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Singular Intersections in Families of Split Semiabelian Varieties

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Introduction

In the beginning of the 20th century, starting from work of Poincaré, Mordell, Weil and Siegel, a new approach was introduced for the study of Diophantine equations, i.e. polynomial equations with integer coefficients of which we are concerned about the integral or rational solutions. Motivated by the (at the time) recent developments in algebraic geometry, it was noticed that Diophantine equations, like any other polynomial equation, define algebraic varieties, with integral and rational solutions corresponding to integral and rational points on these varieties. Finding a link between the geometrical structure of these varieties and the behavior of rational and integral solutions of the equations is one of the main problems of *Diophantine geometry*.

This link is quite clear and well understood in dimension 1, i.e. for algebraic curves, where the number of rational points is controlled by the genus. More precisely, if C is a smooth curve of genus g defined over $\mathbb{Q}$, then:

- if $g = 0$, then either C has no rational point, or, by work of Poincaré [Poi01], it is isomorphic over $\mathbb{Q}$ to $\mathbb{P}^1$, and hence it has infinitely many rational points;
- if $g = 1$, then again C could have no rational points; in case it has, then C is an elliptic curve, and by work of Mordell [Mor22] the set of rational points forms a finitely generated abelian group;
- if $g \geq 2$, it was originally conjectured by Mordell [Mor22], and later proven by Faltings [Fal83], that C can only have at most finitely many rational points.

All of these results actually hold for K-rational points and curves defined over K, with K any number field.

The original proof of Mordell Conjecture given by Faltings followed a strategy that involved reducing the problem to the Shafarevich conjecture via earlier work of Parshin. Other proofs have since been formulated using different techniques: Vojta [Voj91] gave a proof using Diophantine approximation, later simplified by Faltings [Fal91] and Bombieri [Bom90], and more recently Lawrence and Venkatesh [LV20] obtained a proof relying on p-adic Hodge theory. A uniform version of the same statement has also been proven by Dimitrov, Gao and Habegger [DGH21].

Another approach to proving Mordell Conjecture, carrying ideas already present in the works of Poincaré [Poi01] and then developed by Siegel [Sie29] and Chabauty [Cha41], begins by observing that any curve C defined over a number field K and of genus $g \geq 2$ that has a K-rational point P can be embedded into its Jacobian variety J_C via the Abel-Jacobi map relative to P. Since P is K-rational, the corresponding Abel-Jacobi map is defined over K, hence it preserves K-rationality. Identifying C with the image of this embedding yields

$$C(K) = C \cap J_C(K) \subseteq J_C,$$

where $C(K)$ is the set of K-rational points of C. By the Mordell-Weil Theorem, the group of K-rational points $J_C(K)$ is finitely generated. Moreover, the dimension of J_C as a variety is equal to the genus g of C; in particular, C is properly contained in J_C as by assumption $g \geq 2$. Thus, the set of K-rational points of C can be viewed as the intersection of a subvariety of an abelian variety with a finitely generated subgroup of the ambient variety; we expect such an intersection to be finite in general. This perspective naturally invites generalizations, which motivated much of the development of Diophantine geometry. One may consider to replace $J_C(K)$ with other subgroups, such as other finitely generated subgroups, the set of torsion points, or finite rank subgroups; replace C with higher dimensional subvarieties; and consider the problem in more general ambient spaces, such as abelian varieties that are not Jacobians, algebraic tori or semiabelian varieties. This led to the formulation of many classical conjectures, such as the Manin-Mumford Conjecture [Lan65] and the Mordell-Lang Conjecture (ibidem), which we will be reviewing in Chapter 1.

This interpretation of finiteness questions in terms of intersections between subvarieties and subsets of arithmetic interest of group varieties, and the success that it had for example in the proof of the Manin-Mumford Conjecture by Raynaud [Ray83a, Ray83b], led to a broader framework, usually called the theory of *unlikely intersections*. The key heuristic is that, whenever we have two (algebraic) varieties V and W of dimension r and s respectively in a space of dimension n such that $r + s < n$, then we expect that the intersection $V \cap W$ is empty. When this is not the case, we say that we have an *"unlikely intersection"*. Fixing V and letting W vary in a suitable countable family satisfying this dimensional condition, we cannot expect that all the intersections will be empty, but we still expect the union of such intersections not to be dense in V, unless some geometric relationship exists between V and the family of the Ws.

The main conjecture in this framework is the *Zilber-Pink* Conjecture, independently formulated in different contexts by Bombieri, Masser and Zannier [BMZ99, BMZ07], Zilber [Zil02], and Pink [Pin05b]. Suppose that the ambient space S is a Shimura variety; such varieties naturally come with a countable family of distinguished subvarieties, known as *special subvarieties*. If a subvariety V of S is not contained in any of these special subvarieties, then the Zilber-Pink Conjecture predicts that its intersection with the union of all the special subvarieties of codimension bigger or equal to $\dim V + 1$ is not Zariski dense in V. While the conjecture remains open, much progress has been made towards it in the last 30 years, which will be reviewed in Chapter 1.

More recently, a variation of these problems has appeared, where one also takes into account the multiplicity of intersection. Indeed, whenever two (algebraic) varieties V and W of dimension r and s respectively lie in a space of dimension n such that $r + s = n$, then not only do we expect that the irreducible components of the intersection $V \cap W$ are zero dimensional, but also that the intersection at those points happens in a transversal way, i.e.

that the tangent spaces of V and W, seen as subspaces of the tangent space of the ambient variety, have minimal intersection. So, by a similar heuristic to unlikely intersections, if we let W vary over a suitable countable family, we expect that the set of points in which the intersection happens in a non-transverse, or singular, way is not Zariski dense in V. In particular, in analogy with the Zilber-Pink Conjecture, we are concerned about the case in which the ambient space is a Shimura variety and the family of Ws is that of the special subvarieties of codimension bigger or equal to $\dim V$.

This kind of problem has been considered in the literature in some special cases. Marché and Maurin [MM23], motivated by a question on character varieties, proved the non-Zariski denseness for the points of singular intersection between a curve in $\mathbb{G}_{\mathrm{m}}^2$ not contained in any proper algebraic subgroup and algebraic subgroups of dimension 1. The analogous result for curves in a 1-parameter, non-isotrivial family of elliptic curves has also been proven, independently by Corvaja, Demeio, Masser and Zannier [CDMZ21], and Ulmer and Urzùa [UU20, UU21]. The first ones also extend the result to powers of non-isotrivial elliptic curves, with very mild condition on the curve. On the other hand, the result by Ulmer and Urzùa also holds in the isotrivial case, and gives a completely explicit bound for the number of points involved. Outside algebraic groups, similar questions have also been studied in the modular setting, where theorems in the same direction have been proven by Spence [Spe19], Aslanyan [Asl22] and Ballini [Bal24, Thm. 6.0.2].

In this thesis, we investigate some generalizations of these results in the context of singular intersections, in particular for algebraic groups and families. After reviewing the history of the conjectures and the main results in Chapter 1, we focus at first on the case of split semiabelian varieties in Chapter 3, including powers of elliptic curves and $\mathbb{G}_{\mathrm{m}}^n$, and then in Chapter 4 to the case of 1-parameter families of abelian varieties. The proofs of these two cases use substantially different techniques. In the case of families, the statement is first reduced from the setting of abelian schemes to powers of non-isotrivial elliptic schemes. Here the required finiteness result is obtained by using a variation of the Pila-Zannier strategy [PZ08], using a height bound due to Silverman [Sil83]. On the other hand, in the semiabelian setting, we have a similar height bound only under the stricter hypothesis that the curve is not contained into any translate of a subgroup. What we show instead is that, under this more restrictive hypothesis, the tangency condition implies a degree bound *a priori* (i.e. without using the height bound) for a broader set of intersections, namely the singular intersections with all cosets. This more general degree bound, in turn, allows us to deduce a height bound holding without further assumption on the curve, other than it not being contained in a proper subgroup.

Similar to the results by Ulmer and Urzùa [UU20, UU21], we also show an application of our results to divisibility sequences, in particular to elliptic ones.

Structure of the thesis

We conclude this introduction with a short overview of the content of each chapter.

In Chapter 1 we provide a historical review of the process that lead to the Zilber-Pink Conjecture in the context of commutative algebraic groups, and the state of the art in the process of proving it, focusing on the case of abelian varieties, algebraic tori and families of abelian varieties. In the same chapter, we also introduce in more details the problem of singular intersections and the known results in that direction.

Chapter 2 is dedicated to introducing the main tools used in the proofs of unlikely (and singular) intersections results, in particular the theory of heights, o-minimality and point counting, and how these tools are combined in the Pila-Zannier strategy, introduced for the first time in [PZ08] and since become the main proof technique in the field.

In Chapter 3 we study the singular intersections problem for curves defined over a number field K in split semiabelian varieties. Theorem 3.1.1, the main result of this chapter, concerns the finiteness of points of singular intersection between a curve defined over a number field K and the algebraic subgroups of codimension 1. In particular, the same finiteness result holds for curves in $\mathbb{G}_{\mathrm{m}}^n$, generalizing the work of Marché and Maurin [MM23], and curves in powers of elliptic curves. The material of this chapter is part of a work in progress with Ballini and Capuano.

In Chapter 4 we continue the study of the singular intersections problem for a curve defined over a number field K not contained in any proper subgroup scheme of a 1-parameter family of abelian varieties. The main result presented here is Theorem 4.1.3, which proves the finiteness (hence non-Zariski density) of the points of singular intersection between the curve and 1-codimensional subgroup schemes of the family. This generalizes the result of Corvaja, Demeio, Masser and Zannier [CDMZ21], and Ulmer and Urzùa [UU20, UU21]. Furthermore, we show how this implies a result about reduced elements for shifted elliptic divisibility sequences, extending a similar result in the non-shifted case due to Ulmer and Urzùa [UU20]. This chapter is based on the preprint [Ott25], currently submitted for publication.

1 - Generalities on Unlikely Intersections

In this chapter we want to give an introduction, both historical and mathematical, to the topic of unlikely intersections, that has been an object of attention in number theory, especially in the last thirty years. In the following sections, we will provide an overview of some results and methods used in general in this context, including some recent results of intersections with tangency conditions, and we will end by presenting the results obtained in this thesis. some fundamental references on the topic of unlikely intersections are the books [Zan12] and [Pil22], which provide a comprehensive treatment of the main results, conjectures, and techniques. Some very recent survey articles have also been written, detailing the newest development in the topic, in particular [Cap23], for the results in the context of semiabelian varieties, and [Daw25] for the results in the Shimura setting.

The expression *"unlikely intersections"* refers, in general, to the study of intersections between varieties that we expect to be "improbable". If we have, in fact, two (algebraic) varieties V and W of dimension $r, s \geq 0$ in a space of dimension n, then we expect that every component of the intersection $V \cap W$ will have dimension $r + s - n$. In particular, unless there are some particular (geometrical) conditions between the two varieties, we expect that the intersection $V \cap W$ is empty whenever $r + s < n$. When that is not the case, we say that we are in the presence of an "unlikely intersection".

The type of problem usually studied is the following one. Suppose that we have a (fixed) subvariety V in an ambient space X, of dimensions r and n respectively, and a countable family $\mathcal{W}$ of subvarieties of X, all of dimension strictly smaller that $n-r$; this family usually comes from some extra structure attached to the ambient space X, and is chosen independently of V. By the above discussion, if we consider $V \cap W$ with W varying in $\mathcal{W}$ we expect that it will be nonempty only for a *small* subset of $\mathcal{W}$, unless there are some special relations between V and the family $\mathcal{W}$. The question is to clarify *what* are these relations, usually in term of the extra structure on X, and to study the set of intersections $\bigcup_{W \in \mathcal{W}}(V \cap W)$ as a subset of V. More specifically, one asks the following questions: is this set "*small*" in V? When is it finite? In this case, is it possible to actually compute it?

During the last years it has been pointed out that a lot of known problems

involving arithmetic and geometry can be stated using the previous language: usually, the varieties in the family $\mathcal{W}$ are described by equations of growing degrees depending on discrete parameters. In this Chapter we are going to describe some of the main problems studied in this context.

1.1 The beginning of Diophantine geometry

The term "Diophantine geometry" was minted by Lang as a title to one of his book [Lan62], in which he describes some results about the solvability of Diophantine equations from the viewpoint and through the techniques of algebraic geometry.

The use of geometry in a broader sense in the study of the solution of Diophantine equations dates back at least to Newton, and his "method of the secant", to construct from two solutions of a cubic equation a third one. This technique is what then led to the study of the group law on elliptic curves. Even before Newton, a similar process was derived, through purely algebraic means, by Bachet, obtaining formulas in some special cases by what could be described as a "method of the tangent".

The idea that techniques from (classical) algebraic geometry could be useful in the study of rational solutions of equations is due to Hilbert and Hurwitz [HH90], who remarked that searching for rational points on algebraic curves is a problem that is invariant under a birational transformation with rational coefficients. Hence, the correct way to classify and discuss the solvability of a Diophantine equation in two variables is not by degree, as was commonly done before, but by the *genus* of the plane curve described by it.

The same observation was later also made independently by Poincaré [Poi01]. In the same paper he also noted that the tangent and secant constructions give to the set of rational points of genus 1 equations a structure, that in modern terms we would call a module or abelian group. He defines the rank of such a module, and asks for what kind of values (finite or infinite) it can assume. Moreover, in the last section of [Poi01], he also observe that the same kind of construction can be made on curves of higher genus by replacing rational points with rational divisors of degree equal to the genus, obtaining a structure related to the Jacobian of the curve.

1.2 Mordell, Weil and Siegel

The problem by Poincaré was first considered by Mordell. In a paper published in 1922 [Mor22] he proved that the rank of the $\mathbb{Z}$-module of rational solution of a genus 1 equation is always finite. His result, and the techniques that he uses, lead him to pose a wider conjecture, concerning a qualitative answer to the solution of all Diophantine equations in two variables. According to it, whenever the equation defines a smooth curve of genus bigger or equal to 2, the number of rational solutions has to be at most finite. As one can expect,

such conjecture attracted much interest, and much of the later development of Diophantine geometry was motivated by attempts of proving Mordell Conjecture or generalizing it. However it was not until 1983 that Faltings gave a proof of it [Fal83], using techniques and ideas of different kind compared to unlikely intersections. An independent proof, using diophantine approximation, was later given by Vojta [Voj91], and later simplified by Faltings [Fal91] and Bombieri [Bom90].

In 1928, in his PhD thesis [Wei28], Weil extended the result of Mordell to the module of degree g divisors defined over any number field of a curve of genus g. This is what is now known as the Mordell-Weil Theorem. For his proof, Weil introduced the basis for the theory of height, later developed by Siegel, Northcott and Weil; this construction still plays a crucial role in the proofs of many results in Diophantine geometry, including the ones presented in this thesis. Moreover, he realized that the study of rational divisors, in particular relating to the problem of them being effective or not, could be helpful in proving Mordell Conjecture.

Less then one year later, another landmark work was published by Siegel in [Sie29] (see also [Zan14b]). Here, he proved that, given any affine curve of genus bigger or equal to one, the number of integral points on it is finite. This result provided a very broad qualitative result for integral solutions of Diophantine equations, and gave further evidence for Mordell Conjecture. The proof by Siegel works as follows: first, he embeds the curve considered into its Jacobian. The question then becomes to study the intersection of the curve with the integral points of the Jacobian. These are a subset of the rational points, which, by the above mentioned result by Weil, form a finitely generated group. Using some transcendence properties of the abelian functions, he shows that having an infinite number of integral points would give rise to an algebraic point with infinitely many "good" rational approximation, contrasting a variation of Thue's theorem.

This proof introduced many new ideas in the study of Diophantine problems, such as using the theory of abelian varieties by embedding a curve into its Jacobian, a further development of the theory of heights, and a renewed emphasis on the use of Diophantine approximation of algebraic numbers.

These three theorem mentioned in this section are what Lang later described as "the prototypes of the Diophantine results and methods" in his 1962 book [Lan62].

1.3 Lang, and the first unlikely intersections problems

In 1960, while studying the work of Siegel and Chabauty [Cha38, Cha41], Lang [Lan60] was led to formulate the following conjecture, generalizing an earlier conjecture of Chabauty [Cha38]. He also remarked that such conjecture would imply Mordell Conjecture via the Mordell-Weil Theorem.

Conjecture 1.3.1. *Let G be torus (i.e. $\mathbb{G}_{\mathrm{m}}^n$) (resp. an abelian variety) in characteristic 0. Let V be a subvariety of G, having an infinite intersection with a finitely generated subgroup Γ of G. Then V contains a finite number of translations of group subvarieties of G which contain all but a finite number of points of $V \cap \Gamma$.*

Later, in [Lan65], he reports on a conjecture formulated independently by Manin and Mumford.

Conjecture 1.3.2 (Manin-Mumford, original formulation)**.** *Let $C \subseteq J$ be a curve in its Jacobian, and assume that C contains infinitely many points of finite order*[1]*. Then, the curve has genus 1.*

These two conjectures where then combined by Lang into what is now known as the Mordell-Lang Conjecture.

Moreover, we can see how these two statements appear as "unlikely intersections" statement: we are intersecting a variety, in the second case a curve, with a countable set of "special" subvarieties of the ambient space, a finitely generated subgroup in the first case and the torsion points in the second. The conjectures amount to saying that having an infinite (hence Zariski-dense) intersection implies that something geometrically special happens: either the intersections concentrate on finitely many translations of group subvarieties, or the variety is the whole space (as is the case for a curve of genus 1 in its Jacobian).

As a special case of his conjecture, Lang posed the following problem: let $C : f(X, Y) = 0$ be an irreducible plane curve passing through $(1, 1)$, and assume that $f(x, y) = 0$ for infinitely many couples (x, y) of roots of unity. What can we say about the polynomial $f(X, Y)$?

According to the conjectures above, the curve C must be a subgroup variety of $\mathbb{G}_{\mathrm{m}}^2$, so $f(X, Y)$ is necessarily either of the form $X^m - Y^n$ or $X^m Y^n - 1$. Such a result was proven independently by Ihara, Serre and Tate, and their proofs are reported by Lang in [Lan65].

Both the Manin-Mumford Conjecture and the more general Mordell-Lang Conjecture have been proven between the 80's and 90's. In proving them, people realized that more general results hold. Such results are also known, by analogy, as Manin-Mumford and Mordell-Lang Conjecture. Before stating them, we introduce some notation.

Let G be a torus (i.e. $\mathbb{G}_{\mathrm{m}}^n$), or an abelian variety, or more in general an algebraic group, defined over a field of characteristic zero. Let Γ be a subgroup of G (usually, not algebraic). We call the *division group* of Γ the group

$$\Gamma_{div} := \{\gamma \in G \mid \exists k \in \mathbb{N} \text{ such that } \gamma^k \in \Gamma\}.$$

We say that an irreducible subvariety W of G is *special* if W is an irreducible component of an algebraic subgroup of G, and we say that it is Γ_{div}*-special* if it

[1] That is, torsion points

is an irreducible component of the translate of a subgroup of G by an element of Γ_{div}.

Conjecture 1.3.3 (Manin-Mumford, modern formulation 1)**.** *Let G be a torus (i.e. $\mathbb{G}_{\mathrm{m}}^n$), an abelian variety or a semiabelian variety defined over $\mathbb{C}$, and let $V \subseteq G$ be a subvariety. Then, V contains only finitely many maximal special subvarieties.*

An equivalent formulation, that highlights the similarities with the original formulation and with the unlikely intersections ideas, is the following.

Conjecture 1.3.4 (Manin-Mumford, modern formulation 2)**.** *Let G be a torus (i.e. $\mathbb{G}_{\mathrm{m}}^n$), an abelian variety or a semiabelian variety defined over $\mathbb{C}$, and let $V \subseteq G$ be a subvariety not contained in any proper algebraic subgroup of G. Then, the set of points of finite order[2] contained in V is not Zariski dense in V.*

Indeed, torsion points are special subvarieties. So, assuming Conjecture 1.3.3, there is a closed set containing all of the torsion points of V, namely the union of all special subvarieties contained in V. Such a set is strictly contained in V, unless V is itself special, leading to Conjecture 1.3.4.

Vice versa, torsion points are dense in all of the special subvarieties, so the closure of the set of points of finite order contained in V contains all of the algebraic subvarieties of V, and, assuming Conjecture 1.3.4, all the irreducible component of it are special subvarieties, leading to Conjecture 1.3.3.

Conjecture 1.3.5 (Mordell-Lang)**.** *Let G be a torus (i.e. $\mathbb{G}_{\mathrm{m}}^n$), an abelian variety or a semiabelian variety defined over $\mathbb{C}$, let $V \subseteq G$ be a subvariety, and let Γ be a finitely generated subgroup of G. Then, V contains only finitely many maximal Γ_{div}-special subvarieties.*

In all of these conjectures the fundamental case is when the (semi)abelian varieties are defined over $\overline{\mathbb{Q}}$ (or, equivalently, over a number field). This case implies, via specialization arguments, the general case of characteristic zero. In positive characteristic all these results are in general false: over $\overline{\mathbb{F}}_p$ all the points of an abelian variety are torsion, but special subvarieties keep being the irreducible components of algebraic subgroups.

In the case of powers of the multiplicative group, $\mathbb{G}_{\mathrm{m}}^n$, the Manin-Mumford Conjecture has been proven by Laurent [Lau84], based on previous work by Liardet [Lia75]. Actually, Laurent's paper proves directly the Mordell-Lang Conjecture. An alternative proof of the Manin-Mumford Conjecture has been later given by Sarnak and Adams [SAR94].

In the context of abelian varieties, Raynaud [Ray83a] first proved the original Manin-Mumford Conjecture, for curves in an abelian variety, and then extended the result [Ray83b] to arbitrary subvarieties. Multiple alternative proofs have been given since, by Hindry [Hin88], whose proof is in the more general semiabelian setting, Hrushovski [Hru01], using model theory, and Szpiro,

[2]That is, torsion points

Ullmo and Zhang ([SUZ97, Ull98, Zha98]). Szpiro, Ullmo and Zhang actually proved a stronger statement, conjectured by Bogomolov, where the torsion points are replaced by all points of sufficiently small Néron-Tate height; these include the torsion points, as these are exactly the ones of height zero. Especially interesting is the proof given by Pila and Zannier [PZ08], which we will explore in more details in Subsection 2.3 as the techniques that they introduce have since become the main strategy in the field.

The Mordell-Lang Conjecture for abelian varieties is due to Faltings [Fal91] and Vojta [Voj96], and extended to the case of semiabelian variety by McQuillan [McQ95].

1.4 Unlikely intersections and the Zilber-Pink Conjecture

In both the Manin-Mumford and Mordell-Lang Conjectures, we are intersecting the variety V with a countable union of zero dimensional subvarieties. Since $n - \dim V > 0$ by assumption, we are dealing with unlikely intersections.

This problem can be generalized to study the intersection between V and the union of all the algebraic subgroups of a fixed dimension d; we still remain in the case of unlikely intersections if $d < n - \dim V$.

For $d > 0$, this type of problem was firstly studied by Schinzel in the '80s [Sch00], but with different motivations: he obtained general results for a curve V in a space of dimension $n \leq 3$. One more general positive answer to this problem was settled in the paper of Bombieri, Masser and Zannier [BMZ99]; in this work they proved two results in the case of an irreducible curve V in $\mathbb{G}_{\mathrm{m}}^n$ defined over $\overline{\mathbb{Q}}$. The general condition required is that the curve V is not contained in any translate of a proper subgroup, meaning that the coordinate functions $x_1, \ldots, x_n$ are multiplicatively independent modulo constants.

Theorem 1.4.1 (Bombieri-Masser-Zannier, 1999)**.** *If V is an irreducible curve defined over $\overline{\mathbb{Q}}$ and not contained in any translate of a proper algebraic subgroup of $G = \mathbb{G}_{\mathrm{m}}^n$, then*

1. *the Weil Height on the set*

$$V^{[1]} := \bigcup_{\dim H \leq n-1} (V \cap H)$$

is uniformly bounded, where H varies over the proper algebraic subgroups of G;

2. *the set*

$$V^{[2]} = \bigcup_{\dim H \leq n-2} (V \cap H)$$

is finite.

As shown in the same article, the condition of V not being in a translate of an algebraic subgroup is necessary for the first statement. As for the second one, the authors needed it in order to apply part 1, but they remarked that the same statement may hold if one asks for V not to be contained just in a proper subgroup of G. Such a statement has later been proven by Maurin [Mau08], using some methods introduced by Rémond and inequalities of Vojta's type in the theory of heights; his argument works only over $\overline{\mathbb{Q}}$. A deduction of an extension of his theorem to general complex curves appeared in [BMZ08b].

Again in [BMZ99], Bombieri, Masser and Zannier also conjectured that the same finiteness result of Theorem 1.4.1 could hold in more general settings, for higher dimensional varieties V and even algebraic groups or families of abelian varieties.

At the same time, Zilber [Zil02], studying the formal theory of exponentiation from the point of view of model theory and Schanuel's conjecture, stated a conjecture generalizing the second part of Theorem 1.4.1. Before presenting it, we introduce some further definitions.

Definition 1.4.2. Let G be $\mathbb{G}_{\mathrm{m}}^n$, an abelian variety or a semiabelian variety, and let V be a subvariety of G. We say that $W \subseteq G$ is an *algebraic coset* if it is a translate of an algebraic subgroup of G by any element of G.

Definition 1.4.3. Let G be as above and let $W \subseteq G$ be any subvariety. Let S be an irreducible component of $W \cap V$. We say that S is an *atypical component* of the intersection if

$$\dim S > \dim V + \dim W - \dim G,$$

that is, the dimension of S is *larger* than the expected dimension of intersection between V and W. We say that a subvariety V' of V is *atypical (for V in G)* if it is an atypical component of an intersection between V and an algebraic subgroup of G.

For example, if V is a proper subvariety of G, then all the (irreducible components of) algebraic subgroups contained in V are atypical subvarieties of V.

Conjecture 1.4.4 (Conjecture on Intersections with Tori)**.** *For any algebraic variety $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ defined over $\mathbb{Q}$ there exists a finite collection*

$$\tau(V) = \{T_1, \dots, T_k\}$$

of proper algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n$ such that for any algebraic subgroup T of $\mathbb{G}_{\mathrm{m}}^n$ and any atypical component S of $V \cap T$ we have

$$S \subseteq T_i \quad \text{for some } T_i \in \tau(V).$$

In other words, V contains only finitely many maximal atypical subvarieties.

Such conjecture implies both the Manin-Mumford and the Mordell-Lang Conjectures for tori.

In the same paper, Zilber also formulates a similar conjecture for semiabelian varieties, with a slightly weaker content.

Conjecture 1.4.5 (Conjecture on Intersections with Subgroups)**.** *Let G be an abelian or semiabelian variety and let $V \subseteq G$ be an algebraic subvariety. there exists a finite collection*

$$\tau(V) = \{W_1, \dots, W_k\}$$

of proper algebraic cosets *of G such that for any algebraic subgroup H of G and any atypical component S of $V \cap H$ we have*

$$S \subseteq W_i \quad \text{for some } W_i \in \tau(V).$$

Independently of Bombieri, Masser and Zannier, and Zilber, also Pink [Pin05b] formulated a conjecture of similar content, in the much more general context of mixed Shimura variety. These are complex manifolds that arise as quotients, and carry an extra structure. Part of this structure is a countable set $\mathcal{S}$ of distinguished subvarieties, called *special subvarieties*, such that the whole variety is special and for every two special subvarieties S, S' every irreducible component of $S \cap S'$ is also special. A general description of Shimura varieties and their special subvarieties is quite complicated in general, and beyond the scope of this work. A detailed account can be found in [Del71].

Special subvarieties play a similar role of algebraic subgroups for tori and for abelian varieties. Indeed, it turns out that both $\mathbb{G}_{\mathrm{m}}^n$ and abelian varieties have a structure as Shimura varieties, where the special subvarieties are exactly the irreducible components of subgroups.

While trying to generalize the André-Oort Conjecture, an analogue of the Manin-Mumford Conjecture for Shimura varieties, Pink formulated the following.

Conjecture 1.4.6. *Consider a mixed Shimura variety S over $\mathbb{C}$ and an irreducible closed subvariety V. Let S_0 be the smallest special subvariety containing V. Then the intersection of V with the union of all special subvarieties of S of dimension $< \dim S_0 - \dim V$ is not Zariski dense in V.*

Soon after, also Bombieri, Masser and Zannier [BMZ07] formalized their ideas from [BMZ99] into two conjectures, and also introducing some terminology. We will suppose throughout that G is either $\mathbb{G}_{\mathrm{m}}^n$, an abelian variety or a semiabelian variety.

Definition 1.4.7. Let V be an irreducible subvariety of G. An irreducible subvariety V' of V is called *anomalous* if it has positive dimension and lies in an algebraic coset W in G satisfying

$$\dim W \leq \dim G - \dim V + \dim V' - 1$$

It is called *torsion-anomalous* if the coset in question is a torsion coset, i.e. the translate of an algebraic subgroup by a torsion point.

Definition 1.4.8. We denote by V^{oa} what remains of V after removing all the anomalous subvarieties, and by V^{ta} what remains after removing all the torsion-anomalous subvarieties.

For example, if V is an irreducible curve, then its only positive dimensional subvariety is V itself, so V^{oa} and V^{ta} are either empty or the whole curve. In particular, V^{ta} is empty if and only if V is contained in a proper algebraic subgroup, while V^{oa} is empty if and only if V is contained in any translate of a proper algebraic subgroup.

In general, it clearly holds that $V^{oa} \subseteq V^{ta} \subseteq V$. The set V^{oa} is better understood. In [BMZ07] Bombieri, Masser and Zannier proved that it is an open subset of V. However, while V^{ta} is empty only when V is contained into a proper algebraic subgroup of G, the set V^{oa} is empty more often: for example, that is the case for curves that are contained into a proper algebraic coset but not into a proper algebraic subgroup. This motivated Bombieri, Masser and Zannier to formulate the following.

Conjecture 1.4.9 (Torsion Openness Conjecture)**.** *Let V be an irreducible variety in G defined over $\mathbb{C}$. Then V^{ta} is Zariski-open in V.*

Conjecture 1.4.10 (Torsion Finiteness Conjecture)**.** *Let V be an irreducible variety in G defined over $\mathbb{C}$ and of dimension r. Then*

$$\bigcup_{\substack{H<G \\ \dim H<\dim G-r}} V^{ta} \cap H$$

is a finite set.

As remarked in [BMZ07] for $G = \mathbb{G}_{\mathrm{m}}^n$, the Torsion Openness Conjecture implies the Torsion Finiteness Conjecture. More specifically, the Torsion Finiteness Conjecture for a variety V is implied by the Torsion Openness Conjecture for both V and $V \times \mathbb{G}_{\mathrm{m}}$. Indeed, if we denote by

$$Z := \bigcup_{\substack{H<G \\ \dim H<\dim G-r}} V^{ta} \cap H,$$

then $(V \times \mathbb{G}_{\mathrm{m}})^{ta} = (V^{ta} \setminus Z) \times \mathbb{G}_{\mathrm{m}}$. By the two instances of Torsion Openness Conjecture, we deduce that $Z \times \mathbb{G}_{\mathrm{m}}$, and hence Z, is a closed set. Since by construction Z is an at most countable set, it follows that it must be finite. The same reasoning may be done also in the context of abelian (and possibly semiabelian) varieties, by taking the product with an elliptic curve.

Bobieri, Masser and Zannier also showed that these conjectures together imply the Conjecture on Intersections with Tori (Conjecture 1.4.4), and also Conjecture 1.4.6, by Pink, in the appropriate cases.

As is apparent, while all this conjectures point in the same direction, they all have slight differences in the formulation. Following the book of Pila

[Pil22] we take "the" Zilber-Pink Conjecture to be the strongest possible form, the analogous of the Conjecture on Intersection with Tori and the Torsion Finiteness Conjecture in the setting of mixed Shimura varieties.

Conjecture 1.4.11 (Zilber-Pink Conjecture)**.** *Let S be a Shimura variety defined over $\mathbb{C}$, and let $V \subseteq S$ be an irreducible subvariety of S. Then, V contains only finitely many many maximal atypical subvarieties.*

This statement is sometimes referred to as the "strong version", to distinguish it from Conjecture 1.4.6, which would be the weak version of the Zilber-Pink Conjecture. However, it has been proven by Barroero and Dill [BD25, Theorem 12.4] that Conjecture 1.4.6 for *all* Shimura varieties implies Conjecture 1.4.11.

1.5 Some known cases

Having outlined the general formulation of the Zilber–Pink Conjecture, we now review some specific instances in particular settings in which it has been proven, focusing on the semiabelian cases.

1.5.1 The hypersurface case

The first instances proven of the Zilber-Pink Conjecture were those in which the variety V is a hypersurface. Here the statement is equivalent to the Manin-Mumford Conjecture for semiabelian varieties and the André-Oort Conjecture for more general Shimura varieties.

As we have already seen, the Manin-Mumford Conjecture has been proven, by Laurent [Lau84], Raynaud [Ray83b] and Hindry [Hin88].

The more general André-Oort Conjecture has also been proven. At first, Klinger, Ullmo and Yafaev [KY14, UY14] gave a proof conditional on the Generalized Riemann Hypothesis, using ideas by Edixhoven [Edi05]. It was later proved unconditionally for $\mathbb{C}^n$ by Pila [Pil11] and for $\mathbb{A}_g$ by Tsimerman [Tsi18]. It was finally proved in full generality in 2022 by Pila, Shankar and Tsimerman [PST$^+$22], building on work of Binyamini, Schmidt and Yafaev [BSY22] and Gao [Gao16].

1.5.2 The curve case

The other mostly known case is that of curves, i.e., $\dim V = 1$.

When the ambient space is $\mathbb{G}_{\mathrm{m}}^n$, this was proven by Bombieri, Masser and Zannier [BMZ99], with the additional condition that V is not contained in any translate of a subgroup (see Theorem 1.4.1 above). An alternative proof under this hypothesis, using the Pila-Zannier Method (see Subsection 2.3) was given in [CMPZ16]. This condition was later removed by Maurin [Mau08] for curves defined over $\overline{\mathbb{Q}}$, and the result was extended to curves defined over $\mathbb{C}$ by Bombieri, Masser and Zannier [BMZ08b]. Additionally, in collaboration with

Habegger, they provided an alternative proof in [BHMZ10] which is effective when combined with the effective height bound proven in [Hab17].

The analogue of Theorem 1.4.1 for powers of elliptic curves is due to Viada [Via03].

Theorem 1.5.1 (Viada, 2003)**.** *Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$ and $n \in \mathbb{N}$. If $V \subseteq E^n$ is an irreducible curve defined over $\overline{\mathbb{Q}}$ and not contained in any translate of a proper algebraic subgroup of E^n, then*

1. *the Weil Height on the set*

$$V^{[1]} := \bigcup_{\substack{H < E^n \\ \dim H \leq n-1}} (V \cap H)$$

   is bounded;

2. *if we further assume that E has complex multiplication, then the set*

$$X^{[2]} = \bigcup_{\dim H \leq n-2} (X \cap H)$$

   is finite.

This restriction on V in the second point was later removed in joint work with Rémond [RV03]. Ratazzi [Rat08] extended the result to the case of curves in A where A is isogenous to a power of a simple abelian variety with complex multiplication. Carrizosa's lower bounds for the Néron–Tate height [Car08, Car09], combined with Rémond's upper bounds [Rém07], led to a proof of the conjecture for all abelian varieties with complex multiplication. The case of powers of arbitrary elliptic curves was treated by Galateau and Viada [Gal10, Via08].

Finally, the Zilber-Pink Conjecture for curves defined over $\overline{\mathbb{Q}}$ in arbitrary abelian varieties was established by Habegger and Pila [HP16], using a height bound due to Habegger [Hab09]. The condition of the curve being defined over $\overline{\mathbb{Q}}$ was more recently relaxed to $\mathbb{C}$ by Barroero and Dill [BD22].

These results were extended to semiabelian varieties by Barroero, Kühne and Schmidt [BKS23] for curves defined over $\overline{\mathbb{Q}}$.

It is worth mentioning that for curves in more general mixed Shimura varieties, the Zilber-Pink Conjecture is completely open, save for some very special cases, like *asymmetric* curves in $Y(1)^n$ [HP12] or curves with prescribed intersections with the boundary of $Y(1)^n$ [DO25, Pap24, DOP25]. A more comprehensive list of the known results can be found in a recent survey paper by Daw [Daw25].

1.5.3 The codimension 2 case

Beyond the cases of curves and hypersurfaces, significant progress has also been made in studying subvarieties of codimension 2, at least in tori and abelian varieties.

In the case of tori, Bombieri, Masser and Zannier [BMZ07] proved the following result.

Theorem 1.5.2. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible subvariety of dimension $n-2$, defined over $\overline{\mathbb{Q}}$ and not contained in any proper algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n$. Then,*

$$V \cap (\mathbb{G}_{\mathrm{m}})^{[n-1]} = \bigcup_{\substack{H < \mathbb{G}_{\mathrm{m}}^n, \\ \dim H \leq n-1}} V \cap H$$

is not Zariski dense in V.

