# ESERCITAZIONE 20/3/2024

## • INTEGRALI DI FUNZIONI SU CURVE IN $\mathbb{C}$

SIA $f: \Omega \to \mathbb{C}$ UNA FUNZIONE CONTINUA $f(\underset{z=x+iy}{z}) = u(x,y) + iv(x,y)$

SIA $\gamma: [a,b] \to \mathbb{C}$ UNA CURVA REGOLARE $\gamma(t) = z(t) = x(t) + iy(t)$

$$\int_\gamma f(z)\,dz \overset{def}{=} \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt = \int_a^b f(x(t)+iy(t)) \cdot (x'(t)+iy'(t))\,dt$$

$$= \int_\gamma \underbrace{(u\,dx - v\,dy)} + i\,\underbrace{(v\,dx + u\,dy)}$$

1- FORME DIFFERENZIALI

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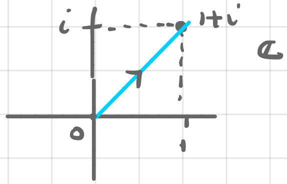
**• ESERCIZIO 1** CALCOLARE I SEGUENTI INTEGRALI:

a) $\int_\sigma x\,dz$ $\quad \sigma =$ SEGMENTO ORIENTATO DA 0 A $1+i$

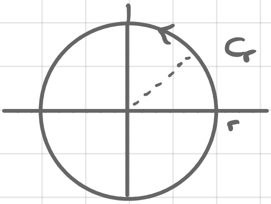



$\sigma(t) = t + it \quad t \in [0,1]$

$$\Rightarrow \int_\sigma x\,dz = \int_0^1 \underset{x(t)}{\underset{\shortparallel}{t}}\;\underbrace{(1+i)}_{\sigma'(t)}\,dt = (1+i)\frac{t^2}{2}\Big|_0^1 = \frac{1}{2}(1+i)$$

b) $\displaystyle\int_{+C_r} x\,dz$ $\qquad +C_r = \{z\in\mathbb{C} : |z| = r\}$

ORIENTATA IN VERSO ANTIORARIO




<u>1° MODO</u>: UNA PARAMETRIZZAZIONE DI $+C_r$ È DATA DA

$$\gamma(t) = re^{it} \quad t\in[0,2\pi] \rightsquigarrow \gamma'(t) = ire^{it}$$

QUINDI: $$\int_{+C_r} x\,dz = \int_0^{2\pi} r\cos t\, \underbrace{ir(\cos t + i\sin t)}_{\gamma'(t)}\,dt$$

$$= \int_0^{2\pi} \left(ir^2(\cos t)^2 - r^2\sin t\cos t\right)dt =$$

$$= ir^2 \underbrace{\int_0^{2\pi}(\cos t)^2\,dt}_{\pi} - r^2\underbrace{\int_0^{2\pi}\sin t\cos t\,dt}_{0} = i\pi r^2$$

<u>2° MODO</u>: OSSERVIAMO CHE $x = \frac{z+\bar z}{2} = \frac12\left(z+\frac{r^2}{z}\right)$

SU $C_r$: $\bar z = \frac{r^2}{z}$

(IN QUANTO $z\bar z = |z|^2 = r^2$)

$$\int_{+C_r} x\,dz = \frac12\left(\int_{+C_r} z\,dz + \int_{+C_r}\frac{r^2}{z}\,dz\right) = \frac12\int_0^{2\pi} re^{it}\,ire^{it}\,dt + \frac12\int_0^{2\pi}\frac{r^2}{\cancel{re^{it}}}\,i\cancel{re^{it}}\,dt$$

$$= \frac{ir^2}{2}\int_0^{2\pi} e^{2it}\,dt + \frac12 ir^2\int_0^{2\pi}dt = \frac{ir^2}{2}\underbrace{\left(\frac{e^{2it}}{2i}\right)\Big|_0^{2\pi}}_{0} + \frac{\cancel{2}\pi ir^2}{\cancel{2}} = \pi i r^2$$

c) SIA $\gamma : [a,b] \to \mathbb{C}$ UNA CURVA REGOLARE, SEMPLICE E CHIUSA, PERCORSA IN SENSO ANTIORARIO.
SIA $\Omega$ LA REGIONE RACCHIUSA DA $\gamma$

VERIFICARE CHE $\displaystyle\int_\gamma \bar{z}\, dz = 2i\, \underbrace{|\Omega|}_{\text{AREA DI } \Omega}$

