ESERCITAZIONE 15/5/2024

ES 1 CALCOLARE IL LIMITE NEL SENSO DELLE DISTRIBUZIONI DI:

$$f_m(x) = m^2 \chi_{(0,\frac{1}{m})}(x) - m^2 \chi_{(-\frac{1}{m},0)}$$

SIA $\varphi \in \mathcal{D}$: $\langle f_m, \varphi \rangle = m^2\left(\int_0^{\frac{1}{m}} \varphi(x)\,dx - \int_{-\frac{1}{m}}^{0} \varphi(x)\,dx\right)$

$$= m^2 \int_0^{\frac{1}{m}} \left(\varphi(0) + \varphi'(0)x + o(x)\right)dx -$$

$\uparrow$

$\varphi(x) = \varphi(0) + \varphi'(0)x + o(x)$

$$- m^2 \int_{-\frac{1}{m}}^{0} \left(\varphi(0) + \varphi'(0)x + o(x)\right)dx =$$

$$= m^2 \left[\cancel{\varphi(0)\frac{1}{m}} + \frac{1}{2m^2}\varphi'(0) + o\left(\frac{1}{m^2}\right) - \cancel{\varphi(0)\frac{1}{m}} + \frac{1}{2m^2}\varphi'(0) \right]$$

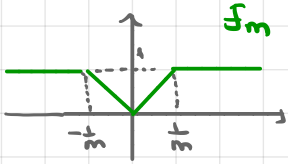
$$= \varphi'(0) + o(1) \xrightarrow{m \to +\infty} \varphi'(0) = -\langle \delta', \varphi \rangle$$

$\Rightarrow \quad f_m \to -\delta'$ in $\mathcal{D}'$

ES 2 SIA $f_m(x) = \min\{m|x|, 1\}$

(A) CALCOLARE f'_m e f''_m NEL SENSO DISTRIBUZIONI

f_m SONO CONTINUE E DERIVABILI A TRATTI



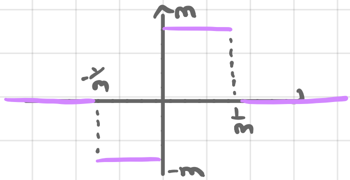
$\Rightarrow f'_m$ COINCIDE QUASI OVUNQUE CON DERIVATA PUNTUALE

$$f'_m = \begin{cases} 0 & \text{SE } |x| > 1/m \\ m & \text{SE } x \in (0, 1/m) \\ -m & \text{SE } x \in (-\frac{1}{m}, 0) \end{cases} = m\chi_{(0,\frac{1}{m})} - m\chi_{(-\frac{1}{m},0)}$$

f'_m È COSTANTE A TRATTI

$\downarrow$

$$f''_m = -m\,\delta_{-\frac{1}{m}} + 2m\,\delta - m\,\delta_{\frac{1}{m}}$$



(B) CALCOLARE I LIMITI NEL SENSO DELLE DISTRIBUZIONI DI f_m, f'_m e f''_m

- SIA $\varphi \in \mathcal{D}$: $\operatorname{supp}\varphi \subseteq [-R, R]$ (supponiamo $R > 1$)

$$\langle f_m, \varphi \rangle = \int_{-R}^{-1/m} \varphi(x)\,dx + \int_{\frac{1}{m}}^{R} \varphi(x)\,dx + \int_{-\frac{1}{m}}^{\frac{1}{m}} m|x|\varphi(x)\,dx = (*)$$

$$\lim_{m\to\infty} \int_{-R}^{-\frac{1}{m}} \varphi(x)\,dx = \int_{-R}^{0} \varphi(x)\,dx = \int_{-\infty}^{0} \varphi(x)\,dx$$

$$\lim_{m\to\infty} \int_{\frac{1}{m}}^{R} \varphi(x)\,dx = \int_{0}^{R} \varphi(x)\,dx = \int_{0}^{\infty} \varphi(x)\,dx$$

$$\lim_{m\to\infty} \int_{-\frac{1}{m}}^{\frac{1}{m}} m|x|\varphi(x)\,dx = \lim_{m\to\infty} \left[\int_0^{\frac{1}{m}} mx\varphi(x)\,dx - \int_{-\frac{1}{m}}^{0} mx\varphi(x)\,dx \right]$$

$$= \frac{1}{m}\int_0^1 y\,\varphi\left(\frac{y}{m}\right)dy - \frac{1}{m}\int_{-1}^{0} y\,\varphi\left(\frac{y}{m}\right)dy \xrightarrow{m\to\infty} 0$$

