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Operads and moduli spaces of genus zero curves

PhD thesis

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Dedicato ai miei nonni

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Contents

Introduction

As the title suggests, this thesis is about operads arising from algebraic geometry: let $\mathcal{M}_{0,n}$ be the moduli space of genus zero Riemann surfaces with n marked points and consider its Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$. For any $i = 1, \ldots, n$ there are gluing maps $\circ_i : \overline{\mathcal{M}}_{0,n} \times \overline{\mathcal{M}}_{0,m} \to \overline{\mathcal{M}}_{0,n+m-2}$ which endow the collection of complex algebraic varieties $\overline{\mathcal{M}} \coloneqq \{\overline{\mathcal{M}}_{0,n}\}_{n \geq 3}$ with a (cyclic) operad structure. Its homology $Hycom \coloneqq \{H_*(\overline{\mathcal{M}}_{0,n})\}_{n\geq 3}$ is called the Hypercommutative operad. The Koszul dual of this operad is known as the gravity operad $Grav \coloneqq \{sH_*(\mathcal{M}_{0,n})\}_{n\geq 3}$. This work is divided in three parts:

Part 1: combinatorial models for some moduli spaces

In the first chapter we give an overview of the theory of operads. Then we introduce the Gravity and the Hypercommutative operad (Chapter 2), discussing the main results already present in the literature. In Chapter 3 we recall two combinatorial models for $\mathcal{M}_{0,n}$ due to Salvatore [47] and Nakamura [41]. The main result of this chapter is the proof that the two models are actually the same (Theorem 3.13), as conjectured in [47]. Then we construct a family of CW-decompositions of $\overline{\mathcal{M}}_{0,n+1}$, following the ideas of Salvatore [47]: for any choice of $a_1, \ldots, a_n > 0$ we define a homeomorphism between $\overline{\mathcal{M}}_{0,n+1}$ and a regular CW-complex $N_n^{\sigma}(\mathcal{C}/S^1)$ (Theorem 4.13). Chapter 5 deals with the combinatorics of these cells: if $C_*^{dual}(\overline{\mathcal{M}}_{0,n})$ denotes the cellular chain complex of $\overline{\mathcal{M}}_{0,n}$ with the dual cell decomposition, then we prove that the collection of chain complexes $C_*^{dual}(\overline{\mathcal{M}}) \coloneqq \{C_*^{dual}(\overline{\mathcal{M}}_{0,n})\}_{n\geq 3}$ carries a natural operad structure and that its homology is precisely the Hypercommutative operad (Theorem 5.10). We conjecture that this operad is quasi-isomorphic to the operad of singular chains $C_*(\overline{\mathcal{M}})$, but I was not able to prove this statement.

Part 2: homology operations for gravity algebras

In the theory of $H_*(\mathcal{D}_2)$-algebras a key role is played by the Dyer-Lashof operations, which correspond to classes in $H_*^{\Sigma_n}(\mathcal{D}_2(n);\mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{D}_2(n);\mathbb{F}_p(\pm 1))$ (here $\mathbb{F}_p(\pm 1)$ denotes the sign representation). Similarly, classes in $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p(\pm 1))$ give rise to homology operations for Gravity algebras. The main result of this part is the explicit computation of $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p(\pm 1))$ for any $n \in \mathbb{N}$ and any prime number p. The key observation to do this computation is the following: the homotopy quotients $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ and $C_n(\mathbb{C})_{S^1}$ (where $C_n(\mathbb{C})$ is the unordered configuration space) are both models for the classifying space of $B_n/Z(B_n)$, the quotient of the braid group by its center. This allows us to do the computation of $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$ by looking at the Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \longrightarrow BS^1 \tag{1}$$

which is much simpler than

$$\mathcal{M}_{0,n+1} \hookrightarrow (\mathcal{M}_{0,n+1})_{\Sigma_n} \longrightarrow B\Sigma_n$$

because in the first case there is not any monodromy. Moreover, in the (homological) Serre spectral sequence associated to (6.1) the homology of the fiber is well known, and the differential of the second page is given by the BV-operator Δ. So everything is now quite explicit and the main result is the following (Theorem 7.16 and Theorem 7.19):

Theorem 0.1. *Let $n \in \mathbb{N}$ and p a prime number. Then:*

- *If $n = 0,1$ mod p we have $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p) = H_*(C_n(\mathbb{C});\mathbb{F}_p) \otimes H_*(BS^1;\mathbb{F}_p)$.*
- *Otherwise $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ is isomorphic to $H_*(C_n(\mathbb{C});\mathbb{F}_p)/Im(\Delta)$.*

The computation of $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p(\pm 1))$ involves different methods, based on the work of Cohen, Bödigheimer and Peim [7]. To explain the strategy we need some notation: let $\lambda : E \to B$ be a fiber bundle with fiber F and consider the (ordered) fiberwise configuration space

$$E(\lambda, n) \coloneqq \{(e_1, \dots, e_n) \in E^n \mid e_i \neq e_j \text{ and } \lambda(e_i) = \lambda(e_j) \text{ if } i \neq j\}$$

Now let X be a connected CW-complex with base point $*$. The space of *fiberwise configurations with label in X* is defined as

$$E(\lambda; X) \coloneqq \bigsqcup_{n=0}^{\infty} E(\lambda, n) \times_{\Sigma_n} X^n / \sim$$

where $\sim$ is the equivalence relation determined by

$$(e_1, \dots, e_n) \times (x_1, \dots, x_n) \sim (e_1, \dots, \hat{e}_i, \dots, e_n) \times (x_1, \dots, \hat{x}_i, \dots, x_n)$$

when $x_i = *$. Now the idea is the following: if $\mathbb{C} \hookrightarrow E \xrightarrow{\lambda} \mathbb{C}P^\infty$ is the tautological line bundle, then $E(\lambda; n)/\Sigma_n$ is a model for the classifying space of $B_n/Z(B_n)$. In this situation one can prove that $H_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$ can be described as a certain subspace of $H_*(E(\lambda; S^{2q+1});\mathbb{F}_p)$ (Proposition 8.6). Therefore it suffices to compute $H_*(E(\lambda; S^{2q+1});\mathbb{F}_p)$, and this is done looking at the fibration

$$C(\mathbb{C}; S^{2q+1}) \hookrightarrow E(\lambda; S^{2q+1}) \to \mathbb{C}P^\infty$$

where $C(\mathbb{C}; S^{2q+1})$ is the labelled configuration space of points in the plane. The result is the following (Theorem 8.7):

Theorem 0.2. *For any $q \in \mathbb{N}$ and p a prime number*

$$H_*(E(\lambda; S^{2q+1});\mathbb{F}_p) = H_*(C(\mathbb{C}; S^{2q+1});\mathbb{F}_p) \otimes H_*(\mathbb{C}P^\infty;\mathbb{F}_p)$$

Part 3: on the topology of $\mathcal{M}_{0,n+1}/\Sigma_n$ In this part we study the topology of the quotients $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$. The main results are the following:

- $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$ are not topological manifolds for any $n \geq 4$ (Theorem 11.8).
- $\mathcal{M}_{0,n+1}/\Sigma_n$ can be realized as the complement of an algebraic variety inside the weighted projective space $\mathbb{P}(n, n-1, \dots, 2)$ (Proposition 11.9).
- $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$ are simply connected (Theorems 11.16 and 11.17).
- $\mathcal{M}_{0,n+1}/\Sigma_n$ has the same rational homology of the point (Theorem 11.19).
- $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{F}_p)$ is computed explicitly when $n \neq 0, 1 \mod p$ (Theorem 12.7.
- $\tilde{H}_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{F}_p) = 0$ if $n = p, p+1$ (Theorem 12.10).
- $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Z})$ is computed for small values of n: in particular we prove that $\mathcal{M}_{0,n+1}/\Sigma_n$ is contractible for $n \leq 5$, while for $n = 6$ we get the first non contractible space (Section 12.4).

0.1 List of symbols and conventions

$\simeq$ means homotopy equivalent.

$\cong$ means homeomorphic.

Δ^n is the standard n-simplex $\{(t_0, \dots, t_n) \in [0,1]^{n+1} \mid t_i + \cdots + t_n = 1\}$.

Σ_n is the symmetric group on n letters.

Shift of a chain complex: if (C_*, d) is a chain complex, we will denote by sC the chain complex where we raise the degree of the elements by 1, i.e. $(sC)_n \coloneqq C_{n-1}$. Similarly, $s^{-1}C$ will denote the chain complex where we decrease the degree of the elements by 1, i.e. $(s^{-1}C)_n \coloneqq C_{n+1}$.

Degree of a variable: if a is an element of a graded vector space V, we we usually denote by $|a|$ the degree of a. When dealing with signs, we sometimes write $(-1)^a$ insted of $(-1)^{|a|}$ to ease the notation.

Koszul sign convention: every time two symbols of degree n and m are permuted, the result is multiplied by $(-1)^{nm}$.

Classifying space: let G be a discrete group. We can see it as a category $\mathcal{G}$ with one object $*$ and $Hom_{\mathcal{G}}(*, *) = G$. We will denote by BG the geometric realization of the simplicial set $N(\mathcal{G})$, where N is the nerve functor.

X_G denotes the homotopy quotient of a G-space X. In this thesis G will be either S^1 or a finite cyclic group.

$C_*(X)/C^*(X)$ are the singular chains/cochains of a topological space X.

$C_*^{cell}(X)/C^*_{cell}(X)$ are the cellular chains/cochains of a CW-complex X.

$\mathcal{D}_n$ denotes the little n disk operad.

$f\mathcal{D}_n$ denotes the framed little n disk operad.

$\mathbb{F}_p$ denotes the field with p elements, with p a prime number.

$F_n(X)$ denotes the ordered configuration space on a space X.

$C_n(X)$ denotes the unordered configuration space on a space X.

LX denotes the free loop space on X.

$Ch(R)$ is the category of chain complexes of R-modules, with R a commutative ring.

Top is the category of topological spaces.

Part I

Combinatorial models for some moduli spaces

Chapter 1

Basic facts on operads

Operads where introduced in the early seventies by Boardman-Vogt [5] and May [38] in the context of iterated loop spaces. Interest in operads was considerably renewed in the nineties thanks to the work of Kontsevich-Manin [33], Getzler-Jones [24], Ginzburg-Kapranov [26] and many others. In this chapter we review very quickly some well known facts about operads. This is the outline of the chapter:

Section 1.1 contains May's definition of topological operad and presents the main examples of topological operads.

Section 1.2 generalizes the definition of operad to any symmetric monoidal category, in the spirit of Getzler-Jones [24].

Section 1.3 presents the construction of the free (co)operad generated by an $\mathbb{S}$-object.

Section 1.4 is about the Bar-Cobar adjunction for dg-operads.

1.1 Operads

Definition 1.1 (May, [38])**.** A (topological) **operad** $\mathcal{P}$ is a sequence of topological spaces $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ with $\mathcal{P}(0) = \{*\}$, together with the following data:

- For any $n \in \mathbb{N}$ the symmetric group Σ_n acts on $\mathcal{P}(n)$.
- For any $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \mathbb{N}$ we have continuous maps

$$\gamma : \mathcal{P}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots + n_k)$$

  that are associative and equivariant in a very precise sense (for details see [38]).
- There is a special point $1 \in \mathcal{P}(1)$ such that $\gamma(1, x) = x$ and $\gamma(x, 1, \dots, 1) = x$ for any $x \in \mathcal{P}(n), n \in \mathbb{N}$. We will call it the **unit**.

Remark 1.1 (Markl's definition of operad)**.** An equivalent definition of operad is due to Markl: he defined maps

$$\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n + m - 1) \qquad i = 1, \dots, n$$

by setting $a \circ_i b \coloneqq \gamma(a, 1, \dots, 1, b, 1, \dots, 1)$, where b is placed on the i-th entry. The advantage of this definition is that the associativity and equivariance of the maps γ can be expressed more easily by the following formulas:

Associativity: for any $a \in \mathcal{P}(n), b \in \mathcal{P}(m), c \in \mathcal{P}(k)$ we have

$$\begin{aligned} (a \circ_i b) \circ_{j+m-1} c &= (a \circ_j c) \circ_i b \qquad 1 \leq i < j \leq n \\ a \circ_i (b \circ_j c) &= (a \circ_i b) \circ_{i+j-1} c \qquad 1 \leq i \leq n, 1 \leq j \leq m \end{aligned}$$

Equivariance: given $a \in \mathcal{P}(n), b \in \mathcal{P}(m), \sigma \in \Sigma_n, \tau \in \Sigma_m$ we have

$$(a \cdot \sigma) \circ_i (b \cdot \tau) = (a \circ_i b) \cdot (\sigma \circ_i \tau)$$

where $\sigma \circ_i \tau \in \Sigma_{n+m-1}$ is the permutation exchanging by σ m consecutive blocks having all one element except for the i-th block which contains m-elements and acting with τ on the i-th block.

