Report on "A Mean Field Game Approach to Optimal Transport Problems" by Gabriele Bocchi.

The purpose of this thesis is to apply techniques developed for mean field game systems of partial differential equations to optimal transport problems. Since my research has mainly been in mean field games for the past several years, it is in this context that I will comment on the significance of the results of this thesis. Mean field games are modeled by systems of PDE, consisting of a forward-in-time Fokker-Planck (or continuity) equation governing the evolution of a density coupled with a backward-in-time Hamilton-Jacobi equation for the value function of an individual optimal control problem. It was noticed right from the beginning of mean field game theory that similar systems of PDE had already appeared in dynamical optimal transport (the famous "Benamou-Brenier" characterization). Indeed, the so-called planning problem in mean field games can be considered as a generalization–or, in many cases, a regularization–of optimal transport problems. Because of this, over the past several years many authors have used variational techniques to study the solutions of mean field games and associated planning problems, as opposed to (or in addition to) more customary fixed point arguments that come from the theory of quasilinear parabolic or elliptic PDE. Given this context, I tend to see this thesis as a valuable contribution to the theory of variational mean field games, although I in no way intend to downplay its significance for optimal transport theory.

The thesis is divided into three parts. The first part deals with entropy penalized dynamic optimal transport on a Riemannian manifold. This problem can be thought of as a mean field planning problem, distinguished from a mean field game only in the sense that the terminal distribution, rather than cost, is given. From a mathematical point of view, this seems to be a significant technical contribution because, as far as I know, mean field games have rarely if ever been studied on Riemannian manifolds. In the context of optimal transport, this seems like a natural choice.

The main object of study in this first part is a system of optimality for optimal curves in the Wasserstein space. The entropy penalization introduces a $\log(m)$ term on the right-hand side of the Hamilton-Jacobi equation. It is known in the "flat case" that this term has a strong regularizing effect on the mean field game system; here similar results are proved on Riemannian manifolds. Chapter 2 contains some interesting results on displacement convexity, which has a rich history in the field of optimal transport. It has also proved useful in the context of variational mean field games; for example, Santambrogio and co-authors have used it to prove extra regularity of minimizers. The estimates in Proposition 2.3 appear to hold only for classical solutions, but based on the words just mentioned, I wonder if they could also hold for weak solutions. Chapter 3 contains very useful and highly technical estimates for smooth solutions. The key point here is that the system of optimality can be written as an elliptic PDE in space-time, an idea that goes back to the lectures of P.-L. Lions and has been used by others such as Muñoz and Porretta to prove the existence of smooth solutions for variational mean field games and the planning problem. What is impressive to me in the present context is handling these estimates on a Riemannian manifold, which adds a nontrivial layer of technical difficulty. Chapter 4 deals with solutions, both classical and weak, to the optimality system–or, equivalently, optimal curves for the transport problem. Theorem 4.4 on the existence of classical solutions is a nice application of the regularity theory established by Chapter 3. I am also quite impressed by the technical analysis in Lemma 4.6, which demonstrates a mastery of the use of truncation functions, leading to Theorem 4.10 on the well-posedness of weak solutions. Finally, Chapter 5 shows that optimal curves to the entropy-penalized transport problem converge to geodesics in the Wasserstein space. Of particular interest is the convergence of smooth solutions to the geodesic; this argument shows the power of displacement convexity estimates.

The second part of this thesis is devoted to studying displacement monotonicity in mean field games. This part is a bit different thematically from the other two part, in that, strictly speaking, it is not about applying variational methods to mean field games PDE systems. Instead, the purpose is essentially to prove uniqueness for certain classes of mean field games. Displacement monotonicity is getting more and more attention ever since the results of Gangbo et al., who proved the well-posedness of the master equation for mean field games under the displacement monotonicity assumption. However, up until now most arguments from displacement monotonicity rely on examining flows of individual agents–particles rather than densities. In this part of the thesis we find a systematic approach to "coupling curves," which are essentially solutions to a Fokker-Planck type equation that doubles the variables, whose solution is therefore a coupling between solutions of two different Fokker-Planck equations. From this systematic approach one has a new PDE method for proving uniqueness of solutions to mean field games. Essentially, if ρ_1 and ρ_2 represent two possible population densities, one takes a coupling curve $\boldsymbol{\rho}_c$ between them and finds an estimate on $\int |x-y|^2 \, \boldsymbol{\rho}_c(\mathrm{d}x, \mathrm{d}y, t)$, ultimately deriving an inequality $W_2(\rho_1(t), \rho_2(t))^2 \lesssim W_2(\rho_1(0), \rho_2(0))^2$ from which uniqueness and indeed stability is clear. This approach appears to be quite useful and should have applications to other more general models, for which the more standard Lasry-Lions monotonicity condition may not be applicable but where displacement monotonicity is reasonable. The method will surely attract PDE analysts who are less inclined to shift point of view to stochastic trajectories, as done in the work of Meszaros and Mou, for example.

The third part of the thesis consists in analyzing a mean field Schrödinger bridge problem. The classical Schrödinger bridge problem can be cast as a sort of mean field game planning problem, namely the optimal control of a Fokker-Planck equation whose optimal control is given by the vector field $V = \nabla v$ for a potential v (the Schrödinger potential) that satisfies a Hamilton-Jacobi equation. In this thesis, the classical problem is modified by assuming the presence of a nonlocal interaction potential. The optimality system in this case corresponds to a nonlocal mean field planning problem. The basic assumption is that the coupling is displacement monotone, which allows for the use of many ideas from the second part of the thesis, for instance to prove uniqueness in Chapter 14. In this regard, I would say Proposition 10.5 presents a nice technical contribution, showing that there is a natural family of examples for which the interaction potential induces a displacement monotone coupling. I was particularly interested to see how displacement monotonicity was so naturally connected to the estimates on the Hessian of the Schrödinger potential from Chapter 13. This was a very nice chapter in its own right; the doubling-of-variables approach to proving semiconvexity is quite satisfying from the PDE point of view. Finally, Chapter 15 analyzes the corresponding ergodic problem and proves the turnpike property for solutions to the mean field Schrödinger problem, and here again we see the benefit of displacement monotonicity.

Overall, I find this to be an impressive thesis. The mathematical arguments show a high level of technical prowess, ranging from advanced techniques in elliptic and parabolic PDE to the calculus of variations. In terms of originality, each part of the thesis contains features that are clearly novel. In the first part, the adaptation of planning problem techniques to a Riemannian manifold is a significant technical feat. In the third part, there is novelty found for example in the nonlocal interaction potential for the Schrödinger problem. The second part, although it is the shortest, contains what in my opinion is the most innovative mathematical development for mean field games. Having this systematic method of studying the transport of Wasserstein distances between population densities should prove to be an incredibly useful tool for many of us studying mean field games from a PDE point of view. Indeed, the thesis itself is in some sense a proof of just how useful it can be to understand displacement monotonicity, even though up until now most results on mean field games have relied instead on the Lasry-Lions monotonicity condition.

In short, I have no hesitation about calling this thesis acceptable for a Ph.D.