ESERCITAZIONE 17 APRILE 2024

ES 1 $\displaystyle\int_0^{\pi} \frac{\sin^2\theta}{\alpha+\cos\theta}\,d\theta \qquad \alpha>1$

$$\int_0^{\pi} \frac{\sin^2\theta}{\alpha+\cos\theta}\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\sin^2\theta}{\alpha+\cos\theta}\,d\theta = (*)$$

PARITÀ DELLA FUNZIONE INTEGRANDA

$z = e^{i\theta}$: $\quad \sin^2\theta = \left(\frac{z-1/z}{2i}\right)^2 = -\frac{1}{4}\frac{(z^2-1)^2}{z^2}$

$$\alpha+\cos\theta = \alpha + \frac{z+\frac{1}{z}}{2} = \frac{z^2+2\alpha z+1}{2z}$$

$$(*) = \frac{1}{2}\int_{+C_1} \frac{-\frac{1}{4z^2}(z^2-1)^2}{\frac{z^2+2\alpha z+1}{2\cancel{z}}}\,\frac{dz}{i\cancel{z}} = -\frac{1}{4i}\int_{+C_1}\frac{(z^2-1)^2}{z^2(z^2+2\alpha z+1)}\,dz$$

CIRCONF. UNITARIA CENTRO $(0,0)$ ORIENTATA VERSO ANTIORARIO

$$f(z) = \frac{(z^2-1)^2}{z^2(z^2+2\alpha z+1)}$$

SINGOLARITÀ:
- $z=0$ ✓
- $z^2+2\alpha z+1=0$

$\iff z_\pm = -\alpha \pm \sqrt{\alpha^2-1}$

$|z_-| = \alpha+\sqrt{\alpha^2-1} > 1$ (NON LO CONSIDERO)

$|z_+| = \frac{1}{\alpha+\sqrt{\alpha^2-1}} < 1$ ✓

$z=0$ POLO DI ORDINE 2

$$\frac{(z^2-1)^2}{z^2(z^2+2\alpha z+1)} = \frac{1}{z^2}\left(1+O(z^2)\right)\left(1-2\alpha z+O(z^2)\right)$$

$(z^2-1)^2$

$$g(z)=\frac{1}{z^2+2\alpha z+1} = 1+g'(0)z+O(z^2)$$

$$g'(0)=-\frac{(2z+2\alpha)}{(z^2+2\alpha z+1)^2}\Big|_{z=0} = -2\alpha$$

$$= \frac{1}{z^2}\left(1-2\alpha z+O(z^2)\right)$$

$$= \frac{1}{z^2}-\frac{2\alpha}{z}+O(1)$$

$$\mathrm{Res}(f,0) = -2\alpha$$

$z=z_+$ POLO SEMPLICE (ORDINE 1)

$$f(z)=\frac{(z^2-1)^2}{z^2(z-z_+)(z-z_-)}$$

$$\Rightarrow \mathrm{Res}(f,z_+) = \left(z_+-\frac{1}{z_+}\right)^2\frac{1}{z_+-z_-} =$$

$$= 4(\alpha^2-1)\frac{1}{2\sqrt{\alpha^2-1}} = 2\sqrt{\alpha^2-1}$$

$$z_+ = -\alpha+\sqrt{\alpha^2-1}$$

$$= -\alpha+\sqrt{\alpha^2-1}\ \frac{(-\alpha-\sqrt{\alpha^2-1})}{-\alpha-\sqrt{\alpha^2-1}}$$

$$= \frac{-1}{\alpha+\sqrt{\alpha^2-1}}$$

$$\Rightarrow \frac{1}{z_+} = -(\alpha+\sqrt{\alpha^2-1})$$

$$\Downarrow$$

$$z_+-\frac{1}{z_+} = 2\sqrt{\alpha^2-1}$$

$$\Rightarrow \int_0^{\pi}\frac{\sin^2 x}{\alpha+\cos x}\,dx = -\frac{1}{4i}\,2\pi i\left[-2\alpha+2\sqrt{\alpha^2-1}\right]$$

$$= \pi\left[\alpha-\sqrt{\alpha^2-1}\right]$$

ES2

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx \qquad n, m \in \mathbb{N}$$

