ESERCITAZIONE 18/4/2024

TRASFORMATA DI LAPLACE

$f : [0, +\infty) \to \mathbb{C}$

f SI DICE LAPLACE TRASFORMABILE SE $\exists\, s \in \mathbb{C}$: $e^{-st} f(t)$ È ASSOLUTAMENTE INTEGRABILE SU $[0, +\infty)$

• SIA f LAPLACE TRASFORMABILE:

$\rightarrow$ $\sigma(f) := \inf \{ Re(s) : e^{-st} f(t)$ È ASSOLUTAMENTE INTEGRAB. SU $[0,+\infty) \} \in \mathbb{R} \cup \{-\infty\}$

ASCISSA DI CONVERGENZA DI f

$\rightarrow$ $\boxed{\mathcal{L}[f](s) = \int_0^{+\infty} e^{-st} f(t)\, dt}$ $\qquad \underbrace{s \in \mathbb{C} : \ Re(s) > \sigma(f)}$

TRASFORMATA DI LAPLACE DI f

semipiano in cui è definita e olomorfa

• SE $\exists\, C > 0$ E $a \in \mathbb{R}$: $|f(t)| \leq C e^{at}$ $\quad \forall t \in [0, +\infty) \implies \sigma(f) \leq a$

PROPRIETÀ: SIANO $f, g : [0, +\infty) \to \mathbb{C}$ CON ASCISSE DI CONVERGENZA $\sigma(f)$ e $\sigma(g)$

• SE $\alpha, \beta \in \mathbb{C}$:

$$\mathcal{L}[\alpha f + \beta g](s) = \alpha \mathcal{L}[f](s) + \beta \mathcal{L}[g](s) \qquad Re(s) > \max\{\sigma(f), \sigma(g)\}$$

(LINEARITÀ)

• SE $a \in \mathbb{C}$: $\mathcal{L}[e^{at}](s) = \dfrac{1}{s-a}$ $\qquad Re(s) > Re(a)$

- SE $a \in \mathbb{R}$: $\mathcal{L}[\sin(at)](s) = \dfrac{a}{s^2+a^2}$ $\quad Re(s) > 0$

- SE $a \in \mathbb{R}$: $\mathcal{L}[\cos(at)](s) = \dfrac{s}{s^2+a^2}$ $\quad Re(s) > 0$

- SE $a \in \mathbb{R}$: $\mathcal{L}[\sinh(at)](s) = \dfrac{a}{s^2-a^2}$ $\quad Re(s) > |a|$

- SE $a \in \mathbb{R}$: $\mathcal{L}[\cosh(at)](s) = \dfrac{s}{s^2-a^2}$ $\quad Re(s) > |a|$

- $\mathcal{L}[t^n](s) = \dfrac{n!}{s^{n+1}}$ $\quad Re(s) > 0 \quad n \geq 0$

SIA $f : [0, +\infty) \longrightarrow \mathbb{C}$ CON ASCISSA $\sigma(f)$

- $\mathcal{L}[e^{at} f(t)](s) = \mathcal{L}[f](s-a)$ $\quad Re(s) > \sigma(f) + Re(a)$

(TRASLAZIONE)

- $\mathcal{L}[H(t-a) f(t-a)](s) = e^{-as} \mathcal{L}[f](s)$ $\quad Re(s) > \sigma(f)$

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

FUNZIONE DI HEAVISIDE / GRADINO UNITARIO

- SE $\alpha>0$: $\mathcal{L}[f(\alpha t)](s) = \frac{1}{\alpha}\mathcal{L}[f]\left(\frac{s}{\alpha}\right)$ $\quad Re(s) > \alpha\, \sigma(f)$

(CAMBIO DI SCALA)

- $\mathcal{L}[t^m f(t)](s) = (-1)^m \frac{d^m}{ds^m}\mathcal{L}[f](s)$ $\quad Re(s) > \sigma(f)$

