Lectures on Conformal Nets

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Part I

One Particle Structure

Chapter 1

The Möbius group

In this chapter we describe the structure of our basic symmetry group, the Möbius group, and of its positive energy representations.

1.1 Basic structure

The group $\mathrm{SL}(2,\mathbb{R})$ of 2×2 real matrices with determinant one acts on the compactified real line $\bar{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}$ line by linear fractional transformations: $g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts as

$$g : x \mapsto gx \equiv \frac{ax+b}{cx+d} \tag{1.1.1}$$

The kernel of this action is $\{\pm 1\}$. We identify $\bar{\mathbb{R}}$ with the circle $S^1 \equiv \{z \in \mathbb{C} : |z| = 1\}$ by the Cayley map

$$C : x \in \mathbb{R} \mapsto -\frac{x-i}{x+i} \in S^1 \ ,$$

inverse of the stereographic map $z \to -i(z-1)(z+1)^{-1}$, (setting $C(\infty) = -1$), so $\mathrm{PSL}(2,\mathbb{R}) \equiv \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ is identified with a group of diffeomorphisms of S^1, the *Möbius group* that is also denoted by $\mathbf{G}$ in these lectures.

Indeed, by the transformation C the group $\mathrm{SL}(2,\mathbb{R})$ can be identified with the group $SU(1,1)$ of complex 2×2 matrices

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1 \ ,$$

acting on S^1 is as

$$z \to \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$$

and so $\mathbf{G} \simeq PSU(1,1) \equiv SU(1,1)/\{\pm 1\}$ (where $\simeq$ means isomorphic).

We shall frequently change from the "*circle picture*" to the "*real line picture*" as some structure is more manifest in one description rather than in the other. It will be clear from the context whether we are considering elements of $\mathbf{G}$ as elements of $PSU(1,1)$ (acting on S^1) or of $\mathrm{PSL}(2,\mathbb{R})$ (acting on $\bar{\mathbb{R}}$).

The following three one-parameter subgroups of $\mathbf{G}$ play an important rôle: the *rotation* subgroup $R(\cdot)$, the *dilation* subgroup $\delta(\cdot)$ and the *translation* subgroup $\tau(\cdot)$; they are defined as quotient of the corresponding subgroup in $\mathrm{SL}(2,\mathbb{R})$

$$R(\vartheta) = \begin{pmatrix} \cos\vartheta/2 & \sin\vartheta/2 \\ -\sin\vartheta/2 & \cos\vartheta/2 \end{pmatrix}, \quad \delta(s) = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix}, \quad \tau(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Note that R is periodic with period 2π in $\mathrm{PSL}(2,\mathbb{R})$ (and 4π in $\mathrm{SL}(2,\mathbb{R})$). Geometrically, the actions are the following:

$$\begin{aligned} R(\vartheta)z &= e^{i\vartheta}z \qquad \text{on } S^1 \\ \delta(s)x &= e^s x \qquad \text{on } \mathbb{R} \\ \tau(t)x &= x + t \qquad \text{on } \mathbb{R}\,. \end{aligned} \tag{1.1.2}$$

Denote by $\mathbf{K}$ (resp. $\mathbf{A}$, $\mathbf{N}$) the rotation (resp. dilation, translation) subgroup, i.e. the range of R (resp. δ, τ).

The isotropy group of the point ∞ for the action (0.1.1) is the translation-dilation subgroup $\mathbf{P} = \mathbf{AN}$ (the "$ax + b$" group). The subgroup of elements fixing both $\{0\}$ and $\{\infty\}$ is the dilation subgroup $\mathbf{A}$.

It follows from the spectral theorem that every one-parameter subgroup of $\mathbf{G}$ is conjugate, up to rescaling, to one of the above three ones (consider the Lie algebra generators, see below). Note that $\mathbf{G}$ acts transitively on the ordered triple of points of S^1 and the stabilizer subgroup of a single point (resp. of two points) of S^1 is conjugate to $\mathbf{P}$ (resp. to $\mathbf{A}$).

By an *interval* I of S^1 we mean an open[1], connected, non-empty, non-dense subset of S^1. The set of all intervals of S^1 will be denoted by $\mathcal{I}$. Note that $\mathbf{G}$ acts transitively on $\mathcal{I}$. If $I \in \mathcal{I}$, we denote by I' the interior of the complement of I in S^1, which is an interval.

[1]As we shall see, there is no advantage to consider closed intervals at this point as Möbius covariant nets will automatically extend to closed intervals

Given any interval I, we now define two one-parameter subgroups of $\mathbf{G}$, the *dilation* δ_I and the *translation* group τ_I associated with I. Let I_1 be the upper semi-circle, i.e. the interval $\{e^{i\vartheta}, \vartheta \in (0, \pi)\}$, that corresponds to the positive real line $\mathbb{R}_+$ in the real line picture. We set $\delta_{I_1} \equiv \delta$, and $\tau_{I_1} \equiv \tau$. Then, if I is any interval, we chose $g \in \mathbf{G}$ such that $I = gI_1$ and set

$$\delta_I \equiv g\delta_{I_1}g^{-1}, \qquad \tau_I \equiv g\tau_{I_1}g^{-1}.$$

The choice of g is unique modulo the subgroup that fixes the two endpoints of I, namely by right multiplication by an element $\mathbf{A}$. As $\mathbf{A}$ is abelian and $\delta(s) \in \mathbf{A}$, δ_I is well defined; while the one parameter group τ_I is defined only up to a rescaling of the parameter due to the commutation relation

$$\delta(s)\tau(t)\delta(-s) = \tau(e^s t) \ . \tag{1.1.3}$$

We note also that $\tau_{I'}(t)$ is a one-parameter subgroup of $\mathbf{G}$ mapping I into itself iff $t \leq 0$. We shall also set $\tau'_I(t) = \tau_{I'}(t)$ an $\tau' \equiv \tau_{(-\infty,0)}$.

If I is an open interval or half-line of $\mathbb{R}$ we write τ_I or δ_I to denote the translation or dilation group associated with $C(I)$ thus, for example, $\tau_{(0,\infty)} = \tau_{I_1} = \tau$.

1.2 KAN decomposition and the universal cover

We now describe the basic internal structure of $\mathbf{G}$. The group $\mathbf{P}$ is the semi-direct product of $\mathbf{A}$ and $\mathbf{N}$ (cf. eq. (0.1.3)) and in particular $\mathbf{A}$ and $\mathbf{N}$ intersect only at the identity and every $p \in \mathbf{P} = \mathbf{AN}$ is uniquely written as $p = an$ with $a \in \mathbf{A}$, $n \in \mathbf{N}$. More generally the following decomposition for elements of $\mathbf{G}$ holds.

Proposition 1.2.1 (Iwasawa decomposition)**.** *We may write* $\mathbf{G} = \mathbf{KAN}$ *uniquely, namely every element* $g \in \mathbf{G}$ *can be written uniquely as a product* $g = kan$ *where* k *belongs to the rotation group* $\mathbf{K}\,(\simeq \mathbb{T})$*,* a *to the dilation group* $\mathbf{A}\,(\simeq \mathbb{R})$ *and* n *to the translation group* $\mathbf{N}\,(\simeq \mathbb{R})$*. Similarly* $\mathbf{G} = \mathbf{ANK}$*.*

Proof. As noted, every $p \in \mathbf{P} = \mathbf{AN}$ is uniquely written as $p = an$ with $a \in \mathbf{A}$, $n \in \mathbf{N}$. Now $\mathbf{G}$ acts on S^1 and the stabilizer of the point -1 is $\mathbf{P}$. Fix $g \in \mathbf{G}$, then g maps -1 to a point $g(-1) \in S^1$; let $k \in \mathbf{K}$ be the rotation such that $k(-1) = g(-1)$. Then $p \equiv k^{-1}g$ preserves -1, so $p \in \mathbf{P}$. Therefore $g = kp$ as above. The decomposition is unique, indeed any decomposition $g = kp$ has to satisfy the equality $g(-1) = k(-1)$ that thus determines p. Therefore $g = kan$ uniquely. Starting with g^{-1} instead of g we get the decomposition $\mathbf{G} = \mathbf{ANK}$ as $\mathbf{AN} = \mathbf{NA}$. □

Note now that $\mathbf{G}$ acts on the upper complex plane $\Im z > 0$ (let x be complex in eq. (0.1.1)). The action is transitive. Indeed the $\mathbf{A}$-orbits are the open half-lines from the origin and the action is dilating, the $\mathbf{N}$-orbits are the horizontal lines and the action is translating. The stabilizer of the point i is $\mathbf{K}$. So we can identify the homogeneous space $\mathbf{G}/\mathbf{K}$ with $\Im z > 0$ which has an invariant measure for the $\mathbf{G}$-action given by $(\Im z)^{-1}\mathrm{d}z$. By the Iwasawa decomposition we may also identify $\mathbf{G}/\mathbf{K}$ with $\mathbf{P}$.

Corollary 1.2.2. *The Haar measure of* $\mathbf{G} = \mathbf{PK}$ *is the product* $\mathrm{d}g = \mathrm{d}p\mathrm{d}k$ *of the Haar measures of* $\mathbf{P}$ *and* $\mathbf{K}$.

Proof. As we have seen, the action of $\mathbf{G}$ on $\mathbf{P} = \mathbf{G}/\mathbf{K}$ has an invariant measure $\mathrm{d}p$. Take a continuos function f with compact support on $\mathbf{G}$. Then $F(g) \equiv \int_{\mathbf{K}} f(gk)\mathrm{d}k$ is a function on $\mathbf{G}/\mathbf{K}$ thus $\int_{\mathbf{P}}\int_{\mathbf{K}} f(pk)\mathrm{d}k\mathrm{d}p$ is a right invariant integration. □

Indeed it is not difficult to see that also the decomposition $\mathrm{d}g = \mathrm{d}a\mathrm{d}n\mathrm{d}k$ holds true.

Notice now that the Iwasawa decomposition is also topological, namely the map

$$(k, a, n) \in \mathbf{K} \times \mathbf{A} \times \mathbf{N} \simeq \mathbb{T} \times \mathbb{R} \times \mathbb{R} \mapsto kan \in \mathbf{G} \tag{1.2.1}$$

is a homeomorphism. This map is indeed clearly continuous and invertible and to show that its inverse is continuous too we have to show that the projections of $\mathbf{G}$ onto $\mathbf{K}$, $\mathbf{A}$ and $\mathbf{N}$ given by the decompostion $g = kan$ are continuous. Now the equality $g(-1) = k(-1)$ (proof of Lemma 0.2.1) shows that the projection onto $\mathbf{K}$ is continuous so (multiplying on the left by k^{-1}) also the projection of $\mathbf{G}$ onto $\mathbf{P}$ is continuous. Now if $g = an \in \mathbf{P}$ then $g = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$ with $a = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ and $n = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ so the projections of $\mathbf{P}$ onto $\mathbf{A}$ and $\mathbf{N}$ are continuous. By the same argument the identification (0.2.1) is *smooth*.

As $\mathbf{G}$ is topologically isomorphic to $\mathbb{T}\times\mathbb{R}\times\mathbb{R}$, $\mathbf{G}$ is not simply connected. The *universal cover* $\bar{\mathbf{G}}$ of $\mathbf{G}$ is thus a group homeomorphic to $\mathbb{R}^3$. The center of $\bar{\mathbf{G}}$ is isomorphic to $\mathbb{Z}$ and the central extension gives an exact sequence

$$\iota \to \mathbb{Z} \to \bar{\mathbf{G}} \to \mathbf{G} \to \iota$$

Of course $\mathbf{G}$ admits also *finite covers* $\mathbf{G}^{(n)}$ corresponding to the n covers of $\mathbf{K} \simeq \mathbb{T}$, with center $\mathbb{Z}_n$ and $\bar{\mathbf{G}}/\mathbb{Z}_n = \mathbf{G}$.

1.3 The Lie algebra

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of **G** consists of 2×2 real matrices with zero trace. A basis is given by

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad E = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with commutation relations

$$[T,S] = -2E,\ [E,T] = T,\ [E,S] = -S. \tag{1.3.1}$$

A convenient basis for the complexification $\mathfrak{sl}_{\mathbb{C}}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) + i\ \mathfrak{sl}(2,\mathbb{R})$ of $\mathfrak{sl}(2,\mathbb{R})$ is

$$\begin{aligned} L_1 &= \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} = E - \frac{i}{2}(T - S), \\ L_{-1} &= \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = -E - \frac{i}{2}(T - S), \\ L_0 &= -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -i\frac{1}{2}(T + S) \end{aligned}$$

so that

$$[L_1, L_{-1}] = -2L_0, \quad [L_0, L_{-1}] = L_{-1}, \quad [L_0, L_1] = -L_1\ .$$

Note that T is the generator of the one-parameter group τ, $S = -\operatorname{Ad}R(\pi)(T)$ of τ', $E = \frac{1}{2}(L_1 + L_{-1})$ of δ and iL_0 of R.

A direct calculation shows that the *Casimir operator* defined by

$$\lambda \equiv E(E-1) - TS = L_0(L_0 - 1) - \frac{1}{4}L_{-1}L_1 \tag{1.3.2}$$

is a central element of the universal enveloping Lie algebra, thus its value in an irreducible unitary representation of $\bar{\mathbf{G}}$ is a scalar, indeed $\lambda \in \mathbb{R}$ (λ is selfadjoint) and $\lambda \geq -1/4$ ($\lambda + 1/4$ is sum of squares of selfadjoint elements).

1.4 Positive energy condition

Let U be a unitary representation of **G** on a Hilbert space $\mathcal{H}$. Then we have a corresponding representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, thus of $\mathfrak{sl}_{\mathbb{C}}(2,\mathbb{R})$.

To simplify notations we always denote the by the same symbols of the Lie algebra elements and the corresponding operators on $\mathcal{H}$. Thus, for example L_0 also denote the infinitesimal generator of the rotation one-parameter group $U(R(\cdot))$ on $\mathcal{H}$, which is called the *conformal Hamiltonian* of U.

We shall be mainly interested in *positive energy representation*, namely any of the two conditions in the following Prop. 0.4.1 holds.

We denote by $\mathcal{D}$ the *Gärding domain* for a unitary representation U, namely the dense linear subspace of vectors of the form $U(f)\xi$ with f a smooth function with compact support on $\mathbf{G}$ where $U(f) \equiv \int f(g)U(g)\mathrm{d}g$. Then $\mathcal{D}$ is an invariant core for all Lie algebra generators.

Proposition 1.4.1. *Let U be a unitary representation of* $\mathbf{G}$*, The following are equivalent:*

(i) *The conformal Hamiltonian L_0 of U is positive;*

(ii) *The generator P ($\equiv -iT$) of the translation one parameter subgroup $U(\tau(\cdot))$ is positive.*

If either the spectrum of P or of L_0 is bounded below, then both P and L_0 are positive.

Proof. Note that P and P' have the same spectrum, where $P' = U(R(\pi))PU(R(-\pi)) = -iS$ is the generator of the translation unitary group $U(\tau'(\cdot)) \equiv U(\tau_{(-\infty,0)}(\cdot))$. Moreover on $\mathcal{D}$

$$L_0 = \frac{1}{2}(P + P') \tag{1.4.1}$$

because of the corresponding Lie algebra relation.

If P is positive, by (0.4.1) we then have $2(L_0\xi, \xi) \geq (P\xi, \xi)$, $\xi \in \mathcal{D}$, so also L_0 is positive as $\mathcal{D}$ is a core for L_0. Thus $(i) \Rightarrow (ii)$.

To prove the converse implication note first that

$$U(\delta(s))PU(\delta(s))^{-1} = e^s P\,; \quad U(\delta(s))P'U(\delta(s))^{-1} = e^{-s}P'\,. \tag{1.4.2}$$

Therefore, assuming $L_0 \geq 0$, by (0.4.1) we have the identity

$$U(\delta(s))L_0U(\delta(s))^{-1} = \frac{1}{2}(e^s P + e^{-s}P')$$

on $\mathcal{D}$ that implies $P \geq 0$ because

$$(\xi, P\xi) = \lim_{s\to\infty} e^{-s}(U(\delta(s))^{-1}\xi, L_0U(\delta(s))^{-1}\xi)/2 \geq 0 \tag{1.4.3}$$

for all ξ in $\mathcal{D}$, and $\mathcal{D}$ is a core for P. Thus $(ii) \Rightarrow (i)$.

Finally note that the spectrum of P is dilation invariant by the first equation in (0.4.2) so it has to be non-negative if it is bounded below. Now if the spectrum of L_0 is bounded below also the spectrum of P is bounded below by (0.4.3). So the last statement follows by the shown equivalence $(i) \Leftrightarrow (ii)$. □

1.5 Classification of positive energy representations

Let U be a unitary representation of $\mathbf{G}$ on a Hilbert space $\mathcal{H}$. As above, let L_0, L_1, L_{-1} be the operators on $\mathcal{H}$ corresponding corresponding to the elements Lie algebra $\mathfrak{sl}_{\mathbb{C}}(2,\mathbb{R})$ with the same symbol. As U is unitary, L_0 is selfadjoint and L_1, L_{-1} are one the adjoint of the other.

We denote $\mathcal{H}_n$ the n-eigenspace of L_0, i.e. $\mathcal{H}_n \equiv \{\xi \in D(L_0) : L_0\xi = n\xi\}$ and by $\mathcal{H}_{\text{fin}}$ the finite-energy subspace, namely the linear span $\mathcal{H}_{\text{fin}} \equiv \sum_n \mathcal{H}_n$. Let U have positive energy, then there exists an integer $m > 0$ such that $\mathcal{H}_m \neq \{0\}$ and $\mathcal{H}_n = \{0\}$, $n < m$; this m is called the *lowest weight* of U. A positive energy representation is also called lowest weight representation.

Lemma 1.5.1. *Let U be an irreducible representation of* $\mathbf{G}$ *with lowest weight m. Then $\mathcal{H}_{\text{fin}}$ is contained in the domain of $L_{\mp 1}$ and $L_{\mp 1}$ are raising/lowering operators: $L_{-1}\mathcal{H}_n \subset \mathcal{H}_{n+1}$, $L_1\mathcal{H}_n \subset \mathcal{H}_{n-1}$. In particular $\mathcal{H}_{\text{fin}}$ is stable for the Lie algebra representation. Moreover $\mathcal{H}_n$ is one-dimensional for all $n \geq m$.*

Proof. Let E_n be the orthogonal projection onto $\mathcal{H}_n$ for a given n:

$$E_n = \frac{1}{2\pi}\int_0^{2\pi} U(R(\vartheta))e^{-in\vartheta}\mathrm{d}\vartheta = \int_K \chi_n(k)U(k)\mathrm{d}k$$

where χ_n is the corresponding character of $\mathbf{K}$.

We shall first show that $\mathcal{D} \cap \mathcal{H}_n$ is dense in $\mathcal{H}_n$ for all n. Let f be a smooth function on $\mathbf{P}$ with compact support that we extend to a function f_n on $\mathbf{G} = \mathbf{PK}$ by $f_n(pk) \equiv f(p)\chi_n(k)$. As $\mathrm{d}g = \mathrm{d}p\mathrm{d}k$ we have

$$U(f_n) = \int_{\mathbf{G}} f_n(g)U(g)\mathrm{d}g = \int_{\mathbf{K}}\int_{\mathbf{P}} f(p)\chi_n(k)U(k)U(p)\mathrm{d}k\mathrm{d}p = E_n\int_{\mathbf{P}} f(p)U(p)\mathrm{d}p$$

Thus if $\xi \in \mathcal{H}$ the vector $U(f_n)\xi \in \mathcal{D} \cap \mathcal{H}_n$. By considering an approximate identity $f^{(i)}$ on $\mathbf{P}$ in place of f we see $U(f_n^{(i)})\xi$ converges to $E_n\xi$, therefore $\mathcal{D} \cap \mathcal{H}_n$ is dense in $\mathcal{H}_n$.

Let now $\xi \in \mathcal{H}_n$ be a vector in $\mathcal{D}$. We have:

$$L_0 L_{-1}\xi = [L_0, L_{-1}]\xi + L_{-1}L_0\xi = L_{-1}\xi + nL_{-1}\xi = (n+1)L_{-1}\xi.$$

and

$$L_0 L_1\xi = [L_0, L_1]\xi + L_1 L_0\xi = -L_1\xi + nL_1\xi = (n-1)L_1\xi$$

namely $L_{-1}\xi \in \mathcal{H}_{n+1}$ and $L_1\xi \in \mathcal{H}_{n-1}$.

We thus fix a unit vector ξ_m in $\mathcal{D} \cap \mathcal{H}_m$ and define $\xi_n \in \mathcal{H}_n$, $a_n \in \mathbb{R}$ recursively by

$$\xi_{n+1} = L_{-1}\xi_n \qquad a_n \equiv \|\xi_n\|,$$

Note that a_n is positive because

$$\begin{aligned} a_{n+1}^2 &= (\xi_{n+1}, \xi_{n+1}) = (L_{-1}\xi_n, L_{-1}\xi_n) = (L_1 L_{-1}\xi_n, \xi_n) \\ &= ([L_1, L_{-1}]\xi_n, \xi_n) + (L_{-1}L_1\xi_n, \xi_n) = 2(L_0\xi_n, \xi_n) + (L_1\xi_n, L_1\xi_n) \geq 2na_n^2 \,. \end{aligned}$$

$\{\xi_n\}$ is an orthogonal family of vectors and its linear span is clearly stable for L_0, L_{-1}; we now show it is also stable for L_1. Indeed since $L_1\xi_m = 0$ we have

$$L_1\xi_{m+1} = L_1 L_{-1}\xi_m = (L_1 L_{-1} - L_{-1}L_1)\xi_m = 2L_0\xi_m = 2m\xi_m$$

so $L_1\xi_{m+1} \in \mathbb{C}\xi_m$ and

$$L_1\xi_{m+2} = L_1 L_{-1}\xi_{m+1} = (L_1 L_{-1} - L_{-1}L_1)\xi_{m+1} + L_{-1}L_1\xi_{m+1} \in \mathbb{C}\xi_{m+1} \,;$$

by iterating the argument $L_1\xi_n \in \mathbb{C}\xi_{n-1}$ for all $n \geq 0$.

We now show that the action of $\mathfrak{sl}_{\mathbb{C}}(2, \mathbb{R})$ on the space linearly generated by the ξ_n is completely determined. Let $c_n \in \mathbb{C}$ be defined by $c_m = 0$ and

$$L_1\xi_n = c_n\xi_{n-1}.$$

Then

$$2n\xi_n = 2L_0\xi_n = [L_1, L_{-1}]\xi_n = (L_1 L_{-1} - L_{-1}L_1)\xi_n = c_{n+1}\xi_n - c_n\xi_n \,,$$

therefore

$$c_{n+1} - c_n = 2n \,,$$

so c_n is uniquely recursively determined. Indeed $c_{m+k} = 2km + k(k-1)$.

On the other hand

$$a_{n+1}^2 = (L_{-1}\xi_n, \xi_{n+1}) = (\xi_n, L_1\xi_{n+1}) = c_{n+1}a_n^2$$

so also a_n is uniquely recursively determined as $a_m = 1$, indeed $a_n = \sqrt{c_{m+1}c_{m+2}\cdots c_n}$.

We now show that $\mathcal{H}_m$ is one dimensional. Let a $\{\xi_m^k\}_k$ be an orthonormal basis for $\mathcal{H}_m$ with $\xi_m^k \in \mathcal{D}$; then $L_{-1}\xi_m^k$ and $L_{-1}\xi_m^{k'}$are orthogonal if $k \neq k'$ because

$$(L_{-1}\xi_m^k, L_{-1}\xi_m^{k'}) = (L_1L_{-1}\xi_m^k, \xi_m^{k'}) = 2(L_0\xi_m^k, \xi_m^{k'}) = 2m(\xi_m^k, \xi_m^{k'}) = 0\ .$$

With V_k the linear span of $\{L_{-1}^n\xi_m^k,\ n \in \mathbb{N}\}$, extending the above argument we see that the V_k's form an orthogonal family and each of them is stable for L_0, L_1, L_{-1}. With $\mathcal{K}_k \equiv \bar{V}_k$, the operators $L_0|_{\mathcal{K}_k}$ and $(L_1 \pm L_{-1})|_{\mathcal{K}_k}$ are essentially selfadjoint on V_k [2]. Thus each $\mathcal{K}_k$ is U-invariant, hence the family $\{\mathcal{K}_k\}_k$ has only one element because U is irreducible, which amounts to say that $\mathcal{H}_m$ is one-dimensional. Similarly each $\mathcal{H}_n$ is one-dimensional, $n \geq m$, and every $\mathcal{H}_n$ is contained in $\mathcal{D}$. So $\mathcal{H}_{\text{fin}} \subset \mathcal{D}$.

As U is irreducible then the linear span of $\{\xi_n\}$ must be dense and we have thus determined the irreducible representations as $c_m = 0$. □

If U is a reducible positive energy representation, the above argument show that the closed linear span of the $\{\xi_n\}$ (with $\xi_m \in \mathcal{D} \cap \mathcal{H}_m$) carries an irreducible subrepresentation of U. Repeating the argument on the orthogonal complement we see that U is direct sum of irreducibles.

Therefore, by the proof of the above lemma, we have shown:

Theorem 1.5.2. *For each non negative integer m there exists a unique irreducible representation of* **G** *with lowest weight m.*

Proof. The uniqueness has been shown. Concerning the existence, we shall later consider the irreducible lowest weight representation U of **G** in Sec. 0.21. Alternatively one reverse the argument in the proof of Thm. 0.5.1, define the so determined operators L_1, L_{-1}, L_0, and exponentiate them to a representation of **G** because of the dense analytic vector domain (see the remark at the end of this section). □

Remark. The lowest weight unitary irreducible representations of **G**, and the corresponding conjugate highest weight representations, are said to form the *discrete series* representations because they are contained in the regular representation of **G**. It is easy to extend the arguments in the proof of Lemma 0.5.1 to list all unitary

[2] If $\mathcal{K} = \oplus_k\mathcal{K}_k$ is a Hilbert space direct sum, A is a selfadjoint operator on $\mathcal{K}$ and $\mathcal{D} = \sum_k \mathcal{D}_k$ is a core for a A with $A\mathcal{D}_k \subset \mathcal{D}_k$ and $\mathcal{D}_k \subset \mathcal{K}_k$, then each $\mathcal{K}_k$ is an invariant subspace for the exponential of A. Indeed $\overline{(A \pm i)\mathcal{D}_k}$ is contained in $\mathcal{K}_k$ so it is equal to $\mathcal{K}_k$ because $\overline{(A \pm i)\mathcal{D}} = \mathcal{K}$. Thus $A|_{\mathcal{D}_k}$ is essentially selfadjoint on $\mathcal{K}_k$

irreducible representations U of $\mathbf{G}$. Let indeed U be irreducible, non-trivial, not in the discrete series and, say, $\mathcal{H}_0 \neq \{0\}$. Choose a unit vector $\xi_0 \in \mathcal{H}_0$ and define again $\xi_n \in \mathcal{H}_n$ by $L_{-1}\xi_n = \xi_{n+1}$. Then $L_1\xi_n = c_n\xi_{n-1}$ (use the fact that λ is a scalar in eq, (0.3.2)) and as above $c_{n+1} - c_n = 2n$. So the U is determined by the choice of $c_0 > 0$. The case $c_0 \geq 1$ gives the *principal series*, the case $c_0 \in (0, 1)$ the *complementary series* of representations, which is Bargamann classification.

