Introduction to Homogenization and Gamma-convergence

Andrea Braides

These Lecture Notes contain the abstract of five lectures conceived as an introduction to Γ-convergence methods in the theory of Homogenization, and delivered on September 8–10, 1993 as part of the "School on Homogenization" at the ICTP, Trieste. Their content is strictly linked and complementary to the subject of the courses held at the same School by A. Defranceschi and G. Buttazzo. Prerequisites are some basic knowledge of functional analysis and of Sobolev spaces. An enlarged version of these Lecture Notes is contained in A. Braides and A. Defranceschi *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998

Section One.
Gamma-convergence for Integral Functionals

1.1. Introduction

The subject of these notes is the study of the asymptotic behaviour as ε goes to 0 of integral functionals of the form

$$\mathcal{F}_\varepsilon(u) = \int_\Omega f(\frac{x}{\varepsilon}, Du(x))\, dx, \tag{1.1}$$

defined on some (subset of a) Sobolev space $W^{1,p}(\Omega;\mathbb{R}^N)$ (in general, of vector-valued functions), when $f = f(y,\xi)$ is a Borel function, (almost) periodic in the variable y, and satisfying the so-called "natural growth" conditions with respect to the variable ξ. Integrals of this form model various phenomena in Mathematical Physics in the presence of microstructures (the vanishing parameter ε gives the microscopic scale of the media). The function f represents the energy density at this lower scale. As an example we can think of u as representing a deformation, and $\mathcal{F}_\varepsilon$ being the stored energy of a cellular elastic material with Ω as a reference configuration. In other applications u is a difference of potential in a condenser composed of periodically distributed material, occupying the region Ω, etc.

The main question we are going to answer is: *does the (medium modeled by the) energy $\mathcal{F}_\varepsilon$ behave as a homogeneous medium in the limit?* (and if so: can we say something about this homogeneous limit?)

First we have to give a precise meaning to this statement. The "behaviour" of the media described by the integral in (1.1) is given by the behaviour of boundary value problems of the Calculus of Variations of the form

$$\min\Big\{\int_\Omega f(\frac{x}{\varepsilon}, Du(x))\, dx + \int_\Omega gu\, dx \ : u = \phi \text{ on } \gamma_0\Big\}, \tag{1.2}$$

where g is some fixed function, and γ_0 is a portion of $\partial\Omega$ (here we suppose Ω sufficiently smooth). If our media behave as a homogeneous medium when ε tends to 0, we expect that there exists a function $f_{\rm hom}$

(representing the energy density of the latter), which is now "homogeneous", that is, independent on the variable x, such that the minima of the problems in (1.2) converge as $\varepsilon \to 0$ to the minimum of the problem

$$\min\Big\{\int_\Omega f_{\rm hom}(Du(x))\,dx + \int_\Omega gu\,dx \ : u = \phi \text{ on } \gamma_0\Big\}, \tag{1.3}$$

and, what is important, the function $f_{\rm hom}$ does not depend on Ω and on the particular choice we make of g, ϕ and γ_0.

The convergence of these minimum values (and, in some weak sense, also of the minimizing functions in (1.2) to the minimizer of (1.3)) will be obtained as a consequence of the convergence of the functionals $\mathcal{F}_\varepsilon$ to the *homogenized functional*

$$\mathcal{F}_{\rm hom}(u) = \int_\Omega f_{\rm hom}(Du(x))\,dx \tag{1.4}$$

in the variational sense of Γ-convergence, which was introduced by E. De Giorgi in the 70s exactly for dealing with problems of this kind. Special relevance will be given to the illustration of the general method, which can be applied, with the due changes, to the study of other types of functionals, different than those defined on Sobolev spaces of the form (1.1) (for example, with essentially the same proof we can obtain a homogenization result for functionals with volume and surface energies (see [11])). In this spirit, many results have been simplified for expository purposes; more technical and general theorems can be found in the papers cited as references.

1.2. Γ-convergence

The notion of Γ-convergence was introduced in a paper by E. De Giorgi and T. Franzoni in 1975 [18], and was since then much developed especially in connection with applications to problems in the Calculus of Variations. We refer to the recent book by Dal Maso [15] for a comprehensive introduction to the subject. Here we are interested mainly in applications to the asymptotic behaviour of minimum problems for integral functionals defined on Sobolev spaces.

First we give an abstract definition of Γ-convergence on a metric space.

Definition 1.1. Let $X = (X, d)$ be a metric space, and for every $h \in \mathbb{N}$ let $F_h : X \to [0, +\infty]$ be a function defined on X. We say that the sequence (F_h) $\Gamma(d)$-*converges* in $x_0 \in X$ to the value $r \in [0, +\infty]$ (and we write $r = \Gamma(d)\text{-}\lim_h F_h(x_0)$) if we have:

(i) for every sequence x_h such that $d(x_h, x_0) \to 0$ we have

$$r \le \liminf_h F_h(x_h); \tag{1.5}$$

(ii) there exists a sequence $\overline{x}_h$ such that $d(\overline{x}_h, x_0) \to 0$, and we have

$$r = \lim_h F_h(x_h) \tag{1.6}$$

(or, equivalently, $r \ge \limsup_h F_h(x_h)$).

If the $\Gamma(d)$-limit $\Gamma(d)\text{-}\lim_h F_h(x)$ exists for all $x \in X$, and the function $F : X \to [0, +\infty]$ satisfies $F(x) = \Gamma(d)\text{-}\lim_h F_h(x)$ for all $x \in X$, then we say that the sequence (F_h) $\Gamma(d)$-*converges* to F (on X) and we write $F = \Gamma(d)\text{-}\lim_h F_h$.

Remark 1.2. Note that if $F = \Gamma(d)\text{-}\lim_h F_h$, then F is a *lower semicontinuous* function with respect to the distance d; *i.e.,*

$$\forall x \in X \quad \forall (x_h): \quad d(x_h, x) \to 0 \quad F(x) \leq \liminf_h F(x_h). \tag{1.7}$$

Remark 1.3. (More remarks on Γ-limits) 1) It can easily be seen, with one-dimensional examples, that the Γ-convergence of a sequence (F_h) is independent from its pointwise convergence. In particular a constant sequence $F_h = F$ $\Gamma(d)$-converges to its constant value F if and only if the function $F : X \to [0, +\infty]$ is *lower semicontinuous* with respect to the distance d.

2) If $F_h = F$ is not lower semicontinuous then we have

$$\Gamma(d)\text{-}\lim_h F_h = \overline{F}, \tag{1.8}$$

where the function $\overline{F}$ is the *d-lower semicontinuous envelope* (or *relaxation*) of F; *i.e.,* the greatest d-lower semicontinuous function not greater than F, whose abstract definition can be expressed as

$$\overline{F}(x) = \inf\Big\{\liminf_h F(x_h) \ : \ d(x_h, x) \to 0\Big\}. \tag{1.9}$$

3) If a sequence Γ-converges, then so does its every subsequence (to the same limit).

4) If $F = \Gamma(d)\text{-}\lim_h F_h$ and G is any d-continuous function then $\Gamma(d)\text{-}\lim_h(F_h + G) = F + G$ (this remark will be extremely useful in applications).

5) The Γ-limit of a sequence of convex functions is convex (here and in the following, we suppose that (X, d) is a topological vector space).

6) The Γ-limit of a sequence of quadratic forms (*i.e.,* $F_h(x+y) + F_h(x-y) = 2F_h(x) + 2F_h(y)$) is a quadratic form.

7) Let $\alpha > 0$; then the Γ-limit of a sequence of positively α-homogeneous functions (*i.e.,* $F_h(tx) = t^\alpha F_h(x)$ for all $t \geq 0$) is positively α-homogeneous.

We easily obtain the property of convergence of minima we are looking for in the case of sequences of equicoercive Γ-converging functionals.

We recall that a subset K of X is *d-compact* if from every sequence (x_h) in K we can extract a subsequence (x_{h_k}) converging to an element $x \in K$.

