{7.58}$$

(i.e., $c_{A,B} = \Gamma\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u_{A,B}, Q_\nu)$*).*

Proof Functions u with finite energy for the limiting problem are necessarily piecewise-affine deformations, whose interfaces are hyperplanes with normal ν (see the notes by Müller [131]). Despite the simple form of the limit deformations, a number of difficulties arises in the construction of a recovery sequence. In particular De Giorgi's trick to glue together low-energy sequences does not work as such, and more properties of those sequences must be exploited such as that no rotations of the gradient are allowed. We refer to [77] for details. □

To tackle the physical case more refined results must be used such as rigidity properties of low-energy sequences (see Theorem 9.8). We refer to the work of Conti and Schweizer [78] for a detailed proof.

Remark 7.22 The computation of the Γ-limit in the higher-order scalar case, when we consider energies of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega (1 - |Du|^2)\, dx + \varepsilon \int_\Omega |D^2 u|^2\, dx, \qquad u \in W^{1,1}(\Omega), \tag{7.59}$$

where the 'rigidity' of the gradient is missing, is an interesting open problem. For this problem we have a equi-coerciveness property and lower estimates (see [19, 96]). A key observation is that the 'Modica-Mortola trick' as such is not applicable, but other lower bounds can be obtained in the same spirit.

8 Concentration problems

In the previous sections we have examined the behaviour of functionals defined on Sobolev spaces, first (Sections 4 –6) in some cases where the Γ-limit is automatically defined on some Sobolev space and its form must be described (with some notable exception when the 'dimension' of the domain of the limit increases as in Section 6.4), and then for phase-transition limit energies, when the weak coerciveness on Sobolev spaces fails and the limit can be defined on (functions equivalent to) sets of finite perimeter. It must be noted that the actual object on which the final phase-transition energies depend is the measure $\mathcal{H}^{n-1} \llcorner \partial^* E$ which can be seen as the limit of the gradients of the recovery sequences. In this section we examine other cases where concentration occurs in a more evident way. In such cases, the limit relevant objects can and will again be thought as measures, and their connection to the corresponding limit functions will be less relevant than in the previous cases.

8.1 Ginzburg-Landau

We consider the simplified Ginzburg-Landau energy

$$\frac{1}{\varepsilon^2}\int_\Omega (|u|-1)^2\,dx + \int_\Omega |Du|^2\,dx, \tag{8.1}$$

where $\Omega\subset\mathbb{R}^n$ and $u:\Omega\to\mathbb{R}^2$. The formal analogy with the Cahn-Hilliard model (up to scaling) is apparent, but it must be immediately noted that in this case the zeroes of the potential $W(u) = (|u|-1)^2$ are the whole set S^1 and that the functional coincides with the Dirichlet integral on the space $H^1(\Omega;S^1)$. This latter space is not trivial contrary to the space $H^1(\Omega;\{-1,1\})$. However, in some problems (e.g., when a boundary datum is added with non-zero degree) a further scaling of this energy is necessary. We will give a heuristic derivation of this scaling and a description of the Γ-limit of the scaled energies in dimension two, and give an idea of the extension to higher dimensions.

8.1.1 The two-dimensional case

An in-depth study of the behaviour of minimizers for the energy above subject to non-trivial boundary data is contained in the book by Bethuel, Brezis and Hélein [35] (we also refer to the monograph by Sandier and Serfaty [140] for the case with magnetic field). Some of the results therein can be rephrased in the language of Γ-convergence. The first step towards an energetic interpretation is the derivation of the correct scaling of the energies. The heuristic idea is that a boundary datum with non-zero degree will force minimizing sequences (u_ε) to create a finite number of singularities $\{x_i\}_i$ in the interior of Ω so that their limit will belong to $H^1_{\rm loc}(\Omega\setminus\{x_i\}_i;S^1)$, and the non-zero degree condition on $\partial\Omega$ will be balanced by some 'vortices' centred at x_i.

We now fix a sequence (u_ε) with fixed non-zero degree on $\partial\Omega$. We note that for fixed $\delta<1$ the set $T_\varepsilon=\{|u_\varepsilon|<\delta\}$ is not empty (otherwise we could 'project' u_ε on S^1 obtaining a homotopy with a map with zero degree). The limit of such sets T_ε is a candidate for the set $\{x_i\}_i$ above. In order to estimate the scale of the energy contribution of such a sequence we assume that T_ε is composed of disks $B^\varepsilon_i = B_{\rho^\varepsilon_i}(x_i)$ on the boundary of which the degree of u_ε is some non-zero integer d_i. We also assume that the gradient $|Du_\varepsilon|$ is of order $1/\rho^\varepsilon_i$ in B^ε_i. Note that the restriction of u_ε to $\partial B_r(x_i)$ for each $0<r<R$, with R such that $B_R(x_i)$ are pairwise disjoint and contained in Ω, has degree d_i so that the integral of the square of its tangential derivative on $\partial B_r(x_i)$ is at least $2\pi d_i^2/r$. We then obtain

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_\Omega (|u_\varepsilon|-1)^2\,dx + \int_\Omega |Du_\varepsilon|^2\,dx \quad&\geq\quad \sum_i \Big(\pi(\rho^\varepsilon_i)^2\Big(\frac{(1-\delta)^2}{\varepsilon^2}+\frac{C}{(\rho^\varepsilon_i)^2}\Big)+2\pi d_i^2\int_{\rho^\varepsilon_i}^R\frac{dr}{r}\Big)\\ &\approx\quad \pi\Big(C\#(\{x_i\}_i)+\sum_i\Big((1-\delta)^2\frac{(\rho^\varepsilon_i)^2}{\varepsilon^2}+2d_i^2(|\log\rho^\varepsilon_i|+\log R)\Big)\Big).\end{aligned}$$

Optimizing in ρ^ε_i gives $\rho^\varepsilon_i = d_i\varepsilon/(1-\delta)$, so that we get a lower bound with $2\pi|\log\varepsilon|\sum_i d_i^2+O(1)$. This estimate suggests the desired scaling, and gives the functionals

$$F_\varepsilon(u)=\frac{1}{\varepsilon^2|\log\varepsilon|}\int_\Omega(|u_\varepsilon|-1)^2\,dx+\frac{1}{|\log\varepsilon|}\int_\Omega|Du_\varepsilon|^2\,dx. \tag{8.2}$$

Note that, as in the case of Section 7.4.3, the leading part of the energy is logarithmic and is due to the 'far-field' away from the singularities.

At this point, we have to define the correct notion of convergence, that will give in the limit the 'vortices' x_i. To this end, it must be noted (see Jerrard [116]) that the relevant quantity 'concentrating' at x_i is the *distributional Jacobian* defined as $Ju = D_1(u^1 D_2 u^2) - D_2(u^1 D_1 u^2)$, which coincides with the usual Jacobian determinant $\det(Du)$ if $u \in H^1(\Omega;\mathbb{R}^2)$, and is a measure of the form $\sum_i \pi d_i \delta_{x_i}$ if u is regular outside a finite set $\{x_i\}$ with degree d_i around x_i. The notion of convergence that makes our functionals equi-coercive is the *flat convergence* (i.e, testing against C^1 functions). If (u_ε) is such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ then, up to subsequences, Ju_ε converge flat to some measure μ of the form $\sum_i \pi d_i \delta_{x_i}$. This defines the convergence $u_\varepsilon \to \{(x_i, d_i)\}_i$, for which we have the following result.

Theorem 8.1 (energy of vortices) *The* Γ*-limit of the functionals above is given by*

$$F_0(\{(x_i, d_i)\}_i) = 2\pi \sum_i |d_i| \tag{8.3}$$

Proof We do not include the details of the proof for which we refer to [35]. We only remark that the lower bound above is almost sharp, giving $\sum_i d_i^2$ in the place of $\sum_i |d_i|$. This is easily made optimal by approximating a vortex of degree d_i with d_i vortices of degree $\operatorname{sign}(d_i)$ in the limsup inequality. Moreover, for each vortex of degree ± 1 a recovery sequence is obtained by mollifying $(x - x_i)/|x - x_i|$. □

Remark 8.2 (interaction of vortices) Note that this results implies that minimizers with boundary datum of degree $d \neq 0$ will generate $|d|$ vortices of degree $\operatorname{sign}(d)$, but does not give any information about the location of such vortices. To this end we have to look for the behaviour of the *renormalized energies* (in the terminology of [35]).

In terms of higher-order Γ-limits we fix a boundary datum $g : \partial\Omega \to S^1$ with $\deg(g) > 0$, and consider the functionals

$$G_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon^2}\displaystyle\int_\Omega (|u_\varepsilon| - 1)^2\,dx + \int_\Omega |Du_\varepsilon|^2\,dx - 2\pi d\,|\log\varepsilon| & \text{if } u = g \text{ on } \partial\Omega \\ +\infty & \text{otherwise;} \end{cases} \tag{8.4}$$

Then the functionals G_ε Γ-converge with respect to the convergence defined above to a limit G_0 that describes the interactions between the vortices. We refer to [35] Theorem 1.7 for the description of this renormalized limit energy via the Green's function of some auxiliary boundary value problem.

Finally, we note that Γ-convergence has also been applied to the asymptotic study of the gradient flows of Ginzburg-Landau energies by Sandier and Serfaty [139].

8.1.2 The higher-dimensional case

The analogue of Theorem 8.1 in three dimensions (or higher) is more meaningful, as we expect the distributional Jacobian to give rise to a limit with a more complex geometry than a set of points. Indeed, it can be seen that these problems have a two-dimensional character that forces concentration on sets of codimension two (hence, lines in three dimensions). Actually, we expect limit objects with some multiplicity defined, taking the place of the degree in dimension two. These objects are indeed *currents*, whose treatment is beyond the scope of these notes and for which we refer to the introductory book by Morgan [126]. We just mention the following result due to Jerrard and Soner [117] and Alberti, Baldo and Orlandi [8].

Theorem 8.3 *The Γ-limit of the functionals F_ε with respect to the flat convergence of currents is defined on integer (up to a factor π) rectifiable currents T and is equal to $F_0(T) = 2\pi\|T\|$, where $\|T\|$ is the mass of the current T.*

8.2 Critical-growth problems

Another class of variational problems where concentration occurs are problems related to the critical growth for the Sobolev embedding. It is well known that the best constant in the Sobolev inequality is not achieved on domains different from the whole space, due to a scaling-invariance property that implies that optimal sequences concentrate at a point. The techniques of *concentration-compactness* type of P.-L. Lions are the classical tool to study such phenomena for a wide class of variational problems of the same nature. Some of these concentration phenomena can be also treated within the theory of Γ-convergence. We give some applications to a large class of variational problems that exhibit concentration and include critical-growth problems.

We will study the behaviour of the family of maximum problems depending on a small parameter $\varepsilon > 0$

$$S_\varepsilon^\Psi(\Omega) = \varepsilon^{-2^*} \sup\left\{ \int_\Omega \Psi(\varepsilon u)\,dx \ : \ \ u \in H_0^1(\Omega)\,, \quad \int_\Omega |\nabla u|^2 dx \le 1 \right\}, \tag{8.5}$$

where Ω is a bounded open set in $\mathbb{R}^n$ with $n \ge 3$ and $2^* = \frac{2n}{n-2}$ is the usual critical *Sobolev exponent*, through some limit of the corresponding functionals

$$F_\varepsilon(u) = \begin{cases} \varepsilon^{-2^*} \displaystyle\int_\Omega \Psi(\varepsilon u)\,dx & \text{if } \int_\Omega |Du|^2 dx \le 1 \\ 0 & \text{otherwise in } L^{2^*}(\Omega). \end{cases} \tag{8.6}$$

We assume: $0 \le \Psi(t) \le c|t|^{2^*}$ for every $t \in \mathbb{R}$; $\Psi \not\equiv 0$ and upper semi-continuous, and, in order to simplify the exposition we also assume that the following two limits

$$\Psi_0(t) = \lim_{t\to 0} \frac{\Psi(t)}{|t|^{2^*}} \quad \text{and} \quad \Psi_\infty(t) = \lim_{t\to\infty} \frac{\Psi(t)}{|t|^{2^*}}$$

exist.

Remark 8.4 (1) Within this class we recover the *capacity* problem (see the Introduction), with

$$\Psi(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \ge 1. \end{cases} \tag{8.7}$$

(2) If Ψ is smooth and $\Psi' = \psi$ then the functional is linked to the study of the asymptotic properties of solutions of

$$\begin{cases} -\Delta u = \lambda\psi(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.8}$$

where $\lambda \to +\infty$ and $0 \le \psi(t) \le c|t|^{2^*-1}$.

Note that the functionals F_ε are weakly equi-coercive in $H_0^1(\Omega)$. If $Du_\varepsilon \rightharpoonup Du$, we may also assume that there exists a measure $\mu \in \mathcal{M}(\overline{\Omega}) := (C(\overline{\Omega}))'$ such that $|Du_\varepsilon|^2\mathcal{L}^n \rightharpoonup^* \mu$ in $\mathcal{M}(\overline{\Omega})$. In general, by the lower semi-continuity of the norm, we get $\mu \geq |Du|^2\mathcal{L}^n$. Thus we can isolate the atoms of μ, $\{x_i\}_{i\in J}$, and rewrite μ as follows

$$\mu = |Du|^2\mathcal{L}^n + \sum_{i\in J}\mu_i\delta_{x_i} + \tilde{\mu}\,, \tag{8.9}$$

where μ_i denotes the positive weight of the atom x_i and $\tilde{\mu}$ is the non-atomic part of $\mu - |Du|^2\mathcal{L}^n$.

