PhD admissions exam

40^{th} cycle – 8^{th} July 2024

This exam consists of 14 exercises divided into seven different topics, each in a separate section of this document. The exercises are independent and can be solved in any order.

- Applicants are required to submit written solutions of **at most four exercises**.
- Each exercise is worth up to **9 points**. However, the score within each topic/section can total up to **15 points** maximum. The maximum total score for the exam is capped at **30 points**.
- Solutions should be written in **English**, except for candidates applying for positions as Italian public servants, who may write their solutions in **Italian**.
- During the exam, it is **forbidden** to consult notes, communicate with others, or use any external resources.

Candidates are encouraged to select a manageable number of exercises that can be completed within the allotted time while maintaining high-quality solutions. This exam serves as an opportunity for candidates to showcase their mathematical prowess. Solutions should be well-communicated, technically rigorous, and fully reasoned, highlighting the candidates' abilities among other desirable qualities.

Contents

1 Algebra

Exercise 1.1. Consider the vector space $\mathbb{R}^3$, endowed with the canonical basis $\mathcal{E} = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ and let (x, y, z) be coordinates with respect to the basis $\mathcal{E}$. Let $\Phi \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^3)$ be the endomorphism defined by the following conditions:

$$\Phi(\mathbf{e}_1) = 2\mathbf{e}_2 + 2\mathbf{e}_3,\ \Phi(\mathbf{e}_1 + 2\mathbf{e}_2) = \mathbf{0},\ \Phi(\mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3) = \mathbf{0}.$$

(i) Compute the characteristic polynomial $P_\Phi(t) \in \mathbb{R}[t]$ of Φ and, for any eigenvalue $\lambda \in \mathbb{R}$ of Φ, determine a basis and linear equations (in the given coordinates (x, y, z)) of the corresponding eigenspace $V_\lambda(\Phi) \subseteq \mathbb{R}^3$.

(ii) Deduce that Φ is not diagonalizable but that Φ admits a Jordan normal form. Explicitely determine such a Jordan normal form as well as the minimal polynomial $m_\Phi(t) \in \mathbb{R}[t]$ of the endomorphism Φ.

(iii) Write down an explicit Jordan basis $\mathcal{J}$ for Φ, expressing each vector of such a Jordan basis $\mathcal{J}$ as a linear combination of the vectors in the canonical basis $\mathcal{E}$.

Exercise 1.2. Consider the symmetric group $(S_5, \circ)$ which acts as a permutation group on the set $X := \{1, 2, 3, 4, 5\}$. Let $Y := \{1, 3, 5\} \subsetneq X$.

(i) Prove that the subset

$$H := \big\{\sigma \in (S_5, \circ) \mid \sigma(y) = y,\ \forall\, y \in Y\big\} \subset S_5$$

is a subgroup of $(S_5, \circ)$ and determine further cardinality of H.

(ii) Prove that the subset

$$K := \big\{\sigma \in (S_5, \circ) \mid \sigma(y) \in Y,\ \forall\, y \in Y\big\} \subset S_5$$

is a subgroup of $(S_5, \circ)$ and determine further cardinality of K.

(iii) Prove that the subset

$$T := \big\{\sigma \in (S_5, \circ) \mid \sigma(x) = x,\ \forall\, x \in X \setminus Y\big\} \subset S_5$$

is a normal subgroup of $(K, \circ)$ and determine, up to isomorphism, the group structure of the quotient group K/T, enstablishing also if one can have a surjective group homomorphism from K/T onto H.

2 Analysis

Exercise 2.1. Let (X, d) be a complete metric space.

(i) Let A a nonempty compact subset of X. Prove that if $f : A \to A$ is such that

$$d(f(x), f(y)) < d(x, y), \qquad \text{for every } x, y \in A, x \neq y,$$

then f has a unique fixed point in A.

(ii) Is the result still true if A is unbounded? And if A is not closed?

Let $g : X \to X$ be a contraction mapping [1] and let $h : X \to X$ be a function commuting with g (that is, such that $g \circ h = h \circ g$).

(iii) Prove that h has a fixed point.

(iv) Is the fixed point of h necessarily unique?