- PROPRIETÀ DELLA DERIVATA m-SIMA

SE f È DERIVABILE m VOLTE IN $[0, +\infty)$ E $f^{(k)}$ HA ASCISSA DI CONVERGENZA $\sigma(f^{(k)})$ $\quad k = 0, 1, \dots, m-1$

$$\Rightarrow \mathcal{L}[f^{(m)}](s) = s^m F(s) - \sum_{k=0}^{m-1} s^{m-1-k} f^{(k)}(0+)$$

CON $Re(s) > \max\limits_{k=0\dots m-1} \{\sigma(f^{(k)})\}$

- PROPRIETÀ DEL VALORE INIZIALE E DEL VALORE FINALE

SE I LIMITI ESISTONO, ALLORA:

$$\lim_{t\to 0^+} f(t) = \lim_{s\to +\infty} s\,\mathcal{L}[f](s)$$

$$\lim_{t\to +\infty} f(t) = \lim_{s\to 0^+} s\,F(s)$$

- PROPRIETÀ DELL'INTEGRALE

$$\mathcal{L}\left[\int_0^t f(x)\,dx\right](s) = \frac{1}{s}\mathcal{L}[f](s) \qquad Re(s) > \max\{\sigma(f), 0\}$$

ESERCIZIO 1

CALCOLARE LE TRASFORMATE DI LAPLACE DELLE SEGUENTI FUNZIONI:

1) $$\mathcal{L}[t^m e^{at}](s) = (-1)^m \frac{d^m}{ds^m}\mathcal{L}[e^{at}](s) =$$

$$= \underbrace{(-1)^m \frac{d^m}{ds^m}\left[\frac{1}{s-a}\right]}_{= (-1)^m \frac{m!}{(s-a)^{m+1}}} = \frac{m!}{(s-a)^{m+1}}$$

II MODO:

$$\mathcal{L}[t^m e^{at}](s) = \mathcal{L}[t^m](s-a) = \frac{m!}{(s-a)^{m+1}}$$

$$\mathcal{L}[t^m](s) = \frac{m!}{s^m}$$

2) $a \in \mathbb{C}$ $\quad \omega \in \mathbb{R}$

- $\mathcal{L}[e^{at}\sin(\omega t)](s) = \mathcal{L}[\sin \omega t](s-a) =$

$$= \frac{\omega}{(s-a)^2+\omega^2}$$

$\mathcal{L}[\sin \omega t](s) = \frac{\omega}{\omega^2+s^2}$

- $\mathcal{L}[e^{at}\cos \omega t](s) = \mathcal{L}[\cos \omega t](s-a) =$

$$= \frac{(s-a)}{(s-a)^2+\omega^2}$$

$\mathcal{L}[\cos \omega t](s) = \frac{s}{s^2+\omega^2}$

3) $\omega \in \mathbb{R}$ $\quad \mathcal{L}[t \sin \omega t](s) = -\frac{d}{ds}\mathcal{L}[\sin \omega t](s) =$

$$= -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = + \frac{2\omega s}{(s^2+\omega^2)^2}$$

4) $\omega \in \mathbb{R}$ $\quad \mathcal{L}[t \cos \omega t](s) = -\frac{d}{ds}\mathcal{L}[\cos \omega t](s) =$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = -\frac{s^2+\omega^2 - s \cdot 2s}{(s^2+\omega^2)^2} =$$

$$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

5) $f(t) = \begin{cases} 2 & t \in [1,4] \\ 0 & \text{ALTRIMENTI} \end{cases}$

$$f(t) = 2\left[H(t-1) - H(t-4)\right] \longrightarrow H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[f](s) = 2\mathcal{L}[H(t-1)](s) - 2\mathcal{L}[H(t-4)](s)$$

$$= 2 \cdot e^{-s}\mathcal{L}[1](s) - 2e^{-4s}\mathcal{L}[1](s) =$$

$$\mathcal{L}[H(t-a)\,g(t-a)](s) = e^{-as}\mathcal{L}[g](s)$$

$$\mathcal{L}[1](s) = \frac{1}{s}$$

$1 = t^0$

$$= 2\frac{e^{-s}}{s} - 2\frac{e^{-4s}}{s} = \frac{2}{s}\left(e^{-s} - e^{-4s}\right)$$