In the abelian setting, for codimension 2 subvarieties defined over $\overline{\mathbb{Q}}$, the Zilber-Pink Conjecture has been proven by Checcoli, Veneziano and Viada [CVV14] for subvarieties in powers of elliptic curves with complex multiplication; by Hubschmid and Viada [HV19] in the non-CM case; by Checcoli and Viada [CV14] for arbitrary products of CM elliptic curves; and finally by Barroero and Dill [BD22] for arbitrary abelian varieties. Indeed, from their Corollary 1.6 follows the following result.

Theorem 1.5.3 (Barroero-Dill)**.** *Let $A/\mathbb{C}$ be an abelian variety of dimension n and $V \subseteq A$ an irreducible subvariety of dimension $n-2$, defined over $\mathbb{C}$ and not contained in any proper special subvariety of A. Then,*

$$V \cap A^{[n-1]} = \bigcup_{\substack{H < A \\ \dim H \leq n-1}} V \cap H$$

is not Zariski dense in V.

As before, for subvarieties of codimension 2 in general mixed Shimura varieties the Zilber-Pink Conjecture is still open.

1.5.4 General subvarieties

Other than the above-mentioned ones, some partial results are known for subvarieties of arbitrary dimension but with additional conditions. These are related to the set V^{oa} (see Definition 1.4.8), obtained by removing from V all of its anomalous subvarieties. This set is related to V^{ta} appearing in the Conjectures 1.4.9 and 1.4.10, predicting that for any subvarieties $V \subseteq G$ of a torus, abelian or semiabelian variety G, the set V^{ta} is open and

$$V^{ta} \cap G^{[\dim V+1]} = \bigcup_{\substack{H < G \\ \dim H \leq \dim G - \dim V - 1}} V^{ta} \cap H$$

is finite. Such conjectures are equivalent to the Zilber-Pink Conjecture in this setting.

The main result of [BMZ07] is that whenever V is a subvariety of $G = \mathbb{G}_{\mathrm{m}}^n$ defined over $\overline{\mathbb{Q}}$, then V^{oa} is an open, but possibly empty, subset of V. The same result for G an abelian variety is due to Rémond [Rém09].

Next, Habegger proved [Hab08, Hab09] that in both the toric and the abelian case, the set

$$V^{oa} \cap G^{[\dim V]} = \bigcup_{\substack{H<G \\ \dim H \leq \dim G - \dim V}} V^{oa} \cap H$$

is a set of bounded Weil (resp. Néron-Tate) height (See Chapter 2 for the definitions of Weil and Néron-Tate heights). This is an analogue of part 1 of Theorem 1.4.1 and Theorem 1.5.1.

Combining this height bound with Lemma 8.1 of [BMZ08a] we get the finiteness of

$$V^{oa} \cap G^{[\dim V+1]} = \bigcup_{\substack{H<G \\ \dim H \leq \dim G - \dim V - 1}} V^{oa} \cap H$$

whenever $G = \mathbb{G}_{\mathrm{m}}^n$. In the abelian setting the same finiteness result is due to Habegger and Pila [HP16].

In particular, it follows that the Zilber-Pink Conjecture holds for all subvarieties V of a torus or abelian variety G that are defined over $\overline{\mathbb{Q}}$ and such that V^{oa} is not empty. Unfortunately, this last condition fails for many subvarieties.

1.6 Unlikely intersections in families of abelian varieties

So far, we have mostly considered the case of tori or fixed abelian varieties. However, all the conjectures and results presented above have natural analogues in the context of families of abelian varieties or, more formally, abelian schemes. This will be the setting of the problem studied in Chapter 4. Throughout this section we will assume that all the varieties mentioned are defined over $\mathbb{C}$, unless otherwise stated.

Let S be a regular, irreducible, quasi-projective variety and let $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$, that is, a proper smooth group scheme such that, for every $s \in S$, the fiber $\mathcal{A}_s := \pi^{-1}(s)$ is an abelian variety of dimension g.

Example 1.6.1. Let $S = Y(2) := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. An important example of abelian scheme, which will be considered in Chapter 4, is the Legendre family of elliptic curves $\lambda : \mathcal{L} \to S$, defined as the scheme with fibers given by

$$\mathcal{L}_\lambda : Y^2 Z = X(X - Z)(X - Z\lambda)$$

for each $\lambda \in S$. We will denote by $\mathcal{L}^n \to Y(2)$ the n-th fold fibered power $\mathcal{L} \times_{Y(2)} \cdots \times_{Y(2)} \mathcal{L}$.

Note that n sections $P_1, \dots, P_n : Y(2) \to \mathcal{L}$ define a curve

$$\mathcal{C} = (P_1, \dots, P_n)(Y(2)) \subseteq \mathcal{L}^n,$$

which dominates the base $Y(2)$.

Let $\pi : \mathcal{A} \to S$ be an abelian scheme. A subgroup scheme of $\mathcal{A}$ is a closed subvariety H of $\mathcal{A}$ such that the restriction to every fiber over a closed point $s \in S$, i.e. $H_s = H \cap \mathcal{A}_s$, is a subgroup of $\mathcal{A}_s$. Equivalently, a subgroup scheme is a closed subvariety H that contains the image of the zero section $O : S \to \mathcal{A}$, is closed under the inversion morphism $[-1] : \mathcal{A} \to \mathcal{A}$ and contains the image of $H \times_S H$ under the addition map $\mathcal{A} \times_S \mathcal{A} \to \mathcal{A}$. We will write $H < \mathcal{A}$ to mean that H is a subgroup scheme of $\mathcal{A}$.

A subgroup scheme $H \subseteq \mathcal{A}$ is called *flat* if $\pi|_H$ is a flat morphism; equivalently, if $\dim H_s$ is constant over all the closed points $s \in S$. We note that, if S is a curve, this is also equivalent (see [Har77, Proposition III.9.7]) to saying that every irreducible component of H dominates the base S.

Remark 1.6.2. It is worth noting that certain abelian schemes, such as those appearing as universal families over Shimura varieties, can be realized as connected mixed Shimura varieties. A general construction is given in [Pin05a, Construction 2.9], while [BD25, Section 13] discusses the specific case of a fibered power of the Legendre family. In particular, if S is a curve and $\mathcal{A} \to S$ is the fibered power of a non-isotrivial elliptic scheme, then the special subvarieties of $\mathcal{A}$ are precisely the irreducible components of flat subgroup schemes or irreducible components of algebraic subgroups of the CM fibers; see [Hab13a, p. 144].

The first conjecture in this setting is a relative version of the Manin-Mumford Conjecture 1.3.4. In the original Manin-Mumford Conjecture, we consider the set of intersections between a subvariety V and the set of torsion points of the ambient algebraic group. In the case of abelian schemes, we consider the torsion points of every fiber. Let $\mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 2$, defined over $\mathbb{C}$, and let us denote by $\mathcal{A}_{s,tors}$ the set of torsion points over the fiber $\mathcal{A}_s$, with $s \in S$ a closed point. We define

$$\mathcal{A}_{tors} := \bigcup_{s \in S} \mathcal{A}_{s,tors} = \bigcup_{N \in \mathbb{Z}_{>0}} \ker[N],$$

where the last equality is due to the fact that torsion points all lie by definition in the kernel of some multiplication-by-N map, and all the points lying in such kernels are torsion points. In particular, $\mathcal{A}_{tors}$ is a countable union of subvarieties of codimension g. Hence, for a subvariety $V \subseteq \mathcal{A}$ with $\dim V < g$, we expect the intersection $V \cap \ker[N]$ to be unlikely. This motivates the following result.

Theorem 1.6.3 (Relative Manin-Mumford Conjecture)**.** *Let S be a regular, irreducible, quasiprojective variety and let $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$, both defined over $\mathbb{C}$. Let $V \subseteq \mathcal{A}$ be an irreducible*

subvariety such that $\bigcup_{N\in\mathbb{Z}_{>0}}[N]V$ *is Zariski dense in* $\mathcal{A}$*. If* $V(\mathbb{C})\cap\mathcal{A}_{tors}$ *is Zariski dense in* V*, then* $\dim V\geq g$*.*

The Relative Manin-Mumford Conjecture originated from ideas outlined by Zhang in his ICM talk [Zha98]. It was later formulated explicitly by Pink in [Pin05b, Conjecture 6.2], and recently proven by Gao and Habegger [GH23].

Prior to Gao and Habegger, some partial results were known. In a series of paper, Masser and Zannier first established the conjecture for curves in abelian schemes of relative dimension 2, at first isogenous to a power of elliptic schemes [MZ08, MZ10, MZ12, MZ14], then for schemes defined over $\overline{\mathbb{Q}}$ [MZ15], and relaxed to $\mathbb{C}$ in collaboration with Corvaja [CMZ17]. The case of curves in abelian schemes of arbitrary dimension defined over $\overline{\mathbb{Q}}$ was subsequently treated by Masser and Zannier [MZ20], who also applied this result to the solvability of Pell's equations in polynomials. Meanwhile, the case of surfaces was addressed by Habegger [Hab13b] and by Corvaja, Tsimerman, and Zannier [CTZ23]. Finally, for general subvarieties of fibered products of elliptic schemes, the conjecture was proved by Kühne [Küh23].

In the setting of families of abelian varieties, Pink, inspired by the conjecture for mixed Shimura varieties, proposed also another conjecture.

For an abelian scheme $\mathcal{A}\to S$, a point $s\in S(\mathbb{C})$ and an integer n let

$$\mathcal{A}_s^{[n]}:=\bigcup_{\substack{H<\mathcal{A}_s\\ \dim H\leq\dim\mathcal{A}-n}}H$$

where the union runs over all the algebraic subgroups H of the fiber $\mathcal{A}_s$. We also set $\mathcal{A}^{[n]}$ to be the union over all the closed fibers of $\mathcal{A}_s^{[n]}$.

Conjecture 1.6.4 ([Pin05b], Conjecture 6.1)**.** *Let* $\mathcal{A}\to S$ *be an abelian scheme defined over* $\mathbb{C}$ *and* $V\subseteq\mathcal{A}$ *be an irreducible closed subvariety that is not contained in any proper closed subgroup scheme of* A*, not even after a finite base change. Then,* $V\cap\mathcal{A}^{[dimV+1]}$ *is not Zariski dense in* V*.*

Remark 1.6.5. The original statement conjectured by Pink is in the broader context of families of semiabelian varieties. However, this more general statement was shown to be false by Bertrand [Ber11], who exhibited an explicit counterexample based on the existence, in some semiabelian families, of so-called *Ribet sections*. Subsequently, Bertrand, Masser, Pillay and Zannier [BMPZ16] proved that the existence of Ribet section is the only obstruction to the conjecture's validity in the case of one-parameter families. A complete and published account of Bertrand's original counterexample has recently been provided by Bertrand and Edixhoven [BE20].

In the case of abelian schemes, Conjecture 1.6.4 is implied by the Zilber-Pink Conjecture, as shown in [Pin05b, Theorem 6.3].[3] Note that the Zilber-Pink

[3]The original statement of this theorem is for families of semiabelian varieties. However, in this more general case the proof by Pink is incorrect, but is correct for abelian schemes, as discussed in Remark 5.4(4) of [BE20].

Conjecture predicts the non Zariski denseness of the intersection of V with a larger set then $\mathcal{A}^{[\dim V+1]}$. Namely, one should consider also some subvarieties of some fibers with codimension (with respect to $\mathcal{A}$) bigger that $\dim V+1$ that are not accounted for by Conjecture 1.6.4.

Partial results towards this conjecture has been made in the recent years. For $\mathcal{A} \to S$ an abelian scheme and $n \in \mathbb{N}$ we introduce the notation

$$\mathcal{A}^{\{n\}} := \bigcup_{\substack{H<\mathcal{A} \text{ flat} \\ \dim H \leq \dim \mathcal{A}-n}} H.$$

Note that $\mathcal{A}^{\{n\}} \subseteq \mathcal{A}^{[n]}$.

The first results in the direction of Conjecture 1.6.4 were the already mentioned results by Masser and Zannier for curves in squares of elliptic schemes [MZ08, MZ10, MZ12, MZ14]. Note that for abelian schemes of relative dimension 2 Conjecture 1.6.3 and Conjecture 1.6.4 coincide.

These results were later extended by Barroero and Capuano [BC16].

Theorem 1.6.6 (Barroero-Capuano [BC16])**.** *Let S be a smooth, irreducible, quasi-projective curve, let $\mathcal{E} \to S$ be a non-isotrivial elliptic scheme, both defined over $\overline{\mathbb{Q}}$, and let us denote by $\mathcal{E}^n \to S$ the n-th fibered power of $\mathcal{E}$. Let $C \subseteq \mathcal{E}^n$ be an irreducible curve defined over $\overline{\mathbb{Q}}$, not contained in a proper subgroup scheme of $\mathcal{E}^n$, not even after a finite base change. Then, $C \cap (\mathcal{E}^n)^{\{\dim V+1\}}$ is a finite set.*

Since the intersection of a flat subgroup scheme with a fiber yields an algebraic subgroup of the same codimension in that fiber, Theorem 1.6.6 provides evidence towards Conjecture 1.6.4. However, it does not fully establish the conjecture in this setting, as it does not account for the fact that some fibers may have subgroups not coming from intersections with flat subgroup schemes. This happens precisely for those fibers whose endomorphism ring is larger than the generic endomorphism ring. This gap was later addressed by Barroero.

Theorem 1.6.7 (Barroero [Bar17])**.** *Let S, C and $\mathcal{E}^n$ be as in Theorem 1.6.6. Then, there exist only finitely many $\boldsymbol{c} \in C(\mathbb{C})$ such that $\mathcal{E}_{\pi(\boldsymbol{c})}$ has complex multiplication and $\boldsymbol{c}$ lies in a proper algebraic subgroup of $\mathcal{E}_{\pi(\boldsymbol{c})}$.*

Note that the conclusion of Theorem 1.6.7 is stronger than what is required by Conjecture 1.6.4. Indeed, the conjecture only considers subgroups of codimension 2 in each fibers, while the theorem deals also with some subgroup of codimension 1. Subgroups of CM fibers however are part of the special subvarieties of $\mathcal{E}^n$ (seen as a mixed Shimura variety), so Theorem 1.6.7 falls into the prediction of the Zilber-Pink Conjecture.

Notice also that, if $C \subseteq \mathcal{E}^n$ is contained in a fixed fiber, or if $\mathcal{E}$ is isotrivial, then the analogues of Theorems 1.6.6 and 1.6.7 reduce to the case of a curve in a fixed abelian variety, which was already covered by a result of Galateau and Viada [Gal10, Via08]. Therefore, by combining these, we obtain a proof of the Zilber–Pink Conjecture for a curve in a fibered power of an elliptic scheme

when everything is defined over $\overline{\mathbb{Q}}$. This was extended to $\mathbb{C}$ by Barroero and Dill [BD22, BD25]

Theorem 1.6.6 was later extended by Barroero and Capuano [BC17] to curves in products of powers of (non-isotrivial) elliptic schemes, and more recently [BC20] to curves in arbitrary abelian schemes, again considering only the intersections with flat subgroup schemes.

The analogue of Theorem 1.6.7 in more general contexts is still open, but some partial results have been studied by Ferrigno [Fer25, Fer26].

1.7 Singular Intersections

As we saw at the beginning of this chapter, the idea behind the problems of unlikely intersections is the geometric expectation that two subvarieties V, W of an ambient variety X with $\dim V + \dim W < \dim X$ are *expected* not to intersect in general.

If we now consider the case when $\dim V + \dim W = \dim X$, the so-called case of "just likely" intersection, then of course we expect that V and W do intersect, and that the set of intersection $V \cap W$ is a finite union of points. Moreover, we also carry the expectation that, at each point $P \in V \cap W$, the intersection is *transverse*, that is, in differential geometric terms, that P is a smooth point for both V and W, and that the tangent spaces $T_P V$ and $T_P W$, seen as vector subspaces of $T_P X$, intersect only at the origin. Equivalently, in algebraic geometric terms, we expect that P is an irreducible component of $V \cap W$ and the multiplicity of intersection of V and W at P is 1. When this does not happen, we say that P is a point of *singular intersection* between V and W. We adopt the following notation

$$\begin{aligned} V \cap_{sing} W &= \{P \in V \cap W \mid \dim (T_P V \cap T_P W) > 0\} \\ &= \{P \in V \cap W \mid P \text{ isolated }, i(P; V, W; X) > 1\} \cup \bigcup_{\substack{D \subseteq V \cap W \\ \dim D > 0}} D \end{aligned}$$

where $i(P; V, W; X)$ denotes the intersection multiplicity.

For subvarieties of tori, abelian varieties and families of abelian varieties, we expect conjectures similar to Zilber-Pink for singular intersections. We formulate here such conjectures in the case of curves.

Conjecture 1.7.1. *Let $\mathcal{G}$ be a split semiabelian variety, and let C be an irreducible subvariety of $\mathcal{G}$ of dimension 1, not contained in any proper algebraic subgroup, where everything is defined over $\mathbb{C}$. Then, the set*

$$C^{[1,sing]} := \bigcup_{\substack{H < \mathcal{G} \\ \dim H \leq \dim \mathcal{G} - 1}} C \cap_{sing} H$$

is a finite set.

Conjecture 1.7.2. *Let S be a smooth, irreducible and quasi-projective curve defined over $\mathbb{C}$. Let $\mathcal{A} \to S$ be an abelian scheme defined over $\mathbb{C}$ and $C \subseteq \mathcal{A}$ be an irreducible closed curve that is not contained in any proper closed subgroup scheme of A, not even after a finite base change. Then, the set*

$$C^{\{1,sing\}} := \bigcup_{\substack{H<\mathcal{A} \text{ flat} \\ \dim H \leq \dim \mathcal{A}-1}} C \cap_{sing} H$$

is a finite set.

Partial results in the direction of these conjectures have already been investigated in the literature. More specifically, the case of curves which are images of sections in a power of a non-isotrivial elliptic scheme follows by works of Corvaja, Demejo, Masser and Zannier [CDMZ21]. Ulmer and Urzúa [UU20, UU21] give an independent proof of the case of curves that are image of sections in a single power of an elliptic scheme, and their proof provide an explicit upper bound to the the size of $C^{\{1,sing\}}$, holding also in the isotrivial case. In addition, in the same papers they also show an application to elliptic divisibility sequences.

In the case of curves in $\mathbb{G}_m^2$, the question was studied by Marché and Maurin [MM23], who proved the finiteness of $C^{[1,sing]}$, while providing effective bounds for both the degree and the height of its points.

Questions related to Conjectures 1.7.1 and 1.7.2 have also been studied in the modular setting, where theorems in the same direction have been proven by Spence [Spe19], Aslanyan [Asl22] and Ballini [Bal24, Thm. 6.0.2], using an Ax-Schanuel type result by Pila and Tsimerman [PT16].

In this thesis, we complete the proof of Conjectures 1.7.1 and 1.7.2, at least in the case where everything is defined over $\overline{\mathbb{Q}}$. Under this condition, the main theorem of Chapter 3, which comprise a joint work with Ballini and Capuano, covers Conjecture 1.7.1. On the other hand, Chapter 4, whose content already appeared as a preprint [Ott25] with an additional condition on the schemes considered, concerns Conjecture 1.7.2. Here, the statement is reduced from the setting of abelian schemes to powers of either non-isotrivial elliptic schemes or constant ones. An interesting remark is that the proofs in the two cases use substantially different techniques. For non-isotrivial elliptic schemes, the required finiteness result is obtained by using a variation of the Pila-Zannier strategy, using a height bound due to Silverman [Sil83]. On the other hand, in the constant elliptic setting (as well as in the toric one), we have a similar height bound only under the stricter hypothesis that the curve is not contained into any translate of a subgroup. What we show instead is that, under this more restrictive hypothesis, the tangency condition implies a degree bound *a priori* (i.e. without using the height bound) for a broader set of intersections, namely the singular intersections with all cosets. This more general degree bound, in turn, allows us to deduce a height bound holding without further assumption on the curve, other than it not being contained in a proper subgroup.

We conclude this section by showing, as already observed by Marché and Maurin [MM23], and Ulmer and Urzúa [UU20, UU21], how the contents of Conjectures 1.7.1 and 1.7.2 can be reframed as statements about unlikely intersections in a different ambient variety. To ease the notation, we will focus on the case of Conjecture 1.7.1, but the exact same construction also works for families of abelian varieties. Keeping the same notation, we will denote by $\mathcal{G}$ a (split) semiabelian variety of dimension n. Let $T\mathcal{G}$ be the tangent bundle of $\mathcal{G}$, and $\mathbb{P}T\mathcal{G}$ the associated projective bundle; thus, $\mathbb{P}T\mathcal{G} \to \mathcal{G}$ is a $\mathbb{P}^{n-1}$-bundle[4] whose fibers parametrize lines in the corresponding tangent space, and the total space $\mathbb{P}T\mathcal{G}$ is a smooth quasi-projective variety of dimension $2n-1$.

Given a curve $C \subseteq \mathcal{G}$, there is a canonical lift of C to $\widetilde{C} \subseteq \mathbb{P}T\mathcal{G}$, constructed in the following way. The inclusion $\iota : C \to \mathcal{G}$ induces a map $\tilde{\iota} : C^{sm} \to \mathbb{P}T\mathcal{G}$ from the smooth locus of C, sending a point $P \in C^{sm}$ to the class of its tangent line $T_P C \subseteq T_P\mathcal{G}$ in $\mathbb{P}T\mathcal{G}$. We take $\widetilde{C}$ to be the closure of the image of this map. Note that, in this way, $\widetilde{C}$ is also a curve.

In a similar way, for $W \subseteq \mathcal{G}$ a smooth subvariety we can define $\widetilde{W} \subseteq \mathbb{P}T\mathcal{G}$ as

$$\widetilde{W} = \{(P, [\Delta]) \mid P \in W, \Delta \subseteq T_P W\}.$$

By this construction, $\dim \widetilde{W} = 2 \dim W - 1$.

A curve C and a subvariety W intersecting at a point P do so tangentially if and only if their lifts $\widetilde{C}$ and $\widetilde{W}$ intersect at a point in $\mathbb{P}T\mathcal{G}$ over P. Now, whenever $\dim W \leq \dim \mathcal{G} - 1$ we have $\dim \widetilde{W} \leq \dim \mathbb{P}TG - 2$, so the intersection between $\widetilde{W}$ and $\widetilde{C}$ is unlikely.

We now recast Conjecture 1.7.1 using this new interpretation.

Conjecture 1.7.3. *Let $\mathcal{G}$ be split semiabelian variety, and let C be an irreducible subvariety of $\mathcal{G}$ of dimension 1, not contained in any proper algebraic subgroup, where everything is defined over $\mathbb{C}$. Let $\widetilde{C}$ and, for any $W \subseteq \mathcal{G}$ smooth subvariety, $\widetilde{W}$ be as above. Then, the set*

$$\widetilde{C}^{[2]} := \bigcup_{\substack{H < \mathcal{G} \\ \dim H \leq \dim \mathcal{G} - 1}} \widetilde{C} \cap \widetilde{H} \subseteq \mathbb{P}T\mathcal{G}$$

is a finite set.

Similarly, we also recast Conjecture 1.7.2 using this interpretation.

Conjecture 1.7.4. *Let $\mathcal{A} \to S$ be an abelian scheme, and let C be an irreducible curve in $\mathcal{A}$, dominant over S and not contained in a proper subgroup scheme of $\mathcal{A}$, even after a finite base change, with everything defined over $\mathbb{C}$. Let $\widetilde{C}$ and, for any $W \subseteq \mathcal{A}$ smooth subvariety, $\widetilde{W}$ be as above. Then, the set*

$$\widetilde{C}^{\{2\}} = \bigcup_{\substack{H \text{ flat} \\ \text{codim } H \geq 1}} \widetilde{C} \cap \widetilde{H},$$

where H runs through all the flat subgroup schemes of codimension at least 1, is a finite set.

[4] actually, it is isomorphic to the trivial one.

1.7.1 Future works

The contents of Conjectures 1.7.1 and 1.7.2 lend themselves to some generalizations, which could be interesting to study in the future.

First of all, Conjecture 1.7.1 may be stated for more general semiabelian varieties. The structure of subgroups here is more complicated compared to the abelian and toric settings, so some of the arguments of Chapter 3 fail. It should be however possible to complement those with the arguments of [BKS23] to recover the claim.

The formulation as unlikely intersections (see Conjecture 1.7.3) suggests that Conjectures 1.7.1 and 1.7.2 have a higher dimensional generalization, analogue to the Zilber-Pink Conjecture in the Pink's formulation (see Conjecture 1.4.6).

Conjecture 1.7.5. *Let $\mathcal{G}$ be a semiabelian variety, and let V be an irreducible smooth subvariety of $\mathcal{G}$, not contained in any proper algebraic subgroup, where everything is defined over $\mathbb{C}$. Let, for any $W \subseteq \mathcal{G}$ subvariety, $\widetilde{W}$ be the lifting of W to $\mathbb{P}T\mathcal{G}$ and let*

$$\widetilde{V}^{[2\dim V]} := \bigcup_{\substack{H<\mathcal{G} \\ \dim H \leq \dim \mathcal{G} - \dim V}} \widetilde{V} \cap \widetilde{H} \subseteq \mathbb{P}T\mathcal{G};$$

then, the projection of $\widetilde{V}^{[2\dim V]}$ onto V is non-Zariski dense in V.

In the proofs of Conjecture 1.7.1 and 1.7.2 in Chapters 3 and 4 we rely on the fact that the Zilber-Pink Conjecture for curves in abelian varieties and tori is established. Since, however, the Zilber-Pink Conjecture is open in general, we can expect a proof of Conjecture 1.7.5 to be approachable only in the cases in which Zilber-Pink is known, in particular hypersurfaces and varieties with nonempty open anomalous locus. In alternative, it may be interesting to prove a statement conditional on Zilber-Pink.

2 - Preliminaries

2.1 Weil heights and canonical heights

In this section we introduce, in increasing generality, the notion of "height". This can be thought of as an arithmetic complexity. As we saw in the previous chapter, this is an essential tool in Diophantine geometry, and having a control of it is an essential step in most of the proofs.

2.1.1 The Weil height over $\mathbb{Q}$

We start by defining the height of a rational number. As mentioned, this will be a measure of complexity for it. Let then $q \in \mathbb{Q}$ be a rational number; either q is zero or we can write q uniquely as the quotient of two integers a and b, such that they are coprime and b is positive. We define the *absolute Weil height* of q as 0 if q is zero or the maximum between b and the absolute value of a otherwise.

While this is a good definition, it does not generalize easily to more general number fields and to $\overline{\mathbb{Q}}$. This is due to the fact that, in general, an element x of a number field K does not admit a preferential representation as a quotient a/b with a and b in its ring of integers $\mathcal{O}_K$. In the following, we will give an alternative definition of the absolute Weil height that, while more complicated and technical, will more easily generalize to $\overline{\mathbb{Q}}$.

We recall that an *absolute value* on a field K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that, for every $x, y \in K$,

1. $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$;
2. $|xy| = |x||y|$;
3. $|x + y| \leq |x| + |y|$ (triangle inequality).

We say that it is *non-Archimedean* if it further satisfies the ultrametric inequality

$$|x + y| \leq \max\{|x|, |y|\},$$

and *Archimedean* otherwise. Every absolute value on K induces a distance $d(x, y) = |x - y|$ on K, and hence a topology. We say that two absolute values $|\cdot|_1$ and $|\cdot|_2$ on K are *equivalent* if they induce the same topology.

On any field K we can define an absolute value, called the *trivial* one, sending every non-zero element of K to 1. This absolute value does not give many information on the elements of K, so we will often ask for an absolute value not to be the trivial one.

We ask ourselves what are the possible absolute values on $\mathbb{Q}$. Of course, we have the usual Archimedean absolute value $|\cdot|_\infty$, inherited by the immersion $\mathbb{Q} \subseteq \mathbb{R}$. Let now $p \in \mathbb{Z}$ be a prime number. We can associate to it an absolute value $|\cdot|_p$, called the *p-adic absolute value*, as follows. We set $|0|_p := 0$. For every $x \in \mathbb{Q}$ non-zero, we can write it in a unique way

$$x = p^{v_p(x)} \frac{a}{b},$$

with $a, b, v_p(x) \in \mathbb{Z}$ and $p \nmid ab$. We call $v_p(x)$ the *p-adic valuation* of x, and set

$$|x|_p := p^{-v_p(x)}.$$

It is easy to check that this is indeed an absolute value, and that it is non-Archimedean.

A bespoke theorem, due to Ostrowski, is that, up to equivalence, the usual absolute value and the p-adic ones are essentially the only possible ones.

Theorem 2.1.1 (Ostrowski, see for example [Neu99, Prop. II.3.7])**.** *Every absolute value on $\mathbb{Q}$ is either the trivial absolute value or equivalent to $|\cdot|_\infty$ or $|\cdot|_p$ for some prime number p.*

We will denote by $M_\mathbb{Q}$ the set of "standard" non-trivial absolute values on $\mathbb{Q}$, consisting of the Archimedean absolute value $|\cdot|_\infty$ and the p-adic absolute values $|\cdot|_p$, and, with slight abuse of notation, we will write $|\cdot|_v$ for the absolute value corresponding to $v \in M_\mathbb{Q}$. As a consequence of the unique factorization in $\mathbb{Z}$ we get the so-called *product formula* (Prop 1.4.1 of [BG06], Prop. II.2.1 of [Neu99]):

$$\prod_{v \in M_\mathbb{Q}} |x|_v = 1 \qquad \forall x \in \mathbb{Q} \setminus \{0\}.$$

We are now ready to give an alternative definition of Weil height over $\mathbb{Q}$.

Definition 2.1.2. Let $x \in \mathbb{Q}$. We define its *absolute Weil height* as

$$H(x) := \prod_{v \in M_\mathbb{Q}} \max\{1, |x|_v\}.$$

It is easy to check, by multiplying by the denominator and using the product formula, that this new height coincides with the one defined previously.

It is often useful to work with the logarithm of the absolute Weil height, so we also cast the following.

Definition 2.1.3. Let $x \in \mathbb{Q}$. We define its *logarithmic Weil height* as

$$h(x) := \log H(x) = \sum_{v \in M_\mathbb{Q}} \log \max\{1, |x|_v\}.$$

We observe that the product (and the sum) is always finite, since $|x|_v = 1$ for all but finitely many $v \in M_\mathbb{Q}$.

2.1.2 The Weil height over $\overline{\mathbb{Q}}$

We will define the Weil height over $\overline{\mathbb{Q}}$ by defining a Weil height over any number field K in such a way that it is compatible over field extensions, that is, the height of an element $x \in K$ does not actually depend on K, but only on x. The definition will be similar to Definition 2.1.2. Before giving such a definition we will need to discuss absolute values over general number fields.

Fix K a number field. We want to fix a set of representatives M_K for all the non-trivial equivalence classes of absolute values defined over K. We observe that any absolute value on K restricts to an absolute value of $\mathbb{Q}$. Moreover, as $K/\mathbb{Q}$ has finite degree, the only absolute value restricting to the trivial one on $\mathbb{Q}$ is the trivial absolute value on K (see for example [Wal00, Section 3.1.2]). So, we decide to pick the representatives in M_K in such a way that they extend the absolute values in $M_{\mathbb{Q}}$. We will now give an explicit description of these absolute values.

We start by describing the Archimedean ones, which are the ones restricting to $|\cdot|_\infty \in M_{\mathbb{Q}}$. For each embedding $\sigma : K \hookrightarrow \mathbb{C}$, we define an Archimedean absolute value associated to it by $|x|_\sigma = |\sigma(x)|_\infty$, where $|\cdot|_\infty$ denotes the usual Archimedean absolute value on $\mathbb{C}$. Since $|z|_\infty = |\bar{z}|_\infty$ for every $z \in \mathbb{C}$, it follows that $|\cdot|_\sigma = |\cdot|_\tau$ whenever $\tau = \bar{\sigma}$.

In light of this, we list the $n := [K : \mathbb{Q}]$ embeddings $K \hookrightarrow \mathbb{C}$ as

$$\sigma_1, \dots, \sigma_{r_1}, \sigma_{r_1+1}, \dots, \sigma_{r_1+r_2}, \overline{\sigma_{r_1+1}}, \dots, \overline{\sigma_{r_1+r_2}},$$

where $\sigma_1, \dots, \sigma_{r_1}$ are the real embeddings and $\sigma_{r_1+1}, \dots, \sigma_{r_1+r_2}$ are the complex ones, up to conjugation. Thus, we get $r_1 + r_2$ non-trivial Archimedean absolute values on K, $|\cdot|_{\sigma_1}, \dots, |\cdot|_{\sigma_{r_1+r_2}}$, which are pairwise non-equivalent. One can also show that any non-trivial Archimedean absolute value on K is equivalent to one of these; see for example [Wal00, Section 3.1.3]. Moreover, if v is an Archimedean absolute value of the number field K, then the completion K_v of K with respect to it is isomorphic to $\mathbb{R}$ whenever v corresponds to a real embedding, and to $\mathbb{C}$ otherwise. Since the completion $\mathbb{Q}_v$ of $\mathbb{Q}$ is isomorphic to $\mathbb{R}$ for every Archimedean place, we have

$$d_v := [K_v : \mathbb{Q}_v] = \begin{cases} 1 & \text{if } v \text{ is real} \\ 2 & \text{if } v \text{ is complex.} \end{cases}$$

Next, we describe the non-Archimedean absolute values. These correspond to the non-zero prime ideals of $\mathcal{O}_K$. Analogously to the p-adic absolute values on $\mathbb{Q}$, we define them starting from a valuation. Let $\mathfrak{p}$ be a non-zero prime ideal of $\mathcal{O}_K$ and let p be the prime number such that $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$. For every element $x \in K \setminus \{0\}$ we can write in a unique way

$$x\mathcal{O}_k = \mathfrak{p}^{v_\mathfrak{p}(x)}\mathcal{I},$$

for $v_\mathfrak{p}(x)$ an integer and $\mathcal{I}$ a fractional ideal such that $\mathcal{I} \not\subseteq \mathfrak{p}$. We call $v_\mathfrak{p}(x)$ the $\mathfrak{p}$-adic valuation of x, and set

$$|x|_\mathfrak{p} := p^{-v_\mathfrak{p}(x)/e(\mathfrak{p}|p)},$$

where $e(\mathfrak{p}|p) = v_{\mathfrak{p}}(p)$ is the ramification index of $\mathfrak{p}$ over p. The factor $e(\mathfrak{p}|p)$ ensures that $|p|_{\mathfrak{p}} = p^{-1}$, so that $|\cdot|_{\mathfrak{p}}$ extends $|\cdot|_p$. It can be verified that absolute values corresponding to different prime ideals are non equivalent and that any non-trivial non-Archimedean absolute value on K is equivalent to one of these. Furthermore, if v is the non-Archimedean absolute value of K associated to $\mathfrak{p}$, we have that $d_v := [K_v : \mathbb{Q}_v]$ is equal to $e(\mathfrak{p}|p)f(\mathfrak{p}|p)$, where K_v is the completion of K with respect to v and $f(\mathfrak{p}|p) = [\mathcal{O}_K/\mathfrak{p} : \mathbb{F}_p]$ is the inertia degree of $\mathfrak{p}$ over p. Similar to $\mathbb{Q}$, we have an analogue of Ostrowski's Theorem 2.1.1 for number fields: in particular, every absolute value on a number field K is either the trivial one, equivalent to an Archimedean absolute induced by an embedding $\sigma : K \hookrightarrow \mathbb{C}$, or equivalent to the $\mathfrak{p}$-adic absolute value, for a certain prime ideal $\mathfrak{p}$.

From now on, we will denote by M_K the set of absolute values of K described above. These absolute values also satisfy a product formula (see for example [HS00, Prop. B.1.2].

Proposition 2.1.4. *Let $x \in K \setminus \{0\}$. Then*

$$\prod_{v \in M_K} |x|_v^{d_v} = 1.$$

In light of the above discussion, we are ready to give the definition for height over any number field K.

Definition 2.1.5. Let x be in a number field K. We define its *Absolute Weil Height* as

$$H(x) := \left(\prod_{v \in M_K} \max\{1, |x|_v^{d_v}\} \right)^{1/[K:\mathbb{Q}]},$$

and the *logarithmic Weil height* as

$$h(x) := \log H(x) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} d_v \log \max\{1, |x|_v\}.$$

We observe that the product (and the sum) is again always finite, since $|x|_v = 1$ for all but finitely many $v \in M_K$. Moreover, the scaling factors ensure that the definition is independent of K, so that it actually extends to a height function defined over $\overline{\mathbb{Q}}$.

We now present several properties of the Weil height.

Proposition 2.1.6.

1. *$h(x) \geq 0$ for any $x \in \overline{\mathbb{Q}}$, and $h(x) = 0$ if and only if $x = 0$ or x is a root of unity (Kronecker);*

2. *for any $x, y \in \overline{\mathbb{Q}}$, $h(xy) \leq h(x) + h(y)$, and moreover, if y is a root of unity, then $h(xy) = h(x)$;*

3. *for any* $x_1, \dots, x_n \in \overline{\mathbb{Q}}$, *we have* $h(x_1 + \cdots + x_n) \leq h(x_1) + \cdots + h(x_n) + \log n$;

4. *for any non-zero* $x \in \overline{\mathbb{Q}}$ *and any* $n \in \mathbb{Z}$, $h(xn) = |n| \cdot h(x)$;

5. *for any* $x \in \overline{\mathbb{Q}}$ *and any* $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, $h(\sigma(x)) = h(x)$.

