4. A weak membrane with randomly distributed defects

The discrete setting provides an easy framework where to model problems with some random choice. In this chapter we will study the description of a two-dimensional square network mixing two types of connections: 'weak' and 'stong' ones. The strong connections are simple quadratic ones, while the weak connections are 'truncated quadratic potentials' that are quadratic below some threshold, and constant above. We may imagine that this network model a two-dimensional membrane, and that the unknown function u represents the vertical displacement of the membrane. A 'strong connection' may be simply viewed as a linear spring between two neighboring nodes of the network. A 'weak spring' behaves in the same way as a 'strong spring' below the fracture threshold, at which it breaks, and the two neighboring nodes get disconnected. The distribution of weak springs can be viewed as a distribution of defects in an otherwise linear material. We will investigate the 'typical' overall behavior of such a model in the hypothesis that the defects be randomly distributed. To this end we will briefly give an overview of some percolation results, and preliminarily treat the case when only weak connections are present.

1 The weak membrane

We consider the energies

$$E_\varepsilon(u) = \frac{1}{2}\sum_{\alpha,\beta} \varepsilon^N f^\varepsilon\Big(\frac{u(\alpha)-u(\beta)}{\varepsilon}\Big)$$

where f^ε are truncated quadratic potentials:

$$f^\varepsilon(z) = \min\Big\{z^2, \frac{1}{\varepsilon}\Big\}.$$

The sum is extended to all nearest neighbors. Note that if

$$\Big|\frac{u(\alpha)-u(\beta)}{\varepsilon}\Big|^2 \le \frac{1}{\varepsilon}$$

for all α,β, then $F_\varepsilon(u)$ is just a discretization of the Dirichlet integral, while if for example u takes just two values, say u_0 and u_1, then for ε small enough we have

$$f^\varepsilon\Big(\frac{u(\alpha)-u(\beta)}{\varepsilon}\Big) = \begin{cases} 0 & \text{if } u(\alpha)=u(\beta) \\ \frac{1}{\varepsilon} & \text{otherwise,} \end{cases}$$

so that, identifying each u with its piecewise-constant extension, we have

$$E_\varepsilon(u) = \varepsilon^{N-1}\#\{\{\alpha,\beta\} : u(\alpha)\neq u(\beta)\} = \mathcal{H}^{N-1}(\partial\{u=u_0\}) + o(1),$$

where the $o(1)$ comes from some boundary corrections.

In general the Γ-limit will be finite on functions that may have a discontinuity set $S(u)$ of dimension $N-1$ and such that this set is rectifiable, and that are otherwise approximately differentiable outside this set. The space of such u is called the space of special functions of bounded variation $SBV(\Omega)$ and is defined as the space of all $u \in BV(\Omega)$ such that the distributional derivative of u can be split in a N-dimensional and a $N-1$-dimensional part. To be more precise, since we will not directly have a bound on BV norms, the u for which our limit energies will make sense are in $GSBV(\Omega)$ (*i.e.*, their truncations $(-T \vee u) \wedge T$ are in $SBV(\Omega)$ for all T).

On this space the Γ-limit of F_ε can be written as

$$F(u) = \int_\Omega |\nabla u|^2\, dx + \int_{S(u)} \|\nu_u\|_1\, d\mathcal{H}^{N-1}.$$

Note that if $u \in H^1(\Omega)$ then $F(u) = \int_\Omega |\nabla u|^2\, dx$. Note moreover that in many problems we will have an a priori bound for the L^∞ norm of the solution, that in this way belongs to $SBV(\Omega)$.

2 A naive view to percolation theory

We want to compute a Γ-limit as in the previous section, of an energy where we randomly mix f^ε as above and simple quadratic interactions. To this end we have to introduce some notions of percolation theory for what is called the 'bond percolation model' (*i.e.*, when the random choice is thought to be performed on the connections. A different model, that can be treated similarly, is the *site percolation model*. In our intuition it would correspond to choosing weak and strong nodes – and to define a weak connection as a connection between two nodes of which at least one is a weak node).

