ESERCITAZIONE 22/5/2024

ES 1 SIA V IL SOTTOSPAZIO DI $L^2(-1,1)$ GENERATO DA $x_1(t)=t$
$x_2(t)=t^2$
$x_3(t)=t^3$

- TROVARE UNA BASE ORTOGONALE DI V

$\langle x_1, x_2 \rangle = 0$ (IL PRODOTTO È UNA FUNZIONE DISPARI $\Rightarrow \int_{-1}^{1} = 0$)

$\langle x_3, x_2 \rangle = 0$ (" " " " " ")

APPLICHIAMO GRAM-SCHMIDT:

$v_1 = x_1 = t$ $\qquad \|x_1\|^2 = \int_{-1}^{1} t^2\,dt = 2\int_0^1 t^2\,dt = \frac{2}{3}$

$v_2 = x_2 = t^2$ $\qquad \|v_2\|^2 = \int_{-1}^{1} t^4 = 2/5$

$$v_3 = x_3 - \frac{\langle x_3, x_1\rangle}{\|x_1\|^2} x_1 - \frac{\overbrace{\langle x_3, x_2\rangle}^{=0}}{\|x_2\|^2} x_2 = t^3 - \frac{2/5}{2/3} t = t^3 - \frac{3}{5}t$$

$$\langle x_3, x_1\rangle = \int_{-1}^{1} t^4\,dt = \frac{2}{5}$$

- CALCOLARE LA PROIEZIONE DI $x(t) = 1 - it^4$ SU V

PROIEZIONE DI 1 SU V:

$$y_1(t) = \underbrace{\langle 1, v_1\rangle}_{=0} \frac{v_1}{\|v_1\|^2} + \overbrace{\langle 1, v_2\rangle}^{\int_{-1}^{1} t^2 dt = \frac{2}{3}} \frac{v_2}{\|v_2\|^2} + \overbrace{\langle 1, v_3\rangle}^{=0} \frac{v_3}{\|v_3\|^2} = \frac{2}{3} \cdot \frac{t^2}{\frac{2}{5}} = \frac{5}{3} t^2$$

PROIEZIONE DI t^4 SU V

$$\int_{-1}^{1} t^6 = \frac{2}{7}$$

$$y_2 = \overbrace{\langle t^4, v_1\rangle}^{=0} \frac{v_1}{\|v_1\|^2} + \overbrace{\langle t^4, v_2\rangle}^{=} \frac{v_2}{\|v_2\|^2} + \overbrace{\langle t^4, v_3\rangle}^{=0} \frac{v_3}{\|v_3\|^2}$$

$$= \frac{5}{7} t^2$$

$\Rightarrow$ proiezione DI $1-t^4$ su V: $y(t) = y_1(t) - i\, y_2(t)$

$$= \left(\frac{5}{3} - i\frac{5}{7}\right) t^2$$

. CALCOLARE DISTANZA L^2 DELLA FUNZIONE $w(t) = i-1$ DA V

LA PROIEZIONE DI $w(t)$ SU V È $(i-1) \cdot \underbrace{\frac{5}{3} t^2}_{\text{proiezione DI } 1} =: z(t)$

$\Rightarrow \| z(t) - w(t) \|^2 =$

$$= \int_{-1}^{1} \left| (i-1)\frac{5}{3}t^2 - (i-1)\right|^2 dt = \underbrace{|i-1|^2}_{\substack{\| \\ 2}} \cdot \underbrace{\int_{-1}^{1} \left|\frac{5}{3}t^2 - 1\right| dt}_{\|} = \frac{16}{9}$$

$$2\int_0^1 \frac{25}{9}t^4 + 1 - \frac{10}{3}t^2$$

$$= 2\left(\frac{5}{9} + 1 - \frac{10}{9}\right) = 2\,\frac{4}{9} = \frac{8}{9}$$

$\Rightarrow$ $\boxed{\|z - w\| = \frac{4}{3}}$

ES2 SIA V IL SOTTOSPAZIO DI $L^2(-\pi,\pi)$ GENERATO DA

$x_1(t) = t \qquad x_2(t) = \cos t \qquad x_3(t) = 1$

- TROVARE BASE ORTOGONALE DI V

$$\langle x_1(t), x_2(t)\rangle = \int_{-\pi}^{\pi} t\cos t\, dt = 0 \qquad (t\cos t \text{ è dispari})$$

$$\langle x_1(t), x_3(t)\rangle = \int_{-\pi}^{\pi} t\, dt = 0$$

$$\langle x_2(t), x_3(t)\rangle = \int_{-\pi}^{\pi} \cos t\, dt = 0$$

QUINDI SONO UNA BASE ORTOGONALE DI V;