Exercise 2.2. Let $\Omega \subset \mathbb{C}$ be an open set. Let $B_r(z) \subset \Omega$ be the closed disk centered in $z \in \mathbb{C}$ of radius $r > 0$. Prove that, for every f holomorphic in Ω, there holds[2]:

(i) $f(z) = \dfrac{1}{\pi r^2} \displaystyle\int_{B_r(z)} f(w)\, dw$;

(ii) $|f(z)| \leq \dfrac{1}{\pi^{1/2} r} \left(\displaystyle\int_{B_r(z)} |f(w)|^2\, dw \right)^{1/2}$.

Let $A^2(\Omega)$ be the vector space of functions holomorphic and square integrable in Ω. Define the scalar product

$$\langle f, g \rangle := \int_\Omega \overline{f(z)} g(z)\, dz,$$

on $A^2(\Omega)$. Prove that:

(iii) with the above defined scalar product, $A^2(\Omega)$ is a complex Hilbert space;

(iv) if $\Omega = \{z \in \mathbb{C} \mid |z| < 1\}$ the functions

$$e_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, \qquad n = 0, 1, 2, \ldots$$

form an orthonormal basis in $A^2(\Omega)$.

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[1] In **Exercise 2.1**: $g : X \to X$ is a **contraction mapping** if there exists $k \in [0, 1)$ such that $d(g(x), g(y)) \leq kd(x, y)$ for every $x, y \in X$

[2] In **Exercise 2.2** integrals are intended with respect to the two dimensional Lebesgue measure.

3 Didactics and history of mathematics

Exercise 3.1. Historical and pedagogical aspects of geometric constructions with ruler and compass, possibly with reference to dynamic geometry software.

Exercise 3.2. The role of digital technologies in mathematics education.

4 Geometry

Exercise 4.1. Consider the complex projective plane $\mathbb{P}^2(\mathbb{C})$, with homogenous coordinates $[x_0, x_1, x_2]$. Recall that a *projective conic* $C \subset \mathbb{P}^2(\mathbb{C})$ is defined to be the vanishing locus of a non-zero, complex quadratic form $F(x_0, x_1, x_2)$. Therefore, if we let $\mathcal{C}$ to be the set of all projective conics, then for any $C \in \mathcal{C}$ there exists a non-zero complex quadratic form $F(x_0, x_1, x_2)$, equivalently a non-zero square symmetric matrix $A \in Sym(3 \times 3;\ \mathbb{C}) \subset M(3 \times 3;\ \mathbb{C})$ s.t.

$$C : F(x_0, x_1, x_2) = (x_0, x_1, x_2)\ A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0.$$

The set of points $p = [p_0, p_1, p_2] \in \mathbb{P}^2(\mathbb{C})$ such that $F(p_0, p_1, p_2) = 0$ is called the *support* of the projective conic C.

(i) Prove that $\mathcal{C}$ identifies with a complex projective space $\mathbb{P}$ and determine the projective dimension of $\mathbb{P}$.

(ii) Let $\mathcal{S} \subsetneq \mathcal{C}$ denote the subset of $\mathcal{C}$ parametrizing *singular* (equivalently *degenerate*) projective conics. Prove that $\mathcal{S}$ identifies with a cubic hypersurface in $\mathbb{P}$, namely with the vanishing locus of a single homogeneous cubic polynomial in the natural homogeneous coordinates of $\mathbb{P}$.

(iii) Let $\mathcal{D} \subsetneq \mathcal{S}$ denote the subset parametrizing *doubly degenerate* projective conics, namely projective conics C whose support is a line in $\mathbb{P}^2(\mathbb{C})$. Prove that $\mathcal{D}$ is a surface in $\mathcal{S}$ which bijectively corresponds to $(\mathbb{P}^2(\mathbb{C}))^*$, the *dual projective plane* of $\mathbb{P}^2(\mathbb{C})$.

Exercise 4.2. Let $\mathbb{E}^2(\mathbb{R})$ be the *Euclidean plane*, namely the 2-dimensional Euclidean space, endowed with origin O and Cartesian coordinates (x, y). Let $C \subset \mathbb{E}^2(\mathbb{R})$ be the plane cubic curve, whose Cartesian equation is given by

$$x^3 - y^2 - 3x + 2 = 0.$$

(i) Determine all the singular points of $C \subset \mathbb{E}^2(\mathbb{R})$ and deduce a *polynomial parametric representation* of C

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\varphi} & C \\ t & \to & \varphi(t) = (x(t), y(t)), \end{array}$$

namely where $x(t),\ y(t) \in \mathbb{R}[t]$.