Therefore one can define an operad to be a sequence of spaces $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ together with maps $\circ_i$ as above and a special element $\iota \in \mathcal{P}(1)$ which is a two-sided unit for the maps $\circ_i$. If we drop the existence of the unit get the definition of **pseudo operad** (see [36] for more details).

Example (Endomorphism operad)**.** For a topological space X the **endomorphism operad** is the operad End_X whose arity n space of operations is $End_X(n) \coloneqq Map(X^n, X)$. The symmetric group Σ_n acts by permuting the inputs, while the operadic composition is defined as follows: given $f \in Map(X^n, X)$, $g \in Map(X^m, X)$ the composite $f \circ_i g \in Map(X^{n+m-1}, X)$ is the function

$$(f \circ_i g)(x_1, \dots, x_{n+m-1}) \coloneqq f(x_1, \dots, x_{i-1}, g(x_i, \dots, x_{i+m-1}), x_{i+m}, \dots, x_{n+m-1})$$

Example (Little n-disk operad)**.** The little n-disk operad was introduced by Boardman and Vogt in [4] and it is one of the most important examples of operad. In arity k we have the space $\mathcal{D}_n(k)$ of k-tuples of self maps of the unit n-disk $f_i : D_n \to D_n$, $i = 1, \dots, k$ obtained by composing translations and dilations and such that $f_i(D_n) \cap f_j(D_n) = \emptyset$ if $i \neq j$. The symmetric group acts by permuting the elements of the k-tuple and the operad composition is defined as follows: given $f = (f_1, \dots, f_k) \in \mathcal{D}_n(k)$ and $g = (g_1, \dots, g_m) \in \mathcal{D}_n(m)$ the i-th composition $f \circ_i g \in \mathcal{D}_n(k + m - 1)$ is the $(k + m - 1)$-tuple $f \circ_i g \coloneqq (f_1, \dots, f_{i-1}, f_i \circ g_1, \dots, f_i \circ g_m, f_{i+1}, \dots, f_k)$.

Example (Framed little n-disk operad)**.** The framed little disk operad $f\mathcal{D}_n$ is a variant of the little disk operad. In arity k we have the space $f\mathcal{D}_n(k)$ of k-tuples of self maps of the unit n-disk $f_i : D_n \to D_n$, $i = 1, \dots, k$ obtained by composing translations, dilations and rotations of $SO(n)$ such that $f_i(D_n) \cap f_j(D_n) = \emptyset$ if $i \neq j$. The symmetric group acts by permuting the elements of the k-tuple and the operad composition is defined as in the case of the little disk operad.

Definition 1.2. A **morphism of operads** $f : \mathcal{P} \to \mathcal{Q}$ is a collection of equivariant maps $\{f_n : \mathcal{P}(n) \to \mathcal{Q}(n)\}_{n\in\mathbb{N}}$ such that:

1. $f(\iota) = \iota$, i.e. f preserves the unit.

2. $f(a \circ_i b) = f(a) \circ_i f(b)$.

Definition 1.3. Let $\mathcal{P}$ be an operad. A **$\mathcal{P}$-algebra** structure on a topological space X is the data of a morphism of operads $f : \mathcal{P} \to End_X$.

As we said before operads were introduced to study iterated loop spaces. The following remarkable Theorem explains the connection between loop spaces and operads:

Theorem 1.2 (May's Recognition Principle)**.** *A connected space X (of the homotopy type of a CW-complex) has the homotopy type of an n-fold loop space if and only if it is a $\mathcal{D}_n$-algebra.*

Remark 1.3. By an abuse of terminology we will use the word *operads* for both referring to *operads* and *pseudo-operads*. The context should make it clear which object we are referring to.

1.2 (co)Operads as (co)monoids

In this section we give an alternative (and more general) definition of operad due to Smirnov [50]. The exposition follows the foundational paper of Getzler-Jones [24]. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category such that:

1. $\mathcal{C}$ has all small limits and colimits.

2. For any object $X \in \mathcal{C}$ the functor $X \otimes -$ preserves colimits.

The most relevant examples for us will be

- $(Top, \times, *)$, the category of topological spaces with the cartesian product as tensor product.

- $(Ch(R), \otimes_R, R)$, the category of chain complexes of R-modules, where R is a commutative ring.

Remark 1.4. $\mathcal{C}$ has all small limits and colimits, so it has an initial object 0.

Let $\mathbb{S}$ be the groupoid whose objects are all finite sets and whose morphisms are bijections. In what follows we will denote by k the set $\{1, \ldots, k\}$.

Definition 1.4. An **$\mathbb{S}$-object** in $\mathcal{C}$ is just a functor $\mathcal{F} : \mathbb{S} \to \mathcal{C}$. We will denote by $Fun(\mathbb{S}, \mathcal{C})$ the category whose objects are the $\mathbb{S}$-objects and whose morphisms are natural transformations between them.

We can define a monoidal structure on $Fun(\mathbb{S}, \mathcal{C})$ as follows: given $\mathcal{F}, \mathcal{G} \in Fun(\mathbb{S}, \mathcal{C})$ and a finite set $X \in \mathbb{S}$ we define

$$(\mathcal{F} \circ \mathcal{G})(X) := \bigoplus_{k=0}^{\infty} \mathcal{F}(k) \otimes_{\Sigma_k} \left(\bigoplus_{f : X \to k} \bigotimes_{i=1}^{k} \mathcal{G}(f^{-1}(i)) \right)$$

where the second coproduct is indexed by all the functions $f : X \to k$. We think such a function as a partition of X into a finite number of disjoint subsets (possibly empty) $\{f^{-1}(i) \mid i = 1, \dots, k\}$. The unit for this tensor product is given by

$$\mathbf{1} : \mathbb{S} \to \mathcal{C}$$
$$X \mapsto \begin{cases} \mathbf{1} \text{ if } |X| = 1 \\ 0 \text{ if } |X| \neq 1 \end{cases}$$

Remark 1.5. The assumption that $X \otimes -$ preserves colimits is necessary to check the associativity of $\circ$.

Definition 1.5. A **(co)operad** is a (co)monoid in the monoidal category $Fun(\mathbb{S}, \mathcal{C})$. We will denote by $Op(\mathcal{C})$ (resp. $coOp(\mathcal{C})$) the full subcategory of $Fun(\mathbb{S}, \mathcal{C})$ spanned by operads (resp. cooperads).

Let us unravel a bit this abstract definition of operad (the case of cooperads is analogous): for any $n \in \mathbb{N}$ we have an object $\mathcal{P}(n)$ equipped with a Σ_n action. The operad multiplication is a morphism $\mu : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$, in particular it is determined by morphisms

$$\mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \cdots + n_k) \tag{1.1}$$

corresponding to the function

$$f : \{1, \dots, n_1 + \cdots + n_k\} \to \{1, \dots, k\}$$

such that $f^{-1}(i) = \{n_1 + \cdots + n_{i-1}, n_1 + \cdots + n_{i-1} + 1, \dots, n_1 + \cdots + n_{i-1} + n_i\}$. Since μ is an associative morphism of $\mathbb{S}$-objects it turns out that the morphisms 1.1 are associative and equivariant is a suitable way and we recover May's definition of operad (for more details see [24]).

1.3 Free (co)operads

Forgetting the structure of (co)monoid we can define a forgetful functor from the category of (co)operads to the category of $\mathbb{S}$-objects. In both cases it turns out that there is a left adjoint functor, which takes an $\mathbb{S}$-object to the free (co)operad generated by it. So we get a pair of adjuncions

$$\mathbb{T} : Fun(\mathbb{S}, \mathcal{C}) \rightleftarrows Op(\mathcal{C}) : U_1 \qquad \mathbb{T}^c : Fun(\mathbb{S}, \mathcal{C}) \rightleftarrows CoOp(\mathcal{C}) : U_2$$

In the rest of this section we sketch an explicit construction of these free functors. Let us focus on the free operad functor $\mathbb{T}$, the case of $\mathbb{T}^c$ is similar. We start with a few combinatorial definitions:

Definition 1.6. A **tree** is an oriented contractible graph $\mathcal{S}$ (with possibly open edges) such that:

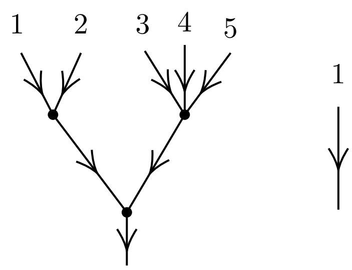


Figure 1.1: On the left we see a tree labelled by the set $\{1, 2, 3, 4, 5\}$. On the right it is depicted the tree with no vertices and only one open edge.

1. For any vertex there is only one outgoing edge.
2. There is only one outgoing open edge.

We also include the possibility that there are no vertices and only one open edge. The edges of $\mathcal{S}$ which are not open are called **internal edges**. For some examples see Figure 1.1. We will use the following notation:

- $V(\mathcal{S})$ will denote the set of vertices of $\mathcal{S}$.
- $in(\mathcal{S})$ (resp. $out(\mathcal{S})$) will be the set of open edges which are incoming (resp. outgoing).
- For any vertex $v \in V(\mathcal{S})$ we denote by $in(v)$ (resp. $out(v)$) the set of incoming (resp. outgoing) edges incident to v.

Definition 1.7. Let X be a finite set. A **tree with labels in** X is a tree $\mathcal{S}$ with a bijection between $in(\mathcal{S})$ and X. We will denote by $T(X)$ the set of trees with label in X up to isomorphism.

Definition 1.8. Let X be a finite set and $\{X_b\}_{b\in B}$ be a partition of X into a finite number n of disjoint subsets. If $\mathcal{F} : \mathbb{S} \to \mathcal{C}$ is an $\mathbb{S}$-object, we define

$$\bigotimes_{b\in B} \mathcal{F}(X_b) \coloneqq \left(\bigoplus_{f:n\to B} \mathcal{F}(X_{f(1)}) \otimes \cdots \otimes \mathcal{F}(X_{f(n)}) \right)_{\Sigma_n}$$

where the coproduct is indexed by all bijections from $\{1, \dots, n\}$ to B.

Now fix $\mathcal{F} \in Fun(\mathbb{S}, \mathcal{C})$ and let X be a finite set. We define $\mathbb{T}(\mathcal{F}) \in Fun(\mathbb{S}, \mathcal{C})$ by the following formula:

$$\mathbb{T}(\mathcal{F})(X) \coloneqq \bigoplus_{\mathcal{S}\in T(X)} \bigotimes_{v\in V(\mathcal{S})} \mathcal{F}(in(v))$$

Remark 1.6. Sometimes we will denote $\otimes_{v\in V(\mathcal{S})}\mathcal{F}(in(v))$ by $\mathcal{F}(\mathcal{S})$. In particular observe that if $\mathcal{S}$ is the tree with no vertices and one open edge then $\mathcal{F}(\mathcal{S}) = \mathbf{1}$.

Remark 1.7. When $(\mathcal{C}, \otimes, \mathbf{1})$ is a concrete category, for example $(Top, \times, *)$ or $(Ch(R), \otimes, R)$ it is customary to think elements of $\mathbb{T}(\mathcal{F})(X)$ as trees with labels in X such that each vertex v is decorated by an element of $\mathcal{F}(in(v))$.

$\mathbb{T}(\mathcal{F})$ can be equipped with a natural operad structure: the unit $\eta : \mathbf{1} \to \mathbb{T}(\mathcal{F})$ is just the inclusion of the summand corresponding to the tree with no vertices and one open edge. The multiplication $\mu : \mathbb{T}(\mathcal{F}) \circ \mathbb{T}(\mathcal{F}) \to \mathbb{T}(\mathcal{F})$ is defined by grafting of trees. This gives an explicit description of the free operad generated by the $\mathbb{S}$-object $\mathcal{F}$. Dually we can also define a natural cooperad structure on $\mathbb{T}(\mathcal{F})$: the counit $\epsilon : \mathbb{T}(\mathcal{F}) \to \mathbf{1}$ is just the projection on the summand $\mathbf{1}$ of $\mathbb{T}(\mathcal{F})(1)$, while the comultiplication $\Delta : \mathbb{T}(\mathcal{F}) \to \mathbb{T}(\mathcal{F}) \circ \mathbb{T}(\mathcal{F})$ is given by degrafting of trees. This gives an explicit description of the free operad generated by the $\mathbb{S}$-object $\mathcal{F}$.