Note now that essentially the same proof of Lemma 0.5.1 shows that a Theorem 0.5.2 holds true for an irreducible, positive energy unitary representation U of the universal covering group $\bar{\mathbf{G}}$ of $\mathbf{G}$. Note that $e^{2\pi i L_0}$ is a multiple of the identity as it commutes with U. Thus $\mathrm{sp}(L_0) \subset \{\ell, \ell + 1, \ell + 2, \dots\}$ and the lowest weight is defined as the lowest point in the spectrum. We thus have:

Theorem 1.5.3. *For each $\ell > 0$ there exists a unique irreducible representation U^ℓ of the universal cover $\bar{\mathbf{G}}$ with lowest weight ℓ.*

Note however that a positive energy representation U of $\bar{\mathbf{G}}$ is not, in general, a direct sum of irreducibles (it could be a direct integral over ℓ). However, if $e^{2\pi i L_0} \in \mathbb{C}$, then U is indeed a direct sum of irreducibles (this is the case of a factorial representation).

Remark. The value of the Casimir operator in the irreducible representation U^ℓ is

$$\lambda = \ell(\ell - 1)\ . \tag{1.5.1}$$

Indeed, if ξ is the lowest weight vector, we have

$$\lambda\xi = \Big(L_0(L_0 - 1) - \frac{1}{4}L_{-1}L_1\Big)\xi = L_0(L_0 - 1)\xi = \ell(\ell - 1)\xi\ .$$

Another immediate corollary is the following:

Corollary 1.5.4.

$$U^\ell \otimes U^{\ell'} = \bigoplus_{k=0}^{\infty} U^{\ell+\ell'+k}$$

Proof. Let L_0^ℓ be the unique (modulo unitary equivalence) selfadjoint operator with simple spectrum equal to $\{\ell, \ell + 1, \ell + 2, \dots\}$. So L_0^ℓ is the conformal Hamiltonian of U^ℓ. The thesis follows from the equality

$$L_0^\ell \otimes 1 + 1 \otimes L_0^{\ell'} = \bigoplus_{k=0}^{\infty} L_0^{\ell+\ell'+k}\ .$$

□

Remark. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $e_m, e_{m+1}, e_{m+2}, \ldots$. On the linear span of the e_k's define the linear operators L_0, L_1, L_{-1} by

$$L_0 e_n = n e_n, \quad L_{-1} e_n = \sqrt{c_{n+1}} e_{n+1}, \quad L_1 e_n = \sqrt{c_n} e_{n-1},$$

and $L_1 e_m = 0$, where $c_{m+k} = 2km + k(k-1)$. As shown in Lemma 0.5.1 ($e_n = a_n^{-1} \xi_n$), the above operators define a representation of $\mathfrak{sl}(2, \mathbb{R})$ that exponentiates to the irreducible unitary representation of $\mathbf{G}$ with lowest weight m.

Now the above expression shows that the e_k's are *analytic vectors* for L_0, L_1, L_{-1}, in fact for $L_0^2 + L_1^2 + L_{-1}^2$. By Nelson theorem, the Lie algebra representation exponentiates, without the a priori knowledge of the irreducible unitary representation of $\mathbf{G}$ with lowest weight m, that can be defined in this way.

1.6 Representations of related groups

We begin to recall von Neumann theorem on the uniqueness of the representation of the Weyl commutation relations. Let U and V be two one-parameter groups on a Hilbert space $\mathcal{H}$. We shall say that they obey the Weyl commutation relations if

$$U(t)V(s) = e^{its} V(s)U(t) , \quad t, s \in \mathbb{R} .$$

Von Neumann theorem states that there is only one representation of the Weyl commutation relations which is irreducible, that is no proper closed invariant subspace. In other words if U, V and U', V' obeys the Weyl commutation relations on the Hilbert spaces $\mathcal{H}, \mathcal{H}'$, both irreducibly and not zero, there exists a unitary $W : \mathcal{H} \to \mathcal{H}'$ such that $WUW^* = U'$, $WVW^* = V'$. Every non-zero representation of the Weyl commutation relations is then a multiple of the unique irreducible one.

A realization of the irreducible representation is the *Schrödinger representation*: $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$, U is the translation group, i.e. $(U(t)f)(x) = f(x - t)$, and $V(s)$ is the multiplication by e^{isx}, i.e. $(V(s)f)(x) = e^{isx} f(x)$.

1.6.1 Representations of the "$ax + b$" group

Let U be a unitary representation of $\mathbf{P}$ on a Hilbert space $\mathcal{H}$. Setting

$$u(t) \equiv U(\tau(t)), \qquad v(s) \equiv U(\delta(s)), \quad t, s \in \mathbb{R}$$

we get two one-parameter unitary groups u and v on $\mathcal{H}$ satisfying the commutation relations

$$v(s)u(t)v(-s) = u(e^s t)\,. \tag{1.6.1}$$

Conversely, given two one-parameter unitary groups u and v on $\mathcal{H}$ satisfying the above commutation relations, we get a unitary representation of $\mathbf{P}$ setting $U(\tau(t)\delta(s)) \equiv u(t)v(s)$.

We shall say that U has positive (resp. negative) energy if the generator P of u is positive (resp. negative). If furthermore P is non-singular, namely u has no non-zero fixed vector, we shall say that u has strictly positive (resp. negative) energy.

Theorem 1.6.1. *There exists exactly one irreducible unitary representation of* $\mathbf{P}$ *with strictly positive energy, up to unitary equivalence. Every unitary representation of* $\mathbf{P}$ *with strictly positive energy is a multiple of the irreducible one. The analogous statement holds for strictly negative energy.*

Let U be any unitary representation of $\mathbf{P}$. *Then U decomposes uniquely in a direct sum $U = U_+ \oplus U_- \oplus U_0$ where $U_\pm$ has strictly positive/negative energy and* $U_0(\tau(t)) = 1$.

Proof. Let u and v be a s above and P the selfadjoint generator of u, thus

$$v(s)Pv(s)^* = e^s P\,. \tag{1.6.2}$$

Assume now that P is strictly positive, thus $\log P$ is defined. We then have $v(s)\log P v(-s) = \log P + s$, thus

$$v(s)e^{it\log P} = e^{its}e^{it\log P}v(s)\,, \tag{1.6.3}$$

namely the two above unitary one-parameter groups satisfy Weyl commutation relations. By von Neumann uniqueness theorem we have the first part of the statement for strictly positive P. The case P strictly negative is obtained by considering $u(-t)$ instead of $u(t)$.

Concerning the second part of the statement, note that by (0.6.2) the spectral subspace $\mathcal{H}_-, \mathcal{H}_+, \mathcal{H}_0$ of P corresponding to $(-\infty, 0)$, $(0, \infty)$ and $\{0\}$ are U-invariant; in other words U is the direct sum of a representations with $P > 0$, $P < 0$ (P non-singular) and $P = 0$. □

The above proof also shows the follwing:

Corollary 1.6.2. *Let U be an irreducible, positive energy unitary representation of* $\mathbf{G}$*. Then the restriction of U to* $\mathbf{P}$ *is also irreducible. If U is non-trivial, $U|_{\mathbf{P}}$ is the unique representation with P positive and non-singular.*

Proof. If U is a unitary irreducible non-trivial lowest weight representation of $\mathbf{G}$, then the selfadjoint generator P of the translation unitary group is positive and non-singular, see Cor. 0.7.3.

Then we have a representation of the Weyl commutation relations (0.6.3). The restriction of U to $\mathbf{P}$ has to be irreducible because any bounded operator commuting with E and T also commutes with S due to the formula (0.3.2).

□

The above corollary is not true if the positive energy condition is dropped.

1.6.2 Representations of $\mathbf{G}_2$ and of $\mathbf{P}_2$

Let I_1 be the upper semicircle and consider the reflection $r_{I_1} : z \to \bar{z}$ of S^1 where $\bar{z}$ is the complex conjugate of z.

For a general $I \in \mathcal{I}$ we choose $g \in \mathbf{G}$ such that $I = gI_0$, set

$$r_I = g r_{I_1} g^{-1}$$

and call r_I the reflection associated with I (it is well defined because r_{I_1} commute with dilations c.f. Section 0.1).

Let r be an orientation reversing isometry of S^1 with $r^2 = 1$ (e.g. r_{I_1}). Let σ_r be the action of r on $\mathbf{G}$ by conjugation and denote by $\mathbf{G}_2$ the semidirect product of $\mathbf{G}$ with $\mathbb{Z}_2$ via σ_r. Note that $\mathbf{G}_2$ is a group of diffeomorphisms of S^1 that contains also elements that do not preserve the orientation.

We call (anti-)unitary a representation U of $\mathbf{G}$ with operators on $\mathcal{H}$ such that $U(g)$ is unitary, resp. anti-unitary, when g is orientation preserving, resp. orientation reversing.

Theorem 1.6.3. *Every unitary, positive energy representation U of* $\mathbf{G}$ *on a Hilbert space $\mathcal{H}$ extends to a (anti-)unitary representation $\tilde{U}$ of* $\mathbf{G}_2$ *on the same Hilbert space $\mathcal{H}$. Every (anti-)unitary, positive energy representation of* $\mathbf{G}_2$ *arises in this way. U_1 is equivalent to U_2 iff $\tilde{U}_1$ is equivalent to $\tilde{U}_2$. U is irreducible iff $\tilde{U}$ is irreducible and in this case the choice $J \equiv \tilde{U}(r)$ is unique modulo replacing J with zJ for some $z \in \mathbb{T}$.*

Proof. Given unitary, positive energy representation U of $\mathbf{G}$, to find an extension $\tilde{U}$ to $\mathbf{G}_2$ it is sufficient to assume that U is irreducible, otherwise decomposing U into irreducibles and extending each direct summand. Then, if U is irreducible, U has the form given in Lemma 0.5.1. Now the anti-unitary involution Cdetermined by $C\lambda\xi_n = \bar{\lambda}\xi_n$ commutes with L_0, L_1, L_{-1}, so with $CEC = E$, $CTC = -T$, $CSC = -S$. Therefore $CU(g)C = U(rgr)$ as desires (see also Cor. 0.20.3.

We now show the uniqueness of the extension, up to unitary equivalence. Suppose $\tilde{U}'$ be a second extension and set $J \equiv \tilde{U}(r_{I_0})$, $J' \equiv \tilde{U}'(r_{I_0})$. Then the unitary $J'J$ commutes with $U(\mathbf{G})$, thus with the center of $U(\mathbf{G})''$; in particular $J'J$ commutes with the projection onto the lowest weight n representation summand ($n = 0, 1, \dots$). We may thus assume that $U = U^{(n)} \otimes 1_{\mathcal{K}}$, where $U^{(n)}$ is the irreducible unitary representation of $\mathbf{G}$ with lowest weigh n and $\mathcal{K}$ is a Hilbert space. We may further assume that in this decomposition $J = J_0 \otimes J_1$ with anti-unitary involutions J_0, J_1. As $J'J = 1 \otimes u$ with u a unitary in $B(\mathcal{K})$, we also have $J' = J_0 \otimes J_1'$ where $J_1' \equiv uJ_1$. Now we are looking for a unitary $V \in U(\mathbf{G})'$ such that $J' = VJV^*$, thus for a unitary $v \in B(\mathcal{K})$ such that $J_1' = vJ_1v^*$. But such a v exists because any two anti-unitary involutions on a Hilbert space $\mathcal{K}$ are unitary equivalent (choose real orthonormal bases).

The above argument also shows that if U_1 and U_2 are equivalent, also their extension $\tilde{U}_1$ and $\tilde{U}_2$ are equivalent. Conversely, if $\tilde{U}_1$ and $\tilde{U}_2$ are equivalent, their restrictions U_1 and U_2 are clearly equivalent. The rest is clear and our proof is complete. □

By the same proof, Thm. 0.6.3 holds true replacing $\mathbf{G}$ with $\bar{\mathbf{G}}$ and $\mathbf{G}_2$ with $\bar{\mathbf{G}}_2$, the semi-direct product of $\bar{\mathbf{G}}$ with $\mathbb{Z}_2$ by the involutive automorphism of $\bar{\mathbf{G}}$ that correspons to σ_r.

Let now $\mathbf{P}_2$ be the subgroup of $\mathbf{G}_2$ generated by $\mathbf{P}$ and the involution r. Thus $\mathbf{P}_2$ is the a semi-direct product of $\mathbf{P}$ by $\mathbb{Z}_2$ and is generated by $\mathbf{P}$ and an involution r such that $r\tau(t)r = \tau(-t)$, $r\delta(s)r = \delta(s)$. The (anti-)representations of $\mathbf{P}_2$ are thus given by pairs (U, J) where U is a unitary representation of $\mathbf{P}$ on a Hilbert space $\mathcal{H}$ and J is a anti-unitary involution on $\mathcal{H}$ such that

$$JU(\tau(t))J = U(\tau(-t)), \qquad JU(\delta(s))J = U(\delta(s)) .$$

We have seen that $\mathbf{P}$ has only one unitary representation with strictly positive energy. By an argument analogous to the one in above proof we then have the following.

Proposition 1.6.4. *Let U be a unitary positive energy representation of* $\mathbf{P}$ *on a Hilbert space* $\mathcal{H}$*. Then U extends to a (anti-)unitary representation $\tilde{U}$ of* $\mathbf{P}_2$*. The extension is unique up to unitary equivalence. U is irreducible iff $\tilde{U}$ is irreducible and in this case $J \equiv \tilde{U}(r)$ is unique modulo a phase.*

1.7 Vanishing of the matrix coefficients

In this section we prove the vanishing of the matrix coefficient theorem for unitary representations of $\mathbf{G}$, Thm. 0.7.2 below. We begin with the following proposition.

Proposition 1.7.1. *Let U be a unitary representation of* $\mathbf{P}$ *on a Hilbert space* $\mathcal{H}$*.*

(*a*) *If $F \subset \mathcal{H}$ is a finite-dimensional subspace which is globally $U(\delta(\cdot))$-invariant, then F is left pointwise fixed by $U(\tau(\cdot))$.*

(*b*) *If U has no non-zero fixed vector for translations then*

$$\lim_{p\to\infty, p\in\mathbf{P}} (U(p)\xi, \xi) = 0, \quad \forall \xi \in \mathcal{H} .$$

Proof. (*a*): Setting $u(t) \equiv U(\tau(t))$ and $v(s) \equiv U(\delta(s))$ we have two one-parameter unitary groups on $\mathcal{H}$ satisfying the commutation relations (0.6.1).

Since F is finite dimensional, we need to show that $u(t)\xi = \xi$ if ξ is a v-eigenvector, i.e. if there exists a character χ of $\mathbb{R}$ such that

$$v(s)\xi = \chi(s)\xi , \quad s \in \mathbb{R} .$$

Indeed in this case by the formula (0.6.1) implies

$$u(e^s t)\xi = v(s)u(t)v(-s)\xi = \overline{\chi(s)}v(s)u(t)\xi ,$$

hence

$$(u(e^s t)\xi, \xi) = (u(t)\xi, \xi), \quad t, s \in \mathbb{R}.$$

As $s \to -\infty$ we thus have

$$(\xi, \xi) = (u(t)\xi, \xi)$$

that implies $u(t)\xi = \xi$ by the limit case of the Schwarz inequality.

(*b*): As by assumptions u has no non-zero fixed vector, by Theorem 0.6.1 U is the direct sum of a strictly positive and a strictly negative energy representations. By the uniqueness in Theorem 0.6.1 it is sufficient to verify that

$$U(p) \to 0 \quad \text{weakly as} \quad p \to \infty\ (g \in \mathbf{P})$$

for the irreducible strictly positive (negative) energy representation of **P**, namely in the Schrödinger representation, where we now check its validity.

Indeed, in the Schrödinger representation on $L^2(\mathbb{R}, \mathrm{d}x)$, u is the translation on parameter group on and $v(s)$ is the multiplication by e^{ise^x}. It is sufficient to show that

$$(u(t_n)v(s_n)f_1, f_2) \to 0$$

for f_1, f_2 with compact support if $(t_n, s_n) \to \infty$ in $\mathbb{R}^2$. If $t_n \to \infty$ then, for large n, $(u(t_n)v(s_n)f_1, f_2) = 0$; so we may assume that $\{t_n\}$ is bounded. By compactness we may assume that t_n is convergent so, by a 3ε argument, it will be enough to show that $(v(s_n)f_1, f_2) \to 0$ if $s_n \to \infty$. Now $v(s) = e^{isA}$ where the selfadjoint generator A, the multiplication by e^x, has Lebesgue absolutely continuous spectrum, so $v(s)$ converges weakly to zero as $s \to \infty$ by the Riemann-Lebesgue theorem. □

Theorem 1.7.2. *Let U be a unitary representation of* **G** *on a Hilbert space $\mathcal{H}$. If U does not contain the identity representation, then*

$$\lim_{g\to\infty}(U(g)\xi, \eta) = 0, \quad \forall \xi, \eta \in \mathcal{H}\ .$$

Proof. First we observe that it is sufficient to show that

$$\lim_{p\to\infty, p\in\mathbf{P}}(U(p)\xi, \eta) = 0, \ \forall \xi, \eta \in \mathcal{H}. \tag{1.7.1}$$

Indeed assume that equation (0.7.1) holds true and let $g_n \in \mathbf{G}$ be a sequence $g_n \to \infty$. We want to show that

$$(U(g_n)\xi, \eta) \to 0, \quad \forall \xi, \eta \in \mathcal{H}\ .$$

Write $g_n = p_n k_n$ by the Iwasawa decomposition. Then $p_n \to \infty$. As k_n belongs to the compact group **K**, we may assume that $k_n \to k$. Then

$$\begin{aligned}|(U(p_n)\xi_1, \eta) - (U(g_n)\xi, \eta)| &= |(U(p_n)U(k)\xi, \eta) - (U(p_n)U(k_n)\xi, \eta)| \\ &\leq \|U(k)\xi - U(k_n)\xi\|\,\|\eta\| \to 0\end{aligned}$$

where $\xi_1 \equiv U(k)\xi$. As $(U(p_n)\xi_1, \xi) \to 0$, then also $(U(g_n)\xi, \eta) \to 0$.

Therefore, by Prop. 0.7.1, the theorem is proved once we show that there is no non-zero vector that is fixed by $U(\mathbf{P})$.

Assume on the contrary that there exists a non-zero $\xi \in \mathcal{H}$ such that $U(g)\xi = \xi$ for all $g \in \mathbf{P}$ and set

$$f(g) \equiv (U(g)\xi, \xi), \quad g \in \mathbf{G}\ .$$

Then f is a bi-$\mathbf{P}$-invariant function, namely $f(pgq) = f(g)\ \forall p, q \in \mathbf{P}$ and $g \in \mathbf{G}$. Thus f defines a continuous function f_1 on the coset space $\mathbf{G}/\mathbf{P}$, invariant for the left action of $\mathbf{P}$ on $\mathbf{G}/\mathbf{P}$.

Now $\mathbf{G}$ acts on $\bar{\mathbb{R}}$ as in (0.1.1) and the stabilizer of the point ∞ is $\mathbf{P}$. So $\mathbf{G}/\mathbf{P} \simeq \bar{\mathbb{R}}$ and the left action of $\mathbf{G}$ on $\bar{\mathbb{R}}$ is given by eq. (0.1.1). In particular $\mathbb{R}$ is a dense orbit for the action of $\mathbf{P}$ on $\bar{\mathbb{R}}$. So f is constant, namely ξ is a $\mathbf{G}$-invariant vector. □

Corollary 1.7.3. *Let U be a unitary representation of $\mathbf{G}$ on a Hilbert space $\mathcal{H}$. Given $\xi \in \mathcal{H}$, the subgroup $\{g \in \mathbf{G} : U(g)\xi = \xi\}$ is either compact or equal to $\mathbf{G}$.*

Corollary 1.7.4. *Let U be a unitary representation of $\bar{\mathbf{G}}$ that has no fixed vector. If $g_n \in \bar{\mathbf{G}}$ is a sequence such that $q(g_n) \to \infty$, where $q : \bar{\mathbf{G}} \to \mathbf{G}$ is the quotient map, then $U(g_n)$ weakly converges to* 0*.*

Proof. Let $\bar{U}$ the conjugate representation of U. Then $U \otimes \bar{U}$ is a representation of $\mathbf{G}$, thus

$$|(U(g_n)\xi_1, \xi_2)|^2 = (U(g_n)\xi_1, \xi_2)(\bar{U}(g_n)\xi_1, \xi_2) = (U(g_n) \otimes \bar{U}(g_n)\xi_1 \otimes \xi_1, \xi_2 \otimes \xi_2) \to 0\,.$$

□

If in the above corollary U were a representation of a finite cover $\mathbf{G}^{(n)}$ then we could have just assumed $g_n \to \infty$ as this is equivalent to $q(g_n) \to \infty$ (in the case of $\bar{\mathbf{G}}$ we may take g_n central, $g_n \to \infty$ but $q(g_n)$ is fixed).

Chapter 2

Standard subspaces of a Hilbert space

We shall now consider certain real closed linear subspaces of a complex Hilbert space. The emerging structure is definitely richer than what one would expect.

2.1 Basic properties, modular theory

Let $\mathcal{H}$ be a complex Hilbert space and $H \subset \mathcal{H}$ a real linear subspace. The *symplectic complement* H' of H is the real Hilbert subspace

$$H' \equiv \{\xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \ \ \forall \eta \in H\}.$$

Clearly

$$H' = (iH)^{\perp} \tag{2.1.1}$$

where the $\perp$ denotes the real orthogonal complement in $\mathcal{H}$, namely the orthogonal complement with respect to the real scalar product $\Re(\cdot\,,\,\cdot)$. Therefore $H \subset H''$ and $\bar{H} = H''$. Moreover

$$H_1 \subset H_2 \Rightarrow H_1' \supset H_2' \,.$$

A closed real subspace H is called *cyclic* if $H + iH$ is dense in $\mathcal{H}$ and *separating* if $H \cap iH = \{0\}$. Because of eq. (0.8.1) we have

$$(H + iH)' = H' \cap iH' \;,$$

so H is cyclic if and only if H' is separating. A *standard* subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic and separating. Thus a closed subspace H is standard iff H' is standard.

Let H be a standard subspace of $\mathcal{H}$. Define the anti-linear operator $S \equiv S_H : D(S) \subset \mathcal{H} \to \mathcal{H}$, where $D(S) \equiv H + iH$,

$$S : \xi + i\eta \mapsto \xi - i\eta\,, \quad \xi, \eta \in H\,.$$

As H is standard, S is well-defined and densely defined. Clearly $S^2 = 1|_{D(S)}$.

We shall soon see in Prop 0.8.4 that S_H is a closed operator.

Lemma 2.1.1. *Let S be a closed, densely defined, anti-linear involution on $\mathcal{H}$. Then $H \equiv \ker(1 - S)$ is a standard subspace of $\mathcal{H}$.*

Proof. First

$$H \equiv \ker(1 - S) = \{\xi \in D(S) : S\xi = \xi\}$$

is a closed real linear subspace of $\mathcal{H}$, because S is a closed operator. To check that H is standard, note that any $\xi \in D(S)$ can be written as

$$\xi = \frac{1}{2}(\xi + S\xi) + i\frac{1}{2i}(\xi - S\xi) \equiv \xi_1 + i\xi_2 \tag{2.1.2}$$

with $\xi_1, \xi_2 \in H$, thus H is cyclic. Moreover H is separating because if $\xi, \eta \in H$ and $\xi = i\eta$, then applying S to both vectors in this equality we also have $\xi = -i\eta$, thus $\xi = \eta = 0$. □

Proposition 2.1.2. *The map*

$$H \mapsto S_H \tag{2.1.3}$$

is a bijection between the set of standard subspaces of $\mathcal{H}$ and the set of closed, densely defined, anti-linear involutions on $\mathcal{H}$. The inverse of the map (0.8.3) *is*

$$S \mapsto \ker(1 - S)\,.$$

Moreover this map is order-preserving, namely

$$H_1 \subset H_2 \Leftrightarrow S_{H_1} \subset S_{H_2}\,, \tag{2.1.4}$$

and we have

$$S_H^* = S_{H'}\,. \tag{2.1.5}$$

Proof. First we show that every densely defined, closed, anti-linear involution S on $\mathcal{H}$ is in the range of the map (0.8.3). To this end note that $H = \{\xi \in D(S) : S\xi = \xi\}$ is a standard subspace by Lemma 0.8.1. Clearly $S \supset S_H$. By eq. (0.8.2) $D(S) = H + iH = D(S_H)$, so $S = S_H$.

We now prove eq. (0.8.5). With $\xi_1, \xi_2 \in H$, $\xi_1', \xi_2' \in H'$ we have

$$\begin{aligned}(S_H(\xi_1 + i\xi_2), \xi_1' + i\xi_2') = (\xi_1 - i\xi_2, \xi_1' + i\xi_2') = (\xi_1, \xi_1') - (\xi_2, \xi_1') + i\big((\xi_1, \xi_2') + (\xi_2, \xi_1')\big) \\ = (\xi_1' - i\xi_2', \xi_1 + i\xi_2) = (S_{H'}(\xi_1' + i\xi_2'), \xi_1 + i\xi_2)\end{aligned}$$

showing that $S_H^* \supset S_{H'}$.