Observe that it is not possible to replace $\log n$ in part 3 with any smaller constant, as demonstrated by the example $x_1 = \cdots = x_n = 1$. For a proof of these properties see Section 1.5 of [BG06].

The following finiteness theorem is one of the reasons why height functions are so widely used in Diophantine geometry, serving as a foundational result for many finiteness results.

Theorem 2.1.7 (Northcott)**.** *Let* B *and* D *be non-negative real numbers. Then, the set*

$$\{\alpha \in \overline{\mathbb{Q}} \mid h(\alpha) \leq B \text{ and } [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq D\}$$

is finite.

2.1.3 The Weil height on projective spaces

We will now begin to extend the notion of Weil height to a more geometric setting, starting from algebraic points in a projective space $\mathbb{P}^n(\overline{\mathbb{Q}})$.

Definition 2.1.8. Let $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$ and let K be a number field containing $x_0, \dots, x_n$. The *absolute logarithmic (projective) Weil height* of P is defined as:

$$h(P) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} d_v \log \max\{|x_0|_v, \dots, |x_n|_n\}.$$

We define the *absolute multiplicative (projective) Weil height* of P as $H(P) = \exp(h(P))$.

As for the Weil height on $\overline{\mathbb{Q}}$, this definition does not depend on the choice of K. Furthermore, by the product formula, the height also does not depend on the choice of the homogeneous coordinates of P, ensuring that it is well-defined. Moreover, it satisfies $h(\sigma(P)) = h(P)$ for any $\sigma \in \mathcal{G}al(\overline{\mathbb{Q}}/\mathbb{Q})$.

Remark 2.1.9. For $n = 1$ we can recover the previous definition of the height on $\overline{\mathbb{Q}}$ by identifying $\overline{\mathbb{Q}} = \mathbb{A}^1(\overline{\mathbb{Q}}) \subseteq \mathbb{P}^1(\overline{\mathbb{Q}})$. Indeed, for any $\alpha \in \overline{\mathbb{Q}}$, we have $h(\alpha) = h([1 : \alpha])$. More generally, this allows us to define the height of a n-tuple of algebraic numbers, by identifying $\overline{\mathbb{Q}}^n = \mathbb{A}^n(\overline{\mathbb{Q}})$. Thus, we get that the height of a tuple $(\alpha_1, \dots, \alpha_n) \in \overline{\mathbb{Q}}^n$ is

$$h((\alpha_1, \dots, \alpha_n)) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} d_v \log \max\{1, |\alpha_1|_v, \dots, |\alpha_n|_v\},$$

where we can choose $K = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$, as the definition does not depend on the choice of K. We call this the *affine height* for the tuple, to distinguish it from the projective height defined above.

For a point $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we define its field of definition as

$$\mathbb{Q}(P) = \mathbb{Q}\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right),$$

for any index j such that $x_j \neq 0$. In particular, up to permutations and normalization of the coordinates, we may assume that $x_0 = 1$. From this, it follows that $h(P) \geq 0$ and, if $x_0 = 1$, $h(P) \geq h(x_i)$ for every $i = 1, \ldots, n$. Thus, Theorem 2.1.7 implies the following generalization of Northcott's theorem to $\mathbb{P}^n(\overline{\mathbb{Q}})$.

Theorem 2.1.10 (Northcott)**.** *Let B and D be non-negative real numbers. Then, the set*

$$\{P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid h(P) \leq B \text{ and } [\mathbb{Q}(P) : \mathbb{Q}] \leq D\}$$

is finite.

Finally, we generalize Kronecker's theorem (part 1 of Proposition 2.1.6) to projective spaces.

Theorem 2.1.11 (Kronecker)**.** *Let $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$, and assume that $x_0 = 1$, as above. Then $h(P) = 0$ if and only if, for every $j = 1, \ldots, n$, either x_j is zero or it is a root of unity.*

2.1.4 The Weil height on projective varieties

After studying the Weil height on $\mathbb{P}^n(\overline{\mathbb{Q}})$, we now extend this notion to the $\overline{\mathbb{Q}}$-points of arbitrary projective varieties. This extension is provided by the *Weil height machine*, which associates height functions to divisors on projective varieties, allowing us to measure the arithmetic complexity of points on more general varieties.

Let V be a projective variety defined over $\overline{\mathbb{Q}}$. A natural way to define a height on $V(\overline{\mathbb{Q}})$ is to use a morphism from V to a projective space $\varphi : V \to \mathbb{P}^n$.

Definition 2.1.12. Suppose that $\varphi : V \to \mathbb{P}^n$ is defined over $\overline{\mathbb{Q}}$. The height on V relative to φ is defined as

$$h_\varphi(P) := h(\varphi(P)) \qquad \text{for any } P \in V(\overline{\mathbb{Q}}),$$

Where h is the absolute logarithmic Weil height on $\mathbb{P}^n$ defined before.

Since height functions are defined via morphisms into projective spaces, one may worry about their dependence on the choice of such morphisms. Such a dependence indeed exists, but the next proposition ensures that it is well-behaved: if two morphisms come from the same complete linear system, then the associated height functions differ only by a bounded function.

Proposition 2.1.13. *Let V be a projective variety defined over $\overline{\mathbb{Q}}$. Let $H \subseteq \mathbb{P}^n$ and $H' \subseteq \mathbb{P}^m$ be hyperplanes. Let $\varphi : V \to \mathbb{P}^n$ and $\psi : V \to \mathbb{P}^m$ be morphisms such that $\varphi^* H$ and $\psi^* H'$ are linearly equivalent. Then, we have $h_\varphi(P) = h_\psi(P) + O(1)$ for every $P \in V(\overline{\mathbb{Q}})$, and the bounded function $O(1)$ depends on V, φ and ψ but not on P.*

While the height functions we introduced sofar depend on the choice of the morphism into a projective space, a more intrinsic approach exists. Instead of relying on embeddings, one can define heights directly in terms of divisors on the variety. This construction, due to Weil, is also known as *Weil's height machine* and it allows us to translate geometric relations into arithmetical statements about heights.

Theorem 2.1.14 (Weil's height machine)**.** *Let V be a smooth projective variety defined over $\overline{\mathbb{Q}}$. Then, there exists a map*

$$\begin{array}{ccc} \mathrm{Div}(V) & \to & \{\textit{functions } V(\overline{\mathbb{Q}}) \to \mathbb{R}\} \\ D & \mapsto & h_{v,D} \end{array}$$

which is, up to bounded functions $O(1)$, uniquely defined by the following three properties:

1. *(Normalization) Let $H \subseteq \mathbb{P}^n$ be an hyperplane. Then, for every $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we have*
$$h_{\mathbb{P}^n,H}(P) = h(P) + O(1),$$
where h is the absolute logarithmic height on $P^n(\overline{\mathbb{Q}})$.

2. *(Functoriality) Let $\varphi : V \to W$ be a morphism and let $D \in \mathrm{Div}(W)$ be a divisor. Then, for every $P \in V(\overline{\mathbb{Q}})$, we have*
$$h_{V,\varphi^* D}(P) = h_{W,D}(\varphi(P)) + O(1).$$

3. *(Additivity) Let $D, D' \in \mathrm{Div}(V)$. Then, for every $P \in V(\overline{\mathbb{Q}})$, we have*
$$h_{V,D+D'}(P) = h_{V,D}(P) + h_{V,D'}(P) + O(1).$$

Moreover, it satisfies the following additional properties:

4. *(Linear equivalence) If $D, D' \in \mathrm{Div}(V)$ are two linearly equivalent divisors, then*
$$h_{V,D}(P) = h_{V,D'}(P) + O(1)$$
for every $P \in V(\overline{\mathbb{Q}})$.

5. *(Positivity) Let $D \in \mathrm{Div}(V)$ be an effective divisor, and let B be the set of base points of the associated linear system $|D|$. Then, there is a constant $\gamma \in \mathbb{R}$ such that, for every $P \in (V \setminus B)(\overline{\mathbb{Q}})$, we have*
$$h_{V,D}(P) \geq \gamma.$$

6. *(Northcott) Let $D \in \mathrm{Div}(V)$ be an ample divisor. Then, for any constants $B, C \in \mathbb{R}_{\geq 0}$ the set*

$$\{P \in V(\overline{\mathbb{Q}}) \mid [\mathbb{Q}(P) : \mathbb{Q}] \leq B \text{ and } h_{V,D}(P) \leq C\}$$

is finite.

The bounded functions $O(1)$ properties 1 to 4 and the constant γ in property 5 depend on the varieties, divisors and morphisms, but not on the points on the variety.

For a proof of this theorem, we refer to [HS00, Theorem B.3.2].

Remark 2.1.15. If the variety V is not smooth, we can use either Cartier divisors or line bundles instead of Weil divisors. See for example [HS00, Theorem B.3.6].

To end this section, we recall some properties on the behavior of the Weil height between morphisms.

Proposition 2.1.16 ([HS00], Theorem B.2.5(a))**.** *Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map of degree d defined over $\mathbb{Q}$, so that φ is given by a $(n+1)$-tuple $\varphi = (f_0, \ldots, f_m)$ of homogeneous polynomials of degree d. Let $Z \subseteq \mathbb{P}^n$ be the subset of common zeros of the f_i's. Then φ is defined on $\mathbb{P}^n \setminus Z$, and for all $P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \setminus Z$,*

$$h(\varphi(P)) \leq dh(P) + O(1).$$

Proposition 2.1.17 ([BHM16], Theorem 2.3, from [Sil11], Theorem 1)**.** *Let X, Y be irreducible, quasi-affine varieties (where we suppose to fix a projective embedding and hence a height), defined over $\overline{\mathbb{Q}}$, with $\dim X = \dim Y$. Let $g : X \to Y$ be a dominant morphism. Then, there exists a constant $\eta > 0$, depending on X and Y and an open and dense subset U of X such that*

$$h(Q) \leq \eta(1 + h(g(Q))),$$

for all $Q \in U(\overline{\mathbb{Q}})$. Moreover, this estimate holds true with

$$U = \{P \in X : P \text{ is isolated in } \varphi^{-1}(\varphi(P))\},$$

which is Zariski open in X.

2.1.5 Canonical heights

The height functions obtained from Theorem 2.1.14 are well-defined only up to an additive bounded term. In this section, we introduce a refinement that, on varieties with some extra structure, selects a distinguished representative within this equivalence class, known as the canonical height. In particular, we will consider the special case of abelian varieties, in which canonical heights satisfy additional properties related to the group structure.

Theorem 2.1.18 (Néron, Tate)**.** *Let V be a smooth projective variety defined over a number field and $D \in \mathrm{Div}(V)$. Let $\varphi : V \to V$ be a morphism defined over $\overline{\mathbb{Q}}$ such that $\varphi^* D \sim \alpha D$ for some $\alpha > 1$. Then, there exists a unique function $\widehat{h}_{V,\varphi,D} : V(\overline{\mathbb{Q}}) \to \mathbb{R}$, which we call the canonical height on V with respect to φ and D, such that*

1. *$\widehat{h}_{V,\varphi,D}(P) = h_{V,D}(P) + O(1)$ for every $P \in V(\overline{\mathbb{Q}})$;*
2. *$\widehat{h}_{V,\varphi,D}(\varphi(P)) = \alpha \widehat{h}_{V,\varphi,D}(P)$ for every $P \in V(\overline{\mathbb{Q}})$;*
3. *if $D' \in \mathrm{Div}(V)$ is linearly equivalent to D, then $\widehat{h}_{V,\varphi,D} = \widehat{h}_{V,\varphi,D'}$.*

Moreover, $\widehat{h}_{V,\varphi,D}$ can be computed as follows:

$$\widehat{h}_{V,\varphi,D}(P) = \lim_{n\to\infty} \frac{1}{\alpha^n} h_{V,D}(\varphi^n(P)),$$

where $\varphi^n = \varphi \ldots \varphi$ is the n-th iterate of φ.

A proof of this theorem can be found for example in [HS00, Theorem B.4.1] or [BG06, Remark 9.2.9].

To illustrate the construction of canonical heights, we consider the special case of projective spaces.

Example 2.1.19. Let $d \geq 2$ and consider the morphism

$$\begin{aligned} \pi_d : \quad \mathbb{P}^n \quad &\to \quad \mathbb{P}^n \\ [x_0 : \cdots : x_n] &\mapsto [x_0^d : \cdots : x_n^d]. \end{aligned}$$

If $H \subset \mathbb{P}^n$ is any hyperplane, then $\pi_d^* H \sim dH$. Thus, we can define a canonical height $\widehat{h}_{\mathbb{P}^n,\pi_d,H}$.

Since $h_{\mathbb{P}^n,H}(P) = h(P) + O(1)$ and $h(\pi_d(P)) = dh(P)$ for every $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$, we have

$$\widehat{h}_{\mathbb{P}^n,\pi_d,H}(P) = \lim_{N\to\infty} \frac{1}{d^N} h_{\mathbb{P}^n,H}(\pi_d^n(P)) = \lim_{N\to\infty} \frac{1}{d^N}\big(d^n h(p) + O(1)\big) = h(P).$$

Thus, in this case, the canonical height coincides with the absolute logarithmic Weil height on $\mathbb{P}^n$.

We notice from the previous example that the canonical height $\widehat{h}_{\mathbb{P}^n,\pi_d,H}$ is non-negative, as such is the logarithmic Weil height. Moreover, the points of $\mathbb{P}^n$ with Weil height zero are exactly those that are represented by $n+1$-tuples of zeros and roots of unity, i.e. those that have finite orbit under the action of π_d. As the next proposition shows, this is not an isolated case, but a general property of canonical heights.

Proposition 2.1.20. *Let V be a smooth projective variety defined over a number field and let $D \in \mathrm{Div}(V)$ be an ample divisor. Let $\varphi : V \to V$ be a morphism such that $\varphi^* D \sim \alpha D$ for some $\alpha > 1$ and let $\widehat{h}_{V,\varphi,D}$ be the associated*

canonical height defined in the theorem above. Then $\widehat{h}_{V,\varphi,D}(P) \geq 0$ *for every* $P \in V(\overline{\mathbb{Q}})$ *and* $\widehat{h}_{V,\varphi,D}(P) = 0$ *if and only if the forward orbit of* P*,*

$$\{P, \varphi(P), \varphi^2(P), \dots, \varphi^n(P), \dots\},$$

is finite.

Suppose now that $V = A$ is an abelian variety, where we denote the group structure additively. We will use the notation $[n] : A \to A$ for the multiplication-by-n map. We want to define a canonical height on A using the construction of Theorem 2.1.18 with $\varphi = [2]$. To do so, we need to have a divisor D such that $[2]^*D \sim \alpha D$ for some $\alpha \geq 1$.

Proposition 2.1.21 (Mumford's formula, [Mum88, Cor. II.6.3, p.59])**.** *Let* $D \in \mathrm{Div}(A)$ *and* $n \in \mathbb{Z}$*. Then*

$$[n]^*D \sim \frac{n^2+n}{2}D + \frac{n^2-n}{2}[-1]^*D.$$

In particular, if D *is* symmetric*, i.e.* $[-1]^*D \sim D$*, then* $[n]^*D \sim n^2D$*. On the other hand, if* D *is antisymmetric, i.e.* $[-1]^*D \sim -D$*, then* $[n]^*D \sim nD$*.*

We can associate canonical heights to symmetric and antisymmetric divisors.

Theorem 2.1.22. *Let* A *be an abelian variety defined over a number field, and let* D *be a symmetric divisor on* A*. Then, the canonical height* $\widehat{h}_{A,D} := \widehat{h}_{A,[2],D} : A(\overline{\mathbb{Q}}) \to \mathbb{R}$ *satisfies the following properties:*

1. *for every* $n \in \mathbb{Z}$ *and for every* $P \in A(\overline{\mathbb{Q}})$ *we have*

$$\widehat{h}_{A,D}([n]P) = n^2\widehat{h}_{A,D}(P);$$

2. *(Parallelogram law) for all* $P, Q \in A(\overline{\mathbb{Q}})$ *we have*

$$\widehat{h}_{A,D}(P+Q) + \widehat{h}_{A,D}(P-Q) = 2\widehat{h}_{A,D}(P) + 2\widehat{h}_{A,D}(Q).$$

Remark 2.1.23. While $\widehat{h}_{A,D}$ is classically defined as the canonical height on A with respect to [2], replacing the 2 with any integer $m \neq -1, 0, 1$ yields the same canonical height on A.

Remark 2.1.24. If D is an ample and symmetric divisor on A, then Proposition 2.1.20 implies that $\widehat{h}_{A,D}(P) \geq 0$ for every $P \in A(\overline{\mathbb{Q}})$. Moreover, $\widehat{h}_{A,D}(P) = 0$ if and only if the set

$$\{P, [2]P, [4]P, \dots, [2^n]P, \dots\}$$

is finite. This finiteness condition is in turn equivalent to the existence of integers $0 \leq i < j$ such that $[2^i]P = [2^j]P$, which is equivalent to P having finite order, i.e. , P being a torsion point.

For antisymmetric divisors, the canonical heights have analogous properties, which can be found in details in [HS00, Theorem B.5.5] or [BG06, Section 9.2].

Suppose now that D is any divisor on A. We can write $2D$ as a sum of a symmetric and an antisymmetric divisor, as follows:

$$2D = (D + [-1]^*D) + (D - [-1]^*D).$$

This, together with the additivity property, allows us to define a canonical height on A associated to any divisor as follows:

$$\widehat{h}_{A,D} := \frac{\widehat{h}_{A,D+[-1]^*D} + \widehat{h}_{A,D-[-1]^*D}}{2}.$$

2.2 O-minimality and point counting

Starting from pioneering work of Pila and Zannier [PZ08], the use of point-counting results has become essential in Diophantine geometry, through the use of the so-called "Pila-Zannier" strategy, outlined in more details in the next section.

Point-counting is referred to the estimation of the number of rational points of bounded height on non-algebraic sets. For such an estimation to be possible, the sets in question must have some tame structure, for example they could be compact subsets of the graph of an analytic function. More in general, this kind of counting theorems are known for sets that are *definable in an o-minimal structure*.

In this section, we will introduce some general definitions and properties of o-minimal structures, and we will give some examples of point-counting results. For a more in-depth treatment we refer to [Dri98], [DM96] and to Chapter 8 of [Pil22].

Definition 2.2.1. A structure is a sequence $\mathcal{S} = (\mathcal{S}_N), N \geq 1$, where each $\mathcal{S}_N$ is a collection of subsets of $\mathbb{R}^N$ such that, for each $N, M \geq 1$:

1. $\mathcal{S}_N$ is a boolean algebra (under the usual set-theoretic operations);
2. $\mathcal{S}_N$ contains every semi-algebraic subset of $\mathbb{R}^N$;
3. if $A \in \mathcal{S}_N$ and $B \in \mathcal{S}_M$, then $A \times B \in \mathcal{S}_{N+M}$;
4. if $A \in \mathcal{S}_{N+M}$, then $\pi(A) \in \mathcal{S}_N$, where $\pi : \mathbb{R}^{N+M} \to \mathbb{R}^N$ is the projection onto the first N coordinates.

If $\mathcal{S}$ is a structure and, in addition,

5. $\mathcal{S}_1$ consists of all finite union of open intervals and points,

then $\mathcal{S}$ is called an *o-minimal structure*.

Given a structure $\mathcal{S}$, we say that $S \subseteq \mathbb{R}^N$ is a *definable set* if $S \in \mathcal{S}_N$.

Let $U \in \mathbb{R}^M \times \mathbb{R}^N$. For $t_0 \in \mathbb{R}^M$, we set $U_{t_0} = \{x \in \mathbb{R}^N \mid (t_0, x) \in U\}$ and call U a *family of subsets* of $\mathbb{R}^N$, while U_{t_0} is called the *fiber* of U above t_0. If U is a definable set, then we call it a *definable family*, and one can see that the fibers U_{t_0} are definable sets too. Let $S \subseteq \mathbb{R}^N$ and $f : S \to \mathbb{R}^M$ be a function. We call f a *definable function* if its graph $\{(x, y) \in S \times \mathbb{R}^M \mid y = f(x)\}$ is a definable set. It is not hard to see that images and preimages of definable sets via definable functions are still definable. We recall some other properties of definable functions; for a reference see for example [DM96, B.7].

Lemma 2.2.2. *Let $U \subseteq \mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$ be a definable function in a structure $\mathcal{S}$. Then:*

1. *the set $\{x \in U \mid f$ is continuous at $x\}$ is definable in $\mathcal{S}$;*
2. *the set $U' := \{x \in U^\circ \mid f$ is differentiable at $x\}$ is definable in $\mathcal{S}$;*
3. *the derivative $Df : U' \to \mathbb{R}^{m \times n} \cong \mathbb{R}^{mn}$ is definable in $\mathcal{S}$;*
4. *the set of all $x \in U'$ such that f is C^1 on an open neighborhood of x contained in U is definable in $\mathcal{S}$;*
5. *if $f : U \to R^m$ is C^1, then the set $\{x \in U \mid \operatorname{rk}\big(f(x)\big) = i\}$ is definable in $\mathcal{S}$ for every $i = 1, \ldots, m$.*

Definable families have some uniformity properties analogous to algebraic families, as the following proposition highlights.

Proposition 2.2.3 ([DM96, 4.4])**.** *Let U be a definable family; then, there exists a positive integer γ such that each fiber of U has at most γ connected components.*

We give now some examples of o-minimal structures, that are of particular interest to us. For a more comprehensive list, see [Dri98].

Example 2.2.4. The structure $\mathcal{S}$ where $\mathcal{S}_n$ is the set of semi-algebraic subsets of $\mathbb{R}^n$ is an o-minimal structure, which we will denote by $\mathbb{R}_{\text{semialg}}$. It is in a sense the "smallest" possible o-minimal structure, by Point 2 of the definition.

Example 2.2.5. Let $\mathcal{S}$ be the smallest structure containing the graphs of *restricted analytic functions*, that is, functions $f : [0, 1]^n \to \mathbb{R}$ that are analytic on a small neighborhood of the unit cube. The sets definible in this structure are called *finitelly subanalitic*, and as a consequence of a theorem of Gabrielov [Gab68] (see also [Dri86]) they form an o-minimal structure, which is usually denoted as $\mathbb{R}_{\text{an}}$.

Notice that adding the graphs of all unrestricted analytic functions does not lead to an o-minimal structure: indeed, if one adds, for example, the graph of the sine function, then $\mathcal{S}_1$ will contain $\sin^{-1}([0, 1])$, which is an infinite union of disjoint intervals. So $\mathcal{S}$ would not satisfy condition 5 of the definition.

Example 2.2.6. Let $\mathcal{S}$ be the smallest structure containing $\mathbb{R}_{\text{an}}$ and the graph of the *unrestricted* exponential function $\exp : \mathbb{R} \to \mathbb{R}_{>0}$. Then, by a result of van den Dries and Miller [DM94], this is still an o-minimal structure, usually denoted by $\mathbb{R}_{\text{an,exp}}$.

Throughout this thesis we are mostly interested in $\mathbb{R}_{\text{an,exp}}$, as most of the sets required in Diophantine applications can be defined here. Indeed, the complex exponential function $\exp : \mathbb{C} \to \mathbb{G}_{\text{m}}(\mathbb{C})$ restricted to the domain $[0,1] \times i\mathbb{R}$ can be defined using the unrestricted real exponential and restricted sine and cosine functions; the j modular function restricted to the usual fundamental domain $\mathcal{F}$ can be defined via the q-expansion as the composition of exp and a restricted analytic function; and, by work of Peterzil and Starchenko [PS04], the Weierstrass $\wp(\tau, z)$ function restricted to $\{(\tau, z) \mid \tau \in \mathcal{F}, z \in \mathcal{L}_\tau\}$, where $\mathcal{F}$ is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and $\mathcal{L}_\tau$ is a fundamental domain for the action of $\mathbb{Z}+\tau\mathbb{Z}$ on $\mathbb{C}$ by translation, is also definable in $\mathbb{R}_{\text{an,exp}}$.

From now on, whenever we will be talking about "definable" objects it is assumed, unless specified otherwise, that they are definable in the structure $\mathbb{R}_{\text{an,exp}}$.

We note that, if $D \subseteq \mathbb{R}^N$ is a compact definable set, I is an open neighborhood of D and $f : I \to \mathbb{R}^M$ is an analytic function, then $f(D)$ is also definable.

O-minimal structures and definable sets provide an appropriate setting to have point-counting results. The first result in this direction is due to Bombieri and Pila [BP89], who proved the following.

Theorem 2.2.7 (Bombieri-Pila, 1989)**.** *Let Γ be the graph of a transcendental analytic function on $[0,1]$. Then, for every $\varepsilon \geq 0$ there exists a constant $c = c(\Gamma, \varepsilon)$ such that*

$$\left|T\Gamma \cap \mathbb{Z}^2\right| \leq cT^{\varepsilon}$$

for every $T \in \mathbb{R}, T \geq 1$.

This theorem can be recast in terms of points of bounded heights. In order to state it, we first introduce some notations. Let H be the coordinatewise affine absolute Weil height, so that $H(a_1, \ldots, a_n) = \max_i H(a_i)$ with $H(a_i)$ the usual absolute Weil height. For $X \subseteq \mathbb{R}^n$ let $X(\mathbb{Q})$ be the subset of rational points of X, and

$$X(\mathbb{Q}, T) := \left\{P \in X(\mathbb{Q}) \;\middle|\; H(P) \leq T\right\}.$$

Bombieri-Pila Theorem can then be recast as follows.

Theorem 2.2.8 ([Pil91])**.** *Let Γ be the graph of a transcendental analytic function on $[0,1]$. Then, for every $\varepsilon \geq 0$ there exists a constant $c = c(\Gamma, \varepsilon)$ such that*

$$\left|X(\mathbb{Q}, T)\right| \leq cT^{\varepsilon}$$

for every $T \in \mathbb{R}, T \geq 1$.

We would want a similar bound for higher dimensional sets $X \subseteq \mathbb{R}^n$. However, it may happen that, despite X not being a semialgebraic set, it contains some semialgebraic subsets of positive dimension, and these subsets may contain "many", i.e. $\gg T^\delta$ for a fixed δ, rational points of height less then T, which would make a bound $\ll T^\varepsilon$ fail for ε smaller than δ.

This motivates us to divide X into two parts: the *algebraic part* X^{alg}, given by the union of all positively-dimensional semialgebraic subsets of X, and the *transcendental part* X^{tran}, given as the complement $X \setminus X^{alg}$.

The set X^{alg} (and, consequently, X^{tran}) can be very complicated and hard to identify; it usually consists of infinitely may connected components. However, as proven by Pila and Wilkie [PW06], removing X^{alg} (i.e. focusing on X^{tran}) allows us to recover a result of the same quality as Theorem 2.2.8, at least whenever X is definable in an o-minimal structure.

Theorem 2.2.9 (Pila-Wilkie Counting Theorem, [PW06, Thm 1.8])**.** *Let $X \subset \mathbb{R}^n$ be a set definable in an o-minimal structure, and let $\varepsilon > 0$. There is a constant $c(X, \varepsilon)$ such that*

$$\left|X^{tran}(\mathbb{Q}, T)\right| \leq c(X, \varepsilon) T^\varepsilon, \tag{2.2.1}$$

In the proof of Theorem 4.1.4 we are going to use an even stronger result, due to Habegger and Pila [HP16]. Before stating it, we introduce some further notation. For a family $Z \subseteq \mathbb{R}^{M_1} \times \mathbb{R}^{M_2} \times \mathbb{R}^N$, a positive real number T and $t \in \mathbb{R}^{M_1}$, we define

$$Z_t^\sim(\mathbb{Q}, T) = \{(y, z) \in Z_t \mid y \in \mathbb{Q}^{M_2}, H(y) \leq T\}. \tag{2.2.2}$$

By π_1 and π_2, we denote the projections of Z_t to the first M_2 and the last N coordinates, respectively.

Proposition 2.2.10 ([HP16, Corollary 7.2])**.** *Let $Z \subseteq \mathbb{R}^{M_1} \times \mathbb{R}^{M_2} \times \mathbb{R}^N$ be a definable family. For every $\varepsilon > 0$, there exists a constant $c = c(Z, \varepsilon)$ with the following property. Fix $t \in \mathbb{R}^{M_1}$ and $T \geq 1$. If $\left|\pi_2(\Gamma)\right| > cT^\varepsilon$ for some $\Gamma \subseteq Z_t^\sim(\mathbb{Q}, T)$, then there exists a continuous definable function $\delta : [0, 1] \to Z_t$ such that*

1. *the composition $\pi_1 \circ \delta : [0, 1] \to \mathbb{R}^{M_2}$ is semi-algebraic and its restriction to $(0, 1)$ is real analytic;*
2. *the composition $\pi_2 \circ \delta : [0, 1] \to \mathbb{R}^N$ is non-constant;*
3. *we have $\pi_2(\delta(0)) \in \pi_2(\Gamma)$.*

2.2.1 Effective point counting

The constants appearing in Theorem 2.2.9 and Proposition 2.2.10 are, in general, not effective. That is, given a set X (or Z) and $\varepsilon > 0$ there is no computable way to bound the constants $c(X, \varepsilon)$ whose existence is ensured by

the above-mentioned facts. However, in the applications, it is very desirable to have an estimation of how large those constants may be. Clearly, to get such effective bounds, we need to restrict ourselves to a framework in which the functions involved can be described in a uniform and effective way, so that we have a notion of "complexity" of the functions and sets involved, with respect to which to give the bound. The way one usually achieves this is to limit oneself to a class of functions that satisfy some systems of polynomial differential equations; the size and degree of these systems provide the required notion of complexity.

One such class of functions, that will be considered here, is that of Pfaffian functions, first introduced by Khovanskiĭ [Kho80, Kho91], and since become one of the most promising candidate for a setting in which to prove effective bounds.

Definition 2.2.11. Let us fix an open subset $U \subseteq \mathbb{R}^n$ (whose coordintates will be denoted by $\boldsymbol{x} = (x_1, \ldots, x_n)$). A sequence $f_1, \ldots, f_l : U \to \mathbb{R}$ of analytic functions is called a *Pfaffian chain* if there are polynomials $P_{i,j} \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_i]$, for $i = 1, \ldots, l$ and $j = 1, \ldots, n$, such that

$$\frac{\partial f_i}{\partial x_j}(\boldsymbol{x}) = P_{i,j}(\boldsymbol{x}, f_1(\boldsymbol{x}), \ldots, f_i(\boldsymbol{x})),$$

for all i, j, and all $\boldsymbol{x} \in U$. In other terms, the derivative of the i-th function in the chain can be expressed as a polynomial in the function itself, all of the function preceeding it in the chain, and the function arguments x_j.

We then say that a function $f : U \to \mathbb{R}$ is *Pfaffian* with chain $f_1, \ldots, f_l$ if $f(\boldsymbol{x}) = P(\boldsymbol{x}, f_1(\boldsymbol{x}), \ldots, f_l(\boldsymbol{x}))$, for all $\boldsymbol{x} \in U$, for some polynomial $P \in \mathbb{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_l]$. In other terms, a Pfaffian function is any function that can be expressed as a polynomial in the argument $\boldsymbol{x}$ and in a Pfaffian chain.
We define the *format* of such a Pfaffian function to be $n + l$ and its *degree* to be $\Sigma_{i,j} \deg(P_{i,j}) + \deg(P)$.

For example, the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is a Pfaffian function, and actually forms a Pfaffian chain, by the well-known identity $\frac{\partial \exp}{\partial x}(x) = \exp(x)$. On the other hand, it turns out that the sine and cosine functions are not Pfaffian.

The first result towards effectivity for sets definable by Pfaffian functions is due to Khovanskiĭ, and already contained in [Kho80]: it shows that Pfaffian functions have some finiteness properties similar to polynomials. More specifically, let $f_1, \ldots, f_l : \mathbb{R}^n \to \mathbb{R}$ be functions forming a Pfaffian chain, and let $Q_1, \ldots, Q_m$ be polynomials in $\mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_l]$. Then, the set $X \subseteq \mathbb{R}^n$ defined by the system

$$\begin{cases} Q_1(\boldsymbol{x}, f_1(\boldsymbol{x}), \ldots, f_l(\boldsymbol{x})) = 0 \\ \vdots \\ Q_m(\boldsymbol{x}, f_1(\boldsymbol{x}), \ldots, f_l(\boldsymbol{x})) = 0 \end{cases}$$

has a finite number of connected components, and the number of such components is effectively bounded in term of n, m, the degrees of the Q_is, and the format and the degree of the Pfaffian chain.

This result was actually crucial in proving that $\mathbb{R}_{\exp}$, the smallest structure containing the graph of the *unrestricted* real exponential function, is an o-minimal structure. This raised the question of whether the same could be true adding the graph of other unrestricted Pfaffian functions. Indeed, Wilkie [Wil99] proved that the smallest structure containing the graphs of all unrestricted Pfaffian functions, $\mathbb{R}_{\text{Pfaff}}$, is o-minimal. In particular, sets definable in $\mathbb{R}_{\text{Pfaff}}$ satisfy the counting theorems, Theorem 2.2.9 and Proposition 2.2.10. As anticipated, however, for these kind of sets we can prove something stronger: we can make the constants appearing effective.

Before stating the effective versions of Theorem 2.2.9 and Proposition 2.2.10, we recall some definitions.

A *semi-Pfaffian set* is a set $X \subseteq \mathbb{R}^n$, for some n, which is defined by a boolean combination of equations of the form $g = 0$ and inequalities of the form $h > 0$, where all the functions g and h involved are Pfaffian and defined on some common U as above. The *format* of X is the maximum of the formats of all the functions g and h appearing in the definition of X, and the *degree* of X is the sum of the degrees of all these g and h.

A *sub-Pfaffian set* is a set $Y \subseteq \mathbb{R}^k$, that is the image via projection on the first k coordinates of a semi-Pfaffian set $X \subseteq \mathbb{R}^n$, for some $n \geq k$. The *format* and *degree* of Y are defined to be those of X. Note that semi-Pfaffian sets are always sub-Pfaffian, but the converse is not true in general.

An effective analogue of Bombieri-Pila Theorem 2.2.8 has been proven by Pila [Pil06].

Theorem 2.2.12. *There is an explicit function $c(r, \alpha)$ with the following property. Suppose that X is the graph of a non-algebraic Pfaffian function $f : \mathbb{R} \to \mathbb{R}$ of format r and degree α restricted to a connected component of its domain. If $T \geq c(r, \alpha)$, then*

$$\big|X(\mathbb{Q}, T)\big| \leq \exp(5\sqrt{\log T}).$$

Later, Binyamini [Bin19] proved an effective constant for the Pila-Wilkie Counting Theorem 2.2.9 for sets definable using Noetherian functions, a larger class that Pfaffian function with a similar characterization using differential equations, but *restricted* to a compact domain. While this result does apply to restrictions of the j modular function and to the Weierstrass $\wp$ function, in actual applications it is often essential not to restrict the domain to a compact set, as the fundamental domains of those functions are not compact.

For *unrestricted* sub-Pfaffian set, an analogous effective point counting result was proven by Binyamini, Jones, Schmidt and Thomas [BJST23]. In the same paper, the authors also show an effective version of Proposition 2.2.10, generalizing also to points of arbitrary degree over $\mathbb{Q}$.

Before stating the theorem, we generalize the notation defined in (2.2.2). Suppose that $Z \subseteq \mathbb{R}^m \times \mathbb{R}^n$. For g a positive integer and $H \geq 1$ a real number

we put

$$Z^{\sim}(g,H) = \{(x,y) \in X : [\mathbb{Q}(x),\mathbb{Q}] \leq g, H(x) \leq H\}.$$

Theorem 2.2.13 ([BJST23, Corollary 4.6], see also end of Section 4 ibidem)**.** *Let ℓ, m and n be nonnegative integers, let g be a positive integer and let $\mathcal{F}, D$ and ε be positive real numbers. Suppose that $Z \subseteq \mathbb{R}^\ell \times \mathbb{R}^m \times \mathbb{R}^n$ is a definable family of format at most $\mathcal{F}$ and degree at most D. Let $\pi_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\pi_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be the natural projections.*

There exists a positive real number $c \leq (D+1)^{\gamma(\mathcal{F},g,\varepsilon)}$, where γ is an effectively computable function independent of Z, with the following property. Suppose that $a \in \mathbb{R}^\ell$, and that the real number $H \geq 1$ and set $\Sigma \subseteq Z_a^{\sim}(g,H)$ are such that

$$\big|\pi_2(\Sigma)\big| > cH^{\epsilon};$$

then, there exists a continuous definable $\beta : [0,1] \to Z_a$ with the following properties:

(i) the map $\pi_1 \circ \beta : [0,1] \to \mathbb{R}^m$ is semialgebraic;

(ii) the map $\pi_2 \circ \beta : [0,1] \to \mathbb{R}^n$ is not constant;

(iii) we have $\pi_2(\beta(0)) \in \pi_2(\Sigma)$;

(iv) the restriction $\beta|_{(0,1)}$ is analytic (and in particular, $\pi_1 \circ \beta|_{(0,1)}$ is analytic).