- CALCOLARE LA DISTANZA DI $y(t) = \sin t$ da V

PROIEZIONE DI $y(t)$ SU V:

$$P_V(y(t)) = \frac{\langle y, x_1\rangle}{\|x_1\|^2} x_1 + \frac{\langle y, x_2\rangle}{\|x_2\|^2} x_2 + \frac{\langle y, x_3\rangle}{\|x_3\|^2} x_3 = \frac{2\pi}{\frac{2}{3}\pi^3} t = \frac{3t}{\pi^2}$$

$$\langle y, x_1\rangle = \int_{-\pi}^{\pi} t\sin t\, dt = -t\cos t \Big|_{-\pi}^{\pi} + \underbrace{\int_{-\pi}^{\pi}\cos t\, dt}_{0} = 2\pi$$

$$\|x_1\|^2 = \int_{-\pi}^{\pi} t^2 = \frac{2}{3}\pi^3$$

$$\langle y, x_2\rangle = \int_{-\pi}^{\pi} \cos t \sin t\, dt = 0 \qquad f \text{ dispari}$$

$$\langle y, x_3\rangle = \int_{-\pi}^{\pi} \sin t\, dt = 0$$

QUINDI: $\operatorname{dist}(y, V) = \sqrt{\|y - \operatorname{Pr}_V(y)\|^2_{L^2}} = \sqrt{\pi - 6/\pi}$

$$\|y - \operatorname{Pr}_V(y)\|^2_{L^2} = \int_{-\pi}^{\pi} \left| \sin t - \frac{3}{\pi^2} t \right|^2 dt$$

$$= \int_{-\pi}^{\pi} \sin^2 t + \frac{9}{\pi^4} t^2 - \frac{6}{\pi^2} t \sin t \, dt$$

$$= \pi + \frac{9}{\pi^4} \frac{2}{3} \pi^3 - \frac{6}{\pi^2} 2\pi = \pi + \frac{6}{\pi} - \frac{12}{\pi} = \pi - \frac{6}{\pi}$$

ES3 SIA V IL SOTTOSPAZIO GENERATO DA $x_1 = e^{it}$, $x_2 = e^{-it}$, $x_3 = \cos t$

- TROVARE BASE ORTOGONALE DI V

OSSERVIAMO CHE $\cos t = \dfrac{e^{it} + e^{-it}}{2} \iff \dfrac{x_1 + x_2}{2} = x_3$

QUINDI SONO LINEARMENTE INDIPENDENTI

$\Rightarrow$ UNA BASE PER V È $\{x_1, x_2\}$

E SONO ORTOGONALI $\langle x_1, x_2 \rangle = \displaystyle\int_{-\pi}^{\pi} e^{it} \cdot \overline{e^{-it}} \, dt = \int_{-\pi}^{\pi} e^{2it} \, dt = 0$

SI POTREBBE ANCHE SCEGLIERE COME BASE $\{\cos t, \sin t\}$

• CALCOLARE LA PROIEZIONE DI $x(t) = e^t$ SU V

$$pr_V(e^t) = \frac{\langle e^t, x_1 \rangle}{\|x_1\|^2} x_1 + \frac{\langle e^t, x_2 \rangle}{\|x_2\|^2} x_2$$

$$\|x_1\|^2 = \|x_2\|^2 = \int_{-\pi}^{\pi} |e^{\pm it}|^2 dt = 2\pi$$

$$\langle e^t, x_1 \rangle = \int_{-\pi}^{\pi} e^t \cdot \overline{e^{+it}} dt = \int_{-\pi}^{\pi} e^{t(1-i)} dt =$$

$$= \frac{1}{1-i} \left[e^{\pi(1-i)} - e^{-\pi(1-i)} \right] =$$

$\sinh(a+b) = \sinh a \cosh b + \sinh b \cosh a$

$$= \frac{2}{1-i} \sinh(\pi(1-i)) = \frac{2i}{1-i} \left[\sinh \pi \cosh(-i\pi) + \sinh(-i\pi) \cosh \pi \right]$$

$$= \frac{2}{1-i} \left[\sinh \pi \cos(-\pi) + i \underset{0}{\sin(-\pi)} \cosh \pi \right]$$