To get the reverse inclusion, notice that S_H^* is a closed anti-linear involution. Setting $K \equiv \{\xi \in D(S_H^*) : S_H^*\xi = \xi\}$, then K is a standard subspace, $K \supset H'$ and $S_H^* = S_K$. With $\xi \in H, \eta \in K$ we have

$$(\xi, \eta) = (\xi, S_K\eta) = (\xi, S_H^*\eta) = (\eta, S_H\xi) = (\eta, \xi)$$

that is $\Im(\xi, \eta) = 0$, so $K \subset H'$, thus $H = K'$.

We have thus proved eq. (0.8.5) so, in particular, S_H is a closed operator. Therefore the range of the map (0.8.3) consists of all densely defined, closed, anti-linear involution on $\mathcal{H}$.

Concerning the injectivity of the map (0.8.3), this follows by the obvious equality $H = \{\xi \in D(S_H) : S_H\xi = \xi\}$.

Last, equation (0.8.4) is immediate. □

Proposition 2.1.3. *Let*

$$S_H = J_H \Delta_H^{1/2}$$

be the polar decomposition of $S = S_H$. Also set $J = J_H$, $\Delta = \Delta_H$. Then:

(a) J is an anti-unitary involution

$$J = J^* = J^{-1}$$

(b) $\Delta \equiv S^*S$ is a positive, non-singular selfadjoint linear operator, and

$$J\Delta J = \Delta^{-1}$$

therefore

$$Jf(\Delta)J = \bar{f}(\Delta^{-1})$$

for every complex Borel function f on $\mathbb{R}$ with complex conjugate $\bar{f}$. In particular J commutes with Δ^{it}.

(c) $J_{H'} = J_H$ *and* $\Delta_{H'} = \Delta_H^{-1}$.

(d) If U *is a unitary operator on* $\mathcal{H}$*, then* $UH = H$ *iff* $U\Delta_H U^* = \Delta_H$ *and* $UJ_H U^* = J_H$.

Proof. The identity $S^2 = 1$ on $D(S)$ shows the S, hence Δ is non-singular. Moreover it gives $J\Delta^{1/2}J\Delta^{1/2} = 1$ on $D(\Delta^{1/2})$, so $J\Delta^{1/2}J = \Delta^{-1/2}$. This implies (b) and in particular $J^2 = 1$.

The identity $S_H^* = S_{H'}$ gives $S_H^* = J_{H'}\Delta_{H'}^{1/2} = \Delta_H^{1/2}J_H = J_H\Delta_H^{-1/2}$ so $J_H = J_{H'}$ and $\Delta_{H'}^{1/2} = \Delta_H^{-1/2}$ by the uniqueness of the polar decomposition. So we have (a) and (c).

Finally, to check (d), note that if U is a unitary we have $S_{UH} = US_HU^*$. So $UH = H$ iff $US_HU^* = S_H$ which is equivalent to $UJ_HU^* = J_H$ and $U\Delta_H^{1/2}U^* = \Delta_H^{1/2}$ again by the uniqueness of the polar decomposition. □

The operator Δ_H is called the *modular operator* and J_H is called the *modular conjugation* of H.

The following theorem is the real Hilbert subspace (easier) version of the fundamental Tomita-Takesaki theorem for von Neumann algebras.

Theorem 2.1.4. *With* $\Delta = \Delta_H$ *and* $J = J_H$ *as above, we have for all* $t \in \mathbb{R}$*:*

$$\Delta^{it}H = H\,, \qquad JH = H'\,.$$

Proof. Δ^{it} commutes with $\Delta^{1/2}$ and J, thus with S. The first relation thus follows because if $\xi \in H$

$$S\Delta^{it}\xi = \Delta^{it}S\xi = \Delta^{it}\xi$$

namely $\Delta^{it}H \subset H$ for any $t \in \mathbb{R}$, thus $\Delta^{it}H = H$.

Concerning the second relation, notice that if $\xi \in H$ then

$$(J\xi, \xi) = (JS\xi, \xi) = (\Delta^{1/2}\xi, \xi) \in \mathbb{R}$$

thus for all $\xi, \eta \in H$

$$(J(\xi + \eta), \xi + \eta) = (J\xi, \xi) + (J\eta, \eta) + (J\xi, \eta) + (J\eta, \xi)$$

is real, so $\Im(J\xi, \eta) = 0$, namely $JH \subset H'$.

As $J_H = J_{H'}$ we also have $JH' \subset H'' = H$, namely $H' \subset JH$. □

Corollary 2.1.5. *Let* $\mathcal{H}$ *be an Hilbert space. There is a bijective correspondence between*

- *Standard subspaces H of $\mathcal{H}$;*

- *Pairs (A, J) where A is a selfadjoint linear operators on $\mathcal{H}$, J is a anti-unitary involution on $\mathcal{H}$ and $JAJ = -A$.*

Up to unitary equivalence, there is a bijective correspondence between

- *Standard subspaces H of some Hilbert space;*

- *Selfadjoint linear operators $B \geq 0$ on some Hilbert space $\mathcal{K}$.*

Proof. For the first equivalence: Given H, the pair $(\log \Delta_H, J_H)$ is the corresponding pair. Conversely, given (A, J), then $S \equiv Je^{\frac{1}{2}A}$ is a anti-linear closed involution and one gets a standard subspace by Prop. 0.8.2. Clearly these constructions are one the inverse of the other.

For the second equivalence: Given a Hilbert space $\mathcal{H}$ and a standard subspace H of $\mathcal{H}$, we define B as the restriction of $\log \Delta_H$ to its spectral subspace corresponding to $[0, \infty)$. Conversely, given B let $\mathcal{K}_0$ and $\mathcal{K}_+$ its spectral subspaces corresponding to $\{0\}$ and $(0, \infty)$, so we have a decomposition $B = 1 \oplus B_+$ on $\mathcal{K}_0 \oplus \mathcal{K}_+$. Choose anti-unitary involutions J_0 and J_+ on $\mathcal{K}_0$, $\mathcal{K}_+$ and set $B \equiv J_+ B_+ J_+ \oplus 1 \oplus B_+$ on $\mathcal{K} \equiv \mathcal{K}_+ \oplus \mathcal{K}_0 \oplus \mathcal{K}_+$. Then $J \equiv V(J_+ \oplus J_0 \oplus J_+)$ is a anti-unitary involution on $\mathcal{K}$ and JBJ, where V is the unitary involution on $\mathcal{K}$ that interchanges the two copies of $\mathcal{K}_+$ and is the identity on $\mathcal{K}_0$. As the choice of J_0, J_+ is unique up to unitary equivalence, this construction is unique up to unitary equivalence too. The rest is clear. □

Corollary 2.1.6. *Let E_λ be the spectral projection of Δ_H relative to the interval (λ^{-1}, λ), $\lambda > 0$. Then $E_\lambda H \subset H$. Therefore $\bigcup_{\lambda>0} E_\lambda H$ is a dense subspace of H and any of its elements ξ has Δ-exponential growth, namely $||\Delta_H^z \xi|| \leq e^{c|\Im z|}$ for some constant $c > 0$ and all $z \in \mathbb{C}$.*

Proof. The characteristic function f of the interval (λ^{-1}, λ) is real and $f(t) = f(t^{-1})$, thus $E_\lambda H \subset H$. Clearly all vectors in $E_\lambda H$ have exponential growth. The rest is clear because $E_\lambda \to 1$ strongly, as $\lambda \to +\infty$ because Δ is non-singular. □

Let H be a real linear subspace of $\mathcal{H}$ and V a one-parameter unitary group of $\mathcal{H}$ leaving H globally invariant. We now consider the following *(one particle) KMS condition* at inverse temperature $\beta > 0$: for every $\xi, \eta \in H$ there exists a function

F (depending on ξ, η), bounded and continuous on $\overline{\mathbb{S}_\beta} = \{z \in \mathbb{C} : 0 \leqslant \Im z \leqslant \beta\}$, analytic in the interior of $\mathbb{S}_\beta$ of $\overline{\mathbb{S}_\beta}$, such that

$$F(t) = (V(t)\xi, \eta), \quad F(t + i\beta) = (\eta, V(t)\xi) \ .$$

As the uniform limit of holomorphic functions is holomorphic, it follows easily that if the KMS condition holds for H, then it holds for $\overline{H}$.

If H is closed, the entire vectors of exponential type are dense (see the proof of Cor. 0.8.6) and it follows that the above KMS condition is equivalent to

$$(\xi, \eta) = (\eta, A^\beta \xi)$$

for a dense set of analytic vectors of exponential type $\xi, \eta \in H$, where $V(t) = A^{-it}$ with A is a non-singular, positive selfadjoint operator.

Proposition 2.1.7. *Let H be a cyclic closed real Hilbert subspace of $\mathcal{H}$.*

If H is standard, then Δ_H^{-it}, H satisfy the KMS condition at inverse temperature 1.

Conversely, if $V(t), H$ as above satisfy the KMS condition at inverse temperature 1*, then H a standard subspace of $\mathcal{H}$ and $V(t) = \Delta_H^{-it}$.*

Proof. Concerning the first assertion, let $\xi, \eta \in H$ be an entire vector as in Cor. 0.8.6. Then, omitting the suffix H, we have in particular $\xi, \eta \in D(\Delta^{1/2})$

$$(\eta, \Delta\xi) = (\Delta^{1/2}\eta, \Delta^{1/2}\xi) = (J\Delta^{1/2}\xi, J\Delta^{1/2}\eta) = (S\xi, S\eta) = (\xi, \eta) \tag{2.1.6}$$

so the KMS condition holds.

For the converse, let $V(t) = A^{-it}$ where A is a non-singular, positive selfadjoint operator. Then by the KMS condition we have

$$(\eta, A\xi) = (\xi, \eta) \tag{2.1.7}$$

in particular for any η and V-entire vectors ξ in H. Now if $\eta \in H \cap iH$ we have

$$(\xi, \eta) = i(i\eta, A\xi) = i(\xi, i\eta) = -(\xi, \eta)$$

for all $\xi \in H$, thus η is in the orthogonal complement of $H + iH$, which is zero as H is cyclic. So H is standard.

As $V(t)H = H$, A commutes with Δ, there is a common set of entire vectors for V and Δ^{it} which is dense in H (extend the argument in the proof of Cor. 0.8.6). On this set we then have by comparing eq. (0.8.6) and (0.8.7) we have

$$(\eta, \Delta\xi) = (\eta, A\xi) \ ,$$

so $A = \Delta$. $\square$

Corollary 2.1.8. *Let H be standard subspaces of $\mathcal{H}$ and $K \subset H$ real Hilbert subspace with $\Delta_H^{it} K = K$, $t \in \mathbb{R}$. Then $\Delta_K^{it} = \Delta_H^{it}|_{\mathcal{K}}$ where $\mathcal{K} \equiv \overline{K + iK}$.*

Proof. Immediate by the KMS condition. □

We now characterize the modular conjugation.

Proposition 2.1.9. *Let H be standard. Then J_H is the unique anti-unitary involution J of $\mathcal{H}$ such that $JH \supset H'$ and*

$$(J\xi, \xi) \geq 0, \quad \forall \xi \in H \, .$$

Proof. The positivity property holds for J_H because $(J_H\xi, \xi) = (\Delta_H^{1/2}\xi, \xi) \geq 0$ for all $\xi \in H$.

On the other hand, let J be a anti-unitary involution that satisfies the positivity condition then

$$(J\xi, \xi) \geq 0$$

for all $\xi \in H$. Then for all $\xi, \eta \in H$

$$(J(\xi + \eta), \xi + \eta) = (J\xi, \xi) + (J\eta, \eta) + (J\xi, \eta) + (J\eta, \xi) = (J\xi, \xi) + (J\eta, \eta) + 2(J\xi, \eta)$$

is real, so $(J\xi, \eta)$ is real. It follows that $JH \subset H'$. Assuming $JH \supset H'$ we then have $JH = H'$.

Moreover for all $\xi + i\eta \in H + iH$ we have

$$\begin{aligned}(JS_H(\xi + i\eta), \xi + i\eta) &= (J\xi + iJ\eta, \xi + i\eta) \\ &= (J\xi, \xi) + (J\eta, \eta) + i(J\eta, \xi) - i(J\xi, \eta) = (J\xi, \xi) + (J\eta, \eta) \geq 0 \, .\end{aligned}$$

So there is a canonical, positive selfadjoint operator Δ on $\mathcal{H}$, with $D(\Delta^{1/2}) \supset D(S_H)$ (use the Friederich extension) such that

$$(JS_H\xi, \xi) = (\Delta^{1/2}\xi, \xi), \quad \xi \in D(\Delta_H^{1/2}) = D(S_H) \, .$$

Now Δ_H^{it} commutes with S_H and with J (because $JH = H'$) so with JS_H. Therefore Δ_H^{it} commutes with $\Delta^{1/2}$. It follows that $\Delta_H^{1/2}$ commutes with $\Delta^{1/2}$, thus they have a common core, so $\Delta^{1/2}$ is selfadjoint on $D(S_H)$. We then have the equality $JS_H = \Delta^{1/2}$ or $S_H = J\Delta^{1/2}$ and, by the uniqueness of the polar decomposition, we get $\Delta = \Delta_H$ and $J = J_H$. □

Proposition 2.1.10. *Let $K \subset H$ be standard subspaces of $\mathcal{H}$. If $\Delta_H^{it} K = K$ for all $t \in \mathbb{R}$, then $K = H$.*

Proof. By assumption, $D(S_K) = K + iK$ is a dense complex subspace of $\mathcal{H}$ which is globally invariant for Δ_H^{it} for all $t \in \mathbb{R}$. As $D(S_K) \subset D(S_H) = D(\Delta_H^{1/2})$, it follows by Prop. 0.8.11 below that $D(S_K)$ is a core for S_H, thus $S_K = S_H$ and $K = H$. □

Proposition 2.1.11. *Let $U(t) = e^{itA}$ be a one-parameter unitary group on a Hilbert space $\mathcal{H}$ and $f : \mathbb{R} \to \mathbb{C}$ a locally bounded Borel function. If $\mathcal{D} \subset D(f(A))$ is a dense, U-invariant linear space, then $\mathcal{D}$ is a core for $f(A)$.*

Proof. By replacing f with $|f|$, we may assume that f is non-negative. Let $\xi \in \mathcal{H}$ be a vector orthogonal to $(f(A) + 1)\mathcal{D}$. We have to show that $\xi = 0$. If g is a function in the Schwartz space $S(\mathbb{R})$, we have

$$((f(A) + 1)g(A)\eta, \xi) = \int \tilde{g}(t)((f(A) + 1)e^{-itA}\eta, \xi)\mathrm{d}t = 0 \,, \tag{2.1.8}$$

for all $\eta \in \mathcal{D}$, where $\tilde{g}$ the Fourier anti-transform of f.

If now g is a bounded Borel function with compact support, we may choose a sequence of smooth functions g_n with compact support such that $g_n(A) \to g(A)$ weakly, thus eq. (0.8.8) holds for such a g.

Let then g be a bounded Borel function with compact support; we may write $g(\lambda) = (f(\lambda)+1)(f(\lambda)+1)^{-1}g(\lambda)$, therefore (0.8.8) with $g(\lambda)$ replaced with $(f(\lambda)+1)^{-1}g(\lambda)$ gives

$$(g(A)\eta, \xi) = 0 \,.$$

As we can choose a sequence g_n of bounded Borel function with compact support such that $g_n(A) \to 1$ strongly, it follows that $(\eta, \xi) = 0$ for all $\eta \in \mathcal{D}$, hence $\xi = 0$ because $\mathcal{D}$ is dense. □

Expectations and abelian subspaces

Let H be a separating real subspace of $\mathcal{H}$ and $K \subset H$ a closed real linear subspace. Then K is separating, so K is standard subspace of $\mathcal{K} \equiv \overline{H + iH}$.

Let E be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. We shall say that E is an *expectation* of H onto K if $EH \subset K$. Then $EH = K$ and $E|_H$ is the real orthogonal projection from H onto K. In this case we shall also say that there exists an expectation from H onto K, although the expectation is unique by definition.

Theorem 2.1.12. *With the above notations, the following are equivalent:*

(i) *There exists an expectation from H onto K;*

(ii) $\Delta_H^{it}K = K$ *for all* $t \in \mathbb{R}$. *In this case* $\Delta_H^{it}|_K = \Delta_K^{it}$.

(iii) $J_H E = EJ_H$. *In this case* $J_H|_{\mathcal{K}} = J_K$.

Proof. Assume (i); then for $\xi, \eta \in H$, so $E\xi, E\eta \in K \subset H$, we have

$$S_H E(\xi + i\eta) = S_H(E\xi + iE\eta) = E\xi - iE\eta = ES_H(\xi + i\eta)$$

thus $S_H E \subset ES_H$, namely E commutes with J_H and $\Delta_H^{1/2}$, and thus with Δ_H^{it}. As $J_H\Delta_H^{1/2}|_{\mathcal{K}} = S_H|_{\mathcal{K}} = S_K$ we then have $\Delta_H^{it}|_{\mathcal{K}} = \Delta_K^{it}$ and $J_H|_{\mathcal{K}} = J_K$. So we have shown (ii) and (iii).

Now notice that E, as an operator $\mathcal{H} \to \mathcal{K}$, is the adjoint of the inclusion map of $\mathcal{K}$ into $\mathcal{H}$. As this is $K - H$-real (see below), it follows that $J_K E J_H$ is $H - K$-real. Assuming (iii), we then have that $E = J_K E J_H$ is $H - K$-real, namely E is an expectation.

Finally, assuming (ii), we have $\Delta_H^{it}|_{\mathcal{K}} = \Delta_K^{it}$ by the KMS condition. If $\xi \in D(S_K)$ we have $S_H\xi = S_K\xi$, therefore $J_H\Delta_K^{1/2}\xi = J_H\Delta_H^{1/2}\xi = J_K\Delta_K^{1/2}\xi$, so $J_H|_{\mathcal{K}} = J_K$. Thus $S_H|_{\mathcal{K}} = S_K$, so $ED(S_H) = D(S_K)$ and so if $\xi \in H$ we have

$$E\xi \in D(S_K) \ \text{ and } \ S_K E\xi = ES_H\xi = E\xi$$

namely $E\xi \in K$ as desired. □

Note that, by replacing $\mathcal{H}$ with $\overline{H + iH}$ we may assume that $\mathcal{H}$ is standard in the definition of the expectation, namely we may replace E with the orthogonal projection from $\overline{H + iH}$ to $\mathcal{K}$.

Proposition 2.1.13. *Let H be a closed real subspace of $\mathcal{H}$. If H is abelian (i.e. $H \subset H'$) then H is separating. Moreover the following are equivalent:*

(i) *H is maximal abelian,*

(ii) *$H = H'$ (i.e. both H and H' are abelian),*

(iii) *H is abelian and cyclic,*

(iv) *H is standard and $\Delta_H^{it} = 1$.*

As a consequence, if $H_1 \subset H_2$ are closed abelian subspaces of $\mathcal{H}$, there exists an expectation from H_2 onto H_1.

All maximal abelian real subspaces of $\mathcal{H}$ are unitarily equivalent.

Proof. Concerning the first assertion, let H be abelian, we want to show that H is separating. Set $\mathcal{K} \equiv \overline{H + iH}$. As $H' = (H' \cap \mathcal{K}) \oplus (\mathcal{H} \ominus \mathcal{K})$, it follows that H is separating in $\mathcal{H}$ iff it is separating in $\mathcal{K}$, so we may assume that $\mathcal{K}$ is cyclic in $\mathcal{H}$. Now H cyclic implies H' separating, but then also $H \subset H'$ is separating.

Concerning the equivalence of the four assertions, $(i) \Leftrightarrow (ii)$ is straightforward.

$(ii) \Rightarrow (iii)$: $(H + iH)^\perp \subset H' = H$, thus $H = H''$ is cyclic.

$(iii) \Rightarrow (iv)$: H is separating because $H' \supset H$, thus H (so H') is standard. As $\Delta^{it}_{H'} H = H$, we then have $H = H'$ by Prop. 0.8.10. As $\Delta^{it}_{H'} = \Delta^{-it}_H$, it follows that Δ^{it}_H is trivial.

$(iv) \Rightarrow (ii)$: H is abelian because if $\xi, \eta \in H$ then $(\xi, \eta) = (S_H\xi, S_H\eta) = (J_H\xi, J_H\eta) = (\eta, \xi)$ is real. By the same argument H' is abelian too.

It remains to prove the uniqueness. If H is maximal abelian then the scalar product of $\mathcal{H}$ is real valued on H. By (iv) $H + iH = D(S_H) = \mathcal{H}$, so $\mathcal{H}$ is the complexification of the real Hilbert space H, which is unique. □

We shall say that a real subspace H of $\mathcal{H}$ is *abelian* if $H \subset H'$, namely $\Im(\xi_1, \xi_2) = 0$ for all $\xi_1, \xi_2 \in H$. If H is a standard subspace of H, its *center* $Z \equiv H \cap H'$ is clearly abelian as $(H \cap H')'$ is the closed linear span of H and H'. Clearly $\Delta^{it}_H Z = Z$ so by Thm. 0.8.12 and Prop.0.8.13 Δ^{it}_H is the identity on Z. Indeed we have

Proposition 2.1.14. $Z = \{\xi \in H : \ \Delta^{it}_H \xi = \xi\}$.

Proof. With $\xi \in H$ a fixed vector for Δ^{it}_H we have to show that $\xi \in H'$. For any vector $\eta \in H \cap D(\Delta_H)$ we have indeed

$$(\xi, \eta) = (S_H\xi, S_H\eta) = (J_H\Delta^{1/2}_H\xi, J_H\Delta^{1/2}_H\eta) = (\Delta^{1/2}_H\eta, \Delta^{1/2}_H\xi) = (\Delta_H\eta, \xi) = (\eta, \xi) \ ,$$

so $\Im(\xi, \eta) = 0$ that entails $\xi \in H'$ because $H \cap D(\Delta_H)$ is dense in H by Cor. 0.8.6. □

2.2 Borchers theorem (one-particle)

We now discuss the standard subspace version of a theorem of Borchers. The original version in the setting of von Neumann algebras will be discussed later. The following proof is adapted from Florig's proof of Borchers original theorem.

Theorem 2.2.1. *Let H be a standard subspace of a Hilbert space $\mathcal{H}$. Let U be a one-parameter group on $\mathcal{H}$, with generator P, satisfying*

$$U(s)H \subset H \ , \quad s \geq 0.$$

If $\pm P > 0$, the following commutation relations hold:

$$\begin{cases} \Delta^{it} U(s) \Delta^{-it} = U(e^{\mp 2\pi t} s), \\ JU(s)J = U(-s), \qquad t, s \in \mathbb{R}\,, \end{cases}$$

where $\Delta \equiv \Delta_H$, $J \equiv J_H$.

Proof. Replacing H with H' we may assume $P \geq 0$. We obtain the adjoint equations of the assumed commutation relations if we replace s by $-s$. Hence, we can assume $s \geq 0$. Let $\xi \in H, \xi' \in H'$. We define a bounded and continuous function on $\overline{\mathbb{S}_{1/2}}$ (as $P \geq 0$ and $\Im e^{2\pi z} s \geqslant 0$ for $z \in \mathbb{S}_{1/2}$)

$$f(z) = (\Delta^{i\bar{z}} \xi', U(e^{2\pi z} s) \Delta^{-iz} \xi)$$

which is analytic in $\mathbb{S}_{1/2}$. Note that $f(t)$ is real if $t \in \mathbb{R}$ because $U(e^{2\pi t} s)\Delta^{-it}\xi \in H$ and $\Delta^{i\bar{z}}\xi' \in H'$.

Set $V(t) = JU(-t)J$, $t \in \mathbb{R}$. Then $V(t)H \subset H$ if $t \geq 0$ because of $JH = H'$ and

$$U(t)H \subset H \Rightarrow U(t)H' \supset H' \Rightarrow U(-t)H' \subset H'\,, \quad t \geq 0\,.$$

Now we have

$$\begin{aligned} f\left(t + \frac{i}{2}\right) &= (\Delta^{-1/2}\Delta^{-it}\xi', U(e^{2\pi t + i\pi} s)\Delta^{-it}\Delta^{1/2}\xi) \\ &= (\Delta^{-1/2}\Delta^{-it}\xi', JV(e^{2\pi t} s)\Delta^{-it}\xi) \\ &= (\Delta^{-it}\xi', (J\Delta^{1/2})V(e^{2\pi t} s)\Delta^{-it}\xi) \\ &= (\Delta^{-it}\xi', V(e^{2\pi t} s)\Delta^{-it}\xi) \end{aligned} \tag{2.2.1}$$

for $t \in \mathbb{R}$. Therefore $f(t + i/2)$ is real if $t \in \mathbb{R}$ because $V(e^{2\pi t} s)\Delta^{-it}\xi \in H$ and $\Delta^{-it}\xi' \in H'$.

Therefore $f(t)$ is bounded continuous on $\overline{\mathbb{S}_{1/2}}$, analytic on $\mathbb{S}_{1/2}$ and real valued on the boundary lines $\Im z = 0$ and $\Im z = 1/2$. By the Schwarz reflection principle f can then be extended to a bounded entire function which has to constant by Liouville theorem. As H and H' are total in $\mathcal{H}$ we then have

$$\Delta^{it} U(e^{2\pi t} s)\Delta^{-it} = \Delta^{i0} U(e^{2\pi 0} s)\Delta^{-i0} = U(s)\,.$$

Moreover $f(0) = f(i/2)$ gives $U(s) = V(s) = JU(-s)J$. □

Remark. Let H be a standard subspace of a Hilbert space $\mathcal{H}$ and $K \subset H$ a standard subspace. We then have

$$J_K J_H H = J_K H' \subset J_K K = K\,,$$

so $\Gamma \equiv J_K J_H$ is a canonical unitary on $\mathcal{H}$ (corresponding to the canonical endomorphism in the von Neumann algebra setting). In particular we have the tunnel

$$H \supset K \supset \Gamma H \supset \Gamma K \supset \Gamma^2 H \supset \Gamma^2 K \supset \ \ldots\ . \tag{2.2.2}$$

If in Borchers theorem one set $K \equiv U(1)H$, then

$$\Gamma = U(2)\ .$$

Indeed $\Gamma = J_K J_H = U(1)J_H U(-1)J_H = U(1)U(1) = U(2)$.

