ESERCITAZIONE 24/04/2024

ANTI-TRASFORMATA DI LAPLACE

INDIVIDUARE QUALE FUNZIONE $f(t)$ HA UNA CERTA TRASFORMATA DI LAPLACE $F(s)$

$$f(t) \xrightarrow{\mathcal{L}} F(s) = \mathcal{L}[f](s)$$
$$\xleftarrow{\mathcal{L}^{-1}}$$

SE $F(s) = \dfrac{P(s)}{Q(s)}$ $\quad P, Q$ POLINOMI CON GRADO $Q >$ GRADO P

$$\Rightarrow f(t) = \sum_{i=1}^{m} \operatorname{Res}(F(s)e^{st}, s_k) \qquad \forall t > 0$$

DOVE $s_1 \dots s_m$ SONO LE SINGOLARITÀ DI F

ESEMPIO: $\quad \mathcal{L}^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$ CON $a > 0$

$F(s) = \dfrac{s}{(s^2+a^2)^2}$ SINGOLARITÀ $\pm ia$ POLI DI ORDINE 2

$$\operatorname{Res}(F(s)e^{st}, +ia) = \frac{d}{ds}\left(\frac{se^{st}}{(s+ia)^2}\right)\Bigg|_{s=ia} =$$

$$= \frac{se^{st}}{(s+ia)^2}\left(\frac{1}{s} + \frac{te^{st}}{e^{st}} - \frac{2(s+ia)}{(s+ia)^2}\right)\Bigg|_{s=ia} = -i e^{iat}\frac{t}{4a}$$

IN MANIERA SIMILE $\operatorname{Res}(F(s)e^{st}, -ia) = \dfrac{i e^{-iat} t}{4a}$

$$\Rightarrow f(t) = -\frac{i e^{iat} t}{4a} + \frac{i e^{-iat} t}{4a} = \frac{t}{2a}\left(\frac{e^{iat} - e^{-iat}}{2i}\right) = \frac{t}{2a}\sin at$$

VERIFICA:

$$\mathcal{L}\left[\frac{t}{2a}\sin at\right](s) = -\frac{d}{ds}\left[\mathcal{L}\left[\frac{\sin at}{2a}\right](s)\right] =$$

$$= -\frac{d}{ds}\left(\frac{\cancel{a}}{2\cancel{a}(s^2+a^2)}\right) = -\frac{1}{2}\left[\frac{-2s}{(s^2+a^2)^2}\right] = \frac{s}{(s^2+a^2)^2} \quad \square$$

ESEMPIO 2 $\quad \mathcal{L}^{-1}\left[\dfrac{2}{s^3+3s^2+2s}\right] \qquad F(s) = \dfrac{2}{s^3+3s^2+2s}$

$$s^3+3s^2+2s = s(s^2+3s+2) = s(s+1)(s+2)$$

SINGOLARITÀ $0, -1, -2$ POLI DI ORDINE 1:

$$\operatorname{Res}(F(s)e^{st}, 0) = \left.\frac{2e^{st}}{(3s^2+6s+2)}\right|_{s=0} = 1$$

$$\operatorname{Res}(F(s)e^{st}, 1) = \left.\left(\frac{2e^{st}}{3s^2+6s+2}\right)\right|_{s=1} = \frac{2e^{-t}}{3-6+2} = -2e^{-t}$$

$$\operatorname{Res}(F(s)e^{st}, -2| = \text{"} \Big|_{s=-2} = \frac{2e^{-2t}}{12-12+2} = e^{-2t}$$

$$\Rightarrow f(t) = 1 - 2e^{-t} + e^{-2t} = (1-e^{-t})^2$$

ALTRO MODO:

$$\frac{2}{s^3+3s^2+2s} = \underbrace{\frac{\overset{=1}{A}}{s}}_{=\mathcal{L}[1]} + \underbrace{\frac{\overset{=-2}{B}}{s+1}}_{=-2\mathcal{L}[e^{-t}]} + \underbrace{\frac{\overset{=1}{C}}{s+2}}_{=\mathcal{L}[e^{-2t}]} = \mathcal{L}[1-2e^{-t}+e^{-2t}]$$