Remark 1.8. The free operad and the free cooperad generated by an $\mathbb{S}$-object $\mathcal{F}$ have the same underlying $\mathbb{S}$-object. To avoid confusion we will use $\mathbb{T}(\mathcal{F})$ to denote the free operad and $\mathbb{T}^c(\mathcal{F})$ to denote the free cooperad.

1.4 Operadic (co)Bar construction

Let $Ch(\mathbb{F})$ the category of chain complexes over a field $\mathbb{F}$. We will call a (co)operad in $Ch(\mathbb{F})$ a **dg-(co)operad**. In this section we recall the basic facts about the operadic (co)bar construction. For more details we refer the reader to [24] or [34].

Definition 1.9. Let $\mathcal{F}$ be an $\mathbb{S}$-object in $Ch(\mathbb{F})$. We will denote by $s\mathcal{F}$ the $\mathbb{S}$-object given by the composition

$$\mathbb{S} \xrightarrow{\mathcal{F}} Ch(\mathbb{F}) \xrightarrow{s} Ch(\mathbb{F})$$

where s is the shift of chain complexes. Similarly one can define $s^{-1}\mathcal{F}$.

Let $\mathcal{P}$ be a dg-operad which is augmented, i.e. it is equipped with a map $\epsilon : \mathcal{P} \to \mathbf{1}$. We will denote by $\overline{\mathcal{P}}$ the kernel of ϵ, i.e. the $\mathbb{S}$-object such that for any finite set X we have

$$\overline{\mathcal{P}}(X) := Ker(\epsilon : \mathcal{P}(X) \to \mathbf{1}(X))$$

The bar cooperad $\mathcal{B}(\mathcal{P})$ is defined by twisting the differential of the free cooperad $\mathbb{T}^c(s\overline{\mathcal{P}})$ using the operad structure: indeed it is possible to construct a degree -1 map

$$\partial : \mathbb{T}^c(s\overline{\mathcal{P}}) \to \mathbb{T}^c(s\overline{\mathcal{P}})$$

such that if d is the differential of $\mathbb{T}^c(s\overline{\mathcal{P}})$ induced by the differential of $\mathcal{P}$, then $\partial + d$ is again a differential and it is compatible with the cooperad structure. Intuitively ∂ acts as follows: pick an element of $\mathbb{T}^c(s\overline{\mathcal{P}})(k)$ and think of it as a tree with k leaves whose vertices v are labelled by elements of $\mathcal{P}(in(v))$. The map ∂ applied to such an element is the sum (with signs) of the trees obtained by contracting an edge and composing the labels with the operad structure. See Figure 1.2 for an example.

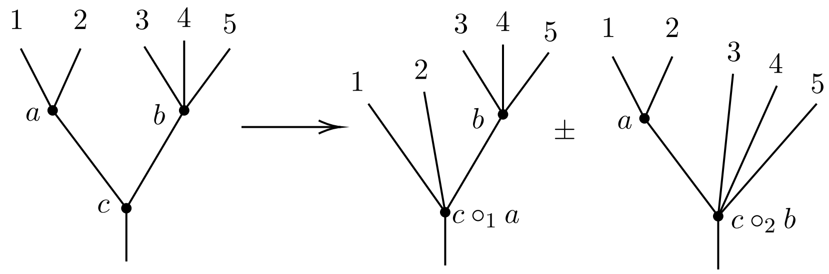


Figure 1.2: In this picture we see how the differential ∂ acts on an element belonging to $\mathbb{T}^c(s\overline{\mathcal{P}})(5)$.

Definition 1.10. The **operadic bar construction** is the functor

$$\mathcal{B} : aug - dgOp \to coaug - dgCoOp$$

which sends an augmented dg-operad $\mathcal{P}$ to $\mathcal{B}(\mathcal{P}) := (\mathbb{T}^c(s\overline{\mathcal{P}}), \partial + d)$.

A remarkable property of the bar construction is that $\mathcal{B}(-)$ has a left adjoint, the so called **operadic cobar construction**: let $\mathcal{P}$ be a coaugmented dg-cooperad, i.e. a dg-cooperad with a morphism $\eta : \mathbf{1} \to \mathcal{P}$. $\Omega(\mathcal{P})$ is defined by twisting the differential of $\mathbb{T}(s^{-1}\overline{\mathcal{P}})$, where $\overline{\mathcal{Q}} := coker(\eta : \mathbf{1} \to \mathcal{P})$ is the coaugmentation ideal. As before, there is a degree -1 map

$$\partial : \mathbb{T}(s^{-1}\overline{\mathcal{P}}) \to \mathbb{T}(s^{-1}\overline{\mathcal{P}})$$

which acts as follows: thinking an element of $\mathbb{T}(s^{-1}\overline{\mathcal{P}})$ as a tree $\mathcal{S}$ whose vertices are labelled by elements of $\mathcal{P}$, ∂ send it to the sum of all trees obtained from $\mathcal{S}$ by vertex expansion and relabelling the vertices using the cooperad structure. To sum up we have an adjunction

$$\Omega : coaug - dgCoOp \rightleftarrows aug - dgOp : \mathcal{B}$$

which plays a crucial role in the theory of algebraic operads.

Chapter 2

Operads and moduli spaces

In this Chapter we just recall the definition of the Hypercommutative and Gravity operad, stating the main results already present in the literature. The outline is the following:

Section 2.1 introduces the Hypercommutative operad.

Section 2.2 introduces the Gravity operad.

2.1 The Hypercommutative operad

2.1.1 Pointed stable curves

Let $\mathcal{M}_{0,n+1}$ be the moduli space of genus zero Riemann surfaces with $n+1$ marked points. Grothendieck and Knudsen [15], [32] defined a canonical compactification $\overline{\mathcal{M}}_{0,n+1}$ of $\mathcal{M}_{0,n+1}$: $\overline{\mathcal{M}}_{0,n+1}$ is the moduli space of stable $(n+1)$-pointed curves of genus 0, i.e. the data $(C, p_0, \dots, p_n)$ of a (possibly reducible) algebraic curve C with at most nodal singularities and smooth points $p_0, \dots, p_n \in C$ (all distinct) such that:

- Each irreducible component of C is isomorphic to $\mathbb{C}P^1$.
- The dual graph is a tree. Recall that the dual graph has one vertex for each irreducible components of C, edges corresponding to intersection points and half edges for each marked point.
- **Stability condition:** each irreducible component of C has at least three special points, where a special point means either one of the p_i, $i = 0, \dots, n$ or a singular point.

Figure 2.1 shows some examples of stable and not stable curves. It turns out that $\overline{\mathcal{M}}_{0,n+1}$ is a smooth complex projective variety of dimension $n-2$ and contains $\mathcal{M}_{0,n+1}$ as an open dense subset. An explicit construction of $\overline{\mathcal{M}}_{0,n+1}$ by a sequence of complex blow ups can be found in [37].

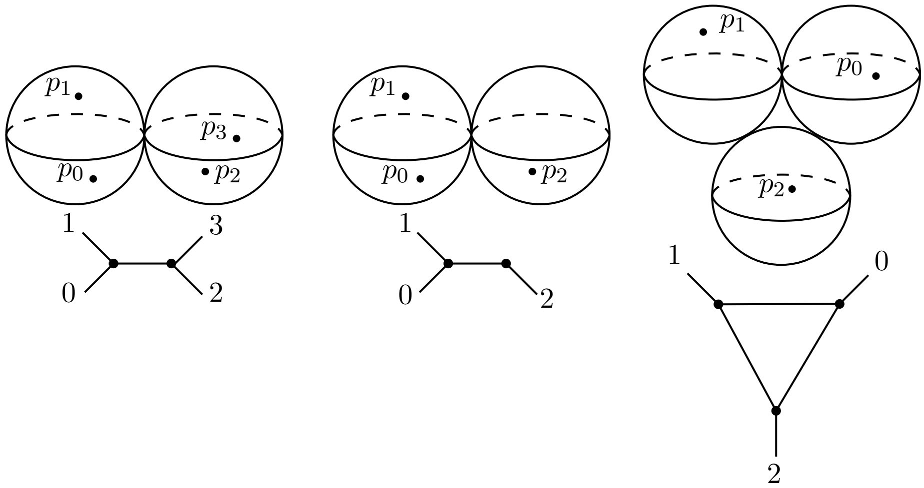


Figure 2.1: In this picture we see tre nodal curves with marked points, and below it is depicted the dual graph: the curve on the left is stable, while the others are not. The middle one does not satisfy the stability condition, while the curve on the right is not stable since its dual graph is not a tree.

2.1.2 The Deligne-Mumford operad

For any $m, n \in \mathbb{N}$ and $i = 1, \dots, n$ we can define *gluing maps*

$$\begin{aligned} \circ_i : \overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1} &\to \overline{\mathcal{M}}_{0,n+m} \\ (C_1, p_0, \dots, p_n) \times (C_2, q_0, \dots, q_m) &\mapsto (C, p_0, \dots, p_{i-1}, q_1, \dots, q_m, p_{i+1}, \dots, p_n) \end{aligned}$$

where C is the curve obtained from $C_1 \sqcup C_2$ identifying p_i and q_0 and introducing a nodal singularity.

Definition 2.1. Let us denote by $\overline{\mathcal{M}}(n) := \overline{\mathcal{M}}_{0,n+1}$. Σ_n acts on $\overline{\mathcal{M}}(n)$ by permuting the marked points $p_1, \dots, p_n$ of a stable curve $(C, p_0, \dots, p_n) \in \overline{\mathcal{M}}_{0,n+1}$. The gluing maps defined above define an operad structure on the collection $\overline{\mathcal{M}} := \{\overline{\mathcal{M}}(n)\}_{n \geq 2}$. We call $\overline{\mathcal{M}}$ the **Deligne-Mumford operad**.

Remark 2.1. We have a natural action of Σ_{n+1} on $\overline{\mathcal{M}}(n) := \overline{\mathcal{M}}_{0,n+1}$ by permuting all the marked points, so $\overline{\mathcal{M}}$ is actually a **cyclic operad**.

A very nice result about this operad is the following, due to Drummond-Cole:

Theorem 2.2 (Drummond-Cole, [17]). *The Deligne-Mumford operad $\overline{\mathcal{M}}$ is the homotopy pushout of the diagram*

$$\begin{array}{ccc} S^1 & \longrightarrow & f\mathcal{D}_2 \\ \downarrow & & \\ * & & \end{array}$$

Remark 2.3. Recently Oancea and Vaintrob proved a higher genus version of this statement, see [44].

2.1.3 The Hypercommutative operad

Since homology is a lax monoidal functor, the operadic structure of $\overline{\mathcal{M}}$ induce an operadic structure on $H_*(\overline{\mathcal{M}};\mathbb{Z})$, giving an operad in graded abelian groups:

Definition 2.2. The **Hypercommutative operad** *Hycom* is a (cyclic) operad in graded abelian groups whose arity n operations are given by

$$Hycom(n) := H_*(\overline{\mathcal{M}}_{0,n+1};\mathbb{Z})$$

As an operad, it is generated by (graded) symmetric operations of degree $2(n-2)$

$$(a_1,\dots,a_n) \in H_*(\overline{\mathcal{M}}_{0,n+1};\mathbb{Z}) \quad n \geq 2$$

which satisfy the following *generalized associativity relations*:

$$\sum_{S_1 \sqcup S_2 = \{1,\dots,n\}} \pm((a,b,x_{S_1}),c,x_{S_2}) = \sum_{S_1 \sqcup S_2 = \{1,\dots,n\}} \pm(a,(b,c,x_{S_1}),x_{S_2}) \tag{2.1}$$

where if $S = \{s_1,\dots,s_k\} \subseteq \{1,\dots,n\}$, x_S is an abbreviation for $x_{s_1},\dots,x_{s_k}$. The sign in front of each summand is determined by the Koszul sign rule. We report below some explicit examples of these relations:

- $n = 0$: we get $((a,b),c) = (a,(b,c))$, i.e. the binary operation is associative (and commutative).
- $n = 1$: $((a,b),c,d) + (-1)^{|c||d|}((a,b,d),c) = (a,(b,c),d) + (a,(b,c,d))$.

Further details can be found in [21] and [33].

Remark 2.4. Geometrically, the n-ary operation $(a_1,\dots,a_n)$ corresponds to the fundamental class $[\overline{\mathcal{M}}_{0,n+1}]$.