Corollary 2.2.2. *Let H be a closed cyclic subspace of a Hilbert space $\mathcal{H}$ and U a one-parameter group on $\mathcal{H}$, with generator P with $\pm P > 0$ (or $P < 0$), such that $U(t)H = H$ for all $t \in \mathbb{R}$. If U has non non-zero fixed vector then $H = \mathcal{H}$.*

Proof. First assume that H is standard. Then $JU(t)J = U(t)$ by Prop. 0.8.3 (d), while $JU(t)J = U(-t)$ by Borchers theorem, so U is the identity and $\mathcal{H} = \{0\}$ because U has no non-zero fixed vectors.

Now in the general let H be cyclic. Then H' is separating, thus H' is standard in $\overline{H' + iH'}$ and U-invariant; so $H' = \{0\}$ as already seen. Therefore H, the symplectic complement of H', must be equal to $\mathcal{H}$. □

Converse of Borchers theorem

Lemma 2.2.3. *Let H_1, H_2 be standard subspaces of the Hilbert space $\mathcal{H}$, and assume that $UH_1 = H_2$, with U a unitary on $\mathcal{H}$. Then $H_2 \subset H_1$ iff $\Delta_{H_1}^{1/2}U^* \subset J_{H_1}U^* J_{H_1}\Delta_{H_1}^{1/2}$.*

Proof. We have $H_2 \subset H_1$ if and only if $S_{H_2} \subset S_{H_1}$. Setting $J_i \equiv J_{H_i}$, $\Delta_i \equiv \Delta_{H_i}$, the following equivalencies hold:

$$H_2 \subset H_1 \Leftrightarrow S_2 \subset S_1 \Leftrightarrow UJ_1\Delta_1^{1/2}U^* \subset J_1\Delta_1^{1/2} \Leftrightarrow \Delta_1^{1/2}U^* \subset J_1U^*J_1\Delta_1^{1/2}.$$

□

Theorem 2.2.4. *Let H be a standard space in the Hilbert space $\mathcal{H}$ and $U(s) = e^{isP}$ a one-parameter group of unitaries on $\mathcal{H}$ satisfying*

$$\Delta^{it} U(s) \Delta^{-it} = U(e^{\mp 2\pi t} s) \tag{2.2.3}$$

where $\Delta = \Delta_H$. The following are equivalent:

(*i*) *$U(s)H \subset H$ for $s \geq 0$;*

(*ii*) *$\pm P$ is positive.*

Proof. We prove the theorem choosing the minus sign in (0.9.3) and the plus sign in (*ii*). The statement with the opposite choice of the signs then follows by considering H' instead of H.

(*ii*) $\Rightarrow$ (*i*): First we show that the relation

$$JU(s)J = U(-s)$$

is valid. Let $\xi \in H, \xi' \in H'$ and $s \geqslant 0$. As $P \geq 0$ and $\Im e^{2\pi z} s \geqslant 0$ for $z \in \mathbb{S}_{1/2}$, the function

$$f(z) = (\Delta^{i\bar{z}} \xi', U(e^{2\pi z} s) \Delta^{-iz} \xi)$$

is bounded continuous on $\overline{\mathbb{S}_{1/2}}$ and analytic in $\mathbb{S}_{1/2}$. Set $V(t) = JU(-t)J$, $t \in \mathbb{R}$. Then we compute as in (0.9.1) that

$$f(t + i/2) = (\Delta^{-it} \xi', V(e^{2\pi t} s) \Delta^{-it} \xi)$$

for $t \in \mathbb{R}$. As by assumption (0.9.3) f is constant on the real axis, then f is constant on $\overline{\mathbb{S}_{1/2}}$ and in particular $f(0) = f(i/2)$, namely

$$(\xi', U(s)\xi) = (\xi', V(s)\xi) \; .$$

As both H and H' are cyclic, we then have $U(s) = V(s) = JU(-s)J$ for $s \geq 0$, hence for all real s (taking adjoints).

In order to prove (*i*), note now that by Thm. 0.6.1 we may assume that $P > 0$ or $P = 0$. When $P = 0$ isotony trivially holds, so we assume that P is positive and non-singular.

By Lemma 0.9.3, we get

$$U(s)H \subset H \Leftrightarrow \Delta^{1/2} U(s)^* \subset U(s) \Delta^{1/2}. \tag{2.2.4}$$

By the uniqueness, up to multiplicity, of the positive energy representation of **P**, the relation on the right hand side of (0.9.4) can be checked in just one non trivial representation, e.g. in the Schrodinger representation, where it holds true (see Sect. 0.21). So the left hand side of (0.9.4) also holds true.

$(i) \Rightarrow (ii)$: The unitary representation of **P** generated by U and Δ^{it} decompose in a direct sum of representations with $P < 0$ and $P = 0$ and $P > 0$. We have to show that the $P < 0$ component does not occur. In other words $P < 0$ and the isotony (i) imply that the underlying Hilbert space is $\{0\}$. Indeed, by what above proved, $P < 0$ implies $U(-s)H \subset H$ for $s \geq 0$, so $U(s)H \supset H$, for $s \geq 0$, and so H is globally U-invariant. Then $U(s)$ commutes with Δ^{it} and eq. (0.9.3) implies that $U(s) = 1$. □

2.3 Real maps

Let $\mathcal{K}$ and $\mathcal{H}$ be (complex) Hilbert spaces and $K \subset \mathcal{K}$, $H \subset \mathcal{H}$ standard subspaces. A bounded linear map $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is said to be $K - H$- *real* if $TK \subset H$.

Theorem 2.3.1. *Let $H \subset \mathcal{H}$, $K \subset \mathcal{K}$ be standard subspaces. Then for $T \in B(\mathcal{K}, \mathcal{H})$ the following conditions $(a) - (g)$ are equivalent* :

(a) *T is $K - H$-real;*

(b) *T^* is $H' - K'$-real;*

(c) *$J_K T^* J_H$ is $H - K$-real;*

(d) *$(T\eta, \xi') \in \mathbb{R}$ for all $\eta \in K$ and $\xi' \in H'$;*

(e) *$TS_K \subset S_H T$;*

(f) *$\Delta_H^{1/2} T \Delta_K^{-1/2}$ is defined on $D(\Delta_K^{-1/2})$ and coincides there with $J_H T J_K$;*

(g) *The map $T(s) \equiv \Delta_H^{-is} T \Delta_K^{is} \in B(\mathcal{K}, \mathcal{H})$, $s \in \mathbb{R}$, extends to a bounded strongly continuous map on $\overline{\mathbb{S}_{1/2}}$, analytic in $\mathbb{S}_{1/2}$ and satisfying*

$$T\Big(\frac{i}{2}\Big) = J_H T J_K. \tag{2.3.1}$$

Moreover, if the above equivalent conditions are satisfied, then we have

$$\|T(z)\| \leq \|T\|\,, \quad z \in \overline{\mathbb{S}_{1/2}}\,, \tag{2.3.2}$$

$$T(z+t) = \Delta_H^{-it}\, T(z)\, \Delta_K^{it}\,, \quad z \in \overline{\mathbb{S}_{1/2}}\,,\ t \in \mathbb{R}\,, \tag{2.3.3}$$

$$T\Big(s + \frac{i}{2}\Big) = J_H T(s) J_K\,, \quad s \in \mathbb{R} \tag{2.3.4}$$

and $T(s)$ is $K - H$-real and $T(s + \frac{i}{2})$ is $K' - H'$-real for all $s \in \mathbb{R}$.

Proof. $TK \subset H$ iff $\Im(T\eta, \xi') = 0$ for all $\eta \in K$, $\xi' \in H'$, thus iff $\Im(\eta, T^*\xi') = 0$. Therefore $(a) \Leftrightarrow (b)$.

Let us assume that (a) holds. Every $\xi \in D(S_K)$ is of the form $\xi = \xi_1 + i\xi_2$ with $\xi_1, \xi_2 \in K$. Hence we get

$$T\xi = T\xi_1 + iT\xi_2 \in H + iH \subset D(S_H),$$
$$S_H T\xi = T\xi_1 - iT\xi_2 = TS_K\xi\ ,$$

proving (e). Conversely, if (e) holds, then we have for every $\xi \in K \subset D(S_K)$

$$T\xi \in D(S_H) \text{ and } S_H T\xi = TS_K\xi = T\xi\,,$$

so $T\xi \in H$. Therefore $(a) \Leftrightarrow (e)$.

Since J_K is involutive and $S_K = \Delta_K^{-1/2} J_K$, $S_H = \Delta_H^{-1/2} J_H$, (e) is equivalent to

$$T\Delta_K^{-1/2} \subset \Delta_H^{-1/2} J_H T J_K\,.$$

This equation is equivalent to the validity of

$$\Delta_K^{1/2} T \Delta_H^{-1/2}\xi = J_H T J_K \xi\,, \quad \xi \in D(\Delta_K^{-1/2})\,,$$

and thus $(d) \Leftrightarrow (f)$.

$(d) \Leftrightarrow (b)$ follows immediately from $J_H H = H'$, $J_K K = K'$.

$(f) \Leftrightarrow (g)$: let $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ be entire vectors of exponential growth (Cor. 0.8.6) respectively for Δ_H and Δ_K. Then $f_{\xi,\eta}(z) \equiv (\Delta_H^{-iz} T \Delta_K^{iz}\eta, \xi) = (T\Delta_K^{iz}\eta, \Delta_H^{\overline{-iz}}\xi)$ is an entire function whose value at $z = t + i/2$ is

$$f_{\xi,\eta}(t + i/2) = (T\Delta_K^{-1/2}\Delta_K^{it}\eta, \Delta_H^{1/2}\Delta_H^{it}\xi) \tag{2.3.5}$$

$(t \in \mathbb{R})$. If (f) holds, then

$$f_{\xi,\eta}(t + i/2) = (J_H T J_K \Delta_K^{it}\eta, \Delta_H^{-it}\xi)$$

so, by the Three Line Theorem, $\|f_{\xi,\eta}(z)\| \leq \|T\|\,\|\xi\|\,\|\eta\|$. By the density of the ξ's in H and the η's in K we get (g).

Conversely, if (g) holds, then by (0.10.5) we get

$$(T\Delta_K^{-1/2}\eta, \Delta_H^{1/2}\xi) = (J_H T J_K \eta, \xi)$$

As the η's for a core for $\Delta_K^{-1/2}$ and the ξ's a core for $\Delta_H^{-1/2}$, it follows that the above equation holds true foll $\xi \in D(S_H)$ and $\eta \in D(S_K)$. This implies $\Delta_H^{1/2} T \Delta_K^{-1/2}\eta = J_H T J_K \eta$ for all $\eta \in D(S_K)$, namely (f) holds.

Now let us assume that the equivalent conditions $(a)-(g)$ are satisfied. Clearly we have $T(s+t) = \Delta_H^{-it}T(s)\Delta_K^{it}$ for $t, s \in \mathbb{R}$, so by (g) we have $T(z+t) = \Delta_H^{-it}T(z)\Delta_K^{it}$ for $t \in \mathbb{R}$ and $z \in \overline{\mathbb{S}_{1/2}}$, namely eq. (0.10.3) holds. Setting $z = -i/2$ we get (0.10.4) by (0.10.1).

So the map $T(\,\cdot\,)$ is bounded and for all $s \in \mathbb{R}$

$$\|T(s)\| = \|\Delta_H^{is} T \Delta_K^{-is}\| = \|T\|, \qquad \left\|T\Big(s + \frac{i}{2}\Big)\right\| = \|J_H T(s) J_K\| = \|T\|,$$

we get also (0.10.2) by the Three Line Theorem. Finally, since K and H are invariant under Δ_K^{is} and Δ_H^{is}, respectively, for every $s \in \mathbb{R}$, $T(s)$ is $K - H$-real and $T(s + \frac{i}{2}) = J_H T(s) J_K$ is $K' - H'$-real. □

Corollary 2.3.2. *Let $K \subset H$ be standard subspaces of $\mathcal{H}$. The map $W(s) = \Delta_H^{-is}\Delta_K^{is}$, $s \in \mathbb{R}$, extends to a strongly continuous map on $\overline{\mathbb{S}_{1/2}}$, analytic in $\mathbb{S}_{1/2}$, such that $W\Big(s + \frac{i}{2}\Big) = J_H W(s) J_K$. Moreover $W\Big(s + \frac{i}{2}\Big)$ is $K' - H'$-real.*

Proof. Immediate setting $T = 1$ in the above theorem. □

We now give a converse of Th. 0.10.1, namely we characterise the map T. Because of a subsequent application we treat the case with an a priori singularity at $z = i/2$, that will turn out to be a removable singularity. This can be generalized to the case with an a priori set of singular values on the strip boundary with one-dimensional Lebesgue measure zero [**?**].

Theorem 2.3.3. *Let $H \subset \mathcal{H}$ and $K \subset \mathcal{K}$ be standard subspaces. Let*

$$z \in \overline{\mathbb{S}_{1/2}}\backslash\{i/2\} \longmapsto T(z) \in B(\mathcal{K}, \mathcal{H})$$

be a bounded, weakly continuous map on $\overline{\mathbb{S}_{1/2}}\backslash\{i/2\}$ which is analytic in $\mathbb{S}_{1/2}$ and satisfies:

- *$T(s)$ is $K-H$-real for all $s \in \mathbb{R}$,*
- *$J_K T(s+i/2) J_H$ is $H-K$-real for all $s \in \mathbb{R}\backslash\{0\}$.*

Then there exists a $K-H$-real operator $T \in B(\mathcal{K}, \mathcal{H})$ such that

$$T(s) = \Delta_H^{-is} T \Delta_K^{is}, \qquad s \in \mathbb{R}. \tag{2.3.6}$$

Hence $T(\cdot)$ extends to a strongly continuous map on $\overline{\mathbb{S}_{1/2}}$ and satisfies

$$T(z+t) = \Delta_H^{-it} T(z) \Delta_K^{it}, \qquad z \in \overline{\mathbb{S}_{1/2}}, \, t \in \mathbb{R}, \tag{2.3.7}$$

$$T(s+i/2) = J_H T(s) J_K, \qquad s \in \mathbb{R}. \tag{2.3.8}$$

Proof. Fix $\xi' \in H'$, $\eta \in K$ and $t \in \mathbb{R}$ and consider the two functions of $s \in \mathbb{R}$

$$F(s) = (\xi', \Delta_H^{is} T(s+t) \Delta_K^{-is} \eta) \, .$$

As $T(\cdot)$ has a bounded continuos extension in the strip $\overline{\mathbb{S}_{1/2}}\backslash\{i/2\}$, analytic in $\mathbb{S}_{1/2}$, it follows that F has a bounded continuos extension in the strip $\overline{\mathbb{S}_{1/2}}\backslash\{i/2-t\}$, analytic in $\mathbb{S}_{1/2}$. Note that, by the assumed reality properties, we have $F(s)$ is real for $s \in \mathbb{R}$.

The upper boundary values are given by

$$\begin{aligned} F(s+i/2) &= (\Delta_H^{-1/2} \xi', \Delta_H^{is} T(s+t+i/2) \Delta_K^{-is} \Delta_K^{1/2} \eta) \\ &= (J_H \xi', \Delta_H^{is} T(s+t+i/2) \Delta_K^{-is} J_K \eta) = (\Delta_H^{-is} J_H T(s+t+i/2) J_K \Delta_K^{is} \eta, \xi') \, , \end{aligned}$$

$s \in \mathbb{R}$, $s+t \neq 0$. Therefore, again by the assumed reality properties, we have that $F(s+i/2)$ is real for all $s \in \mathbb{R}$, $s+t \neq 0$.

So, by the Schwarz reflection principle, we can extend F to a bounded function analytic on the complex plane except for $z \in -t + \frac{i}{2} + i\mathbb{Z}$. As F is bounded, these singularities are removable and so F has to be constant by Liouville theorem. In particular if $t, s \in \mathbb{R}$ we have

$$(\xi', \Delta_H^{is} T(s+t) \Delta_K^{-is} \eta) = (\xi', T(t) \eta)$$

and, by the cyclicity of H' and K, we conclude that

$$T(s+t) = \Delta_H^{-is} T(t) \Delta_K^{is} \, ;$$

so we get eq. (0.10.6) with $T = T(0)$.

2.4 Half-sided modular inclusions of standard subspaces

Let $K \subset H$ be standard subspaces of a Hilbert space $\mathcal{H}$. If

$$\Delta_H^{-it} K \subset K, \quad \text{for } \pm t \geqslant 0,$$

the inclusion $K \subset H$ is called a ±*half-sided modular inclusion* of standard subspaces (if the the ±sign is not specified we shall assume it to be +.

Given the situation described by the statement of Borchers Thm. 0.9.1, the theorem entails that $K \equiv U(1)H \subset H$, $t \geqslant 0$ is a half-sided modular inclusion:

$$\Delta_H^{-it} K = \Delta_H^{-it} U(1)H = U(e^{2\pi t})\Delta_H^{-it} H = \\ U(e^{2\pi t})H = U(1)U(e^{2\pi t} - 1)H \subset U(1)H = K.$$

Moreover

$$\Delta_H^{-it}\Delta_K^{it} = \Delta_H^{-it} U(1)\Delta_H^{it} U(-1) = U(e^{2\pi t})U(-1) = U(e^{2\pi t} - 1) \,,$$

In particular

$$K = U(1)H = \Delta_H^{-it}\Delta_K^{it}|_{t=\frac{1}{2\pi}\log 2} H \,. \tag{2.4.1}$$

On the other hand, given a half-sided modular inclusion $K \subset H$ we can define unitaries $U(t)$, $t \in \mathbb{R}$, by

$$U(e^{2\pi t} - 1) = \Delta_H^{-it}\Delta_K^{it} \,.$$

Thm 0.11.1 shows that U is one-parameter unitary group with positive generator and H, K, U satisfy the conditions of Borchers theorem, thus providing a converse to it.

Wiesbrock theorem (one-particle)

The following theorem is the standard subspace version of a theorem for hsm inclusion of von Neumann algebras that was first point out by Wiesbrock and we shall discuss in a later chapter. The original proof had a serious gap in the proof. A different and complete proof was later given by Araki-Zsido and Borchers.

Theorem 2.4.1. *Let $K \subset H$ be a half-sided modular inclusion of standard subspaces of the Hilbert space $\mathcal{H}$, namely $\Delta_H^{-it} K \subset K$, $t \geq 0$.*

There exists a positive energy unitary representation V of $\mathbf{P}$ *on* $\mathcal{H}$ *determined by*

$$V(\delta(2\pi s)) = \Delta_H^{-is}, \qquad V(\delta_1(2\pi s)) = \Delta_K^{-is} .$$

Here δ_1 *is the one-parameter subgroup of* $\mathbf{P}$ *of dilations of* $(1, \infty)$ *(i.e.* $\delta_1(s) = \tau(-1)\delta(s)\tau(1)$*).*

The translation unitaries $U(t) \equiv V(\tau(t))$ *are defined by* $U(e^{2\pi t} - 1) = \Delta_H^{-it}\Delta_K^{it}$ *and satisfy* $U(s)H \subset H$, $s \geq 0$, *and* $K = U(1)H$.

Proof. By Cor. 0.10.2 we have a strongly continuous map W on $\overline{\mathbb{S}_{1/2}}$, analytic in $\mathbb{S}_{1/2}$, such that $\|W(z)\| \leq 1$, $z \in \overline{\mathbb{S}_{1/2}}$ and

$$W(s) = \Delta_H^{-is}\Delta_K^{is} \text{ and } W\Big(s + \frac{i}{2}\Big) = J_H W(s) J_K , \quad s \in \mathbb{R} . \tag{2.4.2}$$

Now the map W has the following reality properties:

(a_1): $W(z)$ is $K - K$-real for $z \in (0, \infty)$,

(a_2): $W(z)$ is $K' - K'$-real for $z \in (-\infty, 0)$,

(a_3): $W(z)$ is $K' - K'$-real for $z \in \mathbb{R} + \frac{i}{2}$.

Indeed, (a_1) follows by assumed half-sided modular invariance as if $s \geq 0$ we have

$$W(s)K = \Delta_H^{-is}\Delta_K^{is}K \subset \Delta_H^{-is}K = K .$$

Since for $s \in \mathbb{R}$ we have

$$\Delta_H^{is}K' \subset K' \Leftrightarrow \Delta_H^{-is}K' \supset K' \Leftrightarrow \Delta_H^{-is}K \subset K ,$$

we can see that (a_2) follows from (a_1) because if $s \leq 0$

$$W(s)K' = \Delta_H^{-is}\Delta_K^{is}K' = \Delta_H^{-is}K' \subset K' .$$

Concerning (a_3), this holds true because by Cor. 0.10.2 because $W(s+i/2)K' \subset H'$ and $H' \subset K'$.

Now we make a change of variable. We consider the logarithm in the complex cut plane $-\pi < \mathrm{Arg} z \leq \pi$, set

$$h(z) \equiv \frac{1}{2\pi}\log(1 + e^{2\pi z}), \quad z \in \overline{\mathbb{S}_{1/2}} ,$$

and note that h is maps conformally $\mathbb{S}_{1/2}$ onto $\mathbb{S}_{1/2}$ and sets bijections on the boundary as follows

$$
\begin{gathered}
(-\infty, \infty) \leftrightarrow (0, \infty)\,, \\
(-\infty, 0) + \frac{i}{2} \leftrightarrow (-\infty, 0)\,, \\
(0, \infty) + \frac{i}{2} \leftrightarrow (-\infty, \infty) + \frac{i}{2}\,,
\end{gathered}
$$

thus

$$\overline{\mathbb{S}_{1/2}} \backslash \{i/2\} \xleftrightarrow{h} \overline{\mathbb{S}_{1/2}} \backslash \{0\}\,.$$

Set

$$T(z) \equiv W(h(z)) = W(\frac{1}{2\pi}\log(1 + e^{2\pi z})), \quad z \in \overline{\mathbb{S}_{1/2}} \backslash \{i/2\}\,.$$

Then T is a bounded, strongly continuous map on $\overline{\mathbb{S}_{1/2}} \backslash \{i/2\}$, analytic in $\mathbb{S}_{1/2}$. By Cor. 0.10.2 we have

$$
\begin{aligned}
\|T(z)\| \le 1\,, &\quad z \in \overline{\mathbb{S}_{1/2}} \setminus \{i/2\}\,, \\
T(s) \text{ is } K - K\text{-real}\,, &\quad s \in \mathbb{R}\,, \\
J_K T(s + i/2) J_K \text{ is } K - K\text{-real}, &\quad s \in \mathbb{R}\,.
\end{aligned}
\tag{2.4.3}
$$

By Th. 0.10.3 T extends to a strongly continuous map on $\overline{\mathbb{S}_{1/2}}$ and we have the relation $\Delta_K^{is} T(z) \Delta_K^{-is} = T(z - s)$, thus

$$\Delta_K^{is} W\Big(\frac{1}{2\pi}\log(1 + e^{2\pi t})\Big)\Delta_K^{-is} = W\Big(\frac{1}{2\pi}\log(1 + e^{2\pi(t-s)})\Big)\,. \tag{2.4.4}$$

Multiplying the above equation from the left by Δ_H^{-is} and from the right by Δ_K^{is} we get with $e^{2\pi a} = e^{2\pi s} + 1$

$$
\begin{aligned}
\Delta_H^{-is}\Delta_K^{is}\Delta_H^{-ia}\Delta_K^{ia} &= W\Big(\frac{1}{2\pi}\log((1 + e^{2\pi(t-s)}) + t\Big) \\
&= W\Big(\frac{1}{2\pi}\log(e^{2\pi s} + e^{2\pi t})\Big) = W\Big(\frac{1}{2\pi}\log(e^{2\pi a} + e^{2\pi t} - 1)\Big)
\end{aligned}
$$

Since the above expression is symmetric in a and t, the $W(t)$'s form a commutative family. We set

$$U(e^{2\pi t} - 1) \equiv W(t)\,,$$

then the above equation reads

$$U(e^{2\pi t}-1)U(e^{2\pi a}-1)=U(e^{2\pi(t+a)}-2)$$

showing that U is additive for positive arguments hence, by analytic continuation, for any real arguments. Thus U is an continuous one-parameter group of unitaries, which allows a strongly continuous extension in the upper half plane $\Im z \geq 0$, analytic in $\Im z > 0$ and norm bounded by 1. Consequently

$$U(s)=e^{isP}\,,\quad s\in\mathbb{R}\,, \tag{2.4.5}$$

for some positive selfadjoint operator P in $\mathcal{H}$.

Note that, by eq. (0.11.4), we obtain $\Delta_K^{-it}U(s)\Delta_K^{it} = U(e^{2\pi t}s)$ for all $s,t\in\mathbb{R}$, hence

$$\begin{aligned}\Delta_H^{-it}\,U(s)\,\Delta_H^{it} &= (\Delta_H^{-it}\Delta_K^{it})\,\Delta_K^{-it}\,U(s)\,\Delta_K^{it}(\Delta_K^{-it}\Delta_H^{it}) = \\ &= U(1-e^{2\pi t})\,U(e^{2\pi t}s)\,U(1-e^{2\pi t})^* = U(e^{2\pi t}s)\,,\end{aligned} \tag{2.4.6}$$

By (0.11.6) the map

$$V:\mathbf{P}\ni\tau(s)\delta(t)\mapsto U(s)\Delta_H^{-it/2\pi}$$

is a unitary representation on $\mathcal{H}$ of $\mathbf{P}$.

It remains to prove property that $K=U(1)$ and $U(s)H\subset H$ for $s\geq 0$. Since

$$\Delta_H^{-it}\Delta_K^{it}K=\Delta_H^{-it}K\subset\Delta_H^{-it}H=H\,,\quad t\geq 0,$$

we have from the definition of U that

$$U(s)K\subset H\,,\quad s\geq -1, \tag{2.4.7}$$

and in particular $U(1)H\supset K$. Now

$$\Delta_{U(1)H}^{it}=U(1)\Delta_H^{it}U(-1)=\Delta_H^{it}U(1-e^{2\pi t})=\Delta_H^{it}\Delta_H^{-it}\Delta_K^{it}=\Delta_K^{it}$$

so by Prop. 0.8.10

$$K=U(1)H\,, \tag{2.4.8}$$

hence

$$U(s)H=U(-1)U(s)U(1)H=U(-1)U(s)K\subset U(-1)K=H$$

as desired. □

We give two immediate corollaries.