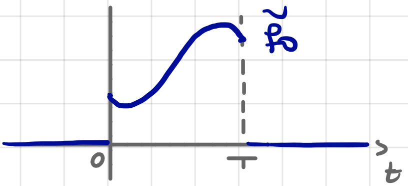
TRASFORMATA DI FUNZIONI PERIODICHE

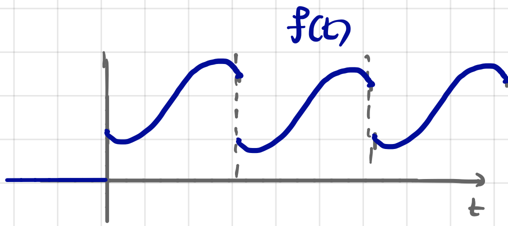
SIA $f_0(t)$ UNA FUNZIONE SU $[0,T]$ ED ESTENDIAMOLA A 0 ALTROVE:

$$\tilde{f}_0(t) = \begin{cases} f_0(t) & \text{IN } [0,T] \\ 0 & \text{IN } \mathbb{R} \setminus [0,T] \end{cases}$$

LA FUNZIONE PERIODICA GENERATA DA f_0:

$$f(t) = \sum_{m=0}^{\infty} \tilde{f}_0(t - mT)$$





$$\Rightarrow \mathcal{L}[f(t)](s) = \sum_{m=0}^{\infty} \mathcal{L}[\tilde{f}_0(t-mT)](s) =$$

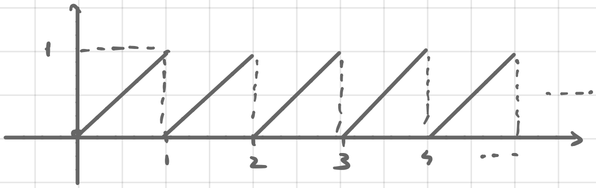
$$= \sum_{m=0}^{\infty} e^{-mTs} \mathcal{L}[\tilde{f}_0](s) =$$

$$= \mathcal{L}[\tilde{f}_0](s) \sum_{m=0}^{\infty} \left(e^{-Ts}\right)^m =$$

$$= \frac{\mathcal{L}[\tilde{f}_0](s)}{1 - e^{-Ts}}$$

ESERCIZIO 2

CALCOLARE LA TRASFORMATA DELLA FUNZIONE f:



$$\tilde{f}_0(t) = t\,(H(t) - H(t-1))$$

$$\mathcal{L}[f(t)] = \frac{\mathcal{L}[\tilde{f}_0(t)](s)}{1-e^{-s}} = \frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-s})}$$

FORMULA DELLA TRASF. DI FUNZIONI PERIODICHE CON $T=1$

$$\mathcal{L}[\tilde{f}_0(t)](s) = \mathcal{L}[t\,(H(t)-H(t-1))](s)$$

$$= -\frac{d}{ds}\left[\underbrace{\mathcal{L}[H(t)](s)}_{=1} - \mathcal{L}[H(t-1)](s)\right] =$$

$$= -\frac{d}{ds}\left[\underbrace{\frac{1}{s} - \frac{e^{-s}}{s}}_{\frac{1-e^{-s}}{s}}\right] = -\frac{e^{-s}s-(1-e^{-s})}{s^2}$$

$$= \frac{1-e^{-s}-se^{-s}}{s^2}$$

$$\mathcal{L}[H(t-a)\,g(t-a)](s) = e^{-as}\mathcal{L}[g](s)$$

DELTA DI DIRAC

$\delta(t)$ DISTRIBUZIONE

$$\int_{\mathbb{R}} f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

$\forall f: \mathbb{R} \to \mathbb{R}$ continua

$$\mathcal{L}[\delta(t-t_0)](s) = \int_0^{\infty} \delta(t-t_0)\,e^{-st}\,dt = e^{-st_0}$$

PRODOTTO DI CONVOLUZIONE

$$f, g: [0,+\infty) \to \mathbb{C} \implies f * g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$$\boxed{\mathcal{L}[f*g](s) = \mathcal{L}[f](s)\cdot\mathcal{L}[g](s)}$$