We say that a function $F : X \to [0, +\infty]$ is *d-coercive* if there exists a d-compact set K such that

$$\inf\{F(x) : \ x \in X\} = \inf\{F(x) : \ x \in K\}. \tag{1.10}$$

Let us also recall here *Weierstrass' Theorem*, which is the fundamental tool of the so-called direct methods of the calculus of variations: *if F is d-coercive and d-lower semicontinuous then there exists a minimizer for F on X.* (*Proof*: by (1.10) there exists a sequence x_h in K such that $F(x_h) \to \inf F$. By the d-compactness of K we can suppose that $x_h \to \overline{x} \in K$. By the d-lower semicontinuity of F we have then $F(\overline{x}) \leq \lim_h F(x_h) = \inf F$; *i.e.,* $\overline{x}$ is a minimizer of F).

We say that a sequence $F_h : X \to [0, +\infty]$ is *d-equicoercive* if there exists a d-compact set K (independent of h) such that

$$\inf\{F_h(x) : \ x \in X\} = \inf\{F_h(x) : \ x \in K\}. \tag{1.11}$$

Theorem 1.4. (The Fundamental Theorem of Γ-convergence) *Let (F_h) be a d-equicoercive sequence $\Gamma(d)$-converging on X to the function F. Then we have the convergence of minima*

$$\min\{F(x)\ :\ x\in X\}=\lim_h \inf\{F_h(x)\ :\ x\in X\}. \tag{1.12}$$

Moreover we have also convergence of minimizers: if $x_h\to x$ and $\lim_h F_h(x_h)=\lim_h \inf F_h$, then x is a minimizer for F.

Proof. Let (h_k) be a sequence of indices such that $\lim_k \inf F_{h_k}=\liminf_h \inf F_h$. Let (x_k) be a sequence in K (K as in (1.11)) satisfying

$$\lim_k F_{h_k}(x_k)=\liminf_k F_{h_k}=\liminf_h \inf F_h. \tag{1.13}$$

By the d-compactness of K we can suppose (possibly passing to a further subsequence) that $x_k\to x\in K$. We have then by (1.5)

$$F(x)\le \liminf_k F_{h_k}(x_k)=\liminf_h \inf F_h, \tag{1.14}$$

so that

$$\inf F\le \inf\{F(x)\ :\ x\in K\}\le \liminf_h \inf F_h. \tag{1.15}$$

Since F is d-lower semicontinuous there exists (by Weierstrass' Theorem) a minimum point $\overline{x}$ for F on K. By (1.6) there exists a sequence x_h such that $x_h\to\overline{x}$, and

$$\min\{F(x)\ :\ x\in K\}=F(\overline{x})=\lim_h F_h(x_h)\ge \limsup_h \inf F_h. \tag{1.16}$$

Hence

$$\min\{F(x)\ :\ x\in K\}=\liminf_h F_h. \tag{1.17}$$

In order to prove (1.12) it will be sufficient to show that K satisfies the coercivity property (1.10). Suppose that (1.10) is not satisfied, then we must have, by (1.17), $\inf F<\lim_h \inf F_h$, so that there exists $x\in X$ such that $F(x)<\lim_h \inf F_h$. This inequality contradicts (1.6), and hence (1.12) is proven.

The convergence of minimizers is a direct consequence of (1.5) and (1.12). □

Note that if F_h is an integral functional with smooth strictly convex integrand, then we obtain from the Γ-convergence of the sequence (F_h) the G-convergence of the corresponding Euler equations. It will be clear in the sequel that no regularity of the integrands is in general necessary for Γ-convergence.

Remark 1.5. The Γ-limit of an arbitrary sequence of functions does not always exist. It will be convenient then to introduce, beside the Γ-limit already studied, also the Γ-limsup and Γ-liminf. Let us define then for $x\in X$

$$\Gamma(d)\text{-}\liminf_h F_h(x)=\inf\{\liminf_h F_h(x_h)\ :\ d(x_h,x)\to 0\}, \tag{1.18}$$

$$\Gamma(d)\text{-}\limsup_h F_h(x)=\inf\{\limsup_h F_h(x_h)\ :\ d(x_h,x)\to 0\}. \tag{1.19}$$

We have $\Gamma(d)\text{-}\liminf_h F_h(x)=\Gamma(d)\text{-}\limsup_h F_h(x)=r$ if and only if there exist the $\Gamma(d)\text{-}\lim_h F_h(x)=r$.

1.3. A Class of Integral Functionals

We have at our disposal now a powerful tool to obtain the desired convergence of minima in (1.2) and (1.3). The next, crucial point now is to understand what the right choice for the space (X,d) is, and how to define the functionals F_h.

At this point, we have to specify the conditions we require on the function f. We suppose $p>1$, and $f:\mathbb{R}^n\times \mathrm{M}^{n\times N}\to[0,+\infty[$ be a Borel function satisfying the so-called "standard growth conditions of order p": there exist constants $c_1\geq 0$, $C_1>0$ such that

$$|\xi|^p-c_1\leq f(x,\xi)\leq C_1(1+|\xi|^p),\qquad \text{for all } x\in\mathbb{R}^n,\xi\in \mathrm{M}^{n\times N} \tag{1.20}$$

(here and afterwards $\mathrm{M}^{n\times N}$ will denote the space of $n\times N$ real matrices) so that the functionals $\mathcal{F}_\varepsilon$ in (1.1) are well-defined on $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ for every Ω open subset of $\mathbb{R}^n$.

Let us now turn to the choice of the space (X,d); the topology of the metric d should be weak enough to obtain equicoercivity for minimum problems, but strong enough to allow for Γ-convergence. For the sake of simplicity let us suppose that $\phi\equiv 0$, $\gamma_0=\partial\Omega$, and Ω itself being sufficiently smooth and bounded (some of these hypotheses may be weakened). Let us recall then the following fundamental theorems on Sobolev spaces.

Theorem 1.6. (Poincaré's Inequality) *Let Ω be a bounded open subset of $\mathbb{R}^n$; then there exist a constant $C'>0$ such that*

$$\int_\Omega |u|^p\,dx\leq C'\int_\Omega |Du|^p\,dx \tag{1.21}$$

for all $u\in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$.

Theorem 1.7. (Rellich's Theorem) *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^n$, and (u_h) be a bounded sequence in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$. Then there exists a subsequence of u_h converging with respect to the $\mathrm{L}^p(\Omega;\mathbb{R}^N)$ metric.*

Theorem 1.7 can be stated also: "the sets $\{u\in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)\ :\ \|u\|_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)}\leq C\}$ (C any constant) are $\mathrm{L}^p(\Omega;\mathbb{R}^N)$-compact".

By Theorems 1.6 and 1.7 we obtain that the whole family of functionals $(\mathcal{F}_\varepsilon)$ is $\mathrm{L}^p(\Omega;\mathbb{R}^N)$-equicoercive on $\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$: it is sufficient to set $c_2=C_1|\Omega|\geq \int_\Omega f(\frac{x}{\varepsilon},0)\,dx$, and to notice that the set

$$E=\{u\in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)\ :\ \mathcal{F}_\varepsilon(u)\leq c_2\}$$

is not empty (the constant 0 belongs to E), and by (1.20) is contained in the set

$$K=\{u\in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)\ :\ \|u\|_{\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)}\leq (1+C')^{1/p}(c_1+c_2)^{1/p}\},$$

which is $\mathrm{L}^p(\Omega;\mathbb{R}^N)$-compact (by Theorem 1.7). In fact by (1.20) and Theorems 1.6, if $u\in E$, then

$$\int_\Omega (|u|^p+|Du|^p)\,dx\leq (1+C')\int_\Omega |Du|^p\,dx\leq (1+C')(\mathcal{F}_\varepsilon(u)+c_1)\leq (1+C')(c_1+c_2).$$

With the same kind of computations we obtain that for each fixed $g\in \mathrm{L}^{p'}(\Omega;\mathbb{R}^N)$ the family of functionals $\mathcal{F}_\varepsilon(u)+\int_\Omega gu\,dx$ is equicoercive on $\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$.

We are led then to consider $X=\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$, and d the restriction of the $\mathrm{L}^p(\Omega;\mathbb{R}^N)$-distance to $\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$.