Corollary 2.4.2. *Let $K \subset H$ be a hsm standard inclusion of standard subspaces of $\mathcal{H}$. Then K and H have the same center Z and we have the direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0$, where $\mathcal{H}_0 = \overline{Z + iZ}$, with $K = K_1 \oplus Z$, $H = H_1 \oplus Z$.*

Then $K_1 \subset H_1$ is a hsm inclusion of standard subspaces and both $\Delta^{it}_{H_1}$ and $\Delta^{it}_{K_1}$ weaky tend to zero as $t \to \pm\infty$.

Proof. By Prop. 0.7.1 a vector $\xi \in \mathcal{H}$ is fixed by Δ^{it}_H iff it is fixed by Δ^{it}_K. By Cor. 0.8.14 the center Z of H and of K then coincide. Denoting by E the expectation onto Z we have the decomposition as stated with $H_1 \equiv (1 - E)H$, $K_1 \equiv (1 - E)K$. The rest is clear by Prop. 0.7.1. □

Corollary 2.4.3. *Let $K \subset H$ be an inclusion of standard subspaces of $\mathcal{H}$. Then $K \subset H$ is positive half-sided modular iff $H' \subset K'$ is negative half-sided modular.*

Proof. $K \subset H$ is ±half-sided modular iff $K = U(1)H$ with U a one-parameter group with positive generator such that $U(\pm t)H \subset H$ for $t \geq 0$. Then $H' = U(\mp 1)K' \subset K'$ is ∓half-sided modular. □

2.5 Appendix. von Neumann algebras and real Hilbert subspaces

We give here a flash forward by mentioning the relation with the von Neumann algebra context, which is our motivational setting.

Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a vector. Clearly

$$H_M \equiv \overline{M_{\mathrm{sa}}\Omega} \tag{2.5.1}$$

is a real Hilbert subspace of H, where M_{sa} denotes the selfadjoint part of M. It follows immediately from the definitions that

$$\Omega \text{ is cyclic} \Leftrightarrow H_M \text{ is cyclic} \tag{2.5.2}$$

$$\Omega \text{ is separating} \Leftrightarrow H_M \text{ is separating} \tag{2.5.3}$$

With a fixed vector Ω, the map $M \mapsto H_M$ is non injective. Indeed H_M contains only a part of the information on M (further order structure on H_M recovers the full algebraic structure of M by a theorem of Connes). However H_M gives the full knowledge of the modular operator and modular conjugation of M

$$\Delta_M = \Delta_{H_M}, \quad J_M = J_{H_M}$$

because the KMS property and Prop. 0.8.10. In particular

$$H'_M = H_{M'}$$

because $H'_M = J_{H_M} H_M = J_M \overline{M_{\mathrm{sa}}\Omega} = \overline{J_M M_{\mathrm{sa}} J_M \Omega} = \overline{M'_{\mathrm{sa}}\Omega} = H_{M'}$.

Proposition 2.5.1. *Let M be a von Neumann algebra with cyclic and separating vector Ω.*

(*a*) *If $N_1, N_2 \subset M$ are a von Neumann subalgebras with $H_{N_1} \subset H_{N_2}$, then $N_1 \subset N_2$.*

(*b*) *If N is a von Neumann algebra commuting with M and $H_N = H'_M$, then $N = M'$.*

Proof. (*a*): Let x be a selfadjoint element of N_1. As $N_{1\mathrm{sa}}\Omega \subset \overline{N_{2\mathrm{sa}}\Omega}$, there exists a sequence of of elements $x_n \in N_{2\mathrm{sa}}$ such that $x_n\Omega \to x\Omega$. Then $x_n x'\Omega \to xx'\Omega$ for all $x' \in M'$. Now x_n and x are all bounded and selfadjoint and x_n tends to x strongly on a dense set (common core), thus in the strong resolvent sense. We conclude that x is affiliated to N_2 and indeed $x \in N_2$ as x is bounded.

(*b*) We have $N \subset M'$ and $H_N = H'_M = H_{M'}$, so $N = M'$ by (*a*). □

Chapter 3

Möbius covariant nets of standard subspaces

We now begin the study of the nets of standard subspaces that are covariant with respect to a positive energy representation of $\mathbf{G}$. All the information on the one-particle Hilbert space is contained in this structure.

3.1 Definition

Let $\mathcal{H}$ be a complex Hilbert space. A *local Möbius covariant net H of real linear subspaces* of $\mathcal{H}$ on the intervals of S^1 is a map

$$I \to H(I)$$

that associates to each interval $I \in \mathcal{I}$ a closed, real linear subspace of a $\mathcal{H}$, that verifies the following properties 1,2,3,4,5:

1. Isotony : *If I_1, I_2 are intervals and $I_1 \subset I_2$, then*

$$H(I_1) \subset H(I_2) .$$

2. Möbius covariance**:** *There is a unitary representation U of $\mathbf{G}$ on $\mathcal{H}$ such that*

$$U(g)H(I) = H(gI) , \quad g \in \mathbf{G},\ I \in \mathcal{I}.$$

3. Positivity of the energy : *U is a positive energy representation.*

4. Cyclicity : *The complex linear span of all the $H(I)$'s is dense in $\mathcal{H}$.*

5. Locality : *If I_1 and I_2 are disjoint intervals then*

$$H(I_1) \subset H(I_2)'$$

3.2 Reeh-Schlieder theorem (one-particle)

Theorem 3.2.1. *Let H be a local Möbius covariant net of real linear subspaces of $\mathcal{H}$ on S^1. Then $H(I)$ is a standard subspace of $\mathcal{H}$ for each $I \in \mathcal{I}$.*

Proof. Fix then an interval I and let $\eta \in \mathcal{H}$ be orthogonal to $H(I)$. We want to show that $\eta = 0$.

Choose an interval I_0 with $\bar{I}_0$ contained in I. Then for all real t in a neighbourhood of 0 such that $\tau_I(t)I_0 \subset I$ we have

$$f(t) \equiv (\eta, U(\tau_I(t))\xi) = (\eta, U(\tau_I(t))\xi) = 0 ,$$

for all $\xi \in H(I_0)$. As the generator of the translation unitary group is positive, f extends to a continuos function on the upper half-plane $\Im z \geq 0$, analytic in $\Im z > 0$, thus $f = 0$ identically. Thus η is orthogonal to $H(\tau_I(t)I_0) = U(\tau_I(t))H(I_0)$ for all $t \in \mathbb{R}$.

By the same argument, with τ_I replaced by $\tau_{I'}$, we see that η is orthogonal to $H(\tau_I'(t)I_0)$for all $t \in \mathbb{R}$. We may now repeat the argument replacing $I_0 \subset I$ with $\tau_I(s)I_0 \subset \tau_I(s)I$ and see that η is orthogonal to $H(\tau_{I'}(t)\tau_I(s)I_0)$ for all $t, s \in \mathbb{R}$.

As the subgroup of $\mathbf{G}$ generated by $\tau_I(t)$ and $\tau_{I'}(s)$, $t, s \in \mathbb{R}$, is equal to $\mathbf{G}$, iterating the above argument we get that η is orthogonal to $U(g)H(I_0) = H(gI_0)$ for all $g \in \mathbf{G}$. As $\mathbf{G}$ acts transitively on $\mathcal{I}$, $\eta = 0$ by the assumed cyclicity of H.

So $H(I)$ is cyclic. As by locality $H(I)' \supset H(I')$ and $H(I')$ is also cyclic, it follows that $H(I)$ is separating too. □

Because of the Reeh-Schlieder theorem we shall refer to a local Möbius covariant net of real linear subspaces also as a local *Möbius covariant net of standard subspaces*.

3.3 Bisognano-Wichmann property and Haag duality (one-particle)

Theorem 3.3.1. *Let $I \in \mathcal{I}$ and Δ_I, J_I be the modular operator and the modular conjugation of $H(I)$. Then we have:*

- Bisognano-Wichmann property*:*

$$\Delta_I^{it} = U(\delta_I(-2\pi t)), \quad t \in \mathbb{R}\,; \tag{3.3.1}$$

- *U extends to an (anti-)unitary representation of $\mathbf{G}_2$ determined by*

$$U(r_I) = J_I,\ I \in \mathcal{I}\,,$$

acting covariantly on $\mathcal{A}$, namely

$$U(g)H(I) = H(gI) \quad g \in \mathbf{G}_2,\ I \in \mathcal{I}\,;$$

- Haag duality*: For every interval I*

$$H(I') = H(I)'\,.$$

Proof. As $U(\delta_I(-2\pi t))$ leaves $H(I)$ globally invariant, $U(\delta_I(-2\pi t))$ and Δ_I^{is} commute. Thus $z_I(t) \equiv \Delta_I^{it}U(\delta_I(2\pi t))$ is a one-parameter unitary group.

By Borchers theorem $U(\delta_I(-2\pi t))$ and Δ_I^{it} have the same commutation relations with the translation unitaries $U(\tau_I(s))$, namely $U(\tau_I(s))$ commutes with $z_I(t)$. By the same reason $U(\tau_{I'}(\cdot))$ commutes with z_I. Now τ_I and $\tau_{I'}$ generate $\mathbf{G}$, thus z_I commutes with $U(\mathbf{G})$. Since $U(g)z_I(t)U(g)^* = z_{gI}(t)$, it follows that z_I is independent of I. But $z_{I'}(t) = z_I(-t)$, thus z_I is trivial being a one-parameter group and we have shown (0.15.1).

Now Δ_I^{-1} is the modular operator of $H(I)'$. By the above geometric action the modular group Δ_I^{-it} of $H(I)'$ leaves globally invariant the standard subspace $H(I')$. By Prop. 0.8.10 we then have $H(I') = H(I)'$, namely Haag duality holds.

It remains to show the $\mathbf{G}_2$-covariance. Again by applying Borchers theorem to the commutation relations between $U(\tau_I(\cdot))$, or $U(\tau_{I'}(\cdot))$, and J_I we see that

$$J_I U(g) J_I = U(r_I g r_I)$$

for all $g \in \mathbf{G}$. We can thus extend U from $\mathbf{G}$ to $\mathbf{G}_2$ by setting $U(gr_I) \equiv U(g)J_I$ for all $g \in \mathbf{G}$.

Now any interval I_0 is of the form $I_0 = gI$ for some $g \in \mathbf{G}$, thus

$$J_I H(I_0) = J_I H(gI) = J_I U(g)H(I) = U(r_I g r_I)J_I H(I)$$
$$= U(r_I g r_I)H(r_I I) = H(r_I gI) = H(r_I I_0)\ , \quad (3.3.2)$$

that completes the proof. □

Corollary 3.3.2. *The representation U of $\mathbf{G}$ is unique.*

Proof. Immediate by the equation (0.15.1). □

It follows that if H_1, H_2 are two local Möbius covariant nets and V is a unitary between the underlying Hilbert spaces such that $VH_1(I) = H_2(I)$, then V automatically intertwines the unitary Möbius group actions.

Note also that the *continuity* property holds true: if $I \in \mathcal{I}$ and I_n is an increasing sequence of intervals such that $\cup_n I_n = I$, then $H(I)$ is equal to $\overline{\cup_n H(I_n)}$. Indeed let $g \in \mathbf{G}$ be such that $\overline{gI} \subset I$. Then $H(I_n) \supset H(gI)$ for some n. As g converges to the identity in $\mathbf{G}$ any vector $\xi \in H(I)$ is the limit of vectors $U(g)\xi \in \cup_n H(I_n)$.

Corollary 3.3.3 (Additivity)**.** *Let I and I_i be intervals with $I \subset \cup_i I_i$. Then $H(I)$ is contained in the closed linear span of the $H(I_i)$'s.*

Proof. First we prove the statement in the case $\cup_i I_i \supset \bar{I}$. As $\bar{I}$ is compact, there exists a finite subfamily of $\{I_i\}$ that covers I, so we may assume that $\{I_i\}$ is itself a finite family. We can then choose a non-empty, open interval $L \subset I$ such that, for any fixed $s \in \mathbb{R}$, $\delta_I(s)L$ is contained in a least one of the I_i's. Let K be the real Hilbert space linearly spanned by the $H(\delta_I(s)L) = \Delta_I^{-2i\pi s}H(L)$ as s varies in $\mathbb{R}$. Clearly $K \subset H(I)$ is cyclic and K is globally invariant under the modular group of $H(I)$. So $K = H(I)$. As the closed linear span of the $H(I_i)$'s contains K, it also contains $H(I)$.

It remains to remove the assumption $\cup_i I_i \supset \bar{I}$. But for every interval I_0 with $\bar{I}_0 \subset I$ the closed linear span of the $H(I_i)$'s contains $H(I_0)$, so it contains $H(I)$ by the above continuity property. □

3.4 Non-degenerate nets

We shall say that the net H is *non-degenerate* if the real linear span generated by all the $H(I)$'s is dense in $\mathcal{H}$.

The property of being non-degenerate is indeed equivalent to several other requirements.

Proposition 3.4.1. *Assume all properties* $1-5$ *for* H*. The following are equivalent:*

(i) *U does not contain the identity representation.*

(ii) *$H(I) \cap H(I)' = \{0\}$ (factoriality).*

(iii) *For any given two points p_1, p_2 the real linear span generated by the $H(I)$, $p_1, p_2 \notin I$, is dense in $\mathcal{H}$.*

(iv) *H is non-degenerate.*

(v) $\bigcap_{I\in\mathcal{I}} H(I) = \{0\}$.

Proof. $(i) \Rightarrow (ii)$: Let I be in an interval and $\xi \in H(I) \cap H(I)'$. By Cor 0.8.8 and Prop. 0.8.13 $\Delta_I^{it}|_{H(I)\cap H(I)'}$ is the identity on $H(I) \cap H(I)'$. So ξ is fixed by $U(\delta_I(-2\pi t)) = \Delta_I^{it}$. By the vanishing of the matrix coefficient theorem $U(g)\xi = \xi$, thus $\xi = 0$ by assumption.

$(ii) \Rightarrow (iii)$: By Haag duality $H(I) + iH(I') = H(I) + iH(I)'$ is dense and (iii) follows by taking I with endpoints p_1, p_2.

$(iii) \Rightarrow (iv)$: Obvious.

$(iv) \Leftrightarrow (v)$: Immediate by Haag duality.

$(v) \Rightarrow (i)$: Let ξ be U-invariant vector. Then $\Delta_I^{it}\xi = U(\delta_I(-2\pi t))\xi = \xi$ for any interval I. Then $\eta \equiv \xi \pm J_I\xi$ are vectors fixed by the extension of U to $\mathbf{G}_2$, thus $S_{I_0}\eta = \eta$ for all $I_0 \in \mathcal{I}$, namely $\eta \in \cap_{I\in\mathcal{I}} H(I) = \{0\}$. So $\xi = 0$. □

Corollary 3.4.2 (One-particle spin-statistics relation)**.** *If H is defined by all the above properties* $1-5$ *but with* $\bar{\mathbf{G}}$ *in place of* $\mathbf{G}$*, then automatically U is indeed a representation of* $\mathbf{G}$.

Proof. Note that in all the above discussion we may have used $\bar{\mathbf{G}}$ in place of $\mathbf{G}$, in particular it follows by the Bisognano-Wichmann property that $U(R(\pi)) = J_{I_1}J_{I_2}$ with I_1, I_2 the right and the upper semicircle. As $J_{I_1}H(I_2) = H(I_2)$, J_{I_1} and J_{I_2} commute, so $U(R(2\pi)) = U(R(\pi))^2 = 1$. □

Corollary 3.4.3 (Equivalence between positive energy and KMS for dilation)**.** *Let H be defined by all the above properties $1-5$ except positivity of the energy. Then positivity of the energy follows by the equality $\Delta_I^{it} = U(\delta_I(-2\pi t))$.*

Proof. Immediate by the converse of Borchers theorem (Thm. 0.9.4). □

3.5 Modular construction of nets on S^1

Let's fix a (anti)-unitary, positive energy representation U of $\mathbf{G}_2$ on a Hilbert space $\mathcal{H}$. Namely $U|_{\mathbf{G}}$ is a unitary, positive energy representation U of $\mathbf{G}$ on $\mathcal{H}$ and a there is a anti-unitary involution J on $\mathcal{H}$ such that

$$JU(g)J = U(rgr) , \quad g \in \mathbf{G} ,$$

where $r : z \mapsto \bar{z}$ is the reflection on S^1 associated with the upper semicircle. Then the involution J_I associated with any interval $I \in \mathcal{I}$ is given by the formula

$$J_I \equiv U(g)JU(g)^* ,$$

where $g \in \mathbf{G}$ is any element of $\mathbf{G}$ mapping the upper semicircle onto I. In other words

$$J_I \equiv U(r_I) ,$$

where r_I is the reflection of S^1 associated with I.

With I any interval, define Δ_I to be the positive, non-singular selfadjoint opertaor on $\mathcal{H}$ such that

$$\Delta_I^{it} \equiv U(\delta_I(-2\pi t)), \quad t \in \mathbb{R} ,$$

namely $-\frac{1}{2\pi}\log\Delta_I$ is the infinitesimal generator of the one-parameter unitary group $U(\delta_I(\cdot))$ of dilations associated with I.

Note that

$$J_I\Delta_I J_I = \Delta_I^{-1} \tag{3.5.1}$$

because $r_I\delta_I(t)r_I = \delta_I(-t)$.

Finally, define the anti-linear operator on $\mathcal{H}$

$$S_I \equiv J_I\Delta_I^{1/2} : \mathcal{H} \to \mathcal{H} .$$

Proposition 3.5.1. *S_I is a densely defined, antilinear, closed operator on $\mathcal{H}$ with $S_I^2 = 1$ on $D(S_I)$.*

Proof. Density and closeness follow from the corresponding property of Δ_I, anti-linearity from the anti-linearity of J_I. Now, $R(S_I) \subset D(S_I) \equiv D(\Delta_I^{1/2})$, where R denotes the range; indeed by eq. (0.17.1) we have that $J_I\Delta_I^{1/2}\xi = \Delta_I^{-1/2}J_I\xi \in D(\Delta_I^{1/2})$. But we get immediately that $S_I^2 = J_I\Delta_I^{1/2}J_I\Delta_I^{1/2} = \Delta_I^{-1/2}\Delta_I^{1/2} \subset 1$ and therefore if $\xi \in D(S_I)$ then $\xi = S_I(S_I\xi) \in R(S_I)$, so we can conclude. □

Let us now define real subspaces of $\mathcal{H}$ associated with any $I \in \mathcal{I}$ as the one associated with S_I:

$$H(I) = \{\xi \in D(S_I) \ : \ S_I\xi = \xi\}.$$

By the Section 0.8, each $H(I)$ is a standard subspace in $\mathcal{H}$, and J_I, Δ_I are the modular operator and conjugation of $H(I)$. In particular we have $J_I H(I) = H(I)'$.

Theorem 3.5.2. *Let U be a (anti-)unitary, positive energy representation of $\mathbf{G}_2$ on $\mathcal{H}$ and define $H(I)$, $I \in \mathcal{I}$, as above.*

Then H is a local Möbius covariant local net of real Hilbert spaces of $\mathcal{H}$.

Proof. First we show that the representation U acts covariantly on the family $\{H(I) : I \in \mathcal{I}\}$, namely,

$$U(g)H(I) = H(gI)\,, \quad g \in \mathbf{G}\,. \tag{3.5.2}$$

As we have $U(g)\Delta_I^{it}U(g)^* = \Delta_{gI}^{it}$ and $U(g)J_IU(g)^* = J_{gI}$ it follows that that

$$U(g)S_IU(g)^* = S_{gI}$$

so eq. (0.17.2) holds.

Locality, indeed Haag duality, will follow from the identity $S_{I'} = S_I^*$. Indeed, since $r_{I'} = r_I$ we have $J_I = J_{I'}$ and since $\delta_{I'}(t) = \delta_I(-t)$, we have $\Delta_{I'} = \Delta_I^{-1}$. Thus

$$S_I^* = (J_I\Delta_I^{1/2})^* = \Delta_I^{1/2}J_I = J_I\Delta_I^{-1/2} = J_{I'}\Delta_{I'}^{1/2} = S_{I'}\,.$$

It remains to show isotony. This follows Theorem 0.9.4. □

Note that eq. (0.17.2) holds true also for $g \in \mathbf{G}_2$ because $U(r_I) = J_I$.

3.6 Nets on S^1 and nets on $\mathbb{R}$

Let $\mathcal{H}$ be a Hilbert space. A *hsm factorization of real subspaces* is a triple $\{H_i, i \in \mathbb{Z}_3\}$ of standard subspaces of $\mathcal{H}$ such that $H_i \subset H'_{i+1}$ is a hsm inclusion of standard subspaces for each $i \in \mathbb{Z}_3$.

Now, given a local net of standard subspaces of H, we obtain a hsm factorization of standard subspaces by considering the standard subspaces associated to a partition of S^1 by three intervals in the counterclockwise order (up to the boundary points), due to the geometric property of the modular group. We shall see that every factorization of standard subspaces actually arise in this way.

Lemma 3.6.1. *Let G be the universal group (algebraically) generated by 3 one-parameter subgroups $\delta_i(\cdot)$, $i \in \mathbb{Z}_3$, such that δ_i and δ_{i+1} have the same commutation relations of δ_{I_i} and $\delta_{I_{i+1}}$ for each $i \in \mathbb{Z}_3$, where I_0, I_1, I_2 are intervals forming a partition of S^1 in the counterclockwise order. Then G is isomorphic to $\bar{\mathbf{G}}$ and the δ_i's are continuous one parameter subgroups naturally corresponding to δ_{I_i}.*

Proof. Obviously G has a quotient isomorphic to $\bar{\mathbf{G}}$, and we denote by q the quotient map. As the exponential map is a local diffeomorphism of the Lie algebra of a Lie group and the Lie group itself, there exists a neighbourhood $\mathcal{U}$ of the origin $\mathbb{R}^3$ such that the map $(t_0, t_1, t_2) \to \delta_{I_0}(2\pi t_0)\delta_{I_1}(2\pi t_1)\delta_{I_2}(2\pi t_2)$ is a diffeomorphism of $\mathcal{U}$ with a neighbourhood of the identity of $\bar{\mathbf{G}}$. Therefore the map $\Phi : (t_0, t_1, t_2) \in \mathcal{U} \to \delta_0(2\pi t_0)\delta_1(2\pi t_1)\delta_2(2\pi t_2) \in G$ is still one-to-one. It is easily checked that the maps $g\Phi : \mathcal{U} \to G$, $g \in G$, form an atlas on G, thus G is a manifold. In fact G is a Lie group since the group operations are smooth, as they are locally smooth. Now G is connected by construction and q is a local diffeomorphism of G with $\bar{\mathbf{G}}$, hence a covering map, that has to be an isomorphism because of the universality property of $\bar{\mathbf{G}}$. □

Theorem 3.6.2. *Let (H_0, H_1, H_2) be a hsm factorization of standard subspaces and let I_0, I_1, I_2 be intervals forming a partition of S^1 in counter-clockwise order. There exists a unique local Möbius covariant net H of standard subspaces on S^1 such that $H(I_k) = H_k$, $k \in \mathbb{Z}_3$. The (unique) positive energy unitary representation U of $\mathbf{G}$ is determined by $U(\delta_{I_k}(-2\pi t)) = \Delta_k^{it}$ (where $\Delta_k \equiv \Delta_{H(I_k)}$).*

Proof. The subgroup of $\bar{\mathbf{G}}$ generated by the one-parameter subgroups δ_{I_k} and $\delta_{I_{k+1}}$, $k \in \mathbb{Z}_3$, is a two-dimensional Lie group $\mathbf{P}_k$ isomorphic to $\mathbf{P}$. As $H_k \subset H'_{k+1}$ is a hsm standard inclusion, by Thm. 0.11.1 the unitary group generated by Δ_k^{it} and Δ_{k+1}^{is} is isomorphic to $\mathbf{P}$, indeed there exists a unitary representation U_k of $\mathbf{P}_k$ determined by $U_k(\delta_{I_k}(-2\pi t)) = \Delta_i^{it}$ and $U_i(\delta_{I_{i+1}}(-2\pi s)) = \Delta_{k+1}^{is}$, therefore by Lemma 0.18.1, there exists a unitary representation U of $\bar{\mathbf{G}}$, such that $U|_{\mathbf{P}_k} = U_k$.

Let $t_0 = \frac{1}{2\pi}\log 2$. Then we have (see eq. (0.11.1))

$$\Delta_0^{it_0}\Delta_1^{it_0}H_0 = H'_1, \tag{3.6.1}$$

and similarly

$$(\Delta_2^{it_0}\Delta_0^{it_0})(\Delta_1^{it_0}\Delta_2^{it_0})(\Delta_0^{it_0}\Delta_1^{it_0})H_0 = H_0'. \tag{3.6.2}$$

The element $g \equiv \delta_{I_2}(-2\pi t_0)\delta_{I_0}(-2\pi t_0)\delta_{I_1}(-2\pi t_0)\delta_{I_2}(-2\pi t_0)\delta_{I_0}(-2\pi t_0)\delta_{I_1}(-2\pi t_0)$ is easily seen to be conjugate to the π-rotation in $\mathbf{G}$, thus in $\bar{\mathbf{G}}$ ($gI_0 = I_0'$, so g has the form $g = \delta_{I_0}(s)r_{I_0} = \delta_{I_0}(s/2)r_{I_0}\delta_{I_0}(-s/2)$), hence equation (0.18.2) entails that $U(R(2\pi))H_0 = H_0$.

Set $H(I_0) \equiv H_0$. If I is an interval of S^1, then $I = gI_0$ for some $g \in \bar{\mathbf{G}}$, and we set $H(I) = U(g)H(I_0)$. Since the group G_{I_0} of all $g \in \bar{\mathbf{G}}$ such that $gI_0 = I_0$ is generated by $\delta_{I_0}(t)$, $t \in \mathbb{R}$, and by $2k\pi$ rotations, $k \in \mathbb{Z}$, then $U(g)H(I_0) = H(I_0)$ for all $g \in G_{I_0}$ and $H(I)$ is well defined.

The isotony of H follows if we show that $gI_0 \subset I_0 \Rightarrow H(gI_0) \subset H(I_0)$. Indeed any such g is a product of an element in G_{I_0} and translations $\tau_{I_0}(\cdot)$ and $\tau_{I_0'}(\cdot)$ mapping I_0 into itself, hence the isotony follows by the half-sided modular conditions.