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx \underset{|z|=1}{\overset{+C_1}{=}} \int \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)\frac{1}{2}\left(z^m + \frac{1}{z^m}\right)\frac{dz}{iz}$$

$$= \frac{1}{4i}\int_{+C_1} \underbrace{\frac{z^{n+m} + z^{n-m} + z^{m-n} + z^{-n-m}}{z}}_{f} \, dz$$

$$f(z) = \frac{1}{z^{1-n-m}} + \frac{1}{z^{1-n+m}} + \frac{1}{z^{1-m+n}} + \frac{1}{z^{1+n+m}}$$

f HA SINGOLARITÀ IN $z=0$ È GIÀ IN SERIE DI LAURENT

$$\operatorname{Res}(f,0) = \begin{cases} 0 & \text{se } n \neq m \\ 2 & \text{se } n = m \end{cases}$$

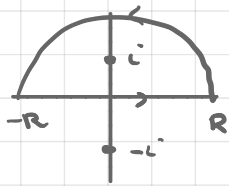
$$\Rightarrow \int_0^{2\pi} \cos(nx)(\cos mx)\,dx = \frac{1}{4i}\int_{C_1} f(z)\,dz = \frac{2\pi i}{4i}\cdot\begin{cases} 0 & n \neq m \\ 2 & n = m \end{cases}$$

$$= \begin{cases} \pi & \text{se } n = m \\ 0 & \text{se } n \neq m \end{cases}$$

ES 3 $\displaystyle\int_0^{+\infty} \frac{\cos mx}{1+x^2}\,dx \qquad m \in \mathbb{Z}$

$$\int_0^{\infty} \frac{\cos mx}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2}\,dx = \frac{1}{2}\,\mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{imx}}{1+x^2}\,dx\right) = (*)$$

$$f(z) = \frac{e^{imz}}{1+z^2}$$ SINGOLARITÀ IN $z = \pm i$



$$(*) = \frac{1}{2}\,\mathrm{Re}\left[2\pi i\,\mathrm{Res}(f, i)\right] =$$

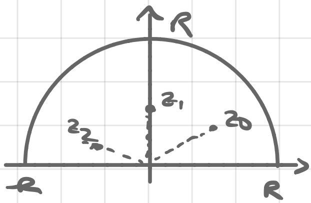
CONSIDERIAMO SOLO i IN QUANTO HA PARTE IMMAGINARIA >0

$$= \frac{1}{2}\,\mathrm{Re}\left[2\pi i\,\frac{e^{imi}}{2i}\right] = \frac{\pi e^{-m}}{2}$$

ES 4 $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$

$$f(z) = \frac{1}{z^6+1}$$ SINGOLARITÀ $z^6 = -1 = e^{i\pi}$

$$z_k = e^{i\left(\frac{\pi}{6} + k\frac{\pi}{3}\right)} \qquad k = 0, 1, 2, \dots 5$$



$$\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = 2\pi i \sum_{k=0}^{2} \text{Res}\left(\frac{1}{z^6+1}, z_k\right) = (*)$$

$$\text{Res}\left(\frac{1}{z^6+1}, z_k\right) = \frac{1}{6 z_k^5} = \frac{z_k}{6 z_k^6} = -\frac{z_k}{6}$$

SE $f(z) = \frac{g(z)}{h(z)}$

con g, h olomorfe e z_0 polo ordine 1.