We do not want to introduce the formal definition of a random variable, but just to look at the relevant elements of percolation theory that will allow us to describe the model of a weak membrane. From now on we will restrict to the two-dimensional case $N = 2$. We start by introducing the dual lattice

$$\mathcal{Z} = \Big\{ \frac{\alpha+\beta}{2} : \alpha, \beta \in \mathbf{Z}^2,\ |\alpha-\beta| = 1 \Big\}.$$

A choice of connections between nodes of $\mathbf{Z}^2$ is a function $\omega : \mathcal{Z} \to \{-1, 1\}$; 1 corresponds to a *strong connection*, and -1 to a *weak connection*. We identify each point $\gamma \in \mathcal{Z}$ with the segment $[\alpha, \beta]$ such that $\alpha, \beta \in \mathbf{Z}^2$ and $2\gamma = \alpha + \beta$. Given ω, we say that two points $\gamma, \gamma' \in \mathcal{Z}$ such that $\omega(\gamma) = \omega(\gamma')$ are connected if there exists a path in $\mathcal{Z}$ (now identified as a set of segments) such that each element of this path γ'' is such that $\omega(\gamma'') = \omega(\gamma)$. Such a path is called a *weak channel* if $\omega(\gamma) = -1$ and a *strong channel* if $\omega(\gamma) = 1$. In this way, we subdivide $\mathcal{Z}$ into 'connected subsets' where either $\omega(\gamma) = 1$ or -1.

We now want to express the fact that

$$\omega(\gamma) = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p. \end{cases}$$

This can be done rigorously by introducing some 'independent identically distributed' random variables. This is not however the scope of our presentation. It suffices to describe the 'almost-sure' properties of such ω.

If $p < 1/2$ then it is 'more probable' to have some γ with $\omega(\gamma) = -1$; not only, it is not likely to have a large number of connected points with $\omega(\gamma) = 1$. This is expressed by the fact that there is one (necessarily unique) infinite connected component of $\{\omega = -1\}$. We call this set the *infinite weak cluster* (or simply weak cluster). Of course, the situation is symmetrical for $p > 1/2$, in which case we have an *infinite strong cluster*.

If two points γ and γ' belong to the weak cluster then there is at least one path L in the cluster (now we identify points with segments) joining γ and γ'. We denote by $|L|$ the length of this path. The *chemical distance* of γ and γ' is defined as

$$d^\omega(\gamma, \gamma') = \min |L|,$$

where the minimum is taken over all such paths L.

This distance is not isotropic (it suffices to think about the trivial case $p = 0$) and depends on ω. Nevertheless, its limit behavior as the points γ and γ' are scaled properly is well defined and independent of ω: we define

$$\varphi^p(\nu) = \liminf_{T\to+\infty} \inf\Big\{ \frac{1}{|T|} d^\omega(\gamma, \gamma') : \gamma - \gamma' = T\nu \Big\}.$$

This limit is finite and independent of ω for all ν, except for a set of ω with zero probability. Note that for $p = 1$ we have $\varphi^p(\nu) = \|\nu\|_1$.

The number $\varphi^p(\nu)$ describes the average distance on the weak cluster in the direction ν (and by symmetry also in the orthogonal direction). Its value cannot be decreased by using 'small portions' of strong connections: if $\delta > 0$ then there exists $T > 0$ and $c = c(\delta) \in (0, 1)$ such that if L is a path joining γ and $\gamma' = \gamma + T\nu$ and $|L| < (\varphi^p(\nu) - \delta)T$, then there are at least $c(\delta)T$ strong connections in the path L.

The weak cluster (and the strong cluster for $p > 1/2$) are 'well distributed'. This can be expressed in the following way (*channel property*): there exist constants $c(p) > 0$ and $c_1(p) > 0$ such that a.s. for any δ, $0 < \delta \le 1$ there is a large enough number $N_0 = N_0(\omega, \delta)$ such that for all $N > N_0$ and any square of size length δN contains at least $c(p)\delta N$ disjoint weak channels which connect opposite sides of the square. Moreover, the length of each such a channel does not exceed $c_1(p)\delta N$.

3 Randomly distributed defects

We fix the probability p of strong connections and choose a *realization* ω (naively, we toss a coin at each lattice connection) and, with this realization fixed, we consider the energy of the corresponding membrane

$$E^\omega_\varepsilon(u) = \frac{1}{2} \sum_{\alpha,\beta} \varepsilon^2 f^\varepsilon_{\omega(\gamma)}\Big(\frac{u(\alpha) - u(\beta)}{\varepsilon}\Big),$$

where

$$\gamma = \frac{\alpha + \beta}{2}, \qquad f^\varepsilon_1(z) = z^2, \qquad f^\varepsilon_{-1}(z) = f^\varepsilon(z),$$

with f^ε the weak membrane energy density defined above.

