ESERCITAZIONE 3/4/2024

ES 1 $f(z) = \dfrac{\operatorname{sen}(i\pi z)}{(z^2+1)^2}$

CALCOLARE $\displaystyle\int_\gamma f(z)\,dz$ NEI SEGUENTI CASI:

a) $\gamma = C(i,1)$ b) $\gamma = C(0,2)$

DOVE $C(z_0, r)$ DENOTA LA CIRCONFERENZA DI CENTRO z_0 E RAGGIO $r>0$ ORIENTATA IN VERSO ANTIORARIO

STUDIAMO LE SINGOLARITÀ DI f

$z^2+1=0 \iff z = \pm i$

STUDIAMO LA NATURA DI QUESTE SINGOLARITÀ:

$\boxed{z_0 = i}$ $\operatorname{sen}(i\pi z) \underset{\substack{\uparrow \\ \text{SVILUPPO DI TAYLOR}}}{=} 0 - \pi i\,(z-i) + O((z-i)^3)$

$$f(z) = \frac{-\pi i\,(z-i) + O(z-i)^3}{(z-i)^2(z+i)^2} = \underbrace{\frac{1}{(z+i)^2}}_{\substack{= \left(\frac{1}{2i}\right)^2 + O(z-i) \\ = -\frac{1}{4}}} \left[\frac{-\pi i}{(z-i)} + O(z-i)\right]$$

$$\frac{1}{4}\,\frac{\pi i}{(z-i)} + O(1)$$

SI TRATTA DI UN POLO SEMPLICE (ORDINE 1) $\operatorname{Res}(f, i) = \dfrac{\pi i}{4}$

IN MANIERA ANALOGA:

$\boxed{z_0 = -i}$ $\qquad \sin(i\pi z) \underset{\uparrow}{=} 0 - \pi i\,(z+i) + O((z+i)^3)$

SVILUPPO DI TAYLOR

$$f(z) = \frac{-\pi i\,(z+i) + O(z+i)^3}{(z-i)^2(z+i)^2} = \frac{1}{(z-i)^2}\left[\frac{-\pi i}{(z+i)} + O(z+i)\right]$$

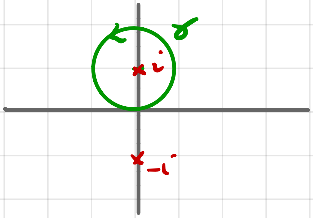
$\left(\frac{-1}{2i}\right)^2 + O(z+i)$

$-\frac{1}{4}$

$$= \frac{1}{4}\,\frac{\pi i}{(z+i)} + O(1)$$

SI TRATTA DI UN POLO SEMPLICE (ORDINE 1) $\qquad \mathrm{Res}(f,-i) = \frac{\pi i}{4}$

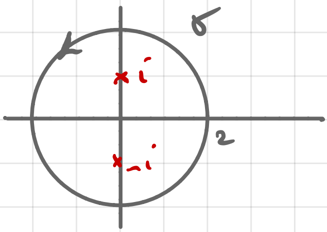
A



$$\int_\gamma f(z) \underset{\uparrow}{=} 2\pi i\,\mathrm{Res}(f,i) = 2\pi i\,\frac{\pi i}{4} = -\frac{\pi^2}{2}$$

TEOREMA DEI RESIDUI

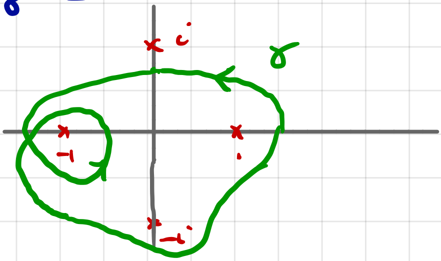
B



$$\int_\gamma f(z)\,dz = 2\pi i\left[\mathrm{Res}(f,i) + \mathrm{Res}(f,-i)\right]$$

$$= 2\pi i\left(\frac{\pi i}{4} + \frac{\pi i}{4}\right) = -\pi^2$$

ES 2 $\displaystyle\int_\gamma \frac{z+2}{z^4-1}\,dz$ DOVE γ È IL PERCORSO IN FIGURA



$f(z)=\dfrac{z+2}{z^4-1}$ HA 4 SINGOLARITÀ : $z=\pm 1, \pm i$ POLI SEMPLICI

$z^4-1=(z-1)(z+1)(z-i)(z+i)$

INDICE DI AVVOLGIMENTO:

$m(\gamma,1)=1$ $\quad m(\gamma,-1)=2$ $\quad m(\gamma,i)=0$ $\quad m(\gamma,-i)=1$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i\left[\operatorname{Res}(f,1)+2\operatorname{Res}(f,-1)+\operatorname{Res}(f,-i)\right]$$

$\boxed{\text{IN } z=1}$

$$\frac{z+2}{z^4-1}=\frac{1}{z-1}\,\frac{z+2}{(z+1)(z^2+1)} \Rightarrow \operatorname{Res}(f,1)=\left.\frac{z+2}{(z^2+1)(z+1)}\right|_{z=1}=\frac{3}{4}$$