In our context, the families considered will be defined using the Weierstrass $\wp$ function. This function has been proven to be Pfaffian due to work of Jones and Schmidt [JS20, Thm. 1], where its degree and format is explicitly computed.

2.3 The Pila-Zannier Method

In this section we give a sketch of the general ideas and steps behind the so-called "Pila-Zannier" method. This was introduced in [PZ08] to provide an alternative proof for the Manin-Mumford Conjecture, and it has since been used and adapted to prove many special cases of the Zilber-Pink Conjecture, such as for curves in general abelian varieties [HP16], the Relative Manin-Mumford Conjecture [MZ08, MZ10, MZ12, MZ14, CMZ17, MZ20, GH23], curves in families of abelian varieties [BC16, Bar17, BC17, BC20], and the André-Oort Conjecture [Pil11, Ull13, Tsi18, PST$^+$22, BSY22]. It will also be used in this thesis, in Chapter 4 to prove Theorem 4.1.4.

We will partially follow the approach of [Pil22], and, for simplicity, we will limit ourself to the case of Manin-Mumford Conjecture 1.3.4 for abelian varieties. For algebraic tori and more general Shimura varieties the approach is similar; an in-depth treatment can be found in the already mentioned book by Pila.

Throughout this section, let G be an abelian variety of dimension $g \geq 2$, and let V be a subvariety of G, defined over a number field k. We are interested

in studying the Manin-Mumford Conjecture for V in G, that is, we want to prove that V contains only finitely many maximal special subvarieties, that is irreducible components of algebraic subgroups (see also the definition before Conjecture 1.3.3).

The Pila-Zannier method is adequate to prove the finiteness of a set of points. However, the Manin-Mumford Conjecture (as well as the Zilber-Pink Conjecture) deals with maximal special (or atypical) subvarieties, of which some could have positive dimension. Therefore, one needs to deal with the positive dimensional ones separately. This was also an issue of previous techniques, and gave rise (in the context of the Zilber-Pink Conjecture) to the notion of anomalous and torsion anomalous loci (see Definition 1.4.8), to the Torsion Openness Conjecture 1.4.9, and to the Structure Theorems for the anomalous locus (See for example [BMZ07, Theorem 1.4], [HP16, Proposition 6.1] and the discussion after Definition 6.4 ibidem, and [BKS23, Theorem 2.1]). In the nomenclature of [Pil22], this is the "Finiteness Ingredient". For the Manin-Mumford Conjecture, it is enough to consider the set V^o obtained by removing from V all of the positive dimensional special subvarieties. This subset was first defined in [BZ95], where it was also proved that it is an open subset of V. Thus, we only need to prove that $V^o \cap G^{tors}$ is a finite set.

Like in many previous proofs in the subject, one proves finiteness by applying the Northcott property (see Theorem 2.1.14). So one needs to prove a bound for the height and for the degree of the points in $V^o \cap G^{tors}$. The height bound in this case is trivial, since the torsion points of G have canonical height all equal to zero. For the more general Zilber-Pink Conjecture, giving a bound for the height is usually the first step (see, for example, Theorem 1.4.1 from [BMZ99], or Theorem 9.15 of [HP16]). Currently, such a bound is known for $V^{oa} \cap G^{[\dim V+1]}$, due to work of Habegger, in both the toric [Hab08] and the abelian case [Hab09]. Such a bound is not known in general for $V^{ta} \cap G^{[\dim V+1]}$, and this is one of the major missing points to establish the Zilber-Pink Conjecture for general subvarieties of tori and abelian varieties.

To get a bound for the degree, we introduce a certain notion of "complexity" $\Delta(P)$ for points P. To prove the Manin-Mumford Conjecture we only need to consider torsion points, and we define their complexity as their multiplicative order. In general, the complexity of P is related to the "smallest" possible coefficients of a group subvariety of G containing P. We then give two estimation for the complexity of a point $P \in V^o \cap G^{tors}$ in terms of its degree, one from above and one from below. Since the upper bound grows asymptotically slower than the lower bound, we thus conclude that the degree must be bounded.

For the upper bound, from a result of Masser [Mas84] (see also [PZ08, Section 3] and [HP16, Section 9.1]), for every $P \in V^o \cap G^{tors}$ one gets that

$$\Delta(P) \leq c_1[k(P) : k]^{c_2} \tag{2.3.1}$$

for some constants $c_1, c_2 > 0$ depending only on G and on V.

The main innovation due to Pila and Zannier is in the lower bound, in particular the use of o-minimal geometry and counting theorems derived from

it, for example the Pila-Wilkie Counting Theorem (Theorem 2.2.9, see also [PW06, Theorem 1.8]) or, for the more general case, its extension due to Habegger and Pila, Proposition 2.2.10 (see also [HP16, Corollary 7.2]).

The first step is to reformulate the problem in terms of a rational points counting problem. To do so, we consider (by slight abuse of notation) G as a complex Lie group and take the exponential map $\exp : T_0G \to G$ from the tangent space at the zero of the group law of G to G itself. Since G is a commutative group, the commutator on T_0G is trivial, hence exp is a group homomorphism, and the preimages of the algebraic subgroups of G are (countable unions of) affine subspaces of T_0G. Moreover, the kernel of exp is a lattice $\Lambda \subset T_0G$ of rank $2g$. We can now chose a base of Λ as a $\mathbb{Z}$-module, which is also a basis of T_0G as a real vector space. This choice induces an isomorphism of real vector spaces $T_0G \cong \mathbb{R}^{2g}$, sending Λ to $\mathbb{Z}^{2g}$. Composing this isomorphism with the exponential map, we get a group homomorphism $\pi : \mathbb{R}^{2g} \to G$, with kernel $\mathbb{Z}^{2g}$. In particular, the torsion points of G are exactly the images via this projection of the rational points of $\mathbb{R}^{2g}$. Moreover, if $\boldsymbol{q} \in \mathbb{Q}^{2g} \cap [0,1)^{2g}$, then the order of the point $\pi(\boldsymbol{q}) \in G^{tors}$, which is also equal to its complexity by definition, is exactly the height $H(\boldsymbol{q})$, where H is the absolute (multiplicative) affine Weil height on $\mathbb{Q}^{2g}$. Notice also that $[0,1)^{2g}$ contains a single representative in each fiber of π.

So, to study the set $V^o \cap G^{tors}$, we need to study the set of rational points in $\pi^{-1}(V^o) \cap [0,1)^{2g}$. This set is not, in general, an algebraic set. However, it turns out that $Z := \pi^{-1}(V) \cap [0,1)^{2g}$ is a set definable in in an o-minimal structure (see Section 2.2 for a definition), to be more specific $\mathbb{R}_{\text{an}}$. For these kind of sets, the structure of their rational points is better understood, due to the Pila-Wilkie Counting Theorem 2.2.9, giving a constant $c(Z,\varepsilon)$ such that

$$\left|\{\boldsymbol{q} \in (Z^{tran}) \cap \mathbb{Q}^n \mid H(\boldsymbol{q}) \leq T\}\right| \leq c(Z,\varepsilon)T^{\varepsilon}, \tag{2.3.2}$$

where $Z^{tran} = Z \setminus Z^{alg}$ is the transcendental part of Z, i.e. what remains of Z after removing all of its positive dimensional semiagebraic subsets.

Suppose now that P is a point in V that is torsion of (exact) order $\Delta(P) = N$, which we imagine to be large. So, by Galois conjugation, V contains at least $[k(P) : k]$ such points. From (2.3.1), we deduce that Z contains $\geq (c_1)^{1/c_2} N^{1/c_2}$ points of height $\leq N$.

However, applying (2.3.2) with $\varepsilon = 1/(2c_2)$, we get that Z^{tran} contains at most $c(Z, 1/(2c_2))N^{1/(2c_2)}$ points of height at most N. So, if N is large enough, we conclude that most of the conjugates of the point P must be contained in $\pi(Z^{alg})$. This leads us to the study of semialgebraic subsets of Z, or, equivalently, their image via π.

A nontrivial fact, proven by Pila and Zannier [PZ08, Proposition 2.1], is that the set Z^{alg} is the same as Z^{ca}, i.e. the union of all the complex algebraic subsets of Z viewed as a subset of the complex vector space T_0G. So the study of Z^{alg} becomes a problem of *functional* transcendence of the (abelian) exponential function $\exp : T_0G \to G$. More specifically, we ask this question: if $X \subset T_0G$ is a (semi)algebraic subset such that $\exp(X)$ is contained in a

proper algebraic subset of G, what can we say about X? Clearly, if X is a complex vector subspace of T_0G such that $\Lambda \cap X$ is a lattice of maximal rank in X (i.e. $\mathrm{rk}(\Lambda \cap X) = 2 \dim X$), then $\exp(X)$ is an abelian subgroup of G, by elementary Lie theory. The same holds for all the translates of one such X. A consequence of a theorem of Ax (see [HP16, Theorem 5.4], or [PZ08, Theorem 2.1]) is that this is essentially the only case possible. More specifically, the following theorem holds, often referred to as the "Ax-Lindemann-Weierstrass" Theorem [1].

Theorem 2.3.1 (Ax-Lindemann-Weierstrass)**.** *Let $X \subset T_0G$ be a semialgebraic subset and assume that $\exp(X)$ is contained in an algebraic subset $W \subseteq G$. Then, there exists an algebraic coset $g + H$ of G such that $g + H \subseteq W$ and $X \subseteq \exp^{-1}(g + H)$.*

Suppose now that we have a torsion point $P \in V$ of large order N, such that P has at least one conjugate in $\pi(Z^{alg})$. Without loss of generality, we may assume that P is one such conjugate. Denote by $\boldsymbol{q}$ the unique element in Z^{alg} such that $\pi(\boldsymbol{q}) = P$, and by $\tilde{P}$ its image in T_0G. Since $\boldsymbol{q} \in Z^{alg} = Z^{ca}$, there exists $X \subset Z$ positive dimensional and semialgebraic such that $\tilde{P} \in X$. In particular, $P \in \exp(X) \subseteq V$. By Ax-Lindemann-Weierstrass, there exists an algebraic coset $g + H$ of G such that $P \in \exp(X) \subseteq g + H \subseteq V$. Moreover, H has to be positive dimensional. Since $P \in g + H$, we have that $g + H = P + H$, and since P is a torsion point, all the irreducible components of $P + H$ are irreducible components of subgroups of G. In particular, $P \in g + H \subseteq V \setminus V^o$. Since V^o is closed under Galois conjugation, all the conjugates of P are also contained in $V \setminus V^o$.

So, we deduce that the torsion points contained in V^o have bounded order, and hence bounded degree. So, by Northcott property, they are finite.

[1] This is due to the similarity with the Lindemann-Weierstrass Theorem in the context of transcendental number theory.

3 - Singular intersections in split semiabelian varieties

3.1 Introduction

In Chapter 1 we saw how a dimensional heuristic for the intersection of subvarieties motivated the Zilber-Pink Conjecture 1.4.11. In a similar way, the expectation of transversality for the intersections led us to formulate Conjectures 1.7.1 and 1.7.2. In this chapter, comprising a joint work with Ballini and Capuano, we aim to prove Conjecture 1.7.1 under the additional hypothesis that everything is defined over $\overline{\mathbb{Q}}$. The main result that we obtain is the following.

Theorem 3.1.1. *Let $\mathcal{G}$ be a split semiabelian variety of dimension n and let $\mathcal{C} \subseteq \mathcal{G}$ be an irreducible curve defined over $\overline{\mathbb{Q}}$ and suppose that $\mathcal{C}$ is not contained in a proper algebraic subgroup. Consider the set $\mathcal{C}^{[1,sing]}$ of points $P \in \mathcal{C}$ such that there is a proper algebraic subgroup H such that:*

1. *P is in $\mathcal{C} \cap H$;*
2. *the tangent space $T_P\mathcal{C}$ of $\mathcal{C}$ at P is included in the tangent space T_PH of H at P (namely, the curve $\mathcal{C}$ is tangent to H at P).*

Then $\mathcal{C}^{[1,sing]}$ is a finite set.

As already mentioned in Section 1.7, in the special case of curves in a two-dimensional torus, this finiteness result was already known due to work of Marché and Maurin [MM23]. Furthermore, this statement is completely trivial if $n = 1$: in this case, $\mathcal{C} = \mathcal{G}$, and the only subgroups of codimension one are torsion points. However, the tangent space of points is trivial, so the set $\mathcal{C}^{[1,sing]}$ is contained in the set of non-smooth points of $\mathcal{C}$, which is finite.

Before giving a sketch of the proof, we add a couple of remarks. First, similarly to the Zilber-Pink case, it is necessary to assume that $\mathcal{C}$ is not contained in a proper algebraic subgroup: if $\mathcal{C}$ was contained in a proper subgroup, say H, any point $P \in \mathcal{C} \cap H$ would satisfy the two conditions above. Second, the tangency condition is actually needed to have finiteness: the set of all the intersections between the curve and algebraic subgroups of codimension one is usually infinite (see [BMZ99] for an example in the toric case). At the same

time, under the more restrictive hypothesis that the curve is not contained in a translate of a proper algebraic subgroup, the set is still "sparse", i.e. is a set of bounded height (see part 1 of Theorems 1.4.1 and 1.5.1, from [BMZ99, Via03]).

Throughout this chapter, we will be treating at the same time abelian and toric situations; it is then useful to have some notation that is uniform in the two cases. To that end, we will from now on refer to "independent" elements to mean either *multiplicatively* or *linearly* independent, based on whether we are working over algebraic tori (usually $\mathbb{G}_{\mathrm{m}}^n$) or abelian varieties (usually powers of elliptic curves).

To prove Theorem 3.1.1, since the statement is invariant under isogenies (see Lemma 3.2.1), by Poincaré Reducibility Theorem, we can assume that $\mathcal{G}$ is the power of a torus and pairwise non-isogenous simple abelian varieties. By [BKS23, Theorem 1.1], the set of points of intersection between the curve $\mathcal{C}$ and the algebraic subgroups of codimension bigger or equal to two, without considering the tangency condition, is finite; hence, we only need to consider the intersections of $\mathcal{C}$ with algebraic subgroups of codimension one. Since we are intersecting with subgroups of codimension one, we can reduce ourselves to the cases in which $\mathcal{G} = G^n$ is a power of either $G = \mathbb{G}_{\mathrm{m}}$ or of an elliptic curve G defined over $\overline{\mathbb{Q}}$.

Since everything is defined over $\overline{\mathbb{Q}}$, the points that lie in the intersections between the curve and proper algebraic subgroups are algebraic; hence, to prove finiteness, it is enough by Northcott property to prove a bound on the height and on the degree of the points. To do this, we first prove Theorem 3.1.1 with the additional condition that the curve $\mathcal{C}$ is not contained in any algebraic coset of $\mathcal{G}$. Under this assumption, we are able to get a bound for the degree of the points of singular intersection between the curve and all proper cosets (see Proposition 3.5.1) using the Gauss map and Bezout's Theorem. Moreover, results of Bombieri, Masser and Zannier [BMZ99] for the toric case and Viada [Via03] for the elliptic case give the bound for the height, which allows us to conclude.

To prove the general case, we can assume that the curve $\mathcal{C}$ is contained in a proper coset W, but not in a proper subgroup. Moreover, we can assume W to be maximal and, after performing a suitable isogeny, that it is of the form $G^d \times \{\boldsymbol{\xi}\}$, for a tuple $\boldsymbol{\xi} \in G^{n-d}$ of independent elements. By projecting on the first d components, we can turn our starting problem into studying the points of singular intersection between a curve not contained in any proper coset and one-codimensional algebraic cosets of a special shape, which we will call $\boldsymbol{\xi}$-*special subvarieties*. This reduction is analogous to the ones performed in [RV03, Pin05b, Mau08]. The same bound for the degree from the previous case applies to these points as well, so to prove finiteness, we only need to get a bound for the height, as the ones from Bombieri, Masser and Zannier and Viada do not apply here. To prove the desired bound, we first use the Gauss map and some properties of the height with respect to morphisms between projective varieties to control the logarithmic height of the points of singular intersection between the curve $\mathcal{C}$ and a $\boldsymbol{\xi}$-special subvariety U in terms of the

logarithmic height of the coefficients of U. A lattice generator bound due to Masser [Mas88] ensures that, for every singular intersection point P, we can choose a U with coefficients that are small with respect to the degree of P and the logarithmic height of P. Combining these two bounds with the degree bound leads to the desired height bound, thus concluding the proof.

3.2 Reduction steps

In this section, similar to part of Section 4.2, we reduce the proof of Theorem 3.1.1 to the case of powers of an elliptic curve or split algebraic tori. To do so, we start by showing that our problem is invariant under isogeny.

Lemma 3.2.1. *Let $\mathcal{G}$ and $\mathcal{G}'$ be split semiabelian varieties, and let $f : \mathcal{G}' \to \mathcal{G}$ be an isogeny. Moreover, let $\mathcal{C} \subseteq \mathcal{G}$ be a curve satisfying the assumptions of Theorem 3.1.1. Then, if the claim of Theorem 3.1.1 holds for all irreducible components of $f^{-1}(\mathcal{C})$, it holds for $\mathcal{C}$.*

Proof. The proof follows from the fact that every isogeny over fields of charachteristic 0 is (analytically) a local isomorphism. Hence, they preserve the tangency conditions required by the statement of Theorem 3.1.1. □

Using Poincaré Reducibility Theorem (see [HS00, Corollary A.5.1.8]), we can then assume that the split semiabelian variety $\mathcal{G}$ of Theorem 3.1.1 is of the form

$$\mathcal{G} = \mathbb{G}_{\mathrm{m}}^k \times \prod_{i=1}^{l} A_i^{n_i},$$

where the A_i are pairwise non-isogenous, simple abelian varieties. For ease of notation, we will adopt the index 0 for the toric component, and we will write $A_0^{n_0}$ instead of $\mathbb{G}_{\mathrm{m}}^k$. We recall that (see, for example, [MW85, Section 5]) all the algebraic subgroups of this split semiabelian variety $\mathcal{G}$ are contained in subgroups of the same dimension and of the form

$$H = H_0 \times H_1 \times \cdots \times H_l,$$

where H_0 is a subgroup of $A_0^{n_0} = \mathbb{G}_{\mathrm{m}}^k$ and H_i is a subgroup of $A_i^{n_i}$. Moreover, all subgroups of codimension at least one will be contained in a subgroup of this form where $H_i = A_i^{n_i}$ for all but one index i.

By Theorem 1.1 of [BKS23], the points of intersection between the curve $\mathcal{C}$ and the algebraic subgroups of codimension bigger or equal to 2, without considering the tangency condition, is finite. Hence, to prove Theorem 3.1.1, we only need to intersect the curve $\mathcal{C}$ with the subgroups of codimension one, so we can assume that they are of the form

$$H = A_0 \times \cdots \times A_{i_\star - 1} \times H_{i_\star} \times A_{i_\star + 1} \times \cdots \times A_l, \tag{3.2.1}$$

where $H_{i_\star} < A_{i_\star}^{n_{i_\star}}$ is a subgroup of codimension 1. This immediately implies that $A_{i_\star}$ is either $\mathbb{G}_{\mathrm{m}}$ (i.e. $i_\star = 0$) or an elliptic curve, and that $n_{i_\star}$ is at least

1. To prove Theorem 3.1.1 it is enough to show that, for every $i_\star$, the set of points of singular intersection between $\mathcal{C}$ and subgroups as in (3.2.1) is finite.

Suppose from now on that $i_\star$ is fixed and that $A_{i_\star}$ is either $\mathbb{G}_{\mathrm{m}}$ (i.e. $i_\star = 0$) or an elliptic curve. Let $\pi : \mathcal{G} \to A_{i_\star}^{n_{i_\star}}$ be the projection map and consider the image $\pi(\mathcal{C}) \subseteq A_{i_\star}^{n_{i_\star}}$.

It may happen that $\pi(\mathcal{C})$ is just a point. If this happens, then such a point cannot be contained in a proper subgroup of $A_{i_\star}^{n_{i_\star}}$, since otherwise the curve $\mathcal{C}$ would be contained in a proper subgroup of $\mathcal{G}$, contradicting the assumptions of Theorem 3.1.1. On the other hand, if P is a point of intersection between $\mathcal{C}$ and a subgroup as in (3.2.1), then $\pi(P) \in H_{i_\star}$. Therefore, we conclude that, under the assumption of Theorem 3.1.1, if $\pi(\mathcal{C})$ is a point, then $\mathcal{C}$ does not intersect subgroups as in (3.2.1).

Suppose then from now on that $\pi(\mathcal{C})$ is a curve and denote by $\tilde{\pi}$ the restriction of π to $\mathcal{C}$. Using an argument similar to the previous paragraph, under the assumptions of Theorem 3.1.1, the curve $\pi(\mathcal{C})$ is not contained in any proper subgroup of $A_{i_\star}^{n_{i_\star}}$. Now, let H be as in (3.2.1) and let $P \in \mathcal{C} \cap H$ be a singular intersection point. We will show that such points are finite. Since $\tilde{\pi}$ is a finite map between curves, it is enough to show that there are only finitely many possible values for $\pi(P)$.

Notice that $H_{i_\star} = \pi(H)$ and $\pi(P) \in \pi(\mathcal{C}) \cap H_{i_\star}$. We have two cases:

1. The differential $d\tilde{\pi}_P$ at P is the trivial map. This is the same as saying that P is a ramification point for the map $\tilde{\pi}_P$. Since this is a non-constant map between curves, it admits only finitely many ramification points.

2. The differential $d\tilde{\pi}_P$ at P is an isomorphism. Hence, the image of $d\tilde{\pi}$, that is, $d\pi_P(T_P\mathcal{C})$, is exactly equal to the tangent space of $\pi(\mathcal{C})$ at $\pi(P)$. Moreover, since by assumption $T_P\mathcal{C} \subseteq T_P H$, we have that $d\pi_P(T_P\mathcal{C}) \subseteq d\pi_P(T_P H) = T_P H_{i_\star}$. In other terms, $\pi(P)$ is a point of singular intersection between $\pi(\mathcal{C})$ and $H_{i_\star}$. The finiteness of these points is ensured by the following result.

Theorem 3.2.2. *Let G be $\mathbb{G}_{\mathrm{m}}$ or E for some elliptic curve E defined over $\overline{\mathbb{Q}}$. Let n be a positive integer, $\mathcal{G} = G^n$, and let $\mathcal{C} \subseteq \mathcal{G}$ be a curve defined over $\overline{\mathbb{Q}}$ and suppose that $\mathcal{C}$ is not contained in a proper special subvariety. Consider the set $\mathcal{C}^{[1,sing]}$ of points $P \in \mathcal{C}$ such that there is a one-codimensional special subvariety H such that:*

1. *P is in $\mathcal{C} \cap H$;*

2. *the tangent space $T_P\mathcal{C}$ of $\mathcal{C}$ at P is included in the tangent space $T_P H$ of H at P (namely, the curve $\mathcal{C}$ is tangent to H at P).*

Then, $\mathcal{C}^{[1,sing]}$ is a finite set.

3.3 Structure of subgroups and cosets

3.3.1 Multiplicative case

Let $G = \mathbb{G}_{\mathrm{m}}$ and let n be a fixed positive integer. We want to characterize the groups and the cosets of $\mathbb{G}_{\mathrm{m}}^n$. This topic is fairly classical, and can be found, for example, in [Zan14a, Section 4.2.3]. We are recalling it here mostly to establish some notation and definitions.

Recall that $\mathrm{End}(\mathbb{G}_{\mathrm{m}}) = \mathbb{Z}$, so the group homomorphisms $\varphi : \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^d$ correspond to matrices in $\mathrm{Mat}_{d\times n}(\mathbb{Z})$, as follows: given a matrix $A = (a_{i,j})$, it induces a homomorphism

$$\varphi_A : (x_1, \dots, x_n) \mapsto \Big((x_1^{a_{1,1}} \cdots x_n^{a_{1,n}}), \dots, (x_1^{a_{d,1}} \cdots x_n^{a_{d,n}}) \Big).$$

It is easy to check that this correspondence is compatible with compositions, that is, if $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ and $B \in \mathrm{Mat}_{d\times m}(\mathbb{Z})$ then $\varphi_B \circ \varphi_A = \varphi_{BA}$.

Proposition 3.3.1.

1. *All the algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n$ of codimension d are of the form $\ker(\varphi_A)$ for some $A \in \mathrm{Mat}_{d\times n}(Z)$ of maximal rank.*

2. *Let $H \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible component of an algebraic subgroup. Then, $H = \omega T$, where T is an irreducible algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n$ and ω is a torsion point.*

3. *Let $W \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible component of an algebraic coset. Then, $W = cT$, where T is an irreducible algebraic subgroup and c is any point in $\mathbb{G}_{\mathrm{m}}^n$.*

This proposition motivates the following definitions.

Definition 3.3.2. We say that $H \subseteq \mathbb{G}_{\mathrm{m}}^n$ is a *special subvariety* or *torsion coset* if it is an irreducible component of an algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n$; we say that $W \subseteq \mathbb{G}_{\mathrm{m}}^n$ is a *weakly special subvariety* if it is an irreducible component of an algebraic coset. Note that weakly special subvarieties are exactly the irreducible algebraic cosets.

Suppose that H is a special subvariety of $\mathbb{G}_{\mathrm{m}}^n$ of codimension d. By the previous proposition, H is an irreducible component of a subgroup $\tilde{H}$, which in turn can be written as $\ker(\varphi_A)$ for some matrix $A \in \mathrm{Mat}_{d\times n}(\mathbb{Z})$ of maximal rank. By performing a base change in both the starting and the arrival spaces, we can turn A into a matrix of the form $(B \mid 0)$, where B is a $d \times d$ diagonal matrix. In this basis, $\tilde{H}$ is then cut by the equations ${x_1}^{b_1} = 1; {x_2}^{b_2} = 1; \dots; {x_d}^{b_d} = 1$. So, H is described by

$$\begin{cases} x_1 &= \omega_1 \\ &\vdots \\ x_d &= \omega_d \end{cases}$$

where ω_i is a b_i-th root of unity, for $i = 1, \dots, d$.

When $d = 1$, i.e. in the case of hypersurfaces, $A \in \mathrm{Mat}_{1\times n}(\mathbb{Z})$ is just a row of integers; we will denote them by $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$. We will adopt for the rest of this paper the following notation:

$$H_{\boldsymbol{a},c} := \varphi_{\boldsymbol{a}}^{-1}(\{c\}),$$

whenever $\boldsymbol{a}$ is a vector of integers and $c \in \mathbb{G}_{\mathrm{m}}$. If $\boldsymbol{a}$ is primitive, that is, the gcd of its elements is 1, then $H_{\boldsymbol{a},c}$ is irreducible, and therefore it is a weakly special subvariety.

3.3.2 Elliptic case

Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$ and let n be a fixed positive integer. We want to study the algebraic subgroups and cosets of E^n. This section is also fairly classical, and most of the results can be found, for example, in [MW83, Section 3.3] and [Via03].

Recall that $\mathrm{End}(E)$ is either $\mathbb{Z}$ or (isomorphic to) an order in a quadratic imaginary field. Similarly to the $\mathbb{G}_{\mathrm{m}}$ case, we have a correspondence between group homomorphisms $\varphi : E^n \to E^d$ and matrices in $\mathrm{Mat}_{d\times n}(\mathrm{End}(E))$, compatible with compositions, which we will also denote by φ.

Proposition 3.3.3.

1. *All algebraic subgroups of E^n of codimension d are* contained *in subgroups of the same codimension of the form* $\ker(\varphi_A)$ *for some $A \in \mathrm{Mat}_{d\times n}(\mathrm{End}(E))$ of maximal rank.*

2. *Let $H \subseteq E^n$ be an irreducible component of an algebraic subgroup. Then, $H = \omega + A$, where A is an irreducible abelian subvariety and ω is a torsion point.*

3. *Let $W \subseteq E^n$ be an irreducible component of an algebraic coset. Then, $W = c + A$, where A is an irreducible abelian subvariety and c is any point in E^n.*

Similarly to the $\mathbb{G}_{\mathrm{m}}^n$ case, we introduce the following definitions.

Definition 3.3.4. We say that $H \subseteq E^n$ is a *special subvariety* or *torsion coset* if it is an irreducible component of algebraic subgroup of E^n; we say that $W \subseteq E^n$ is a *weakly special subvariety* if it is an irreducible component of an algebraic coset. Note again that weakly special subvarieties are exactly the irreducible algebraic cosets.

Recall that any abelian subvariety $A < E^n$ is contained in a codimension one abelian subvariety defined by $a_1P_1 + \cdots + a_nP_n = O$, where the P_i's denote the projections on the i-th factor, O is the zero element of E, and the a_i's are

elements of the endomorphism ring of E. We will adopt the following notation: we write $H_{\boldsymbol{a},c}$ for the subvariety of E^n cut by

$$a_1P_1 + \cdots + a_nP_n = c,$$

for $\boldsymbol{a} = (a_1, \ldots, a_n) \in (\mathrm{End}(E))^n$ and $c \in E$. In particular, every every coset of E^n will be contained in a coset of this shape. Unlike the $\mathbb{G}_{\mathrm{m}}$ case, we can only guarantee containment, as some irreducible subvarieties of E^n may not be expressible in the form $\ker(\varphi_A)$ in those cases where $\mathrm{End}(E)$ is not a PID. For example, let E_5 be a CM elliptic curve with $\mathrm{End}(E_5) = \mathbb{Z}[\sqrt{-5}]$, and consider in $E_5 \times E_5$ the subgroup H defined by

$$\begin{cases} 2P_1 + (1+\sqrt{5})P_2 = 0 \\ (1-\sqrt{5})P_1 + 3P_2 = 0 \end{cases}.$$

Then, H is an irreducible subgroup of codimension one that cannot be described by a single equation. Moreover, it is isogenous but not isomorphic to E_5.

We notice that, for any elliptic curve E, if we identify the elements of $\mathrm{End}(E)$ with the elements of $\mathbb{Z}$ or with the elements of the appropriate order in a quadratic field in the CM case, the abelian subvariety given by the connected component of the identity of $H_{\boldsymbol{p}} = H_{\boldsymbol{p},O}$ is precisely the image via the exponential map of the linear subspace cut by the equation $p_1z_1 + \cdots + p_nz_n = 0$ in the tangent space of E^n at the origin.

Let now $A < E^n$ be an irreducible abelian subvariety of any dimension. By Poincaré Reducibility Theorem (see for example [HS00, Corollary A.5.1.8]), there exist an abelian variety A' over $\overline{\mathbb{Q}}$ and an isogeny $\varphi : A' \times A \to E^n$, both defined over $\overline{\mathbb{Q}}$. By unique factorization, we have that A' is isogenous to E^d and A is isogenous to $E^{(n-d)}$, where $d = \operatorname{codim} A$. Hence, if $A < E^n$ is an irreducible abelian subvariety, up to an isogeny, we can assume that it is the kernel of the first d coordinates. Similarly, if $H \subseteq E^n$ is a special subvariety, we can assume, up to isogeny, that

$$H = \underbrace{E \times \cdots \times E}_{n-d} \times \{\omega_1\} \times \cdots \times \{\omega_d\},$$

with $\omega_1, \ldots, \omega_d$ torsion elements of E and $d = \operatorname{codim} H$, and, if $W \subseteq E^n$ is a weakly special subvariety, we can assume, again up to isogeny, that

$$W = \underbrace{E \times \cdots \times E}_{n-d} \times \{c_1\} \times \cdots \times \{c_d\},$$

with $c_1, \ldots, c_d$ points in $E^d(\overline{\mathbb{Q}})$.

Note that this is not in contrast with the existence of subgroups that are not cut by system of equations of maximal rank: indeed, to write subgroups in that form we need first to perform an isogeny *which depends on the subgroup.*

3.4 The Gauss Map

In this section, we will introduce a technical tool, the Gauss map, that will be crucial in the later proofs. Although we will only need it for curves in $\mathbb{G}_{\mathrm{m}}^n$ or in powers of elliptic curves, we introduce it in a much more general context, that of Lie groups.

The term "Gauss map" is used in general to denote a map associating to every point of a subvariety V of an ambient space X the tangent space $T_P V$. It was first considered by Gauss for orientable surfaces in $\mathbb{R}^3$, mapping to the points of the unit sphere corresponding to the normal directions. He used such a map to study the curvature of surfaces, and ultimately prove the Theorema Egregium in 1827.

Let $\mathcal{L}$ be a Lie group. The group structure allows us to identify the tangent spaces at each point, using the differential of the translation maps. Hence, we get a trivialization of the tangent bundle $T\mathcal{L} \cong \mathcal{L} \times T_e\mathcal{L}$, where $T_e\mathcal{L}$ is the tangent space at the identity. Suppose that $V \subseteq \mathcal{L}$ is a subvariety, and let V^{sm} be the open dense locus of smooth points. Then, for each $P \in V^{sm}$ the tangent space $T_P V$ is well-defined subspace of $T_P\mathcal{L}$ of dimension equal to the dimension of V. Using the above-mentioned trivialization of the tangent bundle of $\mathcal{L}$, we can also identify $T_P V$ with a subspace of $T_e\mathcal{L}$. Hence we get a map

$$\gamma_{\mathcal{L},V} : V^{sm} \to \mathrm{Gr}(\dim V, T_e\mathcal{L}),$$

where $\mathrm{Gr}(\dim V, T_e\mathcal{L})$ is the Grassmanian of $(\dim V)$-dimensional vector subspaces of $T_e\mathcal{L}$. We will often suppress the dependence on the Lie group $\mathcal{L}$, as such datum will be clear from the context. This map has been studied previously, see for example [GH79, Section 4], and is a key ingredient in Raynaud's proof [Ray83a] of the Manin-Mumford Conjecture 1.3.2.

In the case when $V = \mathcal{C}$ is a curve, the Grassmanian $\mathrm{Gr}(\dim V, T_e\mathcal{L})$ is just the projective space $\mathbb{P}^{\dim \mathcal{L}-1}$.

Remark 3.4.1. By definition, we have $\gamma_{\mathcal{C}}(P) = \gamma_{\tau_P^{-1}(\mathcal{C})}(e) = [T_e(\tau_P^{-1}(\mathcal{C}))]$, where τ_P is the left translation by P.

Whenever $\mathcal{L}$ is the analytification of an abelian variety or of $\mathbb{G}_{\mathrm{m}}^n$, then the Gauss map defined above is actually an algebraic map. For abelian varieties, this was proven by Raynaud in [Ray83a]; for $\mathbb{G}_{\mathrm{m}}^n$ we check it by explicitly describing the map.

Let $\mathcal{L} = (\mathbb{G}_{\mathrm{m}}^n)^{an}$. The Gauss map can be written more explicitly as follows. Let $x_1, \ldots, x_n$ denote the coordinates of $\mathbb{G}_{\mathrm{m}}^n$ (one coordinate for each factor); we also see these as coordinate functions on $\mathcal{C}$ and we have the corresponding differentials $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. Endowing $T_1\mathbb{G}_{\mathrm{m}}^n$ with the basis $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, we then have $\gamma_{\mathcal{C}} : \mathcal{C}^{sm} \to \mathrm{Gr}(1, T_1\mathbb{G}_{\mathrm{m}}^n) = \mathbb{P}^{n-1}$ defined by

$$\gamma_{\mathcal{C}} : P = (x_1, \ldots, x_n) \mapsto \left[\frac{\mathrm{d}x_1(t_P)}{x_1} : \cdots : \frac{\mathrm{d}x_n(t_P)}{x_n}\right],$$

where t_P is any non-zero element in $T_P\mathcal{C}$. Since all non-zero elements of $T_P\mathcal{C}$ for P a smooth point of $\mathcal{C}$ are proportional, this is well defined. We notice that this map is just the logarithmic derivative, and therefore we expect multiplicative relations to translate into linear relations. We will prove such a statement, again in a more general setting.

Proposition 3.4.2. *Let $\mathcal{L}$ be an abelian Lie group, $\mathfrak{l} = T_e\mathcal{L}$ the tangent space at the identity, and $\exp : \mathfrak{l} \to \mathcal{L}$ the associated exponential map. Let V be a linear subspace of $\mathfrak{l}$, let $\mathcal{C} \subseteq \mathcal{L}$ be a curve, and let $H = \exp(V)$. Then, for every $P \in \mathcal{C}$ we have that $\mathcal{C}$ is tangent to $P + H$ at P if and only if $\gamma_{\mathcal{C}}(P) \in V$.*

Proof. Notice that, since $\mathcal{L}$ is a commutative Lie group, the exponential map is a group homomorphism, hence H is a subgroup of $\mathcal{L}$. By definition, $\mathcal{C}$ is tangent to $P+H$ in P if and only if $T_P\mathcal{C} \subseteq T_P(P+H)$, when seen as subspaces of $T_P\mathcal{L}$. Since $\mathrm{d}\tau_P$ is an isomorphism, this is the same as $T_e(\tau_P^{-1}(\mathcal{C})) \subseteq T_e(\tau_P^{-1}(P+H)) = T_eH = V$. The claim follows from the fact that $[T_e(\tau_P^{-1}(\mathcal{C}))] = \gamma_P(\mathcal{C})$. □

In the special case in which $\mathcal{L} = \mathcal{G}$ is a power of $\mathbb{G}_{\mathrm{m}}^n$ or of an elliptic curve, we get the following.