(ii) Determine whether the parametrization $\varphi(t)$ as in (i) is a regular parametrization for C and whether $\varphi(t)$ is an injective parametrization for C.

(iii) Classify all singular points of C (namely if they are either *ordinary double points* or *cusps* or *triple points*, etcetera) and deduce that C is *irreducible*, i.e. that C does not split as a union of a line and a (possibly reducible) conic.

5 Mathematical physics

Exercise 5.1. The mass point P moves on a vertical circumference whose parametric expression is $(y, z) = \big(\sin x\,,\, \cos x\big)$, being $x \in (-\pi\,,\, \pi]$ an angle. P experiences the attraction due to gravity, whose acceleration field is constant and always with the same direction, that is oriented as the half-line which starts from the origin O and includes the semi-axis of the ordinates with negative z, with reference to the cartesian frame Oyz. The motion of P is ruled by the differential equation

$$\ddot{x} = -\frac{\mathrm{d}U}{\mathrm{d}x} - \lambda\dot{x}\ , \tag{5.1}$$

where U is the total potential energy experienced by P and $\lambda \geq 0$ is a (constant) friction parameter; the l.h.s. of the equation above is such that the mass of P is equal to 1.

The total potential energy is such that $U = U_{\mathrm{gr}} + U_{\mathrm{el}}$. Here, the gravitational energy U_{gr} is due to the force exerted on P by the gravity when the norm g of its acceleration is equal to 1; moreover, the elastic energy U_{el} is due to a vertical spring that links the point P to the y–axis and it exerts a force on P whose norm is equal to $k|z|$, where $k > 0$ is the elastic constant and (y, z) are the cartesian coordinates of P.

(i) Rewrite the equation of motion (5.1) in the form of a second order ODE, i.e., as $\ddot{x} = f(x, \dot{x}, t)$ for a suitable and explicit expression of function f.

(ii) Determine all the possible stationary solutions as a function of the parameter k.

(iii) Consider the case with $k = 2$ and initial conditions such that $x(0) = 2\pi/3$, $\dot{x}(0) = 0$. Let us recall that the rate of dissipation of the total energy $E = \frac{1}{2}\dot{x}^2 + U(x)$ is such that $\dot{E} = -\lambda\dot{x}^2$.
Prove that

$$\lim_{t\to\infty} x(t) = \frac{\pi}{3} \qquad \forall\ \lambda > \lambda^*\ ,$$

where $\lambda^* \in \mathbf{R}_+$ is a number that the candidate can *choose* according to his/her convenience.

Exercise 5.2. Consider the following expression of a magnetic field (somehow inspired to the so called Parker model):

$$\mathbf{B} = c_1 \frac{\sin\vartheta}{r^2}\mathbf{e}_r - c_2 \frac{\sin^2\vartheta}{r}\mathbf{e}_\varphi \ ,$$

where $c_1 \ge 0$ and $c_2 \ge 0$ are constant real numbers, while $\{\mathbf{e}_r \, , \, \mathbf{e}_\vartheta \, , \, \mathbf{e}_\varphi\}$ are the unit vectors defining the orthogonal basis which refers to the usual sperical coordinates $\big\{(r \, , \, \vartheta \, , \, \varphi) \in \mathbf{R}_+ \cup \{0\} \times [-\pi, \pi] \times [0, 2\pi)\big\}$, such that the coordinates of a cartesian frame $Oxyz$ reads as $(x \, , \, y \, , \, z) = \big(r\cos\vartheta\cos\varphi \, , \, r\cos\vartheta\sin\varphi \, , \, r\sin\vartheta\big)$.

(i) Consider a point P having mass and electric charge equal to m and q, respectively. Focus on the circular periodic orbits which are parallel to the horizontal plane Oxy, with center on the z–axis and of period equal to T.
In the case with $c_1 = 0$, determine the radius and the constant vertical coordinate z of these circular orbits, described by the motion of P when it is subject to the Lorentz force $\mathbf{F}_{\mathrm{L}} = q\mathbf{v} \wedge \mathbf{B}$ and the gravitational force exerted by a body of mass M located in the origin O of the frame.