An algebraic version of Theorem 2.2 is due to Khoroshkin-Markarian-Shandrin:

Theorem 2.5 ([30])**.** *The Hypercommutative operad Hycom is quasi-isomorphic to the homotopy quotient BV/Δ, where BV is the Batalin–Vilkovisky operad.*

Definition 2.3. An **hypercommutative algebra** (in the category of chain complexes) is just an algebra over the Hypercommutative operad. Explicitly, it is a chain complex (A, d_A) equipped by (graded) symmetric products $(-,\cdots,-) : A^{\otimes n} \to A$ of degree $2(n-2)$, such that Equation 2.1 is satisfied for any choice of variables $a,b,c,x_1,\dots,x_n \in A$.

We end this section by giving some examples of Hypercommutative algebras:

- A commutative algebra is a special case of Hypercommutative algebra, where all the higher products $(-,\dots,-) : A^{\otimes n} \to A$ with $n \geq 3$ are zero.
- Barannikov and Kontsevich defined a canonical hypercommutative algebra structure on the cohomology of any compact Calabi-Yau manifold, see [3].
- Kontsevich and Manin have shown in [33] through Gromov-Witten invariants that the rational homology of a smooth projective algebraic variety is a hypercommutative algebra.

2.2 The Gravity operad

The Gravity operad was introduced by Getzler in [22] and [21]. In this section we recall the main facts about the Gravity operad and the algebras over it. There are also chain model versions of the Gravity operad, described in the paper [54] by Westerland and in [23] by Getzler-Kapranov. A comparison between these definitions has been written by Dupont and Horel in [18]. In what follows all the homology groups are taken with integer coefficients, unless otherwise stated. To ease the notation we sometimes write $H_*(X)$ instead of $H_*(X;\mathbb{Z})$.

Definition 2.4. Consider the graded abelian group $Grav(n) := sH_*(\mathcal{M}_{0,n+1};\mathbb{Z})$, where s is the degree shift. The collection $Grav := \{Grav(n)\}_{n\geq 2}$ forms an operad of graded abelian groups, called the **Gravity operad**. The composition maps can be described as follows: first observe that the orbit space $F_n(\mathbb{C})/S^1$ is homotopy equivalent to $\mathcal{M}_{0,n+1}$. The quotient map $p : F_n(\mathbb{C}) \to F_n(\mathbb{C})/S^1$ has a section given by

$$\begin{aligned} j : F_n(\mathbb{C})/S^1 &\to F_n(\mathbb{C}) \\ [z_1,\ldots,z_n] &\mapsto \left(\frac{z_2-z_1}{|z_2-z_1|}\right)^{-1}\cdot(z_1,\ldots,z_n) \end{aligned}$$

Therefore p is a trivial S^1-principal bundle and the transfer map

$$\tau : H_*(F_n(\mathbb{C})/S^1) \to H_*(F_n(\mathbb{C}))$$

is injective. Now let $\circ_i : H_*(F_n(\mathbb{C})) \otimes H_*(F_m(\mathbb{C})) \to H_*(F_{n+m-1}(\mathbb{C}))$ be the map induced in homology by the i-th composition of the little two disk operad. Getzler observed that given two classes $a \in H_*(F_n(\mathbb{C})/S^1)$, $b \in H_*(F_m(\mathbb{C})/S^1)$, the composition $\tau(a)\circ_i\tau(b)$ is the transfer of a unique class of $H_*(F_{n+m-1}(\mathbb{C})/S^1)$. Therefore we can define $a\circ_i b := \tau^{-1}(\tau(a)\circ_i\tau(b))$ and we get the following commutative diagram:

$$\begin{tikzcd} H_*(F_n(\mathbb{C}))\otimes H_*(F_m(\mathbb{C})) \arrow[r, "\circ_i"] & H_*(F_{n+m-1}(\mathbb{C})) \\ H_*(F_n(\mathbb{C})/S^1)\otimes H_*(F_m(\mathbb{C})/S^1) \arrow[u, "\tau\otimes\tau"] \arrow[r, dashed, "\circ_i"] & H_*(F_{n+m-1}(\mathbb{C})/S^1) \arrow[u, "\tau"'] \end{tikzcd}$$

The i-th operadic composition of the gravity operad is then defined to be the dashed arrow of the above diagram. Note that this map raises the degree by 1. If we shift the graded vector spaces $H_*(F_n(\mathbb{C})/S^1)$ by one we get a map of degree zero, justifying the shifting term in the definition of the Gravity operad.

Remark 2.6. The action of Σ_{n+1} on $\mathcal{M}_{0,n+1}$ by relabelling the points induces an action in homology, making $Grav$ a *cyclic operad*.

Unlike many familiar operads, the Gravity operad is not generated by a finite number of operations. However, it has a nice presentation with infinitely many generators:

Theorem 2.7 (Getzler [22]). *As an operad Grav is generated by (graded) symmetric operations of degree one*

$$\{a_1, \ldots, a_n\} \in Grav(n) \quad \textit{for } n \geq 2$$

Geometrically, $\{a_1, \ldots, a_n\}$ corresponds to the generator of $H_0(\mathcal{M}_{0,n+1}, \mathbb{Z})$. These operations (called brackets) satisfy the so called generalized Jacobi relations: for any $k \geq 2$ and $l \in \mathbb{N}$

$$\sum_{1 \leq i < j \leq k} (-1)^{\epsilon(i,j)} \{\{a_i, a_j\}, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k, b_1, \ldots, b_l\} = \{\{a_1, \ldots, a_k\}, b_1, \ldots, b_l\} \tag{2.2}$$

where the right hand term is interpreted as zero if $l = 0$ and $\epsilon(i,j) = (|a_1| + \cdots + |a_{i-1}|)|a_i| + (|a_1| + \cdots + |a_{j-1}|)|a_j| + |a_i||a_j|$.

Some explicit examples of this relations are:

- **Jacobi relation** ($k = 3$, $l = 0$):

  $$\{\{a_1, a_2\}, a_3\} + (-1)^{|a_2||a_3|}\{\{a_1, a_3\}, a_2\} + (-1)^{|a_1|(|a_2|+|a_3|)}\{\{a_2, a_3\}, a_1\} = 0$$

  This shows that the binary bracket $\{-, -\}$ is a Lie bracket (of degree one).

- $k = 3$, $l = 1$:

  $$\{\{a_1, a_2\}, a_3, b_1\} \pm \{\{a_1, a_3\}, a_2, b_1\} \pm \{\{a_2, a_3\}, a_1, b_1\} = \{\{a_1, a_2, a_3\}, b_1\}$$

Definition 2.5. A **Gravity algebra** (in the category of chain complexes) is an algebra over the Gravity operad. To be explicit, it is a chain complex (A, d_A) together with graded symmetric chain maps $\{-, \ldots, -\} : A^{\otimes k} \to A$ of degree one such that for $k \geq 3$, $l \geq 0$ and $a_1, \ldots, a_k, b_1, \ldots, b_l \in A$ Equation 2.2 is satisfied.

Remark 2.8. We can think of a Gravity algebra as a generalization of a (differential graded) Lie algebra. Indeed we have a binary Lie bracket (of degree one) and many other higher brackets which satisfy a higher version of the usual Jacobi relation.

Remark 2.9. Unlike in the case of the Hypercommutative operad, the Gravity operad is not the homology of an operad in topological spaces. Basically this is because we can not define *gluing maps*

$$\circ_i : \mathcal{M}_{0,n+1} \times \mathcal{M}_{0,m+1} \to \mathcal{M}_{0,n+m}$$

as in the case of $\overline{\mathcal{M}}_{0,n+1}$, since we can not introduce singularities. However there is a model for this operad in the category of spectra, as explained by Westerland in [54].

Remark 2.10. The definition of Gravity operad we just gave is the one contained in [22]. In the paper [21] E. Getzler calls *Gravity operad* the operadic suspension

$\Lambda Grav$, i.e. the operad which encode the algebraic structure on the suspension of a gravity algebra. More explicitly, its arity n space is

$$(\Lambda Grav)(n) := s^{n-2} sgn \otimes H_*(\mathcal{M}_{0,n+1};\mathbb{Z})$$

where sgn denote the sign representation of Σ_n. Because of the sign representation, the generators of $\Lambda Grav$ become (graded) antisymmetric operations (of degree $n-2$), so the binary bracket is actually a Lie bracket.

Remark 2.11. (Koszul duality) An old result by D. Quillen [46] estabilished a duality between commutative and Lie algebras. The main result of [21] is an analogue of Quillen's result, replacing commutative algebras by Hypercommutative algebras and Lie algebras by $\Lambda Grav$-algebras. This can be summarized in the sentence:

The Hypercommutative operad $Hycomm$ and $\Lambda Grav$ are Koszul dual.

Here Koszul duality must be interpreted in the sense of Ginzburg-Kapranov [26].

Gravity algebras and BV-algebras are closely related. The idea is that every time we have a BV-algebra structure on the (co)homology of a space/d.g. algebra, then we get a Gravity algebra structure on the S^1-equivariant version of our (co)homology theory. The following list of examples should clarify this last sentence:

1. Let X be a $f\mathcal{D}_2$-algebra. Then $H_*(X)$ is a BV-algebra and $H_*^{S^1}(X)$ is a Gravity algebra (see [54]).

2. Let M be an oriented d-dimensional manifold. The homology $H_*(LM)$ of the free loop space on M carries a rich algebraic structure: the loop product of Chas-Sullivan [9] endow $s^{-d}H_*(LM)$ with a commutative algebra structure. Moreover the S^1 action on LM is compatible with this product, so $s^{-d}H_*(LM)$ is an BV-algebra (Cohen and Jones [14]). As before, if we switch to S^1-equivariant homology (and shift the degree appropriately) we get a Gravity algebra structure on $s^{1-d}H_*^{S^1}(LM)$ (see [54]).

3. If A is a Frobenius algebra, then the Hochschild cohomology $HH^*(A)$ is a BV-algebra. If we switch to the S^1-equivariant version of Hochschild cohomology (i.e. the cyclic cohomology) we get a Gravity algebra structure on $HC^*(A)$. See the paper by Ward [53] for further details and examples.

The following table summarizes what we said in this section:

	BV-algebra	$Grav$-algebra
X $f\mathcal{D}_2$-algebra	$H_*(X)$	$H_*^{S^1}(X)$
M closed oriented	$s^{-d}H_*(LM)$	$s^{1-d}H_*^{S^1}(LM)$
A Frobenius algebra	$HH^*(A)$	$HC^*(A)$

Chapter 3

Combinatorial models for $\mathcal{M}_{0,n+1}$

In this Chapter we compare two combinatorial models for the moduli space of genus zero Riemann surfaces $\mathcal{M}_{0,n+1}$. The main result will be Theorem 3.13. Here is the outline of the chapter:

Section 3.1 is about an open cell decomposition of $\mathcal{M}_{0,n+1}$ based on black and white trees. This model is due to Salvatore [47].

Section 3.2 is a summary of the paper by Nakamura [41], where he constructed an open cell decomposition of $\mathcal{M}_{0,n+1}$.

Section 3.3 contains the proof that the cells of Salvatore and Nakamura are the same (Theorem 3.13). This was conjectured by Salvatore in [47].