INFATTI:

$$\int_\gamma \bar{z}\, dz = \int_\gamma (x - iy)(dx + i\, dy) =$$

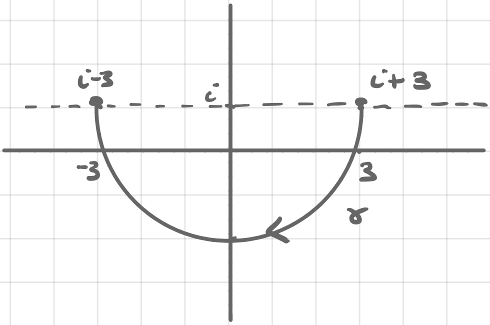
$$= \int_\gamma (x\, dx + y\, dy) + i\,(x\, dy - y\, dx)$$

FORMULE DI GAUSS-GREEN $\longrightarrow$

$$= \iint_\Omega \left( \overset{=0}{\frac{\partial y}{\partial x}} - \overset{=0}{\frac{\partial x}{\partial y}} \right) dx\, dy + i \iint_D \left( \overset{=1}{\frac{\partial x}{\partial x}} + \overset{=1}{\frac{\partial y}{\partial y}} \right) dx\, dy$$

$$= 2i \iint_D dx\, dy = 2i\, |D|.$$

d) $\displaystyle\int_\gamma \frac{1}{(z-i)^2}\, dz$ $\quad\gamma =$ ARCO DI CIRCONFERENZA DI CENTRO $i$ E RAGGIO 3, CONTENUTA NEL SEMIPIANO $|\operatorname{Im} z| \leq 1$, PERCORSA IN SENSO ORARIO.




**1° MODO:**

$$\gamma(t) = i + 3e^{-it} \qquad t \in [0,\pi]$$

$$\int_\gamma \frac{1}{(z-i)^2}\,dz = \int_0^\pi \frac{1}{(3e^{-it})^2}\left(-3ie^{-it}\right)dt =$$

$$= -\frac{i}{3}\int_0^\pi e^{it}\,dt = -\frac{i}{3}\,\frac{e^{it}}{i}\Bigg|_0^\pi = -\frac{1}{3}\Big(\overbrace{e^{i\pi}}^{=-1}-1\Big) = \frac{2}{3}$$

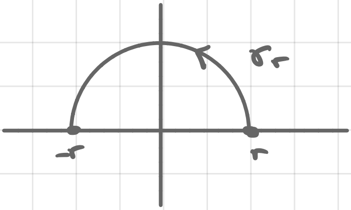
**2° MODO:**

$$\frac{d}{dz}\left(\frac{-1}{(z-i)}\right) = \frac{1}{(z-i)^2}$$

QUINDI LA FUNZIONE AMMETTE UNA PRIMITIVA IN $\mathbb{C}\setminus\{i\}$

$$\Rightarrow \int_\gamma \frac{1}{(z-i)^2}\,dz = \left[-\frac{1}{z-i}\right]_{z=i+3}^{z=i-3} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

e) $\displaystyle\int_{\gamma_r} \left(\frac{Re\, z}{z}\right)^2 dz$ DOVE $\gamma_r$ = SEMICIRCONFERENZA CENTRATA IN 0 E RAGGIO $r>0$ PERCORSA IN SENSO ANTIORARIO DA $z_1 = r$ e $z_2 = -r$




$\gamma(t) = re^{it} \quad t \in [0,\pi]$

1° MODO:

$$\frac{Re\, z}{z} = \left(\frac{z+\bar z}{2}\right)\frac{1}{z} = \frac{1}{2} + \frac{\bar z}{2z} = \frac{1}{2} + \frac{r^2}{2z^2}$$

ma $|z| = r \Rightarrow \bar z = \frac{r^2}{z}$

$$\int_{\gamma_r} \left(\frac{Re(z)}{z}\right)^2 dz = \int_{\gamma_r} \left(\frac{1}{2} + \frac{r^2}{2z^2}\right)^2 dz = \int_0^\pi \left(\frac{1}{2} + \frac{\cancel{r^2}}{2\cancel{r^2}e^{i2t}}\right)^2 e^{it}\, i r\, dt$$

$$= ir\int_0^\pi \left(\frac{1}{4} + \frac{e^{-i2t}}{2} + \frac{e^{-i4t}}{4}\right) e^{it}\, dt =$$

$$= \cancel{i} r \left[ \frac{1}{4} \underbrace{\frac{e^{it}}{\cancel{i}}\Big|_0^\pi} + \underbrace{\frac{e^{-it}}{-2\cancel{i}}\Big|_0^\pi}_{= 2/2 = 1} + \underbrace{\frac{e^{-i3t}}{-12\cancel{i}}\Big|_0^\pi}_{= \frac{2}{12} = \frac{1}{6}} \right] =$$