CAMBIO VARIABILI' $y = mx$

$\downarrow m\to\infty$ 0 $\qquad$ $\downarrow m\to\infty$ 0

IN MANIERA SIMILE

$\Big($ se $M = \max_{\mathbb{R}} |\varphi|$

$$\left| \frac{1}{m}\int_0^1 y\,\varphi\left(\frac{y}{m}\right)dy \right| \le \frac{M}{m}\int_0^1 y\,dy = \frac{M}{2m} \to 0 \Big)$$

QUINDI : $\lim_{m\to\infty} \langle f_m, \varphi\rangle \underset{(*)}{=} \int_{-\infty}^{\infty} \varphi(x)\,dx = \langle 1, \varphi(x)\rangle$

$\Leftrightarrow \quad f_m \to 1 \quad$ IN $\mathcal{D}'$

$\boxed{\text{LIMITE } f_m'}$

I MODO:

- $\varphi \in \mathcal{D}$: $\quad \langle f_m', \varphi\rangle = m\int_0^{\frac{1}{m}} \varphi(x)\,dx - m\int_{-\frac{1}{m}}^{0} \varphi(x)\,dx$

$$= m\left[\cancel{\varphi(0)\frac{1}{m}} + o\left(\frac{1}{m}\right) - \cancel{\varphi(0)\frac{1}{m}} + o\left(\frac{1}{m}\right) \right] \xrightarrow{m\to\infty} 0$$

TAYLOR
$\varphi(x) = \varphi(0) + o(1)$

$\Rightarrow \quad f_m' \to 0 \quad$ IN $\mathcal{D}'$

II MODO:

$\varphi \in \mathcal{D}$: $\quad \langle f_m', \varphi \rangle = - \langle f_m, \varphi' \rangle$
$\cap$
$\mathcal{D}$

$f_m \to 1$ IN $\mathcal{D}'$

$$\Rightarrow \lim_{m\to\infty} \langle f_m', \varphi \rangle = - \lim_{m\to\infty} \langle f_m, \varphi' \rangle =$$

$$= - \langle 1, \varphi' \rangle = - \int_{-R}^{R} \varphi'(x)\,dx = \underset{0}{\underset{\|}{\varphi(-R)}} - \underset{0}{\underset{\|}{\varphi(R)}}$$

supp $\varphi \subseteq [-R,R]$

$= 0$

LIMITE f_m'': I MODO

- SIA $\varphi \in \mathcal{D}$: $\quad \langle f_m'', \varphi \rangle = -m\,\varphi\left(-\frac{1}{m}\right) + 2m\,\varphi(0) - m\,\varphi\left(\frac{1}{m}\right)$

$$= -m\left(\cancel{\varphi(0)} - \cancel{\varphi'(0)\tfrac{1}{m}} + o\left(\tfrac{1}{m}\right)\right) + \cancel{2m\,\varphi(0)}$$

TAYLOR

$\varphi(x) = \varphi(0) + \varphi'(0)x + o(x)$

$$-m\left(\cancel{\varphi(0)} + \cancel{\varphi'(0)\tfrac{1}{m}} + o\left(\tfrac{1}{m}\right)\right)$$

$$= m\, o\left(\tfrac{1}{m}\right) \longrightarrow 0 \quad m\to\infty$$

QUINDI $f_m'' \to 0$ IN $\mathcal{D}'$

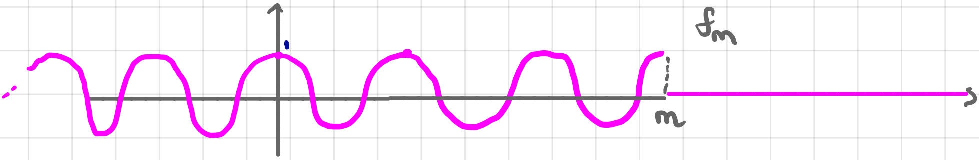
- II MODO $\quad \langle f_m'', \varphi \rangle = - \langle f_m', \varphi' \rangle \xrightarrow{m\to\infty} 0$

$f_m' \to 0$

$\Rightarrow f_m'' \to 0$ IN $\mathcal{D}'$

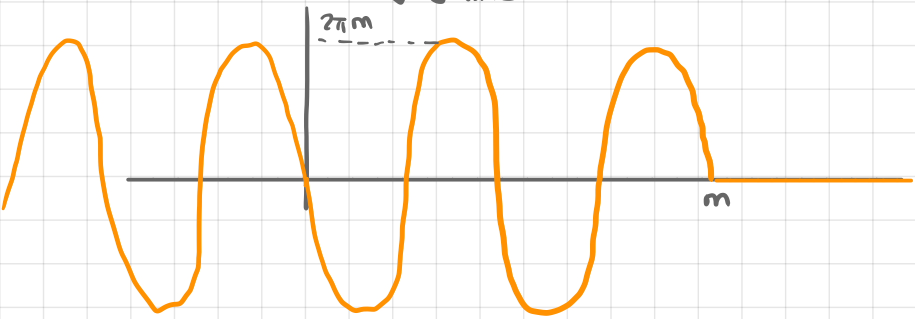
ES 3 Sia $f_m(x) = \chi_{(-\infty,m)}(x)\cos(2\pi m x)$