3.1 Black and white trees

In this section we review very quickly a combinatorial model for the moduli space $\mathcal{M}_{0,n+1}$ based on trees due to Salvatore. We will skip all the details, which can be found in [47]. Before starting let us observe that $\mathcal{M}_{0,n+1}$ can be seen as a quotient of the ordered configuration space $F_n(\mathbb{C})$:

Proposition 3.1. *Let $\mathbb{C} \rtimes \mathbb{C}^*$ acts on $F_n(\mathbb{C})$ by translations, dilations and rotations. Then $\mathcal{M}_{0,n+1}$ is homeomorphic to $F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$.*

Proof. A point in $\mathcal{M}_{0,n+1}$ is just a configuration of points $(p_0, p_1, \ldots, p_n)$ in the Riemann sphere up to biholomorphisms. If we rotate suitably the sphere we can suppose that p_0 is the point at the infinity $\infty \in \mathbb{C} \cup \{\infty\}$. Deleting this point and using the stereographic projection we obtain a configuration of n points in the complex plane $\mathbb{C}$ up to translation, dilatations and rotations, and this proves the claim. $\square$

It will be useful to keep in mind this identification for the rest of this work. The idea to get a combinatorial model of $\mathcal{M}_{0,n+1} \cong F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$ is the following: given a configuration of n points $(z_1, \ldots, z_n) \in F_n(\mathbb{C})$, think each z_i as a negative electric charge of value $-a_i$, where each a_i is a fixed real number strictly greater than 0 and

$\sum_{1=1}^{n} a_i = 1$ (normalization condition). Each charge generates a radial electric field and by superposition we obtain a total electric field

$$E(z) := \sum_{i=1}^{n} -a_i \frac{z - z_i}{|z - z_i|^2}$$

The electric field is conservative, and it is not difficult to find an explicit potential for $E(z)$: indeed it turns out that

$$E(z) = -\nabla U(z)$$

where $U(z) := \log|h(z)|$ and $h(z) := \prod_{i=1}^{n}(z - z_i)^{a_i}$. Now look at the flow lines of $E(z)$, i.e. curves $\gamma(t)$ such that $\gamma'(t) = E(\gamma(t))$: most of them start at one point z_i of the configuration $(z_1, \ldots, z_n)$ and go to infinity; conversely, there are some flow lines which are of finite length, namely those that connects two zeros of $E(z)$ or one zero to a point z_i of the configuration. Starting from these special flow lines we can obtain what we will call an *admissible tree* T with n leaves:

Definition 3.1. An **admissible tree** T with n-leaves is a tree (connected graph without closed paths) such that:

- Its vertices are colored with two colors, black and white. We will write $V(T) = W \sqcup B$, where W is the set of white vertices, and B is the set of black vertices. Moreover we require that there are exactly n white vertices and that they are labelled by the numbers $\{1, \ldots, n\}$. White vertices are also called **leaves**.
- The edges of T are oriented, i.e. they are ordered couples $e = (v, w) \in V(T) \times V(T)$. v is called the source of e, w is called the target. We say that an edge e is incident to a vertex v if v is either the source or the target of e.
- The set E_v of edges incident to a fixed vertex $v \in V(T)$ is equipped with a cyclic ordering. In other words, T is a **ribbon graph**.

Moreover we require the following conditions to be satisfied:

1. A white vertex is not allowed to be a source.
2. If $(v, w) \in E(T)$ is an edge, then $(w, v) \notin E(T)$, i.e. an edge can not be inverted.
3. Any black vertex $v \in B$ is the source of at least two distinct edges. Moreover it can not be the target of two edges that are next to each other in the cyclic ordering of E_v.

We will denote with T_n the set of isomorphism classes of admissible trees with n-leaves. See Figure 3.2 for some examples.

In our case, the black vertices are the zeros of $E(z)$, the white vertices are $\{z_1, \ldots, z_n\}$, and the oriented edges are given by the flow lines of finite lenght (which are naturally oriented). The cyclic order on the set of incident edges E_v is induced by a fixed orientation of the plane (say counterclockwise). See Figure 3.1 for an example. So, we have just assigned a combinatorial object (an admissible tree) to a configuration of points $(z_1, \ldots, z_n)$.

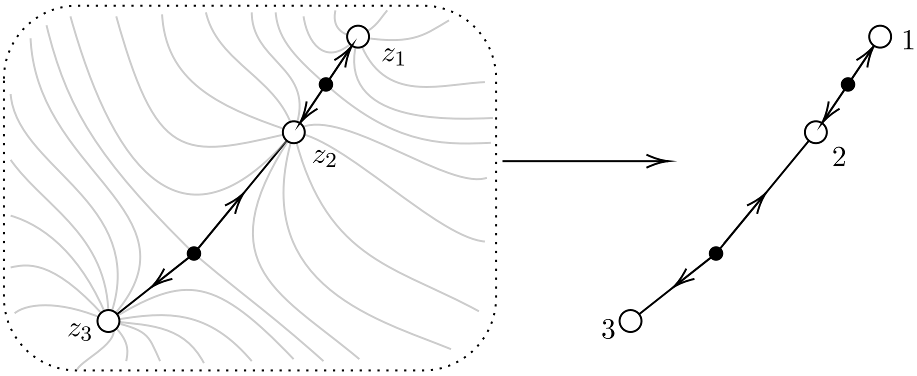


Figure 3.1: This picture shows how to associate an admissible tree to a configuration of points $(z_1, \ldots, z_n) \in \mathcal{M}_{0,n+1}$. The orientation of the grey flow lines is omitted in order to have a clearer picture: their source is the point $\infty \in \mathbb{C} \cup \{\infty\}$, their endpoint is one of the black/white vertices.

Remark 3.2. If we apply a rotation/dilatation/translation to the configuration $(z_1, \ldots, z_n)$, all the flow lines are rotated/dilatated/translated but the resulting tree remains the same! So there is a well defined admissible tree associated to each point in $F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$.

Now we would like to reconstruct the original configuration of charges from the associated tree. Since different points in $F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$ can be associated to the same tree, we have to add some parameters to each admissible tree in order to reconstruct the original configuration.

Definition 3.2. A **labelled tree** is an admissible tree T with the data of the following parameters:

1. A function $f : B \to (0, 1]$ such that:
   - $max(f) = 1$
   - If there is an edge $e = (b, b')$ connecting two black vertices, then $f(b) \geq f(b')$.
2. For each white vertex w, a function $g : E_w \to [0, 1]$ such that $\sum_{e \in E_w} g(e) = 1$. We interpret $2\pi g(e)$ the angle between the edge e and the next edge in the cyclic ordering.

Remark 3.3. Given a configuration $(z_1, \ldots, z_n) \in \mathcal{M}_{0,n+1}$ we associate a labelled tree as follows:

- The underlying admissible tree T is the one constructed with the flow lines.
- Let B be the set of zeroes of $E(z)$ and $M := max_{b \in B} |h(b)|$. Then we set

$$\begin{aligned} f : B &\to (0, 1] \\ b &\mapsto \frac{|h(b)|}{M} \end{aligned}$$

- For any point z_i in the configuration, let E_{z_i} be the set of flow lines of finite lenght that are incident to z_i. Then we put

$$\begin{aligned} g : E_{z_i} &\to [0,1] \\ e &\mapsto \frac{\theta(e)}{2\pi} \end{aligned}$$

  where $\theta(e)$ is the angle between the flow line e and the next one in the cyclic order.

It turns out that a configuration of $\mathcal{M}_{0,n+1}$ is uniquely determined by the correspoding labelled tree. To make this statement a little more precise we need a few definitions:

Definition 3.3. For each tree $T \in T_n$ with set of black vertices B, let σ_B be the topological space of functions $f : B \to (0,1]$ such that:

- $max(f) = 1$
- For any edge $e = (b, b')$ of T we have $f(b) \geq f(b')$

Definition 3.4. For any white vertex $w \in W$ we denote by Δ_w the space of functions $g : E_w \to [0,1]$ such that $\sum_{e \in E_w} g(e) = 1$. This space is a simplex of dimension $|E_w| - 1$.

Definition 3.5. The **stratum** associated to a tree $T \in T_n$ is the space

$$St_T = \sigma_B \times \prod_{w \in W} \Delta_w$$

If we number the white vertices w_i $i = 1, \ldots, m$, then we will denote by $g_i : E_{w_i} \to [0,1]$ an element of Δ_{w_i} and we will write $(f : B \to (0,1], (g_i)_i)_T$ for a generic element of St_T.

Definition 3.6. The space of **labelled trees with n leaves** is the quotient

$$Tr_n \coloneqq (\bigcup_{T \in T_n} St_T)/\sim$$

respect to the following equivalence relation:

1. For $e = (b, b')$ and $f(b) = f(b')$ (b and b' are black vertices)

$$(f : B \to (0,1], (g_i)_i)_T \sim (f' : B/\{b, b'\} \to (0,1], (g_i)_i)_{T/e}$$

   where T/e is the tree obtained from T collapsing the edge e and $f'([x]) \coloneqq f(x)$.

2. If $g_i(e) = 0$ for $e = (b, v_i)$ and the next edge in the cyclic ordering is $e' = (b', v_i)$, then

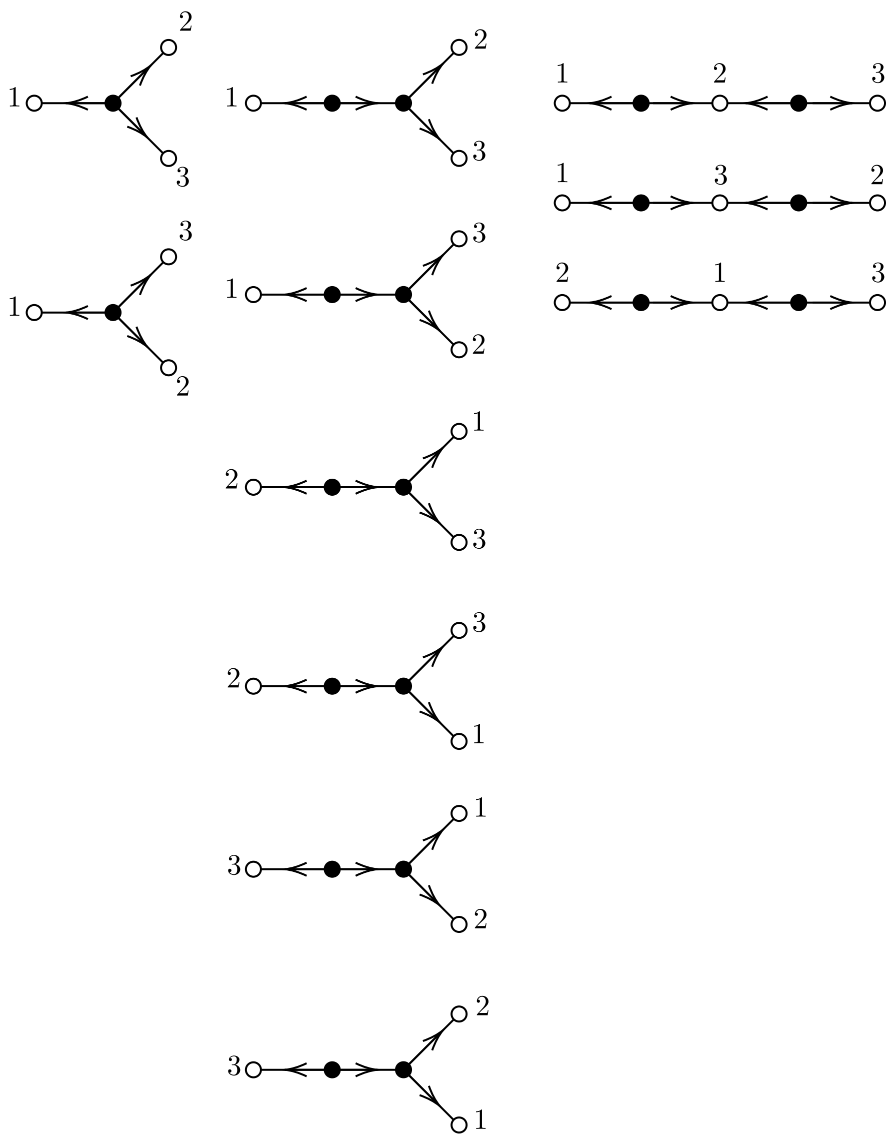


Figure 3.2: This picture shows all the admissible trees with 3 leaves.

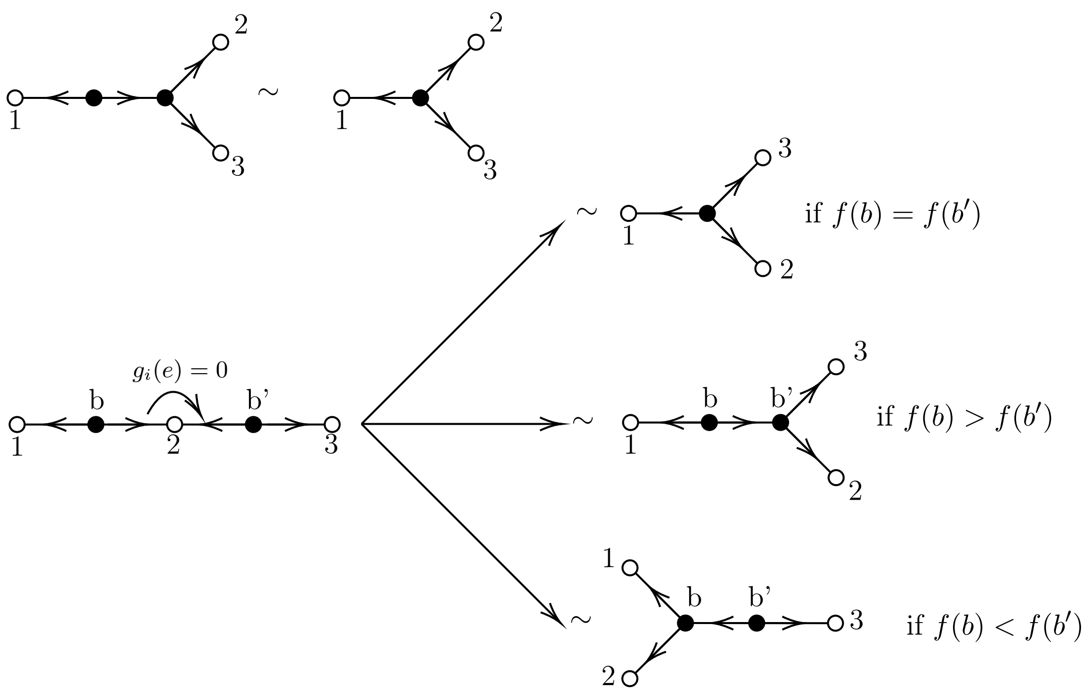


Figure 3.3: This picture shows some identifications that we need to do in the definition of Tr_n.