In order to describe the limit of the problems in (1.2) it is sufficient to consider all limits of problems related to sequences (ε_h) with $\varepsilon_h \to 0$ as $h \to \infty$. Moreover by Remark 1.3(4), since the functionals

$$u \mapsto \int_\Omega ug\,dx \tag{1.22}$$

are continuous (we suppose $g \in \mathrm{L}^{p'}(\Omega;\mathbb{R}^N)$), we can neglect this integral. Hence we have to study the $\Gamma(\mathrm{L}^p(\Omega;\mathbb{R}^N))$-convergence of the functionals

$$F_h(u) = \mathcal{F}^0_{\varepsilon_h}(u) = \begin{cases} \displaystyle\int_\Omega f(\frac{x}{\varepsilon}, Du)\,dx & \text{if } u \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N) \\ +\infty & \text{if } u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N) \setminus \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N). \end{cases} \tag{1.23}$$

We have preferred to define our functionals by (1.23) on the whole $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$, and to deal with the boundary conditions setting the functional to $+\infty$ outside $\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)$ since this is a good illustration of a common procedure for dealing with constraints.

The Γ-convergence of $\mathcal{F}^0_{\varepsilon_h}$ will be deduced from the Γ-convergence of the functionals

$$\mathcal{F}_{\varepsilon_h}(u) = \int_\Omega f(\frac{x}{\varepsilon}, Du)\,dx \qquad \text{if } u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N), \tag{1.24}$$

showing that the boundary condition $u = 0$ on $\partial\Omega$ does not affect the form of the Γ-limit (see Lesson Two).

Exercises

Prove the statements 1)–7) of Remark 1.3 by using the definition of Γ-limit.

1.4. The Localization Method of Γ-convergence. Compactness

The proof of the Γ-convergence of the functionals in (1.1) will follow this line:

(i) prove a *compactness theorem* which allows to obtain from each sequence $(\mathcal{F}_{\varepsilon_h})$ a subsequence Γ-converging to an abstract limit functional;

(ii) prove an *integral representation result*, which allows us to write the limit functional as an integral;

(iii) prove a *representation formula* for the limit integrand which does not depend on the subsequence, showing thus that the limit is well-defined.

The third point is characteristic of homogenization and will be performed in Lesson Three by exploiting the special form of the functionals under examination. Steps (i) and (ii) follow from general theorems in Γ-convergence (see the books by Dal Maso [15] and Buttazzo [13]); here we give briefly an idea of the methods involved in the proof, without entering into details.

Let us fix a sequence of Borel functions $f_h : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0,+\infty]$ satisfying the growth condition

$$|\xi|^p - c_1 \le f_h(x,\xi) \le C_1(1+|\xi|^p) \tag{1.25}$$

(in our case we will have $f_h(x,\xi) = f(\dfrac{x}{\varepsilon_h}, \xi)$, where (ε_h) is a fixed sequence converging to 0), and let us consider the functionals

$$F_h(u) = \int_\Omega f_h(x, Du)\,dx \tag{1.26}$$

defined for $u \in W^{1,p}(\Omega; \mathbb{R}^N)$. We outline the proof of a compactness and integral representation theorem for the sequence (F_h).

Let us first notice that it is easy to obtain, by a diagonal procedure, a compactness theorem for the functionals F_h since the topology of $L^p(\Omega; \mathbb{R}^N)$ has a countable base (see Dal Maso [15] Theorem 8.5). However, the limit functional thus obtained depends *a priori* heavily on the choice of Ω, and it is not possible to obtain directly an integral representation of it. To overcome this difficulty it was introduced a *localization method* characteristic of Γ-convergence. Instead of considering the functionals in (1.26) for a fixed Ω bounded open subset of $\mathbb{R}^n$, we consider

$$F_h(u, A) = \int_A f_h(x, Du)\, dx \tag{1.27}$$

as a function of the pair (u, A) where $A \in \mathcal{A}_n$ (the family of bounded open subsets of $\mathbb{R}^n$) and $u \in W^{1,p}(A; \mathbb{R}^N)$ (this is sometimes called a *variational functional*).

We can now fix a countable dense family $\mathcal{Q}$ of $\mathcal{A}_n$[1] (for example all poly-rectangles with rational vertices), and, again using a diagonal procedure, find an increasing sequence of integers (h_k) such that we have the existence of the Γ-limit

$$F(u, A) = \Gamma(L^p(A; \mathbb{R}^N))\text{-}\lim_k F_{h_k}(u, A) \tag{1.28}$$

for all $A \in \mathcal{Q}$ and $u \in W^{1,p}(A; \mathbb{R}^N)$.

Beside this limit we can consider the upper and lower Γ-limits

$$F^+(u, A) = \Gamma(L^p(A; \mathbb{R}^N))\text{-}\limsup_k F_{h_k}(u, A) \tag{1.29}$$

$$F^-(u, A) = \Gamma(L^p(A; \mathbb{R}^N))\text{-}\liminf_k F_{h_k}(u, A) \tag{1.30}$$

for all $A \in \mathcal{A}_n$ and $u \in W^{1,p}(A; \mathbb{R}^N)$, so that we have

$$F^+(\cdot, A) = F^-(\cdot, A) = F(\cdot, A) \tag{1.31}$$

for all $A \in \mathcal{Q}$.

The next step (which is rather technical, and relies on the growth conditions (1.25) on f; see Section 1.5) is to notice that the increasing set functions $A \mapsto F^+(u, A)$ and $A \mapsto F^-(u, A)$ are *inner-regular*; that is,

$$F^\pm(u, A) = \sup\Big\{F^\pm(u, A') \ : A' \in \mathcal{A}_n,\ \overline{A'} \subset A\Big\} \tag{1.32}$$

for all $A \in \mathcal{A}_n$ and $u \in W^{1,p}(A, \mathbb{R}^N)$.

At this point it suffices to notice that the supremum in (1.32) can be taken for $A' \in \mathcal{Q}$, and to recall (1.31), to obtain

$$F^+(u, A) = \sup\Big\{F(u, A') \ : A' \in \mathcal{Q},\ \overline{A'} \subset A\Big\} = F^-(u, A), \tag{1.33}$$

and then the existence of the Γ-limit in (1.28) for all $A \in \mathcal{A}_n$ and $u \in W^{1,p}(A; \mathbb{R}^N)$.

We have thus obtained a converging subsequence on all $A \in \mathcal{A}_n$.

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[1] We say that $\mathcal{Q}$ is dense in $\mathcal{A}_n$ if for every $A, A' \in \mathcal{A}_n$ with $\overline{A'} \subset A$ there exists $Q \in \mathcal{Q}$ such that $A' \subset Q \subset A$

Theorem 1.8. (Compactness) *Let F_h be defined as in* (1.27)*, with f_h satisfying* (1.25)*; then there exists an increasing sequence of integers (h_k) such that the limit*

$$F(u,A) = \Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\lim_k F_{h_k}(u,A) \tag{1.34}$$

exists for all $A \in \mathcal{A}_n$ and $u \in \mathrm{W}^{1,p}(A;\mathbb{R}^N)$.

It can be proven that, as a set function, the limit F behaves in a very nice way. In fact we have:

(a) (measure property) *for every $\Omega \in \mathcal{A}_n$ and $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ the set function $A \mapsto F(u,A)$ is the restriction to $\mathcal{A}_n(\Omega)$ (the family of all open subsets of Ω) of a regular Borel measure.*

The variational functional F enjoys other properties, which derive from the structure of the Γ-limit:

(b) (semicontinuity) *for every $A \in \mathcal{A}_n$ the functional $F(\cdot,A)$ is $\mathrm{L}^p(A;\mathbb{R}^N)$-lower semicontinuous* (by the lower semicontinuity properties of Γ-limits);

(c) (growth conditions) *we have the inequality*

$$\int_A |Du|^p\,dx - c_1|A| \le F(u,A) \le C_1\Big(|A| + \int_A |Du|^p\,dx\Big)$$

for every $A \in \mathcal{A}_n$ and $u \in \mathrm{W}^{1,p}(A;\mathbb{R}^N)$ (by the growth condition (1.25));

(d) (locality) *if $u = v$ a.e. on $A \in \mathcal{A}_n$, then $F(u,A) = F(v,A)$;*

(e) ("translation invariance") *if $z \in \mathbb{R}^n$ then $F(u,A) = F(u+z,A)$.*

The proofs of the two last statements are trivial since the operation of Γ-limit is local and all functionals F_h are translation invariant.