$$\text{Res}(f, z_0) = \left[(z-z_0)\frac{g(z)}{h(z)}\right]_{z=z_0}$$

$$\underset{\substack{\uparrow \\ h(z_0)=0}}{=} \lim_{z\to z_0} \frac{g(z)}{\frac{h(z)-h(z_0)}{z-z_0}} = \frac{g(z_0)}{h'(z_0)}$$

$$(*) = \frac{2\pi i}{6} \sum_{k=0}^{2} (-z_k) =, \quad z_k = e^{i\left(\frac{\pi}{6} + \frac{k\pi}{3}\right)}$$

$$= \frac{-\pi i}{3} e^{i\frac{\pi}{6}} \sum_{k=0}^{2} \left(e^{i\frac{\pi}{3}}\right)^k$$

$$= \frac{-\pi i}{3} e^{i\frac{\pi}{6}} \, \frac{1-\overbrace{\left(e^{i\frac{\pi}{3}}\right)^3}^{=-1}}{1-e^{i\pi/3}} =$$

$$= \frac{-i\pi}{3} e^{i\frac{\pi}{6}} \, \frac{2}{1-e^{i\pi/3}} =$$

$$= \frac{\pi}{3} \, \frac{2i}{e^{i\pi/6}-e^{-i\pi/6}} = \frac{\pi/3}{\underbrace{\text{sen}\frac{\pi}{6}}_{=\frac{1}{2}}} = \frac{2}{3}\pi$$

ES 5

In maniera analoga $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^{2m}+1}\,dx \qquad m \geq 1$

$f(z) = \dfrac{1}{z^{2m}+1}$ SINGOLARITÀ IN $z_k = e^{i\left(\frac{\pi}{2m} + \frac{k\pi}{m}\right)}$ $k=0\ldots 2m-1$

QUELLE NEL SEMIPIANO $\operatorname{Im} > 0$

z_k $k=0\ldots m-1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{x^{2m}+1}\,dx = 2\pi i \sum_{k=0}^{m-1} \operatorname{Res}(f, z_k) =$$

$$= 2\pi i \sum_{k=0}^{m-1} \frac{1}{2m\, z_k^{2m-1}} = 2\pi i \sum_{k=0}^{m-1} -\frac{z_k}{2m}$$

$$= -\frac{2\pi i}{2m}\, e^{i\frac{\pi}{2m}} \sum_{k=0}^{m-1} \left(e^{i\frac{\pi}{m}}\right)^k$$

$$= -\frac{\pi i}{m}\, e^{i\pi/2m}\, \frac{1-e^{i\pi}}{1-e^{i\pi/m}} = \frac{\pi}{m}\, \frac{2i}{e^{i\pi/2m}-e^{-i\pi/2m}}$$

$$= \frac{\pi}{m \operatorname{sen}\frac{\pi}{m}}$$

ES6

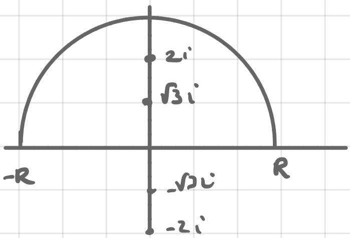
$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6}$$

$$f(z) = \frac{z^2}{z^4+5z^2+6} \qquad z^4+5z^2+6=0 \iff z^2 = \frac{-5\pm\sqrt{25-24}}{2} =$$

$$= \frac{-5\pm 1}{2} = \begin{cases} -2 \\ -3 \end{cases}$$

$$\iff z = \pm\sqrt{2}\,i,\ \pm\sqrt{3}\,i$$

$\uparrow$ Poli semplici



$$\operatorname{Res}(f, \sqrt{2}i) = \left.\frac{z^2}{4z^3+10z}\right|_{z=\sqrt{2}i} = \frac{-2}{-8\sqrt{2}i+10\sqrt{2}i} = \frac{-2}{2\sqrt{2}\,i} = \frac{i}{\sqrt{2}}$$

$$\operatorname{Res}(f, \sqrt{3}i) = \left.\frac{z^2}{4z^3+10z}\right|_{z=\sqrt{3}i} = \frac{-3}{-12\sqrt{3}i+10\sqrt{3}i} = \frac{-3}{-2\sqrt{3}\,i} = -\frac{\sqrt{3}}{2}\,i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6}\,dx = 2\pi i\left[\underbrace{\frac{i}{\sqrt{2}}}_{=\frac{\sqrt{2}}{2}i} + \frac{\sqrt{3}}{2}\,i\right] = \pi\left(\sqrt{3}-\sqrt{2}\right)$$