- For $f(b') = f(b)$ consider the tree T' with black vertex set $B' = B/\{b, b'\}$ and set of edges $E' = E/\{e, e'\}$. Then we identify

$$(f : B \to (0, 1], (g_i)_i)_T \sim (f' : B' \to (0, 1], (g'_i)_i)_{T'}$$

with $g'_i([x]) = g_i(x)$ for $x \in E - \{e\}$ and f' induced by f.
- For $f(b') < f(b)$ consider the tree T^+ with the same vertices of T but the edge $e = (b, v_i)$ replaced by (b, b'). Then

$$(f : B \to (0, 1], (g_i)_i)_T \sim (f : B \to (0, 1], (g_i)_i)_{T^+}$$

- For $f(b') > f(b)$ consider the tree T^- with the same vertices of T but the edge $e' = (b', v_i)$ replaced by (b', b). Then

$$(f : B \to (0, 1], (g_i)_i)_T \sim (f : B \to (0, 1], (g'_i)_i)_{T^-}$$

where $g'_i(e) = g_i(e')$ and g'_i coincides with g_i on the other edges incident to v_i.

See Figure 3.3 for a concrete example of this equivalence relation.

Example. The patient reader could easily verify that Tr_3 is homeomorphic to a sphere with three points deleted. The two trees of the first column of Figure 3.2 are 0-dimensional cells. The six trees of the second column are 1-dimensional (open) cells and the trees of the last column are 2-dimensional (open) cells. See Figure 3.4 for a picture.

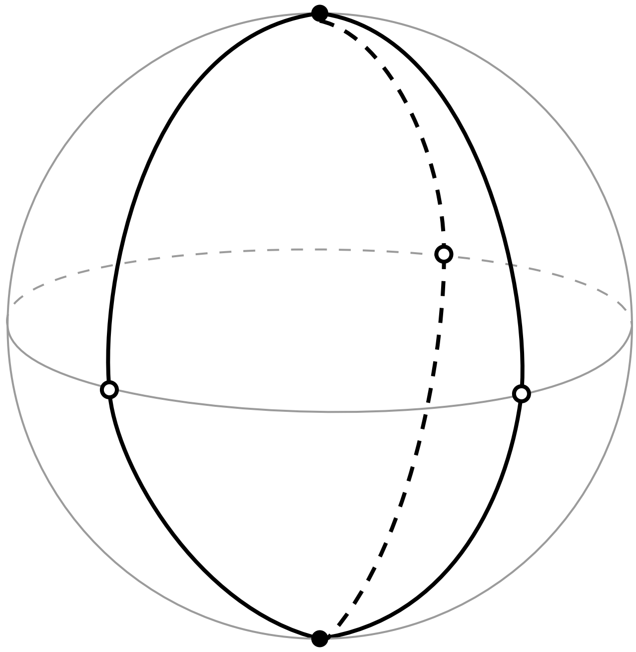


Figure 3.4: The (open) cell decomposition of Tr_3. There are two 0-cells (black vertices), six 1-dimensional open cells (black edges) and three 2-dimensional open cells.

The association we just described, that assign to a point in $\mathcal{M}_{0,n+1}$ a labelled tree with n-leaves turns out to be a homeomorphism, as Salvatore proved in [47]:

Theorem 3.4 (Salvatore, see [47])**.** *For any choice of $a_1, \dots, a_n > 0$ such that $a_1 + \dots + a_n = 1$, there is a homeomorphism*

$$\phi_{a_1,\dots,a_n} : \mathcal{M}_{0,n+1} \to Tr_n$$

The inverse of $\phi_{a_1,\dots,a_n}$ can be described explicitly. We sketch the construction because it will be useful in Section 3.3.

Definition 3.7. Given an admissible tree T we construct a ribbon graph $T \cup \{\infty\}$ as follows: we add another white vertex, which we call ∞, together with edges with source ∞ and target a black vertex so that for any black vertex ingoing and outgoing edges alternate in the cyclic ordering. See Figure 3.5 for an example. To make $T \cup \{\infty\}$ a ribbon graph we have to specify a cyclic ordering on the set E_∞ of edges coming out from ∞. Let e be one of these edges; to find the next edge in the cyclic ordering play the following game: move along e until its end point b, which is a black vertex by construction. Then move along next edge in the cyclic ordering of E_b. It is possible to prove that if you iterate this procedure at the end you came back to ∞ by moving along some edge e'. This edge will be the next to e in the cyclic order of E_∞. The path constructed iteratively in this way is often called the **closed edge path** associated to $e \in E_\infty$.

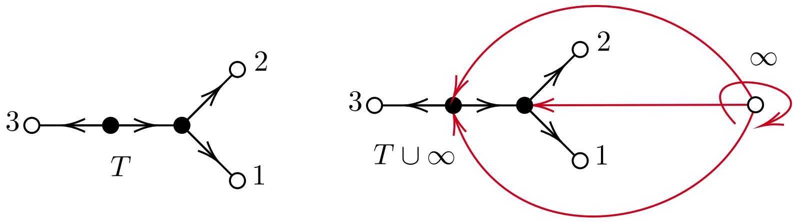


Figure 3.5: On the left there is an admissible tree T with three leaves. The cyclic order of the edges is given by the counterclockwise orientation of the plane. On the right we see the associate ribbon graph: the cyclic order at the black vertices is still induced by the counterclockwise orientation of the plane. Conversely, the cyclic order at ∞ is induced by the opposite orientation of the plane.

Fix a labelled tree $(f : B \to (0,1], g_1, \dots, g_n)_T$ representing a point in Tr_n. Without loss of generality we can suppose that $g_1, \dots, g_n$ are positive functions. We construct a genus zero Riemann surface with n-marked points as follows: consider the ribbon graph $T \cup \{\infty\}$ and realize it geometrically as a one dimensional CW-complex $|T \cup \{\infty\}|$. Then fix a continuous map $\overline{f} : |T \cup \{\infty\}| \to [0, +\infty]$ such that:

- $\overline{f}(b) = f(b)$ for any black vertex $b \in T$.
- $\overline{f}(\infty) = +\infty$, $\overline{f}(w) = 0$ for any white vertex.
- $\overline{f}$ decrease moving along an oriented edge.

For any white vertex w_i $i = 1, \dots, n$ and any $e \in E_{w_i}$ take a strip $[-\infty, +\infty] \times [0, a_i g_i(e)]$. There is a unique closed edge path γ which starts at ∞, passes through w_i and then come back to ∞. Let us denote by γ' the first part of γ (the one from ∞ to w_i) and by γ'' the remaining part. Now attach the strips $[-\infty, +\infty] \times [0, a_i g_i(e)]$ to $|T \cup \{\infty\}|$ according to the following rule:

- Any point $(-\infty, y)$ (resp. $(+\infty, y)$) of the strip is identified with w_i (resp. ∞).
- Any point $(x, a_i g_i(e))$ of the strip is identified with the unique point $p \in |T \cup \{\infty\}|$ such that p is a point of γ' and $log(\overline{f}(p)) = x$.
- Any point $(x, 0)$ of the strip is identified with the unique point $p \in |T \cup \{\infty\}|$ such that p is a point of γ'' and $log(\overline{f}(p)) = x$.

It is possible to prove that the two dimensional CW-complex obtained in this way is actually a genus zero Riemann surface, with $n+1$ marked points corresponding to each white vertex of the tree ([47, Prop. 3.3]). So we get a map

$$\psi_{a_1,\dots,a_n} : Tr_n \to \mathcal{M}_{0,n+1}$$

which is precisely the inverse of $\phi_{a_1,\dots,a_n}$.

3.2 The Nakamura cell decomposition of $\mathcal{M}_{g,n}$

In [41] Nakamura describes a combinatorial model of $\mathcal{M}_{g,n}$ in terms of ribbon graphs, see also [19] for further details about this topic. In this section we will quickly review this combinatorial model.

The Nakamura cell decomposition is based on the following observation: consider a Riemann surface M with n marked points $p_1, \dots, p_n$. Given n real numbers $r_i \neq 0$ $i = 1, \dots, n$ such that $r_1 + \cdots + r_n = 0$, there is a unique abelian differential of the third kind ω that has n simple poles at p_i of residue r_i and has pure imaginary period. This result is due to Giddings and Wolpert [25]. The nice thing is that the real trajectories that pass through a zero of ω forms a ribbon graph (the so called *Nakamura graph*), whose vertices are the poles (i.e. the marked points $p_1, \dots, p_n$) and the zeroes of the differential. Cutting M along this graph one obtains a finite number of strips $\mathbb{R} \times (0, b)$. So, each marked Riemann surface can be obtained by gluing strips to a ribbon graph. The complex structure of M will be encoded in the widths of the strips and the length of the edges of the graph. This leads to a cell decomposition of $\mathcal{M}_{g,n}$ where all the marked Riemann surfaces which are constructed from the same ribbon graph lay in the same cell. In the rest of this section we will review the combinatorics of this cell decomposition.

Remark 3.5. At any point $p \in M$ which is not a pole of ω it is possible to choose local complex coordinates such that locally $\omega = d(z^{k+1})$ with $k \in \mathbb{N}$. If $k \geq 1$ then p is a zero of of order k of ω. In particular there are $2(k+1)$ real trajectories incident to a zero of order k. Outgoing trajectories alternate to incoming trajectories in the cyclic order around a zero which is induced by the orientation of M.

Definition 3.8. A **Nakamura graph** is a finite graph G with the following properties:

1. G is a connected graph with oriented edges.
2. The set of edges incident to each vertex is equipped by a cyclic ordering, i.e. G is a **ribbon graph**.
3. G has two types of vertices, *black vertices* and *white vertices.*
4. White vertices are numbered.
5. Edges around a white vertex are all incoming or all outgoing. Depending on the case, we will call a white vertex incoming or outgoing.
6. Incoming edges and outgoing edges alternate in the cyclic order around a black vertex.
7. If $e = (v, w)$ is an edge, then $v \neq w$.
8. No edge has only white vertices as its endpoints.

9. Every closed edge path of G contains exactly two white vertices, one incoming and one outgoing.

The **genus** of a Nakamura graph G is the number $g(G)$ given by the formula

$$2 - 2g(G) = v - e + f$$

where v is the number of vertices of G, e is the number of edges, and f is the number of closed edge paths.

Remark 3.6. The number of incoming and outgoing vertices of a Nakamura graph G depends on the initial choice of residues $r_1, \dots, r_n$. More precisely, if $r_i > 0$ (resp. $r_i < 0$) then the i-th white vertex will be outgoing (resp. incoming).

Definition 3.9. Consider the following data:

- A Nakamura graph G of genus g with n white vertices. Let us denote by l the number of black vertices, and by $\gamma_1, \dots, \gamma_k$ the closed edge paths of G.

- **Strip width parameters** $b_1, \dots, b_k$, one for each closed edge path such that:

$$b_i \geq 0$$
$$b_{j_1} + \dots + b_{j_m} = |r_j|$$

  where $\gamma_{j_1}, \dots, \gamma_{j_m}$ are the closed edge paths which contains the j-th white vertex.

- **Interaction times** $t_1, \dots, t_l \in \mathbb{R}$, one for each black vertex $z_1, \dots, z_l$ of G such that:

$$t_1 + \dots + t_l = 0$$
$$t_i \leq t_j \text{ if there is an oriented edge from } z_i \text{ to } z_j$$

These parameters define a convex subspace $\mathcal{B}(G) \subseteq \mathbb{R}^{k+l}$ of dimension $k + l - n$.