ESERCIZIO 3

$f = t * t$

$$f(t) = \int_0^t (t-\tau)\cdot\tau\,d\tau = \left[t\frac{\tau^2}{2} - \frac{\tau^3}{3}\right]_{\tau=0}^{\tau=t} = \frac{t^3}{6}$$

$$\mathcal{L}[f](s) = \frac{1}{6}\mathcal{L}[t^3](s) = \frac{-1}{6}\frac{d^3}{ds^3}\left(\frac{1}{s}\right) =$$

$$= -\frac{(-1)^3}{6}\frac{3!}{s^4} = \frac{1}{s^4}$$

$$\mathcal{L}[f](s) = \mathcal{L}[t*t](s)$$

$$\mathcal{L}[t*t](s) = \left(\mathcal{L}[t](s)\right)^2 = \left(-\frac{d}{ds}\frac{1}{s}\right) = \frac{1}{s^4}$$

ESERCIZIO 6

$f(t) = e^{at}$ $\quad g(t) = e^{bt}$ $\quad a,b \in \mathbb{R}$

$$f * g(t) = \int_0^t e^{a(t-\tau)} e^{b\tau} d\tau = e^{at} \int_0^t e^{(b-a)\tau} d\tau$$

$$= \begin{cases} \text{se } a=b: & t e^{at} \\ \text{se } a \neq b & e^{at} \left[\frac{e^{(b-a)\tau}}{b-a} \right]_{\tau=0}^{\tau=t} = \end{cases}$$

$$= \frac{e^{bt} - e^{at}}{b-a}$$

$$\Rightarrow \mathcal{L}[f*g](s) = \mathcal{L}[e^{at}](s) \cdot \mathcal{L}[e^{bt}](s)$$

$$= \frac{1}{(s-a)} \cdot \frac{1}{(s-b)}$$

ESERCIZIO 5 Risolvere il problema di Cauchy

$$(*) \begin{cases} x''(t) + 9x(t) = \cos 2t \\ x(0) = 1 \\ x'(0) = 0 \end{cases}$$

$$\mathcal{L}[f^{(m)}](s) = s^m \mathcal{L}[f](s) - \sum_{k=0}^{m-1} s^{m-1-k} f^{(k)}(0+)$$

SIA $X(s) = \mathcal{L}[x(t)](s)$

$$\mathcal{L}[x''(t)](s) = s^2 X(s) - s\,x(0+) - x'(0)$$

$$= s^2 X(s) - s$$

$$\mathcal{L}[\cos 2t](s) = \frac{s}{s^2+4}$$

$\Rightarrow$

$(*)$: $$s^2 X(s) - s + 9X(x) = \frac{s}{s^2+4}$$

$$(s^2+9)X(s) = \frac{s}{s^2+4} + s$$

$$X(s) = \underbrace{\frac{s}{(s^2+4)(s^2+9)}}_{\shortparallel} + \frac{s}{s^2+9} = \frac{1}{5}\underbrace{\frac{s}{s^2+4}}_{\mathcal{L}[\cos 2t]} + \frac{4}{5}\underbrace{\frac{s}{s^2+9}}_{\mathcal{L}[\cos 3t]}$$

$$\left(\frac{1}{5}\frac{s}{s^2+4} - \frac{1}{5}\frac{s}{s^2+9}\right)$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}[X](t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$$

ESERCIZIO 6

RISOLVERE IL PROBLEMA DI CAUCHY:

$$\begin{cases} x''(t) + x(t) = \delta(t-1) \\ x(0) = 0 \quad x'(0) = 0 \end{cases} \qquad (*)$$