ES7. $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+\alpha^2)^3}\,dx \qquad \alpha \neq 0$

$$\int_0^{\infty} \frac{x^2}{(x^2+\alpha^2)^3}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+\alpha^2)^3}\,dx$$

$f(z) = \dfrac{z^2}{(z^2+\alpha^2)^3}$ POLI IN $z_{\pm} = \pm|\alpha| i$ POLI DI ORDINE 3

CONSIDERIAMO SOLO z_+ IN QUANTO NEL SEMIPIANO $\operatorname{Im} z > 0$

$$f(z) = \frac{z^2}{(z-z_-)^3(z-z_+)^3} \Rightarrow \operatorname{Res}(f, z_+) = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{z^2}{(z-z_-)^3}\right)\Bigg|_{z=z_+}$$

$$\frac{d}{dz}\left(\frac{z^2}{(z-z_-)^3}\right) = \frac{2z(z-z_-)^3 - z^2\,3(z-z_-)^2}{(z-z_-)^6} = \frac{2z(z-z_-) - 3z^2}{(z-z_-)^4} = \frac{-z^2 - 2zz_-}{(z-z_-)^4}$$

$$\frac{d^2}{dz^2}\left(\frac{z^2}{(z-z_-)^3}\right) = \frac{-(2z+2z_-)(z-z_-)^4 + (z^2+2zz_-)\,4(z-z_-)^3}{(z-z_-)^8}$$

$$= \frac{-2(z+z_-)(z-z_-) + 4(z^2+2zz_-)}{(z-z_-)^5}$$

CALCOLATO IN $z = z_+$

(OSSERVIAMO $z_+ + z_- = 0$)

$$\operatorname{Res}(f, z_+) = \frac{1}{2}\frac{4(-\alpha^2+2\alpha^2)}{(2|\alpha| i)^5} = \frac{1}{2}\frac{4\alpha^2}{32|\alpha|^5 i}$$

$$= \frac{-i}{16|\alpha|^3}$$

$$\Rightarrow \int_0^{\infty} \frac{x^2}{(x^2+\alpha^2)^3}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+\alpha^2)^3}\, dx =$$

$$= \frac{1}{2}\, 2\pi i \left(\frac{-i}{16\,|\alpha|^3} \right) = \frac{\pi}{16|\alpha|^3}$$

ES 8. $\displaystyle\int_{-\infty}^{\infty} \frac{x-2}{x^2-4x+5} \sin 2x \, dx$

$$f(z) = \frac{z-2}{z^2-4z+5} e^{i2z} \qquad z^2-4z+5=0 \iff z_\pm = 2 \pm \sqrt{4-5} = 2 \pm i$$

$$\text{Res}(f, 2+i) = \left(\frac{z-2}{z-z_-} e^{i2z} \right)\Big|_{z=z_+} = \frac{i}{2i} e^{i2(2+i)} = \frac{1}{2} e^{4i-2}$$

$$\int_{-\infty}^{\infty} \frac{x-2}{x^2-4x+5} e^{i2x} dx = 2\pi i \, \text{Res}(f, 2+i) = 2\pi i \frac{1}{2} e^{4i-2} = \pi i e^{4i-2}$$

$$= \pi i e^{-2}(\cos 4 + i \sin 4) = -\pi e^{-2} \sin 4 + i e^{-2} \pi \cos 4$$

$$\int_{-\infty}^{\infty} \frac{x-2}{x^2-4x+5} \sin 2x \, dx = \text{Im}\left(\int_{-\infty}^{\infty} \frac{x-2}{x^2-4x+5} e^{i2x} dx \right) = e^{-2} \pi \cos 4 .$$

ES 9.