(ii) It is well known that the vector potential $\mathbf{A}$ is defined so that $\nabla \times \mathbf{A} = \mathbf{B}$, where the curl operator (also known as rotor) is such that in spherical coordinates the previous equation can be expressed as follows:

$$\begin{cases} B_r = \dfrac{1}{r\sin\vartheta}\left(\dfrac{\partial}{\partial\vartheta}\big(A_\varphi \sin\vartheta\big) - \dfrac{\partial A_\vartheta}{\partial\varphi}\right) \\ B_\vartheta = \dfrac{1}{r}\left(\dfrac{1}{\sin\vartheta}\dfrac{\partial A_r}{\partial\varphi} - \dfrac{\partial}{\partial r}\big(rA_\varphi\big)\right) \\ B_\varphi = \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}\big(rA_\vartheta\big) - \dfrac{\partial A_r}{\partial\vartheta}\right) \end{cases} ,$$

with obvious meaning of the symbols, i.e., $A_r = \mathbf{A}\cdot\mathbf{e}_r \, , \, A_\vartheta = \mathbf{A}\cdot\mathbf{e}_\vartheta \, , \, A_\varphi = \mathbf{A}\cdot\mathbf{e}_\varphi \, , \, B_r = \mathbf{B}\cdot\mathbf{e}_r \, , \, B_\vartheta = \mathbf{B}\cdot\mathbf{e}_\vartheta \, , \, B_\varphi = \mathbf{B}\cdot\mathbf{e}_\varphi \, .$

(ii-1) Under the assumpion that the potential vector is axisymmetric, i.e., $\frac{\partial A_r}{\partial\varphi} = \frac{\partial A_\vartheta}{\partial\varphi} = \frac{\partial A_\varphi}{\partial\varphi} = 0$, determine A_φ .

(ii-2) Under the further assumpion that $A_r = 0$, determine A_ϑ .

6 Numerical analysis

Exercise 6.1. Let $\{x_i\}_{i\geq 0}$ be the sequence obtained by applying the Newton method (also referred to as method of tangents) to approximate the zeros of the function

$$f(x) = |x|^\alpha, \quad 0 < \alpha \in \mathbb{R},$$

for a given $x_0 \neq 0$.

(i) Determine, if any, α and x_0 such that $\lim_{i\to\infty} x_i = 0$;

(ii) determine, if any, α and x_0 such that $\lim_{i\to\infty} |x_i| = +\infty$;

(iii) determine, if any, α and x_0 such that $\lim_{i\to\infty} |x_i| = \alpha$;

(iv) determine, if any, $\alpha \neq 1$ and x_0 such that $\lim_{i\to\infty} \frac{|x_{i+1}|}{x_i^2} = c$ where c is a strictly positive constant.

Exercise 6.2. Let us consider the matrix A with entries

$$a_{i,j} := \begin{cases} 2, & i = j, \\ -1, & |i-j| = 1\ , \\ 0, & \text{otherwise}, \end{cases} \qquad i, j = 1, \ldots, n;$$

(i) prove that A^{-1} exists;

(ii) show that the eigenvalues of A are real and belong to the open interval $(0, 4)$;

(iii) provide an upper bound for $\|A\|_2$, where $\|A\|_2 := \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$;

(iv) determine, if any, the values of $\omega \in \mathbb{R}$ such that the iterative method

$$\mathbf{x}^{(i+1)} = (I - \omega A)\mathbf{x}^{(i)} + \mathbf{q}, \quad \mathbf{q} \in \mathbb{R}^n$$

is convergent for any choice of the initial vector $\mathbf{x}^{(0)}$.

7 Probability

Exercise 7.1. For $\lambda > 0$, let X_λ be an exponential Gamma$(1, \lambda)$ distributed r.v. (that is, the pdf of X_λ is $f_\lambda(x) = \lambda e^{-\lambda x} 1\!\!1_{x>0}$). In the following, $\lfloor \cdot \rfloor$ denotes the integer part function.