ESERCIZIO 3 RISOLVERE IL PROBLEMA DI CAUCHY

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = 3 \quad y'(0) = 13 \end{cases}$$

$$\mathcal{L}[y] = u \qquad \mathcal{L}[y'] = s\,\mathcal{L}[y] - y(0) = s\,u(s) - 3$$

$$\mathcal{L}[y''] = s^2\,\mathcal{L}[y] - s\,y(0) - y'(0) = s^2 u(s) - 3s - 13$$

$$\Rightarrow$$

$$(s^2-2s-3)\,u(s) - 3s - 13 + 6 = 0 \Rightarrow$$

$$\rightarrow u(s) = \frac{3s+7}{s^2-2s-3}$$

$$\mathcal{L}^{-1}\left[\frac{3s+7}{s^2-2s-3}\right](t)$$

$$s^2-2s-3=0 \qquad s = 1 \pm \sqrt{1+3} = \begin{cases} 3 \\ -1 \end{cases}$$

$$\Rightarrow \quad \mathrm{Res}\left(\frac{3s+7}{s^2-2s-3}e^{st}, 3\right) = \frac{16e^{3t}}{4} = 4e^{3t}$$

$$\rightsquigarrow \frac{d}{ds}(\ldots) = 2s-2$$

$$\mathrm{Res}\left(\frac{3s+7}{s^2-2s-3}e^{st}, -1\right) = \frac{4e^{-t}}{-4} = -e^{-t}$$

$$\Rightarrow \quad y(t) = \mathcal{L}^{-1}\left[\frac{3s+7}{s^2-2s-3}\right](t) = 4e^{3t} - e^{-t}$$

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$$\begin{cases} y'' + 2y' + y = 1 \\ y(0) = 1 \qquad y'(0) = 2 \end{cases} \qquad \text{e calcolare} \quad \int_0^\infty e^{-t} y(t)\, dt$$

$$u(s) = \mathcal{L}[y](s)$$

$$\to \quad s^2 y(s) + 2s y(s) + y(s) - s - 4 = 1/s$$

$$\Rightarrow \quad (s+1)^2 y(s) = s + 4 + 1/s \qquad \Rightarrow \quad y(s) = \frac{s+4}{(s+1)^2} + \frac{1}{s(s+1)^2}$$

$$u(s) = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{s}$$

$$\Rightarrow \quad A=2 \quad C=1 \quad B=0$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{2}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s}\right] =$$

$$= 2e^{-t}\mathcal{L}\left[\frac{1}{s^2}\right](t) + 1 = 2te^{-t}+1$$

$$\int_0^\infty e^{-t} y(t)\, dt = \mathcal{L}[y](1) = \frac{s^2+4s+1}{(s+1)^2 s} = \frac{3}{2}$$

ESERCIZIO RISOLVERE LA SEGUENTE EQUAZIONE INTEGRALE

$$y(t) = \sin t + \underbrace{\int_0^t \cos(t-z)\, y(z)\, dz}_{= \cos * y(t)}$$

SIA $u(s) = \mathcal{L}[y](s)$

$$u(s) = \underbrace{\mathcal{L}[\sin t](s)}_{= \frac{1}{1+s^2}} + \underbrace{\mathcal{L}[\cos t](s)}_{= \frac{s}{s^2+1}} \cdot u(s)$$

$$u(s)\left(1 - \frac{s}{1+s^2}\right) = \frac{1}{1+s^2}$$

$$\boxed{u(s) = \frac{1}{1-s+s^2}} = \frac{1}{\left(s-\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2 + \frac{3}{4}} \cdot \frac{2}{\sqrt{3}}$$

TRASFORMATA $\Rightarrow$ $\boxed{y(t) = \frac{2}{\sqrt{3}} e^{t/2} \sin \frac{\sqrt{3}}{2} t}$

OPPURE $F(s) = \dfrac{1}{1-s+s^2}$ $\qquad 1-s+s^2 = 0 \qquad s = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm 3i}{2}$

$$= \frac{1}{\left(s - \left(\frac{1+\sqrt{3}i}{2}\right)\right)\left(s - \frac{1-\sqrt{3}i}{2}\right)}$$

$$\operatorname{Res}\left(F(s)e^{st}, \frac{1+\sqrt{3}i}{2}\right) = \frac{e^{\left(\frac{1+\sqrt{3}i}{2}\right)t}}{\frac{1+\sqrt{3}i}{2} - \frac{1-\sqrt{3}i}{2}} = \frac{e^{\left(\frac{1+\sqrt{3}i}{2}\right)t}}{\sqrt{3}i}$$

$$\operatorname{Res}\left(F(s)e^{st}, \frac{1-\sqrt{3}i}{2}\right) = \frac{e^{\left(\frac{1-\sqrt{3}i}{2}\right)t}}{\frac{1-\sqrt{3}i}{2} - \frac{1+\sqrt{3}i}{2}} = -\frac{e^{\left(\frac{1-\sqrt{3}i}{2}\right)t}}{\sqrt{3}i}$$

$$\Rightarrow \mathcal{L}^{-1}[u(s)] = \frac{2}{\sqrt{3}}\left(\frac{e^{\left(\frac{1+\sqrt{3}i}{2}\right)t} - e^{\left(\frac{1-\sqrt{3}i}{2}\right)t}}{2i}\right) =$$

$$= \frac{2e^{\frac{t}{2}}}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t$$