$$= r\left(-\frac{1}{2} + 1 + \frac{1}{6}\right) = \boxed{\frac{2}{3} r}$$

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2° MODO: $\gamma(t) = re^{it}$ $\quad f(\gamma(t)) = \left(\dfrac{r\cos t}{re^{it}}\right)^2$ $\quad (\cos t - i \sin t) = e^{-it}$

$$\int_\gamma \left(\frac{Re\, z}{z}\right)^2 dz = \int_0^\pi \left(\frac{\cancel{r}\cos t}{\cancel{r}e^{it}}\right)^2 r\, i e^{it}\, dt = r i \int_0^\pi \cos^2 t\, e^{-it}\, dt$$

$$= r i \int_0^\pi (\cos^3 t - i \sin t \cos^2 t)\, dt = r i \left( \overbrace{\int_0^\pi \cos^3 t\, dt}^{= 0} + i \frac{\cos^3 t}{3}\Big|_0^\pi \right)$$

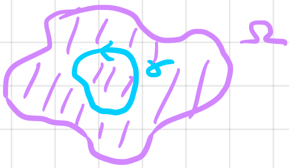
$$= r i \left(-i\frac{2}{3}\right) = \frac{2}{3} r$$

## • TEOREMA DI CAUCHY

• $\Omega \subseteq \mathbb{C}$ APERTO SEMPLICEMENTE CONNESSO $f: \Omega \to \mathbb{C}$ OLOMORFA

• SE $\gamma$ È UNA CURVA CHIUSA, SEMPLICE E REGOLARE CONTENUTA IN $\Omega$

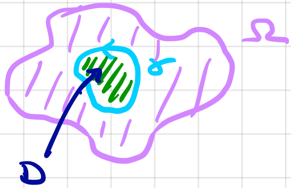
$$\Rightarrow \int_\gamma f(z)\,dz = 0$$




## • FORMULA DI CAUCHY

• $\Omega \subseteq \mathbb{C}$ APERTO SEMPLICEMENTE CONNESSO $f: \Omega \to \mathbb{C}$ OLOMORFA

• SE $\gamma$ È UNA CURVA SEMPLICE, CHIUSA E REGOLARE CONTENUTA IN $\Omega$ E ORIENTATA POSITIVAMENTE.




• SIA $D$ LA REGIONE APERTA DELIMITATA DA $\gamma$

$$\Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}\,dz = \begin{cases} f(z_0) & \text{SE } z_0 \in D \\ 0 & \text{SE } z_0 \notin \overline{D} \end{cases}$$

PIÙ IN GENERALE:

$$\Rightarrow \forall m \in \mathbb{N}: \quad \frac{m!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{m+1}} = \begin{cases} f^{(m)}(z_0) & \text{SE } z_0 \in D \\ 0 & \text{SE } z_0 \notin \overline{D} \end{cases}$$

## • ESERCIZIO 2

CALCOLARE I SEGUENTI INTEGRALI:

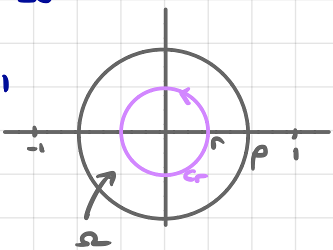
a) $\displaystyle\int_{+C_r} \frac{dz}{z^2-1}\,dz$ $\qquad +C_r = \{|z| = r\}$ ORIENTATA POSITIVAMENTE CON $r \in (0,+\infty)$, $r \neq 1$

$$g(z) = \frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$g$ è olomorfa in $\mathbb{C} \setminus \{\pm 1\}$

• se $r < 1$ $\Rightarrow$ APPLICO IL TEOREMA DI CAUCHY SU

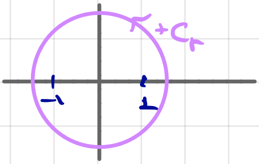
$$\Omega = \{|z| < \rho\} \text{ con } r < \rho < 1$$




$g$ è olomorfa in $\Omega$

$$\Rightarrow \int_{+C_r} \frac{dz}{z^2-1}\,dz = 0$$

• se $r > 1$ $\Rightarrow$ $\displaystyle\int_{+C_r} \frac{dz}{z^2-1} = \frac{1}{2}\left(\underbrace{\int_{+C_r} \frac{dz}{z-1}}_{=2\pi i \cdot 1} - \underbrace{\int_{+C_r} \frac{dz}{z+1}}_{=2\pi i \cdot 1}\right) = 0$