By (0.18.1) we have

$$\Delta_1^{it_0}\Delta_2^{it_0}\Delta_0^{it_0}\Delta_1^{it_0}H_0 = H_2$$

and since the corresponding element in $\bar{\mathbf{G}}$ maps I_0 onto I_2, we get $H_2 = H(I_2)$ and analogously $H_1 = H(I_1)$.

The locality of H now follows by the factorization property.

Finally, the positivity of the energy follows by the Bisognano-Wichmann property (Cor. 0.16.3) and U is a representation of $\mathbf{G}$ by the one-particle spin-statistics relation (Cor. 0.16.2). □

By a net of real Hilbert subspaces on $\mathbb{R}$ we shall mean an isotonous map

$$I \mapsto H(I) \subset \mathcal{H}$$

that associates to each bounded interval of $\mathbb{R}$ a real Hilbert subspace of a Hilbert space $\mathcal{H}$. All properties like locality, translation-dilation covariance, etc. have their obvious meaning.

Starting with a local net on S^1 we get by restriction a local net on $\mathbb{R}$.

Although a local Möbius covariant net satisfies Haag duality on S^1, duality on $\mathbb{R}$ does not necessarily hold.

Lemma 3.6.3. *Let H be a local Möbius covariant net of standard subspaces on S^1. The following are equivalent:*

(i) *The restriction of H to $\mathbb{R}$ satisfies Haag duality:*

$$H(I) = H(\mathbb{R}\backslash I)'$$

(*ii*) *H is* strongly additive*: If I_1, I_2 are the connected components of the interval I with one internal point removed, then*

$$H(I) = \overline{H(I_1) + H(I_2)}$$

(*iii*) *if I, I_1, I_2 are intervals as above*

$$H(I_1)' \cap H(I) = H(I_2)$$

Proof. Note that by Möbius covariance we may suppose that the point removed in (*i*) and (*ii*) is the point ∞. Now (*i*) $\Leftrightarrow$ (*ii*) because $\mathbb{R}\backslash I$ consists of two contiguous intervals in S^1 whose union has closure equal I', and by Haag duality $H(I) = H(I')'$. Similarly (*ii*) $\Leftrightarrow$ (*iii*) because, after taking symplectic complements and renaming the intervals, one relation becomes equivalent to the other one. $\square$

Examples of Möbius covariant nets on S^1 that are not strongly additive, i.e. not Haag dual on the line, will be discussed later. Haag duality on S^1 entails duality for half-lines on $\mathbb{R}$ hence essential duality, namely the dual net of the restriction H_0 to $\mathbb{R}$ is local:

$$I \mapsto H_0^d(I) \equiv H(\mathbb{R}\backslash I)' , \quad I \subset \mathbb{R}.$$

Due to locality the net H_0^d is larger than the original one, namely

$$H_0(I) \subset H_0^d(I) , \quad I \subset \mathbb{R}.$$

The main feature of the dual net H_0^d is that it obeys Haag duality on $\mathbb{R}$. The dual net does not in general transform covariantly under the covariance representation of the starting net.

Theorem 3.6.4. *Let H be a local net of standard subspaces on the intervals of* $\mathbb{R}$*, and U a unitary representation of* $\mathbf{P}$ *acting covariantly on H. The following are equivalent:*

(*i*) *H extends to a Möbius covariant net on S^1.*

(*ii*) *The Bisognano-Wichmann property holds for H, namely*

$$\Delta^{it}_{(0,\infty)} = U(\delta_{(0,\infty)}(-2\pi t)). \tag{3.6.3}$$

Proof. $(i) \Rightarrow (ii)$ follows by (ii) of Thm. 0.11.1.

$(ii) \Rightarrow (i)$: Note first that $\Delta^{it}_{(a,\infty)} = U(\delta_{(a,\infty)}(-2\pi t))$ for all $a \in \mathbb{R}$ because of translation covariance. Hence $H(-\infty, a)$ is a standard subspace of $H(a,\infty)'$ that is cyclic on Ω and globally invariant under the modular group of $H(a,\infty)'$ with respect to Ω, hence, by the modular theory (Prop. 0.8.10), duality for half-lines holds

$$H(a,\infty)' = H(-\infty, a).$$

It follows now that $\Big(H(-\infty,-1), H(-1,1), H(1,\infty)\Big)$ is a hsm factorization of standard subspaces. Indeed $\Big(H(-\infty,-1) \subset H(-1,1)'$ to be a hsm inclusion is equivalent (by Cor. 0.11.3 and duality for half-lines) to $H(-1,1) \subset \Big(H(-1,-\infty)$ to be a -hsm inclusion and this is the case by the geometric action of $\Delta^{it}_{(-1,\infty)} = U(\delta_{(-1,\infty)}(-2\pi t))$; and also the inclusions $H(-1,1) \subset H(1,\infty)'$ and $H(1,\infty) \subset H(-\infty,-1)'$, namely the inclusions $H(-1,1) \subset H(=\infty,1)$ and $H(1,\infty) \subset H(-1,\infty)$ are again hsm by the geometric action of the modular group.

Therefore we get a Möbius covariant net from Theorem 0.18.2. Due to the Bisognano-Wichmann property, the associated unitary representation of $\mathbf{G}$ restricts to a unitary representation of $\mathbf{P}$ that acts covariantly on H. So H is indeed an extension to S^1 of the original net. □

Now, if H is a local Möbius covariant net on S^1, its restriction H_0 to $\mathbb{R}$ does not depend, up to isomorphism, on the point we cut S^1, because of Möbius covariance. The local net on S^1 extending H_0^d is thus well defined up to isomorphism. We call it the dual net of H and denote it by H^d.

Corollary 3.6.5. *The dual net of a local Möbius covariant net on S^1 is a strongly additive Möbius covariant net on S^1.*

Proof. By construction, the dual net satisfies Haag duality on $\mathbb{R}$, hence strong additivity by Lemma 0.18.3. □

The following Corollary summarizes part of the above discussion.

Corollary 3.6.6. *There exists a one-to-one correspondence between:*

- *Half-sided modular inclusions of standard subspaces $K \subset H$.*
- *Pairs (H, U) with H a standard subspace and U is a one-parameter unitary group with positive generator s.t. $U(t)H \subset H,\ t \geq 0$.*

- *Translation-dilation covariant, Haag dual nets of standard subspaces on* $\mathbb{R}$ *with the Bisognano-Wichmann property* $\Delta^{it}_{(0,\infty)} = U(\delta_{(0,\infty)}(-2\pi t))$.
- *Strongly additive local Möbius covariant nets of standard subspaces on* S^1.
- *Unitary representations of* $\mathbf{G}$ *with lowest weight* ≥ 1.

In the above Corollary the unitary representation of $\mathbf{G}$ has lowest weight one iff U has no non-zero fixed vectors, iff the nets are irreducible, iff the hsm inclusion is non-degenerate, see the following Cor. 0.22.2. We shall see that there exists a *unique* irreducible hsm inclusion of standard subspaces and every non-degenerate hsm inclusions of standard subspaces is unitary equivalent to a multiple of the unique irreducible one.

We have also proved the following:

Theorem 3.6.7. *We have a one-to-one correspondence:*

Factorizations (H_0, H_1, H_2)

$\updownarrow$

$\mathbf{P}$ *covariant local nets of standard subspaces on intervals of* $\mathbb{R}$ *with BW property*

$\updownarrow$

Local $\mathbf{G}$ *covariant nets of standard subspaces on* S^1

$\updownarrow$

Unitary, positive energy representations of $\mathbf{G}$

In particular the correspondence between local $\mathbf{G}$ covariant nets of standard subspaces on S^1 and unitary, positive energy representations of $\mathbf{G}$ is given by combining thm. 0.15.1 and Thm. 0.17.2 by using the correspondence between unitary, positive energy representations of $\mathbf{G}$ and unitary, positive energy representations of $\mathbf{G}_2$ given by Thm. 0.6.3.

As the positive energy unitary representations of $\mathbf{G}$ are classified, the above theorem gives in particular a *classification* of all Möbius covariant nets of standard subspaces. We shall see another version of this classification in Cor. 0.22.1.

3.7 Appendix. Locality for irreducible nets

If U is irreducible, the locality property for a Möbius covariant net of real Hilbert spaces is automatic. This is due to the fact that we are assuming U to be a representation of $\mathbf{G}$, not of $\bar{\mathbf{G}}$.

Proposition 3.7.1. *Assume all properties* $1-4$ *for* H *in Sect. 0.13. If* U *is irreducible, then* H *is local (provided* $H(I) \neq \mathcal{H}$ *for some* $I \in \mathcal{I}$*).*

Proof. By the argument in the proof of the Reeh-Schlieder theorem, every $H(I)$ is cyclic. Set $K \equiv \cap_I H(I)$; then $K \cap iK$ is U-invariant, so it is either zero or equal to $\mathcal{H}$ because U is irreducible. In the second case $K = \mathcal{H}$, so $H(I) = \mathcal{H}$, that is not possible by assumption. So K is separating. This is equivalent to say that the net $I \mapsto H(I')'$ is cyclic, so each $H(I')'$ is cyclic, again by the argument in the proof of the Reeh-Schlieder theorem. We conclude that every $H(I)$ is separating.

Now, as in the proof of Th. 0.15.1, $z(t) \equiv \Delta_I^{it} U(\delta_I(2\pi t))$ is independent of I and so belongs to $U(\mathbf{G})'$. As U is irreducible, $z(t) = \chi(t)$ for a one-dimensional character of $\mathbb{R}$. As $z(t)H(I) = H(I)$, $z(t)$ commutes with J_I. So $z(t) = J_I z(t) J_I = J_I \chi(t) J_I = \chi(-t) = z(-t)$ and $z(t) = 1$, namely the Bisognano-Wichmann property (0.15.1) holds true and in particular $\Delta_{I'} = \Delta_I^{-1}$. In fact the proof of Th. 0.15.1 also shows that $J_I = J_{I'}$. Then if $\xi \in H(I), \xi' \in H(I')$ we have

$$\begin{aligned}(\xi, \xi') = (S_I \xi, S_{I'} \xi') = (J_I \Delta_I^{1/2} \xi, J_I' \Delta_{I'}^{1/2} \xi') = (J_I \Delta_I^{1/2} \xi, J_I \Delta_{I'}^{1/2} \xi') \\ = (\Delta_{I'}^{1/2} \xi', \Delta_I^{1/2} \xi) = (\Delta_I^{-1/2} \xi', \Delta_I^{1/2} \xi) = (\xi', \xi)\end{aligned}$$

so $\Im(\xi, \xi') = 0$, namely locality holds. □

Chapter 4

Representations of G and local subspaces

If one pass from a current and one of its derivative, the “real line” structure remains unchanged. Here we describe this phenomenon and, at the same time, provide a concrete description of the local standard subspaces that will give us further information on the associated nets.

4.1 Representations of G with the same restriction to P

We shall now consider pairs of unitary representations of $\bar{\mathbf{G}}$ that coincide when restricted to $\mathbf{P}$ or, equivalently, representations of the amalgamated free product $\bar{\mathbf{G}} *_{\mathbf{P}} \bar{\mathbf{G}}$. As a motivation, recall from Chapter 0.12 that the dual net of Möbius covariant net and the original net have different representations of $\mathbf{G}$, that indeed coincide when restricted to $\mathbf{P}$.

Let U be an irreducible unitary representation of $\bar{\mathbf{G}} *_{\mathbf{P}} \bar{\mathbf{G}}$. We shall denote by U_1 and U_2 the restrictions of U to the two copies of $\mathbf{G}$. We shall say that U has positive energy if $U|_{\mathbf{P}}$ has positive energy, i.e. if $P = -iT$ is a positive operator. Here T is the image of the translation generator in the Lie algebra of $\mathbf{P}$ (we shall often denote by the same symbol both an element of the Lie algebra and its image under the representation).

We shall classify the unitary positive energy representations of $\mathbf{G} *_{\mathbf{P}} \mathbf{G}$ such that either U_1 or U_2 is irreducible or, equivalently, such that $(L_{01} - L_{02})T$ is a scalar, where L_{0k} denotes the conformal Hamiltonian of U_k.

Theorem 4.1.1. *Let U be an irreducible unitary representation of $\mathbf{G} *_{\mathbf{P}} \mathbf{G}$ with positive energy. Then U_k is irreducible for some $k = 1, 2$ if and only if both the U_k are irreducible, and if and only if $(L_{01} - L_{02})T \in \mathbb{C}$. Moreover, such representations are classified by pairs of natural numbers (n_1, n_2), where n_k is the lowest weight of U_k.*

We now describe all lowest weight representations of $\mathbf{G}$ (or its universal covering group $\bar{\mathbf{G}}$) as extensions of the representation of $\mathbf{P}$.

Let us fix now the unitary irreducible representation U of $\mathbf{G}$ with lowest weight 1 and denote by E, T and S Lie algebra generators in the representation space.

Proposition 4.1.2. *Each irreducible unitary representation U^α of $\bar{\mathbf{G}}$ with lowest weight $\alpha \geq 1$ is unitarily equivalent to the representation obtained by exponentiation of the operators $T_\lambda = T$, $E_\lambda = E$, $S_\lambda = S - \lambda T^{-1}$ with $\lambda = \alpha(\alpha - 1)$*[1]

Proof. $U^\alpha|_{\mathbf{P}}$ and $U|_{\mathbf{P}}$ are irreducible and equivalent by Cor. 0.6.2, so we may identify them. Now by (0.5.1) the value of the Casimir operator in the representation U_α is $\lambda = \alpha(\alpha - 1)$, so one gets the formula for S_λ by multiplying on the left by T^{-1} both sides of the equation $\lambda = E(E - 1) - TS_\lambda$ because $E(E - 1) - TS = 0$ in the representation U, see (0.3.2) and (0.5.1).

To show that $T_\lambda = T$, $E_\lambda = E$, $S_\lambda = S - \lambda T^{-1}$ exponentiate to the unitary representation U_α, it is sufficient to show that the domain of the conformal Hamiltonian $L_{0\lambda} \equiv L_0 + i\lambda T^{-1}$ contain a complete orthonormal set of eigenvectors that belong to the domain of T_λ, E_λ and S_λ. Indeed it is enough to find a non-zero α-eigenvector ξ for $L_{0\lambda}$ which is C^∞ for the Lie algebra representation given by T, E, S_λ (cf. Lemma 0.5.1).

To this end, we may assume that E, T are the operators in the Schrödinger representation. Now observe that when $\lambda > 0$, $\lambda = \alpha(\alpha - 1)$, $\alpha \geq 1$,

$$L_{0\lambda} = -\frac{i}{2}\left(e^x - \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x}\frac{\mathrm{d}}{\mathrm{d}x}\right) + \lambda e^{-x}\right).$$

so the function $\xi(x) \equiv e^{\alpha x} e^{-e^x}$ is such a vector. □

Proof of Theorem 0.20.1. If $(L_{01} - L_{02})T$ is a scalar, $e^{itL_{02}}$ belongs to $(U_1(\mathbf{G}))''$ and $e^{itL_{01}}$ belongs to $(U_2(\mathbf{G}))''$, therefore, since U is irreducible, U_k is irreducible too, $k = 1, 2$. On the other hand, if say U_1 is irreducible, we may identify it with one of the representations described in Proposition 0.20.2 for some $\alpha \in \mathbb{R}$. Then,

[1] If A and B are linear operator with closable sum, the closure of their sum is denoted simply by $A + B$.

since U is irreducible and $U_1|_{\mathbf{P}} = U_2|_{\mathbf{P}}$, U_2 too has to be of the form described in Proposition 0.20.2, hence $(L_{01} - L_{02})T$ is a scalar. The rest of the statement is now obvious. □

Corollary 4.1.3. *Given two irreducible unitary representations U^α, U^β of $\bar{\mathbf{G}}$ with lowest weight $\alpha, \beta \geq 1$, with the same restriction to $\mathbf{P}$, they extend to anti-unitary irreducible representations $\tilde{U}^\alpha$, $\tilde{U}^\beta$ of $\bar{\mathbf{G}}_2$ with the same anti-untary involution $C \equiv \tilde{U}^\alpha(r) = \tilde{U}^\beta(r)$, where r is the reflection associated to the upper semi-circle.*

Proof. Let E_λ, T_λ and S_λ be the generators of the representation of lowest weight α as above. $\bar{\mathbf{G}}_2$ is generated by $\bar{\mathbf{G}}$ and an element corresponding to the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ of $SL(2, \mathbb{R})$, which correspond to the change of sign on $\mathbb{R}$, so we look for a anti-unitary C which satisfies $CE_\lambda C = E_\lambda$, $CT_\lambda C = -T_\lambda$ and $CS_\lambda C = -S_\lambda$. Since in the Schrödinger representation the complex conjugation C satisfies the mentioned commutation relations with T_0, S_0 and E_0, it trivially has the prescribed commutation relations with T_λ and E_λ, and the last relation follows by the formula $S_\lambda = S_0 - \lambda T_0^{-1}$. □

Multiplicative perturbations

We now give an alternative way to pass from the representation of lowest weight 1 to the representation with lowest weight $\alpha > 0$. In this subsection we denote by E, T, S the Lie algebra generators in the lowest weight 1 representation, and with E, T, S_α the corresponding generators in the lowest weight α case. Instead of defining the generator S_α as $S - \lambda T^{-1}$, $\lambda = \alpha(\alpha - 1)$, we will define the unitary R_α corresponding to the ray inversion (in the real line picture), namely $R_\alpha \equiv U^\alpha(R(\pi))$ is the π−rotation in the representation U^α with lowest weight α. We set $R \equiv R_1$. As R is given, we will equivalently define

$$\Gamma_\alpha = R_\alpha R = J_\alpha J \tag{4.1.1}$$

where J, resp. J_α is the modular conjugation of $H(-1, 1)$, resp. $H_\alpha(-1, 1)$, as $J = CR$ and $J_\alpha = CR_\alpha$ with the same anti-unitary conjugation commuting with them as in the proof of Cor. 0.20.3. In the examples with α an integer, Γ will be the canonical unitary for the inclusion of algebras $H_\alpha(-1, 1) \subset H(-1, 1)$ and its second quantization will implement the canonical endomorphism for the corresponding algenras given by $(\alpha - 1)$-derivative of the current algebra.

We now make some formal motivation calculations, that may however be given a rigorous meaning. First note that $\Gamma - \alpha$ commutes with E, because both J and J_α commute with E, hence Γ_α must be a bounded Borel function of E because the bounded Borel functions of E form a maximal abelian von Neumann algebra. Indeed, by Cor. 0.6.2, we can easily check this in the Schrödinger representation because E and T exponentiate to an irreducible representation of $\mathbf{P}$ with strictly positive energy.

Set

$$f_\alpha(z) \equiv \frac{(z-1)(z-2)\cdots(z-n+1)}{(z+1)(z+2)\cdots(z+n-1)}\,.$$

Then f_α is a function on $\mathbb{C}$ that satisfies the functional equation

$$\frac{f_\alpha(z-1)}{f_\alpha(z)} = 1 - \frac{\lambda}{z(z-1)}\,, \quad z \in \mathbb{C}\,, \tag{4.1.2}$$

and $|f_\alpha(z)| = 1$ for all $z \in i\mathbb{R}$.

Proposition 4.1.4. *If $\alpha = n$ is an integer, then*

$$\Gamma_\alpha = \frac{(E-1)(E-2)\cdots(E-n+1)}{(E+1)(E+2)\cdots(E+n-1)}. \tag{4.1.3}$$

Proof. Let Γ_α be given by the formula (0.20.4), namely $\Gamma_\alpha \equiv f_\alpha(E)$. In order to check that Γ_α is (up to a phase) the unitary (0.20.1) it is enough to check that

$$\Gamma_\alpha E \Gamma_\alpha^* = R_\alpha E R_\alpha = E \tag{4.1.4}$$

$$\Gamma_\alpha S \Gamma_\alpha^* = R_\alpha T R_\alpha = S_\alpha \tag{4.1.5}$$

because the representation generated by E and S is irreducible by Cor. 0.6.2.

The first equation is obvious because Γ_α is a function of E. To verify the second equation we notice that $-iS$ is positive and non-singular by Cor. 0.6.2. As $SE - ES = S$, we have $SES^{-1} = E + 1$ which implies $Sf(E)S^{-1} = f(E+1)$ for all Borel bounded functions f. The functional equation (0.20.2) for f_α implies

$$f_\alpha(E)Sf_\alpha(E)^* = f_\alpha(E)f_\alpha(E+1)^*S = \left(1 - \lambda(E(E+1))^{-1}\right)S = S - \lambda T^{-1} = S_\alpha\,,$$

where we have used the identity $\left(E(E+1)\right)^{-1}S = T^{-1}$ or, equivalently, the identity $E(E+1) = ST$; the latter is indeed the adjoint of the equation $E(E-1) = TS$ that holds true because the Casimir operator vanishes in the lowest weight one representation.

That the phase in one can be verified by eq. (0.21.3), (0.21.4) in the following. $\square$

Along the same lines, in the case of the irreducible lowest weight α representation of $\bar{\mathbf{G}}$, we have $\Gamma_\alpha = f_\alpha(E)$ with

$$f_\alpha(z) = \frac{\mathbf{\Gamma}(z+1)\mathbf{\Gamma}(z)}{\mathbf{\Gamma}(z+\alpha)\mathbf{\Gamma}(z-\alpha+1)} \tag{4.1.6}$$

where $\mathbf{\Gamma}$ is the Euler Gamma-function.

4.2 Lowest weight representations of G and derivatives of the $U(1)$-current

We now give a concrete realization of the lowest weight representations of **G**, and of the corresponding local Möbius covariant nets of standard subspaces. Namely we describe the one-particle Hilbert space of a free Boson field on the circle, and of its derivatives.

On the space $C^\infty(S^1, \mathbb{R})$ of real valued smooth functions on the circle S^1, we consider the seminorm

$$||\phi||^2 = \sum_{k=1}^{\infty} k|\hat{\phi}_k|^2$$

and the operator $\mathcal{J} : \widehat{\mathcal{J}\phi}_k = -i\text{sign}(k)\hat{\phi}_k$, where the $\hat{\phi}_k$'s denote the Fourier coefficients of ϕ.

Since $\mathcal{J}^2 = -1$ and $\mathcal{J}$ is an isometry w.r.t. $\|\cdot\|$, $(C^\infty(S^1, \mathbb{R}), \mathcal{J}, \|\cdot\|)$ becomes a complex vector space with a positive bilinear form, defined by polarization. Thus, taking the quotient by constant functions and completing, we get a complex Hilbert space $\mathcal{H}$.

We note that the symplectic form ω may be written as

$$\omega(f,g) = \Im(f,g) = \frac{-i}{2}\sum_{k\in\mathbb{Z}} k\hat{f}_{-k}\hat{g}_k = \frac{1}{2}\int_{S^1} g\mathrm{d}f.$$

The natural action of **G** on S^1 gives rise to a unitary representation on $\mathcal{H}$:

$$U(g)\phi(t) = \phi(g^{-1}t)$$

Then, observing that $\mathcal{J}\cos kt = \sin kt$ for $k \geq 1$, it is easy to see that $\cos kt$ is an eigenvector of the rotation subgroup $U(\theta)$:

$$U(\theta)\cos kt = \cos k(t-\theta) = (\cos k\theta + \sin k\theta\,\mathcal{J})\cos kt = e^{ik\theta}\cos kt, \quad k \geq 1,$$

and that all the eigenvectors have this form. Therefore the representation has lowest weight 1.

We need another description of the Hilbert space $\mathcal{H}$ which is more suitable to be generalized. First we pass to the real line picture ($x = \tan(\vartheta/2)$) and so identify $C^\infty(S^1, \mathbb{R})$ with $C^\infty(\bar{\mathbb{R}})$. Since the symplectic form is the integral of a differential form it does not depend on the coordinate:

$$\omega(f, g) = \frac{1}{2}\int_{\mathbb{R}} g(x)\mathrm{d}f(x)$$

A computation shows that the anti-unitary $\mathcal{J}$ applied to a function f coincides up to an additive constant with the convolution of f with the distribution $1/(x+i0)$ on $\mathbb{R}$, therefore, since the symplectic form is trivial on the constants, the (real) scalar product may be written as

$$\begin{aligned}\langle f, g\rangle = \omega(f, \mathcal{J}g) &= \frac{1}{2}\int\left(\frac{1}{x+i0} * g(x)\right)f'(x)\mathrm{d}x \\ &= \frac{1}{4\pi}\int f(x)g(y)\frac{1}{|x-y+i0|^2}\mathrm{d}x\mathrm{d}y = \text{const.}\int_0^\infty p\hat{f}(-p)\hat{g}(p)\mathrm{d}p\end{aligned} \tag{4.2.1}$$

and $\mathcal{H}$ may be identified with the completion of $C^\infty(\bar{\mathbb{R}})$ w.r.t. this norm.

Note that since $\mathcal{J}f = -if$ if $\mathrm{supp}\hat{f} \subset [0, +\infty)$, $\mathcal{H}$ is also the completion of $C^\infty(\mathbb{R}, \mathbb{C})$ modulo $\{f | \hat{f}|_{(-\infty,0]} = 0\}$ with scalar product $(f, g) = \int_0^\infty p\overline{\hat{f}(p)}\hat{g}(p)\mathrm{d}p$.

Let us now consider the space $X^n \equiv C^\infty(\bar{\mathbb{R}}) + \mathbb{R}^{2(n-1)}[x]$, $n \geq 1$, where $\mathbb{R}^p[x]$ denotes the space of real polynomials of degree p, and the bilinear form on it given by

$$\langle f, g\rangle_n = \frac{1}{4\pi}\int f(x)g(y)\frac{1}{|x-y+i0|^{2n}}\mathrm{d}x\mathrm{d}y$$

It turns out that $\langle\,\cdot\,,\,\cdot\,\rangle_n$ is a well defined positive semi-definite bilinear form on X^n which degenerates exactly on $\mathbb{R}^{2(n-1)}[x]$. On this space one may define also a symplectic form by

$$\omega_n(f, g) = \frac{1}{2}\int f(x)g(y)\delta_0^{(2n-1)}(x-y)\mathrm{d}x\mathrm{d}y$$

Here $\delta_0^{(k)}$ denotes the k-derivative of the Dirac measure at zero. This form may be read as the restriction of ω_1 to the n-th derivatives. Therefore we can recognize this symplectic form as coming from the commutation relations for the n-th derivatives of $U(1)$−currents. This form again degenerates exactly on $\mathbb{R}^{2(n-1)}[x]$, and the operator $\mathcal{J}$ defined before connects the positive form with the symplectic form for any n in such a way that $(\cdot,\cdot)_n \equiv \langle\cdot,\cdot\rangle_n + i\omega_n(\cdot,\cdot)$ becomes a complex bilinear form on $(X^n, \mathcal{J})$. We shall denote by $\mathcal{H}^n$ the complex Hilbert space obtained by completing the quotient $X^n/\mathbb{R}^{2(n-1)}[x]$.