$$\int_{-\infty}^{\infty} \frac{\sin^2 \pi x}{x^2+1}\, dx =$$

$$\cos 2\pi x = \cos^2 \pi x - \sin^2 \pi x = 1 - 2 \sin^2 \pi x$$

$$\Rightarrow \quad \sin^2 \pi x = \frac{1}{2} - \frac{1}{2} \cos 2\pi x$$

$$(*) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1 - \cos 2\pi x}{x^2+1}\, dx = \frac{1}{2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx}_{\pi} - \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos 2\pi x}{x^2+1}$$

$$= \frac{\pi}{2} - \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos 2\pi x}{x^2+1}\, dx$$

$$f(z) = \frac{e^{2\pi i x}}{z^2+1} = \frac{e^{2\pi i x}}{(z-i)(z+i)}$$

singolarità in $z = \pm i$ poli semplici

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i x}}{x^2+1}\, dx = 2\pi i \operatorname{Res}(f, i) = 2\pi i \frac{e^{-2\pi}}{2i} = \pi e^{-2\pi}$$

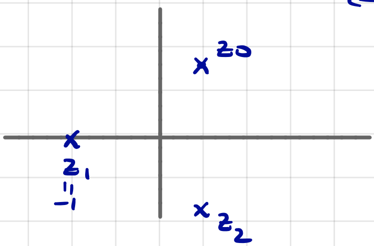
$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos 2\pi x}{x^2+1}\, dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{2\pi i x}}{x^2+1}\, dx \right) = \pi e^{-2\pi}$$

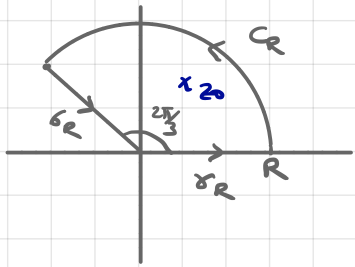
$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 \pi x}{x^2+1}\, dx = \frac{\pi}{2} - \frac{1}{2} \pi e^{-2\pi} = \frac{\pi}{2} \left(1 - e^{-2\pi}\right)$$

Es 10. $$\int_0^{\infty} \frac{1}{1+x^3}\,dx$$

$f(z) = \dfrac{1}{1+z^3}$ HA SINGOLARITÀ IN $z^3+1=0 \iff z^3=-1 = e^{i\pi}$

$$\iff z_k = e^{i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)} \qquad k=0,1,2$$





CONSIDERIAMO IL SEGUENTE CAMMINO Γ_R:

$$\int_{\Gamma_R} f(z)\,dz \overset{R>1}{=} 2\pi i \,\mathrm{Res}(f, z_0)$$

$$= 2\pi i \frac{1}{3 z_0^2} = 2\pi i \frac{z_0}{3 z_0^3} = -\frac{2\pi i}{3} e^{i\pi/3}$$

$\parallel$

$$\int_{\gamma_R} + \int_{C_R} + \int_{\sigma_R}$$

$=0$

FACCIAMO $\lim\limits_{R\to+\infty}$ E OSSERVIAMO CHE $\lim\limits_{R\to\infty} \displaystyle\int_{C_R} \frac{1}{1+z^3} = 0$

$$\int_{\gamma_R} \frac{1}{z^3+1}\,dz = \int_0^R \frac{1}{1+x^3}\,dx$$

$\gamma(x) = x \quad x \in [0, R]$

$$\int_{\sigma_R} \frac{1}{z^3+1}\,dz = -\int_0^R \frac{1}{(xe^{\frac{2\pi i}{3}})^3+1}\, e^{i\frac{2\pi}{3}}\,dx = -\int_0^R \frac{e^{\frac{2\pi i}{3}}}{1+x^3}\,dx$$