In general we say that a sequence u_ε converges to (u,μ) if

$$u_\varepsilon \rightharpoonup u \quad \text{in } H_0^1(\Omega) \qquad \text{and} \qquad |Du_\varepsilon|^2\mathcal{L}^n \rightharpoonup^* \mu \quad \text{in } \mathcal{M}(\overline{\Omega}). \tag{8.10}$$

In view of the study of the asymptotic behaviour of the maxima of F_ε and the corresponding maximizing sequences we introduce the notion of Γ^+-convergence, symmetric to the notion of Γ-convergence used until now for minimum problems.

Definition 8.5 *Let $F_\varepsilon : X \to \overline{\mathbb{R}}$, be a family of functionals. We say that the sequence F_ε* Γ^+-con- verges *to the functional $F : X \to \overline{\mathbb{R}}$ if the following two properties are satisfied for all $x \in X$*

(i) *for every sequence $x_\varepsilon \to x$ we have that* $\limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \leq F(x)$*;*

(ii) *for every $x \in X$, there exists a sequence x_ε, such that $x_\varepsilon \to x$ and* $\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \geq F(x)$.

To define the Γ^+-limit we denote by $S^\Psi := S_1^\Psi(\mathbb{R}^n)$ the *ground-state energy*; i.e.,

$$S^\Psi = \sup\left\{\int_{\mathbb{R}^n}\Psi(u)\,dx \ : \quad u \in D^{1,2}(\mathbb{R}^n)\,, \quad \int_{\mathbb{R}^n}|Du|^2dx \leq 1\right\} \tag{8.11}$$

where $D^{1,2}$ is the closure of C^1 with respect to the L^2-norm of the gradient. We then have the following result proved in Amar and Garroni [17].

Theorem 8.6 (concentration by Γ-convergence) *The sequence F_ε Γ^+-converges with respect to the convergence given by* (8.10) *to the functional*

$$F(u,\mu) := \Psi_0\int_\Omega |u|^{2^*}dx + S^\Psi\sum_{i\in J}(\mu_i)^{\frac{2^*}{2}}\,. \tag{8.12}$$

As a consequence we get the result of concentration known for this type of problem (obtained by Lions in the smooth case and Flucher and Müller in the general case). In fact, by a scaling argument one can see that $S^*\Psi_0 \leq S^\Psi$ (S^* denotes the best Sobolev constant) and that $S_\varepsilon^\Psi \to S^\Psi$, hence by the Γ^+-convergence we get $S^\Psi = \max F(u,\mu)$. On the other hand by the convexity of the function $|t|^{\frac{2^*}{2}}$ we get

$$\begin{aligned} F(u,\mu) \quad &= \quad \Psi_0\int_\Omega |u|^{2^*}dx + S^\Psi\sum_{i\in J}(\mu_i)^{\frac{2^*}{2}} \leq \Psi_0 S^*\left(\int_\Omega |Du|^2dx\right)^{\frac{2^*}{2}} + S^\Psi\sum_{i\in J}(\mu_i)^{\frac{2^*}{2}} \\ &\leq \quad S^\Psi\left[\left(\int_\Omega |Du|^2dx\right)^{\frac{2^*}{2}} + \sum_{i\in J}(\mu_i)^{\frac{2^*}{2}}\right] \leq S^\Psi\mu(\overline{\Omega}) \leq S^\Psi\,. \end{aligned} \tag{8.13}$$

Since the Sobolev constant is not attained in Ω the first inequality is strict unless $u = 0$ and the third inequality is strict unless $\mu = \delta_{x_0}$ for some $x_0 \in \overline{\Omega}$. In other words

$$F(u,\mu) = S^\Psi = \max F \qquad \Longleftrightarrow \qquad (u,\mu) = (0,\delta_{x_0})\,,$$

which corresponds to the concentration of a maximizing sequence at x_0.

Proof *(Theorem* 8.6*)* The Γ^+-'limsup inequality' (i) above is essentially the so-called concentration-compactness lemma in its generalized version proved by Flucher and Müller (see [100] for details), where the asymptotic behaviour of the sequence $\varepsilon^{-2^*}\Psi(\varepsilon u_\varepsilon)$ is given in terms of the limit (u,μ). The optimization of the upper bound is easily achieved on pairs $(0,\delta_x)$ by arguing as in (8.13) above, while if μ does not contain an atomic part it is derived from the strong convergence in L^{2^*} due to the concentration-compactness lemma. For details we refer to [17]. □

We can apply the result above to the case of the capacity, with the choice of Ψ as in Remark 8.4(1). The maximum problem S^Ψ_ε can be rewritten as (V stands for 'volume')

$$S^V_\varepsilon(\Omega) = \max\left\{|A| : \quad \mathrm{Cap}(A,\Omega) \le \varepsilon^2\right\}, \tag{8.14}$$

after identifying any open set A with the level set $\{v \ge 1\}$ of its *capacitary potential* (see (1.19)). We have that the maximizing sets A_ε concentrate at a single point x_0 in $\overline{\Omega}$; i.e., the corresponding capacitary potentials (divided by ε) converge to $(0,\delta_{x_0})$ in the sense of (8.10).

As for the two-dimensional Ginzburg-Landau energies, a classical question in these problems of concentration is the identification of the concentration points. The answer can also be given in terms of the Γ^+-convergence of F_ε suitably scaled. A key role is played by the diagonal of the regular part of the Green's function of Ω for the Laplacian and the Dirichlet problem. This plays the same role as the renormalized energy for the Ginzburg-Landau functionals. It is called the *Robin function* τ_Ω and it is given by

$$\tau_\Omega(x) = H_\Omega(x,x)\,,$$

where $H_\Omega(x,y)$ is the regular (harmonic) part of the Green's function $G_\Omega(x,y)$. In the case of the capacity we have the following result (see [101] and [17]).

Theorem 8.7 (identification of concentration points) *Let Ψ be as in Remark* 8.4(1)*; then the sequence*

$$\frac{F_\varepsilon(u) - S^V}{\varepsilon^2}\,,$$

where $S^V := S^V_1(\mathbb{R}^n)$, Γ^+-converges with respect to the convergence given by (8.10) *to the functional*

$$F^1(u,\mu) = \begin{cases} -\dfrac{n}{n-2} S^V \tau_\Omega(x_0) & \textit{if } (u,\mu) = (0,\delta_{x_0}) \\ -\infty & \textit{otherwise.} \end{cases}$$

Proof The proof of the Γ^+-limsup inequality is based on an asymptotic formula for the capacity of small sets involving the Robin function (see [112, 111]). To prove the converse inequality, if v_ε is the capacitary potential of the level set of the Green's function $A_\varepsilon = \{G_\Omega(x_0,\cdot) > \varepsilon^{-2}\}$, for which it is not difficult to prove that

$$|A_\varepsilon| \ge \varepsilon^{2^*} S^V \left(1 - \frac{n}{n-2}\tau_\Omega(x_0)\varepsilon^2 + o(\varepsilon^2)\right),$$

then a recovery sequence is given by $u_\varepsilon = v_\varepsilon/\varepsilon$. □

As a consequence of this Γ^+-convergence result we deduce that all sequences of *almost maximizers* of problem (8.14); i.e., satisfying

$$\varepsilon^{2^*}|A_\varepsilon| = S^V_\varepsilon(\Omega) + o(\varepsilon^2)\,, \tag{8.15}$$

concentrate at a minimum point of the Robin function (a *harmonic center* of Ω).

More in general one can compute the asymptotic behaviour of

$$\frac{F_\varepsilon(u) - S^\Psi}{\varepsilon^2}$$

under a condition which rules out dilation-invariant problems; e.g., $S^\Psi > \max\{\Psi_0, \Psi_\infty\}S^*$.

Theorem 8.8 *If $S^\Psi > \max\{\Psi_0, \Psi_\infty\}S^*$, then all sequences of almost maximizers for F_ε; i.e., satisfying*

$$\varepsilon^{2^*}\int_\Omega \Psi(\varepsilon u_\varepsilon)\,dx = S^\Psi_\varepsilon(\Omega) + o(\varepsilon^2)\,, \tag{8.16}$$

up to a subsequence, concentrate at a harmonic center of Ω.

9 Dimension-reduction problems

The small parameter ε entering the definition of the functionals F_ε may be some times related to some small dimension of the domain of integration. This happens for example in the theory of thin films, rods and shells. As $\varepsilon \to 0$ some energy defined on a lower-dimensional set is expected to arise as the Γ-limit. We will illustrate some aspects of this passage to the limit for 'thin films' (i.e., for n-dimensional domains whose limit is $n-1$-dimensional, with the case $n = 3$ in mind).

9.1 The Le Dret-Raoult result

We begin with an illuminating result for homogeneous functionals. We consider a domain of the form

$$\Omega_\varepsilon = \omega \times (0, \varepsilon), \qquad \omega \ \text{ bounded open subset of } \ \mathbb{R}^{n-1}, \tag{9.1}$$

and an energy

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} f(Du)\,dx, \qquad u \in W^{1,p}(\Omega_\varepsilon;\mathbb{R}^m), \tag{9.2}$$

where f satisfies a standard growth condition of order p. Note that up to a relaxation argument, we can suppose that f be quasiconvex. The normalization factor $1/\varepsilon$ is a simple scaling proportional to the measure of Ω_ε.

In order to understand in what sense a Γ-limit of F_ε can be defined, we first identify F_ε with a functional defined on a fixed domain

$$G_\varepsilon(v) = \int_\Omega f\Big(D_\alpha v, \frac{1}{\varepsilon}D_n v\Big)\,dx, \qquad v \in W^{1,p}(\Omega;\mathbb{R}^m), \tag{9.3}$$

where $x_\alpha = (x_1, \dots, x_{n-1})$ and $D_\alpha v = (D_1 v, \dots, D_{n-1}v)$, $\Omega = \omega \times (0,1)$ and v is obtained from u by the scaling

$$v(x_\alpha, x_n) = u(x_\alpha, \varepsilon x_n). \tag{9.4}$$

Note that G_ε satisfies a degenerate growth condition

$$G_\varepsilon(v) \geq C \int_\Omega \Big(|D_\alpha v|^p + \frac{1}{\varepsilon^p} |D_n v|^p \Big)\, dx - C. \tag{9.5}$$

From this condition we deduce first that a family with equi-bounded energies $G_\varepsilon(v_\varepsilon)$ and bounded in L^p is weakly precompact in $W^{1,p}(\Omega;\mathbb{R}^m)$; second, that if $v_\varepsilon \rightharpoonup v$ (up to subsequences) then

$$\int_\Omega |D_n v|^p\, dx \leq \liminf_{\varepsilon\to 0} \int_\Omega |D_n v_\varepsilon|^p \leq \liminf_{\varepsilon\to 0} \varepsilon^p C = 0, \tag{9.6}$$

so that v is actually independent of the n-th variable and can therefore be identified with a function $u \in W^{1,p}(\omega;\mathbb{R}^m)$.

With this compactness result in mind, we can define a convergence $u_\varepsilon \to u$ of functions $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon;\mathbb{R}^m)$ to $u \in W^{1,p}(\omega;\mathbb{R}^m)$ if the corresponding v_ε defined above converges to the function $v(x_\alpha, x_n) = u(x_\alpha)$. The functionals F_ε are equi-coercive with respect to this convergence. We then have the following convergence result [119].

Theorem 9.1 (Le Dret-Raoult thin-film limit) *Let F_ε be defined above, and let*

$$F_0(u) = \int_\omega Q\overline{f}(D_\alpha u) dx_\alpha, \qquad\qquad u \in W^{1,p}(\omega;\mathbb{R}^m), \tag{9.7}$$

where Q denotes the quasiconvex envelope *(for energy densities on the space of $m\times(n-1)$ matrices) and*

$$\overline{f}(\xi) = \inf\{ f(\xi\,|\,b) : b \in \mathbb{R}^m \} \tag{9.8}$$

(here $(\xi\,|\,b) \in \mathbb{M}^{m\times n}$ is a matrix whose first $n-1$ columns coincide with the $m\times(n-1)$ matrix ξ, and the last column with the vector b).

This result gives an easy way to characterize the energy density of the limit that highlights the superposition of two optimality processes: first, the minimization in the dependence on the n-th variable disappearing in the limit. This gives the function $\overline{f}$, that may be not $m\times(n-1)$-quasiconvex even though f is $m\times n$-quasiconvex; hence a second optimization in oscillations in the $n-1$ 'planar' coordinates must be taken into account (expressed by the quasiconvexification process).

Proof The lower bound is easily achieved since

$$\begin{aligned}
\liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) &= \liminf_{\varepsilon\to 0} \int_\Omega f\Big(D_\alpha v_\varepsilon, \frac{1}{\varepsilon} D_n v_\varepsilon\Big)\, dx \\
&\geq \liminf_{\varepsilon\to 0} \int_\Omega \overline{f}(D_\alpha v_\varepsilon)\, dx \geq \liminf_{\varepsilon\to 0} \int_\Omega Q\overline{f}(D_\alpha v_\varepsilon)\, dx \\
&\geq \int_\Omega Q\overline{f}(D_\alpha v)\, dx = \int_\omega Q\overline{f}(D_\alpha u)\, dx_\alpha,
\end{aligned}$$

the last inequality due to the lower semicontinuity of $\int_\Omega Q\overline{f}(D_\alpha v)\, dx$.