$\cosh iz = \cos z$

$\sinh iz = i \sin z$

$$= \frac{-2}{1-i} \sinh \pi$$

come sopra

$$\langle e^t, x_2 \rangle = \int_{-\pi}^{\pi} e^t \overline{e^{-it}} dt = \frac{1}{1+i} \sinh(\pi(1+i)) = \frac{-2}{1+i} \sinh \pi$$

$$\Rightarrow pr_V(e^t) = \frac{-2}{2\pi} \sinh \pi \left[\frac{e^{it}}{1-i} + \frac{e^{-it}}{1+i} \right] = \frac{-1}{\pi} \frac{\sinh \pi}{2} \left[e^{it} + i e^{it} + e^{-it} - i e^{-it} \right]$$

$$= \frac{-1}{2\pi} \sinh \pi \left[2\cos t - 2 \sin t \right] = \frac{1}{\pi} \sinh \pi (\sin t - \cos t)$$

ES 6 CALCOLARE LA PROIEZIONE IN $L^2(-\frac{\pi}{2},\frac{\pi}{2})$ DI $x(t)=t$ SUL SOTTOSPAZIO GENERATO DA $x_1(t)=\sin t$ $\quad x_2(t)=e^{it}$

POICHÉ $\sin t = \frac{e^{it}-e^{-it}}{2i} \Rightarrow V$ È GENERATO DA $v_1 = e^{it}$ $\quad v_2 = e^{-it}$

CHE SONO ORTOGONALI.

$$pr_V(t) = \frac{\langle t, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle t, v_2\rangle}{\|v_2\|^2} v_2$$

$$\|v_1\|^2 = \|v_2\|^2 = \int_{-\pi/2}^{\pi/2} |e^{\pm it}|^2 = \pi$$

$$\langle t, v_1\rangle = \int_{-\pi/2}^{\pi/2} t\, e^{-it}\,dt = t\,\frac{e^{-it}}{-i}\Big|_{-\pi/2}^{\pi/2} + \frac{1}{i}\int_{-\pi/2}^{\pi/2} e^{-it}\,dt$$

$$= \underbrace{i\Big(\frac{\pi}{2}\underset{-i}{\underset{\shortparallel}{e^{-i\frac{\pi}{2}}}} + \frac{\pi}{2}\underset{i}{\underset{\shortparallel}{e^{i\frac{\pi}{2}}}}\Big)}_{0} + \frac{1}{i}\;\frac{e^{-it}}{-i}\Big|_{-\pi/2}^{\pi/2} = e^{-i\frac{\pi}{2}} - e^{i\frac{\pi}{2}} = -2i\sin\frac{\pi}{2} = -2i$$

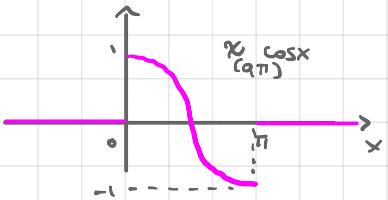
$$\langle t, v_2\rangle = \int_{-\pi/2}^{\pi/2} t\,e^{+it}\,dt \underset{s=-t}{=} \int_{-\pi/2}^{\pi/2} -s\,e^{-is}\,ds = +2i$$

$$\Rightarrow pr_V(t) = \frac{1}{\pi}\Big[-2i\,e^{it} + 2i\,e^{-it}\Big] = \frac{-2i}{\pi}\underset{2i\,\sin t}{\overset{}{\left(e^{it}-e^{-it}\right)}} = \frac{4}{\pi}\sin t$$

Alcuni esercizi sulle distribuzioni

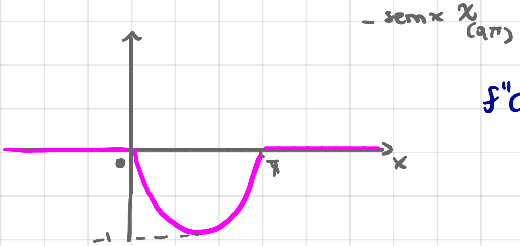
(Alcuni non svolti in classe ma richiesti da molti di voi)

Ess (A) $f(x) = \chi_{(0,\pi)}(x)\cos x$. Calcolare f' e f'' nel senso delle distrib.