Attaching k strips of widths $b_1, \dots, b_k$ to G (taking the interaction times as the real coordinates of the black vertices) we obtain a bijection between $\mathcal{B}(G)$ and a subspace of the Teichmüller space $\mathcal{T}_{g,n}$:

Theorem 3.7 (Nakamura, [41]). *Let $r_1, \dots, r_n$ be non-zero real numbers satisfying $r_1 + \dots + r_n = 0$. Then there exists a cell decomposition of $\mathcal{T}_{g,n}$*

$$\mathcal{T}_{g,n} = \bigsqcup_{G \in \mathcal{G}} \mathcal{B}(G)$$

where $\mathcal{G}$ is the set of genus g Nakamura graphs with n white vertices and such that the i-th white vertex is outgoing (resp. incoming) if $r_i > 0$ (resp. $r_i < 0$).

Moreover this cell decomposition is invariant under the action of the mapping class group, so it induce a cell decomposition of $\mathcal{M}_{g,n}$. The action of the mapping class group is encoded combinatorially by the action of $Aut(G)$ on G, where $Aut(G)$ is the group of automorphisms of G which fix every pole and preserve the cyclic orderings. Indeed it is possible to prove that any element $g \in Aut(G)$ gives rise to a biholomorphism of any marked Riemann surface associated to G which fix each marked point. So we have:

Theorem 3.8 (Nakamura)**.** *Let $r_1, \dots, r_n$ be non-zero real numbers satisfying $r_1 + \cdots + r_n = 0$. Then there is a (orbi)cell decomposition of $\mathcal{M}_{g,n}$*

$$\mathcal{M}_{g,n} = \bigsqcup_{G \in \mathcal{G}} \mathcal{B}(G)/Aut(G)$$

where $\mathcal{G}$ is the set of genus g Nakamura graphs with n white vertices and such that the i-th white vertex is outgoing (resp. incoming) if $r_i > 0$ (resp. $r_i < 0$).

Remark 3.9. Each choice of the residues $r_1, \dots, r_n$ gives different graphs, and therefore a different cell decomposition on $\mathcal{M}_{g,n}$. If we set $r_1 > 0$ and $r_i < 0$ we get the smaller cell decomposition (see [41]).

3.3 Comparison between Salvatore and Nakamura cells

In this section we prove that the Nakamura cell decomposition of $\mathcal{M}_{0,n+1}$ coincides to the one of Salvatore for a particular choice of residues and weights. From now on we will suppose that the residues of the Nakamura cell decomposition satisfy

$$\begin{cases} r_1 > 0 \\ r_i < 0 \text{ for all } i = 1, \dots, n \end{cases}$$

In this case we obtain Nakamura graphs with only one outgoing white vertex, which we call ∞, and all the other white vertices will be incoming.

Lemma 3.10. *Let G be a genus 0 Nakamura graph with $n \geq 2$ incoming white vertices, and one outgoing white vertex. Then $Aut(G) = 1$.*

Proof. Each automorphism ϕ of G can be realized geometrically as a biholomorphism of $\mathbb{C}P^1$ which fix $n+1$ points. But this imply that this biholomorphism is the identity, and so the same holds for ϕ. □

Corollary 3.11. *Let $r_0 > 0$ and $r_1, \dots, r_n < 0$ be real numbers such that $\sum_{i=0}^n r_i = 0$. Then there is a cell decomposition of $\mathcal{M}_{0,n+1}$*

$$\mathcal{M}_{0,n+1} = \bigsqcup_{G \in \mathcal{G}} \mathcal{B}(G)$$

where $\mathcal{G}$ is the set of genus 0 Nakamura graphs with one outgoing white vertex and n incoming white vertices.

Proof. Just combine the previous Lemma and Theorem 3.8.

□

Let G be a genus zero Nakamura graph. If we delete the only outgoing white vertex ∞ together with all the edges incident to it we obtain another ribbon graph $G-\{\infty\}$, with the cyclic order at the vertices induced by the one of G. The next Proposition shows that $G - \{\infty\}$ is an admissible tree.

Proposition 3.12. *If G is a Nakamura graph of genus 0, then $G - \{\infty\}$ is an admissible tree. Conversely, if T is an admissible tree with n leaves then $T \cup \{\infty\}$ is a Nakamura graph of genus 0.*

Proof. Let G be a genus 0 Nakamura graph with $n+1$ white vertices (one outgoing, n incoming). Consider $T := G - \{\infty\}$: the complement $\mathbb{C}P^1 - T$ contracts to ∞ following the real trajectories of the Giddings-Wolpert differential. By Alexander duality T is a finite graph with trivial homology, so it is a tree. Clearly T has two kind of vertices (black and white), oriented edges and a cyclic ordering at each vertex. By hypothesis all the white vertices of G were incoming except for ∞, therefore all the white vertices of T are incoming. If $e = (v, w)$ is an edge of T, then (w, v) is not an edge because T is a tree. Each black vertex of T is the source of at least two edges: indeed each black vertex of T comes from a black vertex of G. By Remark 3.5 each black vertex of G has at least four real trajectories, so it has at least two outgoing real trajectories. Finally, each black vertex b can not be the target of two edges next to each other in the cyclic order: suppose by contradiction there are two such edges e, e'. Since in G outgoing and incoming edges alternate in the cyclic ordering of each black vertex, then there must be an edge of G starting at b, ending at ∞ and which is between e and e' in the cyclic order of edges incident to b. But this is a contradiction because ∞ is an outgoing vertex.
Conversely, let T be an admissible tree, we show that $T \cup \{\infty\}$ is a Nakamura graph: by construction it is connected, its edges are oriented and there is a cyclic ordering on the set of edges incident to a vertex. It has two kind of vertices (black and white); white vertices are all incoming, except for ∞ which is the only outgoing vertex. For each black vertex b, incoming and outgoing edges alternate in the cyclic ordering of E_b because of the definition of $T \cup \{\infty\}$. Conditions (7), (8) and (9) of the definition of a Nakamura graph are clearly satisfied. It remains to prove that $T \cup \{\infty\}$ has genus 0, i.e. that the surface we obtain gluing strips to it has genus zero. But we already know this form Section 3.1, because the surface we obtain gluing strips to $T \cup \{\infty\}$ is the Riemann sphere. □

Now let us fix some weights $a_1, \ldots, a_n$, $a_i > 0$. As we saw in Section 3.1 this choice gives us a homeomorphism

$$\psi_{a_1,\ldots,a_n} : Tr_n \to \mathcal{M}_{0,n+1}$$

and so we obtain an open cell decomposition of $\mathcal{M}_{0,n+1}$. We want to prove that this decomposition is the same of the one by Nakamura for a suitable choice of residues.

Theorem 3.13. *The cell decomposition of* $\mathcal{M}_{0,n+1}$ *given by* $\psi_{a_1,\dots,a_n} : Tr_n \to \mathcal{M}_{0,n+1}$ *coincided with the Nakamura cell decomposition associated to the residues* $r_i := -a_i$ *for* $i = 1, \dots, n$ *and* $r_0 := \sum a_i$.

Proof. Consider the residues as in the statement. This choice determines a cell decomposition of $\mathcal{M}_{0,n+1}$ in terms of Nakamura graphs. We claim that the cell $\mathcal{B}(G)$ associated to a Nakamura graph G coincides with the cell of Salvatore associated to the admissible tree $G - \{\infty\}$. Each cell $\mathcal{B}(G)$ is described by:

- A genus 0 Nakamura graph G with one outgoing white vertex and n incoming white vertices.
- $b_1, \dots, b_k$ widths parameters (one for each closed edge path $\gamma_1, \dots, \gamma_k$ of G) such that:

$$b_i \geq 0 \tag{3.1}$$
$$b_{j_1} + \dots + b_{j_m} = |r_j| \tag{3.2}$$

  where $\gamma_{j_1}, \dots, \gamma_{j_m}$ are the closed edge paths containing the j-th white vertex.
- $t_1, \dots, t_l$ interaction times for each black vertex $z_1, \dots, z_l$ such that:

$$t_1 + \dots + t_l = 0 \tag{3.3}$$
$$t_i \leq t_j \text{ if there is an oriented edge from } z_i \text{ to } z_j \tag{3.4}$$

If we fix $(b_1, \dots, b_k, t_1, \dots, t_l) \in \mathcal{B}(G)$ as above, the marked Riemann surface associated to it is constructed as follows (see [41] for more details): first take the geometric realization $|G|$ of G. Then fix a continuous map $\overline{f} : |G| \to [-\infty, +\infty]$ such that:

- $\overline{f}(z_i) = t_i$ for any black vertex $z_i \in G$.
- $\overline{f}(\infty) = -\infty$, $\overline{f}(w) = +\infty$ for any incoming white vertex.
- $\overline{f}$ increase moving along an oriented edge.

Finally, take strips $[-\infty, +\infty] \times [0, b_i]$ and glue them to $|G|$ as follows: each strip corresponds to a closed edge path γ starting at ∞, passing through a white vertex w and then coming back to ∞. Let us denote by γ' the first part of γ (the one from ∞ to w) and by γ'' the remaining part. Now attach the strips $[-\infty, +\infty] \times [0, b_i]$ to $|G|$ according to the following rule:

- Any point $(-\infty, y)$ (resp. $(+\infty, y)$) of the strip is identified with ∞ (resp. w).
- Any point (x, b_i) of the strip is identified with the unique point $p \in |G|$ such that $p \in \gamma'$ and $\overline{f}(p) = x$.
- Any point $(x, 0)$ of the strip is identified with the unique point $p \in |G|$ such that $p \in \gamma''$ and $\overline{f}(p) = x$.

Let us denote by $M(G, b_1, \dots, b_k, t_1, \dots, t_l) \in \mathcal{M}_{0,n+1}$ the resulting marked Riemann surface. Now consider the following labelled tree:

- A underlying admissible tree we put $T \coloneqq G - \{\infty\}$.
- Let m be the minimum between $t_1, \dots, t_l$. For any black vertex $z_i \in G$ we put $f(z_i) \coloneqq e^{m-t_i} \in (0, 1]$. Note that the inequalities 3.4 translate to the fact that f decrease if we move on a oriented edge.
- Let w_j be the white vertex with residual $r_j = -a_j$ and let $e \in E_{w_i}$ be an incident edge. Let us denote by $e' \in E_{w_j}$ the next edge in the cyclic ordering, and by $b(e)$ the width of the strip corresponding to the unique closed edge path starting at $\infty \in G$, going into w_j through the edge e and then coming back towards ∞ through e'. Then we define

  $$\begin{aligned} g_j : E_{w_j} &\to [0, 1] \\ e &\mapsto b(e)/a_j \end{aligned}$$

  Note that Equation 3.2 implies that $\sum_{e \in E_{w_j}} g_j(e) = 1$.

Now we claim that that the point $\psi_{a_1,\dots,a_n}((f, g_1, \dots, g_n)_{G-\{\infty\}}) \in \mathcal{M}_{0,n+1}$ associated to such data through Salvatore's construction is precisely $M(G; b_1, \dots, b_l, t_1, \dots, t_k)$: by definition $\psi_{a_1,\dots,a_n}((f, g_1, \dots, g_n)_{G-\{\infty\}})$ is obtained by gluing strips $[-\infty, +\infty] \times [0, b_j]$ $j = 1, \dots, l$ to G, and the position of the black vertices $z_1, \dots, z_l$ of $G - \{\infty\}$ along the boundaries of the strips is given by $log f(z_i) = m - t_i$. In the construction of Salvatore the points $(-\infty, y)$ (resp. $(y, +\infty)$) of a strip $[-\infty, +\infty] \times [0, b_j]$ are identified with a white vertex (resp. ∞) of G. The resulting marked Riemann surface is then $M(G; b_1, \dots, b_l, t_1 - m, \dots, t_k - m)$. If we translate the interaction times the resulting marked Riemann surface does not change (it amounts to perform a dilation of the resulting configuration of points in the plane). Therefore

$$\begin{aligned} \psi_{a_1,\dots,a_n}((f, g_1, \dots, g_n)_{G-\{\infty\}}) &= M(G; b_1, \dots, b_l, t_1 - m, \dots, t_k - l) \\ &= M(G, b_1, \dots, b_k, t_1, \dots, t_l) \end{aligned}$$

concluding the proof. $\square$

Chapter 4

Combinatorial models for $\overline{\mathcal{M}}_{0,n+1}$

In [47] Salvatore constructs a CW-decomposition of $FM_2(n)$, the Fulton-Pherson compactification of $F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{R}^{>0}$. The main result of this chapter is the construction of a similar cell decomposition for $\overline{\mathcal{M}}_{0,n+1}$ (Theorem 4.13). The outline of the chapter is the following:

Section 4.1 is a recollection of some known facts about cacti.