APPLICO FORMULA DI CAUCHY AD ENTRAMBI INTEGRALI CON $f(z) = 1$ e $z_0 = \pm 1$, $\Omega = \mathbb{C}$ (LA FUNZIONE $f(z) \equiv 1$ È OLOMORFA OVUNQUE)

b) $\displaystyle\int_{+C_1} \frac{e^z}{z^m}\,dz$ AL VARIARE DI $m \in \mathbb{Z}$

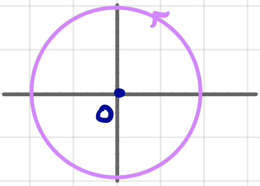
$+C_1 = \{|z|=1\}$ ORIENTATA IN VERSO ANTI-ORARIO

• SE $m \leq 0$ $\quad f(z) = \dfrac{e^z}{z^m} = e^z z^{|m|}$ è OLOMORFA SU $\mathbb{C}$

$\Rightarrow$ TEOREMA DI CAUCHY $\quad \displaystyle\int_{+C_1} \frac{e^z}{z^m} = 0$

• SE $m \geq 1$ $\quad \displaystyle\int_{+C_1} \frac{e^z}{z^m}\,dz = \frac{2\pi i}{(m-1)!}$

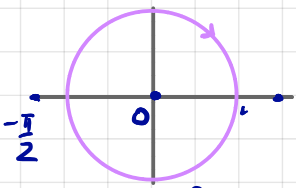
$\uparrow$




FORMULA CAUCHY PER DERIVATE CON $z_0 = 0$

$0 \in$ REGIONE DELIMITATA DA $C_1$

c) $\displaystyle\int_{-C_1} \frac{\cos z}{z(z+\frac{\pi}{2})}\,dz$ $\quad -C_1 = \{|z|=1\}$ ORIENTATA IN VERSO ORARIO




LA FUNZIONE $f(z) = \dfrac{\cos z}{z+\frac{\pi}{2}}$ È OLOMORFA IN $|z| < \frac{\pi}{2}$

$\Rightarrow$ FORMULA DI CAUCHY

$$\int_{-C_1} \frac{\cos z}{z(z+\frac{\pi}{2})}\,dz = -\int_{+C_1} \frac{f(z)}{z}\,dz = -2\pi i\, f(0) = -2\pi i \frac{1}{\frac{\pi}{2}} = -4i$$

d) $\displaystyle\int_{+C_\rho} \frac{1}{|z-a|^2}\,dz$ con $0<\rho\neq|a|$

$+C_\rho := \{|z|=\rho\}$ ORIENTATA IN VERSO ANTI-ORARIO

$$\frac{1}{|z-a|^2} = \frac{1}{(z-a)(\bar{z}-\bar{a})} = \frac{1}{(z-a)\left(\frac{\rho^2}{z}-\bar{a}\right)}$$

$z\in C_\rho \Rightarrow \bar{z}=\frac{\rho^2}{z}$

$$= \frac{z}{(z-a)(\rho^2-\bar{a}z)} = \frac{A}{z-a} + \frac{B}{\rho^2-\bar{a}z} =$$

$$\frac{A}{z-a} + \frac{B}{\rho^2-\bar{a}z} = \frac{A(\rho^2-\bar{a}z)+B(z-a)}{(z-a)(\rho^2-\bar{a}z)}$$

$$= \frac{(B-\bar{a}A)z + (A\rho^2-aB)}{(z-a)(\rho^2-\bar{a}z)}$$

$$\Rightarrow \begin{cases} A\rho^2-aB=0 \\ B-\bar{a}A=1 \end{cases}$$

$$\Rightarrow \quad A=\frac{a}{\rho^2-|a|^2} \qquad B=\frac{\rho^2}{\rho^2-|a|^2}$$

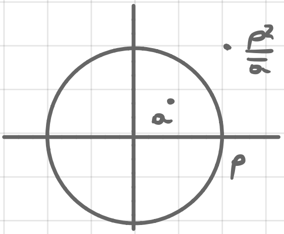
$$= \frac{1}{\rho^2-|a|^2}\left(\frac{a}{z-a} + \frac{\rho^2}{\rho^2-\bar{a}z}\right)$$

$$\int_{+C_\rho} \frac{dz}{|z-a|^2} = \frac{1}{\rho^2-|a|^2}\left(\int_{+C_\rho} \frac{a}{z-a}\,dz + \int_{+C_\rho} \frac{\rho^2}{\rho^2-\bar{a}z}\,dz\right)$$

$$= \frac{1}{\rho^2-|a|^2}\left(a\int_{+C_\rho} \frac{1}{z-a}\,dz - \frac{\rho^2}{\bar{a}}\int_{C_\rho} \frac{dz}{z-\frac{\rho^2}{\bar{a}}}\right)$$