The limsup inequality needs to be shown only for piecewise-affine function by a density argument. It is sufficient to exhibit a recovery sequence for the target function $u(x_\alpha) = \xi x_\alpha$. We may suppose that f be quasiconvex and hence continuous (locally Lipschitz), so that we easily get the existence of $b \in \mathbb{R}^m$ such that $\overline{f}(\xi) = f(\xi \,|\, b)$. Note that if $\overline{f}$ were quasiconvex then

$$u_\varepsilon(x) = \xi x_\alpha + \varepsilon b x_n \tag{9.9}$$

would be a recovery sequence for $F_0(u)$. In general we have to improve this argument: we can fix a 1-periodic smooth $\varphi(x_\alpha)$ such that

$$\int_{(0,1)^{n-1}} \overline{f}(\xi + D\varphi(x_\alpha))\, dx_\alpha \leq Q\overline{f}(\xi) + \eta.$$

Let $b = b(x_\alpha)$ be such that $\overline{f}(\xi + D\varphi(x_\alpha)) = f(\xi + D\varphi(x_\alpha), b(x_\alpha))$. Then a recovery sequence is given by

$$u_\varepsilon(x) = \xi x_\alpha + \varphi(x_\alpha) + \varepsilon x_n b(x_\alpha).$$

Note a few technical issues: the existence of b may be proven by suitable measurable-selection criteria; if b is not differentiable then a mollification argument must be used; we get an extra term $\varepsilon x_n D_\alpha b(x_\alpha)$ in $D_\alpha u_\varepsilon$ that can be neglected due to the local Lipschitz continuity of f. □

Remark 9.2 (the convex case) Note that if f is convex then $\overline{f}$ is convex, and the quasiconvexification process is not necessary. In this case recovery sequences for affine functions are simply given by (9.9).

9.2 A compactness theorem

The localization arguments in Section 3.3 can be adapted to sequences of functionals defined on thin films. It must be noted that in order to obtain a limit functional defined on $W^{1,p}(\omega;\mathbb{R}^m)$ the localization argument must be performed on open sets of $\mathbb{R}^{n-1}$, or, equivalently, on cylindrical sets $A \times (0,1)$ with A open set of $\mathbb{R}^{n-1}$. With this observation in mind the following theorem holds by Braides, Fonseca and Francfort [56], which is an analogue of Theorem 4.2.

Theorem 9.3 (compactness theorem for thin films) *Given a family of Borel functions $f_\varepsilon : \Omega_\varepsilon \times \mathbb{M}^{m\times n} \to [0,+\infty)$, satisfying growth condition (4.2), there exists a Carathéodory function $f_0 : \omega \times \mathbb{M}^{m\times(n-1)} \to [0,+\infty)$, satisfying the same growth conditions, such that, up to subsequences,*

$$\int_A f_0(x_\alpha, D_\alpha u)\, dx_\alpha = \Gamma\text{-}\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{A\times(0,\varepsilon)} f_\varepsilon(x, Du)\, dx \tag{9.10}$$

with respect to the convergence $u_\varepsilon \to u$ as above, for all open subsets A of ω.

Proof Follow the arguments in Section 3.3 applied to the energies

$$G_\varepsilon(v, A) = \int_{A\times(0,1)} f_\varepsilon\Big(x_\alpha, \varepsilon x_n, D_\alpha v, \frac{1}{\varepsilon} D_n v\Big)\, dx.$$

Note that we can use De Giorgi's argument to match boundary conditions on $(\partial A) \times (0,1)$ to prove the fundamental estimate, provided we choose the cut-off functions independent of the n-th variable. □

Remark 9.4 (convergence of minimum problems) Minimum problems of the type

$$m_\varepsilon = \min\Big\{\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} f_\varepsilon(x, Du)\,dx : u = \phi \text{ on } \partial\omega\times(0,1)\Big\} \tag{9.11}$$

can be treated by the usual arguments provided that $\phi = \phi(x_\alpha)$, so that

$$m_\varepsilon = \min\Big\{\int_{\Omega} f_\varepsilon\Big(x_\alpha, \varepsilon x_n, D_\alpha v, \frac{1}{\varepsilon}D_n v\Big)\,dx : v = \phi \text{ on } \partial\omega\times(0,1)\Big\}, \tag{9.12}$$

converge to

$$m = \min\Big\{\int_{\omega} f_0(x_\alpha, D_\alpha u)\,dx_\alpha : u = \phi \text{ on } \partial\omega\Big\}. \tag{9.13}$$

Again, to prove this, it suffices to use the arguments in Section 4.2.1.

In the same way we may treat the case $\omega = (0,1)^{n-1}$ and u satisfying 1-periodic conditions in the x_α variables.

Remark 9.5 (alternate formula for the Le Dret-Raoult result) In the case of integrands that are 'homogeneous' and independent of ε (i.e., $f_\varepsilon(x,\xi) = f(\xi)$) as in Theorem 9.1 we may easily infer that the function f_0 given by the compactness Theorem 9.3 is itself homogeneous (and quasiconvex) so that

$$\begin{aligned}
f_0(\xi) &= \min\Big\{\int_{(0,1)^{n-1}} f_0(\xi + D_\alpha u)dx_\alpha : u = 0 \text{ on } \partial(0,1)^{n-1}\Big\}\\
&= \min\Big\{\int_{(0,1)^{n}} f_0(\xi + D_\alpha u)dx : u = 0 \text{ on } (\partial(0,1)^{n-1})\times(0,1)\Big\}\\
&= \lim_{\varepsilon\to 0}\min\Big\{\int_{(0,1)^{n-1}\times(0,1)} f\Big(\xi + D_\alpha v, \frac{1}{\varepsilon}D_n v\Big)dx : v = 0 \text{ on } (\partial(0,1)^{n-1})\times(0,1)\Big\}\\
&= \lim_{T\to+\infty}\frac{1}{T^{n-1}}\min\Big\{\int_{(0,T)^{n-1}\times(0,1)} f(\xi + D_\alpha v, D_n v)dy : v = 0 \text{ on } (\partial(0,T)^{n-1})\times(0,1)\Big\},
\end{aligned}$$

where we have performed the change of variables $y_\alpha = \varepsilon x_\alpha$ and set $T = 1/\varepsilon$. This is a formula of *homogenization type*, and can be easily extended to cover the case $f_\varepsilon(x,\xi) = f(x_\alpha/\varepsilon, \xi)$ with f 1-periodic in the x_α directions (see [56]). Note moreover that the zero boundary condition can be replaced by periodicity.

Remark 9.6 (thin films with oscillating profiles) It is interesting to note that the fact that the dependence on the n-th variable disappears allows to consider more complex geometries for the sets Ω_ε; for example we can consider

$$\Omega_\varepsilon = \{(x_\alpha, x_n) : 0 < x_n < \varepsilon\,\psi_\varepsilon(x_\alpha)\}. \tag{9.14}$$

The compactness argument must be adequately extended since now Ω_ε cannot be rescaled to a single set Ω. In order not to have a degenerate behaviour we assume that $0 < c \le \psi_\varepsilon \le 1$ uniformly. The scaling in the n-th variable brings Ω_ε into $\Omega^\varepsilon = \{(x_\alpha, x_n) : 0 < x_n < \psi_\varepsilon(x_\alpha)\}$. If $F_\varepsilon(u_\varepsilon)$ is equibounded we can apply the compactness argument to the scaled sequence v_ε on $\omega\times(0,c)$ and define a limit $u \in W^{1,p}(\omega;\mathbb{R}^m)$. We can easily see that we indeed have

$$\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon}|v_\varepsilon - u|^p\,dx = \lim_{\varepsilon\to 0}\int_\omega\int_0^{\psi_\varepsilon(x_\alpha)}|v_\varepsilon - u|^p\,dx_n\,dx_\alpha = 0,$$

by a simple use of Poincaré-Wirtinger's inequality, since $v - u_\varepsilon \to 0$ on $\omega \times (0, c)$ and we have a bound for $\frac{1}{\varepsilon} D_n v_\varepsilon = \frac{1}{\varepsilon} D_n(v_\varepsilon - u)$. The sequence F_ε is then equi-coercive with respect to this convergence, and the compactness Theorem 9.3 can be proven using the same arguments as before.

Note that we have used the fact that the sections of Ω_ε in the n-th direction are connected. This cannot be dropped, as shown by an example by Bhattacharya and Braides [31] (with Ω_ε with an increasing number of small cracks as $\varepsilon \to 0$), showing that otherwise we may have a limit in which the dependence on the n-th variable does not disappear.

Remark 9.7 (equi-integrability for thin films) In the computation of the Γ-limits for thin films, as for the case of n-dimensional objects, the possibility of reducing to sequences with some equi-integrability property is very useful. A result of Bocea and Fonseca [39] shows that for any converging sequence (u_ε) such that $\sup_\varepsilon \int_\Omega (|\nabla_\alpha u_\varepsilon|^p + \frac{1}{\varepsilon^p}|\nabla_n u_\varepsilon|^p)\, dx < +\infty$ there exists an 'equivalent sequence' v_ε such that the sequence $(|\nabla_\alpha v_\varepsilon|^p + \frac{1}{\varepsilon^p}|\nabla_n v_\varepsilon|^p)$ is equi-integrable on Ω. An alternative proof and the generalization to any co-dimension of this result can be found in the paper by Braides and Zeppieri [65].

Note that the general approach outlined above may be generalized to cover the cases when the 'thin directions' are more than one, and in the limit we get objects of codimension more than one. In the case of low-dimensional theories of rods, it must be noted that the one-dimensional nature of the final objects easily allows for more general growth conditions for the energies (see [2]).

9.3 Higher-order Γ-limits

The analysis carried over in the first part of this chapter can be applied to derive low-dimensional theories from three-dimensional (finite) elasticity, where $m = n = 3$ and the function $f : \mathbb{M}^{3\times 3} \to [0, +\infty]$, the *elastic stored energy* of the material, is continuous and *frame indifferent*; i.e., $f(R\xi) = f(\xi)$ for every rotation R and every $\xi \in \mathbb{M}^{3\times 3}$, where $R\xi$ denotes the usual product of 3×3 matrices. We assume that f vanishes on the set $SO(3)$ of rotations in $\mathbb{R}^3$, is of class C^2 in a neighbourhood of $SO(3)$, and satisfies the inequality

$$f(\xi) \geq C\left(\operatorname{dist}(\xi, SO(3))\right)^2 \quad \text{for every } \xi \in \mathbb{M}^{3\times 3}, \tag{9.15}$$

with a constant $C > 0$.

For these energies other scalings than that considered above, provide a variational justification of a number of low-dimensional theories commonly used in Mechanics (see Friesecke, James and Müller [108]). In this section we briefly focus on a derivation of plate theory. In this case the energies F_ε must be further scaled by $1/\varepsilon^2$ obtaining the functionals $\mathcal{F}_\varepsilon$ defined by

$$\mathcal{F}_\varepsilon(v) := \frac{1}{\varepsilon^2}\int_\Omega f\Big(D_1 v \,|\, D_2 v \,|\, \frac{1}{\varepsilon} D_3 v\Big)\, dx \qquad v \in W^{1,2}(\Omega; \mathbb{R}^3),$$

where again $v(x_1, x_2, x_3) = u(x_1, x_2, \varepsilon x_3)$, $u : \Omega_\varepsilon \to \mathbb{R}^3$ is the deformation of Ω_ε and $\xi = (\xi_1 \,|\, \xi_2 \,|\xi_3)$ represents a matrix ξ via its columns.

The Γ-limit of $\mathcal{F}_\varepsilon$ with respect to the L^2-convergence turns out to be finite on the set $\Sigma(\omega; \mathbb{R}^3)$ of all *isometric embeddings* of ω into $\mathbb{R}^3$ of class $W^{2,2}$; i.e., $v \in \Sigma(\omega; \mathbb{R}^3)$ if and only if $v \in W^{2,2}(\omega; \mathbb{R}^3)$ and $(Dv)^T Dv = I$ a.e. on ω. As above, elements of $\Sigma(\omega; \mathbb{R}^3)$ are also regarded as maps from Ω

into $\mathbb{R}^3$ independent of x_3. To describe the Γ-limit we introduce the quadratic form $\mathcal{Q}_3$ defined on $\mathbb{M}^{3\times 3}$ by

$$\mathcal{Q}_3(\xi) := \frac{1}{2} D^2 f(I)[\xi,\xi]\,,$$

which is the density of the linearized energy for the three-dimensional problem, and the quadratic form $\mathcal{Q}_2$ defined on the space of symmetric 2×2 matrices by

$$\mathcal{Q}_2 \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} := \min_{(b_1,b_2,b_3)\in\mathbb{R}^3} \mathcal{Q}_3 \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

The Γ-limit of $\mathcal{F}_\varepsilon$ is the functional $\mathcal{F} : L^2(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by

$$\mathcal{F}(v) := \frac{1}{12}\int_\Omega \mathcal{Q}_2(A(v))\,dx \qquad \text{if } v \in \Sigma(\omega;\mathbb{R}^3),$$

where $A(v)$ denotes the second fundamental form of v; i.e.,

$$A_{ij}(v) := -D_i D_j v \cdot \nu\,, \tag{9.16}$$

with normal vector $\nu := D_1 v \wedge D_2 v$. The proof of this fact can be found in the paper by Friesecke, James and Müller [106].