These properties assure us that it is possible to represent the functional F as an integral.

Theorem 1.9. (Integral Representation Theorem (Buttazzo & Dal Maso; see [13] Chapter 4.3 and [15] Chapter 20)) *If F is a variational integral satisfying* (a)–(e)*, then there exists a Carathéodory integrand $\varphi : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0,+\infty[$ satisfying*

$$|\xi|^p - c_1 \le \varphi(x,\xi) \le C_1(1+|\xi|^p) \qquad \textit{(growth condition)}$$

and

$$|A|\varphi(x,\xi) \le \int_A \varphi(x,\xi + Du(y))\,dy \qquad \textit{(quasiconvexity)}$$

for all $A \in \mathcal{A}_n$, $x \in \mathbb{R}^n$, $\xi \in \mathrm{M}^{n\times N}$, and $u \in \mathrm{W}^{1,p}_0(A,\mathbb{R}^N)$, such that

$$F(u,A) = \int_A \varphi(x,Du)\,dx \tag{1.35}$$

for all $A \in \mathcal{A}_n$ and $u \in \mathrm{W}^{1,p}(A;\mathbb{R}^N)$.

Remark 1.10. Let us recall that quasiconvexity is a well-known necessary and sufficient condition for the L^p-lower semicontinuity of functionals of the form (1.26) with integrands satisfying (1.25) (see Acerbi & Fusco [2]). Convex functions are quasiconvex; the two notions coincide only in the case $n = 1$ or $N = 1$. Examples of quasiconvex non convex functions are *polyconvex* functions: we say that $f : \mathrm{M}^{n\times N} \to \mathbb{R}$ is polyconvex if $f(\xi)$ is a convex function of the vector of all minors of the matrix ξ. In the case $n = N = 2$ this means that $f(\xi) = g(\xi, \det\xi)$, with g convex.

Proof of Theorem 1.9. We will just give an idea of the proof. First of all one can obtain a representation for $F(u,A)$ when $u = \xi x$ is linear (or affine, which is the same because of the translation invariance): since

$F(\xi x, \cdot)$ is a measure (absolutely continuous with respect to the Lebesgue measure), then, by Riesz Theorem, there exists a function g_ξ such that

$$F(\xi x, A) = \int_A g_\xi(x)\,dx$$

for all $A \in \mathcal{A}_n$.

Let us define then $\varphi(x,\xi) = g_\xi(x)$. If u is piecewise affine then we obtain immediately (1.35) since $F(\xi x, \cdot)$ is a measure. If u is general, then the inequality

$$F(u, A) \le \int_A \varphi(x, Du)\,dx$$

follows by approximating u with piecewise affine functions in the $\mathrm{W}^{1,p}$ metric, and then using the lower semicontinuity of F (on the left hand side), and Lebesgue Theorem (on the right hand side).

Fixed u we can define $G(v,A) = F(u+v, A)$. This variational functional still satisfies the hypotheses (a)–(e). Hence we can construct as above a function ψ such that $G(v,A) = \int_A \psi(x, Dv)\,dx$ for v piecewise affine, and

$$G(v, A) \le \int_A \psi(x, Du)\,dx$$

for general v. We obtain then (if u_h is a sequence of piecewise affine functions converging strongly in $\mathrm{W}^{1,p}(A;\mathbb{R}^N)$ to u)

$$\int_A \psi(x,0)\,dx = G(0,A) = F(u,A) \le \int_A \varphi(x,Du)\,dx = \lim_h \int_A \varphi(x, Du_h)\,dx$$

$$= \lim_h F(u_h, A) = \lim_h G(u_h - u, A) \le \lim_h \int_A \psi(x, Du_h - Du)\,dx = \int_A \psi(x,0)\,dx,$$

so that all inequalities are indeed equalities and we get (1.35).

The quasiconvexity of φ follows by the theorem of Acerbi & Fusco. □

We can apply all the machinery above to our functionals. Hence for every fixed sequence (ε_h) there exist a subsequence (ε_{h_k}) and a Carathéodory quasiconvex function φ such that the limit

$$\Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\lim_k \int_A f\Big(\frac{x}{\varepsilon_{h_k}}, Du\Big)\,dx = \int_A \varphi(x, Du)\,dx \tag{1.36}$$

exists for all $A \in \mathcal{A}_n$ and $u \in \mathrm{W}^{1,p}(A, \mathbb{R}^N)$.

1.5. The Fundamental Estimate. Boundary Value Problems

As we have already remarked, the very crucial point in the compactness procedure for integral functionals, described in Section 1.4 is the proof of the properties of the Γ-limit as a set function, namely that it is (the restriction to the family of bounded open sets of) a inner-regular measure. For example, it must be proven the *subadditivity* of $F(u,\cdot)$; that is, for all pairs of sets $A, B \in \mathcal{A}_n$ and $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ we must have

$$F(u, A\cup B) \le F(u,A) + F(u,B).$$

Recalling the definition of Γ-limit, this means that from the knowledge of the "minimizing sequences" for $F(u,A)$ and $F(u,B)$ we must somehow obtain an estimate for $F(u, A\cup B)$. This is done by elaborating a method for "joining" sequences of functions, without increasing in the limit the value of the corresponding integrals. This procedure is not possible in general for arbitrary integral functionals, and indeed there are examples of Γ-limits which are not measures (as set functions). Anyhow, for functionals whose integrands satisfy (1.25) the possibility of inexpensive joints was shown by De Giorgi in [17]; his method was later generalized in many papers (see [16], [15] and the references therein), and remains one of the cornerstones of the theory. A general formulation of this property can be found in [15] Definition 18.2.

Lemma 1.11. (Fundamental Estimate) *Let F_h be the functionals in* (1.25), (1.26). *Then, for every $\eta > 0$, and for every $A, A', B \in \mathcal{A}_n$ with $\overline{A'} \subset A$ there exists a constant $M > 0$ with the property: for every $h \in \mathbb{N}$, for every $w \in W^{1,p}(A;\mathbb{R}^N)$, $v \in W^{1,p}(B;\mathbb{R}^N)$ there exists a cut-off function*[2] *ϕ between A' and A such that*

$$(1.37) \qquad F_h(\phi w + (1-\phi)v, A' \cup B) \leq (1+\eta)\Big(F_h(w,A) + F_h(v,B)\Big) + M\int_{A\cap B} |w-v|^p\,dx.$$

Note that ϕ depends on h, v, and w.

With this property in mind it is not difficult to prove the inner regularity of $F^{\pm}$, and hence that F is a measure (for useful criteria which give conditions on an increasing set function equivalent to being a measure we refer to De Giorgi and Letta [18]). We are not going to prove these consequences, nor Lemma 1.11 (for a proof see [15] Section 19, and also the paper by Fusco [23] where the vector-valued case is dealt with in detail). Let us remark instead how this property allows us also to deduce the Γ-convergence of functionals defined taking into account (homogeneous) Dirichlet boundary conditions.

Lemma 1.12. (Γ-limits and Boundary Conditions) *Let (F_{h_k}) be the converging subsequence of (F_h) given by Theorem 1.8. If we set*

$$(1.38) \qquad F^0_h(u,A) = \begin{cases} \displaystyle\int_A f_h(x,Du)\,dx & \text{if } u \in W^{1,p}_0(A;\mathbb{R}^N) \\ +\infty & \text{elsewhere on } W^{1,p}(A;\mathbb{R}^N), \end{cases}$$

then we have for all $A \in \mathcal{A}_n$ and $u \in W^{1,p}(A,\mathbb{R}^N)$

$$(1.39) \qquad \Gamma(L^p(A;\mathbb{R}^N))\text{-}\lim_k F^0_{h_k}(u,A) = F^0(u,A),$$

where

$$(1.40) \qquad F^0(u,A) = \begin{cases} \displaystyle\int_A \varphi(x,Du)\,dx & \text{if } u \in W^{1,p}_0(A;\mathbb{R}^N) \\ +\infty & \text{elsewhere on } W^{1,p}(A;\mathbb{R}^N), \end{cases}$$

and φ is given by Theorem 1.9.