Corollary 3.4.3. *Let $\mathcal{C}$ be a curve in $\mathcal{G} = \mathbb{G}_{\mathrm{m}}^n$ or $\mathcal{G} = E^n$ for some elliptic curve E, all defined over a number field K, and let $W = H_{\boldsymbol{p},c}$ be a one-codimensional coset. Let P be a point of $\mathcal{C} \cap W$ and assume that $\mathcal{C}$ is tangent to W at P. Then $\gamma_{\mathcal{C}}(P)$ is contained in the hyperplane $L \subseteq \mathbb{P}^{n-1} = \mathbb{P}T_e\mathcal{G}$ given by the equation:*

$$p_1 z_1 + \cdots + p_n z_n = 0,$$

where $z_i = \frac{\partial}{\partial x_i}$ are the (usual) coordinates of $\mathbb{P}T_e\mathcal{G}$.

3.5 Proof for $\mathcal{C}$ not contained in a coset

We recall the notations from Theorem 3.2.2. Let G be $\mathbb{G}_{\mathrm{m}}$ or E for some elliptic curve E defined over $\overline{\mathbb{Q}}$, let n be a positive integer, $\mathcal{G} = G^n$ and let $\mathcal{C} \subseteq \mathcal{G}$ be an irreducible curve defined over $\overline{\mathbb{Q}}$ that is not contained in any proper algebraic subgroup. Let us fix a number field K over which both $\mathcal{G}$ and $\mathcal{C}$ are defined.

Let us further assume in this section that $\mathcal{C}$ is not contained in any proper algebraic coset of $\mathcal{G}$. Under this assumption, we will show a bound for the degree of the field of definition of a point $P \in \mathcal{C} \cap W$, where W is a one-codimensional weakly special subvariety of $\mathcal{G}$ and $\mathcal{C}$ is tangent to P at W; this bound will be independent of W. The statement of the theorem will follow combining this degree bound with height bounds for points in a curve not contained in a weakly special subaviety lying in a proper algebraic subgroup proven by Bombieri, Masser and Zannier (part 1 of Theorem 1.4.1, see also [BMZ99]) in the case of tori and by Viada (part 1 of Theorem 1.5.1, see also [Via03]) in the ellptic curve case.

We remark that this height bound does not hold anymore if the curve is contained in a weakly special subvariety, as shown in [BMZ99] and [Via03]; so this last hypothesis is crucial to perform the argument.

Proposition 3.5.1. *Let $G = \mathbb{G}_{\mathrm{m}}$ or E for some elliptic curve E defined over a number field K, and $\mathcal{G} = G^n$. Let $\mathcal{C} \subseteq \mathcal{G}$ be a curve, also defined over K, and assume that is not contained in any proper algebraic coset. There is a constant $\delta_1 > 0$, depending only on $\mathcal{C}$ (and $\mathcal{G}$), such that, for every one-codimensional coset W and every point $P \in \mathcal{C} \cap W$ with $\mathcal{C}$ tangent to W at P:*

$$[K(P) : K] \leq \delta_1.$$

Proof. Up to removing a finite number of points, we may assume $\mathcal{C}$ to be a smooth curve, so that $\gamma_{\mathcal{C}}$ is defined everywhere. Up to enlarging W, we may assume that $W = H_{\boldsymbol{p},c}$ for some $\boldsymbol{p} \in \mathrm{End}(G)^n$ and $c \in G$. We will denote by L the hyperplane cut by $p_1 z_1 + \cdots + p_n z_n = 0$ in $\mathbb{P}T_0\mathcal{G} = \mathbb{P}^{n-1}$.

Notice that the image $\gamma_{\mathcal{C}}(\mathcal{C})$ cannot be contained in L. Indeed, consider the map $\varphi_{\boldsymbol{p}} : \mathcal{G} \to G$ given by

$$\varphi_{\boldsymbol{p}} : (x_1, \ldots, x_n) \mapsto x_1^{p_1} \cdots x_n^{p_n}$$

(resp. $p_1 x_1 + \cdots + p_n x_n$) and its differential $\mathrm{d}\varphi$. If by contradiction $\gamma_{\mathcal{C}}(\mathcal{C}) \subseteq L$, we have that $\mathrm{d}\varphi$ is constantly zero on $\mathcal{C}$, and this means that $\varphi|_{\mathcal{C}}$ is constant. Hence $\mathcal{C}$ is contained in the weakly special subvariety $H_{\boldsymbol{p},c'}$, for some $c' \in G$, which is in contradiction with the assumption that $\mathcal{C}$ is not contained into any proper coset.

Now, let $P \in \mathcal{C} \cap W$ be a point such that $\mathcal{C}$ is tangent to W at P. Then $\gamma_{\mathcal{C}}(P)$ is in $L \cap \gamma_{\mathcal{C}}(\mathcal{C})$ and the latter is a finite set, as $\gamma_{\mathcal{C}}(\mathcal{C})$ is not contained in L. We have two possibilities:

1. $\gamma_{\mathcal{C}}(\mathcal{C})$ is a point: this means that $\gamma_{\mathcal{C}}(\mathcal{C})$ is exactly $\{\gamma_{\mathcal{C}}(P)\}$ and therefore $\gamma_{\mathcal{C}}(\mathcal{C})$ is contained in L; as already remarked, this cannot happen under our assumptions.

2. $\gamma_{\mathcal{C}}(\mathcal{C})$ is a curve: observe that in this case there is an upper bound on the cardinality $\left|L \cap \gamma_{\mathcal{C}}(\mathcal{C})\right|$ which does not depend on L; such an upper bound is the degree d of $\gamma_{\mathcal{C}}(\mathcal{C})$ with respect to the line bundle $O(1)$ (and the following holds: any hyperplane intersects $\gamma_{\mathcal{C}}(\mathcal{C})$ in at most d points and a generic hyperplane intersects $\gamma_{\mathcal{C}}(\mathcal{C})$ in exactly d points).

   As L is defined over $\mathbb{Q}$ (or over a quadratic field in the CM case, which we may assume to be contained into K) and $\gamma_{\mathcal{C}}(\mathcal{C})$ is defined over K, this means that:
   $$[K(\gamma_{\mathcal{C}}(P)) : K] \leq \left|L \cap \gamma_{\mathcal{C}}(\mathcal{C})\right| \leq d.$$

   Hence $[K(P) : K] \leq d \cdot \deg \gamma_{\mathcal{C}}$, where $\deg \gamma_{\mathcal{C}}$ is the degree of the morphism of curves given by $\gamma_{\mathcal{C}} : \mathcal{C} \to \gamma_{\mathcal{C}}(\mathcal{C})$.

□

By Northcott's Theorem, to prove Theorem 3.2.2 for $\mathcal{C}$ not contained in any proper algebraic coset we only need to have a height bound for the points $P \in \mathcal{C}$ where $\mathcal{C}$ intersects tangentially a proper algebraic coset W. Such bounds are already known by previous work of Bombieri, Masser and Zannier (part 1 of Theorem 1.4.1, see also [BMZ99]) for $G = \mathbb{G}_{\mathrm{m}}$ and by Viada (part 1 of Theorem 1.5.1, see also [Via03]) for $G = E$ an ellptic curve, who proved that the set of points $P \in \mathcal{C}$ such that there exists a one-codimensional special subvariety H with $P \in \mathcal{C} \cap H$ is a set of bounded height.

We finally remark that, differently from the usual proofs in Unlikely Intersections (e.g. [BMZ99, Via03, PZ08, HP16], and all those proven using the Pila-Zannier method outlined in Section 2.3), the bound on the degree given by Proposition 3.5.1 is independent of the bound on the height. This is similar to what Marché and Maurin get in the special case of $\mathcal{G} = \mathbb{G}_{\mathrm{m}}^2$ in [MM23], and will be crucial in the proof of the general case.

3.6 Proof in the general case

The proof of the result under the more general assumption that the curve is not contained in a special subvariety is more difficult to handle; indeed, we do not have that the height of the points in our set is automatically bounded applying the results of Bombieri, Masser and Zannier (part 1 of Theorem 1.4.1, see also [BMZ99]) for $G = \mathbb{G}_{\mathrm{m}}$ and by Viada (part 1 of Theorem 1.5.1, see also [Via03]) for $G = E$ an ellptic curve. Since we already proved the result in the case when the curve $\mathcal{C}$ is not contained in any weakly special subvariety, we can now assume that this is the case. We will now translate our problem (up to isogenies) in an analogous one: namely, we will show that there is a 1-1 correspondence between points of $\mathcal{C}$ which intersect tangentially a special subvariety and points of another curve $\mathcal{C}'$, lying in some G^d with $d \leq n$ and not contained in a weakly special subvariety which intersect tangentially a weakly special subvariety of G^d of a special type. Using this reduction, Theorem 3.2.2 will be a consequence of the (more general) Theorem 3.6.3, which is analogous to [RV03, Theorem 1.6],[Pin05b, Conjecture 5.2] and [Mau08, Théorème 1.6] (see also [BMZ06]).

We recall our setting from Theorem 3.2.2. Let $G = \mathbb{G}_{\mathrm{m}}$ or E for some elliptic curve E defined over a number field K; let n be a positive integer, $\mathcal{G} = G^n$ and let $\mathcal{C} \subseteq \mathcal{G}$ be a curve defined over K and not contained in a proper special subvariety. We want to prove that the set $\mathcal{C}^{[1,sing]}$ of points of $\mathcal{C}$ where $\mathcal{C}$ intersects tangentially a proper special subvariety of $\mathcal{G}$ is a finite set.

In the last section, we proved this under the assumption that $\mathcal{C}$ was contained in no weakly special subvariety. Now, we consider the case in which $\mathcal{C}$ is actually contained in a proper weakly special subvariety W (not special), which we suppose to have minimal dimension. As shown in Section 3.3, up to an isogeny of $\mathcal{G}$ we can assume that $W = G^d \times \{\xi_1\} \times \cdots \times \{\xi_k\}$ with $d + k = n$.

Since $\mathcal{C}$ is contained in W and $\mathcal{C}$ is defined over K, then also $\xi_1, \ldots, \xi_k$ will

be defined over K; moreover, these points will be (multiplicatively or linearly) independent in G^k, since otherwise $\mathcal{C}$ would be contained in a proper special subvariety of $\mathcal{G}$. We put $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k)$.

If $\mathcal{C}^{[1,sing]}$ is empty, there is nothing to prove; therefore, we assume that this is not the case and fix $P \in \mathcal{C}^{[1,sing]}$.

We now wish to restrict our attention to the first d components (with elements $\boldsymbol{y} = (y_1, \dots, y_d)$) and consider the following injection:

$$\begin{array}{rccc} \iota: & G^d & \to & G^n \\ & \boldsymbol{y} & \mapsto & (y_1, \dots, y_d, \xi_1, \dots, \xi_k) \end{array}$$

Notice that $\iota(G^d)$ is exactly W, and since we are assuming that $\mathcal{C} \subseteq W$, we can consider $\iota^{-1}(\mathcal{C})$; this is again a curve that we will denote by $\mathcal{C}'$.

Using the minimality of W, we can show that $\mathcal{C}'$ is not contained in any proper weakly special subvariety of G^d. Indeed, if we assume by contradiction that $\iota^{-1}(\mathcal{C}) \subseteq W'$ for W' a proper weakly special subvariety of G^d, the image $\iota(W')$ would be again a weakly special subvariety strictly contained in W, contradicting the minimality of the dimension of W.

We will be showing that ι induces a one-to-one correspondence between points in $\mathcal{C} \cap H$ where H is a special subvariety of G^n and points lying in the intersection between $\mathcal{C}'$ and a weakly-special subvariety of G^d of a "special shape". Recall that for $\boldsymbol{a} \in (\mathrm{End}(G))^h$, we denote by $\varphi_{\boldsymbol{a},h} : G^h \to G$ the map defined by $\varphi_{\boldsymbol{a},h}(x_1, \dots, x_h) = x_1^{a_1} \cdots x_h^{a_h}$ if $G = \mathbb{G}_{\mathrm{m}}$ and $\varphi_{\boldsymbol{a},h}(x_1, \dots, x_h) = a_1 x_1 + \cdots + a_h x_h$ if $G = E$.
Let $P \in \mathcal{C} \cap H_{\boldsymbol{a}}$, where $H_{\boldsymbol{a}}$ is the algebraic subgroup of G^n defined by $\varphi_{\boldsymbol{a},n} = O$, and O is the identity element of G; since $\mathcal{C} \subseteq W$, in particular $P = (x_1, \dots, x_d, \xi_1, \dots, \xi_k)$. Then, $P \in \mathcal{C} \cap H_{\boldsymbol{a}}$ if and only if

$$\varphi_{\boldsymbol{a}',d}(x_1, \dots, x_d) = \varphi_{-\boldsymbol{a}'',k}(\boldsymbol{\xi}), \tag{3.6.1}$$

with $\boldsymbol{a}' = (a_1, \dots, a_d)$ and $\boldsymbol{a}'' = (a_{d+1}, \dots, a_n)$. This means that the point $\iota^{-1}(P) = (x_1, \dots, x_k)$ lies in the intersection between $\mathcal{C}'$ and the subvariety of G^d cut by (3.6.1) (which is nothing else than $\iota^{-1}(H_{\boldsymbol{a}})$). Notice that this subvariety is union of weakly-special subvarieties of G^d of some special form, since $\boldsymbol{\xi}$ is fixed.

This characterization leads us to the following definition.

Definition 3.6.1. Let k, d be two positive natural numbers and let $\boldsymbol{\xi} \in \mathbb{C}^k$ be a k-tuple of independent non-zero elements $(\xi_1, \dots, \xi_k)$ of G. A *$\boldsymbol{\xi}$-special hypersurface* of G^d is an irreducible component of the zero locus of an equation of shape

$$\varphi_{\boldsymbol{a}',d}(\boldsymbol{y}) = \varphi_{-\boldsymbol{a}'',k}(\boldsymbol{\xi}),$$

with $\boldsymbol{a}'$ and $\boldsymbol{a}''$ tuples of endomorphisms such that $\boldsymbol{a}'$ is not the zero vector.

Notice that $\boldsymbol{\xi}$-special hypersurfaces are weakly special subvarieties of G^d of codimension one, and we showed that $P \in \mathcal{C} \cap H$ with H a proper special

subvariety of G^n if and only if $\iota^{-1}(P)$ lies in the intersection of $\mathcal{C}'$ and a $\boldsymbol{\xi}$-*special hypersurface* of G^d.

Next we will show that, if the first intersection is singular, then also the second one is singular. More specifically, putting all together we have the following result.

Lemma 3.6.2. *Let $\mathcal{C} \subset G^n$ be an irreducible curve of the form $\mathcal{C}' \times \{\xi_1\} \times \cdots \times \{\xi_k\}$, where $\mathcal{C}' \subset G^d$ is not contained in any weakly special subvariety and $\xi_1, \ldots, \xi_k \in G$ are independent. Let H be a one-codimensional special subvariety of G^n and let $P \in \mathcal{C} \cap H$ be such that $\mathcal{C}$ is tangent to H at P. Then, there exists a $\boldsymbol{\xi}$-special hypersurface of G^d, which we denote by U, such that $\mathcal{C}'$ is tangent to U at $\iota^{-1}(P)$.*

Proof. Let us denote by $W = G^d \times \{\xi_1\} \times \cdots \times \{\xi_k\}$ which contains $\mathcal{C}$ by assumption. Let us first prove that $T_P(W \cap H) = T_P W \cap T_P H$. Notice that W and H are both translates of algebraic subgroups; if we denote by W_0 and H_0 the images of W and H by the translation that maps P to the origin, the statement above amounts to saying that the Lie algebra of (the irreducible component of) $W_0 \cap H_0$ (containing the origin) coincides with the intersection of the Lie algebras of W_0 and H_0. This is true both for the complex Lie group $\mathbb{G}_{\mathrm{m}}^n$ and the complex lie group obtained from powers of elliptic curves: their Lie algebra can be identified with some $\mathbb{C}^n$ and its closed complex subgroups (which are all algebraic) correspond to subspaces of $\mathbb{C}^n$ defined over $\mathbb{Q}$, for $\mathbb{G}_{\mathrm{m}}^n$ and regular elliptic curves, or the CM field, for CM elliptic curves; such subgroups are closed by intersection.

We now have that $T_P\mathcal{C} \subseteq T_P H$ by hypothesis and, since $\mathcal{C} \subseteq W$, then $T_P\mathcal{C} \subseteq T_P(W \cap H)$ by the above argument. Observe that ι^{-1} is an isomorphism between W and G^d, therefore $T_{\iota^{-1}(P)}(\iota^{-1}(\mathcal{C}))$ is contained in $T_{\iota^{-1}(P)}(\iota^{-1}(H))$. Let now $\boldsymbol{a} \in \mathrm{End}(G)^n$ be such that H is an irreducible component of the kernel of $\varphi_{\boldsymbol{a},n}$. Then, $\iota^{-1}(H)$ is contained in the subvariety of G^d cut out by

$$\varphi_{\boldsymbol{a}',d}(\boldsymbol{y}) = \varphi_{-\boldsymbol{a}'',k}(\xi),$$

where $\boldsymbol{a}' = (a_1, \ldots, a_d)$ and $\boldsymbol{a}'' = (a_{d+1}, \ldots, a_n)$, and it is not empty since $P \in W \cap H$. Hence, if we take as U the unique irreducible component of $\iota^{-1}(H)$ containing P, this satisfies our claim. □

Thanks to Lemma 3.2.1 and Lemma 3.6.2, Theorem 3.2.2 will now be a consequence of the following result.

Theorem 3.6.3. *Let k and d be positive integers, let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_k)$ be a vector of independent non-zero elements of $G(K)$ with K a number field and let $\mathcal{C} \subseteq G^d$ be a curve defined over K which is not contained in any proper weakly special subvariety. Then, there are only finitely many points $P \in \mathcal{C}$ such that there is a $\boldsymbol{\xi}$-special hypersurface U such that:*

1. *P is in $\mathcal{C} \cap U$;*

2. *the tangent space $T_P\mathcal{C}$ of $\mathcal{C}$ at P is included in the tangent space $T_P U$ of U at P (namely, the curve $\mathcal{C}$ is tangent to U at P).*

This theorem is analogous to [Pin05b, Conjecture 5.2], [RV03, Theorem 1.6] [Mau08, Théorème 1.6]. More generally, one can prove that the general version of the Zilber-Pink Conjecture [Pin05b, Conjecture 5.1] is equivalent to [Pin05b, Conjecture 5.2] (see [Pin05b, Theorem 5.5] and [RV03, Proposition 4.2]).

3.7 A bound for the height

To prove Theorem 3.6.3, we will need a bound for the height of the points of singular intersection. In what follows, we will always assume that our varieties are embedded in a projective space of which they inherit the usual projective logarithmic Weil height. For example, we embed $\mathbb{G}_{\mathrm{m}}^n$ into $\mathbb{P}^n$ via:

$$(x_1, \dots, x_n) \to [x_1 : \cdots : x_n : 1].$$

We will now show that, if a point P lies in the intersection of a curve with a one-codimensional weakly special subvariety of the form $H_{\boldsymbol{p},c}$, with $\boldsymbol{p} \in (\mathrm{End}(E))^n \setminus \{\mathbf{0}\}$ and $c \in G(\overline{\mathbb{Q}})$ a non zero element and the intersection is tangent, then the height of P is bounded in terms of the height of $\boldsymbol{p}$.

Proposition 3.7.1. *Let $G = \mathbb{G}_{\mathrm{m}}$ or $G = E$ with E an elliptic curve defined over a number field K. Let n be a positive integer and let $\mathcal{C} \subseteq \mathcal{G} = G^n$ be a curve again defined over K, not contained in any proper weakly special subvariety of $\mathcal{G}$. Then, there exists a constant $\delta_2 > 0$, depending only on $\mathcal{C}$ (and $\mathcal{G}$) such that, for every $P \in \mathcal{C}(\overline{\mathbb{Q}})$ and for every one-codimensional weakly special subvariety $H_{\boldsymbol{p},c}$ with $\boldsymbol{p} \in (\mathrm{End}(E))^n \setminus \{\mathbf{0}\}$ and $c \in G(\overline{\mathbb{Q}})$ such that $\mathcal{C}$ and $H_{\boldsymbol{p},c}$ intersects at P tangentially, then*

$$h(P) \leq \delta_2(1 + h(\boldsymbol{p})),$$

where $h(\boldsymbol{p})$ is the projective height of $\boldsymbol{p} \in \mathbb{P}^{n-1}$.

In order to prove Proposition 3.7.1, we recall the Gauss map introduced in Section 3.4:

$$\gamma_{\mathcal{C}} : \mathcal{C}^{sm} \to \mathbb{P}^{n-1} = \mathbb{P}T_e\mathcal{G}.$$

Possibly removing the finitely many singular points, we will assume $\mathcal{C}^{sm} = \mathcal{C}$ and recall that, since $P \in \mathcal{C} \cap_{sing} H_{\boldsymbol{p},c}$, by Corollary 3.4.3 $\gamma_{\mathcal{C}}(P)$ is contained in the hyperplane L given by the equation:

$$p_1 z_1 + p_2 z_2 + \cdots + p_n z_n = 0,$$

where $z_1, z_2, \dots, z_n$ are the coordinates of $\mathbb{P}^{n-1}$. As in Section 3.5, if $\gamma_{\mathcal{C}}(\mathcal{C})$ were contained in L, this would imply that $\mathcal{C}$ is contained in a one-codimensional weakly special subvariety. Since by assumption this is not the case, the set

$\gamma_{\mathcal{C}}^{-1}(\gamma_{\mathcal{C}}(\mathcal{C}) \cap L)$ is a finite set containing P (as in Section 3.5, $\gamma_{\mathcal{C}}$ is not constant, since $\gamma_{\mathcal{C}}(\mathcal{C})$ is not contained in L).

By Proposition 2.1.17 we just need to bound the height of any element of $\gamma_{\mathcal{C}}(\mathcal{C}) \cap L$ (notice that $\mathcal{C}$ and $\gamma_{\mathcal{C}}(\mathcal{C})$ are already embedded in $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$, respectively). This amounts to proving the following.

Proposition 3.7.2. *Let s be a positive integer and $\mathcal{C}' \subseteq \mathbb{P}^s$ be an irreducible curve defined over a number field K. Then, there is a constant $\delta' > 0$, depending only on $\mathcal{C}'$, such that, for every hyperplane $L_{\boldsymbol{p}}$ given by the equation:*

$$p_0 z_0 + p_1 z_1 + \cdots + p_s z_s = 0,$$

with $\boldsymbol{p} = (p_0, p_1, \ldots, p_s) \in \overline{\mathbb{Q}}^{s+1} \setminus \{\mathbf{0}\}$, not containing $\mathcal{C}'$, and for every point $P \in \mathcal{C}' \cap L$, one has

$$h(P) \leq \delta'(1 + h(\boldsymbol{p})),$$

where $h(\boldsymbol{p})$ is the projective height of $\boldsymbol{p} \in \mathbb{P}^s$.

Proof. Recall that the Grassmannian $(\mathbb{P}^s)^\vee$ parameterizing hyperplanes in $\mathbb{P}^s$, also known as its dual, is isomorphic to $\mathbb{P}^s$. This isomorphism is induced by the choice of a projective basis: we choose the canonical one, and denote by $z_0, \ldots z_s$ the coordinates (with respect to this basis) of the starting $\mathbb{P}^s$ and with $p_0, \ldots, p_s$ the coordinates in the Grassmanians $(\mathbb{P}^s)^\vee$ with respect to the dual basis. In other words, the point $[p_0 : \cdots : p_s] \in (\mathbb{P}^s)^\vee$ represents the hyperplane $L_{\boldsymbol{p}}$ of $\mathbb{P}^s$ given by the equation $p_0 z_0 + p_1 z_1 + \cdots + p_s z_s = 0$.

Let $V \subseteq (\mathbb{P}^s)^\vee$ be the subset consisting of $\boldsymbol{p} \in (\mathbb{P}^s)^\vee$ such that $L_{\boldsymbol{p}}$ does not contain $\mathcal{C}'$; V is an open subset of $(\mathbb{P}^s)^\vee$ (since the requirement of containing a prescribed point is a closed condition and therefore $(\mathbb{P}^s)^\vee \setminus V$ is an intersection of closed sets).

Our strategy consists in constructing some morphisms which behave well with respect to height in order to bound the height of P from the height of $\boldsymbol{p}$.

Let us consider $Sym^d(\mathcal{C}') = (\mathcal{C}')^d / S_d$ the d-th symmetric power of $\mathcal{C}'$. From [Knu71, Prop IV.1.5, p.180], $Sym^d(\mathcal{C}')$ is a quasi-projective variety, so we can assume that it is embedded into a projective space, from which it inherits a height function.

We can suppose that $\mathcal{C}'$ is closed in $\mathbb{P}^s$ (this would just enlarge $\mathcal{C}' \cap L_{\boldsymbol{p}}$). By Bézout's Theorem, there is a positive integer d such that, for any $\boldsymbol{p} \in V$, we have that $L_{\boldsymbol{p}}$ intersects $\mathcal{C}'$ in precisely d points (counted with multiplicity). This gives rise to the following morphism, defined over K:

$$\begin{array}{rccc} \phi_1 : & V & \to & Sym^d(\mathcal{C}') \\ & (\boldsymbol{p}) & \mapsto & \mathcal{C}' \cap_m L_{\boldsymbol{p}}, \end{array}$$

where $\mathcal{C}' \cap_m L_{\boldsymbol{p}}$ denotes the schematic intersection of $\mathcal{C}'$ and $L_{\boldsymbol{p}}$, taking into account multiplicities. Since $Sym^d(\mathcal{C}') = (\mathcal{C}')^d / S_d$ is the quotient of $(\mathcal{C}')^d$ by

the finite group S_d, the quotient map defines a finite morphism, defined over K:

$$\begin{array}{rccc} \phi_2 : & (\mathcal{C}')^d & \to & Sym^d(\mathcal{C}') \\ & (P_1, \dots, P_d) & \mapsto & (P_1, \dots, P_d)_{no} \end{array}$$

where the d-uple is ordered in the LHS and non-ordered in the RHS (we use the subscript $_{no}$ to denote this).

Finally, we consider the projection on the first coordinate:

$$\begin{array}{rccc} \phi_3 : & (\mathcal{C}')^d & \to & \mathcal{C}' \\ & (P_1, \dots, P_d) & \mapsto & P_1 \end{array}$$

To summarize, we have the following diagram:

$$\begin{array}{ccccc} & & (\mathcal{C}')^d & & V \subseteq (\mathbb{P}^s)^\vee \\ & \swarrow^{\phi_3} & & \searrow^{\phi_2} & \swarrow^{\phi_1} \\ \mathcal{C}' & & & Sym^d(\mathcal{C}') & \end{array}$$

We will now see that these maps behave well with respect to height.

Since V is an open set of $(\mathbb{P}^s)^\vee$, the map ϕ_1 extends (formally) to a rational map between projective spaces. Therefore, by Proposition 2.1.16, there exists a constant $\eta_1 > 0$ such that, for all $\boldsymbol{p} \in V(\overline{\mathbb{Q}})$, we have

$$h(\phi_1(\boldsymbol{p})) \leq \eta_1(1 + h(\boldsymbol{p})).$$

If we consider $(\mathcal{C}')^d \subseteq (\mathbb{P}^s)^d \subseteq \mathbb{P}^{(s+1)^d-1}$, then the map ϕ_3 can also be extended to a rational map between projective spaces such that it is defined on $(\mathcal{C}')^d$, and in a similar way we get that there exists a constant $\eta_3 > 0$ such that, for all $(P_1, \dots, P_d) \in (\mathcal{C}')^d(\overline{\mathbb{Q}})$,

$$h(P_1) = h(\phi_3((P_1, \dots, P_d))) \leq \eta_3(1 + h((P_1, \dots, P_d)).$$

Let us now focus on ϕ_2; this is the quotient map by a finite group, so $(\mathcal{C}')^d$ and $Sym^d(\mathcal{C}')$ have the same dimension, ϕ_2 is surjective, and all the fibers are finite; in particular, all the points $P \in (\mathcal{C}')^d$ are isolated in their fiber. We can then apply Propostition 2.1.17 with $U = (\mathcal{C}')^d$, which gives us a constant $\eta_2 > 0$ such that

$$h((P_1, \dots, P_d)) \leq \eta_2(1 + h(\phi_2((P_1, \dots, P_d))) = \eta_2(1 + h((P_1, \dots, P_d)_{no})).$$

Let now be $P \in \mathcal{C}' \cap L_{\boldsymbol{p}}$. We can choose points $Q_1, \dots, Q_{d-1}$ in $\mathcal{C}'$ such that the element $(P, Q_1, \dots, Q_{d-1})_{no} = \mathcal{C}' \cap_m L_{\boldsymbol{p}}$. We have:

$$\begin{aligned} \phi_1(\boldsymbol{p}) &= (P, Q_1, \dots, Q_{d-1})_{no} \\ \phi_2(P, Q_1, \dots, Q_{d-1}) &= (P, Q_1, \dots, Q_{d-1})_{no} \\ \phi_3(P, Q_1, \dots, Q_{d-1}) &= P. \end{aligned}$$

We thus have

$$\begin{aligned} h(P) &\leq \eta_3(1 + h((P, Q_1, \ldots, Q_{d-1}))) \\ &\leq \eta_3(1 + \eta_2(1 + h((P, Q_1, \ldots, Q_{d-1})_{no}))) \\ &\leq \eta_3(1 + \eta_2(1 + \eta_1(1 + h(\boldsymbol{p})))) \end{aligned}$$

Putting $\delta' = (\eta_1 + 1)(\eta_2 + 1)(\eta_3 + 1)$ we conclude.

□

3.8 Proof of Theorem 3.6.3

This section is dedicated to proving Theorem 3.6.3. We recall the notation introduced in the statement. Let $G = \mathbb{G}_{\mathrm{m}}$ or $G = E$, for E an elliptic curve defined over a number field K. Let k and d be positive integers, let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_k)$ be a vector of independent non-zero elements of $G(K)$ and let $\mathcal{C} \subseteq \mathcal{G} = G^d$ be a curve defined over K which is not contained in any proper weakly special subvariety. Fix a point $P \in \mathcal{C}$ such that there is some $\boldsymbol{\xi}$-special hypersurface U with $\mathcal{C}$ tangent to U at P.

Let us assume that U is contained in the zero locus of

$$\varphi_{\boldsymbol{p}_U, d}(\boldsymbol{y}) = \varphi_{-\boldsymbol{q}_U, k}(\boldsymbol{\xi}),$$

where $\boldsymbol{p}_U, \boldsymbol{q}_U$ are tuples of endomorphisms of G such that not all of the p_i are zero. We want to show that we can assume $\boldsymbol{p}_U$ and $\boldsymbol{q}_U$ to be sufficiently small in an appropriate sense. Let Γ be the submodule of $\mathrm{End}(G)^{d+k}$ consisting of the tuples $(a_1, \ldots, a_d, b_1, \ldots, b_k)$ such that:

$$\varphi_{\boldsymbol{a}, d}(P) = \varphi_{-\boldsymbol{b}, k}(\boldsymbol{\xi}),$$

and let Γ_{sing} be the submodule of Γ consisting of the tuples $(a_1, \ldots, a_d, b_1, \ldots, b_k)$ such that the curve $\mathcal{C}$ is tangent at P to the coset cut by the equation

$$\varphi_{\boldsymbol{a}, d}(\boldsymbol{y}) = \varphi_{-\boldsymbol{b}, k}(\boldsymbol{\xi}).$$

Recall that, for a module M over an integral domain $\mathcal{O}$ with field of fractions K, the *rank* of M is defined as the rank of the maximal free $\mathcal{O}$-submodule contained in M, or, equivalently, with the K-dimension of $M \otimes_{\mathcal{O}} K$.

Notice that the ranks of Γ and Γ_{sing} are at least one, since $P \in U$ and the intersection of the curve with U at P is tangent, so $(\boldsymbol{p}_U, \boldsymbol{q}_U) \in \Gamma_{\mathrm{sing}} \subseteq \Gamma$. There are then two possibilities:

1. If $\mathrm{rk}(\Gamma) \geq 2$, then there are finitely many possible values as a consequence of Zilber-Pink-like statements for a curve in a power of $\mathbb{G}_{\mathrm{m}}$, due to Maurin [Mau08, Théorème 1.6], or in a power of an elliptic curve, due to Viada [Via03, Via08], Rémond and Viada [RV03, Théorème 1.5-1.6] and Galateau [Gal10, Théorème 1.10].

2. If, on the other hand, $\mathrm{rk}(\Gamma) = \mathrm{rk}(\Gamma_{\mathrm{sing}}) = 1$, let $r = [\mathrm{End}(G) : \mathbb{Z}]$, so $r = 1$ for $G = \mathbb{G}_{\mathrm{m}}$ or $G = E$ for a regular elliptic curve, and $r = 2$ for $G = E$ a CM elliptic curve. We also define

$$\widetilde{h}_{\mathrm{aff}}(P) := \max\{h(\xi_1), \dots, h(\xi_k), h(y_1(P)), \dots, h(y_d(P))\}.$$

By a consequence of a result of Masser [Mas88] (see [BC17, Lemmas 5.1 and 5.2] and the proof of [Bar17, Lemma 6.1]), for an absolute constant $\eta_{d+k} > 0$, the module Γ contains an element $(p_1, \dots, p_d, q_1, \dots, q_k)$ with

$$\begin{aligned} \max\{|p_1|, \dots, |p_d|, |q_1|, \dots, |q_k|\} \leq \\ \leq \eta_{d+k}[K(P) : K]^{r(d+k)+1}(\widetilde{h}_{\mathrm{aff}}(P))^{r(d+k)-1}. \end{aligned} \tag{3.8.1}$$

We will need a similar small element in Γ_{sing}. This is ensured by the following.

Lemma 3.8.1. *Let F be the field of fractions of* $\mathrm{End}(G)$. *Then $F^{d+k} \supseteq F\Gamma_{sing} \cap \Gamma = \Gamma_{sing}$. In particular, the element $(p_1, \dots, p_d, q_1, \dots, q_k)$ above lies in Γ_{sing}.*

Proof. Clearly, $F\Gamma_{sing} \cap \Gamma \supseteq \Gamma_{sing}$, so we only need to prove the reverse inclusion. Recall that in Section 3.6 we defined the map ι by

$$\begin{array}{rccc} \iota: & G^d & \to & G^d \times G^k \\ & \boldsymbol{y} & \mapsto & (\boldsymbol{y}, \boldsymbol{\xi}). \end{array}$$

We will, throughout this proof, tacitly apply Lemma 3.6.2.

Let $(a_1, \dots, a_d, b_1, \dots, b_k) \in F\Gamma_{sing} \cap \Gamma$; then, $\iota(P) = (P, \boldsymbol{\xi})$ lies in the hypersurface $U_{\boldsymbol{a},\boldsymbol{b}}$ of $G^d \times G^k$ cut out by $\varphi_{\boldsymbol{a},d}(\boldsymbol{y}) = \varphi_{-\boldsymbol{b},k}(\boldsymbol{z})$. Moreover, there exists $\mu \in F^\times$ such that the curve $\iota(\mathcal{C})$ is tangent to the hypersurface $U_{\mu\boldsymbol{a},\mu\boldsymbol{b}}$ cut by $\varphi_{\mu\boldsymbol{a},d}(\boldsymbol{y}) = \varphi_{-\mu\boldsymbol{b},k}(\boldsymbol{z})$ in $G^d \times G^k$ at $(P, \boldsymbol{\xi})$. We want to prove that, actually, $\iota(\mathcal{C})$ is tangent to $U_{\boldsymbol{a},\boldsymbol{b}}$ at $(P, \boldsymbol{\xi})$.

Let $T = T_{(P,\boldsymbol{\xi})}(G^{d+k})$ be the tangent space in $\iota(P)$, which we identify with the tangent space at the origin, with coordinates $\partial y_1, \dots, \partial y_d, \partial z_1, \dots, \partial z_k$. Now, the curve $\iota(\mathcal{C})$ is tangent to $U_{\boldsymbol{a},\boldsymbol{b}}$ at $(P, \boldsymbol{\xi})$ if and only if we have that the line $T_{(P,\boldsymbol{\xi})}(\iota(\mathcal{C}))$ is contained in the hyperplane $a_1\partial y_1 + \cdots + a_d\partial y_d + b_1\partial z_1 + \cdots + b_k\partial z_k = 0$. By assumption, $T_{(P,\boldsymbol{\xi})}\iota(\mathcal{C})$ is contained in $\mu a_1\partial y_1 + \cdots + \mu a_d\partial y_d + \mu b_1\partial z_1 + \cdots + \mu b_k\partial z_k = 0$, and clearly these two are the same hyperplane, so our claim is proven. □

Notice that, by Proposition 3.5.1, we have an absolute upper bound on $[K(P) : K]$; combining this with (3.8.1) we obtain:

$$\max\{|p_1|, \dots, |p_d|, |q_1|, \dots, |q_k|\} \leq \rho(1 + h(P))^{r(m+n)-1}, \tag{3.8.2}$$

where $\rho > 0$ is a constant depending only on $\mathcal{C}$ and $\boldsymbol{\xi}$. By Proposition 3.7.1 we have

$$h(P) \leq \rho'(1 + \log\max\{|p_1|, \dots, |p_d|\}), \tag{3.8.3}$$

for a constant $\rho' > 0$ depending only on $\mathcal{C}$. Combining (3.8.2) and (3.8.3) we obtain an upper bound on $|p_1|, \ldots, |p_n|$ and hence an upper bound on $h(P)$. Finally, together with the upper bound for $[K(P) : K]$ given by Proposition 3.5.1, we conclude the proof of Theorem 3.6.3 applying Northcott's theorem.