SIA $X(s) = \mathcal{L}(x(t))(s)$

$$\mathcal{L}(x''(t)) = s^2 X(s)$$

$$\mathcal{L}(\delta(t-1)) = e^{-s}$$

$$\Rightarrow \quad (*) \iff \quad s^2 X(s) + X(s) = e^{-s}$$

$$\iff \quad X(s) = \frac{e^{-s}}{1+s^2} = \underbrace{e^{-s} \cdot \mathcal{L}[\sin t](s)}_{= \, \mathcal{L}[H(t-1)\sin(t-1)](s)}$$

$$\mathcal{L}[H(t-a)f(t-a)](s) = e^{-as}\mathcal{L}[f](s)$$

$$\Rightarrow \quad x(t) = H(t-1)\sin(t-1) \qquad \text{PER } t \geq 0$$

ES 7

CALCOLARE LA TRASF. DI LAPLACE DI

$$f(t) = \int_0^t (t-s)^{10} \cos^2 s \, ds = h * g(t)$$

DOVE $h(t) = t^{10}$ $\quad g(t) = \cos^2 t$

$$\mathcal{L}[f](s) = \mathcal{L}[h * g](s) = \underbrace{\mathcal{L}[h](s)}_{\frac{10!}{s^{11}}} \cdot \underbrace{\mathcal{L}[g(s)]}_{''}$$

$$(\cos t)^2 = \left(\frac{e^{it} + e^{-it}}{2}\right)^2 = \frac{e^{2it} + e^{-2it} + 2}{4} = 2 + \frac{e^{2it} + e^{-2it}}{4}$$

$$\mathcal{L}[\cos^2 t] = \mathcal{L}\left[2 + \frac{e^{2it} + e^{-2it}}{4}\right] = \frac{1}{2s} + \frac{1}{4}\left(\frac{1}{s-2i} + \frac{1}{s+2i}\right) =$$

$$= \frac{1}{2s} + \frac{1}{4}\,\frac{2s}{s^2+4} = \frac{s^2+2}{(s^2+4)s}$$

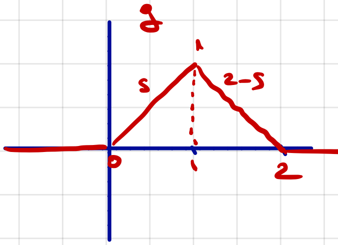
$$\Rightarrow \mathcal{L}[f](s) = \frac{10!}{s^{12}}\,\frac{s^2+2}{(s^2+4)}$$

ES 8

$$f(t) = \int_0^t \cos^2(t-\tau)\, g(\tau)\, d\tau \qquad g(\tau) = \max\{1-|\tau-1|, 0\}$$

$$= h * g(t) \qquad h(t) = \cos^2 t$$

$$\Rightarrow \mathcal{L}[f](s) = \underbrace{\mathcal{L}[h](s)}_{= \frac{s^2+2}{s(s^2+4)}} \cdot \mathcal{L}[g](s)$$



$$g(t) = t\,\chi_{[0,1]} + (2-t)\,\chi_{[1,2]} = t\,(1-H(t)) + (2-t)\,(H(t-1) - H(t-2))$$

$$= t - 2(t-1)H(t-1) + (t-2)H(t-2)$$

$$\mathcal{L}[g(t)](s) = \frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} = \frac{(1+e^{-s})^2}{s^2}$$

$$\Rightarrow \mathcal{L}[f](s) = \frac{s^2+2}{s^3(s^2+4)}\,(1+e^{-s})^2$$

ES 9 RISOLVERE IL SISTEMA

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -x'(t) \end{cases} \qquad x(0)=0 \quad y(0)=1 \qquad (*)$$

$$u(s) = \mathcal{L}[x](s)$$

$$v(s) = \mathcal{L}[y](s)$$

$$\rightsquigarrow \quad (*): \begin{cases} s\,u(s) = v(s) \\ s\,v(s) - 1 = -u(s) \end{cases}$$

$$\downarrow$$

$$\begin{cases} v(s) = s\,u(s) \\ s^2 u(s) - 1 = -u(s) \end{cases} \Rightarrow u(s) = \frac{1}{s^2+1}$$

$$\Rightarrow \begin{cases} u(s) = \dfrac{1}{1+s^2} \\ v(s) = \dfrac{s}{1+s^2} \end{cases}$$

$$\Rightarrow \begin{cases} x(t) = \sin t \\ y(t) = \cos t \end{cases}$$