• SE $|a|<\rho$: $\displaystyle\int_{+C_\rho} \frac{a}{z-a}\,dz = 2\pi i$ (FORMULA CAUCHY)

$$\int_{C_\rho} \frac{dz}{z-\frac{\rho^2}{\bar{a}}} = 0 \quad \text{(TEOREMA DI CAUCHY)}$$




se $|a|<\rho \Rightarrow \left|\frac{\rho^2}{\bar{a}}\right| > \frac{\rho^2}{\rho} = \rho$

$\uparrow$ $|\bar{a}|<\rho$

IN MANIERA SIMILE:

• se $|a|>\rho$: $\displaystyle\int_{C_\rho} \frac{dz}{z-\frac{\rho^2}{\bar{a}}} = 2\pi i$

$$\int_{+C_\rho} \frac{a}{z-a}\,dz = 0$$

$$\Rightarrow \int_{+C_\rho} \frac{dz}{|z-a|^2} = \frac{1}{\rho^2-|a|^2}\cdot\begin{cases} 2\pi i a & \text{se } |a|<\rho \\ -\frac{\rho^2}{\bar{a}}2\pi i & \text{se } |a|>\rho \end{cases} = \begin{cases} \frac{2\pi i\, a}{\rho^2-|a|^2} & \text{se } |a|<\rho \\ \frac{2\pi i\,\rho^2}{\bar{a}(|a|^2-\rho^2)} & \text{se } |a|>\rho \end{cases}$$

e) $\displaystyle\int_{+C_1} \frac{\sin z}{z^m}\, dz$ AL VARIARE DI $m \in \mathbb{Z}$

$+C_1 = \{|z|=1\}$ ORIENTATO IN VERSO ANTI-ORARIO

. SE $m \leq 0$ $f(z) = \dfrac{\sin z}{z^m}$ È OLOMORFA IN $\mathbb{C}$

$$\Rightarrow \int_{+C_1} \frac{\sin z}{z^m}\, dz = 0 \quad \text{(TEOREMA DI CAUCHY)}$$

. SE $m \geq 1$

$$\int_{+C_1} \frac{\sin z}{z^m}\, dz = \frac{2\pi i}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (\sin z)\Big|_{z=0}$$

FORMULA DI CAUCHY APPLICATA A $f(z) = \sin z$ E $z_0 = 0$

$$\boxed{\frac{d^m}{dz^m} f(z_0) = \frac{m!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-z_0)^{m+1}}}$$

$$= \begin{cases} 0 & m \text{ DISPARI} \\ (-1)^{\frac{m}{2}-1} & m \text{ PARI} \end{cases}$$

$$\frac{d^k}{dz^k} \sin z = \begin{cases} (-1)^{k/2} \sin z & k \text{ PARI} \\ (-1)^{\frac{k-1}{2}} \cos z & k \text{ DISPARI} \end{cases}$$

## ESERCIZIO 3

SIA $f: \mathbb{C} \to \mathbb{C}$ UNA FUNZIONE INTERA TALE CHE $Re(f(z)) \geq 0 \quad \forall z$

DEDURRE CHE $f(z) \equiv$ COSTANTE

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SOLUZIONE: CONSIDERIAMO $g(z) = e^{-f(z)}$

$g: \mathbb{C} \to \mathbb{C}$ ED È OLOMORFA IN QUANTO COMPOSIZIONE DI FUNZIONI OLOMORFE

$$|g(z)| = |e^{-f(z)}| = |e^{-Re(f(z))} \cdot e^{-i Im(f(z))}|$$

$$= e^{-Re(f(z))} \cdot \underbrace{|e^{-i Im(f(z))}|}_{= 1}$$

$$= e^{-Re(f(z))} \leq 1 \quad \forall z$$

QUINDI $g$ È UNA FUNZIONE INTERA E LIMITATA

$\Rightarrow$ (TEOREMA DI LIOUVILLE) $\quad g(z) \equiv c \neq 0$

$\Rightarrow \quad f(z) \equiv \log|c| + i Arg(c) + 2k\pi i \quad$ PER QUALCHE $k \in \mathbb{Z}$

Poiché $f$ è continua, $k$ è lo stesso $\forall z \in \mathbb{C}$

$\Rightarrow f \equiv$ COSTANTE $\quad \square$