4 - Singular intersections in families of abelian varieties

4.1 Introduction

In this chapter, we continue the study of singular intersections for curves, tackling the case of one-parameter families of abelian varieties.

In this context, as we saw in Section 1.6, the finiteness of the set of unlikely intersections with *flat* subgroups and everything defined over $\overline{\mathbb{Q}}$ is known due to work of Masser and Zannier [MZ08, MZ10, MZ12, MZ14], Habegger and Pila [HP16] and Barroero and Capuano [BC16, BC17, BC20]. The final statement that one gets is the following.

Theorem 4.1.1 ([BC20], Theorem 1.1)**.** *Let S be a smooth irreducible curve defined over $\overline{\mathbb{Q}}$. Consider an abelian scheme $\mathcal{A} \to S$ and $\mathcal{C}$ an irreducible curve in $\mathcal{A}$ not contained in a proper subgroup scheme of $\mathcal{A}$, even after a finite base change. Let*

$$\mathcal{C}^{\{2\}} = \bigcup_{\substack{G \text{ flat} \\ \operatorname{codim} G \geq 2}} \mathcal{C} \cap G,$$

where G runs through all the flat *subgroup schemes of codimension at least 2, that is, all the subgroup schemes of codimension at least 2 whose irreducible components dominate the base S; then, $\mathcal{C}^{\{2\}}$ is a finite set.*

On the other hand, the finiteness of the set of unlikely intersections with non-flat special subvarieties, namely the so-called *special fibers*, is only known in limited cases, by work of Barroero [Bar17] and Ferrigno [Fer25, Fer26].

Our aim for this chapter is to prove that analogous finiteness results hold for the set of singular intersections; in particular, we are interested to prove Conjecture 1.7.2. Similar to the Zilber-Pink case, we consider separately the intersections with flat subgroups and the intersections with the special fibers. The finiteness in this second cases is implied by the following statement.

Theorem 4.1.2. *Let S be a smooth irreducible curve defined over $\mathbb{C}$. Consider $\mathcal{A} \to S$ be an abelian scheme and let $\mathcal{C}$ be an irreducible curve in $\mathcal{A}$, not contained in any fiber. Let*

$$\mathcal{C}^{\{1,ram\}} = \bigcup_{s \in S} \mathcal{C} \cap_{sing} \mathcal{A}_s.$$

Then, $\mathcal{C}^{\{1,ram\}}$ is a finite set.

The proof of this result follows from the fact that the set $\mathcal{C}^{\{1,ram\}}$ is exactly the ramification locus of the map $\mathcal{C} \to S$, which is dominant as $\mathcal{C}$ is not contained in any fiber. This, in turn, can be seen in two ways. The first, more algebraic in nature, is that the schematic intersection $\mathcal{C} \cap \mathcal{A}_s$ is exactly the schematic preimage of s via the map $\mathcal{A} \to S$. The second is more differential in nature. If $\boldsymbol{c} \in \mathcal{C} \cap \mathcal{A}_s$ is a point of intersection, then the tangent space to the fiber $T_{\boldsymbol{c}}\mathcal{A}_s$, seen as a subspace of the tangent space at $\boldsymbol{c}$, is exactly the kernel of the differential at $\boldsymbol{c}$ of the structural map $\mathcal{A} \to S$. So, $\mathcal{C}$ is tangent to $\mathcal{A}_s$ at $\boldsymbol{c}$ if and only if the differential of $\mathcal{C} \to S$ vanishes at $\boldsymbol{c}$, i.e. $\boldsymbol{c}$ is a ramification point.

The rest of this chapter is then dedicated to prove the finiteness of the singular intersections with flat subgroups.

Theorem 4.1.3. *Let S be a smooth irreducible curve defined over $\overline{\mathbb{Q}}$. and let us consider $\mathcal{A} \to S$ an abelian scheme, also defined over $\overline{\mathbb{Q}}$. Let $\mathcal{C}$ be an irreducible curve in $\mathcal{A}$, dominant over S and not contained in a proper subgroup scheme of $\mathcal{A}$, even after a finite base change. Let*

$$\mathcal{C}^{\{1,sing\}} = \bigcup_{\substack{G \text{ flat} \\ \operatorname{codim} G \geq 1}} \mathcal{C} \cap_{sing} G,$$

where G runs through all the flat subgroup schemes of codimension at least 1. Then, $\mathcal{C}^{\{1,sing\}}$ is a finite set.

Some special cases of this theorem are already known from the literature. In particular, the case of curves which are images of sections (so, after a base change reduction like Proposition 4.2.3, all curves) intersecting tangentially the torsion cosets in a power of an elliptic scheme follows from works of Corvaja, Demejo, Masser and Zannier [CDMZ21]. Ulmer and Urzúa [UU20, UU21] give an independent proof of the case of curves that are image of sections in a single power of an elliptic scheme. In the same papers they also show an application to elliptic divisibility sequences.

To prove Theorem 4.1.3, similarly to [BC20], we start by showing that we can substitute $\mathcal{A}$ with an isogenous abelian scheme $\mathcal{A}'$. Therefore we can assume, up to isogeny, that $\mathcal{A}$ is the product $\prod_i \mathcal{A}_i^{n_i}$ of fibered powers of pairwise non-isogenous simple abelian schemes. The finiteness of the intersection of $\mathcal{C}$ with the flat subgroups of codimension ≥ 2 is proven in [BC20], so we can restrict ourselves to consider only codimension 1 subgroups. Those will all be of the shape

$$H = \mathcal{A}_1^{n_1} \times_S \cdots \times_S \mathcal{A}_{i-1}^{n_{i-1}} \times_S H_i \times_S \mathcal{A}_{i+1}^{n_{i+1}} \cdots \times_S \mathcal{A}_r^{n_r},$$

where $H_i < \mathcal{A}_i$ is a codimension 1 subgroup of $\mathcal{A}_i$, and $\mathcal{A}_i$ is a power of an elliptic scheme. Therefore, it is enough to consider the case where $\mathcal{A}$ is itself a power of an elliptic scheme. If $\mathcal{A}$ is isotrivial, than the statement reduces to

Theorem 3.2.2; so we will suppose from now on that it is not. Adapting some work of Habegger [Hab13a] we can reduce to the case where $\mathcal{A}$ is a power of the Legendre scheme, i.e. the elliptic scheme $\lambda : \mathcal{L} \to \mathbb{C} \setminus \{0, 1\}$, where for every $\lambda_0 \in \mathbb{C} \setminus \{0, 1\}$ the fiber $\mathcal{L}_{\lambda_0}$ is an elliptic curve defined by

$$Y^2 = X(X - 1)(X - \lambda_0).$$

The full reduction is performed in the second section.

We are then left to prove the following statement.

Theorem 4.1.4. *Let $\lambda : \mathcal{L} \to S = \mathbb{P} \setminus \{0, 1, \infty\}$ be the Legendre elliptic scheme with fibers $\mathcal{L}_\lambda$, and let $\mathcal{L}^n$ be its n-th fibered power, with coordinates $P_1, \dots, P_n$. For $\boldsymbol{a} \in \mathbb{Z}^n$ let $\varphi_{\boldsymbol{a}} : \mathcal{L}^n \to \mathcal{L}$ be the fiberwise linear map given by*

$$\varphi_{\boldsymbol{a}}(P_1, \dots, P_n) = a_1 P_1 + \cdots + a_n P_n.$$

Let $\mathcal{C} \subseteq \mathcal{L}^n$ be an irreducible curve defined over $\overline{\mathbb{Q}}$ and not contained in a fiber over S. Assume that, for every $j = 1, \dots, n$, there is no $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\varphi_{\boldsymbol{a}} = O \circ \lambda,$$

identically on $\mathcal{C}$, where O is the zero section of $\mathcal{L}$. Then, there are at most finitely many $\boldsymbol{c} \in \mathcal{C}$ such that there exists $\boldsymbol{a} \in \mathbb{Z}^n \setminus \{0\}$ with

$$\varphi_{\boldsymbol{a}}(\boldsymbol{c}) = O(\lambda(\boldsymbol{c})) \tag{4.1.1}$$

and such that $\mathcal{C}$ and $\ker(\varphi_{\boldsymbol{a}})$ intersect tangentially at $\boldsymbol{c}$.

This result follows from [CDMZ21, Thm. 2.10] and the invariance of the problem under base change, proven in Lemma 4.2.3. However, we provide an independent proof of it following Pila-Zannier method, a strategy introduced by Pila and Zannier in [PZ08] to give an alternative proof of the Manin-Mumford conjecture (already proved by Raynaud in [Ray83a]), and used in several works in the field. This method may lead to an effective result for the number of points of singular intersection, using an effective version of Proposition 2.2.10, such as Theorem 2.2.13 (see also [BJST23, Corollary 4.6]).

To prove finiteness, we prove that $\mathcal{C}^{\{1,sing\}}$ is a set of bounded height and degree, and then conclude by Northcott property. In particular, the height bound comes from Silverman Specialization Theorem [Sil83], while for the degree we compare two estimates for the growth of the size of the set of singular intersections of $\mathcal{C}$ with subgroups defined by linear conditions with bounded coefficients. To get an upper bound, we consider the elliptic logarithms $z_1, \dots, z_n$ of $P_1, \dots, P_n$ restricted to the curve $\mathcal{C}$, their derivatives with respect to λ and the equations

$$\begin{cases} z_j = u_j f + v_j g, \\ \frac{\partial z_j}{\partial \lambda} = \frac{\partial u_j}{\partial \lambda} f + u_j \frac{\partial f}{\partial \lambda} + \frac{\partial v_j}{\partial \lambda} g + v_j \frac{\partial g}{\partial \lambda}, \end{cases}$$

for $j = 1, \dots, n$, where f and g are suitably chosen basis elements of the period lattice of $\mathcal{L}_\lambda$. If we consider the coefficients $u_j, v_j, \frac{\partial u_j}{\partial \lambda}, \frac{\partial v_j}{\partial \lambda}$ as functions of λ and restrict them to a compact set, we obtain a subanalytic surface Σ in $\mathbb{R}^{4n}$. The points where $\mathcal{C}$ has singular intersection with a flat subgroup scheme give rise to two linear relations with integer coefficients, one for the z_js and another one (actually the same) for the derivatives. Consequently, the points will correspond to points of Σ lying on linear varieties defined by equations whose coefficients are the same as those of the linear (on the elliptic curve) relations that the points satisfy. Combining an o-minimality result by Habegger and Pila [HP16] with transcendence of the elliptic logarithms and their derivatives proven by Bertrand [Ber90], we get that the number of points lying on Σ and these linear varieties with coefficients bounded by T grows slower than any power of T.
On the other side, if a point of degree d lies in $\mathcal{C}^{\{1,sing\}}$, it brings all of its Galois conjugates, and we can prove that all of them need to satisfy a linear equation with coefficients bounded by a constant times a certain power of d. Combining this with the upper bound forces the degree to be bounded, concluding the argument.

We conclude this introduction with a final remark on Theorem 4.1.3. As already observed in Section 1.7, the result is equivalent to an unlikely intersections statement in the projectivized tangent bundle $\mathbb{P}T\mathcal{A}$. It is important to notice that in this case we are not considering the relative tangent bundle of $\mathcal{A}$ with respect to $\pi : \mathcal{A} \to S$, but the global one.

4.2 Reduction to powers of the Legendre elliptic scheme

In this section we will reduce the proof of Theorem 4.1.3 to the case of powers of the Legendre scheme. Similarly to [BC20], we prove that our problem is invariant under finite base changes and isogenies. Combining this with Poincaré Reducibility Theorem we reduce to two cases: the first one is the case of subgroups of codimension at least two, which is covered by a theorem of Habegger and Pila [HP16] in the case of isotrivial components and by the aforementioned result by Barroero and Capuano [BC20] for the non-isotrivial components. The second case is that of codimension one subgroups in powers of an elliptic scheme, which is not isotrivial by assumption. Part of this section is similar to Section 3.2.

In the second part of this section we further reduce ourselves to consider powers of the Legendre scheme and flat subgroups defined by a single linear condition with integer coefficients.

This section is inspired by the second sections of [BC16] and [BC20], and by [Hab13a].

For V, W subvarieties of a variety X, and $P \in V \cap W$ an isolated intersection point we denote by $i(P; V, W; X)$ the intersection multiplicity of V and W in P. Notice that this number is by definition 0 whenever $\dim V \neq \operatorname{codim}_X W$. We say that P is a *proper component* of the intersection, or that V and W *intersect properly at* P, if P is an isolated intersection point and $\dim V = \operatorname{codim}_X W$. The main tool we will use in our reduction steps is the following.

Proposition 4.2.1 ([Now98], p.173)**.** *Let $f : X' \to X$ be a dominant morphism between smooth varieties defined over an algebraically closed field k, let V' and W' be two subvarieties of X' that intersect properly at a closed point P', and let $V := f(V'), W := f(W')$ and $P := f(P')$. If the morphism f is étale at P', then*

$$i(P'; V', W'; X') = i(P; V, W; X).$$

In other words, the intersection multiplicity is invariant under étale morphisms.

This will help us show that our problem is preserved by moving back and forward through étale morphisms.
We also state a corollary, that will be required in Section 4.7.

Corollary 4.2.2. *Let $f : X \to X$ be an isomorphism between smooth varieties over an algebraically closed field k, and let V and W be two subvarieties of X invariant under f, that intersect properly at a closed point P. Then,*

$$i(P; V, W; X) = i(f(P); V, W; X).$$

In particular, this holds when $k = \overline{\mathbb{Q}}$, the varieties X, V and W are defined over a number field and f is induced by a Galois action.

We also note that, since we are proving a finiteness result, we are always allowed to replace the base curve by a non-empty open subset, and we will tacitly do so. This allows us to pass from an abelian variety defined over a function field of a curve to the corresponding abelian scheme over (a non-empty open subset of) the curve.

We recall our setting and some definitions. Let $\mathcal{A}$ be an abelian scheme over a smooth irreducible curve S, where everything is defined over $\overline{\mathbb{Q}}$. We call $\pi : \mathcal{A} \to S$ the structural morphism. A subgroup scheme G of $\mathcal{A}$ is a closed subvariety, possibly reducible, which contains the image of the zero section $O : S \to \mathcal{A}$, is mapped to itself by the inversion morphism and such that the image of $G \times_S G$ under the addition morphism is in G. A subgroup scheme G is called *flat* if $\pi|_G : G \to S$ is flat. By [Har77, Proposition III.9.7], since S has dimension 1, this is equivalent to require that all irreducible components of G dominate the base curve S.

4.2.1 Reduction to powers of simple abelian schemes

We want to show that, in order to prove Theorem 4.1.3, we can perform finite base changes and isogenies.

Lemma 4.2.3. *Let $\mathcal{C}$ be as in Theorem 4.1.3. Let $\mathcal{A}' = \mathcal{A} \times_S S'$ for some finite cover $S' \to S$ and let f be the projection $\mathcal{A}' \to \mathcal{A}$. Then, if the conclusion of Theorem 4.1.3 holds for all irreducible components of $f^{-1}(\mathcal{C})$, it holds for $\mathcal{C}$.*

Proof. First, we see that f is flat because it is a fibered product of two flat morphisms $\mathcal{A} \to \mathcal{A}$ and $S' \to S$. Moreover, since $\mathcal{A}$ and $\mathcal{A}'$ have the same dimension, by [Har77, Corollary III.9.6], it follows that f is quasi-finite, and finite, since it is also proper. Up to substituting S' and S with an open dense subset, we can also assume that $S' \to S$ is unramified, and therefore $\mathcal{A}' \to \mathcal{A}$ as well, as it is a fibered product of unramified maps. Hence, the projection $\mathcal{A}' \to \mathcal{A}$ is étale. By [Har77, Corollary III.9.6], we have that if $X \subseteq \mathcal{A}$ is an irreducible variety dominating S, as f is finite and flat, each component of $f^{-1}(X)$ is a variety of the same dimension dominating S'. It is clear now that, if the assumptions of Theorem 4.1.3 hold for $\mathcal{C}$, then they must hold for all components of $f^{-1}(\mathcal{C})$. Finally, the preimage of any point of $\mathcal{C}$ lying in a flat subgroup scheme of $\mathcal{A}$ of codimension at least 1 must lie in a flat subgroup scheme of the same codimension, namely the preimage. Moreover, by Proposition 4.2.1, the tangency condition is also preserved. □

Lemma 4.2.4. *Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian schemes over the same irreducible curve S, let η be the generic point of S, and let $f_\eta : \mathcal{A}'_\eta \to \mathcal{A}_\eta$ be an isogeny between the generic fibers defined over $k(S)$. Moreover, let $\mathcal{C} \subseteq \mathcal{A}$ be a curve satisfying the assumptions of Theorem 4.1.3. Then, if the claim of Theorem 4.1.3 holds for all irreducible components of $f^{-1}(\mathcal{C})$, it holds for $\mathcal{C}$.*

Proof. By the proof of [BC20, Lemma 2.2], we have that f_η extends to a map $f : \mathcal{A}' \to \mathcal{A}$ that is finite, flat, and is an isogeny on every fiber. So, to conclude in the same way as the previous lemma, we only need to show that f is also unramified, and thus étale.
Since f is finite and $\mathcal{A}$ and $\mathcal{A}'$ are locally noetherian, by [Liu02, Chapter 4, Lemma 3.20] f is unramified if and only if, for every $p \in \mathcal{A}$, the fibers $\mathcal{A}'_p$ are finite. By the same argument, for any $s \in S$, the map $f_s : \mathcal{A}'_s \to \mathcal{A}_s$ is unramified if and only if for any $p \in \mathcal{A}_s$ the fiber $(\mathcal{A}'_s)_p$ is finite. By properties of the base change, $(\mathcal{A}'_s)_p = \mathcal{A}'_p$, so it is enough to show that f_s is unramified for every $s \in S$. This follows from [Mum88, Section II.7, Corollary 1 of Thm. 4, p.74] and the fact that f is fiberwise an isogeny. □

Now we can proceed as in [BC20] to show that it is enough to prove Theorem 4.1.3 for products of simple abelian schemes. Let us consider the generic fiber $\mathcal{A}_\eta$ of $\mathcal{A}$ as an abelian variety defined over $k(S)$. Since every abelian variety is isogenous to a product of simple abelian varieties (see for instance [HS00, Corollary A.5.1.8]), there are geometrically simple, pairwise non-isogenous abelian varieties $B_1, \dots, B_m$, such that $\mathcal{A}_\eta$ is isogenous to $A' := \prod_i B_i^{n_i}$. Note that the abelian variety A' and the isogeny might not be defined over $k(S)$, but over a finite extension of it, which is $k'(S')$ for some irreducible, non-singular curve S' covering S. However, up to enlarging the base field we can (and do) assume $k' = k$, and by Lemma 4.2.3 we can (and do) assume $S' = S$. For the same

reason, we can assume that the endomorphism ring of A' is defined over $k(S)$. Up to shrinking S, the abelian varieties B_i extend to abelian schemes $\mathcal{B}_i \to S$. We define $\mathcal{A}'$ as $\prod_i \mathcal{B}_i^{n_i}$, where the product and the powers are intended as fibered product and fibered powers over S. Lemma 4.2.4 then allows us to reduce the proof of Theorem 4.1.3 to the case of products of simple abelian schemes.

We are now going to describe flat subgroup schemes of $\mathcal{A}$, which is a fibered product $\mathcal{A}_1 \times_S \cdots \times_S \mathcal{A}_m$, where $\mathcal{A}_i$ is the n_i-th fibered power of $\mathcal{B}_i$ and $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are abelian schemes whose generic fibers B_i are pairwise non-isogenous geometrically simple abelian varieties. Moreover, we let R_i be the endomorphism ring of B_i, which we can suppose to be defined over $k(S)$.
Fix i_0, with $1 \leq i_0 \leq m$. For every $\boldsymbol{a} = (a_1, \ldots, a_{n_{i_0}}) \in R_{i_0}^{n_{i_0}}$, we have a morphism $\varphi_{\boldsymbol{a}} : \mathcal{A}_{i_0} \to \mathcal{B}_{i_0}$ defined by

$$\varphi_{\boldsymbol{a}}(P_1, \ldots, P_{n_{i_0}}) = a_1 P_1 + \cdots + a_{n_{i_0}} P_{n_{i_0}}.$$

In a similar way, square matrices of size n_{i_0} with entries in R_{i_0} will encode endomorphisms of $\mathcal{A}_{i_0}$, and m-tuples in $\prod_i Mat_{n_i}(R_i)$ will encode endomorphisms of $\mathcal{A}$. These endomorphisms form a ring, which we call R.
If $\alpha \in R$, the kernel of α is the fibered product of $\alpha : \mathcal{A} \to \mathcal{A}$ and the zero section $S \to \mathcal{A}$. We consider it as a closed subscheme of $\mathcal{A}$. It is quite easy to see that $\ker(\alpha)$ is a subgroup scheme.

Let g_i be the relative dimension of $\mathcal{B}_i$ over S. If $\alpha = (\alpha_1, \ldots, \alpha_m) \in R$, we define the rank $r(\alpha)$ of α to be $\sum_i \mathrm{rk}(\alpha_i) g_i$, where $\mathrm{rk}(\alpha_i)$ is the matrix rank of $\alpha_i \in Mat_{n_i}(R_i)$. In practice, $\ker \alpha$ is a closed subscheme of $\mathcal{A}$ obtained by imposing $\mathrm{rk}(\alpha_i)$ independent equations on each factor $\mathcal{A}_i$.

Lemma 4.2.5 ([BC20], Lemma 2.3)**.** *Let G be a flat subgroup scheme of $\mathcal{A}$ of codimension d with $1 \leq d \leq n$. Then, there exists an $\alpha \in R$ of rank d such that $G \subseteq \ker(\alpha)$. Moreover, $\ker(\alpha)$ is a flat subgroup scheme of $\mathcal{A}$ of codimension $r(\alpha)$.*

From this lemma, we can deduce that each flat subgroup scheme of $\mathcal{A}$ is contained in a flat subgroup scheme of the same dimension and of the form

$$G = G_1 \times_S \cdots \times_S G_m,$$

where, for every $i = 1, \ldots, m$, G_i is a flat subgroup scheme of $\mathcal{A}_i$.
This implies that any flat subgroup scheme of $\mathcal{A}$ of codimension at least 1 is contained in a $G = G_1 \times_S \cdots \times_S G_m$ of codimension at least 1. Moreover, up to enlarging G, and to possibly increasing its dimension, we can assume that all $G_i = \mathcal{A}_i$ except for one index i_0, and that G_{i_0} is a maximal flat subgroup of $\mathcal{A}_{i_0}$, In this way, G_{i_0}, and therefore G, has codimension exactly g_{i_0}. Notice that, in order to prove Theorem 4.1.3, it is enough to show that the set of points of singular intersections between the curve $\mathcal{C}$ and the subgroups of this shape is finite. Let then

$$G = \mathcal{A}_1 \times \cdots \times \mathcal{A}_{i_0 - 1} \times G_{i_0} \times \mathcal{A}_{i_0 + 1} \times \cdots \times \mathcal{A}_m, \tag{4.2.1}$$

let $\pi : \mathcal{A} \to \mathcal{A}_{i_0}$ be the projection map, and consider the image $\pi(\mathcal{C}) \subset \mathcal{A}_{i_0}$.

It may happen that $\pi(\mathcal{C})$ is just a point. In that case, such a point cannot be contained in a proper flat subgroup of $\mathcal{A}_{i_0}$, as otherwise the curve $\mathcal{C}$ would be contained in a proper flat subgroup of $\mathcal{A}$, contradicting the assumptions of Theorem 4.1.3. On the other hand, if $\boldsymbol{c}$ is a point of intersection between $\mathcal{C}$ and G, then $\pi(\boldsymbol{c}) \subseteq G_{i_0}$. Therefore, we conclude that, under the assumption of Theorem 4.1.3, if $\pi(\mathcal{C})$ is a point, then $\mathcal{C}$ does not intersect subgroups as in (4.2.1).

Suppose then from now on that $\pi(\mathcal{C})$ is a curve and denote by $\tilde{\pi}$ the restriction of π to $\mathcal{C}$. Using an argument similar to the previous paragraph, under the assumptions of Theorem 4.1.3, the curve $\pi(\mathcal{C})$ is not contained in any proper flat subgroup of $\mathcal{A}_{i_0}$. Now, let G be as in (4.2.1) and let $\boldsymbol{c} \in \mathcal{C} \cap G$ be a singular intersection point.

Notice that $G_{i_0} = \pi(G)$ and $\pi(\boldsymbol{c}) \in \pi(\mathcal{C}) \cap G_{i_0}$. We have two cases:

1. The differential $d\tilde{\pi}_{\boldsymbol{c}}$ at $\boldsymbol{c}$ is the trivial map. This is the same as saying that $\boldsymbol{c}$ is a ramification point for the map $\tilde{\pi}$. Since this is a non-constant map between curves, it admits only finitely many ramification points.

2. The differential $d\tilde{\pi}_{\boldsymbol{c}}$ at $\boldsymbol{c}$ is an isomorphism. Hence, the image of $d\tilde{\pi}$, that is, $d\pi_{\boldsymbol{c}}(T_{\boldsymbol{c}}\mathcal{C})$, is exactly equal to the tangent space of $\pi(\mathcal{C})$ at $\pi(\boldsymbol{c})$. Moreover, since by assumption $T_{\boldsymbol{c}}\mathcal{C} \subseteq T_{\boldsymbol{c}}G$, we have that $d\pi_{\boldsymbol{c}}(T_{\boldsymbol{c}}\mathcal{C}) \subseteq d\pi_{\boldsymbol{c}}(T_{\boldsymbol{c}}G) = T_{\boldsymbol{c}}G_{i_0}$. In other terms, $\pi(\boldsymbol{c})$ is a point of singular intersection between $\pi(\mathcal{C})$ and G_{i_0}. Moreover, since $\tilde{\pi}$ is a finite map between curves, only finitely many points of $\mathcal{C}$ lie above $\pi(\boldsymbol{c})$.

So, to prove Theorem 4.1.3 we can assume that $\mathcal{A} = \mathcal{B}^n$ is the n-th fibered power of a geometrically simple abelian scheme $\mathcal{B}$ of relative dimension g over the base S.

The case $\mathcal{B}$ geometrically simple can be further divided in these four cases:

1. $g_1 \geq 2$ and $\mathcal{B}$ is isotrivial;

2. $g_1 \geq 2$ and $\mathcal{B}$ is not isotrivial;

3. $g_1 = 1$ and $\mathcal{B}$ is isotrivial;

4. $g_1 = 1$ and $\mathcal{B}$ is not isotrivial.

In the first two cases, G needs to have codimension ≥ 2, so the claim follows from [HP16] in the isotrivial case and from [BC20] in the non isotrivial one, while in the third case the claim follows from Theorem 3.2.2. Therefore, we are only left to prove the fourth case.

4.2.2 Reduction to the Legendre elliptic scheme

In the last subsection we showed that, to prove Theorem 4.1.3, it is enough to prove the following special case:

Theorem 4.2.6. *Let S be an irreducible and smooth curve and let $\mathcal{E} \to S$ be a non isotrivial elliptic scheme over S both defined over $\overline{\mathbb{Q}}$. Let $\mathcal{A} = \mathcal{E}^n$ be the n-th fibered power of $\mathcal{E}$ over S, and let $\mathcal{C}$ be an irreducible curve in $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$ and not contained in any proper subgroup schemes of $\mathcal{A}$. Let*

$$\mathcal{C}^{\{1,sing\}} = \bigcup_{\substack{G<\mathcal{A} \text{ flat} \\ codim\ G\geq 1}} \mathcal{C} \cap_{sing} G.$$

Then, $\mathcal{C}^{\{1,sing\}}$ is a finite set.

We will show that this is implied by Theorem 4.1.4. To do so, we will need to use some results from a work of Habegger [Hab13a].
Consider the Legendre family defined by

$$Y^2 = X(X-1)(X-\lambda).$$

This gives an example of a non isotrivial elliptic scheme, which we call the Legendre elliptic scheme and denote by $\mathcal{L}$, over the modular curve $Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. We write $\mathcal{A}_L$ for the n-fold fibered power of $\mathcal{L}$ and π_L for the structural morphism $\pi_L : \mathcal{A}_L \to Y(2)$.

Lemma 4.2.7 ([Hab13a, Lemma 5.4])**.** *Let $\mathcal{A}$ be as above. After possibly replacing S by a Zariski open, nonempty subset, there exists an irreducible, non-singular, curve S' defined over $\overline{\mathbb{Q}}$ such that we have a commutative diagram*

$$\begin{array}{ccccc} \mathcal{A} & \xleftarrow{f} & \mathcal{A}' & \xrightarrow{e} & \mathcal{A}_L \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow{\scriptstyle \pi_L} \\ S & \xleftarrow[l]{} & S' & \xrightarrow[\lambda]{} & Y(2) \end{array}$$

where l is finite, λ is quasi-finite, $\mathcal{A}'$ is the abelian scheme $\mathcal{A} \times_S S'$, f is finite and flat and e is quasi-finite and flat. Moreover, the restriction of f and e to any fiber of $\mathcal{A}' \to S'$ is an isomorphism of abelian varieties.

The following lemma, already stated in [BC16], follows from the first part of Lemma 5.5 of [Hab13a].

Lemma 4.2.8 ([BC16, Lemma 2.4])**.** *If G is a flat subgroup scheme of $\mathcal{A}$, then $e(f^{-1}(G))$ is a flat subgroup scheme of $\mathcal{A}_L$ of the same dimension. Moreover, let X be a subvariety of $\mathcal{A}$ dominating S and not contained in a proper flat subgroup scheme of $\mathcal{A}$, let X'' be an irreducible component of $f^{-1}(X)$ and let X' be the Zariski closure of $e(X'')$ in $\mathcal{A}_L$. Then, X' has the same dimension as X, dominates $Y(2)$ and is not contained in a proper flat subgroup scheme of $\mathcal{A}_L$.*

Using these two lemmas we can now prove how Theorem 4.2.6 follows from Theorem 4.1.4.

Proof of Theorem 4.2.6 assuming Theorem 4.1.4. First, note that, since $\mathcal{C}$ is not contained in any proper subgroup scheme, it is in particular not contained in any flat subgroup scheme. Moreover, it is also not contained in any fiber, as each fiber with the zero section forms a subgroup scheme.

To ease the notation, we also extend the definition of $\mathcal{C}^{\{1,sing\}}$ to non-irreducible curves as follows:

$$\mathcal{C}^{\{1,sing\}} := \bigcup_{\mathcal{D}\subseteq\mathcal{C} \text{ irred}} \bigcup_{\substack{G \text{ flat} \\ \operatorname{codim} G\geq 1}} \mathcal{D} \cap_{sing} G$$

where $\mathcal{D}$ runs through the irreducible components of $\mathcal{C}$.

Keeping the notation from Lemma 4.2.7, since e is a quasi-finite morphism, if $e(f^{-1}(\mathcal{C}^{\{1,sing\}}))$ is finite, then $\mathcal{C}^{\{1,sing\}}$ is finite. By Lemma 4.2.8 and by Proposition 4.2.1, we have that

$$\mathcal{Z} := e(f^{-1}(\mathcal{C}^{\{1,sing\}})) \setminus e(f^{-1}(\mathcal{C}))^{\{1,sing\}}$$

is contained in the fibers over the branch locus of λ and over the image by λ of the ramification locus of l. Hence, $\mathcal{Z}$ is contained in the intersection between the dominant curve $e(f^{-1}(\mathcal{C}))$ and a finite number of fibers of π_L. Then, it is a finite set, as by Lemma 4.2.8 no irreducible component of $e(f^{-1}(\mathcal{C}))$ can be contained in a fiber of π_L. Hence, it is enough to prove that $e(f^{-1}(\mathcal{C}))^{\{1,sing\}}$ is a finite set, since $e(f^{-1}(\mathcal{C}^{\{1,sing\}})) \subseteq \mathcal{Z} \cup e(f^{-1}(\mathcal{C}))^{\{1,sing\}}$.

Let us consider the Zariski closure $\mathcal{C}'$ of $e(\mathcal{C}'')$ for an irreducible component $\mathcal{C}''$ of $f^{-1}(\mathcal{C})$. Then, $\mathcal{C}'$ is irreducible, and by Lemma 4.2.8, $\mathcal{C}'$ is a curve in $\mathcal{A}_L$ dominating $Y(2)$ and not contained in a proper flat subgroup scheme. Moreover, $e(f^{-1}(\mathcal{C}))$ is a finite union of such $\mathcal{C}'$s. Proving that $(\mathcal{C}')^{\{1,sing\}}$ is finite will make us conclude the proof. By Lemma 4.2.5, each flat subgroup scheme of codimension at least 1 of $\mathcal{A}_L$ is contained in $\ker(\varphi_{\boldsymbol{a}})$ for some $\boldsymbol{a} \in \mathbb{Z}^n$. So, it is enough to show that $\bigcup \mathcal{C}' \cap_{sing} \ker(\varphi_{\boldsymbol{a}})$ is finite, where the union is taken over all $\boldsymbol{a} \in \mathbb{Z}^n \setminus \{0\}$. The claim now follows by applying Theorem 4.1.4 since $\mathcal{C}'$ is not contained in a proper flat subgroup scheme. □

4.3 Periods and elliptic logarithms

In this section, we recall some tools and results from [MZ12] and [BC16] regarding the elliptic logarithms.

We start by recalling our setting. We defined $\lambda : \mathcal{L} \to Y(2) = \mathbb{P} \setminus \{0, 1, \infty\}$ to be the Legendre elliptic scheme, as a subscheme of $\mathbb{P}^2 \times Y(2)$, with fibers $\mathcal{L}_\lambda$ defined in affine coordinates by the equation

$$Y^2 = X(X-1)(X-\lambda).$$

It naturally comes with functions X and Y defined on it. We denote by $\mathcal{L}^n$ its n-th fibered power, with coordinates $P_1, \ldots, P_n$, and with corresponding functions $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$.

We have an irreducible curve $\mathcal{C}$, defined over $\overline{\mathbb{Q}}$ and embedded in $\mathcal{L}^n$, not contained in a fiber over $Y(2)$, and not contained in any proper flat subgroup of $\mathcal{L}^n$.

Let $\widehat{\mathcal{C}}$ be the subset of points $\boldsymbol{c} \in \mathcal{C}$ such that for every $j = 1, \dots, n$ the values $x_j(\boldsymbol{c})$ are different from $0, 1, \infty$ and $\lambda(\boldsymbol{c})$, and such that $\boldsymbol{c}$ is not a singular point or a point on which the differential of λ vanishes. Note that, in this way, we exclude finitely many $\boldsymbol{c} \in \mathcal{C}$, and these are algebraic points of $\mathcal{C}$. Moreover, on $\widehat{\mathcal{C}}$, the coordinate function λ has a local inverse everywhere.

Fix a point $\boldsymbol{c}_* \in \widehat{\mathcal{C}}$ and a (analytic) simply connected neighborhood $N_{\boldsymbol{c}_*}$ on which λ has an inverse. On $N_{\boldsymbol{c}_*}$ it is possible to define analytic functions f, g that form a basis of the local system of periods $\Pi_{\mathcal{E}_\lambda}$. More specifically, we are choosing an identification between the relative tangent bundle $T_{O \circ \lambda(N_{\boldsymbol{c}_*})}\mathcal{L}$ and $N_{\boldsymbol{c}_*} \times \mathbb{C}$ and functions $f, g : N_{\boldsymbol{c}_*} \to \mathbb{C}$ such that for every $\boldsymbol{c} \in N_{\boldsymbol{c}_*}$ the complex numbers $f(\boldsymbol{c})$ and $g(\boldsymbol{c})$ form a $\mathbb{Z}$-basis for the kernel of the exponential map for $\mathcal{L}_{\lambda(\boldsymbol{c})}$ induced by the identification above. In particular, $\mathcal{L}_{\lambda(\boldsymbol{c})} \cong \mathbb{C}/f(\boldsymbol{c})\mathbb{Z} + g(\boldsymbol{c})\mathbb{Z}$ for any $\boldsymbol{c} \in N_{\boldsymbol{c}_*}$. For ease of argument, we will be assuming in the following that the identification chosen is the one induced by the affine model $Y^2 = X(X-1)(X-\lambda)$ and that f and g are the analytic continuations of hypergeometric functions in $\lambda(\boldsymbol{c})$. With a more general choice the proofs carries through with minor variations in the shapes of the formulae.