Equi-coerciveness for problems involving the functionals $\mathcal{F}_\varepsilon$ in $L^2(\Omega;\mathbb{R}^3)$ is not trivial; it follows from (9.15) through the following lemma which is due to Friesecke, James and Müller (see [107]).

Lemma 9.8 (geometric rigidity estimate) *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain; then there exists a constant $C(\Omega)$ such that*

$$\min_{R\in \mathrm{SO}(n)} \int_\Omega |Du - R|^2\,dx \le C(\Omega)\int_\Omega \big(\mathrm{dist}(Du, SO(3))\big)^2\,dx$$

for all $u \in H^1(\Omega;\mathbb{R}^n)$.

10 Approximation of free-discontinuity problems

"Free-discontinuity problems", following a terminology introduced by De Giorgi, are those problems in the Calculus of Variations where the unknown is a pair (u,K), with K varying in a class of (sufficiently smooth) closed hypersurfaces contained in a fixed open set $\Omega \subset \mathbb{R}^n$ and $u : \Omega\setminus K \to \mathbb{R}^m$ belonging to a class of (sufficiently smooth) functions. Such problems are usually of the form

$$\min\{E_v(u,K) + E_s(u,K) + \text{ "lower-order terms"}\}, \tag{10.1}$$

with E_v, E_s being interpreted as *volume* and *surface* energies, respectively. Several examples can be described in this setting, among which: image and signal reconstruction problems (linked to the Mumford and Shah functional, see below), fracture in brittle hyperelastic media (where E_v denotes the elastic energy, and E_s the surface energy due to the crack), equilibrium problems for drops of liquid crystals, or even simply prescribed curvature problems (for which $u = \chi_E$ and $K = \partial E$, $E_v(u,K) = \int_E g(x)\,dx$ and $E_s(u,K) = \mathcal{H}^{n-1}(\partial E)$).

Despite the existence theory developed in SBV-spaces, functionals arising in free-discontinuity problems present some serious drawbacks. First, the lack of differentiability in any reasonable norm implies the impossibility of flowing these functionals, and dynamic problems can be tackled only in an indirect way. Moreover, numerical problems arise in the detection of the unknown discontinuity surface. To bypass these difficulties, a considerable effort has been spent recently to provide variational approximations of free discontinuity problems, and in particular of the Mumford-Shah functional MS defined in (10.8) below, with differentiable energies defined on smooth functions.

10.1 Special functions with bounded variation

The treatment of free-discontinuity problems following the direct methods of the Calculus of Variations presents many difficulties, due to the dependence of the energies on the surface K. Unless topological constraints are added, it is usually not possible to deduce compactness properties from the only information that such kind of energies are bounded. An idea of De Giorgi has been to interpret K as the set of discontinuity points of the function u, and to set the problems in a space of discontinuous functions. The requirements on such a space are of two kinds:

(a) *structure properties*: if we define K as the set of discontinuity points of the function u then K can be interpreted as an hypersurface, and u is "differentiable" on $\Omega \setminus K$ so that bulk energy depending on ∇u can be defined;

(b) *compactness properties*: it is possible to apply the direct method of the Calculus of Variations, obtaining compactness of sequences of functions with bounded energy.

The answer to the two requirements above has been De Giorgi and Ambrosio's space of *special functions of bounded variation*: a function u belongs to $SBV(\Omega)$ if and only if its distributional derivative Du is a bounded measure that can be split into a bulk and a surface term. This definition can be further specified: if $u \in SBV(\Omega)$ and $S(u)$ (the *jump set* or *discontinuity set* of u) stands for the complement of the set of the Lebesgue points for u then a *measure-theoretical normal* ν_u to $S(u)$ can be defined $\mathcal{H}^{n-1}$-a.e. on $S(u)$, together with the *traces* $u^{\pm}$ on both sides of $S(u)$; moreover, the *approximate gradient* ∇u exists a.e. on Ω, and we have

$$Du = \nabla u\, \mathcal{L}_n + (u^+ - u^-)\nu_u\, \mathcal{H}^{n-1} \llcorner S(u)\,. \tag{10.2}$$

Replacing the set K by $S(u)$ we obtain a weak formulation for a class of free-discontinuity problems, whose energies take the general form

$$\int_\Omega f(x, u, \nabla u)\, dx + \int_{S(u)} \vartheta(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{n-1}\,. \tag{10.3}$$

An existence theory for problems involving these kinds of energies has been developed by Ambrosio. Various regularity results show that for a wide class of problems the weak solution u in $SBV(\Omega)$ provides a solution to the corresponding free-discontinuity problem, taking $K = \overline{S(u)}$ (see [20, 125, 89]).

10.1.1 A density result in SBV

It is useful to ensure the existence of 'dense' sets in SBV spaces to use a density argument in the computation of the limsup inequality. To this end, we can show that 'piecewise-smooth functions' are dense as in the following lemma (see [49]) derived from the regularity results in [93].

We recall that $\mathcal{M}^{n-1}(E)$ is the $n-1$ dimensional Minkowski content of E, defined by

$$\mathcal{M}^{n-1}(E) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} |\{x \in \mathbb{R}^n : \operatorname{dist}(x, E) < \varepsilon\}| \,, \tag{10.4}$$

whenever the limit in (10.4) exists.

Lemma 10.1 *Let Ω be a bounded open set with Lipschitz boundary. Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ with $\int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{n-1}(S(u)) < +\infty$ and let Ω' be a bounded open subset of $\mathbb{R}^n$ such that $\Omega \subset\subset \Omega'$. Then u has an extension $z \in SBV(\Omega') \cap L^\infty(\Omega')$ such that $\int_{\Omega'} |\nabla z|^2\,dx + \mathcal{H}^{n-1}(S(z)) < +\infty$, $\mathcal{H}^{n-1}(S(z) \cap \partial\Omega) = 0$, and $\|z\|_{L^\infty(\Omega')} = \|u\|_{L^\infty(\Omega)}$. Moreover, there exists a sequence (z_k) in $SBV(\Omega') \cap L^\infty(\Omega')$ such that (z_k) converges to z in $L^1(\Omega')$, (∇z_k) converges to ∇z in $L^2(\Omega'; \mathbb{R}^n)$, $\|z_k\|_{L^\infty(\Omega')} \le \|u\|_{L^\infty(\Omega)}$,*

$$\lim_k \mathcal{H}^{n-1}(S(z_k)) = \mathcal{H}^{n-1}(S(z)) \,, \tag{10.5}$$

$$\lim_k \mathcal{H}^{n-1}(S(z_k) \cap \overline{\Omega}) = \mathcal{H}^{n-1}(S(z) \cap \Omega) = \mathcal{H}^{n-1}(S(u)) \,, \tag{10.6}$$

$\mathcal{H}^{n-1}(\overline{S(z_k)} \setminus S(z_k)) = 0$, and $\mathcal{H}^{n-1}(S(z_k) \cap K) = \mathcal{M}^{n-1}(S(z_k) \cap K)$ for every compact set $K \subseteq \Omega'$.

Remark 10.2 (approximations with polyhedral jump sets) The result above has been further refined in a very handy way by Cortesani and Toader [79], who have proved that a dense class of SBV-functions are those which jump set $S(u)$ is composed of a finite number of polyhedral sets.

10.1.2 The Mumford-Shah functional

The prototype of the energies in (10.3) derives from a model in Image Reconstruction due to Mumford and Shah [133], where we minimize

$$\int_{\Omega \setminus K} |\nabla u|^2\,dx + c_1 \int_{\Omega \setminus K} |u - g|^2\,dx + c_2 \mathcal{H}^1(K) \tag{10.7}$$

on an open set $\Omega \subset \mathbb{R}^2$. In this case g is interpreted as the input picture taken from a camera, u is the "cleaned" image, and K is the relevant contour of the objects in the picture. The constants c_1 and c_2 are contrast parameters. Note that the problem is meaningful also adding the constraint $\nabla u = 0$ outside K, in which case we have a minimal partitioning problem. Note moreover that very similar energies are linked to Griffith's theory of fracture, where $|\nabla u|^2$ is substituted by a linear elastic energy.

In the framework of $SBV(\Omega)$ functions, with Ω a bounded open subset of $\mathbb{R}^n$, the *Mumford-Shah functional* is written as

$$\mathrm{MS}(u) = \alpha \int_\Omega |\nabla u|^2\,dx + \beta\, \mathcal{H}^{n-1}(S(u)) \,. \tag{10.8}$$

Comparing with (10.7) note that we drop the term $\int |u - g|^2\,dx$ since it is continuous with respect to the $L^2(\Omega)$-convergence.

Weak solutions for problems involving energies (10.7) are obtained by applying the following compactness and lower-semicontinuity theorem by Ambrosio [20].

Theorem 10.3 (SBV compactness) *Let (u_j) be a sequence of functions in $SBV(\Omega)$ such that $\sup_j MS(u_j) < +\infty$ and $\sup \|u_j\|_\infty < +\infty$. Then, up to subsequences, there exists a function $u \in SBV(\Omega)$ such that $u_j \to u$ in $L^2(\Omega)$ and $\nabla u_j \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^n)$. Moreover $\mathcal{H}^{n-1}(S(u)) \le \liminf_j \mathcal{H}^{n-1}(S(u_j))$.*

Remark 10.4 (existence of optimal segmentation) Let $g \in L^\infty(\Omega)$ and consider the minimum problem

$$\min\Big\{\mathrm{MS}(u) + \int_\Omega |u-g|^2\,dx : u \in SBV(\Omega)\Big\}. \tag{10.9}$$

If (v_j) is a minimizing sequence for this problem, note that such is also the truncated sequence $u_j = ((-\|g\|_\infty)\vee v_j)\wedge \|g\|_\infty$. To this sequence we can apply the compactness result above to obtain a solution.

If $(u_\varepsilon) \to u$ in $L^1(\Omega)$, $u_\varepsilon \in SBV(\Omega)$ and $\sup_\varepsilon \mathrm{MS}(u_\varepsilon) < +\infty$ then u need not be a SBV-function. To completely describe the domain of MS we should introduce the following extension of $SBV(\Omega)$

Definition 10.5 (generalized special functions of bounded variation) *A function u belongs to $GSBV(\Omega)$ if all its truncations $u_T = (u \vee (-T))\wedge T$ belong to $SBV(\Omega)$.*

In the rest of the section it will not be restrictive to describe our functionals on SBV since we will always be able to reduce to that space by truncations.

10.1.3 Two asymptotic results for the Mumford-Shah functional

1) The 'complete' Mumford-Shah functional (10.7) with given g may be scaled to understand the role of the contrast parameter c_1, c_2. This has been done by Rieger and Tilli [137], who have shown that an interesting scaling gives the energy

$$F_\varepsilon(u) = \frac{1}{\varepsilon^2}\int_\Omega \Big(|\nabla u|^2 + |u-g|^2\Big)\,dx + \varepsilon\mathcal{H}^1(S(u)). \tag{10.10}$$

In this case the convergence taken into account is the weak* convergence of measures applied to the measures $(\mathcal{H}^1(S(u)))^{-1}\mathcal{H}^1 \llcorner S(u)$, so that the limit is defined on probability measures μ on $\overline{\Omega}$. If $g \in H^1(\Omega)$ in this space it takes the form

$$F_0(\mu) = \Big(\frac{9}{16}\int_\Omega \frac{|\nabla g|^2}{(d\mu/dx)^2}\,dx\Big)^{1/3}, \tag{10.11}$$

where $d\mu/dx$ is the density of the absolutely continuous part of μ with respect to the Lebesgue measure.

2) The presence of two competing terms with different growth and scaling properties makes asymptotic problems for energies of the Mumford-Shah type technically more challenging. The homogenization of functionals of this form has been computed in [55], while a very interesting variant of the Γ-limit of functionals on perforated domains as in Section 6, with the Mumford-Shah functional in place of the Dirichlet integral in (6.5), has been performed by Focardi and Gelli [102]. In that case, the scaling is given by $\delta_\varepsilon = \varepsilon^{n/n-1}$ and the limit is represented as

$$F_0(u) = \alpha\int_\Omega |\nabla u|^2\,dx + \beta\mathcal{H}^{n-1}(S(u)) + \beta'|\{u \neq 0\}|, \tag{10.12}$$

with β' depending only on K and β and defined through the minimization of the perimeter among sets containing K (see [102] for details and generalizations).

10.2 The Ambrosio-Tortorelli approximation

The first approximation by Γ-convergence of the Mumford-Shah functional was given by Ambrosio and Tortorelli in [21]. Following the idea developed by Modica and Mortola for the approximation of the perimeter functional by elliptic functionals, they introduced an approximation procedure of $\mathrm{MS}(u)$ with an auxiliary variable v, which in the limit approaches $1-\chi_{S(u)}$. As the approximating functionals are elliptic, even though non-convex, numerical methods can be applied to them. It is clear, though, that the introduction of an extra variable v can be very demanding from a numerical viewpoint.