Proof. Let us apply the definition of Γ-convergence. Let us consider a converging sequence $u_k \to u$ in $L^p(A;\mathbb{R}^N)$. If $u \notin W^{1,p}_0(A;\mathbb{R}^N)$ then we must have $F_{h_k}(u_k,A) \to +\infty$; otherwise (by the growth conditions (1.25)) (u_k) would be a bounded sequence in $W^{1,p}_0(A;\mathbb{R}^N)$, so that (a subsequence of it) converges weakly in $W^{1,p}_0(A;\mathbb{R}^N)$ to u, obtaining thus a contradiction. Hence $F^0(u,A) = +\infty$. If $u \in W^{1,p}_0(A;\mathbb{R}^N)$ we have trivially

$$F(u,A) \leq \liminf_k F_{h_k}(u_k,A) \leq \liminf_k F^0_{h_k}(u_k,A)$$

for all $u_k \to u$; that is,

$$(1.41) \qquad \Gamma(L^p(A;\mathbb{R}^N))\text{-}\liminf_k F^0_{h_k}(u,A) \geq F^0(u,A).$$

Vice versa, let $u_k \to u$ be such that $F(u,A) = \lim_k F_{h_k}(u_k,A)$. Let us fix a compact subset K of A, $A' \subset A$, $\eta > 0$, choose in Lemma 1.11 $B = A \setminus K$, $w = u_k$, $v = u$, and define $v_k = \phi u_k + (1-\phi)u \in W^{1,p}_0(A;\mathbb{R}^N)$, where ϕ is given by Lemma 1.11. We have then $v_k \to u$, and

$$F^0_{h_k}(v_k,A) = F_{h_k}(v_k,A) \leq (1+\eta)\Big(F_{h_k}(u_k,A) + F_{h_k}(u,A\setminus K)\Big) + M\int_{A\setminus K} |u_k - u|^p\,dx.$$

---

[2] We say that ϕ is a *cut-off function* between A' and A if $\phi \in C^\infty_0(A)$ and $\phi = 1$ on a neighbourhood of $\overline{A'}$.

Letting $k \to +\infty$, and recalling (1.25), we obtain

$$\limsup_k F^0_{h_k}(v_k, A) \le (1+\eta)F(u,A) + (1+\eta)\int_{A\setminus K} C_1(1+|Du|^p)\,dx,$$

hence by the arbitrariness of K, and letting $\eta \to 0$,

$$\Gamma(\mathrm{L}^p(A;\mathbb{R}^N))\text{-}\limsup_k F^0_{h_k}(u,A) \le F^0(u,A). \tag{1.42}$$

This inequality completes the proof. □

Exercises

1. State and prove the analog of Lemma 1.12 for the boundary condition $u = \phi$ on γ_0, under appropriate assumptions on the data.
2. Prove (1.32) using (1.37).
3. Prove that the Dirichlet integral $\int_A |Du|^2\,dx$ satisfies the fundamental estimate.

Section Two. Homogenization Formulas

2.1. The Asymptotic Homogenization Formula

We have reduced the problem of Γ-convergence of the functionals $\mathcal{F}_\varepsilon$ to the description of the function φ in (1.36). In order to deduce the convergence of the whole sequence it is sufficient now to prove that φ is independent of the sequence (ε_{h_k}). This will be done by proving an asymptotic formula for φ.

We shall make a weaker assumption on f than periodicity, namely a sort of uniform *almost periodicity* (see the book by Besicovitch [5] for a study of different types of almost periodic functions). The motivation for the introduction of this kind of hypothesis lies in its greater flexibility compared to mere periodicity:

(a) sum and product of almost periodic functions are almost periodic (this happens for periodic functions only if they have a common period; think of $\sin x + \sin(\pi x)$);

(b) restriction of an almost periodic function to an affine subspace is still almost periodic (this is not true for periodic functions; think as above of the function $\sin x + \sin y$ restricted to the line $y = \pi x$);

(c) almost periodic functions are "stable under small perturbations" (this concept will be explained and studied later).

Moreover, the techniques are essentially of the same type as in the periodic case, so that we get a stronger result for free.

Let us recall that a continuous function $a : \mathbb{R}^n \to \mathbb{R}$ is *uniformly almost periodic* if the following property holds: for every $\eta > 0$ there exists an *inclusion length* $L_\eta > 0$ and a set $T_\eta \subset \mathbb{R}^n$ (which will be called the set of η-*almost periods* for a) such that $T_\eta + [0, L_\eta]^n = \mathbb{R}^n$, and if $\tau \in T_\eta$ we have

$$|a(x+\tau) - a(x)| \le \eta \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.1}$$

Of course if a is periodic then we can take for all η the lattice of all periods of a as $T = T_\eta$, and $L = L_\eta$ equal to the mesh size of the lattice. Particular uniformly almost periodic functions are *quasiperiodic* functions;

that is, functions of the form $a(x) = b(x, \dots, x)$, where b is a continuous periodic function of a higher number of variables. The set of uniformly almost periodic functions can be seen also as the closure of all trigonometric polynomials in the uniform norm.

We can model our hypotheses on functionals of the form

$$\int_\Omega a(\frac{x}{\varepsilon})|Du|^p\,dx, \tag{2.2}$$

with the coefficient a uniformly almost periodic. We say then that a Borel function $f : \mathbb{R}^n \times \mathbb{M}^{n\times N} \to [0, +\infty]$ is *p-almost periodic* (see [7]) if for every $\eta > 0$ there exists $L_\eta > 0$ and a set $T_\eta \subset \mathbb{R}^n$ such that $T_\eta + [0, L_\eta]^n = \mathbb{R}^n$, and if $\tau \in T_\eta$ we have

$$|f(x+\tau,\xi) - f(x,\xi)| \le \eta(1+|\xi|^p) \qquad \text{for all } x \in \mathbb{R}^n. \tag{2.3}$$

Notice that we do not require any continuity of f since it will not be necessary in the proofs; hence all Borel functions $f = f(x, \xi)$ periodic in x (with period independent of ξ) satisfy the hypothesis of p-almost periodicity.

The first result we will obtain by exploiting the almost periodicity of f will be the "homogeneity" of the function φ.

Proposition 2.1. *Let us suppose that f be p-almost periodic and satisfy the growth condition* (1.25). *Then the function $\varphi = \varphi(x, \xi)$ in* (1.36) *can be chosen independent of x.*

Proof. (Let us remark that we follow the line of the proof of the corresponding statement in the periodic case by Marcellini [26]; see also [7] Proposition 5.1) Let us fix $x_0, y_0 \in \mathbb{R}^n$, $r > 0$, $K \in \mathbb{N}$, and $\eta > 0$. Let $B = B(x_0, r)$, $B_K = B(x_0, r(1-1/K))$, and (τ_k) be a sequence of points of T_η such that $\lim_k \varepsilon_{h_k}\tau_k = y_0 - x_0$. Let (u_k) be a sequence in $W^{1,p}(B;\mathbb{R}^N)$ with $u_k \to 0$ and

$$\int_B \varphi(x,\xi)\,dx = \lim_k \int_B f(\frac{x}{\varepsilon_{h_k}}, Du_k + \xi)\,dx. \tag{2.4}$$