At the two extreme cases we have:

• $p = 0$ (zero probability of strong connections) then we almost surely are in the case of the weak membrane, and the Γ-limit is

$$F^0(u) = \int_\Omega |\nabla u|^2\, dx + \int_{S(u)} \|\nu_u\|_1\, d\mathcal{H}^{N-1}$$

defined on $SBV(\Omega)$;

• $p = 1$ (strong connections with probability one) then we almost surely are in the case of the 'strong' membrane, and the Γ-limit is simply

$$F^1(u) = \int_\Omega |\nabla u|^2 \, dx$$

on $H^1(\Omega)$.

We will show that

(1) the *percolation threshold* $p = 1/2$ separates two different regimes. Fracture may appear only below this threshold;

(2) below the percolation threshold the Γ-limit is of fracture type, and the surface interaction energy is described by the asymptotic chemical distance φ^p only, being independent of ω, and is given by

$$F^p(u) = \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)} \varphi^p(\nu_u) \, d\mathcal{H}^{N-1}$$

defined on $SBV(\Omega)$. This means that fracture essentially occurs on the weak cluster, and that the energy density is simply obtained by minimizing the length of the fracture paths;

(3) above the percolation threshold the effect of the weak connections is negligible, and the overall behavior is simply described by the Dirichlet integral, independently of $p > 1/2$.

4 The supercritical case

We first treat the case $p > 1/2$. In this case we already have an upper bound since

$$E^\omega_\varepsilon(u) \leq E^1_\varepsilon(u) := \frac{1}{2} \sum_{\alpha,\beta} (u(\alpha) - u(\beta))^2 = \frac{1}{2} \sum_{\alpha,\beta} \varepsilon^2 \Big(\frac{u(\alpha) - u(\beta)}{\varepsilon} \Big)^2,$$

and the Γ-limit of the latter is the Dirichlet integral. We only have to prove the lower bound inequality.

To this end, we use an indirect argument: first, we use the coerciveness of the discrete energies for the weak membrane to deduce that we may suppose that the limit of a sequence such that $E^\omega_\varepsilon(u_\varepsilon)$ is equi-bounded is indeed in $SBV(\Omega)$; subsequently, we use the percolation properties to show that the discontinuity set of the limit function u must be negligible and hence $u \in H^1(\Omega)$. The final equality then follows since on $H^1(\Omega)$ all limits coincide with the Dirichlet integral.

First step. We use the inequality $f^\varepsilon_{\omega(\gamma)} \geq f^\varepsilon$ to check that

$$E^\omega_\varepsilon(u) \geq E^0_\varepsilon(u) := \frac{1}{2} \sum_{\alpha,\beta} \varepsilon^2 f^\varepsilon \Big(\frac{u(\alpha) - u(\beta)}{\varepsilon} \Big).$$

Let $u_\varepsilon \to u$; we deduce that $\sup_\varepsilon E^0_\varepsilon(u_\varepsilon)$ is equi-bounded and then that $u \in GSBV(\Omega)$. Moreover we have the lower bound

$$\liminf_\varepsilon E^\omega_\varepsilon(u_\varepsilon) \geq F^0(u).$$

Since $F^0 = F^1$ on $H^1(\Omega)$, it will suffice to prove that $u \in H^1(\Omega)$.

Second step. We now prove that $\mathcal{H}^1(S(u)) = 0$. This shows that $u \in H^1(\Omega)$ and concludes the proof. We will actually prove more: for any fixed any $c > 0$ the number of points in $S(u)$ such that $|u^+(x) - u^-(x)| \geq c$ is finite.

Take any N such points $x_1, \ldots, x_N$. Let $\nu_i = \nu_u(x_i)$, and fix $\rho > 0$ such that the cubes $Q^{\nu_i}_\rho(x_i)$ of side length ρ, center x_i and one side orthogonal to ν_i have disjoint closures.

We now estimate the contribution to the total energy due to the interactions contained in $Q^{\nu_1}_\rho(x_1)$. Upon a translation we can suppose $x_1=0$. We may take ρ small enough so that it is not restrictive to suppose that $|u(x)-u(y)|\ge c/2$ if $x\in S^+_\rho$ and $y\in S^-_\rho$, where $S^\pm_\rho$ are the two sides of $Q^{\nu_i}_\rho$ orthogonal to ν; we may also suppose that $|u_\varepsilon(x)-u_\varepsilon(y)|\ge c/4$ for such x,y. We now use the channel property of the strong cluster (after scaling) to deduce that for ε small enough there are at least $c(p)\rho/\varepsilon$ disjoint strong channels C_j joining S^-_ρ and S^+_ρ, of length at most $c_1\rho$. If x,y are points in S^-_ρ and S^+_ρ belonging to the same strong channel, we can estimate