10.2.1 An approximation by energies on set-function pairs

A common pattern in the approximation of free-discontinuity energies (often, in the liminf inequality) is the substitution of the sharp interface $S(u)$ by a 'blurred' interface, of small but finite Lebesgue measure, shrinking to $S(u)$. In this section we present a simple result in that direction, which will be used in the proof of the Ambrosio-Tortorelli result in the next one, and formalizes an intermediate step present in many approximation procedures (see, e.g., [44]). The result mixes Sobolev functions and sets of finite perimeter and is due to Braides, Chambolle and Solci [48].

Theorem 10.6 (set-function approximation) *Let δ_ε be a sequence of positive numbers converging to 0, and let F_ε be defined on pairs (u,E), where E is a set of finite perimeter and u is such that $u(1-\chi_E)\in SBV(\Omega)$ and $S(u(1-\chi_E))\subseteq\partial^*E$ (i.e., $u\in H^1(\Omega\setminus E)$ if E is smooth) by*

$$F_\varepsilon(u,E)=\begin{cases}\alpha\displaystyle\int_{\Omega\setminus E}|\nabla u|^2\,dx+\frac{1}{2}\beta\,\mathcal{H}^{n-1}(\partial^*E) & \text{if } |E|\le\delta_\varepsilon\\ +\infty & \text{otherwise.}\end{cases}\tag{10.13}$$

Then F_ε Γ-converges to the functional (equivalent to MS*)*

$$F_0(u,E)=\begin{cases}\mathrm{MS}(u) & \text{if } |E|=0\\ +\infty & \text{otherwise}\end{cases}\tag{10.14}$$

if $u\in SBV(\Omega)$, with respect to the $L^2(\Omega)$-convergence of u and χ_E. Clearly, the functional F_0 is equivalent to MS *as far as minimum problems are concerned.*

Proof The proof is achieved by slicing. The one-dimensional case is easily achieved as follows: if $u_\varepsilon\to u$ and $\sup_\varepsilon F_\varepsilon(u_\varepsilon,E_\varepsilon)<+\infty$, then $\chi_{E_\varepsilon}\to 0$ and E_ε are composed by an equi-bounded number of segments. We can therefore assume, up to subsequences, that E_ε shrink to a finite set S. Since $u_\varepsilon\rightharpoonup u$ in $H^1_{\rm loc}(\Omega\setminus S)$ and $\int_{\Omega\setminus S}|\nabla u|^2\,dx\le\liminf_{\varepsilon\to 0}\int_{\Omega\setminus E_\varepsilon}|\nabla u_\varepsilon|^2\,dx$ we deduce that $u\in SBV(\Omega)$ and $S(u)\subset S\cap\Omega$. Since for each point $x\in S\cap\Omega$ we have at least two families of points in ∂E_ε converging to x, we deduce that $2\#(S(u))\le\liminf_{\varepsilon\to 0}\#(\partial E_\varepsilon)$, and then the liminf inequality. The limsup inequality is then achieved by taking $u_\varepsilon=u$ and E_ε any sequence of open sets containing $S(u)$ with $|E_\varepsilon|\le\delta_\varepsilon$. The n-dimensional case follows exactly the proof of Theorem 7.3 with the due changes. □

Note that in the previous result we may restrict the domain of F_ε to pairs (u,E) with E with Lipschitz boundary and $u\in H^1(\Omega)$, up to a smoothing argument for the recovery sequence.

10.2.2 Approximation by elliptic functionals

We now can introduce the Ambrosio-Tortorelli approximating energies

$$F_\varepsilon(u,v) = \alpha \int_\Omega v^2 |\nabla u|^2\,dx + \frac{\beta}{2}\int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{1}{4\varepsilon}(1-v)^2\Big)\,dx\,, \tag{10.15}$$

defined on functions u, v such that $v \in H^1(\Omega)$, $uv \in H^1(\Omega)$ and $0 \le v \le 1$. The idea behind the definition of F_ε is that the variable v tends to take the value 1 almost everywhere due to the last term in the second integral, and the value 0 on $S(u)$ in order to make the first integral finite. As a result, the interaction of the two terms in the second integral will give a surface energy concentrating on $S(u)$. We present a proof which uses the result in the previous section; in a sense, v^2 is replaced by $1-\chi_E$ and the second integral by $\mathcal{H}^{n-1}(\Omega\cap\partial^* E)$.

Theorem 10.7 (Ambrosio-Tortorelli approximation) *The functionals F_ε defined in* (10.15) *Γ-converge as $\varepsilon\to 0$ with respect to the $(L^1(\Omega))^2$-topology to the functional*

$$F(u,v) = \begin{cases} \mathrm{MS}(u) & \text{if } v=1 \text{ a.e. on } \Omega \\ +\infty & \text{otherwise,} \end{cases} \tag{10.16}$$

defined on $(L^1(\Omega))^2$.

Proof Let $u_\varepsilon \to u$ and $v_\varepsilon \to v$ be such that $F_\varepsilon(u_\varepsilon, v_\varepsilon) \le c < +\infty$. We first observe that $v=1$ a.e. since $\int_\Omega (v_\varepsilon - 1)^2\,dx \le C\varepsilon$. For every A open subset of Ω we denote by $F_\varepsilon(\cdot,\cdot;A)$ the energy obtained by computing both integrals on A. The 'Modica-Mortola trick' and an application of the coarea formula gives:

$$\begin{aligned} F_\varepsilon(u_\varepsilon, v_\varepsilon; A) &\ge \alpha\int_A v_\varepsilon^2|\nabla u_\varepsilon|^2\,dx + \beta\int_A |1-v_\varepsilon||\nabla v_\varepsilon|\,dx \\ &\ge \alpha\int_A v_\varepsilon^2|\nabla u_\varepsilon|^2\,dx + \beta\int_0^1 (1-s)\mathcal{H}^{n-1}(\partial\{v_\varepsilon < s\}\cap A)\,ds. \end{aligned}$$

Now, we fix $\delta\in(0,1)$. The Mean Value Theorem ensures the existence of $t_\varepsilon^\delta \in (\delta,1)$ such that

$$\int_\delta^1 (1-s)\mathcal{H}^{n-1}(\partial\{v_\varepsilon<s\}\cap A)\,ds \ge \int_\delta^1 (1-s)ds\ \mathcal{H}^{n-1}(\partial E_\varepsilon^\delta\cap A) = \frac{1}{2}(1-\delta)^2\mathcal{H}^{n-1}(\partial E_\varepsilon^\delta \cap A),$$

where $E_\varepsilon^\delta = \{v_\varepsilon < t_\varepsilon^\delta\}$; hence

$$F_\varepsilon(u_\varepsilon, v_\varepsilon; A) \ge \alpha\delta^2 \int_{A\setminus E_\varepsilon^\delta} |\nabla u_\varepsilon|^2\,dx + \frac{\beta}{2}(1-\delta)^2\ \mathcal{H}^{n-1}(\partial E_\varepsilon^\delta\cap A). \tag{10.17}$$

An application of Theorem 10.6 gives

$$\begin{aligned} &\liminf_{j\to+\infty}\Big(\alpha\delta^2\int_{A\setminus E_\varepsilon^\delta} |\nabla u_\varepsilon|^2\,dx + \frac{\beta}{2}(1-\delta)^2\mathcal{H}^{n-1}(\partial E_\varepsilon^\delta\cap A)\Big) \\ &\quad \ge \alpha\delta^2\int_A |\nabla u|^2 dx + \beta(1-\delta)^2\mathcal{H}^{n-1}(S(u)\cap A). \end{aligned}$$

An easy application of Lemma 3.1 with $\lambda = \mathcal{L}^n + \mathcal{H}^{n-1} \llcorner S(u)$ gives the lower bound.

By Lemma 10.1 it suffices to prove the limsup inequality when the target function u is smooth and $S(u)$ is essentially closed with Minkowski content equal to its $\mathcal{H}^{n-1}$ measure. The proof of the optimality of the lower bound is obtained through the construction of recovery sequences $(u_\varepsilon, v_\varepsilon)$ with $u_\varepsilon = u$ and

$$v_\varepsilon(x) = \begin{cases} v((\mathrm{dist}\,(x, S(u)) - \varepsilon^2)/\varepsilon) & \text{if } \mathrm{dist}(x, S(u)) \geq \varepsilon^2 \\ 0 & \text{if } \mathrm{dist}(x, S(u)) < \varepsilon^2 \end{cases}$$

with a 'one-dimensional' structure, as in the proof of the Modica-Mortola result. The function v_ε equals to 0 in a neighbourhood of $S(u)$ so that the first integral converges to $\int_\Omega |\nabla u|^2\,dx$. The function v is chosen to be an optimal profile giving the equality in the Modica-Mortola trick, with datum $v(0) = 0$. In the case above the computation is explicit, giving $v(t) = 1 - e^{-t/2}$. □

Remark 10.8 (approximate solutions of the Mumford-Shah problem) A little variation must be made to obtain coercive functionals approximating MS, by adding a perturbation of the form $k_\varepsilon \int_\Omega |\nabla u|^2\,dx$ to $G_\varepsilon(u, v)$ with $0 < k_\varepsilon = o(\varepsilon)$. In this way, with fixed $g \in L^\infty(\Omega)$, for each ε we obtain a solution to

$$m_\varepsilon = \min\Big\{ F_\varepsilon(u, v) + k_\varepsilon \int_\Omega |\nabla u|^2\,dx + \int_\Omega |u - g|^2\,dx : u, v \in H^1(\Omega) \Big\}, \tag{10.18}$$

and, up to subsequences, these solutions $(u_\varepsilon, v_\varepsilon)$ converge to $(u, 1)$, where u is a minimizer of (10.9).

10.3 Other approximations

In this section we briefly illustrate some alternative ideas, for which details can be found in [45]. A simpler approach to approximate MS is to try an approximation by means of local integral functionals of the form

$$\int_\Omega f_\varepsilon(\nabla u(x))\,dx\,, \tag{10.19}$$

defined in the Sobolev space $H^1(\Omega)$. It is clear that such functionals cannot provide any variational approximation for MS. In fact, if an approximation existed by functionals of this form, the functional MS(u) would also be the Γ-limit of their lower-semicontinuous envelopes; i.e., the convex functionals

$$\int_\Omega f_\varepsilon^{**}(\nabla u(x))\,dx\,, \tag{10.20}$$

where f_ε^{**} is the convex envelope of f_ε (see Proposition 2.5(2) and Remark 4.8(4)), in contrast with the lack of convexity of MS. However, functionals of the form (10.19) can be a useful starting point for a heuristic argument. We can begin by requiring that for every $u \in SBV(\Omega)$ with ∇u and $S(u)$ sufficiently smooth we have

$$\lim_{\varepsilon \to 0^+} \int_\Omega f_\varepsilon(\nabla u_\varepsilon(x))\,dx = \alpha \int_\Omega |\nabla u|^2\,dx + \beta \mathcal{H}^{n-1}(S(u))$$

if we choose u_ε very close to u, except in an ε-neighbourhood of $S(u)$ (where the gradient of u_ε tends to be very large). It can be easily seen that this requirement is fulfilled if we choose f_ε of the form

$$f_\varepsilon(\xi) = \frac{1}{\varepsilon} f(\varepsilon|\xi|^2), \text{ with } f'(0) = \alpha \text{ and } \lim_{t \to +\infty} f(t) = \frac{\beta}{2}\,; \tag{10.21}$$

The simplest such f is $f(t) = \alpha t \wedge \frac{\beta}{2}$.

10.3.1 Approximation by convolution functionals

Non-convex integrands of the form (10.21) can be used for an approximation, provided we slightly modify the functionals in (10.19). This can be done in many ways. For example, the convexity constraint in ∇u can be removed by considering approximations of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f\Big(\varepsilon \fint_{B_\varepsilon(x)\cap\Omega} |\nabla u(y)|^2\,dy\Big)\,dx\,, \tag{10.22}$$

defined for $u \in H^1(\Omega)$, where f is a suitable non-decreasing continuous (non-convex) function and $\fint_B$ denotes the *average* on B. These functionals, proposed by Braides and Dal Maso, are non-local in the sense that their energy density at a point $x \in \Omega$ depends on the behaviour of u in the whole set $B_\varepsilon(x)\cap\Omega$, or, in other words, on the value of the convolution of u with $\frac{1}{|B_\varepsilon|}\chi_{B_\varepsilon}$. More general convolution kernels with compact support may be considered. Note that, even if the term containing the gradient is not convex, the functional F_ε is weakly lower semicontinuous in $H^1(\Omega)$ by Fatou's Lemma. These functionals Γ-converge, as $\varepsilon \to 0$, to the Mumford-Shah functional MS in (10.8) if f satisfies the limit conditions in (10.21) (see [52]).