Furthermore, we can define functions $z_1, \dots, z_n$, again analytic on $N_{\boldsymbol{c}_*}$, such that $\exp_{\lambda(\boldsymbol{c})}(z_j(\boldsymbol{c})) = P_j(\boldsymbol{c})$, for any $\boldsymbol{c} \in N_{\boldsymbol{c}_*}$, where exp is the exponential map induced by the identification above. In other words, they are elliptic logarithms for the P_i's. The explicit construction of these maps can be found in [MZ12] and [BC16].

Notice that, since we are avoiding singular points and points on which the differential of λ vanishes, λ gives an analytic isomorphism $\lambda : N_{\boldsymbol{c}_*} \to \lambda(N_{\boldsymbol{c}_*})$. Therefore, we can view $f, g, z_1, \dots, z_n$ as analytic functions on $\lambda(N_{\boldsymbol{c}_*})$, and, in particular, it makes sense to consider $\frac{\partial f}{\partial \lambda}$, $\frac{\partial g}{\partial \lambda}$ and $\frac{\partial z_i}{\partial \lambda}$.

4.3.1 A functional transcendence result

Keeping the same notation as the previous subsection, we prove the following transcendence result, that will be needed later.

Lemma 4.3.1. *The functions $z_1, \dots, z_n, \frac{\partial z_1}{\partial \lambda}, \dots, \frac{\partial z_n}{\partial \lambda}$ are algebraically independent over $\mathbb{C}\left(\lambda, f, g, \frac{\partial f}{\partial \lambda}, \frac{\partial g}{\partial \lambda}\right)$ on $N_{\boldsymbol{C}_*}$.*

Proof. The functions $z_1, \dots, z_n, f, g$ are analytic functions of λ, linearly independent over $\mathbb{Z}$. Indeed, a relation $a_1 z_1 + \cdots + a_n z_n = a_{n+1} f + a_{n+2} g$, with integer coefficients, would map via $\exp_\lambda$ to a relation of the form (4.1.1) on $N_{\boldsymbol{C}_*}$, and therefore on all of $\mathcal{C}$, which cannot hold by the hypothesis of the theorem. Moreover, if $\wp_\lambda$ is the Weierstrass $\wp$-function associated with $\Pi_{\mathcal{E}_\lambda} = f(\lambda)\mathbb{Z} + g(\lambda)\mathbb{Z}$, the $\wp_\lambda(z_i)$ are algebraic functions of λ because $\wp_\lambda(z_j) = x_j - \frac{1}{3}(\lambda+1)$ (see [MZ10,

(3.8), p. 1683]). Therefore, applying [Ber90, Théorème 5, p. 136] we have that $z_1, \dots, z_n, \frac{\partial z_1}{\partial \lambda}, \dots, \frac{\partial z_n}{\partial \lambda}$ are algebraically independent over $\mathbb{C}\left(\lambda, f, g, \frac{\partial f}{\partial \lambda}\right)$.
To conclude, we will prove that $\frac{\partial g}{\partial \lambda} \in \mathbb{C}\left(\lambda, f, g, \frac{\partial f}{\partial \lambda}\right)$, and so actually we have that $\mathbb{C}\left(\lambda, f, g, \frac{\partial f}{\partial \lambda}\right) = \mathbb{C}\left(\lambda, f, g, \frac{\partial f}{\partial \lambda}, \frac{\partial g}{\partial \lambda}\right)$. This part of the proof was suggested to us by Daniel Bertrand.
Since f and g are hypergeometric series, they satisfy the hypergeometric differential equation (see [Hus87, p. 182]):

$$\lambda(1-\lambda)\frac{d^2\omega}{d\lambda^2} + (1-2\lambda)\frac{d\omega}{d\lambda} - \frac{1}{4}\omega = 0.$$

Since they are linearly independent, they form a basis of the space of solution of such a differential equation. We can therefore consider the Wronskian associated with them:

$$W(\lambda) := f\frac{\partial g}{\partial \lambda} - g\frac{\partial f}{\partial \lambda}.$$

By explicit computation, we get that

$$\frac{dW}{d\lambda} = -\frac{2\lambda - 1}{\lambda(\lambda-1)}W;$$

solving this differential equation gives $W = k(\lambda^2 - \lambda)^{-1}$ for some $k \in \mathbb{C}$, hence $W \in \mathbb{C}(\lambda)$. We conclude that

$$\frac{\partial g}{\partial \lambda} = \left(W + g\frac{\partial f}{\partial \lambda}\right)\frac{1}{f} \in \mathbb{C}\left(\lambda, f, g, \frac{\partial f}{\partial \lambda}\right),$$

proving the claim. □

4.4 The main estimate

Let us fix a $\boldsymbol{c}_* \in \widehat{\mathcal{C}}$ and a neighborhood $N_{\boldsymbol{c}_*}$ of $\boldsymbol{c}_*$ in $\widehat{\mathcal{C}}$. Moreover, let us fix a closed disc $D_{\boldsymbol{c}_*}$ inside $\lambda(N_{\boldsymbol{c}_*})$, centered in $\lambda(\boldsymbol{c}_*)$.

In Section 4.3 we defined the analytic functions f, g and z_i on a small neighborhood of $\boldsymbol{c}_*$, which we may assume to be $N_{\boldsymbol{c}_*}$ up to shrinking, as a basis for the local system of periods of $\mathcal{L}_\lambda$ and elliptic logarithms of the P_j. For the rest of this section, we suppress the dependence on $\boldsymbol{c}_*$ and D in the notation, since they are fixed.
We use Vinogradov's $\ll$ notation. The implied constants will always depend on D.

Given $\boldsymbol{a} \in \mathbb{Z}^n \setminus \{0\}$, we define $D(\boldsymbol{a})$ to be the set of $\lambda \in D$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^{n} a_i z_i(\lambda) = a_{n+1} f(\lambda) + a_{n+2} g(\lambda), \\ \displaystyle\sum_{i=1}^{n} a_i \frac{\partial z_i}{\partial \lambda}(\lambda) = a_{n+1}\frac{\partial f}{\partial \lambda}(\lambda) + a_{n+2}\frac{\partial g}{\partial \lambda}(\lambda), \end{cases} \tag{4.4.1}$$

for some $a_{n+1}, a_{n+2} \in \mathbb{Z}$. Notice that $\lambda(\boldsymbol{c}) \in D(\boldsymbol{a})$ corresponds to $\mathcal{C}$ being tangent to the subgroup $\sum a_i P_i = O$ in $\boldsymbol{c}$.

For a vector of integers $\boldsymbol{a}$ we denote by $|\boldsymbol{a}| = \max\{|a_1|, \dots, |a_n|\}$.

Proposition 4.4.1. *Under the hypotheses of Theorem 4.1.4, for every $\varepsilon > 0$ we have $\left|D(\boldsymbol{a})\right| \ll_\varepsilon |\boldsymbol{a}|^\varepsilon$, for every non-zero $\boldsymbol{a} \in \mathbb{Z}^n$.*

Before proving this estimate, let us collect a few more preliminary facts.

Let us define

$$\Delta = f\overline{g} - \overline{f}g,$$

which does not vanish on D, since $f(\lambda)$ and $g(\lambda)$ are $\mathbb{R}$-linearly independent for every $\lambda \in D$. Moreover, for every $j = 1, \dots, n$ let u_j, v_j be the functions from D to $\mathbb{C}$ defined by

$$u_j = \frac{z_j\overline{g} - \overline{z_j}g}{\Delta}, \qquad v_j = \frac{z_j\overline{f} - \overline{z_j}f}{\Delta}.$$

One can easily check that these functions are real-valued and, furthermore, that we have

$$z_j = u_j f + v_j g;$$

u_j and v_j are therefore the so-called *Betti coordinates* of P_j. Taking derivatives on both sides yields

$$\frac{\partial z_j}{\partial \lambda} = \frac{\partial u_j}{\partial \lambda} f + \frac{\partial v_j}{\partial \lambda} g + u_j \frac{\partial f}{\partial \lambda} + v_j \frac{\partial g}{\partial \lambda}.$$

If we view D as a subsets of $\mathbb{R}^2$, then $u_j, v_j, \frac{\partial u_j}{\partial \lambda}, \frac{\partial v_j}{\partial \lambda}$ are real analytic functions on a neighborhood of D.

Define now $\Theta : D \to \mathbb{R}^{4n}$ by

$$\lambda \mapsto \left(u_1(\lambda), v_1(\lambda), \dots, u_n(\lambda), v_n(\lambda), \frac{\partial u_1}{\partial \lambda}(\lambda), \frac{\partial v_1}{\partial \lambda}(\lambda), \dots, \frac{\partial u_n}{\partial \lambda}(\lambda), \frac{\partial v_n}{\partial \lambda}(\lambda)\right)$$

and set $\Sigma = \Theta(D)$. Since Θ is analytic and D is a closed disc, we have that Σ is definable in $\mathbb{R}_{\text{an}}$, as image of a definable set under a definable function.

Lemma 4.4.2 ([BC17], Lemma 4.2)**.** *Under the hypotheses of Theorem 4.1.4, there exists a constant γ (depending only on D) such that, for every choice of integers $a_1, \dots, a_{n+2}$, not all zero, the number of λ in D with*

$$a_1 z_1(\lambda) + \cdots + a_n z_n(\lambda) = a_{n+1} f(\lambda) + a_{n+2} g(\lambda),$$

is at most γ.

In what follows $(u_1, v_1, \dots, r_n, s_n)$ will denote the coordinates in $\mathbb{R}^{4n}$.
For $T > 0$, let us denote by $\Sigma^{(1)}(\boldsymbol{a}, T)$ the set of points of Σ of coordinates $(u_1, v_1, \dots, r_n, s_n)$ such that there exist $a_{n+1}, a_{n+2} \in \mathbb{Z} \cap [-T, T]$ with

$$\begin{cases} a_1u_1 + \cdots + a_nu_n = a_{n+1}, \\ a_1v_1 + \cdots + a_nv_n = a_{n+2}, \\ a_1r_1 + \cdots + a_nr_n = 0, \\ a_1s_1 + \cdots + a_ns_n = 0. \end{cases} \tag{4.4.2}$$

Lemma 4.4.3. *Under the hypotheses of Theorem 4.1.4, for every $\varepsilon > 0$, we have*

$$\left|\Sigma^{(1)}(\boldsymbol{a}, T)\right| \ll_\varepsilon (\max\{|\boldsymbol{a}|, T\})^\varepsilon.$$

Proof. Let us set $T' = \max\{|\boldsymbol{a}|, T\}$ and fix $\varepsilon > 0$.
Let us define W to be the set of $(\alpha_1, \dots, \alpha_{n+2}, u_1, \dots, s_n) \in \mathbb{R}^{n+2} \times \Sigma$ such that

$$\begin{cases} \alpha_1u_1 + \cdots + \alpha_nu_n = \alpha_{n+1}, \\ \alpha_1v_1 + \cdots + \alpha_nv_n = \alpha_{n+2}, \\ \alpha_1r_1 + \cdots + \alpha_nr_n = 0, \\ \alpha_1s_1 + \cdots + \alpha_ns_n = 0; \end{cases} \tag{4.4.3}$$

this is a definable set in $\mathbb{R}_{\mathrm{an}}$. Using the notation introduced in (2.2.2), we denote by $W^\sim(\mathbb{Q}, T')$ the set of tuples $(\alpha_1, \dots, \alpha_{n+2}, u_1, \dots, s_n) \in W$ with rational $\alpha_1, \dots, \alpha_{n+2}$ of height at most T'. Note that $\pi_2(W^\sim(\mathbb{Q}, T')) \supseteq \Sigma^{(1)}(\boldsymbol{a}, T)$, where $\pi_2 : W \to \Sigma$ is the projection to Σ. Then, $\left|\Sigma^{(1)}(\boldsymbol{a}, T)\right| \leq \left|\pi_2(W^\sim(\mathbb{Q}, T'))\right|$. We claim that $\left|\pi_2(W^\sim(\mathbb{Q}, T'))\right| \ll_\varepsilon (T')^\varepsilon$.

Assume by contradiction that this is not the case; then, by Proposition 2.2.10, there exists a continuous definable $\delta : [0, 1] \to W$ such that $\delta_1 := \pi_1 \circ \delta : [0, 1] \to \mathbb{R}^{n+2}$ is semi-algebraic and $\delta_2 := \pi_2 \circ \delta : [0, 1] \to \Sigma$ is non-constant. Therefore, there is a connected infinite subset $E \subseteq [0, 1]$ such that $\delta_1(E)$ is contained in a real algebraic curve and $\delta_2(E)$ has positive dimension. Then, there exists a connected infinite $D' \subseteq D$ with $\Theta(D') \subseteq \delta_2(E)$.
The coordinate functions $\alpha_1, \dots, \alpha_{n+2}$ on D' satisfy $n + 1$ independent algebraic relations with coefficients in $\mathbb{C}$. Moreover, the relations given by (4.4.3) translate to

$$\begin{cases} \alpha_1z_1 + \cdots + \alpha_nz_n = \alpha_{n+1}f + \alpha_{n+2}g, \\ \alpha_1\dfrac{\partial z_1}{\partial \lambda} + \cdots + \alpha_n\dfrac{\partial z_n}{\partial \lambda} = \alpha_{n+1}\dfrac{\partial f}{\partial \lambda} + \alpha_{n+2}\dfrac{\partial g}{\partial \lambda}, \end{cases}$$

adding 2 algebraic relations among the $\alpha_1, \dots, \alpha_{n+2}$ and the $z_j, \frac{\partial z_j}{\partial \lambda}, f, g, \frac{\partial f}{\partial \lambda}, \frac{\partial g}{\partial \lambda}$. Thus, on D', and therefore by continuation on the whole D, the $n + 2 + 2n$ functions $\alpha_1, \dots, \alpha_{n+2}, z_1, \dots, z_n, \frac{\partial z_1}{\partial \lambda}, \dots, \frac{\partial z_n}{\partial \lambda}$ satisfy $n + 1 + 2$ independent algebraic relations over $F = \mathbb{C}\left(\lambda, f, g, \frac{\partial f}{\partial \lambda}, \frac{\partial g}{\partial \lambda}\right)$. Therefore,

$$\mathrm{trdeg}_F F\left(z_1, \dots, z_n, \frac{\partial z_1}{\partial \lambda}, \dots, \frac{\partial z_n}{\partial \lambda}\right) \leq 2n - 1.$$

This contradicts Lemma 4.3.1, and proves the claim and the lemma.

□

Proof of Proposition 4.4.1. If $t \in D(\boldsymbol{a})$, then $\Theta(t)$ satisfies (4.4.2) for some integers a_{n+1} and a_{n+2}. Now, since D is compact, we have that the sets $z_i(D), f(D), g(D)$ are bounded, and therefore we can choose a_{n+1}, a_{n+2} bounded solely in terms of $|\boldsymbol{a}|$. Therefore, we have $\Theta(t) \in \Sigma^{(1)}(\boldsymbol{a}, T_0)$, with $T_0 \ll |\boldsymbol{a}|$. Now, by Lemma 4.4.2, we have $|D(\boldsymbol{a})| \ll |\Sigma^{(1)}(\boldsymbol{a}, T_0)|$ and the claim follows from Lemma 4.4.3. □

4.5 Small generators of the relations lattices

In this section we recall general facts about linear relations on elliptic curves. Recall from Section 2.1.3 that for a point $(\xi_1, \dots, \xi_N) \in \overline{\mathbb{Q}}^N$, the (affine) logarithmic Weil height $h(\xi_1, \dots, \xi_N)$ is defined by

$$h(\xi_1, \dots, \xi_N) = \frac{1}{[\mathbb{Q}(\xi_1, \dots, \xi_N) : \mathbb{Q}]} \sum_v \log \max\left\{1, |\xi_1|_v, \dots, |\xi_N|_v\right\},$$

where v runs over a suitably normalized set of valuations of $\mathbb{Q}(\xi_1, \dots, \xi_N)$.

We recall our running setting. We have an irreducible curve $\mathcal{C}$ embedded in the nth fibered power of the Legendre elliptic scheme $\mathcal{L}^n \to Y(2)$, not contained in a fiber or proper flat subgroup of $\mathcal{L}^n$. We considered the subset $\widehat{\mathcal{C}}$, obtained by removing from $\mathcal{C}$ finitely many points, and showed that for any point $\boldsymbol{c} \in \widehat{\mathcal{C}}$, we can define generators of the period lattice $f(\boldsymbol{c}')$ and $g(\boldsymbol{c}')$, and the elliptic logarithms z_i of the projections $P_i(\boldsymbol{c}') \in \lambda(\boldsymbol{c}')$, for $\boldsymbol{c}'$ in a neighborhood $N_{\boldsymbol{c}}$ of $\boldsymbol{c}$. Moreover, we can look at such elliptic logarithms as a function of λ, so it does make sense to write $\frac{\partial z_i}{\partial \lambda}$.

We define, for $\boldsymbol{c} \in \widehat{\mathcal{C}}$, the lattice of relations:

$$\begin{aligned} L(\boldsymbol{c}) &= \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid a_1 P_1(\boldsymbol{c}) + \cdots + a_n P_n(\boldsymbol{c}) = O(\lambda(\boldsymbol{c})) \in \mathcal{L}_{\lambda(\boldsymbol{c})}\} \\ &= \{\boldsymbol{a} \in \mathbb{Z}^n \mid \exists a_{n+1}, a_{n+2} \in \mathbb{Z} : \sum_{i=1}^{n} a_i z_i(\boldsymbol{c}) = a_{n+1} f(\boldsymbol{c}) + a_{n+2} g(\boldsymbol{c})\}. \end{aligned}$$

This is a sublattice of $\mathbb{Z}^n$, and it is of some positive rank r whenever $\boldsymbol{c}$ is contained in a proper subgroup of $\mathcal{L}^n$.

We can also define the lattice of *singular* relations:

$$\begin{aligned} L^{sing}(\boldsymbol{c}) &= \\ &= \{(a_1,\dots,a_n)\in\mathbb{Z}^n \mid a_1P_1(\boldsymbol{c})+\cdots+a_nP_n(\boldsymbol{c})=O(\lambda(\boldsymbol{c})) \text{ tangentially}\} \\ &= \left\{ \boldsymbol{a}\in\mathbb{Z}^n \;\middle|\; \exists a_{n+1},a_{n+2}\in\mathbb{Z} : \sum_{i=1}^n a_i z_i(\boldsymbol{c}) = a_{n+1}f(\boldsymbol{c})+a_{n+2}g(\boldsymbol{c}), \right. \\ &\qquad\qquad \left. \sum_{i=1}^n a_i\frac{\partial z_i}{\partial\lambda}(\boldsymbol{c}) = a_{n+1}\frac{\partial f}{\partial\lambda}(\boldsymbol{c})+a_{n+2}\frac{\partial g}{\partial\lambda}(\boldsymbol{c}) \right\}. \end{aligned}$$

Suppose now that $\boldsymbol{c}\in\widehat{\mathcal{C}}$ is one of the points considered in Theorem 4.1.4, i.e. there exits $\boldsymbol{a}\in\mathbb{Z}^n\setminus\{0\}$ with $\varphi_{\boldsymbol{a}}(\boldsymbol{c})=O(\lambda(\boldsymbol{c}))$ and $\mathcal{C}$ and $\ker(\varphi_{\boldsymbol{a}})$ intersect tangentially at $\boldsymbol{c}$.
In this situation we have that the ranks of $L^{sing}(\boldsymbol{c})$ and $L(\boldsymbol{c})$ are both positive, and $\lambda(\boldsymbol{c})$ is an algebraic number, as $\mathcal{C}$ is defined over $\overline{\mathbb{Q}}$ and not contained in any proper algebraic subgroup of $\mathcal{L}^n$. Under this condition, we would like to show that $L^{sing}(\boldsymbol{c})$ has a set of generators with small norm $|\boldsymbol{a}|=\max\{a_1,\dots,a_n\}$. Unfortunately we are able to prove such a result only in the case when $\mathrm{rk}(L^{sing}(\boldsymbol{c}))=\mathrm{rk}(L(\boldsymbol{c}))$, but we will see that this will suffice.

By slight abuse of notation we will omit in the following the dependence on $\boldsymbol{c}$, writing λ for $\lambda(\boldsymbol{c})$ and $P_1,\dots,P_n$ for $P_1(\boldsymbol{c}),\dots,P_n(\boldsymbol{c})$.
Suppose that $P_1,\dots,P_n\in\mathcal{L}_\lambda$ are defined over some finite extension K of $\mathbb{Q}(\lambda)$ of degree $d=[K:\mathbb{Q}]$. Furthermore, assume that $P_1,\dots,P_n$ have Néron–Tate height $\hat{h}$ at most $q\geq 1$ (for the definition of Néron–Tate height, see for example p. 255 of [Mas88]). In case the P_i are all torsion, we have $\hat{h}(P_i)=0$, and we put $q=1$.

We recall the following lemma.

Lemma 4.5.1 ([BC16], Lemma 6.1, based on [Mas88])**.** *Under the above hypotheses, there are generators $\boldsymbol{a}_1,\dots,\boldsymbol{a}_r$ of L with*

$$|\boldsymbol{a}_i|\leq\delta_1 d^{\delta_2}(h(\lambda)+1)^{2n}q^{\frac{1}{2}(n-1)},$$

for some positive constants δ_1,δ_2 depending only on n.

Before stating and proving the next result we notice that, by definition, $L^{sing}\subseteq L$. In general we do not expect them to be equal.

Lemma 4.5.2. *One has that*

$$\mathbb{Q}L^{sing}\cap L = L^{sing};$$

in particular, if $\mathrm{rk}(L^{sing})=\mathrm{rk}(L)$*, then* $L^{sing}=L$*.*

Proof. Clearly we have $\mathbb{Q}L^{sing} \cap L \supseteq L^{sing}$, so we only have to prove the other inclusion.

Let $\boldsymbol{a} \in \mathbb{Q}L^{sing} \cap L$. Since $\boldsymbol{a} \in L$, there exist some integers a_{n+1}, a_{n+2} such that $\sum_{i=1}^{n} a_i z_i(\lambda) = a_{n+1} f(\lambda) + a_{n+2} g(\lambda)$, which are unique because f and g are linearly independent over $\mathbb{R}$.
Since $\boldsymbol{a} \in \mathbb{Q}L^{sing}$, then for some integer μ we have $\mu\boldsymbol{a} \in L^{sing}$, hence for some $b_{n+1}, b_{n+2} \in \mathbb{Z}$, we have

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \mu a_i z_i(\lambda) - b_{n+1} f(\lambda) - b_{n+2} g(\lambda) = 0 \\ \displaystyle\sum_{i=1}^{n} \mu a_i \frac{\partial z_i}{\partial \lambda}(\lambda) - b_{n+1} \frac{\partial f}{\partial \lambda}(\lambda) - b_{n+2} \frac{\partial g}{\partial \lambda}(\lambda) = 0 \end{cases}$$

We know, moreover, that $\sum_{i=1}^{n} \mu a_i z_i(\lambda) - \mu a_{i+1} f(\lambda) - \mu a_{i+2} g(\lambda) = 0$; since the b_i's are unique again by the linear independence of f and g, we have that $\mu a_i = b_i$ for $i = n+1, n+2$. Hence we have

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \mu a_i \frac{\partial z_i}{\partial \lambda}(\lambda) - \mu a_{n+1} \frac{\partial f}{\partial \lambda}(\lambda) - \mu a_{n+2} \frac{\partial g}{\partial \lambda}(\lambda) \\ &= \mu \left(\sum_{i=1}^{n} a_i \frac{\partial z_i}{\partial \lambda}(\lambda) - a_{n+1} \frac{\partial f}{\partial \lambda}(\lambda) - a_{n+2} \frac{\partial g}{\partial \lambda}(\lambda) \right) \end{aligned}$$

so

$$\sum a_i \frac{\partial z_i}{\partial \lambda}(\lambda) - a_{n+1} \frac{\partial f}{\partial \lambda}(\lambda) - a_{n+2} \frac{\partial g}{\partial \lambda}(\lambda) = 0,$$

from which we deduce that actually $\boldsymbol{a} \in L^{sing}$, as desired. □

Combining the two lemmas we get the following.

Corollary 4.5.3. *Let $\boldsymbol{c}, \lambda, q, d$ be as in Lemma 4.5.1; assume* $\mathrm{rk}(L^{sing}(\boldsymbol{c})) = \mathrm{rk}(L(\boldsymbol{c}))$. *Then, there are generators $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ of $L^{sing}(\boldsymbol{c})$ with*

$$|\boldsymbol{a}_i| \leq \delta_1 d^{\delta_2} (h(\lambda) + 1)^{2n} q^{\frac{1}{2}(n-1)},$$

for some positive constants δ_1, δ_2 depending only on n.

4.6 Bounded height

Let $\mathcal{C}'$ be the set of points $\boldsymbol{c}$ of $\widehat{\mathcal{C}}$ considered by Theorem 4.1.4, i.e. such that $P_1(\boldsymbol{c}), \ldots, P_n(\boldsymbol{c})$ satisfy a linear relation with integer coefficients on the specialized elliptic curve $\mathcal{L}_{\lambda(\boldsymbol{c})}$, and fix $\boldsymbol{c}_0 \in \mathcal{C}'$. Since $\mathcal{C}$ is defined over $\overline{\mathbb{Q}}$, the $x_j(\boldsymbol{c}_0)$ and $\lambda(\boldsymbol{c}_0)$ must be algebraic, unless the P_j are identically linearly dependent on $\mathcal{C}$, which we excluded by hypothesis. We want to show that $\mathcal{C}$ is a set of bounded height. This will follow from a theorem of Silverman [Sil83], which we will now state.

Theorem 4.6.1 ([Sil83, Theorem C])**.** *Let S be an irreducible quasi-projective smooth curve and let $\pi : \mathcal{A} \to S$ be an abelian scheme with no isotrivial component, where everything is defined over a number field K. Then, the set*

$$\{s \in S(\overline{K}) \mid \exists \sigma : S \to \mathcal{A} \text{ section }, \sigma \neq O, \sigma(s) = O(s) \in \mathcal{A}_s\}$$

is a set of bounded height in S. This bound is effective in principle.

Suppose now that we are in the situation of Theorem 4.1.4, with the notation established at the beginning of this section. We therefore have a curve $\mathcal{C}$ in $\mathcal{L}^n \to Y(2)$. The coordinates of this curve $P_1, \dots, P_n$ will in general be given by multisections of $\mathcal{L} \to Y(2)$, that is, sections of a finite base change of the Legendre scheme. Hence, there exists a finite base change

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\mu^\sharp} & \mathcal{L} \\ {\scriptstyle\pi}\downarrow & \lrcorner & \downarrow{\scriptstyle\lambda} \\ S & \xrightarrow{\mu} & Y(2) \end{array}$$

such that $P_1, \dots, P_n$ are given by sections $\overline{P_1}, \dots, \overline{P_n} : S \to \mathcal{E}$. In this way, the curve $\mathcal{C}$ is induced by a section $\overline{\boldsymbol{P}} : S \to \mathcal{E}^n$. We will denote by $\boldsymbol{P} : S \to \mathcal{L}^n$ the composite $\mu^\sharp \circ \overline{\boldsymbol{P}}$, and by slight abuse of notation, with P_i the projection on each component of $\mathcal{L}$.

Now, for every $\boldsymbol{a} \in \mathbb{Z}^n \setminus 0$, the coordinates of $\boldsymbol{c} \in \mathcal{C}$ satisfy the linear relation with coefficients $\boldsymbol{a}$ if and only if the section

$$a_1\overline{P_1} + \cdots + a_n\overline{P_n} : S \to \mathcal{E}$$

sends $\boldsymbol{P}^{-1}(\boldsymbol{c})$ to the zero element of the corresponding fiber. Hence, by Theorem 4.6.1, there exists a constant $\overline{\gamma_1}$ such that the height of every element of $\boldsymbol{P}^{-1}(\boldsymbol{c}_0)$ is less that $\overline{\gamma_1}$.

To obtain a bound for $\lambda(\boldsymbol{c}_0)$ we apply a height inequality due again to Silverman.

Theorem 4.6.2 ([Sil11, Theorem 1], special case of curves)**.** *Let $\varphi : W \dashrightarrow V$ be a dominant rational map between quasi-projective curves, all defined over $\overline{\mathbb{Q}}$. Fix height functions h_V and h_W on V and W, respectively, corresponding to ample divisors. Then there are constants $C_1 > 0$ and C_2, effective in principle, such that*

$$h_V(\varphi(P)) \geq C_1 h_W(P) - C_2 \text{ for all } P \in W(\overline{\mathbb{Q}}).$$

Combining these, we obtain a constant $\gamma_1 > 0$, effectively computable in principle, such that

$$h(\lambda(\boldsymbol{c}_0)) \leq \gamma_1. \tag{4.6.1}$$

This bound has some important consequences that we will see in the rest of this section.

First, we will show a "Large Galois Orbit" type of statement. Let k be a

number field on which $\mathcal{C}$ is defined. We can also assume that the finitely many points we excluded from $\mathcal{C}$ to get $\widehat{\mathcal{C}}$, which are algebraic, are defined over k. In the proof of Theorem 4.1.4 we want to get a bound on the degree of points in $\mathcal{C}^{\{1,sing\}}$, using Proposition 4.4.1 and the fact that every point in $\mathcal{C}^{\{1,sing\}}$ has many Galois conjugates also in $\mathcal{C}^{\{1,sing\}}$. However, Proposition 4.4.1 gives a bound only in a small neighborhood of a point. The next lemma shows that a good proportion of Galois conjugates are actually in the same neighborhood. Let $\delta > 0$ be a small real number, and let

$$\mathcal{C}^{\delta} = \left\{ \boldsymbol{c} \in \mathcal{C} \mid \|\boldsymbol{c}\| \leq \frac{1}{\delta}, \|\boldsymbol{c} - \boldsymbol{c}'\| \geq \delta \text{ for all } \boldsymbol{c} \in \mathcal{C} \setminus \widehat{\mathcal{C}} \right\},$$

where $\|\cdot\|$ is the standard norm of $\mathbb{C}^{2n+1}$.

Lemma 4.6.3. *There is a positive δ such that there are at least $\frac{1}{2}[k(\boldsymbol{c}_0) : k]$ different k-embeddings σ of $k(\boldsymbol{c}_0)$ in $\mathbb{C}$ such that $\sigma(\boldsymbol{c}_0)$ lies in $\mathcal{C}^{\delta}$ for all $\boldsymbol{c}_0 \in \mathcal{C}'$.*

Proof. See Lemma 8.2 of [MZ14]. □

Remark 4.6.4. We would like to point out that it would be possible to avoid the restriction to a smaller domain and the use of the previous lemma by either constructing the Betti coordinates u_j, v_j in a more global way, following [Gao20], or by using the results from Peterzil and Starchenko [PS04] and Jones and Schmidt [JS20], who proved that the Weierstrass $\wp$ function, and hence its derivative, can be defined globally in the structure $\mathbb{R}_{\text{an,exp}}$.

Secondly, we show a bound for the Néron-Tate heights and the degree of the coordinates of the points in $\mathcal{C}^{\{1,sing\}}$, needed in order to apply Corollary 4.5.3.

Lemma 4.6.5 ([BC16, Lemma 7.2])**.** *There exist positive constants γ_2, γ_3 such that, for every $\boldsymbol{c}_0 \in \mathcal{C}'$ and every $i = 1, \ldots, n$ we have*

$$\hat{h}(P_i(\boldsymbol{c}_0)) \leq \gamma_2,$$

and the $P_i(\boldsymbol{c}_0)$'s are defined over some number field $K \supseteq k(\lambda(\boldsymbol{c}_0))$, with

$$[K : \mathbb{Q}] \leq \gamma_3 [k(\lambda(\boldsymbol{c}_0)) : k].$$

Remark 4.6.6. In the assumptions of Lemma 7.2 of [BC16] the authors assume that $\boldsymbol{c}_0$ satisfies two linearly independent conditions on the specialized elliptic curve $\mathcal{L}_{\lambda(\boldsymbol{c}_0)}$. However, the proof uses this only to get a height bound like (4.6.1), so the same proof works in our situation as well.

4.7 Proof of Theorem 4.1.4

We want to show that there are at most finitely many $\boldsymbol{c}$ on the curve such that $P_1(\boldsymbol{c}), \dots, P_n(\boldsymbol{c})$ satisfy a linear relation with integer coefficients on $\mathcal{L}_{\lambda(\boldsymbol{c})}$ with multiplicity ≥ 2. Let $\mathcal{C}^{\{1,sing\}}$ be the set of such points. Since $\mathcal{C} \setminus \widehat{\mathcal{C}}$ is finite, as observed in Section 4.3, we only need to prove the finiteness of

$$\mathcal{C}'' := \mathcal{C}^{\{1,sing\}} \cap \widehat{\mathcal{C}}.$$

In the previous section we showed that for $\boldsymbol{c} \in \mathcal{C}'' \subseteq \mathcal{C}'$ the height of $\lambda(\boldsymbol{c})$ is bounded by a constant γ_1. By Northcott's theorem [Nor49] we then only need to bound the degree d of $\lambda(\boldsymbol{c})$ over k, a number field over which $\mathcal{C}$ and the points of $\mathcal{C} \setminus \widehat{\mathcal{C}}$ are defined.

Fix one $\boldsymbol{c}_0 \in \mathcal{C}''$ and $d_0 := [k(\boldsymbol{c}_0) : k]$ which we suppose large. It is enough to prove that d_0 is bounded, as $\lambda(\boldsymbol{c}_0) \in k(\boldsymbol{c}_0)$. By Theorem 1.1 of [BC16], up to removing a finite number of points from $\mathcal{C}''$ we can assume that the $P_i(\boldsymbol{c}_0)$'s do not satisfy two linearly independent linear relations, that is, $\mathrm{rk}(L(\boldsymbol{c}_0)) = 1$. Since they satisfy a linear relation with tangency condition, we also have $\mathrm{rk}(L^{sing}(\boldsymbol{c}_0)) = 1$.

By Lemma 4.6.3, we can choose δ, independently of $\boldsymbol{c}_0$, such that $\boldsymbol{c}_0$ has at least $\frac{1}{2}d_0$ conjugates in $\mathcal{C}^\delta$. Now, since $\mathcal{C}^\delta$ is compact, there are $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{\gamma_4} \in \widehat{\mathcal{C}}$ with corresponding neighborhoods $N_{\boldsymbol{c}_1}, \dots, N_{\boldsymbol{c}_{\gamma_4}}$ and $D_{\boldsymbol{c}_1}, \dots, D_{\boldsymbol{c}_{\gamma_4}} \subseteq \pi(\widehat{\mathcal{C}})$, where $D_{\boldsymbol{c}_i} \subseteq \lambda(N_{\boldsymbol{c}_i})$ contains $\lambda(\boldsymbol{c}_i)$ and is analytically isomorphic to a closed disc, and we have that the $\lambda^{-1}(D_{\boldsymbol{c}_i}) \cap N_{\boldsymbol{c}_i}$ for $i = 1, \dots, \gamma_4$ cover $\mathcal{C}^\delta$.

We can assume that $D_{\boldsymbol{c}_1}$ contains $t_0^\sigma = \lambda(\boldsymbol{c}_0^\sigma)$ for at least $\frac{1}{2\gamma_4}d_0$ conjugates $\boldsymbol{c}_0^\sigma$. Since each $t \in \lambda(\mathcal{C})$ has a uniformly bounded number of preimages $\boldsymbol{c} \in \mathcal{C}$, we can suppose that there are at least $\frac{1}{\gamma_5}d_0$ distinct such t_0^σ in $D_{\boldsymbol{c}_1}$.

Now, the corresponding points $P_1(\boldsymbol{c}_0^\sigma), \dots, P_n(\boldsymbol{c}_0^\sigma)$ satisfy the same relations. Moreover, by Corollary 4.2.2, the tangency of these relations is also preserved, so we have that $\mathrm{rk}(L^{sing}(\boldsymbol{c}_0^\sigma)) = \mathrm{rk}(L(\boldsymbol{c}_0^\sigma)) = 1$.

By Lemma 4.6.5, $\hat{h}(P_i(\boldsymbol{c}_0^\sigma)) \leq \gamma_2$, and the $P_i(\boldsymbol{c}_0^\sigma)$ are defined over a field K of degree $\leq \gamma_3 d_0$ over $\mathbb{Q}$. Hence, by applying Corollary 4.5.3 and recalling (4.6.1), there exist $\boldsymbol{a} \in L^{sing}(\boldsymbol{c}_0^\sigma)$ such that

$$|\boldsymbol{a}| \leq \gamma_7 d_0^{\gamma_8}. \tag{4.7.1}$$

Recall that in Section 4.3 we defined on $D_{\boldsymbol{c}_1}$ the functions f, g to be generators of the period lattice of $\mathcal{L}_\lambda$, and the z_i as the elliptic logarithms of the P_i. Since $\boldsymbol{a} \in L^{sing}(\boldsymbol{c}_0^\sigma)$, we have

$$\begin{cases} a_1 z_1(t_0^\sigma) + \cdots + a_n z_n(t_0^\sigma) = a_{n+1} f(t_0^\sigma) + a_{n+2} g(t_0^\sigma), \\ a_1 \dfrac{\partial z_1}{\partial \lambda}(t_0^\sigma) + \cdots + a_n \dfrac{\partial z_n}{\partial \lambda}(t_0^\sigma) = a_{n+1} \dfrac{\partial f}{\partial \lambda}(t_0^\sigma) + a_{n+2} \dfrac{\partial g}{\partial \lambda}(t_0^\sigma). \end{cases}$$

Notice also that the same relationships are satisfied by all the other Galois conjugates of t_0^σ in $D_{\boldsymbol{c}_1}$, by Corollary 4.2.2.