With $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on $\bar{\mathbb{R}}$ as in (0.1.1), for any $n \geq 1$ consider the operators $U^n(g)$ on X^n:

$$U^n(g)f(x) = (cx - a)^{2(n-1)} f(g^{-1}x).$$

It turns out that U^n is a representation of $\mathbf{G}$, $n \geq 1$, and that the positive form is preserved as well as the symplectic form and the operator $\mathcal{J}$, therefore U^n extends to a unitary representation of $\mathbf{G}$ on $\mathcal{H}^n$.

We remark that while X^n and $\mathbb{R}^{2(n-1)}[x]$ are globally preserved by U^n, the space $C^\infty(\bar{\mathbb{R}})$ is not, and that explains why the space X^n had to be introduced.

By definition the space $\mathcal{H}^1$ coincides with the space $\mathcal{H}$ and the representation U^1 with the representation U, which we proved to be lowest weight 1. We observe that, for functions in $C^\infty(\bar{\mathbb{R}})$, one gets

$$\begin{aligned}
\langle f, g\rangle_n &= \frac{1}{2}\int_{\mathbb{R}} |p|^{2n-1} \hat{f}(-p)\hat{g}(p)\mathrm{d}p \\
\omega(f, g)_n &= \frac{1}{2}\int_{\mathbb{R}} p^{2n-1} \hat{f}(-p)\hat{g}(p)\mathrm{d}p
\end{aligned}$$

hence

$$(f, g)_n = (D^{n-1}f, D^{n-1}g)_1,$$

i.e. D^{n-1} is a unitary between $\mathcal{H}^n$ and $\mathcal{H}^1 \equiv \mathcal{H}$, where D is the derivative operator. The following holds:

Theorem 4.2.1. *The representation U^n has lowest weight n.*

Proof. Making use of the results of Proposition 0.20.4, we have to show that

$$R_n R = \prod_{k=1}^{n-1}\left(\frac{E-k}{E+k}\right), \qquad n \geq 1$$

where $R_n = D^{n-1}U^n(r)(D^{n-1})^*$ with r the ray inversion, $R = R_1$. This amounts to prove

$$D^{n-1}U^n(r) = \prod_{k=1}^{n-1}\left(\frac{E-k}{E+k}\right)U(r)D^{n-1}. \tag{4.2.2}$$

Now we take equation (0.21.2) as an inductive hypothesis. Then, equation(0.21.2) for $n+1$ can be rewritten, using the inductive hypothesis and the relation $U^{n+1}(r) = x^2 U^n(r)$, as

$$D^n(x^2 U^n(r)) = \left(\frac{E-n}{E+n}\right) D^{n-1} U^n(r) D. \tag{4.2.3}$$

Finally we observe that $U^n(r)D = x^2 D U^n - 2(n-1)xU^n$, hence equation (0.21.3) is equivalent to

$$(E+n)D^n(x^2\cdot) = (E-n)D^{n-1}(x^2 D \cdot - 2(n-1)x\cdot). \tag{4.2.4}$$

Since $E = -xD$, equation (0.21.4) follows by a straightforward computation. □

Proposition 4.2.2. *The unitary representations of* **G** *on* $\mathcal{H}$ *given by*

$$D^{n-1} U^n (D^{n-1})^*, \quad n \geq 1,$$

coincide on **P**.

Proof. We have to prove that $D^{n-1}U^n(g) = U(g)D^{n-1}$ when g is a translation or a dilation. For translations, $U^n(t)f(x) = f(x-t)$, and the equality is obvious; for dilations, $g = \begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix}$, $U^n(g)f(x) = e^{\lambda n} f(e^{-\lambda}x)$, hence $D^{n-1}U^n(g)f(x) = f^{(n)}(e^{-\lambda}x) = U(g)D^n f(x)$. □

4.2.1 Local spaces and regularity

Let fix $n \geq 1$ and, for any interval I of $\bar{\mathbb{R}}$, we set

$$X^n(I) = \{f \in X^n : f|_{I'} \equiv 0\}.$$

It is easy to check that that the immersion $i_n^I : X^n(I) \to \mathcal{H}^n$ is injective and the spaces $H^n(I) \equiv (i_n^I X^n(I))^-$, where the closure is taken w.r.t. $\|\cdot\|_n$, form a local Möbius covariant net of standard subspaces of $\mathcal{H}^n$; moreover

$$\overline{\text{lin.span}_{I\subset\bar{\mathbb{R}}} H^n(I)} = \mathcal{H}^n .$$

Now we identify $\mathcal{H}^n$ with $\mathcal{H}$ via the unitary D^{n-1}, and set $H_n(I) \equiv D^{n-1}H^n(I)$. Then, if I is a bounded interval of $\mathbb{R}$ and $f \in H_n(I)$, f may be integrated $n-1$ times, giving a function which has still support in I, therefore

$$H_n(I) = \left\{[f] \in \mathcal{H} : f|_{I'} = 0,\ \int t^k f = 0,\ k = 0, \ldots, n-2\right\}, \tag{4.2.5}$$

where $[f]$ denotes the equivalence class of f modulo polynomials.

If I is a half line in $\mathbb{R}$, $H_n(I)$ is an invariant subspace of the dilation subgroup, which is the modular group of $H(I)$. Hence, see Prop. 0.8.10, this implies that

$$H_n(I) = H(I), \qquad I \text{ half-line.} \tag{4.2.6}$$

Finally we observe that by the unique correspondence in Thm. 0.18.7, these nets coincide with those abstractly constructed in Section 0.17.

Now, we fix a bounded interval I in $\mathbb{R}$, e.g. $(-1, 1)$, and consider the family $H_n(I)$.

Proposition 4.2.3. *We have $H_m(I) \subset H_n(I)$ and* $\operatorname{codim}(H_m(I) \subset H_n(I)) = m - n$, *$m \geq n$.*

Theabove concrete characterization of H_n shows that $H_m(I) \subset H_n(I)$ if $m \geq n$. Before proving the codimension formula, we discuss some of its consequences.

A net H is said to be *$n-$regular* if, for any partition of S^1 into n intervals $I_1, \ldots, I_n$, the linear span of the $H(I_k)$, $k = 1, \ldots n$, is dense in $\mathcal{H}$.

All irreducible local Möbius covariant nets are 2-regular, because Haag duality holds and local spaces are factors. Obviously, strong additivity implies n-regularity for all n.

Corollary 4.2.4. *H_1 is n-regular for any n. H_2 is* 3*-regular but it is not* 4*-regular. H_n, $n \geq 3$, is not* 3*-regular. Moreover H_n is strongly additivity iff $n = 1$, therefore H_1 is the dual net of H_n for every n.*

Proof. First we recall that a net is strongly additive if and only if it coincides with its dual net. Then, the net $H \equiv H_1$ is strongly additive because its dual net should be of the form H_n (cf. Prop. 0.20.2) and should satisfy $H^d(-1, 1) \supset H(-1, 1)$. As a consequence, H is n-regular for any n.

Then, since the spaces for the half-lines do not depend on n, the dual net of H_n does not depend on n either, hence coincides with H.

Since **G** acts transitively on the ordered triples of distinct points, we may study 3-regularity for the special triple $(-1, 1, \infty)$ in $\mathbb{R} \cup \{\infty\}$. Then,

$$\begin{aligned}(H_n(\infty, -1) + H_n(-1, 1) + H_n(1, \infty))' &= (H_1(\infty, -1) + H_n(-1, 1) + H_1(1, \infty))' \\ &= (H_1(-1, 1)' + H_n(-1, 1))' = H_n(-1, 1)' \cap H_1(-1, 1)\end{aligned}$$

where we used strong additivity and duality for H_1. By Theorem 0.21.3, 3-regularity holds if and only if $n = 1, 2$.

Violation of 4-regularity for H_2 may be proved by exhibiting a function which is localized in the complement of any of the intervals $(\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$, i.e. belongs to $H_2(-1, 0)' \cap H_2(0, 1)' \cap H_1(-1, 1)$:

$$\phi(x) = \begin{cases} 1 + x & \text{if} \quad -1 \geq x \geq 0, \\ 1 - x & \text{if} \quad 0 \geq x \geq 1, \\ 0 & \text{if} \quad |x| \geq 1 \end{cases}$$

In the same way we may construct a function which violates 3-regularity for H_3, namely

$$\phi(x) = \begin{cases} x^2 - 1 & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geq 1 \end{cases}$$

Clearly, $\phi \in H_3(\infty, -1)' \cap H_3(-1, 1)' \cap H_3(1, \infty)' = H_3(I)' \cap H_1(I)$. □

Proof of Prop. 0.21.3: It is sufficient to show that $\operatorname{codim}(H_{m+1}(I) \subset H_m(I)) = 1$. Since $H_{m+1}(I) = \{\phi \in H_m(I) : \int x^{m-1}\phi(x)dx = 0\}$, and we may find a function $\psi_{m-2} \in C^\infty(\mathbb{R})$ such that $\psi'_{m-1}(x) = x^{m-1}$ for $x \in (-1, 1)$, we get $H_{m+1}(I) = \{\phi \in H_m(I) : \omega(\psi_{m-1}, \phi) = 0\}$. As the functional $\phi \to \omega(\psi_m, \phi)$ is continuous and non zero on H_m, the thesis follows. □

4.3 Modular construction of nets on $\mathbb{R}$ and their extensions to S^1

Let U be the unique irreducible positive energy unitary representation of $\mathbf{P}$ with no non-zero translation fixed vector on a Hilbert space $\mathcal{H}$. Denote by the symbol U its extension to a (anti-)unitary representation of $\mathbf{P}_2$. With J the anti-unitary involution corresponding to the map $x \mapsto -x$ on $\mathbb{R}$ and K the generator of the dilation one-paramenter unitary group, set $S \equiv Je^{-\pi K}$ and $H(0, \infty) \equiv \ker(1 - S)$; then we get a net H on the half-lines $I \subset \mathbb{R}$ given by

$$H(a, \infty) \equiv U(\tau(a))H(0, \infty), \quad H(\infty, a) \equiv H(a, \infty)' \,.$$

The following Corollary summarizes part of the above discussion.

Corollary 4.3.1. *Let H the above net on the half-lines of $\mathbb{R}$. There exists a bijective correspondence between*

- *Extensions of H to a local Möbius covariant net on the intervals of S^1.*

- *Standard subspaces* $K \subset H(-\infty, 1)' \cap H(0, \infty)$ *such that* $U(\delta_{(0,\infty)}(s))K \subset K$ *and* $U(\delta_{(\infty,1)}(-s))K \subset K$ *for* $s \geq 0$.

- *The real linear spaces* $H_n(0, 1)$*,* $n \in \mathbb{N}$.

Proof. Combine the above discussion with Thm. 0.18.2. □

We shall say that the hsm inclusion of standard subspaces $K \subset H$ is trivial if $K = H$ and *non-degenerate* if it has no trivial, non-zero direct summand.

In the above corollary $K \subset H$ is non degenerate iff U has lowest weight 1 iff $K' \cap H = 0$, iff U has no non-zero fixed vector, iff there is no nonzero joint fix vector for Δ_H and Δ_K etc.. We shall also say that $K \subset H$ is irreducible if it is not the direct sum of two non-zero hsm inclusions of standard subspaces, i.e. if U is irreducible. We have an example of irreducible hsm inclusion by the above corollary by considering the irreducible positive energy representation of **G** with lowest weight 1. This is indeed the unique one:

Corollary 4.3.2. *All irreducible, non-zero, non-degenerate hsm inclusions of standard subspaces are unitarily equivalent.*

If $K \subset H$ *is a non-degenerate hsm inclusion of standard subspaces, then* $K \subset H$ *is unitary equivalent to a direct sum of copies of the unique irreducible one.*

Every hsm inclusion of standard subspaces is canonically the direct sum of a non-generate one and (possibly) of a trivial one.

Proof. By Wiesbrock theorem a hsm inclusion $K \subset H$ of standard subspaces gives rise to a positive energy representation U of **P**. Then, by Prop. 0.7.1, a vector is fixed by U iff it is fixed by Δ_H^{it} iff it is fixed by Δ_K^{it}. So we get the decomposition in a direct sum of a trivial inclusion (the fixed points) and a non-degenerate hsm inclusion. The rest is clear. □

Chapter 5

Nuclearity properties and $\mathrm{SL}(2,\mathbb{R})$ operator inequalities

An important condition for a positive energy representation of $\mathbf{G}$ is the trace class property, namely $\mathrm{Tr}(e^{-\beta L_0}) < \infty$ for all $\beta > 0$, that is there exists a "partition function". We will study how this condition is related to other nuclearity properties for a net of standard subspaces, We shall apply our analysis in later chapters in relation with the split properties.

Most of the analysis in this chapter will consist in set a number of remarkable identities and inequalities for operators associated with a positive energy unitary representation of $\mathbf{G}$.

5.1 A first operator identity associated with $\mathrm{SL}(2,\mathbb{R})$

If $I, \tilde{I}$ are intervals of S^1 we shall write $I \Subset \tilde{I}$ if the closure of I is contained in the interior of $\tilde{I}$.

Let U be a unitary positive energy representation of $\bar{\mathbf{G}}$ on a Hilbert space $\mathcal{H}$. We shall denote by K_I the infinitesimal generator of $U(\delta_I(s))$. Given intervals $I \Subset \tilde{I}$ we shall set

$$T_{\tilde{I},I}(\lambda) \equiv e^{2\pi i\lambda K_{\tilde{I}}} e^{-2\pi i\lambda K_I} .$$

Suppose U is a representation of $\mathbf{G}$ and let H be the associated local Möbius covariant net of real Hilbert subpaces of $\mathcal{H}$. Then $K_I = -2\pi \log \Delta_I$, where $\Delta_I \equiv \Delta_{H(I)}$ is the modular operator of $H(I)$, so

$$T_{\tilde{I},I}(\lambda) \equiv \Delta_{\tilde{I}}^{-i\lambda}\Delta_I^{i\lambda}$$

(the closure on the right hand side operator is omitted as usual); by Cor. 0.10.2 the map $\lambda \mapsto T_{\tilde{I},I}(\lambda)$ is holomorphic in the strip $\mathbb{S}_{1/2}$, continuous on its closure and $||T_{\tilde{I},I}(i\lambda)|| \le 1$.

The case $\lambda = i/4$ is of particular relevance and we set

$$T_{\tilde{I},I} \equiv T_{\tilde{I},I}(i/4) \,.$$

In this chapter the upper and the right semicircle will be denoted respectively by I_1 and I_2. We put $K_k \equiv K_{I_k}$ and $\Delta_k \equiv \Delta_{I_k}$.

We now prove a formula pointed out by Schroer and Wiesbrock.

Theorem 5.1.1. *Let U be a positive energy unitary representation of* $\bar{\mathbf{G}}$. *For every $s \ge 0$, the following identity holds*

$$\Delta_1^{1/4}\Delta_2^{-is}\Delta_1^{-1/4} = e^{-2\pi s L_0} \tag{5.1.1}$$

where $\Delta_k \equiv e^{-\pi K_k}$, $k = 1,2$ and L_0 are associated with U.

More precisely the domain of $\Delta_1^{1/4}\Delta_2^{-is}\Delta_1^{-1/4}$ is a core for $\Delta_1^{-1/4}$ and the closure of $\Delta_1^{1/4}\Delta_2^{-is}\Delta_1^{-1/4}$ is equal to $e^{-2\pi s L_0}$.

Proof. We assume that U is a representation of $\mathbf{G}$; for the general case see the remark below.

We denote here by k_1, k_2 and l_0 the elements of the Lie algebra $\mathfrak{sl}_{\mathbb{C}}(2,\mathbb{R})$ corresponding to $K_1 \equiv K_{I_1}$, $K_2 \equiv K_{I_2}$ and L_0 (thus $k_1 = \frac{i}{2}E$ and $k_2 = \frac{i}{2}(T - S)$). Then

$$[k_1, k_2] = -il_0, \quad [k_1, l_0] = k_2 \,.$$

Denoting by $\mathrm{Ad}(g)$ the adjoint action of $g \in \mathbf{G}$ on $\mathfrak{sl}_{\mathbb{C}}(2,\mathbb{R})$ we then have

$$\mathrm{Ad}(e^{-2\pi i t k_1})(k_2) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\delta_{k_1}^n(k_2) = \cosh(2\pi t)k_2 - \sinh(2\pi t)l_0$$

where $\delta_{k_1} \equiv 2\pi[k_1, \cdot\,]$, therefore for all $s, t \in \mathbb{R}$ we have the identity in $\mathbf{G}$

$$e^{-2\pi i t k_1} e^{2\pi i s k_2} e^{2\pi i t k_1} = e^{2\pi i s\left(\cosh(2\pi t)k_2 - \sinh(2\pi t)l_0\right)}$$

that of course gives the operator identity

$$e^{-2\pi i t K_1} e^{2\pi i s K_2} e^{2\pi i t K_1} = e^{2\pi i s\left(\cosh(2\pi t)K_2 - \sinh(2\pi t)L_0\right)} \tag{5.1.2}$$

Consider the right hand side of eq. (0.23.2) that we denote by $W_s(t)$. Given $r > 1/2$, by Lemma 0.23.2 there exist $s_0 > 0$, a dense set $\mathcal{V} \subset \mathcal{H}$ of joint analytic vectors for K_1, K_2, L_0 such that, for any fixed $s \in \mathbb{R}$ with $|s| \leq s_0$ and $\eta \in \mathcal{V}$, the vector-valued function

$$t \mapsto W_s(t)\eta \equiv e^{2\pi i s\left(\cosh(2\pi t)K_2 - \sinh(2\pi t)L_0\right)}\eta$$

has a bounded analytic continuation in the ball $B_r \equiv \{z \in \mathbb{C} : |z| < r\}$.

Consider now the left hand side of eq. (0.23.2). By definition it is equal to $\Delta_1^{it}\Delta_2^{-is}\Delta_1^{-it}$.

Now the map

$$t \in \mathbb{R} \mapsto \Delta_1^{it}\Delta_2^{-is}\Delta_1^{-it} \tag{5.1.3}$$

has a uniformly bounded, strongly continuous, analytic extension in the strip $\mathbb{S}_{-1/2}$. Indeed $\Delta_1^{\lambda}\Delta_2^{-is}\Delta_1^{-\lambda} = T_{I_1,I_{1,s}}(-\lambda)\Delta_2^{-is}$ where $I_{1,s} = \delta_{I_2}(2\pi s)I_1 \subset I_1$, so the analyticity of (0.23.3) follows from Lemma 0.10.2.

Taking matrix elements

$$(\eta, \Delta_1^{it}\Delta_2^{-is}\Delta_1^{-it}\xi) = (W_s(t)^*\eta, \xi) = (W_{-s}(t)\eta, \xi)$$

with $\eta \in \mathcal{V}$ and ξ an entire vector for Δ_1, both the functions defined by the left and the right side of the above equation have an analytic extension in $\mathbb{S}_{-1/2} \cap B_r$.

Taking the value at $t = -i/4$ we have

$$(\eta, \Delta_1^{1/4}\Delta_2^{-is}\Delta_1^{-1/4}\xi) = (e^{-2\pi s L_0}\eta, \xi) = (\eta, e^{-2\pi s L_0}\xi)$$

hence the closure of $\Delta_1^{1/4}\Delta_2^{-is}\Delta_1^{-1/4}$ is equal to $e^{-2\pi s L_0}$ if s is real and $|s| \leq s_0$, hence for all $s \in \mathbb{R}$ by the group property. This ends the proof. □

Remark. Equation (0.23.1) is clearly equivalent to the identity $\Delta_2^{-is}\Delta_1^{-1/4} \subset \Delta_1^{-1/4}e^{-2\pi s L_0}$. Note that we have a closed operator on both sides. If the representations U_1 and U_2 have lowest weight α_1 and α_2, then $U \equiv U_1 \otimes U_2$ has lowest weight $\alpha = \alpha_1 + \alpha_2$. So, if α is an integer, formula (0.23.1) holds for U. It easy to infer then that formula (0.23.1) holds for any lowest weight representation. In the following we prove other operator inequalities for unitary lowest weight representations of **G**. A similar argument then shows that they are all valid for any lowest weight unitary representation.

We now recall the following result by Nelson used in the proof of Thm. 0.23.1.

Lemma 5.1.2 ([?]). *Let U be a unitary representation of a Lie group on a Hilbert space $\mathcal{H}$ and $X_1, X_2, \dots X_n$ a basis for the associated Lie algebra generators. There exist a neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^n$ and a dense set of vectors $\mathcal{V} \subset \mathcal{H}$ of smooth vectors for U such that*

$$\sum_{k=0}^{\infty} \frac{\|(u_1 X_1 + u_2 X_2 + \cdots + u_n X_n)^k \eta\|}{k!} < \infty$$

for all $(u_1, u_2, \dots u_n) \in \mathcal{U}$ and $\eta \in \mathcal{D}$.

5.2 Trace class property and L^2-nuclearity for representations of G

Second formula

If $A \in B(\mathcal{H})$, the nuclear norm $\|A\|_1$ of A is the L^1 norm, namely $\|A\|_1 \equiv \mathrm{Tr}(|A|)$ where $|A| \equiv \sqrt{A^*A}$; the Hilbert-Schmidt norm is given by $\|A\|_2 = \mathrm{Tr}(A^*A)^{1/2}$.

We shall consider the property that $\|T^U_{\tilde{I},I}\|_1 < \infty$, that we call *$L^2$-nuclearity* (with respect to $I \Subset \tilde{I}$). Note that $\|T^U_{\tilde{I},I}\|_1$ depends only on $\ell(\tilde{I}, I)$, namely $\|T^U_{\tilde{I},I}\|_1$ does not change if we replace $I \Subset \tilde{I}$ by $hI \Subset h\tilde{I}$ with $h \in \mathbf{G}$.

Given intervals $I \Subset \tilde{I}$ we consider the *inner distance* $\ell(\tilde{I}, I)$, see Appendix 0.26.

Proposition 5.2.1. *In every positive energy unitary representation U, we have*

$$T^U_{\tilde{I},I} = e^{-sL_0} \Delta_2^{is/2\pi}$$

where $\tilde{I} = I_1$, $I \Subset \tilde{I}$ is symmetric w.r.t. the vertical axis, $s = \ell(\tilde{I}, I)$ and Δ_2 is as above. Therefore

$$\|T^U_{\tilde{I},I}\|_1 = \|e^{-sL_0}\|_1 \tag{5.2.1}$$

for any inclusion $I \Subset \tilde{I}$ such that $s = \ell(\tilde{I}, I)$.

Proof. By multiplying both sides of formula (0.23.1) by Δ_2^{is} on the right, we get the equality

$$\begin{aligned} e^{-2\pi s L_0} \Delta_2^{is} &= \Delta_1^{1/4} (\Delta_2^{-is} \Delta_1^{-1/4} \Delta_2^{is}) \\ &= \Delta_1^{1/4} (U(\delta_{I_2}(2\pi s)) \Delta_1^{-1/4} U(\delta_{I_2}(-2\pi s))) \\ &= \Delta_1^{1/4} \Delta_{I_{1,2\pi s}}^{-1/4} \end{aligned}$$

where $I_{1,s} \equiv \delta_{I_2}(s)I_1$; that is to say $T^U_{\tilde{I},I} = e^{-sL_0}\Delta_2^{is/2\pi}$ where $s = \ell(\tilde{I}, I)$. Since then $\Delta_2^{is/2\pi}$ is unitary we are done. □

As a consequence we have a key equation for the T operator

$$T_{\tilde{I},I}T^*_{\tilde{I},I} = e^{-2sL_0}, \quad s \equiv \ell(\tilde{I}, I) \tag{5.2.2}$$

with $\tilde{I}$ the upper semicircle and I symmetric w.r.t. the vertical axis. Of course by Möbius covariance we have a general formulation of the above proposition and the above equation for arbitrary inclusions $I \Subset \tilde{I}$. In particular formula (0.24.1) holds true for any inclusion of intervals $I \Subset \tilde{I}$ with $s = \ell(\tilde{I}, I)$.

Third formula

let $I \in \mathfrak{I}$ be an interval and $I_{a',a} \equiv \tau'(-a')\tau(a)I$ with $a, a' > 0$, so that $I_{a',a} \Subset I$. Let U be a positive energy unitary representation of $\mathbf{G}$ and denote by P_I and P'_I the positive selfadjoint generators of the one-parameter unitary subgroups corrsponding to τ_I and τ'_I. We have:

Proposition 5.2.2.

$$T^U_{I,I_{a',a}} = e^{-a'P'_I}e^{-aP_I}e^{-iaP_I}e^{ia'P'_I} .$$

Proof. Indeed

$$\begin{aligned} T^U_{I,I_{a',a}} &\equiv \Delta_I^{1/4}\Delta_{I_{a',a}}^{-1/4} \\ &= \Delta_I^{1/4}e^{-ia'P'_I}e^{iaP_I}\Delta_I^{-1/4}e^{-iaP_I}e^{ia'P'_I} \\ &= \Delta_I^{1/4}e^{-ia'P'_I}\Delta_I^{-1/4}\Delta_I^{1/4}e^{iaP_I}\Delta_I^{-1/4}e^{-iaP_I}e^{ia'P'_I} \\ &= e^{-a'P'_I}e^{-aP_I}e^{-iaP_I}e^{ia'P'_I} \end{aligned} \tag{5.2.3}$$

where we have used the commutation relation $\Delta_I^{is}e^{iaP_I}\Delta_I^{-is} = e^{i(e^{-2\pi s})aP_I}$ and the analogous one with P'_I instead of P_I. If $a > 0$, the above equation holds true for all complex $s \in \overline{\mathbb{S}_{-1/2}}$ and we have applied it with $s = -i/4$ (see the proof of 0.9.1). □

As a consequence we have another key equation for the T operator:

$$T_{I,I_{a',a}}T^*_{I,I_{a',a}} = e^{-aP_I}e^{-2a'P'_I}e^{-aP_I} . \tag{5.2.4}$$

Note also that by Prop. 0.24.1 we also have

$$\|T_{I,I_{a',a}}\|_1 = \mathrm{Tr}(e^{-\ell(I,I_{a',a})L_0}) = \mathrm{Tr}(e^{-2\sinh^{-1}(\ell'(I,I_{a',a}))L_0}) = \mathrm{Tr}(e^{-2\sinh^{-1}(\sqrt{aa'})L_0})$$

where $\ell'(I, I_{a',a}) = \sqrt{a'a}$ is the *second inner distance* (Appendix 0.26), thus $\ell' = \sinh(\frac{\ell}{2})$ by Prop. 0.26.1. We now have some of our basic formulas.