$$f'(x) = \underbrace{\chi_{(0,\pi)}(x)\,(-\operatorname{sen} x)}_{\downarrow\ \text{derivata}} + \delta(x) + \delta(x+\pi))$$

$$f''(x) = -\cos x\,\chi_{(0,\pi)} + \delta'(x) + \delta'(x+\pi))$$

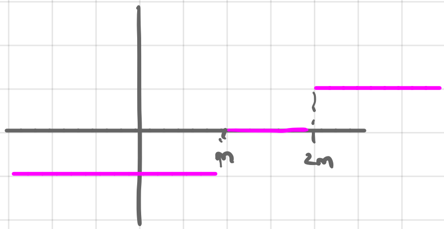


(B) Sia $f_m(x) = (H(x-m) - H(2m-x))\,|\cos mx|$

calcolare limite f_m nel senso distribuzioni

$$H(x-m) = \begin{cases} 1 & x > m \\ 0 & x < m \end{cases} \qquad H(2m-x) = \begin{cases} 0 & x > 2m \\ 1 & x < 2m \end{cases}$$

$$\Rightarrow \quad H(x-m) - H(2m-x) = \begin{cases} -1 & x < m \\ 0 & m < x < 2m \\ 1 & x > 2m \end{cases}$$



$$\Rightarrow \langle f_n, \varphi \rangle = \int_{-k}^{k} -|\cos nx|\, dx = \langle -|\cos nx|, \varphi \rangle$$

$\uparrow$ $\varphi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\varphi \subseteq [-k,k]$

$\uparrow$ $n > k$

$$\langle -|\cos nx|, \varphi \rangle \xrightarrow{n \to +\infty} \frac{2}{\pi}$$

infatti

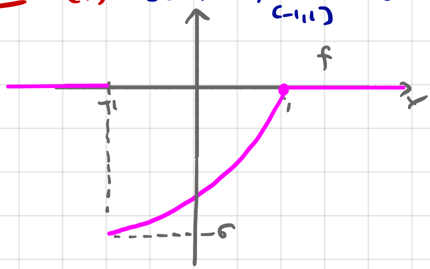
$-|\cos nx| = h(nx)$

$h(t) = -|\cos t|$

funzione periodica di periodo π

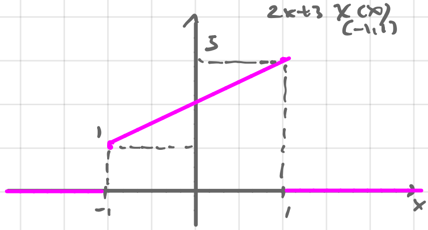
$$\Rightarrow h(n\cdot) \xrightarrow{\text{in } \mathcal{D}'} \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -|\cos t|\, dt = -\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\, dt = -\frac{2}{\pi}$$

ES6 **(A)** $f(x) = \chi_{(-1,1)}(x)\, \overbrace{(x-1)(x+4)}^{x^2+3x-4}$ CALCOLARE f' e f''



$$f'(x) = \underbrace{(2x+3)\chi_{(-1,1)}(x)}_{} - 6\,\delta(x+1)$$

DERIVATA



$$f''(x) = 2\chi_{(-1,1)}(x) + \delta(x+1) - 5\,\delta(x-1) - 6\,\delta'(x+1)$$

(B) $f_m(x) = m\left(\delta_{\frac{1}{m}} - \delta_{-\frac{1}{m}}\right) + \delta_m$. CALCOLARE $\lim_{m\to+\infty}$ IN $\mathcal{D}'$

$\varphi \in C_0^\infty(\mathbb{R})$ $\quad \operatorname{supp}\varphi \subseteq [-k,k]$

$$\langle f_m, \varphi\rangle = m\underbrace{\left(\varphi(\tfrac{1}{m}) - \varphi(-\tfrac{1}{m})\right)}_{\substack{\shortparallel \\ \cancel{\varphi(0)} + \varphi'(0)\frac{1}{m} - \cancel{\varphi(0)} \\ -\varphi'(0)(-\frac{1}{m}) + o(\frac{1}{m})}} + \underbrace{\varphi(m)}_{\substack{\downarrow m\to+\infty \\ 0}}$$

IN QUANTO $\operatorname{supp}\varphi \subseteq [-k,k]$

$$= 2\varphi'(0) + o(1) \xrightarrow{m\to+\infty} 2\varphi'(0) = \langle -2\delta', \varphi\rangle$$

$$\Rightarrow \quad f_m \to -2\delta'$$

ES 7 CALCOLARE IL LIMITE NEL SENSO DELLE DISTRIB. DI $f_m(x) = \dfrac{\cos mx}{2+\cos mx}$