A) Calcolare f_m' e f_m'' nel senso delle distribuzioni



f_m è derivabile tranne in $x = m$ (ha un salto)

$$f_m' = \underbrace{\chi_{(-\infty,m)}(-2\pi m \operatorname{sen} mx)}_{\text{GRAFO}} - \delta_m$$



$$f_m'' = -4\pi^2 m^2 \cos(2\pi m x)\,\chi_{(-\infty,m)} - \delta_m'$$

B) CALCOLARE $\lim_{n\to\infty} f_n$ nel senso distribuzioni:

$\varphi \in \mathcal{D}$ $\quad$ supp $\varphi \subseteq [-k, k]$

$$\langle f_n, \varphi \rangle \underset{\forall n > k}{=} \int_{-k}^{k} \cos(2\pi n x)\, \varphi(x)\, dx =$$

$$= \langle \cos 2\pi(nx), \varphi \rangle \xrightarrow{n \to +\infty} 0$$

$\Rightarrow f_n \to 0$ in $\mathcal{D}'$

C) CALCOLARE $\lim_{n\to\infty} f_n'$ NEL SENSO DELLE DISTRIBUZIONI

SIA $\varphi \in \mathcal{D}$

$$\langle f_n', \varphi \rangle = -\langle f_n, \varphi' \rangle \xrightarrow[f_n \to 0 \text{ in } \mathcal{D}']{n \to +\infty} 0$$

$\Rightarrow f_n' \to 0$ in $\mathcal{D}'$

ES3 CALCOLARE LA DERIVATA PRIMA E SECONDA, NEL SENSO DELLE DISTRIBUZIONI DI $f(x) = \operatorname{ARCTG}\left(\frac{2}{x}\right)$

$f(x)$ È CONTINUA A TRATTI: UNICO PUNTO DI DISCONTINUITÀ IN $x=0$

$$\lim_{x\to 0^+} f(x) = \frac{\pi}{2} \qquad \lim_{x\to 0^-} f(x) = -\frac{\pi}{2} \longleftarrow$$

HA UN SALTO IN $x=0$

$f(0^+) - f(0^-) = \pi$

$$f'(x) = \frac{1}{\left(\frac{2}{x}\right)^2+1}\left(-\frac{2}{x^2}\right) + \pi\delta =$$

$$= \frac{-2}{x^2+4} + \pi\delta$$

$$f''(x) = \frac{4x}{(x^2+4)^2} + \pi\delta'$$

ES 6 SIA $\psi \in C^\infty(\mathbb{R})$ E SIA T UNA DISTRIBUZIONE

DEFINIAMO LA DISTRIBUZIONE $(\psi \cdot T)$ NEL SEGUENTE MODO:

$$\forall \varphi \in \mathcal{D}: \quad \langle (\psi \cdot T), \varphi \rangle = \langle T, \underbrace{\psi \cdot \varphi}_{\in \mathcal{D}} \rangle$$

Ⓐ DIMOSTRARE CHE $(\psi \cdot T)' = \psi' \cdot T + \psi \cdot T'$ $\quad (\psi' \in C^\infty)$

INFATTI: SE $\varphi \in \mathcal{D}$:

- $\langle (\psi \cdot T)', \varphi \rangle \underset{\text{DEF. DERIVATA}}{=} - \langle \psi \cdot T, \varphi' \rangle \overset{\text{DEF DI } \psi \cdot T}{=} - \langle T, \psi \varphi' \rangle$

- $\langle (\psi' T + \psi \cdot T'), \varphi \rangle \underset{\text{LINEARITÀ}}{=} \langle \psi' T, \varphi \rangle + \langle \psi \cdot T', \varphi \rangle =$

$$= \langle T, \psi' \cdot \varphi \rangle + \langle T', \psi \varphi \rangle =$$

$$= \langle T, \psi' \cdot \varphi \rangle - \langle T, \underbrace{(\psi \varphi)'}_{= \psi' \varphi + \psi \varphi'} \rangle$$

$$= \langle T, \cancel{\psi'} \cdot \varphi \rangle - \langle T, \cancel{\psi'} \cdot \varphi \rangle - \langle T, \psi \varphi' \rangle$$