(i) Find the law of $\lfloor X_\lambda \rfloor$ and $X_\lambda - \lfloor X_\lambda \rfloor$ and prove that $\lfloor X_\lambda \rfloor$ and $X_\lambda - \lfloor X_\lambda \rfloor$ are independent.

(ii) Set $\lambda = n \in \mathbb{N} \setminus \{0\}$. Study the convergence (in law, in probability, in L^p and a.s.) of X_n and of $Y_n = X_n - \lfloor X_n \rfloor$ as $n \to \infty$.

(iii) Set now $\lambda = \frac{1}{n}$, $n \in \mathbb{N} \setminus \{0\}$. Does $U_n = X_{1/n}$ and/or $W_n = X_{1/n} - \lfloor X_{1/n} \rfloor$ converge in law as $n \to \infty$?

Exercise 7.2. [3] Let $X = (X_1, \ldots, X_{2d})$ be a random vector in $\mathbb{R}^{2d}$ such that $X \sim \mathrm{N}(0, C)$. We define the random vectors $\bar{X} = (\bar{X}_1, \ldots, \bar{X}_d)$ and $\hat{X} = (\hat{X}_1, \ldots, \hat{X}_d)$ in $\mathbb{R}^d$ as follows:

$$\bar{X} = (X_1, \ldots, X_d) \text{ and } \hat{X} = (X_{d+1}, \ldots, X_{2d}).$$

(i) Prove that $\bar{X}$ and $\hat{X}$ are still Gaussian distributed in $\mathbb{R}^d$ and find the associated covariance matrices. Prove moreover that $\bar{X}$ and $\hat{X}$ are independent if and only if

$$C = \begin{pmatrix} \bar{C} & O \\ O & \hat{C} \end{pmatrix} \tag{7.1}$$

where $\bar{C}, \hat{C}, O \in \mathrm{Mat}(d \times d)$ and O denotes the null matrix.

[3] In **Exercise 7.1**:

- The notation $\mathrm{Mat}(n \times n)$ stands for the set of all the $n \times n$ matrices over $\mathbb{R}$.
- For a vector $b \in \mathbb{R}^n$ and a symmetric positive semi-definite matrix $\Gamma \in \mathrm{Mat}(n \times n)$, we recall that if $X = (X_1, \ldots, X_n) \sim \mathrm{N}(b, \Gamma)$ then $b = \mathbb{E}(X)$, Γ is the covariance matrix ($\Gamma_{ij} = \mathrm{Cov}(X_i, X_j)$, $i, j = 1, \ldots, n$) and the characteristic function of X is given by
$$\varphi_X(\theta) = \exp\Big(\mathrm{i}\langle \theta, b\rangle - \frac{1}{2}\langle \Gamma\theta, \theta\rangle\Big), \quad \theta \in \mathbb{R}^n,$$
where i denotes the imaginary unit and $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product.
- A function $f : \mathbb{R} \to \mathbb{R}$ is said to be symmetric and positive semi-definite if for every $n \in \mathbb{N} \setminus \{0\}$ and for every $x_1, \ldots, x_n \in \mathbb{R}$ then the $n \times n$ matrix $(f(x_i - x_j))_{i,j=1}^n$ is symmetric and positive semi-definite.

(ii) Assume that C is of the form (7.1) and let $Y = (Y_1, \dots, Y_d)$ be the random vector in $\mathbb{R}^d$ defined by

$$Y_i = \bar{X}_i \hat{X}_i, \quad i = 1, \dots d.$$

Prove that $\mathrm{Cov}(Y_i, Y_j) = \bar{C}_{ij} \hat{C}_{ij}$, $i, j = 1, \dots, d$.

(iii) For $A, B \in \mathrm{Mat}(d \times d)$, let $A \odot B$ be defined as the following matrix product:

$$(A \odot B)_{ij} = A_{ij} B_{ij}, \quad i, j = 1, \dots, d.$$

Using (ii), prove that if A and B are both symmetric and positive semi-definite then $A \odot B$ is symmetric and positive semi-definite as well. As a consequence, prove that if $f, g : \mathbb{R} \to \mathbb{R}$ are symmetric and positive semi-definite then the product fg is still symmetric and positive semi-definite.