$$\begin{aligned}\frac{c}{4} &\le |u_\varepsilon(x)-u_\varepsilon(y)|\le\sum_{x',y'}\varepsilon\Big|\frac{u_\varepsilon(x')-u_\varepsilon(y')}{\varepsilon}\Big|\\ &\le \sqrt{c_1\rho}\sqrt{\sum_{x',y'}\varepsilon\Big|\frac{u_\varepsilon(x')-u_\varepsilon(y')}{\varepsilon}\Big|^2},\end{aligned}$$

where the sum is performed over ordered neighboring x',y' along the same strong channel, so that

$$\frac{c^2\varepsilon}{16c_1\rho}\le\sum_{x',y'}\varepsilon^2\Big|\frac{u_\varepsilon(x')-u_\varepsilon(y')}{\varepsilon}\Big|^2.$$

We sum up over all disjoint strong channels to obtain

$$\frac{c^2c(p)}{16c_1}\le\sum_{C_j}\sum_{x',y'}\varepsilon^2\Big|\frac{u_\varepsilon(x')-u_\varepsilon(y')}{\varepsilon}\Big|^2\le\frac12\sum_{\alpha,\beta}\varepsilon^2 f^\varepsilon_\omega\Big(\frac{u(\alpha)-u(\beta)}{\varepsilon}\Big),$$

where the sum is performed over all pairs α,β in $Q^{\nu_1}_\rho$.

Since the $Q^{\nu_i}_\rho(x_i)$ are disjoint we can repeat the reasoning for all $i=1,\dots,N$, and deduce the estimate

$$N\le\frac{16c_1}{c^2c(p)}E^\omega_\varepsilon(u_\varepsilon)$$

on the number of such points, as desired.

5 The subcritical case

In the subcritical case we have to prove both an upper and a lower bound. Again, we can use a comparison argument with the weak-membrane energies to deduce that the limit of a sequence with equi-bounded energy is a function in $GSBV(\Omega)$.

We want to give a 'local' estimate on the limit energy. To this end, we define

$$E^\omega_\varepsilon(u,A)=\frac12\sum_{\alpha,\beta\in A}\varepsilon^2 f^\varepsilon_{\omega(\gamma)}\Big(\frac{u(\alpha)-u(\beta)}{\varepsilon}\Big),$$

where we limit to interactions such that $\alpha,\beta\in A$.

Let $u_\varepsilon\to u$. We may suppose that $u\in L^\infty(\Omega)$, upon a truncation argument, and hence that $u\in SBV(\Omega)$. Fix $c>0,\delta>0$ and take $x\in S(u)$ such that $|u^+(x)-u^-(x)|\ge c$ and for all ρ small enough

$$\liminf_\varepsilon E^\omega_\varepsilon\big(u_\varepsilon,R^{\nu_u(x)}_{\delta,\rho}(x)\big)\le\rho(\varphi^p(\nu_u(x))-3\delta),\qquad(1)$$

where $R^\nu_{\delta,\rho}(x)$ is the rectangle of center x, one side of length ρ orthogonal to ν and the other side of length $\delta\rho$. With fixed ρ we may suppose that $u_\varepsilon\to u$ on the two sides $S^\pm_\rho(x)$ of $R^{\nu_u(x)}_{\delta,\rho}(x)$ that are orthogonal to $\nu_u(x)$.

We can give an estimate on the size of the set of indices γ such that the corresponding $\alpha, \beta \in R^{\nu_u(x)}_{\delta,\rho}(x)$, and

$$\omega(\gamma) = -1, \qquad \Big|\frac{u_\varepsilon(\alpha) - u_\varepsilon(\beta)}{\varepsilon}\Big|^2 > \frac{1}{\varepsilon}.$$

If we denote by $I_\varepsilon(\rho)$ such set of indices, by (1) we have (upon passing to a subsequence of ε)

$$\#(I_\varepsilon(\rho)) \le \frac{\rho}{\varepsilon}(\varphi^p(\nu_u(x)) - 2\delta)$$

for ρ small enough. In the complement of this set the interactions are quadratic (either because $\omega(\gamma) = 1$ or because the difference quotient is below the threshold $1/\sqrt{\varepsilon}$).