The proof of this result is rather technical in the liminf part, reducing to a 'non-local slicing procedure'. It must be remarked that, on the other side, the limsup inequality is easily obtained as the Γ-limit coincides with the limit (at least for regular-enough $S(u)$) of mollified u_ε with a mollifier with support on a scale finer than ε. In fact for such u_ε

$$\begin{aligned}
\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon\to 0}\Bigg(\frac{1}{\varepsilon}\int_{\Omega\cap\{\mathrm{dist}(x,S(u))>\varepsilon\}} f\Big(\varepsilon\fint_{B_\varepsilon(x)\cap\Omega}|\nabla u_\varepsilon(y)|^2\,dy\Big)\,dx \\
&\qquad + \frac{1}{\varepsilon}\int_{\Omega\cap\{\mathrm{dist}(x,S(u))\le\varepsilon\}} f\Big(\varepsilon\fint_{B_\varepsilon(x)\cap\Omega}|\nabla u_\varepsilon(y)|^2\,dy\Big)\,dx\Bigg) \\
&= \lim_{\varepsilon\to 0}\Bigg(\frac{1}{\varepsilon}\int_{\Omega\cap\{\mathrm{dist}(x,S(u))>\varepsilon\}} \alpha\varepsilon\fint_{B_\varepsilon(x)\cap\Omega}|\nabla u(y)|^2\,dy\,dx \\
&\qquad + \beta\frac{|\Omega\cap\{\mathrm{dist}(x,S(u))\le\varepsilon\}|}{2\varepsilon}\Bigg) \\
&= \lim_{\varepsilon\to 0}\Bigg(\int_\Omega \alpha\fint_{B_\varepsilon(x)\cap\Omega}|\nabla u(y)|^2\,dy\,dx + \beta\frac{|\Omega\cap\{\mathrm{dist}(x,S(u))\le\varepsilon\}|}{2\varepsilon}\Bigg) \\
&= \alpha\int_\Omega |\nabla u(y)|^2\,dy + \beta\mathcal{H}^{n-1}(\Omega\cap S(u)).
\end{aligned}$$

10.3.2 A singular-perturbation approach

A different path can be followed by introducing a second-order singular perturbation. Dealing for simplicity with the 1-dimensional case, we have functionals of the form (f again as in (10.21))

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega f(\varepsilon|u'|^2)\,dx + \varepsilon^3\int_\Omega |u''|^2\,dx \tag{10.23}$$

on $H^2(\Omega)$. Note that the Γ-limit of these functionals would be trivial without the last term, and that the convexity in u'' assures the weak lower semicontinuity of F_ε in $H^2(\Omega)$. Alicandro, Braides and Gelli [14] have proven that the family (F_ε) Γ-converges to the functional defined on $SBV(\Omega)$ by

$$F(u) = \alpha \int_\Omega |u'|^2\,dx + C \sum_{t\in S(u)} \sqrt{|u^+(t) - u^-(t)|}\,, \tag{10.24}$$

with C explicitly computed from β through the optimal profile problem

$$C = \min_{T>0} \min\Big\{\beta T + \int_{-T}^{T} |v''|^2\,dt :\; v(\pm T) = \pm 1/2, v'(\pm T) = 0\Big\} \tag{10.25}$$

corresponding to minimizing the contribution of the part of the energy $F_\varepsilon(u_\varepsilon)$ concentrating on an interval where $f(\varepsilon|u'_\varepsilon|^2) = \beta/2$, centred in some x_ε, after the usual scaling $v(t) = u_\varepsilon(x_\varepsilon + \varepsilon t)$.

In contrast with those in (10.15), functionals (10.23) possess a particularly simple form, with no extra variables. The form of the approximating functionals gets more complex if we want to use this approach to recover in the limit other surface energies (as, for example, in the Mumford-Shah functional) in which case we must substitute f by more complex f_ε.

10.3.3 Approximation by finite-difference energies

A sequence of functionals proposed by De Giorgi provides another type of non-local approximation of the Mumford-Shah functional, proved by Gobbino [115], namely,

$$F_\varepsilon(u) = \frac{1}{\varepsilon^{n+1}} \int_{\Omega\times\Omega} f\Big(\frac{(u(x) - u(y))^2}{\varepsilon}\Big)\, e^{-|x-y|^2/\varepsilon}\,dx\,dy\,, \tag{10.26}$$

defined on $L^1(\Omega)$, with f as in (10.21). In this case the constants α and β in MS must be replaced by two other constants A and B.

The idea behind the energies in (10.26) derives from an analogous result by Chambolle [71] in a discrete setting, giving an anisotropic version of the Mumford-Shah functional (see Section 11.2.2 below); F_ε are designed to eliminate such anisotropy. This procedure is particularly flexible, allowing for easy generalizations, to approximate a wide class of functionals. The main drawback of this approach is the difficulty in obtaining coerciveness properties. In this case the Γ-limit coincides with the pointwise limit. In particular, if $u(x) = x_1$ we get

$$\begin{aligned}\lim_{\varepsilon\to 0} F_\varepsilon(u) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{n+1}} \int_{\Omega\times\Omega} f\Big(\frac{(x_1 - y_1)^2}{\varepsilon}\Big)\, e^{-|x-y|^2/\varepsilon}\,dx\,dy \\ &= \alpha|\Omega| \int_{\mathbb{R}^n} |\xi_1|^2 e^{-|\xi|^2} d\xi,\end{aligned}$$

and similarly, if we choose $u(x) = \mathrm{sign}(x_1)$ we get

$$\begin{aligned}\lim_{\varepsilon\to 0} F_\varepsilon(u) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{n+1}} \int_{\Omega\times\Omega} f\Big(\frac{\mathrm{sign}(x_1) - \mathrm{sign}(y_1))^2}{\varepsilon}\Big)\, e^{-|x-y|^2/\varepsilon}\,dx\,dy \\ &= \beta\,\mathcal{H}^{n-1}(\Omega\cap S(u)) \int_{\mathbb{R}^n} |\xi| e^{-|\xi|^2} d\xi,\end{aligned}$$

from which we obtain the value of the constants A and B.

10.4 Approximation of curvature functionals

We close this chapter with a brief description of the approximation for another type of functional used in Visual Reconstruction; namely,

$$\mathcal{G}(u,C,P) = \#(P) + \int_C (1+\kappa^2)\, d\mathcal{H}^1 + \int_{\Omega\setminus(C\cup P)} |\nabla u|^2\, dx$$

(to which a 'fidelity term' $\int_\Omega |u-g|^2\, dx$ can be added), where C is a family of curves, P is the set of the endpoints of the curves in C, and $\#(P)$ is the number of points in P. The functional $\mathcal{G}$ has been proposed in this form by Anzellotti, and has a number of connections with similar energies proposed in Computer Vision. Existence results for minimizers of $\mathcal{G}$ can be found in a paper by Coscia [80]. They are relatively simple, since energy bounds imply a bound on the number of components of C and on their norm as $W^{2,2}$-functions.

We now illustrate an approximation due to Braides and March [63]. This result will use at the same time many of the arguments introduced before. One will be the introduction of intermediate energies defined on set-function triplets (as in Section 10.2.1), another will be the iterated use of a gradient approach as in the Modica-Mortola case to generate energies on sets of different dimensions, and finally the modification of the Ambrosio-Tortorelli construction to obtain recovery sequences in H^2.

We first show the idea for an approximation by energies on triplets function-sets. The first step is to construct a variational approximation of the functional $\#(P)$ that simply counts the number of the points of a set P by another functional on sets, whose minimizers are discs of small radius ε (additional conditions will force these discs to contain the target set of points). Such a functional is given by

$$\mathcal{E}^{(1)}_\varepsilon(D) = \frac{1}{4\pi}\int_{\partial D}\Big(\frac{1}{\varepsilon} + \varepsilon\kappa^2(x)\Big)\, d\mathcal{H}^1(x),$$

where κ denotes the curvature of ∂D. The number $1/4\pi$ is a normalization factor that derives from the fact that minimizers of $\mathcal{E}^{(1)}_\varepsilon(D)$ are given by balls of radius ε. This functional may be interpreted, upon scaling, as a singular perturbation of the perimeter functional by a curvature term.

The next step is then to construct another energy defined on sets, that approximates the functional $\int_C (1+\kappa^2) d\mathcal{H}^1$, where C is a (finite) union of $W^{2,2}$-curves with endpoints contained in P. To this end we approximate C away from D by sets A, whose energy is defined as

$$\mathcal{E}^{(2)}_\varepsilon(A,D) = \frac{1}{2}\int_{(\partial A)\setminus D} (1+\kappa^2)\, d\mathcal{H}^1, \qquad |A| \le \delta_\varepsilon,$$

where $0 < \delta_\varepsilon = o(1)$ play the same role as in Theorem 10.6, and force A to shrink to C. As in Section 10.2.1, the factor $1/2$ depends on the fact that, as A tends to C, each curve of C is the limit of two arcs of ∂A.

The intermediate function-set approximation is thus constructed by assembling the pieces above and the simpler terms that account for u:

$$\mathcal{E}_\varepsilon(u,A,D) = \mathcal{E}^{(1)}_\varepsilon(D) + \mathcal{E}^{(2)}_\varepsilon(A,D) + \int_{\Omega\setminus(A\cup D)} |\nabla u|^2\, dx, \qquad |A| \le \delta_\varepsilon$$

defined for A and D compactly contained in Ω. Note that $A \cup D$ contains the singularities of u. By following a recovery sequence for the Γ-limit of $\mathcal{E}_\varepsilon$, a triplet (u, C, P) is approximated by means of triplets $(u_\varepsilon, A_\varepsilon, D_\varepsilon)$.

To obtain an energy defined on functions, we again use a gradient-theory approach as by Modica and Mortola, where it is shown that the perimeter measures $\mathcal{H}^1 \llcorner \partial A$ and $\mathcal{H}^1 \llcorner \partial D$ are approximated by the measures $\mathcal{H}^1_\varepsilon(s, \nabla s)\, dx$ and $\mathcal{H}^1_\varepsilon(w, \nabla w)\, dx$ where

$$\mathcal{H}^1_\varepsilon(s, \nabla s) = \zeta_\varepsilon |\nabla s|^2 + \frac{s^2(1-s)^2}{\zeta_\varepsilon}, \qquad \mathcal{H}^1_\varepsilon(w, \nabla w) = \zeta_\varepsilon |\nabla w|^2 + \frac{w^2(1-w)^2}{\zeta_\varepsilon},$$

$\zeta_\varepsilon \to 0$ as $\varepsilon \to 0$, and s and w are optimal-profile functions approximating $1 - \chi_A$ and $1 - \chi_D$, respectively. We define the curvature of s and w as

$$\kappa(\nabla s) = \begin{cases} \operatorname{div}\Big(\dfrac{\nabla s}{|\nabla s|}\Big) & \text{if } \nabla s \neq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad \kappa(\nabla w) = \begin{cases} \operatorname{div}\Big(\dfrac{\nabla w}{|\nabla w|}\Big) & \text{if } \nabla w \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

respectively. The next step is formally to replace the characteristic functions $1 - \chi_A$ and $1 - \chi_D$ by functions s and w. The terms $\mathcal{E}^{(1)}_\varepsilon(D)$ and $\mathcal{E}^{(2)}_\varepsilon(A, D)$ are then substituted by

$$\begin{aligned} \mathcal{G}^{(1)}_\varepsilon(w) &= \int_\Omega \Big(\frac{1}{\varepsilon} + \varepsilon \kappa^2(\nabla w)\Big) \mathcal{H}^1_\varepsilon(w, \nabla w)\, dx, \\ \mathcal{G}^{(2)}_\varepsilon(s, w) &= \int_\Omega w^2 (1 + \kappa^2(\nabla s)) \mathcal{H}^1_\varepsilon(s, \nabla s)\, dx, \end{aligned}$$

respectively, and the constraint that $|A| \leq a_\varepsilon$ by an integral penalization

$$\mathcal{I}_\varepsilon(s, w) = \frac{1}{\mu_\varepsilon} \int_\Omega ((1-s)^2 + (1-w)^2)\, dx,$$

(where $\mu_\varepsilon \to 0$) that forces s and w to be equal to 1 almost everywhere in the limit as $\varepsilon \to 0$, so that we construct a candidate functional

$$\mathcal{G}_\varepsilon(u, s, w) = \frac{1}{4\pi b_0} \mathcal{G}^{(1)}_\varepsilon(w) + \frac{1}{2b_0} \mathcal{G}^{(2)}_\varepsilon(s, w) + \int_\Omega s^2 |\nabla u|^2\, dx + \mathcal{I}_\varepsilon(s, w),$$

where b_0 is a normalization constant.

The following result shows that these elliptic energies are indeed variational approximations of the energy $\mathcal{G}$, for a suitable choice of ζ_ε and μ_ε.

Theorem 10.9 (approximation of curvature functionals) *The functionals $\mathcal{G}_\varepsilon$ Γ-converge to $\mathcal{G}$ as $\varepsilon \to 0^+$ with respect to the convergence $(u_\varepsilon, s_\varepsilon, w_\varepsilon) \to (u, C, P)$ defined as the convergence of a.a. level sets (see* [63] *for a precise definition).*

A technical but important difference with the Ambrosio-Tortorelli approach is that there the double-well potential $s^2(1-s)^2$ in the approximation of the perimeter is replaced by the single-well potential $(1-s)^2$. This modification breaks the symmetry between 0 and 1 and forces automatically s to tend to 1 as $\varepsilon \to 0^+$. Unfortunately, it also forbids recovery sequences to be bounded in H^2: with this substitution the curvatures terms in $\mathcal{G}^{(1)}_\varepsilon$ and $\mathcal{G}^{(2)}_\varepsilon$ would necessarily be unbounded. In our case, the necessary symmetry breaking is obtained by adding the 'lower-order' term $\mathcal{I}_\varepsilon$. We note that the complex form of the functionals $\mathcal{G}_\varepsilon$, in particular of $\mathcal{G}^{(1)}_\varepsilon$, seems necessary despite the simple form of the target energy. Indeed, in order to describe an energy defined on points in the limit, it seems necessary to consider degenerate functionals. Our approach may be compared with that giving vortices in the Ginzburg-Landau theory or concentration of energies for functionals with critical growth in Section 8.