Let us set $y_k = x_0 + \varepsilon_{h_k}\tau_k$; if k is large enough we have $y_0 + B_K \subset y_k + B$. We have then (using (2.3) and the definition of Γ-limit)

$$\int_B \varphi(x,\xi) \ge \liminf_k \int_B f(\frac{x}{\varepsilon_{h_k}} + \tau_k, Du_k+\xi)\,dx - \eta\limsup_k \int_B (1+|Du_k+\xi|^p)\,dx$$

$$= \liminf_k \int_{y_k+B} f(\frac{y}{\varepsilon_{h_k}}, Du_k(y-y_k)+\xi)\,dy - \eta c$$

$$\ge \liminf_k \int_{y_0+B_K} f(\frac{y}{\varepsilon_{h_k}}, Du_k(y-y_k)+\xi)\,dy - \eta c \ge \int_{y_0+B_K} \varphi(x,\xi)\,dx - \eta c$$

(c a constant depending on (u_k)). By the arbitrariness of η and K we have

$$\int_B \varphi(x,\xi)\,dx \ge \int_{y_0+B} \varphi(y,\xi)\,dy = \int_B \varphi(x+y_0,\xi)\,dx, \tag{2.5}$$

and then by symmetry the equality

$$\int_B \varphi(x,\xi) = \int_B \varphi(x+y_0,\xi)\,dx. \tag{2.6}$$

It is easy to see that from this equality we can conclude the proof. □

The independence from the space variable is essential for expressing the value $\varphi(\xi)$ as the solution of a minimum problem. In fact by the quasiconvexity of φ we have

$$|\Omega|\varphi(\xi) = \min\Big\{\int_\Omega \varphi(\xi + Du(y))\,dy \ : \ u \in W_0^{1,p}(\Omega;\mathbb{R}^N)\Big\}$$

for every $\Omega \in \mathcal{A}_n$; in particular we can choose $\Omega =]0,1[^n$ so that

$$\begin{aligned}\varphi(\xi) &= \min\Big\{\int_{]0,1[^n} \varphi(\xi + Du(y))\,dy \ : \ u \in W_0^{1,p}(]0,1[^n;\mathbb{R}^N)\Big\}\\ &= \min\Big\{F^0(u+\xi x,]0,1[^n) \ : \ u \in W^{1,p}(]0,1[^n;\mathbb{R}^N)\Big\}.\end{aligned} \tag{2.7}$$

We can use now the Γ-convergence of $F^0_{h_k}$ to F^0 (Lemma 1.12), the equicoercivity of these functionals (Section 1.3), and the Fundamental Theorem of Γ-convergence (Theorem 1.4), to obtain

$$\varphi(\xi) = \lim_k \inf\Big\{\int_{]0,1[^n} f(\frac{y}{\varepsilon_{h_k}}, Du(y)+\xi)\,dy : \ u \in W_0^{1,p}(]0,1[^n;\mathbb{R}^N)\Big\}. \tag{2.8}$$

At this point is is clear that our next step must be the proof of the independence of this limit of the sequence (ε_{h_k}).

Proposition 2.2. (Asymptotic Homogenization Formula) *Let f be as above. Then the limit*

$$f_{\rm hom}(\xi) = \lim_{t\to+\infty} \frac{1}{t^n} \inf\Big\{\int_{]0,t[^n} f(x, Du(x)+\xi)\,dx : \ u \in W_0^{1,p}(]0,t[^n;\mathbb{R}^N)\Big\} \tag{2.9}$$

exists for every $\xi \in M^{n\times N}$.

Proof. (Let us remark that we follow the line of the proof of the corresponding statement in the periodic case in [6]; see also [7]) Let us fix $\xi \in M^{n\times N}$ and define for $t > 0$

$$g_t = \frac{1}{t^n} \inf\Big\{\int_{]0,t[^n} f(x, Du(x)+\xi)\,dx : \ u \in W_0^{1,p}(]0,t[^n;\mathbb{R}^N)\Big\}; \tag{2.10}$$

moreover let $u_t \in W_0^{1,p}(]0,t[^n;\mathbb{R}^N)$ satisfy

$$\frac{1}{t^n}\int_{]0,t[^n} f(x, Du_t(x)+\xi)\,dx \le g_t + \frac{1}{t}. \tag{2.11}$$

Let $\eta > 0$. If $s \ge t + L_\eta$ (the inclusion length related to η and f) we can construct $u_s \in W_0^{1,p}(]0,s[^n;\mathbb{R}^N)$ as follows: for every $\mathbf{i} = (i_1,\dots,i_n) \in \mathbf{Z}^n$ with $0 \le (t+L_\eta)i_j \le s$ for all $j = 1,\dots,n$, we choose $\tau_{\mathbf{i}} \in T_\eta$ with $\tau_{\mathbf{i}} \in (t+L_\eta)\mathbf{i} + [0,L_\eta]^n$, and we define

$$u_s(x) = \begin{cases} u_t(x-\tau_{\mathbf{i}}) & \text{if } x \in \tau_{\mathbf{i}} + [0,t]^n\\ 0 & \text{otherwise.}\end{cases} \tag{2.12}$$

Let us also define $Q_s =]0,s[^n \setminus \bigcup_{\mathbf{i}} (\tau_{\mathbf{i}} + [0,t]^n)$; we have $|Q_s| \le s^n - (s-t-L_\eta)^n \Big(\dfrac{t}{t+L_\eta}\Big)^n$.

We can estimate g_s by using u_s:

$$g_s \le \frac{1}{s^n}\int_{]0,s[^n} f(x, Du_s(x)+\xi)\,dx$$

$$= \frac{1}{s^n}\Big(\sum_{\mathbf{i}}\int_{\tau_i+[0,t]^n} f(x, Du_t(x-\tau_{\mathbf{i}})+\xi)\,dx + \int_{Q_s} f(x,\xi)\,dx\Big)$$

$$\le \frac{1}{s^n}\Big(\sum_{\mathbf{i}}\int_{[0,t]^n} f(y+\tau_{\mathbf{i}}, Du_t+\xi)\,dy + |Q_s|C_1(1+|\xi|^p)\Big) \tag{2.13}$$

$$\le \frac{1}{s^n}\Big(\sum_{\mathbf{i}}\int_{[0,t]^n}\Big(f(y, Du_t+\xi)+\eta(1+|Du_t+\xi|^p)\Big)\,dy + |Q_s|C_1(1+|\xi|^p)\Big)$$

$$\le (1+\eta)\frac{1}{(t+L_\eta)^n}t^n\Big(g_t+\frac{1}{t}\Big)+\eta+\Big(1-\Big(\frac{s-t-L_\eta}{s}\Big)^n\Big(\frac{t}{t+L_\eta}\Big)^n\Big)C_1(1+|\xi|^p).$$

Taking the limit first in s and then in t we get

$$\limsup_{s\to+\infty} g_s \le (1+\eta)\liminf_{t\to+\infty} g_t + \eta.$$

By the arbitrariness of η we conclude the proof. $\square$

Note that our growth hypotheses guarantee by a density argument that the infima in (2.9) can be computed on smooth functions; hence we can write also

$$f_{\rm hom}(\xi) = \lim_{t\to+\infty}\frac{1}{t^n}\inf\Big\{\int_{]0,t[^n} f(x, Du(x)+\xi)\,dx :\ u\in C_0^\infty(]0,t[^n;\mathbb{R}^N)\Big\}. \tag{2.14}$$

exists for every $\xi\in \mathrm{M}^{n\times N}$.

We can conclude now the proof of our homogenization result by simply remarking that the limit in (2.8) can be transformed in the form (2.9) by the change of variables $y=\varepsilon_{h_k}x$ (when $t=1/\varepsilon_{h_k}$), so that $\varphi(\xi)=f_{\rm hom}(\xi)$ is independent of (ε_{h_k}).

Remark 2.3. By an use of the Fundamental Estimate as in the proof of Lemma 1.13 it is easy to see that an equivalent formula for $f_{\rm hom}$ is the following:

$$f_{\rm hom}(\xi) = \inf_{k\in\mathbb{N}}\frac{1}{k^n|Q|}\inf\Big\{\int_{kQ} f(x, Du(x)+\xi)\,dx :\ u\in \mathrm{W}^{1,p}_{\#}(kQ;\mathbb{R}^N)\Big\}, \tag{2.15}$$

where Q is any non-degenerate open parallelogram in $\mathbb{R}^n$, and $\mathrm{W}^{1,p}_{\#}(kQ;\mathbb{R}^N)$ denotes the space of functions in $\mathrm{W}^{1,p}_{\rm loc}(\mathbb{R}^n;\mathbb{R}^N)$ which are Q-periodic. This formula may be useful in the case of f periodic in x with period Q.