$\Longrightarrow \langle (\psi \cdot T)', \varphi \rangle = \langle (\psi' T + \psi \cdot T'), \varphi \rangle \quad \forall \varphi \in \mathcal{D}$

$\Longrightarrow (\psi \cdot T)' = \psi' T + \psi T'$ $\quad \square$

(B) CALCOLARE $x\delta$, $(x\delta)'$ E $x\delta'$

$$\langle x\delta, \varphi\rangle \underset{\forall \varphi \in \mathcal{D}}{=} \langle \delta, x\varphi\rangle = (x\varphi(x))\Big|_{x=0} = 0 \implies \boxed{x\delta = 0}$$

$$(x\delta)' \underset{(A)}{=} \delta + x\delta'$$

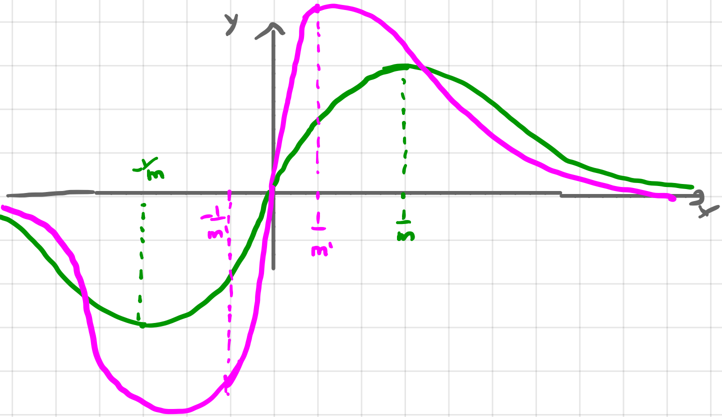
D'ALTRO CANTO $(x\delta) = 0 \implies \boxed{(x\delta)' = 0}$

$$\implies \boxed{x\delta' = -\delta}$$

ES 5 CALCOLARE IL LIMITE NEL SENSO DELLE DISTRIBUZIONI DI:

$$f_m(x) = \frac{m^2 x}{1+m^2x^2}$$

← FUNZIONE DISPARI CON MAX IN $x = \frac{1}{m}$ DOVE VALE $f\left(\frac{1}{m}\right) = \frac{m}{2}$



$m' > m$

DIMOSTRIAMO CHE $f_m \to pv\left(\frac{1}{x}\right)$ NEL SENSO DELLE DISTRIBUZIONI

SIA $\varphi \in \mathcal{D}$, $\operatorname{supp}\varphi \subseteq [-k,k]$

$$\langle f_m, \varphi \rangle = \int_{-k}^{k} \frac{m^2 x}{1+m^2 x}\varphi(x)\,dx =$$

$\int_{-k}^{k} \frac{m^2 x}{1+m^2x^2} = 0$ IN QUANTO DISPARI

$$= \int_{-k}^{k} \frac{m^2 x}{1+m^2x^2}\left(\varphi(x)-\varphi(0)\right)dx$$

$$= \int_{-k}^{k} \frac{m^2x^2}{1+m^2x^2}\left(\frac{\varphi(x)-\varphi(0)}{x}\right)dx \longrightarrow$$

OSSERVIAMO CHE

$$\left|\underbrace{\frac{m^2x^2}{1+m^2x^2}}_{\leq 1} \frac{\varphi(x)-\varphi(0)}{x}\right| \leq \left|\frac{\varphi(x)-\varphi(0)}{x}\right|$$

← LIMITATA IN $x=0$ QUINDI INTEGRABILE IN $[-k,k]$

APPLICANDO IL TEOREMA DELLA CONVERGENZA DOMINATA:

$$\lim_{m\to\infty} \langle f_m, \varphi \rangle = \lim_{m\to\infty} \int_{-K}^{K} \frac{m^2x^2}{1+m^2x^2}\left(\frac{\varphi(x)-\varphi(0)}{x}\right) dx$$

$$\overset{\text{CONV. DOMINATA}}{=} \int_{-K}^{K} \lim_{m\to\infty}\left[\overbrace{\frac{m^2x^2}{1+m^2x^2}}^{\to 1}\left(\frac{\varphi(x)-\varphi(0)}{x}\right)\right] dx =$$

$$= \int_{-K}^{K} \frac{\varphi(x)-\varphi(0)}{x}\, dx = \left\langle pv\left(\frac{1}{x}\right), \varphi \right\rangle$$

$\square$