11 Continuum limits of lattice systems

In this last section we touch a subject of active research, with connections with many issues in Statistical Mechanics, Theoretical Physics, Computer Vision, computational problems, approximation schemes, etc. Namely, that of the passage from a variational problem defined on a discrete set to a corresponding problem on the continuum as the number of the points of the discrete set increases. Some of these problems naturally arise in an atomistic setting, or as finite-difference numerical schemes. An overview of scale problems for atomistic theories can be found in the review paper by Le Bris and Lions [118], an introduction to some of their aspects in Computational Materials Science can be found in that by Blanc, Le Bris and Lions [37]. An introduction to the types of problems treated in this section in a one-dimensional setting can be found in the notes by Braides and Gelli [60] (see also [46] Chapters 4 and 11).

The setting for discrete problems in which we have a fairly complete set of results is that of *central interactions* for *lattice systems*; i.e., systems where the reference positions of the interacting points lie on a prescribed lattice, whose parameters change as the number of points increases, and each point of the lattice interacts separately with each other point. In more precise terms, we consider an open set $\Omega \subset \mathbb{R}^n$ and take as reference lattice $Z_\varepsilon = \Omega \cap \varepsilon\mathbb{Z}^n$. The general form of a *pair-potential* energy is then

$$F_\varepsilon(u) = \sum_{i,j\in Z_\varepsilon} f^\varepsilon_{ij}(u(i), u(j)), \tag{11.1}$$

where $u : Z_\varepsilon \to \mathbb{R}^m$. The analysis of energies of the form (11.1) has been performed under various hypotheses on f_{ij}. The first simplifying assumption is that F_ε is invariant under translations (in the target space); that is,

$$f^\varepsilon_{ij}(u, v) = g^\varepsilon_{ij}(u - v). \tag{11.2}$$

Furthermore, an important class is that of homogeneous interactions (i.e., invariant under translations in the reference space); this condition translates into

$$f^\varepsilon_{ij}(u, v) = g^\varepsilon_{(i-j)/\varepsilon}(u, v). \tag{11.3}$$

If both conditions are satisfied, we may rewrite the energies F_ε above as

$$F_\varepsilon(u) = \sum_{k\in\mathbb{Z}^n} \sum_{i,j\in Z_\varepsilon, i-j=\varepsilon k} \varepsilon^n \psi^\varepsilon_k\Big(\frac{u(i) - u(j)}{\varepsilon}\Big), \tag{11.4}$$

where $\psi^\varepsilon_k(\xi) = \varepsilon^{-n} g^\varepsilon_k(\varepsilon\xi)$. In this new form the interactions appear through the (discrete) difference quotients of the function u. Upon identifying each function u with its piecewise-constant interpolation (extending the definition of u arbitrarily outside Ω), we can consider F_ε as defined on (a subset of) $L^1(\Omega; \mathbb{R}^m)$, and hence compute the Γ-limit with respect to the $L^1_{\rm loc}$-topology. Under some coerciveness conditions the computation of the Γ-limit will give a continuous approximate description of the behaviour of minimum problems involving the energies F_ε for ε small.

11.1 Continuum energies on Sobolev spaces

Growth conditions on energy densities ψ_k^ε imply correspondingly boundedness conditions on gradient norms of piecewise-affine interpolations of functions with equi-bounded energy. The simplest type of growth condition that we encounter is on *nearest neighbours*; i.e., for $|k|=1$. If $p>1$ exists such that

$$c_1|z|^p - c_2 \le \psi_k^\varepsilon(z) \le c_2(1+|z|^p) \tag{11.5}$$

($c_1, c_2 > 0$ for $|k|=1$), and if $\psi_k^\varepsilon \ge 0$ for all k then the energies are *equi-coercive*: if (u_ε) is a bounded sequence in $L^1(\Omega;\mathbb{R}^m)$ and $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$, then from every sequence (u_{ε_j}) we can extract a subsequence converging to a function $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. In this section we will consider energies satisfying this assumption. Hence, their Γ-limits are defined in the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^m)$.

First, we remark that the energies F_ε can also be seen as an integration with respect to measures concentrated on Dirac deltas at the points of $Z_\varepsilon \times Z_\varepsilon$. If each ψ_k^ε satisfies a growth condition $\psi_k^\varepsilon(z) \le c_k^\varepsilon(1+|z|^p)$, then we have

$$F_\varepsilon(u) \le \int_{\Omega\times\Omega} (1+|u(x)-u(y)|^p)\,d\mu_\varepsilon,$$

where

$$\mu_\varepsilon = \sum_{k=1}^{\infty} \sum_{i-j=\varepsilon k,\ i,j\in Z_\varepsilon} c_k^\varepsilon \frac{1}{\varepsilon^p}\delta_{(i,j)}.$$

A natural condition for the finiteness of the limit of F_ε is the equi-boundedness of these measures (as $\varepsilon \to 0$) for every fixed set Ω. However, under such assumption, we can have a non-local Γ-limit of the form

$$F(u) = \int_\Omega f(Du(x))\,dx + \int_{\Omega\times\Omega} \psi(u(x)-u(y))\,d\mu(x,y),$$

where μ is the weak*-limit of the measures μ_ε outside the 'diagonal' of $\mathbb{R}^n \times \mathbb{R}^n$ (just as we obtain Dirichlet forms from degenerate quadratic functionals). Under some decay conditions, such long-range behaviour may be ruled out. The following compactness result proved by Alicandro and Cicalese [15] shows that a wide class of discrete systems possesses a 'local' continuous limit (an analogue for linear difference operators can be found in Piatnitski and Remy [135]). We state it in a general 'space-dependent' case.

Theorem 11.1 (compactness for discrete systems) *Let $p>1$ and let ψ_k^ε satisfy:*

(i) (coerciveness on nearest neighbours) *there exists $c_1, c_2 > 0$ such that for all $(x,z) \in \Omega \times \mathbb{R}^m$ and $i \in \{1,\dots,n\}$*

$$c_1|z|^p - c_2 \le \psi_{e_i}^\varepsilon(x,z) \tag{11.6}$$

(ii) (decay of long-range interactions) *for all $(x,z) \in \Omega\times\mathbb{R}^m$, and $k \in \mathbb{Z}^n$*

$$\psi_k^\varepsilon(x,z) \le c_k^\varepsilon(1+|z|^p), \tag{11.7}$$

where c_k^ε satisfy

(H1): $\limsup\limits_{\varepsilon\to 0^+} \sum\limits_{k\in\mathbb{Z}^n} c_k^\varepsilon < +\infty$*;*

(H2): *for all $\delta > 0$, $M_\delta > 0$ exists such that* $\limsup\limits_{\varepsilon\to 0^+} \sum_{|k|>M_\delta} c_k^\varepsilon < \delta$.

Let F_ε be defined by

$$F_\varepsilon(u) = \sum_{k\in\mathbb{Z}^n} \sum_{i\in R^k_\varepsilon} \varepsilon^n \psi^\varepsilon_k \left(i, \frac{u(i+\varepsilon k) - u(i)}{\varepsilon |k|} \right),$$

where $R^k_\varepsilon := \{i \in Z_\varepsilon :\ i+\varepsilon k \in Z_\varepsilon\}$. Then for every sequence (ε_j) of positive real numbers converging to 0, there exist a subsequence (not relabeled) and a Carathéodory function $f : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ satisfying

$$c(|\xi|^p - 1) \le f(x,\xi) \le C(|\xi|^p + 1),$$

with $0 < c < C$, such that (F_{ε_j}) Γ-converges with respect to the $L^p(\Omega;\mathbb{R}^m)$-topology to the functional F defined as

$$F(u) = \int_\Omega f(x, Du)\, dx, \qquad \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m). \tag{11.8}$$

Proof The proof of this theorem follows the general compactness procedure in Section 3.3. It must be remarked, though, that discrete functionals are non-local by nature, so that all arguments where locality is involved must be carefully adapted. The non-locality disappears in the limit thanks to condition (ii). For a detailed proof we refer to [15]. □

Remark 11.2 (homogenization) In the case of energies defined by a scaling process; i.e., when

$$\psi^\varepsilon_k(x,z) = \psi_k\Big(\frac{z}{\varepsilon}\Big), \tag{11.9}$$

then the limit energy density $\varphi(M) = f(x,M)$ is independent of x and of the subsequence, and is characterized by the *asymptotic homogenization formula*

$$\varphi(M) = \lim_{T\to+\infty} \frac{1}{T^N} \min\left\{\mathcal{F}_T(u),\ u|_{\partial Q_T} = Mi\right\}, \tag{11.10}$$

where $Q_T = (0,T)^N$,

$$\mathcal{F}_T(u) = \sum_{k\in\mathbb{Z}^n} \sum_{i\in R^k_1(Q_T)} \psi_k \left(\frac{u(i+k) - u(i)}{|k|} \right)$$

and $u|_{\partial Q_T} = Mi$ means that “close to the boundary” of Q_T the function u is the discrete interpolation of the affine function Mx (see [15] for further details). In the one-dimensional case this formula was first derived in [61], and it is the discrete analogue of the nonlinear asymptotic formula for the homogenization of nonlinear energies of the form $\int_\Omega f(x/\varepsilon, Du)\, dx$.

Note however that the two formulas differ in two important aspects: the first is that (11.10) transforms functions depending on difference quotients (hence, vectors or scalars) into functions depending on gradients (hence, matrices or vectors, respectively); the second one is that the boundary conditions in (11.10) must be carefully specified, since we have to choose whether considering or not interactions that may ‘cross the boundary’ of Q_T.

It must be noted that formula (11.10) does not simplify even in the simplest case of three levels of interactions in dimension one, thus showing that this effect, typical of nonlinear homogenization, is really due to the lattice interactions and not restricted to vector-valued functions as in the case of homogenization on the continuum.

It is worth examining formula (11.10) in some special cases. First, if all ψ_k are *convex* then, apart from a possible lower-order boundary contribution, the solution in (11.10) is simply $u_i = Mi$. In this case the Γ-limit coincides with the pointwise limit. Note that convexity in a sense always 'trivializes' discrete systems, in the sense that their continuous counterpart, obtained by simply substituting difference quotients with directional derivatives is already lower semicontinuous, and hence provides automatically an optimal lower bound.

Next, if only *nearest-neighbour interactions* are present then it reduces to

$$\varphi(M) = \sum_{i=1}^{n} \psi_i^{**}(Me_i),$$

where $\psi_i = \psi_{e_i}$ and ψ_i^{**} denotes the lower semicontinuous and convex envelope of ψ_i. Note that convexity is not a necessary condition for lower semicontinuity at the discrete level: this convexification operation should be interpreted as an effect due to oscillations at a 'mesoscopic scale' (i.e., much larger than the 'microscopic scale' ε but still vanishing as $\varepsilon \to 0$). If not only nearest neighbours are taken into account then the mesoscopic oscillations must be coupled with microscopic ones (see [61, 134, 46] and the next section).

Finally, note that also in the non-convex case (the relaxation of) the pointwise limit always gives an upper bound for the Γ-limit and is not always trivial (see, e.g., the paper by Blanc, Le Bris and Lions [36]).

11.1.1 Microscopic oscillations: the Cauchy-Born rule

One issue of interest in the study of discrete-to-continuous problems is whether to a 'macroscopic' gradient there corresponds at the 'microscopic' scale a 'regular' arrangement of lattice displacement. For energies deriving from a scaling process as in (11.9) this can be translated into the asymptotic study of minimizers for the problems defining $\varphi(M)$; in particular whether $u_i = Mi$ is a minimizer (in which case we say that the (strict) *Cauchy-Born rule* holds at M), or if minimizers tend to a periodic perturbation of Mi; i.e. ground states are periodic (in which case we say that the *weak Cauchy-Born rule* holds at M). Note that the strict Cauchy-Born rule can be translated into the equality

$$\varphi(M) = \sum_{k \in \mathbb{Z}^N} \psi_k\Big(\frac{Mk}{|k|}\Big), \tag{11.11}$$

and that it always holds if all ψ_k are convex, as remarked above.

A simple example in order to understand how the validity and failure of the Cauchy-Born rule can be understood in terms of the form of φ is given by the one-dimensional case with *next-to-nearest neighbours*; i.e. when only ψ_1 and ψ_2 are non zero. In this case $\varphi = \psi^{**}$, where

$$\psi(z) = \psi_2(2z) + \frac{1}{2}\min\Big\{\psi_1(z_1) + \psi_1(z_2) : z_1 + z_2 = 2z\Big\}. \tag{11.12}$$

The second term, obtained by minimization, is due to oscillations at the microscopic level: nearest neighbours rearrange so as to minimize their interaction coupled with that between second neighbours (see [46] for a simple treatment of these one-dimensional problems). In this case we can read the microscopic behaviour as follows (for the sake of simplicity we suppose that the minimum problem in (11.12) has a unique solution, upon changing z_1 into z_2):

(i) first case: ψ *is convex at* z (i.e., $\psi(z) = \varphi(z)$). We have the two cases

(a) $\psi(z) = \psi_1(z) + \psi_2(z)$; in this case $z = z_1 = z_2$ minimizes the formula giving φ and (11.11) holds; hence, the strict Cauchy-Born rule applies;

(b) $\psi(z) < \psi_1(z) + \psi_2(z)$; in this case we have a 2-periodic ground state with 'slopes' z_1 and z_2, and the weak Cauchy-Born rule applies;

(ii) second case: ψ *is not convex at* z (i.e., $\psi(z) > \varphi(z)$). In this case the Cauchy-Born rule is violated, but a finer analysis (see below) shows that minimizers are fine mixtures of states satisfying the conditions above; hence the condition holds 'locally'.