Remark 2.4. We will see in the next section that a simpler formula, which involves a single minimization problem on the periodicity cell, can be obtained in the *convex and periodic* case. It is important to note that in the (vector-valued) non convex case formula (2.9) *cannot be simplified*, as shown by a counterexample by S.Müller [27]: in a sense homogenization problems in the vector-valued case have an almost periodic nature even if the integrand is periodic.

2.2. The Convex and Periodic Case

In this section we will suppose in addition to the previous hypotheses that for a.e. $x \in \mathbb{R}^n$ the function $f(x,\cdot)$ is *convex* on $\mathrm{M}^{n\times N}$. This is no restriction in the scalar case $N=1$ since it can be seen that in this case an equivalent convex integrand to f (that is, giving the same infima) may be constructed by "convexification" (see Ekeland & Temam [21]). Moreover, we suppose that f is 1-*periodic* in the first variable:

$$f(x+e_i,\xi)=f(x,\xi) \qquad \text{for all } x\in\mathbb{R}^n, \xi\in \mathrm{M}^{n\times N}, i=1,\dots,n, \tag{2.16}$$

where $\{e_1,\dots,e_n\}$ denotes the canonical base of $\mathbb{R}^n$ (every periodic function can be reduced to this case by a change of variables).

We can choose $Q=]0,1[^n$ in (2.15) to obtain the formula

$$f_{\mathrm{hom}}(\xi)=\inf_{k\in\mathbb{N}}\frac{1}{k^n}\inf\Big\{\int_{]0,k[^n} f(x,Du(x)+\xi)\,dx:\ u\in \mathrm{W}^{1,p}_{\#}(]0,k[^n;\mathbb{R}^N)\Big\}. \tag{2.17}$$

If we define the function $f_{\#}:\mathrm{M}^{n\times N}\to[0,+\infty[$ by setting

$$f_{\#}(\xi)=\inf\Big\{\int_{]0,1[^n} f(x,Du(x)+\xi)\,dx:\ u\in \mathrm{W}^{1,p}_{\#}(]0,1[^n;\mathbb{R}^N)\Big\} \tag{2.18}$$

we have obviously

$$f_{\mathrm{hom}}(\xi)\le f_{\#}(\xi). \tag{2.19}$$

Thanks to the convexity of f we can reverse this inequality and obtain the following result.

Proposition 2.5. (Convex Homogenization Formula) *Let f be convex and periodic as above. Then we have $f_{\mathrm{hom}}(\xi)=f_{\#}(\xi)$ for all $\xi\in \mathrm{M}^{n\times N}$.*

Proof. Let u_k^ξ be a solution to the minimum problem

$$\frac{1}{k^n}\inf\Big\{\int_{]0,k[^n} f(x,Du(x)+\xi)\,dx:\ u\in \mathrm{W}^{1,p}_{\#}(]0,k[^n;\mathbb{R}^N)\Big\}=f^k_{\#}(\xi), \tag{2.20}$$

which exists by the coerciveness and lower semicontinuity of the functional $\mathcal{F}_1$ (see Remark 1.10). Let I_h be the set of $\mathbf{i}=(i_1,\dots,i_n)\in\mathbf{Z}^n$ with $0\le i_j<k$, and let us define

$$u^\xi(x)=\frac{1}{k^n}\sum_{\mathbf{i}\in I_h}u_k^\xi(x+\mathbf{i}) \tag{2.21}$$

a convex combination of the translated functions $u_h^\xi(\cdot+\mathbf{i})$. The function u^ξ is 1-periodic, and we have

$$\begin{aligned} f_{\#}(\xi) &\le \int_{]0,1[^n} f(x,Du^\xi(x)+\xi)\,dx=\frac{1}{k^n}\int_{]0,k[^n} f(x,Du^\xi(x)+\xi)\,dx \\ &\le \frac{1}{k^n}\sum_{\mathbf{i}\in I_h}\frac{1}{k^n}\int_{]0,k[^n} f(x,Du_k^\xi(x+\mathbf{i})+\xi)\,dx \\ &= \frac{1}{k^n}\sum_{\mathbf{i}\in I_h}\frac{1}{k^n}\int_{]0,k[^n} f(x,Du_k^\xi(x)+\xi)\,dx=f^k_{\#}(\xi). \end{aligned} \tag{2.22}$$

Since obviously we have $f_{\#}(\xi)=f^1_{\#}(\xi)\ge f^k_{\#}(\xi)$, by (2.22) and (2.17) we have $f_{\#}(\xi)=\inf_k f^k_{\#}(\xi)=f_{\mathrm{hom}}(\xi)$, and we can conclude the proof. □

Remark 2.6. Let us remark that in the convex and periodic case the homogenization formula and the Γ-convergence of the functionals $\mathcal{F}_\varepsilon$ can be proven under the weaker growth hypothesis

$$0 \le f(x,\xi) \le C_1(1+|\xi|^p) \tag{2.23}$$

(see [6] and [15]). Of course, no convergence of minima can be deduced in these hypotheses. The Γ-convergence of the functionals $\mathcal{F}_\varepsilon$ under only the growth hypothesis (2.23) in the general vector-valued case is to my knowledge still an open problem.

2.3. Stability of Homogenization

A natural requirement in the study of oscillating media seems to be the stability of the limit under small perturbations. For example we would like our results to remain unchanged if we add to f a function with compact support (we expect the overall properties of a medium to be maintained in the presence of an impurity in a very small and confined region).

Theorem 2.6. (Stability for Homogenization) *Let f be a homogenizable*[3] *quasiconvex Borel function, and let $\psi : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0,+\infty[$ be a quasiconvex Borel function. Let us suppose that both functions satisfy the growth condition* (1.25)*, and that we have for every $r>0$*

$$\limsup_{t\to+\infty} \frac{1}{t^n} \int_{]0,t[^n} \sup_{|\xi|\le r} |f(x,\xi)-\psi(x,\xi)|\,dx = 0. \tag{2.24}$$

Then also ψ is homogenizable and $\psi_{\rm hom} = f_{\rm hom}$.

Proof. Let us prove that for every $\xi \in \mathrm{M}^{n\times N}$ there exists $\psi_{\rm hom}(\xi) = f_{\rm hom}(\xi)$. Let $\varepsilon > 0$, and let us consider a solution u^ξ_ε to the minimum problem (which exists since by the quasiconvexity and growth conditions the integral functional is lower semicontinuous and coercive)

$$\min\Big\{ \int_{]0,1[^n} f\big(\frac{x}{\varepsilon}, Du(x)+\xi\big)\,dx : \ u \in \mathrm{W}^{1,p}_0(]0,1[^n;\mathbb{R}^N)\Big\} = f^\varepsilon_{\rm hom}(\xi). \tag{2.25}$$

Let us recall that $\lim_{\varepsilon\to 0} f^\varepsilon_{\rm hom}(\xi) = f_{\rm hom}(\xi)$.

We use a partial regularity result which tells us that the solutions to the minimum problems are bounded in some Sobolev space with exponent larger than p (in some sense they behave as if they were Lipschitz continuous).

Theorem 2.7. (Partial Regularity Theorem; Meyers & Elcrat [28]) *There exist $\eta > 0$ and a constant $C > 0$ such that we have*

$$\int_{]0,1[^n} |Du^\xi_\varepsilon + \xi|^{p+\eta}\,dx \le C \tag{2.26}$$

for every $\varepsilon > 0$.

Let us fix $r > 0$ and define

$$E_\varepsilon = \Big\{ x \in]0,1[^n : \ |Du^\xi_\varepsilon + \xi| > r \Big\}.$$

[3] We say in general that $f : \mathbb{R}^n \times \mathrm{M}^{n\times N} \to [0,+\infty[$ is homogenizable if the function $f_{\rm hom}$ gives the integrand of the Γ-limit in (1.36) for all converging sequences. Notice that in this theorem we do not make any hypotheses of periodicity or almost periodicity on f.