We then deduce that we may find $c(\delta)\frac{\rho}{\varepsilon}$ paths in the complement of $I_\varepsilon(\rho)$. In fact, upon scaling and setting $T = \frac{\rho}{\varepsilon}$, if this were not true then we could find a path L connecting two points γ, γ' with $\gamma - \gamma' = T\nu$ such that $|L| \le (\varphi^p(\nu_u(x)) - \delta)T$ and with a percentage of strong connection less than $c(\delta)$.

At this point, we have a fixed percentage of paths where we can reason as in the supercritical case, to deduce in particular that for all $c > 0$

$$\mathcal{H}^1(\{x \in S(u) : |u^+(x) - u^-(x)| \ge c,\ (1) \text{ holds}\}) = 0,$$

and hence that for $\mathcal{H}^1$-almost all $x \in S(u)$

$$\liminf_\varepsilon E^\omega_\varepsilon\big(u_\varepsilon, R^{\nu_u(x)}_{\delta,\rho}(x)\big) \ge \rho(\varphi^p(\nu_u(x)) - 3\delta). \tag{2}$$

By a covering argument of (compact sets of) $S(u)$ by rectangles $R^{\nu_u(x)}_{\delta,\rho}(x)$, (2) and the lower bound coming from the weak membrane (applied to the complement of such sets), for all $\sigma > 0$ we deduce that

$$\liminf_\varepsilon E^\omega_\varepsilon(u_\varepsilon) \ge \int_\Omega |\nabla u|^2\,dx + \int_{S(u)} (\varphi^p(\nu_u(x)) - 3\delta)\,d\mathcal{H}^1 - \sigma,$$

and the lower bound by the arbitrariness of δ and σ.

It remains to check the upper bound. As usual we do not consider a general target function u, but limit our analysis to u in a 'dense class', the general case being obtained by approximation. In this case, we can consider u such that $S(u)$ is a finite union of segments, and $u \in C^1(\Omega \setminus \overline{S(u)}) \cap H^1(\Omega \setminus \overline{S(u)})$.

We only treat the case when $S(u) = (-1/2, 1/2) \times \{0\}$, since our argument is local and can be easily extended to all orientations of $S(u)$. We fix $\delta > 0$ consider the rectangle $\frac{1}{\varepsilon}([-1/2, 1/2] \times [0, \delta]))$. We apply our percolation properties in a slightly different 'dual' way: we identify each point/segment $\gamma \in \mathcal{Z}$ with the orthogonal segment with the same middle point. This identification defines a 'dual' lattice $\mathcal{Z}'$, in which we may find a path L'_ε of weak connections (*i.e.*, still $\omega(\gamma) = -1$) joining the two opposite 'vertical' sides of $\frac{1}{\varepsilon}((-1/2, 1/2) \times (0, \delta))$ and with $|L'_\varepsilon| \le (\varphi^p(e_2) + \delta)$.

The path L'_ε divides $\frac{1}{\varepsilon}([-1/2, 1/2] \times [0, \delta])$ in two connected components that we denote by R^+_ε and R^-_ε (the latter the one containing $\frac{1}{\varepsilon}[-1/2, 1/2] \times \{0\}$). We then simply define:

$$u_\varepsilon(\alpha) = \begin{cases} u(\alpha_1, 0) & \text{if } \alpha/\varepsilon \in R^-_\varepsilon \\ u(\alpha) & \text{otherwise.} \end{cases}$$

Note that for ε small enough the set of $\gamma \in \varepsilon\mathbf{Z}^2$ such that

$$\Big|\frac{u_\varepsilon(\alpha) - u_\varepsilon(\beta)}{\varepsilon}\Big|^2 > \frac{1}{\varepsilon}$$

is contained in $\varepsilon L'_\varepsilon \cup (\{-1/2\} \times (0,\delta)) \cup (\{1/2\} \times (0,\delta))$, and hence we have

$$\limsup_\varepsilon E^\omega_\varepsilon(u_\varepsilon) \le \int_\Omega |\nabla u|^2\,dx + \mathcal{H}^1(S(u))\varphi^p(e_2) + 2\delta.$$

Now we may further extract a diagonal subsequence in δ and obtain a sequence, still denoted by (u_ε), such that $u_\varepsilon \to u$ and

$$\limsup_\varepsilon E^\omega_\varepsilon(u_\varepsilon) \le \int_\Omega |\nabla u|^2\,dx + \mathcal{H}^1(S(u))\varphi^p(e_2) = \int_\Omega |\nabla u|^2\,dx + \int_{S(u)} \varphi^p(\nu_u)\,d\mathcal{H}^1,$$

as desired.

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