For energies in higher-dimensions this analysis is more complex. A similar argument as in the one-dimensional case is used by Friesecke and Theil [109] to show the non-validity of the Cauchy-Born rule even for some types of very simple lattice interactions in dimension two, with nonlinearities of geometrical origin.

11.1.2 Higher-order developments: phase transitions

In the case of failure of the Cauchy-Born rule, non-uniform states may be preferred as minimizers, and surface energies must be taken into account in their description. A first attempt to rigorously describe this phenomena can be found in Braides and Cicalese [51], again in the simplest nontrivial case of next-to-nearest neighbour interactions of the form independent of ε. In that case, we may infer that (under some technical assumptions) the discrete systems are equivalent to the perturbation of a non-convex energy on the continuum, of the form

$$\int_\Omega \psi(u')\,dt + \varepsilon^2 C \int_\Omega |u''|^2\,dt,$$

thus recovering the well-known formulation of the gradient theory of phase transitions. This result shows that a surface term (generated by the second gradient) penalizes high oscillations between states locally satisfying some Cauchy-Born rule.

11.1.3 Homogenization of networks

We have stated above that convex discrete problems are 'trivial' since they can immediately be translated into local integral functionals. However, in some cases constraints are worse expressed in the continuous translation rather than in the original lattice notation, so that a direct treatment of the discrete system is easier. A striking and simple example is the computation of bounds for composite linear conducting networks in dimension two, as shown by Braides and Francfort [57]. This is the discrete analogue of the problem presented in Section 5.4, that translates into the computation of homogenized matrices given by

$$\begin{aligned}\langle A\xi,\xi\rangle \quad = \quad & \frac{1}{N^2}\min\Big\{\sum_{i\in\{1,\dots,N\}^2} h_i(\xi_1+\varphi(i_1+1,i_2)-\varphi(i_1,i_2))^2 \\ & \qquad + \sum_{i\in\{1,\dots,N\}^2} v_i(\xi_2+\varphi(i_1,i_2+1)-\varphi(i_1,i_2))^2 : \varphi:\mathbb{Z}^2\to\mathbb{R}\ N\text{-periodic}\Big\}\end{aligned}$$

where $h_i, v_i \in \{\alpha, \beta\}$ (h_i stands for the horizontal connection at i, v_i for vertical), N is arbitrary, and the percentage θ of α-connections is given; i.e., $\#\{i : v_i = \alpha\} + \#\{i : h_i = \alpha\} = 2N^2\theta$. The set of such matrices $\mathcal{H}_d(\theta)$ is the analogue of the set $\mathcal{H}(\theta)$ in Section 5.4.1.

We do not discuss the computation of optimal bounds for $\mathcal{H}(\theta)$, for which we refer to [57], but just remark that the additional microscopic dimension brings new micro-geometries. In fact, it is easily seen that $\mathcal{H}(\theta) \subset \mathcal{H}_d(\theta)$ since coefficients h_i, v_i such that $h_i = v_i$ correspond to discretizing continuous coefficients, but we can also construct discrete laminates at the same time in the horizontal and vertical directions, providing a much larger set of homogenized matrices. As an example, in the discrete case the set of all reachable matrices (see Remark 5.10) contains all diagonal matrices with eigenvalues λ_1, λ_2 with $\alpha \leq \lambda_i \leq \beta$, while in the continuous case we are restricted by the bounds (5.20).

11.2 Continuum energies on discontinuous functions

In many cases discrete potentials related to atomic theories do not satisfy the hypotheses of the compactness result above, and the limits are defined on spaces of discontinuous functions.

11.2.1 Phase transitions in discrete systems

The easiest example of a discrete system exhibiting a phase transition is that of nearest-neighbour interactions for an elementary Ising system. We can consider energies of the form

$$-\sum_{|i-j|=\varepsilon} u_i u_j \qquad \text{(ferromagnetic interactions)}, \tag{11.13}$$

defined on functions $u : \varepsilon\mathbb{Z}^n \cap \Omega \to \{-1, 1\}$, and $u_i = u(i)$. Upon a scaling of the energies we can equivalently consider the functionals

$$F_\varepsilon(u) = \sum_{|i-j|=\varepsilon} \varepsilon^{n-1}(1 - u_i u_j) \tag{11.14}$$

Note that we have $1 - u_i u_j = 0$ if $u_i = u_j$ and $1 - u_i u_j = 2$ if $u_i \neq u_j$. With this observation in mind we may identify each u with the function which takes the value u_i in the coordinate cube Q_i^ε with centre εi and side length ε. In this way we can rewrite

$$F_\varepsilon(u) = 2\int_{S(u)\cap\Omega} \|\nu\|_1 d\mathcal{H}^{n-1} + O(\varepsilon) \tag{11.15}$$

(the error term comes from the cubes intersecting the boundary of Ω), where $\|\nu\|_1 = \sum_i |\nu_i|$. Here we have taken into account that the interface between $\{u = 1\}$ and $\{u = -1\}$ is composed of the common boundaries of neighbouring cubes Q_i^ε and Q_j^ε where $u_i \neq u_j$ and that these boundaries are orthogonal to some coordinate direction (note that $\|\nu\|_1$ is the greatest convex and positively homogeneous function of degree one with value 1 on the coordinate directions). This heuristic derivation can be turned into a theorem (see the paper by Alicandro, Braides and Cicalese [12]). Note that the symmetries of the limit energy density are derived from those of the square lattice, and that the reasoning above also provides an equi-coerciveness result.

Theorem 11.3 (continuum limit of a binary lattice system) *The* Γ*-limit in the* L^1*-convergence of the functional* F_ε *defined in* (11.14) *is given by*

$$F_0(u) = 2\int_{\Omega\cap S(u)} \|\nu\|_1 d\mathcal{H}^{n-1} \qquad (11.16)$$

on characteristic functions of sets of finite perimeter.

Proof The argument above gives the lower bound by the lower semicontinuity of F_0, an upper bound is obtained by first considering smooth interfaces, for which we may define $u_\varepsilon = u$, and then reason by density. □

Similar results can also be obtained for long-range interactions, and a more subtle way to define the limit phases must be envisaged in the case of long-range *anti-ferromagnetic* interactions (i.e., when we change sign in (11.13), so that microscopic oscillations are preferred to uniform states). Details are found in [12].

11.2.2 Free-discontinuity problems deriving from discrete systems

The pioneering example for this case is due to Chambolle [70, 71], who treated the limit of some finite-difference schemes in Computer Vision (see [65]), producing as the continuum counterpart the one-dimensional version of the Mumford-Shah functional.

Theorem 11.4 (Blake-Zisserman approximation of the Mumford-Shah functional) *Let* F_ε *be defined by the* truncated quadratic energy

$$F_\varepsilon(u) = \sum_{|i-j|=\varepsilon} \varepsilon^n \min\Big\{\Big(\frac{u_i - u_j}{\varepsilon}\Big)^2, \frac{1}{\varepsilon}\Big\} \qquad (11.17)$$

defined on functions $u : \varepsilon\mathbb{Z}^n \cap \Omega \to \mathbb{R}$*; then the* Γ*-limit in the* L^1*-convergence of* F_ε *is the* anisotropic Mumford-Shah functional

$$F_0(u) = \int_\Omega |\nabla u|^2\,dx + \int_{\Omega\cap S(u)} \|\nu\|_1 d\mathcal{H}^{n-1}. \qquad (11.18)$$

on $GSBV(\Omega)$.

Proof In the one-dimensional case it suffices to identify each u with a discontinuous piecewise-affine interpolation v for which $F_\varepsilon(u) = F_0(v)$ (see [46] Section 8.3). A more complex interpolation can be used in the two-dimensional case (see [71]), while the general case can be achieved by adapting the slicing method (see [58, 72]). Note that for functions $u_i \in \{-1, 1\}$ the functional coincides (up to a factor 2) with the one studied in the previous section. □

11.2.3 Lennard-Jones potentials

The case of more general convex-concave potentials such as Lennard-Jones or Morse potentials brings additional problems due to the degenerate behaviour at infinity. In the one-dimensional case we consider energies

$$F_\varepsilon(u) = \sum_i \varepsilon J\Big(\frac{u_i - u_{i-1}}{\varepsilon}\Big), \qquad J(z) = \frac{1}{z^{12}} - \frac{1}{z^6}, \qquad (11.19)$$

defined on $u : \varepsilon\mathbb{Z} \cap (0,1) \to \mathbb{R}$ with the constraint $u_i > u_{i-1}$ (*non-interpenetration*). It must be noted that the Γ-limit of such an energy is equal to

$$F_0(u) = \int_0^1 J^{**}(u')\, dt, \qquad u \text{ increasing} \tag{11.20}$$

(u' here denotes the absolutely continuous part of the distributional derivative Du). Note that the singular part of Du does not appear here so that u may in particular have infinitely-many (increasing) jumps at 'no cost'.

Note that $J^{**}(z)$ takes the constant value $\min J$ for $z \geq z^* := \operatorname{argmin} J = 2^{1/6}$, and hence $F_0(u) = \min J$ for all increasing u with $u' \geq z^*$. Some other method must be used to derive more information. These problems can be treated in different ways, which we mention briefly.

1. Γ-*developments*. We can consider the scaled functionals $F^1_\varepsilon(u) = \frac{1}{\varepsilon}(F_\varepsilon(u) - \min J)$. In this case we obtain as a Γ-limit the functional whose domain are increasing (discontinuous) piecewise-affine functions with $u' = z^*$, on which the limit is $F^1(u) = C\#(S(u))$, and $C = -\min J$. This limit function can be interpreted as a *fracture energy* for a rigid body. (see [144] and [46] Section 11.4 for a complete proof).

2. *Scaling of convex-concave potentials*. The study of these types of energies have been initiated in a paper by Braides, Dal Maso and Garroni [53], who consider potentials of convex-concave type and express the limit in terms of different scalings of the two parts, expressing the limit in the space of functions with bounded variation. In Mechanical terms the limit captures softening phenomena and size effects. The general case of nearest-neighbour interactions has been treated by Braides and Gelli in [59]. The fundamental issue here is the *separation of scale effect* (see also [46] Chapter 11 for a general presentation).

3. *Renormalization-group approach*. This suggests a different scaling of the energy, and to consider

$$F'_\varepsilon(u) = \sum_i \Big(J\Big(z^* + \frac{u_i - u_{i-1}}{\sqrt{\varepsilon}}\Big) - \min J\Big). \tag{11.21}$$

The Γ-limit of F'_ε has been studied by Braides, Lew and Ortiz [62], and can be reduced to the case studied in Theorem 11.4, obtaining as a Γ-limit

$$F'(u) = \alpha \int_0^1 |u'|^2\, dt + \beta\#(S(u)), \text{ with } u^+ > u^- \text{ on } S(u), \tag{11.22}$$

which is interpreted as a Griffith fracture energy with a unilateral condition on the jumps. Here α is the curvature of J at its minimum and $\beta = -\min J$.

All these approaches can be extended to long-range interactions, but are more difficult to repeat in higher dimension.

11.2.4 Boundary value problems

For the sake of completeness it must be mentioned that according to the variational nature of the approximations all these convergence results lead to the study of convergence of minimum problems. To this regard we have to remark that in the case of the so called 'long-range interactions' for functional allowing for fracture (that is, when the limit energy presents a non-zero surface part) more than one type of boundary-value problem can be formulated and an effect of boundary layer also occurs. The problem was first studied by Braides and Gelli in [58] where two types of problems where treated. The first one is to define discrete functions on the whole $\varepsilon\mathbb{Z}^N$ and to fix the values on the nodes outside the domain Ω equal to a fixed function φ; in this case the interactions 'across the boundary of Ω' give rise to an additional boundary term in the limit energy of the type

$$\int_{\partial\Omega} \mathcal{G}(\gamma(u) - \varphi, \nu_{\partial\Omega}) d\mathcal{H}^{n-1} \tag{11.23}$$

where $\gamma(u)$ is the inner trace of u on $\partial\Omega$.

The second method consists in considering the functions as fixed only on $\partial\Omega$ that is, only a proper subset of pairwise interactions are linked with the constraint; in this case, the boundary term gives a different contribution, corresponding to a boundary-layer effect. Indeed, the additional term is still of type (11.23) but with the surface density $\mathcal{G}'$ strictly less than $\mathcal{G}$, the gap magnifying with the range of interaction considered.

Boundary-layer effects also appear in the definition of the surface energies for fracture in the case of long-range interactions (see [62, 51]) and in the study of *discrete thin films* [13].

Note to the bibliography. The following list of references contains only the works directly quoted in the text, as it would be impossible to write down an exhaustive bibliography. We refer to [85], [54] and [46] for a guide to the literature.

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