SIA $h(t) = \dfrac{\cos t}{2+\cos t}$ $\quad h$ è periodica di periodo 2π

e $f_m(x) = h(mx) \quad \Rightarrow \quad f_m \to \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} h(t)\,dt = 1 - \frac{2}{\sqrt{3}}$

$$\int_0^{2\pi} \underbrace{\frac{\cos t}{2+\cos t}}_{\uparrow\ \frac{\cos t + 2 - 2}{\cos t + 2}}\,dt = 2\pi - 2\underbrace{\int_0^{2\pi} \frac{1}{2+\cos t}\,dt}_{\substack{\shortparallel \\ \frac{2\pi}{\sqrt{3}}}} = 2\pi\left(1 - \frac{2}{\sqrt{3}}\right)$$

$\leftarrow$ AD ES. COL METODO DEI RESIDUI

ES 8 CALCOLARE IL LIMITE NEL SENSO DELLE DISTRIB. DI $f_m(x) = \frac{\sin(mx)}{x}$

SIA $\varphi \in C_0^\infty(\mathbb{R})$ $\quad \mathrm{supp}\,\varphi \subseteq [-k,k]$

$$\langle f_m, \varphi \rangle = \int_{-k}^{k} \frac{\sin mx}{x} \underset{\varphi(0)+\varphi'(0)x+o(x)}{\underset{\shortparallel}{\varphi(x)}}\, dx =$$

$$= \underbrace{\int_{-k}^{k} \frac{\sin mx}{x}\varphi(0)}_{\shortparallel \;\leftarrow y = mx} + \underbrace{\int_{-k}^{k} (\varphi'(0) + o(1)) \sin mx \, dx}_{\downarrow\, m\to+\infty \quad 0}$$

$$\varphi(0)\int_{-mk}^{mk} \frac{\sin y}{y/\cancel{m}} \cdot \frac{dy}{\cancel{m}}$$

$$\xrightarrow{m\to+\infty} \left(\lim_{R\to+\infty} \int_{-R}^{R} \frac{\sin t}{t}\, dt\right)\varphi(0) = \pi\,\varphi(0)$$

$$\Rightarrow f_m \to \pi\,\delta$$

ES 9 CALCOLARE IL LIMITE NEL SENSO DISTRIBUZIONI DI $f_n(x) = \frac{1}{n} x^{\frac{1-n}{n}} \chi_{[0,1]}$

SIA $\varphi \in C_0^\infty(\mathbb{R})$ $\text{supp}\,\varphi \subseteq [-k,k]$ (POSSIAMO ASSUMERE $k>1$)

$$\langle f_n, \varphi \rangle = \frac{1}{n}\int_0^1 x^{\frac{1-n}{n}} \varphi(x)\,dx = \frac{1}{n}\int_0^1 \underset{\underset{x^{-1+1/n}}{\shortparallel}}{x^{\frac{1-n}{n}}} \left(\varphi(0) + \varphi'(0)x + o(x)\right) dx$$

$$= \underbrace{\frac{\varphi(0)}{n}\int_0^1 x^{\frac{1-n}{n}}\,dx}_{\shortparallel} + \boxed{\frac{1}{n}\int_0^1 x^{1/n}\left(\varphi'(0) + o(1)\right) dx}$$

$$\frac{\varphi(0)}{n} \left. \frac{x^{\cancel{-1}+\frac{1}{n}\cancel{+1}}}{\cancel{-1}+\frac{1}{n}\cancel{+1}} \right|_0^1 \qquad\qquad \downarrow \; n \to +\infty \quad \text{infatti}$$

$$0$$

$$\frac{1}{n}\left| \int_0^1 x^{1/n}\left(\varphi'(0) + o(1)\right)\right|$$

$$= \frac{\varphi(0)}{n} \frac{1}{1/n} = \varphi(0) \qquad\qquad \le \frac{1}{n}\|\varphi'(0) + o(1)\|_\infty \xrightarrow{n\to+\infty} 0$$

$$\Rightarrow \quad \langle f_n, \varphi \rangle \longrightarrow \varphi(0) = \langle \delta, \varphi \rangle$$

$$\Rightarrow \quad f_n \to \delta \text{ in } \mathcal{D}'$$