Clearly we have

$$|E_\varepsilon| r^p \le \int_{E_\varepsilon} |Du^\xi_\varepsilon + \xi|^p \, dx \le \int_{]0,1[^n} |Du^\xi_\varepsilon + \xi|^p \, dx$$

and by (1.25)

$$\int_{]0,1[^n} |Du^\xi_\varepsilon + \xi|^p \, dx \le C_1(1 + |\xi|^p) = C_\xi,$$

so that

$$|E_\varepsilon| \le r^{-p} C_1(1 + |\xi|^p) = r^{-p} C_\xi. \tag{2.27}$$

By using Hölder's inequality and (2.26) we get also

$$\begin{aligned} \int_{E_\varepsilon} |Du^\xi_\varepsilon + \xi|^p \, dx &\le |E_\varepsilon|^{\eta/(p+\eta)} \Big(\int_{E_\varepsilon} |Du^\xi_\varepsilon + \xi|^{p+\eta} \, dx \Big)^{p/(p+\eta)} \\ &\le r^{-p\eta/(p+\eta)} C_\xi^{\eta/(p+\eta)} C^{p/(p+\eta)} = C'_\xi r^{-p\eta/(p+\eta)} \end{aligned} \tag{2.28}$$

Using u^ξ_ε as a test function in the definition of

$$\min\Big\{ \int_{]0,1[^n} \psi(\frac{x}{\varepsilon}, Du(x) + \xi) \, dx : \; u \in \mathrm{W}^{1,p}_0(]0,1[^n; \mathrm{IR}^N) \Big\} = \psi^\varepsilon_{\hom}(\xi) \tag{2.29}$$

we have (using (1.25), (2.25), (2.27) and (2.28))

$$\begin{aligned} \psi^\varepsilon_{\hom}(\xi) &\le \int_{]0,1[^n} \psi(\frac{x}{\varepsilon}, Du^\xi_\varepsilon + \xi) \, dx \\ &= \int_{\{|Du^\xi_\varepsilon + \xi| \le r\}} \psi(\frac{x}{\varepsilon}, Du^\xi_\varepsilon + \xi) \, dx + \int_{E_\varepsilon} \psi(\frac{x}{\varepsilon}, Du^\xi_\varepsilon + \xi) \, dx \\ &\le \int_{\{|Du^\xi_\varepsilon + \xi| \le r\}} (\psi(\frac{x}{\varepsilon}, Du^\xi_\varepsilon + \xi) - f(\frac{x}{\varepsilon}, Du^\xi_\varepsilon + \xi)) \, dx \\ &\quad + \int_{]0,1[^n} f(\frac{x}{\varepsilon}, Du^\xi_\varepsilon + \xi) \, dx + \int_{E_\varepsilon} C_1(1 + |Du^\xi_\varepsilon + \xi|^p) \, dx \\ &\le \int_{]0,1[^n} \sup_{|z| \le r} |\psi(\frac{x}{\varepsilon}, z) - f(\frac{x}{\varepsilon}, z)| \, dx + f^\varepsilon_{\hom}(\xi) + C_1(r^{-p} C_\xi + r^{-p\eta/(p+\eta)} C'_\xi). \end{aligned} \tag{2.30}$$

We can pass to the limit first as $\varepsilon \to 0$, and then as $r \to +\infty$, recalling (2.24), obtaining

$$\limsup_{\varepsilon \to 0} \psi^\varepsilon_{\hom}(\xi) \le f_{\hom}(\xi);$$

since f and ψ play symmetric roles, we can interchange $\psi^\varepsilon_{\hom}(\xi)$ and $f^\varepsilon_{\hom}(\xi)$ in (2.30) so that we obtain

$$\liminf_{\varepsilon \to 0} \psi^\varepsilon_{\hom}(\xi) \ge f_{\hom}(\xi).$$

This proves the existence of $\psi_{\hom}(\xi) = \lim_{\varepsilon \to 0} \psi^\varepsilon_{\hom}(\xi) = f_{\hom}(\xi)$. The rest of the proof of Theorem 2.7 follows easily by using a compactness argument and showing that all converging subsequences can be represented by means of $\psi_{\hom}(\xi)$ (the only delicate point is the proof of the homogeneity of the limit integrand, that can be obtained by a similar argument as above; for details see [8] Section 3). □

Remark 2.8. (Stability by Compact Support Perturbation) If for every $r>0$ there exists $T_r>0$ such that $f(x,\xi)=\psi(x,\xi)$ for $|x|>T_r$ and $|\xi|\le r$ then 2.16 is satisfied; hence in this sense the homogenization is stable under compact support perturbations.

Remark 2.9. The hypothesis that ψ satisfies (1.25) is essential. In [8] Section 3 it can be found an example of a functions ψ not homogenizable (the Γ-liminf different from the Γ-limsup) which satisfies (2.24) with $f(x,\xi)=|\xi|^2$.

Remark 2.10. (A Stronger Homogenization Theorem) With the same type of arguments as in Theorem 2.6 we can prove a Closure Theorem for the Homogenization: *let f_h be a sequence of homogenizable Borel functions and let ψ be a Borel function. Let us suppose all these functions be quasiconvex, satisfy* (1.25), *and*

$$\lim_h \limsup_{t\to+\infty} \frac{1}{t^n}\int_{]0,t[^n} \sup_{|\xi|\le r} |f_h(x,\xi)-\psi(x,\xi)|\,dx = 0 \tag{2.31}$$

for all $r>0$. Then also ψ is homogenizable and $\psi_{\rm hom}=\lim_h f_{\rm hom}$.

Using this result and a suitable approximation procedure we can prove a stronger homogenization theorem under the only hypothesis of f satisfying (1.25) and $f(\cdot,\xi)$ being Besicovitch-almost periodic[4] (details in [8] Sections 3 and 4). The class $\mathcal{F}$ of these functions is stable under perturbations as in (2.24); that is, if $f\in\mathcal{F}$ and ψ satisfies (2.24), then $\psi\in\mathcal{F}$.

Exercises

Rewrite the proofs of Propositions 2.1 and 2.2 in the case of f periodic in x, using its periods instead of its almost periods.

Notation

$B(x,r)$ open ball of center x and radius r;

$\{e_1,\dots,e_n\}$ canonical base of $\mathbb{R}^n$: $e_1=(1,0,\dots,0)$, $e_2=(0,1,0,\dots,0),\dots$;

$|E|$ Lebesgue measure of the set E;

$\mathcal{A}_n$ family of all bounded open subsets of $\mathbb{R}^n$;

$\mathcal{A}_n(\Omega)$ family of all bounded open subsets of $\Omega\subset\mathbb{R}^n$;

$\mathrm{W}^{k,p}(\Omega;\mathbb{R}^N)$ Sobolev space of $\mathbb{R}^N$-vaued functions on Ω with p-summable weak derivatives up to the order k (if $N=1$ we write $\mathrm{W}^{k,p}(\Omega)$); $\mathrm{L}^p(\Omega;\mathbb{R}^N)=\mathrm{W}^{0,p}(\Omega;\mathbb{R}^N)$;

$\mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^N)=\mathrm{H}^{1,p}_0(\Omega;\mathbb{R}^N)$ closure in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^N)$ of compactly supported smooth functions;

p' conjugate exponent of p, *i.e.*, $\dfrac{1}{p}+\dfrac{1}{p'}=1$;

$\triangle u$ Laplacian of u.

[4] We say that f is *Besicovitch-almost periodic* if there exists a sequence of trigonometric polynomials (P_h) such that $\lim_h \limsup_{t\to+\infty} t^{-n}\int_{]0,t[^n} |f(x)-P_h(x)|\,dx=0$ (*i.e.*, $P_h\to f$ *in the mean*).

References

We just list the papers directly referred to in the lessons. For a complete reference list see [15] and the book A. Braides and A. Defranceschi *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

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