$\boxed{\text{IN } z=-1}$

$$\frac{z+2}{z^4-1}=\frac{1}{z+1}\,\frac{z+2}{(z-1)(z^2+1)} \Rightarrow \operatorname{Res}(f,-1)=\left.\frac{z+2}{(z-1)(z^2+1)}\right|_{z=-1}=-\frac{1}{4}$$

$\boxed{\text{IN } z=-i}$

$$\frac{z+2}{z^4-1}=\frac{z+2}{(z+i)(z-i)(z^2-1)} \Rightarrow \operatorname{Res}(f,-i)=\left.\frac{z+2}{(z-i)(z^2-1)}\right|_{z=-i}=\frac{-i+2}{(-2i)(-2)}=$$

$$=-\frac{i}{4}(2-i)=-\frac{1}{4}-\frac{i}{2}$$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \left[\cancel{\frac{3}{4}} - \cancel{2\frac{1}{4}} - \cancel{\frac{1}{4}} - \frac{i}{2} \right] = \pi$$

ES 3 CALCOLARE $\displaystyle\int_0^\pi \frac{1}{\alpha + \cos\theta}\,d\theta$ con $\alpha > 1$ (numero reale)

CONSIDERIAMO LA SOSTITUZIONE (PER $|z|=1$):

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{z+\bar{z}}{2} \underset{\uparrow}{=} \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$$

(se $|z|=1$, $\frac{1}{z} = \bar{z}$)

$$\int_0^\pi \frac{1}{\alpha+\cos\theta}\,d\theta \underset{\text{PARITÀ}}{=} \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{\alpha + \cos\theta}\,d\theta \underset{\uparrow}{=} \frac{1}{2}\int_{C(0,1)} \frac{1}{\alpha + \frac{z^2+1}{2z}}\,\frac{1}{iz}\,dz$$

$z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$d\theta = \frac{1}{iz}$

$$= -i\int_{C(0,1)} \frac{dz}{\underbrace{z^2+2\alpha z+1}_{\substack{\shortparallel \\ f(z)}}}$$

$f(z)$ HA POLI SEMPLICI IN

$$\beta_\pm = \frac{-\alpha \pm \sqrt{\alpha^2-1}}{2}$$

OSSERVIAMO CHE $\beta_\pm = -\alpha \pm \sqrt{\alpha^2-1}$ SONO REALI

$$\beta_+ = -\alpha + \sqrt{\alpha^2-1} = \frac{1}{\alpha+\sqrt{\alpha^2-1}} \in (0,1)$$

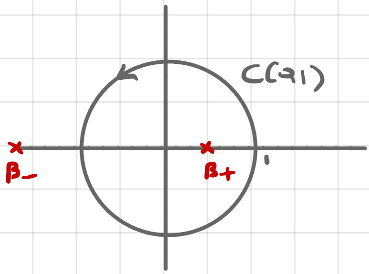
$$\beta_- = -\alpha - \sqrt{\alpha^2-1} < -1$$

← NON È INTERNO AL DISCO UNITARIO

DETERMINIAMO IL RESIDUO IN $z = \beta_+$

$$f(z) = \frac{1}{z^2+2\alpha z+1} = \frac{1}{(z-\beta_+)(z-\beta_-)}$$

$$\mathrm{Res}(f, \beta_+) = \frac{1}{\beta_+-\beta_-} = \frac{1}{2\sqrt{\alpha^2-1}}$$



$$\Rightarrow \int_0^\pi \frac{d\theta}{\alpha+\cos\theta} = -i \int_{C(0,1)} \frac{dz}{z^2+2\alpha z+1} =$$

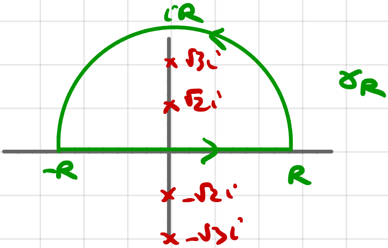
$$= 2\pi i(-i)\,\mathrm{Res}(f, \beta_+) = \frac{\pi}{\sqrt{\alpha^2-1}}$$

TEOREMA DEI RESIDUI

ES 4

$$\int_0^{\infty} \frac{x^2}{x^4+5x^2+6}\,dx$$

CONSIDERIAMO LA FUNZIONE $f(z)=\dfrac{z^2}{z^4+5z^2+6}$ SUL CAMMINO



$$z^4+5z^2+6=0 \iff z^2=\frac{-5\pm\sqrt{25-24}}{2}=\frac{-5\pm 1}{2}= \begin{cases} -3 \\ -2 \end{cases}$$

$$\iff z\in\{\pm\sqrt{3}i,\ \pm\sqrt{2}i\}$$

SE $R>\sqrt{3}$ I POLI INTERNI ALLA REGIONE DELIMITATA DA γ_R SONO $\sqrt{2}i$ e $\sqrt{3}i$