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$$\begin{cases} y'(x) + 5\int_0^x y(x-t)\cos 2t \, dt = 10 \\ y(0) = 0 \end{cases}$$

$$u(s) = \mathcal{L}[y]$$

$$\mathcal{L}[y'] = s\,u(s)$$

$$\rightarrow \quad s\,u(s) + 5\,u(s) \cdot \frac{s}{s^2+4} = \frac{10}{s}$$

$$\Rightarrow u(s)\,\frac{s^3+4s+5s}{s^2+4} = \frac{10}{s} \qquad u(s) = 10 \cdot \frac{s^2+4}{s^2(s^2+9)}$$

$$\mathcal{L}^{-1}\left[10\,\frac{s^2+4}{s^2(s^2+9)}\right] = 10\,\mathcal{L}^{-1}\left[\frac{s^2+4}{s^2(s^2+9)}\right]$$

poli in $s=0$ $\quad s=\pm 3i$

$\uparrow$ poli semplici

$\uparrow$

polo ordine 2

$$\mathrm{Res}\left(F(s)e^{st},0\right) = \frac{4}{9}t$$

$$e^{st}\cdot\frac{s^2+4}{s^2(s^2+9)} = \frac{1}{s^2}\left(1+st+O(s^2)\right)\left(4+O(s^2)\right)\frac{1}{9}\left(1+O(s^2)\right)$$

$$= \frac{1}{9s^2}\left(4+4st+O(s^2)\right) = \frac{4/9}{s^2}+\frac{4/9\,t}{s}+O(1)$$

$$\mathrm{Res}\left(F(s)e^{st},+3i\right) = \frac{(-9+4)e^{3it}}{-9(6i)} = \frac{5}{54i}e^{(3it)}$$

$$\mathrm{Res}\left(F(s)e^{st},-3i\right) = \frac{(-9+4)\,e^{-3i}}{-9(-6i)} = -\frac{5}{54i}\,e^{-3it}$$

$$\mathcal{L}^{-1}\left[\frac{s^2+4}{s^2(s^2+9)}\right] = \frac{5}{27}\sin 3t + \frac{4}{9}t$$

$$\Rightarrow \boxed{y(s) = \frac{50}{27}\sin 3t + \frac{40}{9}t}$$

ES

$$y'' - y = \int_0^x (t-x)e^{2t}\,dt \qquad y(0) = y'(0) = 0$$

$$u(s) = \mathcal{L}[y](s)$$

$$\mathcal{L}[y''] = s^2 u(s)$$

$$\Rightarrow \quad (s^2-1)\,u(s) = \underbrace{\mathcal{L}[t](s)}_{=\frac{1}{s^2}} \cdot \underbrace{\mathcal{L}[e^{2t}]}_{=\frac{1}{s-2}}$$

$$u(s) = \frac{1}{s^2(s^2-1)(s-2)}$$

POLI : 0 ORDINE 2

$\pm 1, 2$ ORDINE 1

$$\text{Res}(u(s)e^{st}, 0) = \frac{2t+1}{4}$$

$$\frac{1}{s^2}\left(1 + st + O(s^2)\right)\left(-1 + O(s^2)\right)\left(-\tfrac{1}{2}\left(1 + \tfrac{s}{2} + O(s^2)\right)\right)$$

$$= \frac{1}{2s^2}\left(1 + st + \frac{s}{2} + O(s^2)\right) = \frac{1}{2s^2} + \frac{\frac{2t+1}{4}}{s} + O(1)$$

$$\text{Res}(u(s)e^{st}, \pm 1) = \frac{\left(\frac{1}{s^2(s-2)}\right)}{2s}\Bigg|_{s=\pm 1} = \begin{cases} -\dfrac{e^t}{2} & s = 1 \\[2ex] \dfrac{e^{-t}}{3} & s = -1 \end{cases}$$

$$\text{Res}(u(s)e^{st}, 2) = \frac{e^{2t}}{4}$$

$$\Rightarrow \quad y(s) = \frac{2t+1}{4} - \frac{e^{t}}{2} + \frac{e^{-t}}{3} + \frac{e^{2t}}{4}$$

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