PARAMETRIZZATO NEL

VERSO OPPOSTO DI PERCORRENZA

$-\sigma_R(x) = x e^{i\frac{2\pi}{3}} \quad x \in [0, R]$

$$\lim_{R\to+\infty} \int_{\Gamma_R} \frac{1}{z^3+1}\,dz = \frac{2}{3}\pi i\, e^{-\frac{2\pi}{3}i}$$

$$\parallel$$

$$\lim_{R\to+\infty} \int_{\gamma_R} \frac{1}{z^3+1}\,dz + \lim_{R\to\infty} \int_{\sigma_R} \frac{1}{z^3+1}\,dz$$

$$= \int_0^\infty \frac{1}{1+x^3}\,dx - e^{\frac{2\pi i}{3}} \int_0^\infty \frac{1}{1+x^3}\,dx$$

$$= \left(1 - e^{\frac{2\pi i}{3}}\right) \int_0^\infty \frac{1}{1+x^3}\,dx$$

$$\left(1-e^{\frac{2\pi i}{3}}\right)\cdot\int_0^{\infty}\frac{1}{1+x^3}\,dx=-\frac{2}{3}\pi i\, e^{\frac{\pi}{3}i}$$

$$\int_0^{\infty}\frac{1}{1+x^3}\,dx=-\frac{2}{3}\pi i\;\frac{e^{\frac{\pi}{3}i}}{1-e^{\frac{2\pi}{3}i}}=-\frac{2}{3}\pi i\;\frac{1}{e^{-\frac{\pi}{3}i}-e^{i\frac{\pi}{3}}}$$

$$=\frac{\pi}{3}\,\frac{1}{\underbrace{\boxed{\operatorname{sen}\frac{\pi}{3}}}_{=\frac{\sqrt{3}}{2}}}=\frac{2\pi}{3\sqrt{3}}$$

ES 11. $$\int_{-\infty}^{\infty}\frac{x}{(x^2+1)^2\,(x^2+2x+2)}\,dx$$

$$f(z)=\frac{z}{(z^2+1)(z^2+2z+2)}$$

SINGOLARITÀ:

$(z^2+1)^2=0 \implies z=\pm i$ POLI DI ORDINE 2

$z^2+2z+2=0 \implies z=-1\pm i$ POLI ORDINE 1

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2(x^2+2x+2)}\, dx = 2\pi i \left[\operatorname{Res}(f, i) + \operatorname{Res}(f, -1+i) \right]$$

$$= (*)$$

$$\operatorname{Res}(f, i) = \frac{d}{dz}\left(\frac{z}{(z+i)^2(z^2+2z+2)} \right)\Bigg|_{z=i} =$$

$$= \frac{(z+i)^2(z^2+2z+2) - z\left(2(z+i)(z^2+2z+2) + (z+i)^2(2z+2)\right)}{(z+i)^4(z^2+2z+2)^2}.$$

$$= \frac{-4(1+2i) - i\left[4i(1+2i) - 8(1+i)\right]}{16(1+2i)^2}$$

$$= \frac{-\cancel{1}-\cancel{2i}+(\cancel{1}+2i)+\cancel{2i}-2}{4(1+2i)^2} = \frac{i-1}{2(1+2i)^2}$$

$$= \frac{i-1}{2\underbrace{(1+2i)^2(1-2i)^2}_{=25}}\,(1-2i)^2 = \frac{7+i}{50}$$

$$(i-1)(1-2i)^2 = (i-1)(\underbrace{1-4}_{=-3}-4i)$$

$$= -3i+4+3+4i = 7+i$$

$$\mathrm{Res}(f,-1+i) = \left.\frac{z}{(z^2+1)^2(z+1+i)}\right|_{z=-1+i} = \frac{-1+i}{(1-2i)^2(2i)} =$$

$$(-1+i)^2+1 = (\cancel{1}-\cancel{1}-2i)+1 \qquad = \frac{-1+i}{2i\,\underbrace{(1-2i)^2\,(1+2i)^2}_{=25}}\,(1+2i)^2 = \frac{-7i-1}{50\,i}$$

$$(-1+i)(1+2i)^2 = (-1+i)(\overbrace{1-4}^{=-3}+4i) \qquad = -\frac{7}{50}+\frac{i}{50}$$

$$= 3-4i-3i-4 = -7i-1$$

$$(*) = 2\pi i\left[\cancel{\frac{7}{50}}+\frac{i}{50}-\cancel{\frac{7}{50}}+\frac{i}{50}\right] = -\frac{4\pi}{50} = -\frac{2\pi}{25}$$