CALCOLIAMO I RESIDUI

$\boxed{\text{in } z=\sqrt{2}i}$

$$f(z)=\frac{z^2}{(z-\sqrt{2}i)(z+\sqrt{2}i)(z^2+3)}$$

$$\mathrm{Res}(f,\sqrt{2}i)=\frac{(\sqrt{2}i)^2}{2\sqrt{2}i\left((\sqrt{2}i)^2+3\right)}=\frac{-2}{2\sqrt{2}i}=\frac{i}{\sqrt{2}}=i\frac{\sqrt{2}}{2}$$

$\boxed{\text{in } z=\sqrt{3}i}$

$$f(z)=\frac{z^2}{(z-\sqrt{3}i)(z+\sqrt{3}i)(z^2+2)}$$

$$\mathrm{Res}(f,\sqrt{3}i)=\frac{-3}{2\sqrt{3}i(-1)}=\frac{-i\sqrt{3}}{2}$$

$$\Rightarrow \int_{\gamma_R} \frac{z^2}{z^4+5z^2+6}\,dz = 2\pi i \left(i\frac{\sqrt{2}}{2} - i\frac{\sqrt{3}}{2} \right) = \pi(\sqrt{3}-\sqrt{2})$$

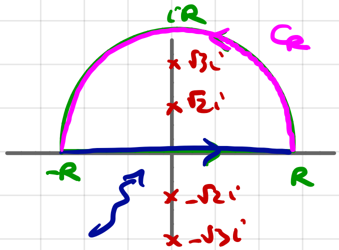
TEOREMA DEI RESIDUI ($R > \sqrt{3}$)

$$\Downarrow$$

$$\underbrace{\lim_{R\to+\infty} \int_{\gamma_R} \frac{z^2}{z^4+5z^2+6}\,dz}_{=\ \pi(\sqrt{3}-\sqrt{2})} = \lim_{R\to+\infty}\left(\int_{C_R} \frac{z^2}{z^4+5z^2+6}\,dz + \int_{\sigma_R} \frac{z^2\,dz}{z^4+5z^2+6} \right)$$

SEMICIRCONF. (C_R)

segmento TRA $-R$ e R (σ_R)



$$= \underbrace{\lim_{R\to\infty}\left(\int_{C_R} \frac{z^2}{z^4+5z^2+6}\,dz \right)}_{=\ 0} + \int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6}\,dx$$

$$\left| \int_{C_R} \frac{z^2}{z^4+5z^2+6}\,dz \right| \le \int_{C_R} \frac{R^2}{R^4-5R^2-6}\,|dz| = \pi R \frac{R^2}{R^4-5R^2-6} \xrightarrow{R\to\infty} 0$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6}\,dx = \pi(\sqrt{3}-\sqrt{2})}$$

INTEGRALI DI FUNZIONI RAZIONALI

$$\boxed{\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx}$$

P, Q POLINOMI T.C. $\boxed{\text{GRADO } Q - \text{GRADO } P > 1}$

$\boxed{Q(x) \neq 0 \quad \forall x \in \mathbb{R}}$

PROCEDENDO COME NELL'ESERCIZIO 6

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} = 2\pi i \sum_{k=1}^{m} \mathrm{Res}\left(\frac{P(z)}{Q(z)}, z_k\right)$$

$z_1 \dots z_m$ SINGOLARITÀ DI $\frac{P(z)}{Q(z)}$ CON $\mathrm{Im}\, z_k > 0$

ANALOGAMENTE:

$$\boxed{\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x)\,dx} \qquad \boxed{\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x)\,dx}$$

P, Q POLINOMI T.C. $\boxed{\text{GRADO } Q - \text{GRADO } P \geq 1}$ e $\boxed{\alpha > 0}$

$\boxed{Q(x) \neq 0 \quad \forall x \in \mathbb{R}}$

PROCEDENDO COME NELL'ES. 4 E RICORDANDO CHE $e^{i\alpha x} = \cos \alpha x + i \sin \alpha x$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \alpha x \,dx = \mathrm{Re}\left[2\pi i \sum_{k=1}^{m} \mathrm{Res}\left(\frac{P(z)}{Q(z)} e^{i\alpha z}, z_k\right)\right]$$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin \alpha x \,dx = \mathrm{Im}\left[2\pi i \sum_{k=1}^{m} \mathrm{Res}\left(\frac{P(z)}{Q(z)} e^{i\alpha z}, z_k\right)\right]$$

$z_1 \dots z_m$ SINGOLARITÀ CON $\mathrm{Im}\, z_i > 0$

ES 5

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx$$

$$f(z) = \frac{1}{z^2+1} e^{+iz}$$

SINGOLARITÀ $z = \pm i$ $\leadsto$ CONSIDERIAMO QUELLE IN $\{\operatorname{Im} z > 0\}$

$$\operatorname{Res}(f, i) = \frac{e^{i \cdot i}}{2i} = \frac{e^{-1}}{2i}$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx = \operatorname{Re}\left[2\pi i \frac{e^{-1}}{2i}\right] = \frac{\pi}{e}$$