In Section 4.4 we defined $D_{\boldsymbol{c}_1}(\boldsymbol{a})$ as the set of $t \in D_{\boldsymbol{c}_1}$ such that relations of the form (4.4.1) (same as the above) hold. In particular, we have that $t_0^\sigma \in D_{\boldsymbol{c}_1}(\boldsymbol{a})$ for all the t_0^σ as above. Therefore, $\left|D_{\boldsymbol{c}_1}(\boldsymbol{a})\right| \geq \frac{1}{\gamma_5} d_0$.
On the other hand, by Proposition 4.4.1 we have $\left|D_{\boldsymbol{c}_1}(\boldsymbol{a})\right| \leq \gamma_9 |\boldsymbol{a}|^\varepsilon$, so applying (4.7.1) we have that $\left|D_{\boldsymbol{c}_1}(\boldsymbol{a})\right| \leq \gamma_{10} d_0^{\gamma_8 \varepsilon}$. Therefore if we choose $\varepsilon = 1/(2\gamma_8)$ we have a contradiction if d_0 is large enough.

We just deduced that d_0 is bounded and, by (4.6.1) and Northcott's Theorem, we have the claim of Theorem 4.1.4.

□

4.8 An application to elliptic divisibility sequences

In this section we discuss an application of Theorem 4.1.4 to geometric elliptic divisibility sequences.

A sequence of positive integers $\{d_n\}_{n\in\mathbb{N}}$ is called a *divisibility sequence* if, for all $m, n \in \mathbb{N}$, whenever $m \mid n$ we have that $d_m \mid d_n$. Examples include the Fibonacci sequence and, for all positive integers a, the sequence $\{a^n - 1\}_n$. Bugeaud, Corvaja and Zannier, in [BCZ03, Theorem 1], proved that if one takes a and b to be multiplicatively independent, then the gcd between $a^n - 1$ and $b^n - 1$ cannot be too large. More specifically, for every $\varepsilon \geq 0$,

$$\gcd\{a^n - 1, b^n - 1\} \ll_\varepsilon e^{\varepsilon n}.$$

Shortly after, it was conjectured by Ailon and Rudnick [AR04, Conjecture A] that a stronger statement should actually hold, namely that for infinitely many $n \in \mathbb{N}$,

$$\gcd\{a^n - 1, b^n - 1\} = \gcd\{a - 1, b - 1\}.$$

As observed by Silverman [Sil05], these statements have a geometric interpretation. Recall that every positive integer can be viewed as divisor over $\operatorname{Spec}\mathbb{Z}$, via its prime factorization. Moreover, the multiplicative group $\mathbb{G}_{\mathrm{m}}$ can be viewed as a group scheme defined over any ring. Suppose now that a and b are fixed positive integers different from 1; they induce in a natural way sections P_a and P_b from $\operatorname{Spec}\mathbb{Z}[\frac{1}{ab}]$ to $\mathbb{G}_{\mathrm{m}}$, seen as a $\mathbb{Z}[\frac{1}{ab}]$ scheme, that are always different from the unitary section P_1. Now, the number $\gcd\{a^n - 1, b^n - 1\}$ corresponds exactly to the pull back via $(P_1, P_1) : \operatorname{Spec}\mathbb{Z}[\frac{1}{ab}] \to \mathbb{G}_{\mathrm{m}}^2$ of the image of (P_a^n, P_b^n). Substituting the multiplicative group with other group schemes over (open subsets of) $\operatorname{Spec}\mathbb{Z}$, naturally leads to other divisibility sequences.

This geometric interpretation led to a function field version of the conjecture by Ailon and Rudnick. Let $\mathcal{C}$ be an irreducible curve; a *divisibility sequence* on $\mathcal{C}$ is a sequence of divisors $(D_n)_{n\geq 1}$ of $\mathcal{C}$ such that $\forall m, n \in \mathbb{N}$, if $m \mid n$ then $D_m \mid D_n$, that is, $D_n - D_m$ is an effective divisor. Notice that this is exactly the function field analogue of an integral divisibility sequence.

Let now $\pi : \mathcal{G} \to \mathcal{C}$ be a group scheme with zero-section O. As noted by Silverman, again in [Sil05], every section $P : \mathcal{C} \to \mathcal{G}$ that is not identically torsion, i.e. $nP \neq O$ for all $n \geq 1$, has an associated divisibility sequence given by

$$D_{nP} := O^*(nP), \tag{4.8.1}$$

i.e., D_{nP} is the pull-back along the zero section O of the divisor nP on $\mathcal{G}$; such divisibility sequences are called *geometric*.

The analogue of the conjecture by Ailen and Rudnick in the function field case was formulated by Silverman in [Sil05], and it predicts that, if $\mathcal{G}$ has relative dimension at least 2, and the images of the multiples of P are dense in the generic fiber, then $D_{nP} = D_P$ for infinitely many $n > 0$. This question has been studied by Silverman [Sil04] and Ghioca, Hsia and Tucker [GHT18] when $\mathcal{G}$ is the product of two elliptic schemes, and by Barroero, Capuano and Turchet [BCT24] in the case of split semiabelian schemes. The results proved in [GHT18] and [BCT24] are stronger than the one mentioned above: under the conditions predicted by Silverman's conjecture, the set of $n \in \mathbb{N}$ for which $D_{nP} = D_P$ is the complement of a finite union of arithmetic progressions.

Generalizing the setting above, if we have two sections P and Q of $\pi : \mathcal{G} \to \mathcal{C}$, we can construct a sequence of divisors by replacing the zero section in (4.8.1) with Q, provided that $nP \neq Q$ for all $n \geq 1$. We then get

$$D_{nP,Q} := Q^*(nP) = \pi_*(nP \cap Q), \tag{4.8.2}$$

where, by slight abuse of notation, we are identifying nP and Q with their image, and the intersection is intended to be the scheme-theoretic intersection as zero-cycles.
These sequences have also been studied by Ghioca, Hsia and Tucker [GHT18] and Barroero, Capuano and Turchet [BCT24], for $\mathcal{G}$ the product of two elliptic schemes and a split semiabelian scheme respectively. They proved that, under the same conditions on P as the above mentioned conjecture of Silverman's, $D_{nP,Q}$ is bounded by a divisor independent of n.

If the relative dimension of $\mathcal{G}$ is 1, then there is no expectation that $D_{nP} = D_P$ infinitely often. However, in this case it is interesting to study when D_{nP} is reduced, that is, it is of the form $D_{nP} = \sum_i t_i$ for $t_i \in \mathcal{C}$ distinct points (equivalently, each non-zero coefficient of D_{nP} is equal to 1). This condition is an analogue for an integer to be square-free.

In [UU21], the authors prove that, if $\mathcal{G} = \mathcal{E}$ is an elliptic scheme and P is not identically torsion, then D_{nP} is reduced for all n outside a finite union of arithmetic progressions. Using Theorem 4.1.4 we prove that a similar statement holds for $D_{nP,Q}$. We start by proving a consequence of Theorem 4.1.4.

Theorem 4.8.1. *Let $\pi : \mathcal{E} \to S$ be an elliptic scheme over a smooth curve S, and let P, Q be sections $S \to \mathcal{E}$. If there exist infinitely many $s \in S(\overline{\mathbb{Q}})$ for which there exists $m_s \in \mathbb{Z}$ such that $m_s P(s) = Q(s)$ tangentially, then, for some $m \in \mathbb{Z}$, we have that $Q = mP$ identically on S.*

Proof. Consider $(s_n)_n$ an infinite sequence of pairwise distinct $s_n \in S(\overline{\mathbb{Q}})$ for which there exists $m_{s_n} \in \mathbb{Z}$ such that

$$m_{s_n} P(s_n) = Q(s_n) \text{ tangentially.} \tag{4.8.3}$$

We have two cases: either the m_{s_n} are bounded, or they are unbounded. We will treat these two cases separately.

If the m_{s_n} are bounded, then, up to taking a subsequence, we have that for a fixed m there exist infinitely many $s \in S(\overline{\mathbb{Q}})$ where $mP(s) = Q(s)$, from which the claim follows.

The case in which the m_{s_n} are unbounded is left to prove, so we will be assuming to be in that case. From Theorem 4.1.4 it follows that, for some $a, b \in \mathbb{Z}$ not both zero,

$$aP = bQ \tag{4.8.4}$$

identically on S. Combining this with (4.8.3), we find that $P(s_n)$ is torsion for infinitely many n, since we have

$$O(s_n) = aP(s_n) - bQ(s_n) = aP(s_n) - bm_n P(s_n) = (a - bm_n)P(s_n) \tag{4.8.5}$$

and $(a - bm_n)$ is non-zero infinitely often, as m_n is unbounded. Moreover, since (4.8.3) is true tangentially and (4.8.4) is true identically, we get that (4.8.5) also holds tangentially. By [UU21, Theorem 1.1] (see also [CDMZ21, Proposition 2.4]), P must be identically torsion, and so the same is true for Q, by (4.8.4). Since by assumption there is a certain $s \in S(\overline{\mathbb{Q}})$ and $m \in \mathbb{Z}$ such that $mP(s) = Q$, such relation must hold for all $s \in S(\overline{\mathbb{Q}})$, and hence identically on S.

□

Theorem 4.8.2. *Let $\pi : \mathcal{E} \to \mathcal{C}$ be an elliptic scheme, and let $P, Q : \mathcal{C} \to \mathcal{E}$ be two sections such that no multiple of P is identically equal to Q. Let $D_{nP,Q}$ be as in* (4.8.2)*. Then the set*

$$\Xi = \{n \in \mathbb{N} \mid D_{nP,Q} \text{ is not reduced}\}$$

is the union of a finite set and a finite union of arithmetic progressions.

Proof. For $s \in \mathcal{C}$ let $\Xi_s = \{n \in \mathbb{N} \mid 2s \leq D_{nP,Q}\}$, that is, the set of $n \in \mathbb{N}$ such that nP and Q intersect tangentially in s. Clearly, $\Xi = \bigcup_{s \in \mathcal{C}} \Xi_s$. Moreover, by Theorem 4.8.1, there are only finitely many $s \in \mathcal{C}$ such that Ξ_s is nonempty. It is then enough to show that for each $s \in \mathcal{C}$ we have that Ξ_s is either empty, a singleton, or a finite union of arithmetic progressions. Suppose that Ξ_s is nonempty, and let n_0 be its smallest element. For any other $n \in \Xi_s$ we have that both $n_0 P$ and nP intersect Q tangentially over s. Hence, $(n - n_0)P$ intersects the zero section tangentially over s. By [UU21, Thm 1.4], we can deduce that $\Xi_s - n_0$ is a finite union of arithmetic progressions starting from 0, so Ξ_s itself is a finite union of arithmetic progressions starting from n_0. □

Remark 4.8.3. The result of Theorem 4.8.2 is weaker than what Ulmer and Urzúa get in the special case $Q = O$ ([UU21, Thm 1.4]). There, the authors prove that $\{n \in \mathbb{N} \mid D_{nP} \text{ is not reduced}\}$ is the finite union of arithmetic progressions *starting from 0*, implying that if D_P is reduced, then D_{nP} is reduced infinitely often. In our case, the arithmetic progressions may not start from 0, so Theorem 4.8.2 does not provide an easy method to check wether $D_{nP,Q}$ is reduced infinitely often.

Bibliography

[AR04] N. Ailon and Z. Rudnick. Torsion points on curves and common divisors of $a^k - 1$ and $b^k - 1$. *Acta Arithmetica*, 113(1):31–38, 2004.

[Asl22] V. Aslanyan. Weak modular Zilber-Pink with derivatives. *Mathematische Annalen*, 383(1-2):433–474, 2022.

[Bal24] F. Ballini. *Diophantine problems of avoiding unlikely intersections*. Ph.D. thesis, University of Oxford, 2024.

[Bar17] F. Barroero. CM relations in fibered powers of elliptic families. *Journal of the Institute of Mathematics of Jussieu*, 18(5):941–956, 2017.

[BC16] F. Barroero and L. Capuano. Linear relations in families of powers of elliptic curves. *Algebra & Number Theory*, 10:195–214, 2016.

[BC17] F. Barroero and L. Capuano. Unlikely intersections in products of families of elliptic curves and the multiplicative group. *The Quarterly Journal of Mathematics*, 68(4):1117–1138, 2017.

[BC20] F. Barroero and L. Capuano. Unlikely intersections in families of abelian varieties and the polynomial Pell equation. *Proceedings of the London Mathematical Society*, 120(2):192–219, 2020.

[BCT24] F. Barroero, L. Capuano, and A. Turchet. Greatest common divisor results on semiabelian varieties and a conjecture of Silverman. *Research in Number Theory*, 10(1):16, 2024. Id/No 17.

[BCZ03] Y. Bugeaud, P. Corvaja, and U. Zannier. An upper bound for the G.C.D. of $a^n - 1$ and $b^n - 1$. *Mathematische Zeitschrift*, 243(1):79–84, 2003.

[BD22] F. Barroero and G. A. Dill. On the Zilber-Pink conjecture for complex abelian varieties. *Annales scientifiques de l'École Normale Supérieure*, 55(1):261–282, 2022.

[BD25] F. Barroero and G. A. Dill. Distinguished categories and the Zilber-Pink conjecture. *American Journal of Mathematics*, 147(3):715–778, 2025.

[BE20] D. Bertrand and B. Edixhoven. Pink's conjecture on unlikely intersections and families of semi-abelian varieties. *Journal de l'École polytechnique — Mathématiques*, 7:711–742, 2020.

[Ber90] D. Bertrand. Extensions de $\mathcal{D}$-modules et groupes de Galois differentiels. In F. B. et al., editor, *p-adic analysis*, pp. 125–141. Springer Berlin Heidelberg, 1990.

[Ber11] D. Bertrand. Special points and Poincaré bi-extensions. *https://arxiv.org/abs/1104.5178v1*, 2011. Preprint.

[BG06] E. Bombieri and W. Gubler. *Heights in Diophantine Geometry*. New Mathematical Monographs. Cambridge University Press, 2006.

[BHM16] M. Bainbridge, P. Habegger, and M. Möller. Teichmüller curves in genus three and just likely intersections in $\mathbf{G}_m^n \times \mathbf{G}_a^n$. *Publications mathématiques de l'IHÉS*, 124(1):1–98, 2016.

[BHMZ10] E. Bombieri, P. Habegger, D. Masser, and U. Zannier. A note on Maurin's theorem. *Rendiconti Lincei, Matematica e Applicazioni*, 21(3):251–260, 2010.

[Bin19] G. Binyamini. Density of algebraic points on Noetherian varieties. *Geometric and Functional Analysis*, 29(1):72–118, 2019.

[BJST23] G. Binyamini, G. O. Jones, H. Schmidt, and M. E. M. Thomas. An effective Pila-Wilkie theorem for sets definable using Pfaffian functions, with some diophantine applications. *https://arxiv.org/abs/2301.09883*, 2023. Preprint.

[BKS23] F. Barroero, L. Kühne, and H. Schmidt. Unlikely intersections of curves with algebraic subgroups in semiabelian varieties. *Selecta Mathematica*, 29(2), 2023.

[BMPZ16] D. Bertrand, D. Masser, A. Pillay, and U. Zannier. Relative Manin-Mumford for semi-abelian surfaces. *Proceedings of the Edinburgh Mathematical Society*, 59(4):837–875, 2016.

[BMZ99] E. Bombieri, D. Masser, and U. Zannier. Intersecting a curve with algebraic subgroups of multiplicative groups. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 7(1):51–80, 1999.

[BMZ06] E. Bombieri, D. Masser, and U. Zannier. Intersecting curves and algebraic subgroups: conjectures and more results. *Transactions of the American Mathematical Society*, 358(5):2247–2257, 2006.

[BMZ07] E. Bombieri, D. Masser, and U. Zannier. Anomalous subvarieties - structure theorems and applications. *International Mathematics Research Notices*, 2007.

[BMZ08a] E. Bombieri, D. Masser, and U. Zannier. Intersecting a plane with algebraic subgroups of multiplicative groups. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V*, 7(1):51–80, 2008.

[BMZ08b] E. Bombieri, D. Masser, and U. Zannier. On unlikely intersections of complex varieties with tori. *Acta Arithmetica*, 133(4):309–323, 2008.

[Bom90] E. Bombieri. The Mordell conjecture revisited. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV*, 17(4):615–640, 1990.

[BP89] E. Bombieri and J. Pila. The number of integral points on arcs and ovals. *Duke Mathematical Journal*, 59(2):337–357, 1989.

[BSY22] G. Binyamini, H. Schmidt, and A. Yafaev. Lower bounds for Galois orbits of special points on Shimura varieties: a point-counting approach. *Mathematische Annalen*, 385(1-2):961–973, 2022.

[BZ95] E. Bombieri and U. Zannier. Algebraic points on subvarieties of $\mathbb{G}_{\mathrm{m}}^n$. *International Mathematics Research Notices*, 1995(7):333, 1995.

[Cap23] L. Capuano. An overview on problems of unlikely intersections in families of abelian varieties. *Expositiones Mathematicae*, 41(3):603–617, 2023.

[Car08] M. Carrizosa. Problème de Lehmer et variétés abéliennes CM. *Comptes Rendus. Mathématique*, 346(23-24):1219–1224, 2008.

[Car09] M. Carrizosa. Petits points et multiplication complexe. *International Mathematics Research Notices*, 2009.

[CDMZ21] P. Corvaja, J. Demeio, D. Masser, and U. Zannier. On the torsion values for sections of an elliptic scheme. *Journal für die Reine und Angewandte Mathematik*, 2022(782):1–41, 2021.

[Cha38] C. Chabauty. Sur les équations diophantiennes liées aux unités d'un corps de nombres algébriques fini. *Annali di Matematica Pura ed Applicata*, 17(1):127–168, 1938.

[Cha41] C. Chabauty. Sur les points rationnels des variétés algébriques dont l'irrégularité est supérieure à la dimension. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 212:1022–1024, 1941.

[CMPZ16] L. Capuano, D. Masser, J. Pila, and U. Zannier. Rational points on Grassmannians and unlikely intersections in tori. *Bulletin of the London Mathematical Society*, 48(1):141–154, 2016.

[CMZ17] P. Corvaja, D. Masser, and U. Zannier. Torsion hypersurfaces on abelian schemes and Betti coordinates. *Mathematische Annalen*, 371(3-4):1013–1045, 2017.

[CTZ23] P. Corvaja, J. Tsimerman, and U. Zannier. Finite orbits in surfaces with a double elliptic fibration and torsion values of sections. *https://arxiv.org/abs/2302.00859*, 2023.

[CV14] S. Checcoli and E. Viada. On the torsion anomalous conjecture in CM abelian varieties. *Pacific Journal of Mathematics*, 271(2):321–345, 2014.

[CVV14] S. Checcoli, F. Veneziano, and E. Viada. On torsion anomalous intersections. *Rendiconti Lincei, Matematica e Applicazioni*, 25(1):1–36, 2014.

[Daw25] C. Daw. Unlikely intersections in Shimura varieties and beyond: a survey. *https://arxiv.org/abs/2506.02900*, 2025.

[Del71] P. Deligne. Travaux de Shimura. In *Séminaire Bourbaki, 23ème année (1970/1971)*, volume 244 of *Lecture Notes in Math.*, pp. 123–165. Springer, Berlin-New York, 1971.

[DGH21] V. Dimitrov, Z. Gao, and P. Habegger. Uniformity in Mordell-Lang for curves. *Annals of Mathematics. Second Series*, 194(1):237–298, 2021.

[DM94] L. van den Dries and C. Miller. On the real exponential field with restricted analytic functions. *Israel Journal of Mathematics*, 85(1-3):19–56, 1994.

[DM96] L. van den Dries and C. Miller. Geometric categories and o-minimal structures. *Duke Mathematical Journal*, 84(2):497–540, 1996.

[DO25] C. Daw and M. Orr. Zilber-Pink in a product of modular curves assuming multiplicative degeneration. *Duke Mathematical Journal*, 174(13), 2025.

[DOP25] C. Daw, M. Orr, and G. Papas. Some new cases of Zilber-Pink in $Y(1)^3$. *https://arxiv.org/abs/2510.09603*, 2025.

[Dri86] L. van den Dries. A generalization of the Tarski-Seidenberg theorem, and some nondefinability results. *American Mathematical Society. Bulletin. New Series*, 15(2):189–193, 1986.

[Dri98] L. van den Dries. *Tame Topology and o-minimal Structures*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1998.

[Edi05] B. Edixhoven. Special points on products of modular curves. *Duke Mathematical Journal*, 126(2), 2005.

[Fal83] G. Faltings. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Inventiones Mathematicae*, 73(3):349–366, 1983.

[Fal91] G. Faltings. Diophantine approximation on abelian varieties. *Annals of Mathematics*, 133(3):549, 1991.

[Fer25] L. Ferrigno. *Some problems of unlikely intersections in families of abelian varieties*. Ph.D. thesis, Università degli studi Roma Tre, 2025.

[Fer26] L. Ferrigno. Isogeny relations in products of families of elliptic curves. *Forum Mathematicum*, 38(3):939–953, 2026.

[Gab68] A. M. Gabriélov. Projections of semi-analytic sets. *Functional Analysis and Its Applications*, 2(4):282–291, 1968.

[Gal10] A. Galateau. Lower bound for the essential minimum on abelian varieties. *Commentarii Mathematici Helvetici*, 85(4):775–812, 2010.

[Gao16] Z. Gao. About the mixed André-Oort conjecture: Reduction to a lower bound for the pure case. *Comptes Rendus. Mathématique*, 354(7):659–663, 2016.

[Gao20] Z. Gao. Generic rank of Betti map and unlikely intersections. *Compositio Mathematica*, 156(12):2469–2509, 2020.

[GH79] P. Griffiths and J. Harris. Algebraic geometry and local differential geometry. *Annales scientifiques de l'École normale supérieure*, 12(3):355–452, 1979.

[GH23] Z. Gao and P. Habegger. The Relative Manin-Mumford Conjecture. *https://arxiv.org/abs/2303.05045*, 2023. Preprint.

[GHT18] D. Ghioca, L.-C. Hsia, and T. Tucker. A variant of a theorem by Ailon-Rudnick for elliptic curves. *Pacific Journal of Mathematics*, 295(1):1–15, 2018.

[Hab08] P. Habegger. On the Bounded Height Conjecture. *International Mathematics Research Notices*, 2008.

[Hab09] P. Habegger. Intersecting subvarieties of abelian varieties with algebraic subgroups of complementary dimension. *Inventiones mathematicae*, 176(2):405–447, 2009.

[Hab13a] P. Habegger. Special points on fibered powers of elliptic surfaces. *Journal für die Reine und Angewandte Mathematik*, 685(685):143–179, 2013.

[Hab13b] P. Habegger. Torsion points on elliptic curves in Weierstrass form. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, pp. 687–715, 2013.

[Hab17] P. Habegger. Effective height upper bounds on algebraic tori. In P. Habegger, G. Rémond, T. Scanlon, E. Ullmo, and A. Yafaev, editors, *Autour de la conjecture de Zilber-Pink*, number 52 in Panoramas et Synthèses, pp. 167–242. Société Mathématique de France, 2017.

[Har77] R. Hartshorne. *Algebraic Geometry*. Springer New York, 1977.

[HH90] D. Hilbert and A. Hurwitz. Über die diophantischen Gleichungen vom Geschlecht Null. *Acta Mathematica*, 14(0):217–224, 1890.

[Hin88] M. Hindry. Autour d'une conjecture de Serge Lang. *Inventiones Mathematicae*, 94(3):575–603, 1988.

[HP12] P. Habegger and J. Pila. Some unlikely intersections beyond André-Oort. *Compositio Mathematica*, 148(1):1–27, 2012.

[HP16] P. Habegger and J. Pila. O-minimality and certain atypical intersections. *Annales Scientifiques de l'École Normale Supérieure*, 49(4):813–858, 2016.

[Hru01] E. Hrushovski. The Manin-Mumford conjecture and the model theory of difference fields. *Annals of Pure and Applied Logic*, 112(1):43–115, 2001.

[HS00] M. Hindry and J. H. Silverman. *Diophantine Geometry: an Introduction*, volume 201 of *Graduate Texts in Mathematics*. Springer, New York, NY, 2000.

[Hus87] D. Husemöller. *Elliptic Curves*, volume 111 of *Graduate Texts in Mathematics 111*. Springer New York, second edition, 1987.

[HV19] P. Hubschmid and E. Viada. An addendum to the elliptic torsion anomalous conjecture in codimension 2. *Rendiconti del Seminario Matematico della Università di Padova*, 141:209–220, 2019.

[JS20] G. Jones and H. Schmidt. Pfaffian definitions of Weierstrass elliptic functions. *Mathematische Annalen*, 379(1-2):825–864, 2020.

[Kho80] A. G. Khovanskiĭ. A class of systems of transcendental equations. *Doklady Akademii Nauk SSSR*, 255(4):804–807, 1980.

[Kho91] A. G. Khovanskiĭ. *Fewnomials*. Number 88 in Translations of mathematical monographs. American Mathematical Society, 1991.

[Knu71] D. Knutson. *Algebraic spaces*. Springer Berlin Heidelberg, 1971.

[Küh23] L. Kühne. The relative Bogomolov conjecture for fibered products of elliptic curves. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 0(0), 2023.

[KY14] B. Klingler and A. Yafaev. The André-Oort conjecture. *Annals of Mathematics*, pp. 867–925, 2014.

[Lan60] S. Lang. Integral points on curves. *Publications Mathématiques de l'IHÉS*, 6:27–43, 1960.

[Lan62] S. Lang. *Diophantine geometry*. Number 11 in Interscience Tracts in Pure and Applied Mathematics. John Wiley & Sons Ltd., 1962.

[Lan65] S. Lang. Division points on curves. *Annali di Matematica Pura ed Applicata*, 70(1):229–234, 1965.

[Lau84] M. Laurent. Equations diophantiennes exponentielles. *Inventiones Mathematicae*, 78(2):299–327, 1984.

[Lia75] P. Liardet. Sur une conjecture de Serge Lang. In *Journées Arithmétiques de Bordeaux (Conf., Univ. Bordeaux, Bordeaux, 1974)*, volume No. 24-25 of *Astérisque*, pp. 187–210. Soc. Math. France, Paris, 1975.

[Liu02] Q. Liu. *Algebraic Geometry and Arithmetic Curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, repr. edition, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[LV20] B. Lawrence and A. Venkatesh. Diophantine problems and p-adic period mappings. *Inventiones Mathematicae*, 221(3):893–999, 2020.

[Mas84] D. W. Masser. Small values of the quadratic part of the Néron-Tate height on an abelian variety. *Compositio Mathematica*, 53(2):153–170, 1984.

[Mas88] D. Masser. Linear relations on algebraic groups. In *New Advances in Transcendence Theory*, pp. 248–262. Cambridge University Press, 1988.

[Mau08] G. Maurin. Courbes algébriques et équations multiplicatives. *Mathematische Annalen*, 341(4):789–824, 2008.

[McQ95] M. McQuillan. Division points on semi-abelian varieties. *Inventiones Mathematicae*, 120(1):143–159, 1995.

[MM23] J. Marché and G. Maurin. Singular intersections of subgroups and character varieties. *Mathematische Annalen*, 386:713–734, 2023.

[Mor22] L. J. Mordell. On the rational solutions of the indeterminate equation of the third and fourth degrees. *Proc. Cambridge Philos. Soc.*, 21:179–192, 1922.

[Mum88] D. Mumford. *Abelian varieties*. Number 5 in Studies in Mathematics. Oxford Univ. Pr., London [u.a.], second edition, 1988.

[MW83] D. W. Masser and G. Wüstholz. Fields of large transcendence degree generated by values of elliptic functions. *Inventiones Mathematicae*, 72(3):407–464, 1983.

[MW85] D. W. Masser and G. Wüstholz. Zero estimates on group varieties. II. *Inventiones Mathematicae*, 80(2):233–267, 1985.

[MZ08] D. Masser and U. Zannier. Torsion anomalous points and families of elliptic curves. *Comptes Rendus Mathematique*, 346(9-10):491–494, 2008.

[MZ10] D. Masser and U. Zannier. Torsion anomalous points and families of elliptic curves. *American Journal of Mathematics*, 132(6):1677–1691, 2010.

[MZ12] D. Masser and U. Zannier. Torsion points on families of squares of elliptic curves. *Mathematische Annalen*, 352:453–484, 2012.

[MZ14] D. Masser and U. Zannier. Torsion points on families of products of elliptic curves. *Advances in Mathematics*, 259:116–133, 2014.

[MZ15] D. Masser and U. Zannier. Torsion points on families of simple abelian surfaces and Pell's equation over polynomial rings. *Journal of the European Mathematical Society*, 17(9):2379–2416, 2015.

[MZ20] D. Masser and U. Zannier. Torsion points, Pell's equation, and integration in elementary terms. *Acta Mathematica*, 225(2):227–313, 2020.

[Neu99] J. Neukirch. *Algebraic number theory*. Springer, 1999.

[Nor49] D. G. Northcott. An inequality in the theory of arithmetic on algebraic varieties. *Mathematical Proceedings of the Cambridge Philosophical Society*, 45(4):502–509, 1949.

[Now98] K. Nowak. On the intersection multiplicity of images under an etale morphism. *Colloquium Mathematicum*, 75(2):167–174, 1998.

[Ott25] N. Ottolini. Singular intersections in families of abelian varieties. *https://arxiv.org/abs/2506.15344*, 2025. Preprint.

[Pap24] G. Papas. Zilber-Pink in $Y(1)^n$: Beyond multiplicative degeneration. *https://arxiv.org/abs/2402.09487*, 2024.

[Pil91] J. Pila. Geometric postulation of a smooth function and the number of rational points. *Duke Mathematical Journal*, 63(2):449–463, 1991.

[Pil06] J. Pila. Note on the rational points of a pfaff curve. *Proceedings of the Edinburgh Mathematical Society*, 49(2):391–397, 2006.

[Pil11] J. Pila. O-minimality and the André-Oort conjecture for $\mathbb{C}^n$. *Annals of Mathematics*, 173(3):1779–1840, 2011.

[Pil22] J. Pila. *Point-counting and the Zilber-Pink Conjecture*. Cambridge Tracts in Mathematics. Cambridge University Press, 2022.

[Pin05a] R. Pink. *A combination of the conjectures of Mordell-Lang and André-Oort*, pp. 251–282. Birkhäuser-Verlag, 2005.

[Pin05b] R. Pink. A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang. *preprint*, 2005.

[Poi01] H. Poincaré. Sur les propriétés arithmétiques des courbes algébriques. *Journal de Mathématiques Pures et Appliquées*, 5e série, 7:161–233, 1901.

[PS04] Y. Peterzil and S. Starchenko. Uniform definability of the Weierstrass $\wp$ functions and generalized tori of dimension one. *Selecta Mathematica. New Series*, 10(4):525–550, 2004.

[PST+22] J. Pila, A. N. Shankar, J. Tsimerman, H. Esnault, and M. Groechenig. Canonical heights on Shimura varieties and the André-Oort Conjecture. *https://arxiv.org/abs/2109.08788*, 2022. Preprint.

[PT16] J. Pila and J. Tsimerman. Ax-Schanuel for the j-function. *Duke Mathematical Journal*, 165(13), 2016.

[PW06] J. Pila and A. J. Wilkie. The rational points of a definable set. *Duke Mathematical Journal*, 133(3), 2006.

[PZ08] J. Pila and U. Zannier. Rational points in periodic analytic sets and the Manin-Mumford Conjecture. *Rendiconti Lincei - Matematica e Applicazioni*, pp. 149–162, 2008.

[Rat08] N. Ratazzi. Intersection de courbes et de sous-groupes et problèmes de minoration de hauteur dans les variétés abéliennes C.M. *Annales de l'Institut Fourier*, 58(5):1575–1633, 2008.

[Ray83a] M. Raynaud. Courbes sur une variété abélienne et points de torsion. *Inventiones Mathematicae*, 71(1):207–233, 1983.

[Ray83b] M. Raynaud. *Sous-variétés d'une variété abélienne et points de torsion*, pp. 327–352. Birkhäuser Boston, 1983.

[Rém07] G. Rémond. Intersection de sous-groupes et de sous-variétés. II. *Journal of the Institute of Mathematics of Jussieu*, 6(2):317–348, 2007.

[Rém09] G. Rémond. Intersection de sous-groupes et de sous-variétés III. *Commentarii Mathematici Helvetici*, 84(4):835–863, 2009.

[RV03] G. Rémond and E. Viada. The Mordell-Lang problem modulo certain abelian subvarieties. *International Mathematics Research Notices*, 2003(35):1915–1931, 2003.

[SAR94] P. Sarnak, S. Adams, and Z. Rudnick. Betti numbers of congruence groups. *Israel Journal of Mathematics*, 88(1-3):31–72, 1994.

[Sch00] A. Schinzel. *Polynomials with special regard to reducibility*. Cambridge University Press, 2000.

[Sie29] C. L. Siegel. Über einige Anwendungen diophantischer Approximationen. *Abhandlungen der Preußischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse*, 1:41–69, 1929.

[Sil83] J. H. Silverman. Heights and the specialization map for families of abelian varieties. *Journal für die Reine und Angewandte Mathematik*, 342:197–211, 1983.

[Sil04] J. H. Silverman. Common divisors of elliptic divisibility sequences over function fields. *Manuscripta Mathematica*, 114(4), 2004.

[Sil05] J. H. Silverman. Generalized greatest common divisors, divisibility sequences, and Vojta's conjecture for blowups. *Monatshefte für Mathematik*, 145(4):333–350, 2005.

[Sil11] J. H. Silverman. Height estimates for equidimensional dominant rational maps. *Journal of the Ramanujan Mathematical Society*, 26:145–163, 2011.

[Spe19] H. Spence. A modular André-Oort statement with derivatives. *Proceedings of the Edinburgh Mathematical Society*, 62(2):323–365, 2019.

[SUZ97] L. Szpiro, E. Ullmo, and S. Zhang. Equirépartition des petits points. *Inventiones Mathematicae*, 127(2):337–347, 1997.

[Tsi18] J. Tsimerman. The André-Oort conjecture for $\mathscr{A}_g$. *Annals of Mathematics*, 187(2), 2018.

[Ull98] E. Ullmo. Positivite et discretion des points algebriques des courbes. *The Annals of Mathematics*, 147(1):167, 1998.

[Ull13] E. Ullmo. Applications du théorème d'Ax–Lindemann hyperbolique. *Compositio Mathematica*, 150(2):175–190, 2013.

[UU20] D. Ulmer and G. Urzúa. Bounding tangencies of sections on elliptic surfaces. *International Mathematics Research Notices*, 2021(6):4768–4802, 2020.

[UU21] D. Ulmer and G. Urzúa. Transversality of sections on elliptic surfaces with applications to elliptic divisibility sequences and geography of surfaces. *Selecta Mathematica. New Series*, 28(2):Paper No. 25, 36, 2021.

[UY14] E. Ullmo and A. Yafaev. Galois orbits and equidistribution of special subvarieties: towards the André-Oort conjecture. *Annals of Mathematics*, 180(3):823–865, 2014.

[Via03] E. Viada. The intersection of a curve with algebraic subgroups in a product of elliptic curves. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V*, 2(1):47–75, 2003.

[Via08] E. Viada. The intersection of a curve with a union of translated codimension-two subgroups in a power of an elliptic curve. *Algebra & Number Theory*, 2(3):249–298, 2008.

[Voj91] P. Vojta. Siegel's theorem in the compact case. *The Annals of Mathematics*, 133(3):509, 1991.

[Voj96] P. Vojta. Integral points on subvarieties of semiabelian varieties, I. *Inventiones Mathematicae*, 126(1):133–181, 1996.

[Wal00] M. Waldschmidt. *Diophantine Approximation on Linear Algebraic Groups*. Springer Berlin Heidelberg, 2000.

[Wei28] A. Weil. *L'arithmétique sur les courbes algébriques*. Ph.D. thesis, Faculté des sciences de Paris, 1928.

[Wil99] A. J. Wilkie. A theorem of the complement and some new o-minimal structures. *Selecta Mathematica. New Series*, 5(4):397–421, 1999.

[Zan12] U. Zannier. *Some Problems of Unlikely Intersections in Arithmetic and Geometry.*, volume 181 of *Annals of Mathematics Studies*. Princeton University Press, 2012.

[Zan14a] U. Zannier. *Lecture Notes on Diophantine Analysis*. Springer eBook Collection. Edizioni della Normale, Pisa, 2014.

[Zan14b] U. Zannier, editor. *On some applications of diophantine approximations*. Publications of the Scuola Normale Superiore Ser. Edizioni della Normale Pisa, Pisa, 2014.

[Zha98] S.-W. Zhang. *Small points and Arakelov theory*, pp. 217–225. EMS Press, 1998.

[Zil02] B. Zilber. Exponential sums equations and the Schanuel conjecture. *Journal of the London Mathematical Society. Second Series*, 65(1):27–44, 2002.