Theorem 5.2.3. *In any positive energy unitary representation of* $\mathbf{G}$ *we have*

$$e^{-2sL_0} = e^{-\tanh(\frac{s}{2})P}e^{-\sinh(s)P'}e^{-\tanh(\frac{s}{2})P} \tag{5.2.5}$$

therefore

$$e^{-2sL_0} \leq e^{-2\tanh(\frac{s}{2})P} \tag{5.2.6}$$

for all $s > 0$.

Proof. Consider an inclusion of intervals $I \Subset \tilde{I}$ with $\tilde{I} = I_1$, I symmetric with respect to the vertical axis and $\ell(\tilde{I}, I) = s$. By Prop. 0.24.1 we have $T_{\tilde{I},I} = e^{-sL_0}\Delta_2^{is/2\pi}$ thus

$$T_{\tilde{I},I}T^*_{\tilde{I},I} = e^{-2sL_0}\ .$$

On the other hand by Prop. 0.24.2 we have

$$T_{\tilde{I},I}T^*_{\tilde{I},I} = e^{-aP}e^{-2a'P'}e^{-aP}$$

where $a > 0$ and $a' > 0$ satisfy $\tau'(-a)\tau(a)\tilde{I} = I$. By equation (0.26.5) we have $a' = \sinh(s)/2$ and $a = \tanh(s/2)$, so we have formula (0.24.5).

Equation (0.24.5) immediately entails $e^{-2sL_0} \leq e^{-2\tanh(\frac{s}{2})P}$. □

Note that the inequality

$$e^{-sL_0} \leq e^{-\tanh(\frac{s}{2})P} \tag{5.2.7}$$

follows from (0.24.6) because the square root is an operator-monotone function.[1] Note also that the equation

$$e^{-2sL_0} = e^{-\tanh(\frac{s}{2})P'}e^{-\sinh(s)P}e^{-\tanh(\frac{s}{2})P'}$$

follows by (0.24.5) by applying a conjugation by a π-rotation on both sides.

Remark. We may formally analytically continue the parameter s in formula (0.24.5) to the imaginary axis and get the equality

$$e^{-2isL_0} = e^{i\tan(\frac{s}{2})P}e^{i\sin(s)P'}e^{i\tan(\frac{s}{2})P}\ , \tag{5.2.8}$$

which correspond to the identity in $\mathbf{G}$: $R(2s) = \tau(\tan\frac{s}{2})\tau'(\sin s)\tau(\tan\frac{s}{2})$. In particular we have $e^{-i\pi L_0} = e^{iP}e^{iP'}e^{iP}$.

[1] The inequality (0.24.7) does not follow from $L_0 \geq \frac{1}{2}P$ because the exponential is not operator monotone.

More general embeddings

With U a positive energy representation of $\mathbf{G}$ as above, we shall need to estimate the nuclear norm of the more general embedding operators

$$T^U_{I,I_0}(i\lambda) = T^U_{I,I_0}(i\lambda) \equiv \Delta_I^\lambda \Delta_{I_0}^{-\lambda}, \quad 0 < \lambda < 1/2,$$

associated with an inclusion of intervals $I_0 \Subset I$. As above $T^U_{I,I_0} = T^U_{I,I_0}(i/4)$.

Proposition 5.2.4. *For an inclusion of intervals $I_t \subset I$ with $I_t \equiv I_{t,t} = \tau'_{-t}\tau_t I$ as above, thus $\ell'(I, I_t) = t$, we have*

$$T_{I,I_t}(i\lambda) = e^{-i\cos(2\pi\lambda)tP'_I}(e^{-\sin(2\pi\lambda)tP'_I}e^{-\sin(2\pi\lambda)tP_I})e^{i\cos(2\pi\lambda)tP_I}e^{-itP_I}e^{itP'_I}\,.$$

Proof.

$$\begin{aligned}
T_{I,I_t}(i\lambda) &\equiv \Delta_I^\lambda \Delta_{I_t}^{-\lambda}\\
&= \Delta_I^\lambda e^{-itP'_I}e^{itP_I}\Delta_I^{-\lambda}e^{-itP_I}e^{itP'_I}\\
&= (\Delta_I^\lambda e^{-itP'_I}\Delta_I^{-\lambda})(\Delta_I^\lambda e^{itP_I}\Delta_I^{-\lambda})e^{-itP_I}e^{itP'_I}\\
&= e^{-i(e^{-2\pi i\lambda})tP'_I}e^{i(e^{2\pi i\lambda})tP_I}e^{-itP_I}e^{itP'_I}\\
&= e^{-i(\cos(2\pi\lambda)-i\sin(2\pi\lambda))tP'_I}e^{i(\cos(2\pi\lambda)+i\sin(2\pi\lambda))tP_I}e^{-itP_I}e^{itP'_I}\\
&= e^{-i\cos(2\pi\lambda)tP'_I}e^{-\sin(2\pi\lambda)tP'_I}e^{-\sin(2\pi\lambda)tP_I}e^{i\cos(2\pi\lambda)tP_I}e^{-itP_I}e^{itP'_I}
\end{aligned} \tag{5.2.9}$$

□

Corollary 5.2.5. $\|T_{I,I_t}(i\lambda)\|_1 = \|T_{I,I_{\sin(2\pi\lambda)t}}\|_1,\ 0 < \lambda < 1/2.$

Proof. Immediate because by Proposition 0.24.4 the operator $T_{I,I_{\sin(2\pi\lambda)t}}$ is obtained by left and right multiplication of $T_{I,I_t}(i\lambda)$ by unitary operators.

A fourth operator inequality

Proposition 5.2.6. *Let U be a positive energy unitary representation of* $\mathbf{G}$. *We have the following inequality:*

$$\|e^{-\tan(2\pi\lambda)d_I P}\Delta_I^{-\lambda}\| \le 1\,, \qquad 0 < \lambda < 1/4\,. \tag{5.2.10}$$

Here P is the translation generator, I is an interval of $\mathbb{R}$ with usual lenght d_I and Δ_I is the modular operator associated with I.

Proof. Fix $a > 0$. Then

$$\Delta^{is}_{(a,\infty)} = U(a)\Delta^{is}U(-a) = \Delta^{is}\Delta^{-is}U(a)\Delta^{is}U(-a) = \Delta^{is}U(e^{2\pi s}a - a)$$
$$\Delta^{is}_{(\infty,-a)} = U(-a)\Delta^{is}U(a) = \Delta^{is}\Delta^{-is}U(-a)\Delta^{is}U(a) = \Delta^{is}U(-e^{2\pi s}a + a)$$

where $\Delta \equiv \Delta_{(0,\infty)}$.

We have for all real z

$$T(z) \equiv \Delta^{-iz}\Delta^{iz}_{(a,\infty)} = U(e^{2\pi z}a - a)$$

thus for all complex $z \in \mathbb{S}_{1/2}$. Indeed the both sides of the above equations define a bounded continuous function on $\overline{\mathbb{S}_{1/2}}$, analytic on $\mathbb{S}_{1/2}$, that are equal for real z. The left side is analytic because of Thm. 0.10.2, and the right side is analytic because $\Im(e^{2\pi z}a - a) \geq 0$ for $z \in \mathbb{S}_{1/2}$ and the translation generator P is positive. In particular, setting $z = i\lambda$, $0 \leq \lambda \leq 1/2$ we have

$$\Delta^{\lambda}\Delta^{-\lambda}_{(a,\infty)} = U(e^{i2\pi\lambda}a - a) = U\big(a\cos(2\pi\lambda) - a + ia\sin(2\pi\lambda)\big)$$
$$= e^{-a\sin(2\pi\lambda)P}U\big(a\cos(2\pi\lambda) - a\big) \quad (5.2.11)$$

Analogously

$$\Delta^{-\lambda}\Delta^{\lambda}_{(-\infty,-a)} = e^{-a\sin(2\pi\lambda)P}U\big(-a\cos(2\pi\lambda) + a\big)$$

therefore

$$e^{-a\sin(2\pi\lambda)P} = \Delta^{\lambda}\Delta^{-\lambda}_{(a,\infty)}U\big(-a\cos(2\pi\lambda) + a\big) = \Delta^{\lambda}U(a)\Delta^{-\lambda}U\big(-a\cos(2\pi\lambda)\big)$$
$$e^{-a\sin(2\pi\lambda)P} = \Delta^{-\lambda}\Delta^{\lambda}_{(-\infty,-a)}U\big(a\cos(2\pi\lambda) - a\big) = \Delta^{-\lambda}U(-a)\Delta^{\lambda}U\big(a\cos(2\pi\lambda)\big)$$

Choosing $a = d_I/\cos(2\pi\lambda)$, that forces to take $\lambda < 1/4$, we have

$$e^{-d_I\tan(2\pi\lambda)P} = \Delta^{\lambda}U(a)\Delta^{-\lambda}U(-d_I) = \Delta^{\lambda}U(a - d_I)\Delta^{-\lambda}_{(d_I,\infty)}$$
$$e^{-d_I\tan(2\pi\lambda)P} = \Delta^{-\lambda}U(-a)\Delta^{\lambda}U(d_I) = \Delta^{-\lambda}U(-a + d_I)\Delta^{\lambda}_{(-d_I,\infty)}$$

Let $E_\pm$ be the spectral projection of Δ relative to the intervals $(1,\infty)$ and $(0,1]$. Then

$$E_-e^{-d_I\tan(2\pi\lambda)P}\Delta^{-\lambda}_I = E_-\Delta^{\lambda}U(a - d_I)\Delta^{\lambda}_{(-\infty,d_I)}\Delta^{-\lambda}_I = E_-\Delta^{\lambda}U(a - d_I)T_{(-\infty,d_I),I}$$
$$E_+e^{-d_I\tan(2\pi\lambda)P}\Delta^{-\lambda}_I = E_+\Delta^{-\lambda}U(-a + d_I)\Delta^{\lambda}_{(-d_I,\infty)}\Delta^{-\lambda}_I = E_+\Delta^{-\lambda}U(-a + d_I)T_{(-d_I,\infty),I}$$

Now $\|E_\mp\Delta^{\pm\lambda}U(\pm(a - d_I))\| \leq 1$. As I is contained in $(-\infty, d_I)$ and in $(-d_I,\infty)$, we also have $\|T_{(-\infty,d_I),I}\| \leq 1$, $\|T_{(-d_I,\infty),I}\| \leq 1$, hence

$$\|e^{-d_I\tan(2\pi\lambda)P}\Delta^{-\lambda}_I\| \leq 2 \quad (5.2.12)$$

We show now that the left hand side is bounded by 1, namely Indeed we may consider the n-tensor product net $U \otimes \cdots \otimes U$ and apply eq. (0.24.12) to it. Then

$$||e^{-\tan(2\pi\lambda)d_{d_I}P}\Delta_I^{-\lambda}||^n \leq 2$$

that proves our inequality as n is arbitrary. □

Note that the inequality (0.24.10) gives

$$\Delta_I^{-\lambda}e^{-2\tan(2\pi\lambda)d_I P}\Delta_I^{-\lambda} \leq 1$$

namely

$$e^{-2\tan(2\pi\lambda)d_I P} \leq \Delta_I^{2\lambda} .$$

In particular, if $I = I_2$ is the interval $(-1, 1)$ of the real line, we have $e^{-2\tan(2\pi\lambda)P} \leq \Delta_2^{2\lambda}$. By conjugating both members of the inequality with the modular conjugation J_{I_2} (ray inversion map) we get $e^{-2\tan(2\pi\lambda)P'} \leq \Delta_2^{-2\lambda}$, namely

$$e^{-2\tan(2\pi\lambda)P} \leq \Delta_2^{2\lambda} \leq e^{2\tan(2\pi\lambda)P'} \tag{5.2.13}$$

and, by rescaling with the dilation unitaries we obtain the inequality

$$e^{-2\tan(2\pi\lambda)d_I P} \leq \Delta_I^{2\lambda} \leq e^{2\tan(2\pi\lambda)\frac{1}{d_I}P'} \tag{5.2.14}$$

In particular, evaluating at $\lambda = 1/8$, we get

$$e^{-2d_I P} \leq \Delta_I^{1/4} \leq e^{\frac{2}{d_I}P'} . \tag{5.2.15}$$

5.3 Modular nuclearity and L^2-nuclearity

Basic abstract setting

Recall that a linear operator $A : X \to Y$ between Banach spaces X, Y is called nuclear if there exist sequences of elements $f_k \in X^*$ and $y_k \in Y$ such that $\sum_k ||f_k||\,||y_k|| < \infty$ and $Ax = \sum_k f_k(x)y_k$. (A linear operator on a Hilbert space is nuclear iff it is of trace class). The infimum $||A||_1$ of $\sum_k ||f_k||\,||y_k||$ over all possible choices of $\{f_k\}$ and $\{y_k\}$ as above is the nuclear norm of A.

Let $\mathcal{H}$ be a Hilbert space and $H \subset \tilde{H}$ an inclusion of standard subspaces of $\mathcal{H}$.

We shall say that $H \subset \tilde{H}$ satisfies L^2*-nuclearity* if the operator

$$T_{\tilde{H},H} \equiv \Delta_{\tilde{H}}^{1/4}\Delta_H^{-1/4}$$

is nuclear.

We shall say that $H \subset \tilde{H}$ satisfies *modular nuclearity* if the operator

$$\Delta_{\tilde{H}}^{1/4} E_H$$

is nuclear. Here E_H is the real orthogonal projection of $\mathcal{H}$ onto H and the operator $\Delta_{\tilde{H}}^{1/4} E_H$ is thus a real linear operator.

L^2 and modular compactness are analogously defined by requiring the compactness of the corresponding operators.

Proposition 5.3.1. *L^2-nuclearity implies modular nuclearity and* $\|\Delta_{\tilde{H}}^{1/4} E_H\|_1 \leq \|T_{\tilde{H},H}\|_1$.

Proof. First note that

$$\|\Delta_H^{1/4}|_H\| \leq 1 \ ,$$

indeed if $\xi \in H$ we have

$$\begin{aligned}\|\Delta_H^{1/4}\xi\|^2 = (\Delta_H^{1/2}\xi, \xi) = (J_H\xi, J_H\Delta_H^{1/2}\xi) = (J_H\xi, S_H\xi) \\ = (J_H\xi, \xi) \leq \|J_H\xi\|\,\|\xi\| = \|\xi\|^2 \ .\end{aligned}$$

Therefore, assuminig L^2-nuclearity for $H \subset \tilde{H}$, we have

$$\|\Delta_{\tilde{H}}^{1/4} E_H\|_1 = \|T_{\tilde{H},H}\Delta_H^{1/4} E_H\|_1 \leq \|T_{\tilde{H},H}\|_1 \|\Delta_H^{1/4} E_H\| \leq \|T_{\tilde{H},H}\|_1$$

□

We shall consider the condition

$$\|T_{\tilde{H},H}(i\lambda)\|_1 < \infty$$

with $T_{\tilde{H},H}(z) \equiv \Delta_{\tilde{H}}^{-iz}\Delta_H^{iz}$, for general exponents $0 < \lambda < 1/2$.

By an immediate extension of the above argument we then have

$$\|\Delta_{\tilde{H}}^{\lambda} E_H\|_1 \leq \|T_{\tilde{H},H}(i\lambda)\|_1 \ .$$

Comparison of the nuclearity conditions

Let H be a Möbius covariant net of real Hilbert subspaces of a Hilbert space $\mathcal{H}$. Consider the following nuclearity conditions for H.

Trace class condition: $\mathrm{Tr}(e^{-sL_0}) < \infty$, $s > 0$;

L^2-nuclearity: $\|T_{\tilde{I},I}(i\lambda)\|_1 < \infty$, $\forall I \Subset \tilde{I}$, $0 < \lambda < 1/2$;

Modular nuclearity: $\Xi_{\tilde{I},I}(\lambda) : \xi \in H(I) \to \Delta_{\tilde{I}}^{\lambda}\xi \in \mathcal{H}$ is nuclear $\forall I \Subset \tilde{I}$, $0 < \lambda < 1/2$;

Buchholz-Wichmann nuclearity: $\Phi_I^{\mathrm{BW}}(s) : \xi \in H(I) \to e^{-sP}\xi \in \mathcal{H}$ is nuclear, I interval of $\mathbb{R}$, $s > 0$ (P the generator of translations);

Conformal nuclearity: $\Psi_I(s) : \xi \in H(I) \to e^{-sL_0}\xi \in \mathcal{H}$ is nuclear, I interval of S^1, $s > 0$.

We shall show the following chain of implications:

$$
\begin{gathered}
\text{Trace class condition}\\
\Updownarrow\\
L^2 - \text{nuclearity}\\
\Downarrow\\
\text{Modular nuclearity}\\
\Downarrow\\
\text{Buchholz-Wichmann nuclearity}\\
\Downarrow\\
\text{Conformal nuclearity}
\end{gathered}
$$

Where all the conditions can be understood for a specific value of the parameter, that will be determined, or for all values in the parameter range.

We have already discussed the implications “Trace class condition $\Leftrightarrow$ L^2-nuclearity $\Rightarrow$ Modular nuclearity”.

Modular nuclearity $\Rightarrow$ BW-nuclearity

Equation (0.24.10) gives $\|e^{-\tan(2\pi\lambda)d_I P}\Delta_I^{-\lambda}\| \le 1$ for all $0 < \lambda < 1/4$, so the following holds:

Proposition 5.3.2. *Let $I_0 \Subset I$ be a an inclusion of intervals of $\mathbb{R}$. We have*

$$\|\Phi_{I_0}^{\mathrm{BW}}(\tan(2\pi\lambda)d_I)\|_1 \le \|\Xi_{I,I_0}(\lambda)\|_1$$

where d_I is the length of I, $0 < \lambda < 1/4$.

Proof. With $\xi \in H(I_0)$ we have

$$\Phi^{\mathrm{BW}}_{I_0}(s)\xi = e^{-sP}\xi = (e^{-sP}\Delta_I^{-\lambda})\Delta_I^{\lambda}\xi$$

thus

$$\Phi^{\mathrm{BW}}_{I_0}(\tan(2\pi\lambda)d_I) = (e^{-\tan(2\pi\lambda)d_I P}\Delta_I^{-\lambda}) \cdot \Xi_{I,I_0}(\lambda)$$

and so $||\Phi^{\mathrm{BW}}_{I_0}(\tan(2\pi\lambda)d_I)||_1 \leq ||\Xi_{I,I_0}(\lambda)||_1$ as desired □

Quantitative and asymptotic estimates

At this point we have the following chain of inequalities:

$$\begin{aligned} ||\Phi^{\mathrm{BW}}_{I_0}(\tan(2\pi\lambda)d_I)||_1 &\leq ||\Xi_{I,I_0}(\lambda)||_1 \\ &\leq ||T_{I,I_0}(\lambda)||_1 \\ &= ||T_{I,I_1}||_1 \qquad \ell'(I,I_1) = \sin(2\pi\lambda)\ell'(I,I_0) \\ &= \mathrm{Tr}(e^{-sL_0}), \qquad s = \ell(I,I_1) \end{aligned}$$

Note that $s = 2\sinh^{-1}(\ell'(I,I_1)) = 2\sinh^{-1}(\sin(2\pi\lambda)\ell'(I,I_0))$ by Prop. 0.26.1.

As $\lambda \to 0^+$ one has $\tan(2\pi\lambda) \sim 2\pi\lambda$ and $s \sim 4\pi\lambda\ell'(I,I_0)$ so we have the asymptotic inequality

$$||\Phi^{\mathrm{BW}}_{I_0}(a)||_1 \leq \mathrm{Tr}(e^{-(2\ell'(I,I_0)/d_I)aL_0}), \quad a \to 0^+ . \tag{5.3.1}$$

Here below we have our last estimate that will give a relation to conformal nuclearity.

BW-nuclearity ⇒ Conformal nuclearity

By equation (0.24.6) there exists a bounded operator B with norm $||B|| \leq 1$ such that

$$e^{-sL_0} = Be^{-\tanh(\frac{s}{2})P}$$

therefore

$$\Psi_I(s) = B\Phi^{\mathrm{BW}}_I(\tanh(s/2)) \tag{5.3.2}$$

and so we have

Proposition 5.3.3. $||\Psi_I(s)||_1 \leq ||\Phi^{\mathrm{BW}}_I(\tanh(s/2))||_1$.

5.4 Appendix. Inner distance

Given intervals $I \Subset \tilde{I}$ we consider the *inner distance* between $\tilde{I}$ and I to be the number $\ell(\tilde{I}, I)$ defined as follows. First, in the real line picture, assume $\tilde{I} = (-1, 1)$. Then there exists $s \in \mathbb{R}$ such that $\delta_{\tilde{I}}(s)I$ is symmetric, i.e. $I = (-a, a)$. Then we set $\ell(\tilde{I}, I) = -\log a$. By definition $\ell(\tilde{I}, I_0) = \ell(\tilde{I}, I)$ if I and I_0 are in the same $\delta_{\tilde{I}}$-orbit.

If now $I \Subset \tilde{I}$ is any inclusion of intervals of S^1, we choose $g \in \mathbf{G}$ such that $g\tilde{I}$ is the right semicircle (that corresponds to $(-1, 1)$ in the real line picture). Then we set $\ell(\tilde{I}, I) \equiv \ell(g\tilde{I}, gI)$. As the choice of g is unique modulo right multiplication by an element $\delta_{\tilde{I}}(s)$, the inner distance is well defined.

Note the following equality

$$\ell(I_2, \delta_{I_1}(-s)I_2) = s \tag{5.4.1}$$

where I_1 and I_2 are the upper and the right semicircles and $s > 0$.

It is easily seen that the inner distance satisfies the following properties:

- *Positivity*: $\ell(\tilde{I}, I) > 0$ if $I \Subset \tilde{I}$ and all positive values are attained.
- *Monotonicity*: If $I_1 \Subset I_2 \Subset I_3$ we have $\ell(I_3, I_1) > \ell(I_3, I_2)$ and $\ell(I_3, I_1) > \ell(I_2, I_1)$.
- *Möbius invariance*: $\ell(\tilde{I}, I) = \ell(g\tilde{I}, gI)$ for all $g \in G$.
- *Super-additivity*: $\ell(I_3, I_1) \geq \ell(I_3, I_2) + \ell(I_2, I_1)$ if $I_1 \Subset I_2 \Subset I_3$.

We now define second inner distance ℓ' as follows. For any $a, a' > 0$ we set

$$\ell'(\tilde{I}, I) = \sqrt{aa'} \quad \text{if} \quad I = \tau'_{\tilde{I}}(-a')\tau_{\tilde{I}}(a)\tilde{I}\,.$$

Since conjugating $\tau_{\tilde{I}}(t)$ and $\tau'_{\tilde{I}}(t)$ by a dilation $\delta_{\tilde{I}}(s)$ by $\lambda = e^s$ gives $\tau_{\tilde{I}}(\lambda t)$ and $\tau'_{\tilde{I}}(\lambda^{-1}t)$, the above formula gives a well defined and Möbius invariant quantity.

Note now that τ'_{-t} is the conjugate of τ_t by the ray inversion

$$\tau'(-t) : x \mapsto \frac{x}{1 + tx}\,,$$

Proposition 5.4.1. $\ell' = \sinh(\ell/2)$.

Proof. Given $t > 0$ and an inclusion of intervals $I \subset \tilde{I}$ with $\ell'(\tilde{I}, I) = t$ we want to calculate $s \equiv \ell(\tilde{I}, I)$. We may assume that $\tilde{I} = \mathbb{R}_+$ and $I = \tilde{I}_{\lambda^{-1}t, \lambda t} \equiv \tau'(\lambda^{-1}t)\tau(\lambda t)\tilde{I}$. Then

$$I = \tau'(-\lambda^{-1}t)\tau(\lambda t)\tilde{I} = \tau'(-\lambda^{-1}t)(\lambda t, \infty) = \left(\frac{\lambda^{-1}t}{1 + t^2}, \frac{1}{\lambda t}\right)\,.$$

We may further choose λ so that I symmetric under ray inversion, namely $\lambda^{-1} = \sqrt{1+t^2}$. thus

$$I = \left(\frac{t}{\sqrt{1+t^2}}, \frac{\sqrt{1+t^2}}{t} \right) . \tag{5.4.2}$$

Now in the real line picture the right semicircle I_2 corresponds to the interval $(-1,1)$, thus by eq. (0.26.1) s is determined by

$$\delta_{(-1,1)}(s)0 = \frac{t}{\sqrt{1+t^2}} . \tag{5.4.3}$$

Now

$$\delta_{(-1,1)}(s) : x \mapsto \frac{x+1-e^{-s}(x-1)}{x+1+e^{-s}(x-1)}$$

thus eq. (0.26.3) gives

$$\frac{t}{\sqrt{1+t^2}} = \frac{1-e^{-s}}{1+e^{-s}} = \tanh(s/2) = \frac{\sinh(s/2)}{\sqrt{1+\sinh^2(s/2)}}$$

and this implies $t = \sinh(s/2)$. □

Note that $\delta_{(-1,1)}(s)\mathbb{R}_+ = I_{\lambda^{-1}t,\lambda t} \equiv \tau'(-\lambda^{-1}t)\tau(\lambda t)I$ with $\lambda = 1/\sqrt{1+t^2}$, namely

$$\delta_{(-1,1)}(s)\mathbb{R}_+ = \tau'(-t\sqrt{1+t^2})\tau(t/\sqrt{1+t^2})\mathbb{R}_+ . \tag{5.4.4}$$

In term of the inner distance s the interval I in (0.26.2) is given by

$$\begin{aligned} I = (\tanh(s/2), \coth(s/2)) &= \tau'(-\sinh(s/2)\cosh(s/2))\tau(\tanh(s/2))\mathbb{R}_+ \\ &= \tau'(-\sinh(s)/2)\tau(\tanh(s/2))\mathbb{R}_+ \end{aligned} \tag{5.4.5}$$