Section 4.2 is about nested trees and their combinatorics.

Section 4.3 deals with the definition of a regular CW-complex, the space of nested cacti $N_n^\sigma(\mathcal{C}/S^1)$.

Section 4.4 compares nested cacti and labelled trees.

Section 4.5 contains the proof that for each choice of weights $a_1, \ldots, a_n > 0$, $\sum a_i = 1$ there is a homeomorphism $\overline{\phi}_{a_1,\ldots,a_n} : \overline{\mathcal{M}}_{0,n+1} \to N_n^\sigma(\mathcal{C}/S^1)$.

4.1 Cacti

In this section we recall the definition and properties of the space of cacti. A combinatorial construction of this space was introduced by McClure and Smith [39], while geometric constructions are due to Voronov [52] and Kaufmann [28]. Salvatore compared the two approaches in [49]. Another good reference where the reader can find many connections with string topology is the book by Cohen, Hess, Voronov [13].

Definition 4.1. [The space of cacti] Let $\mathcal{C}_n$ be the set of partitions x of S^1 into n closed 1-manifolds $I_1(x), \ldots, I_n(x)$ such that:

- They have equal measure.
- They have pairwise disjoint interiors.
- There is no cyclically ordered sequence of points (z_1, z_2, z_3, z_4) in S^1 such that $z_1, z_3 \in I_j(x)$, $z_2, z_4 \in I_k(x)$ and $j \neq k$.

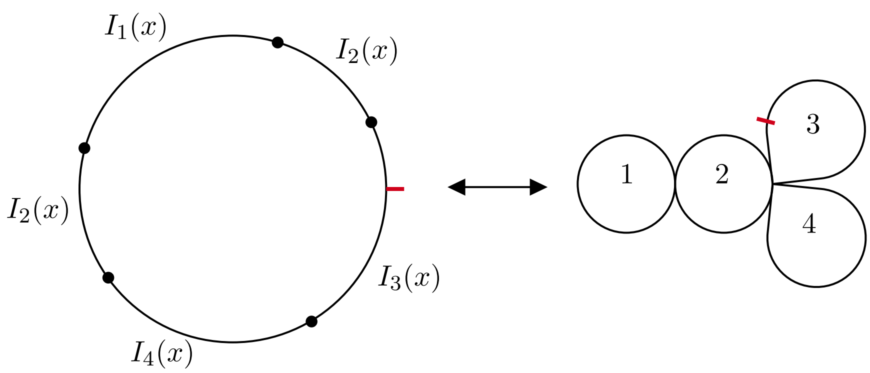


Figure 4.1: On the left there is an element $x \in \mathcal{C}_4$, on the right its associated cactus $c(x)$. The base point of the circle S^1 is depicted in red and corresponds to a base point on the cactus $c(x)$ (which we depict as a red spine).

We can equip this set with a topology by defining a metric on it: for any $x, y \in \mathcal{C}_n$ we set

$$d(x,y) = \sum_{j=1}^{n} \mu(I_j(x) - \mathring{I}_j(y))$$

where μ denotes the measure. We will call $\mathcal{C}_n$ (with this topology) the **space of based cacti with n-lobes**. See Figure 4.1 for an example.

Remark 4.1. $\mathcal{C}_n$ is called the space of cacti for the following reason: given $x \in \mathcal{C}_n$, let us define a relation $\sim$ on S^1: two points $z_1, z_2 \in S^1$ are equivalent if there is an index $j \in \{0, \dots, n\}$ such that z_1 and z_2 are the boundary points of the same connected component of $S^1 - \mathring{I}_j$. The quotient space $c(x) := S^1/\sim$ by this relation is a pointed space (the base point is just the image of $1 \in S^1$) called the **cactus** associated to x: topologically it is a configuration of n-circles in the plane, called lobes, whose dual graph is a tree. The dual graph is a graph with two kind of vertices: a white vertex for any lobe and a black vertex for any intersection point between two lobes. An edges connects a white vertex w to a black vertex b if b represent the intersection point of the lobe corresponding to w with some other lobe. See Figure 4.2 for some examples. Note that the lobes are the image of the 1-manifolds $I_1(x), \dots, I_n(x)$ under the quotient map $S^1 \to S^1/\sim$. In what follows we will freely identify a partition $x \in \mathcal{C}_n$ and its associated cactus $c(x)$.

Cell decomposition: to any point $x \in \mathcal{C}_n$ we can associate a sequence $(X_1, \dots, X_l)$ of numbers in $\{1, \dots, n\}$ by the following procedure: start from the point $1 \in S^1$ and move along the circle clockwise. The sequence $(X_1, \dots, X_l)$ is obtained by writing one after the other the indices of all the 1-manifolds you encounter and has the following properties:

- All values between 1 and n appear.
- $X_i \neq X_{i+1}$ for every $i = 1, \dots, l-1$.

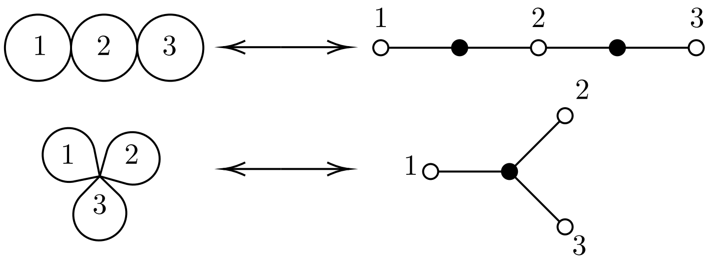


Figure 4.2: On the left there are some cacti (without base point), on the right the corresponding dual graphs.

- There is no subsequence of the form (i, j, i, j) with $i \neq j$.

This sequence is just a combinatorial way to encode the shape of the cactus $c(x)$. By an abuse of notation we will also denote by $(X_1, \dots, X_l)$ the subspace of $\mathcal{C}_n$ consisting of all partitions whose associated sequence is $(X_1, \dots, X_l)$. This subspace turns out to be homeomorphic to a product of simplices. More precisely, if m_i is the cardinality of $\{j \in \{1, \dots, l\} \mid X_j = i\}$, $i = 1, \dots, n$, then

$$(X_1, \dots, X_l) \cong \prod_{i=1}^{n} \Delta^{m_i - 1}$$

For an example look at the cactus of Figure 4.1: it belongs to the cell $(3, 4, 2, 1, 2, 3) \cong \Delta^0 \times \Delta^1 \times \Delta^1 \times \Delta^0$. Intuitively all the cacti $c(x)$ associated to partitions $x \in (X_1, \dots, X_l)$ have the same shape. So we will represent pictorially a cell by drawing a cactus, meaning that the cell contains all the partitions $x \in \mathcal{C}_n$ whose associated cactus $c(x)$ has that shape. From this point of view, the parameters of a cell $(X_1, \dots, X_l)$ can be thought as the lengths of the arcs between two consecutive *marked points*, where a marked point is an intersection point of lobes or the base point. The boundary of a cell is obtained by collapsing some of these arcs. See Figure 4.3 for some examples. This gives a regular CW-decomposition of $\mathcal{C}_n$.

Remark 4.2. There are two relevant groups acting on $\mathcal{C}_n$: S^1 acts by rotating the base point of a cactus, Σ_n acts by relabelling the lobes.

Proposition 4.3 (Salvatore, [47])**.** *The projection $p : \mathcal{C}_n \to \mathcal{C}_n/S^1$ is a trivial principal S^1-bundle.*

The important thing about cacti is that they are a very small cellular model for the configuration space $F_n(\mathbb{C})$:

Theorem 4.4 (Salvatore, [47])**.** *The space of cacti $\mathcal{C}_n$ is $(S^1 \times \Sigma_n)$-equivariantly homotopy equivalent to $F_n(\mathbb{C})$.*

Actually there is more, since it is possible to define cellular maps

$$\theta_{n_1, \dots, n_k} : \mathcal{C}_k \times \mathcal{C}_{n_1} \times \dots \times \mathcal{C}_{n_k} \to \mathcal{C}_{n_1 + \dots + n_k}$$

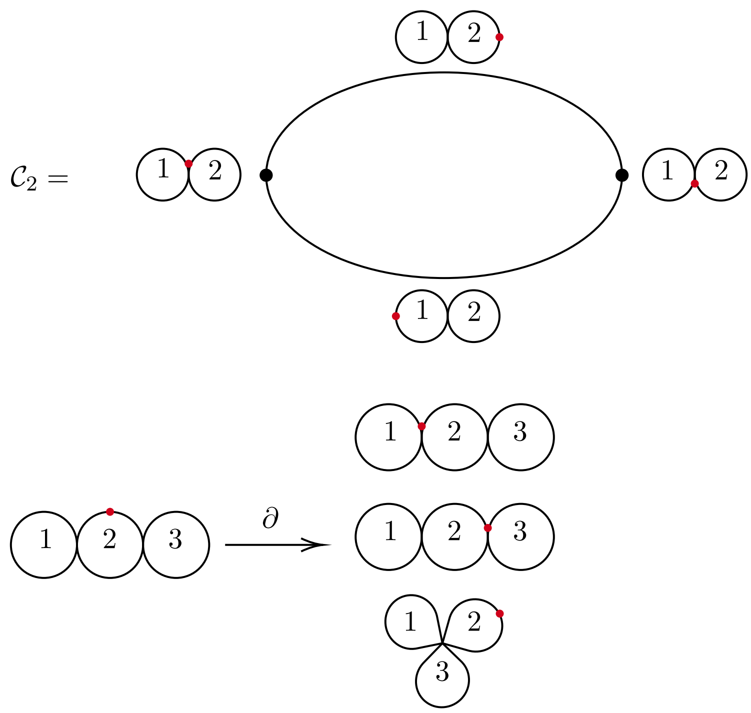


Figure 4.3: On top where is a full description of $\mathcal{C}_2 \cong S^1$: there are two zero cells $(2,1)$ (on the left) and $(1,2)$ (on the right). The 1-cells are $(2,1,2)$ (on the top) and $(1,2,1)$ (on the bottom). Below we see the cell $(2,3,2,1,2) \cong \Delta^0 \times \Delta^2 \times \Delta^0$ of $\mathcal{C}_3$ and the codimension one cells in its boundary.

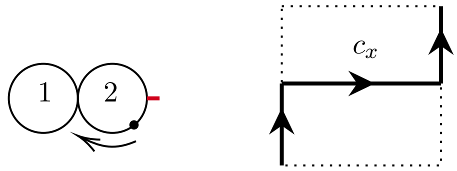


Figure 4.4: On the left we see a cactus x, whose associated sequence is $(2,1,2)$. On the right we see the image of the corresponding cactus map $c_x : [0,2] \to [0,1]^2$

which give a structure of operad up to homotopy on the sequence of spaces $\{\mathcal{C}_n\}_{n\geq 1}$. As usual we can also define partial compositions $\circ_i : \mathcal{C}_n \times \mathcal{C}_m \to \mathcal{C}_{n+m-1}$ by the formula

$$x \circ_i y \coloneqq \theta(1, \dots, 1, y, 1, \dots, 1)$$

where 1 denotes the unique point of $\mathcal{C}_1$. In what follows we construct $\theta_{n_1,\dots,n_k}$, following [47, Sec. 4]. First we need a preliminary definition:

Definition 4.2. Let $x \in \mathcal{C}_n$ be a cactus and let $(X_1, \dots, X_l)$ be its associated sequence. If we think S^1 as the quotient of $[0,n]$ by its endpoints we can pullback the partition x along the quotient map $p : [0,n] \to [0,n]/\sim$, obtaining a decomposition of $[0,n]$ in l closed intervals $[0,y_1],[y_1,y_2],\dots,[y_{l-1},n]$. The **cactus map** $c_x = (c_x^1, \dots, c_x^n) : [0,n] \to [0,1]^n$ is defined as follows:

- $c_x(0) = (0, \dots, 0)$
- If $y \in [y_i, y_{i+1}]$ then $c_x^j(y) = \begin{cases} c_x^j(y_i) \text{ if } j \neq X_i \\ c_x^{X_i}(y_i) + (y - y_i) \text{ if } j = X_i \end{cases}$

Intuitively the curve c_x describes the motion of a point that moves along the cactus clockwise (starting at the base point): when the points passes through the i-th lobe the i-th coordinate of c_x increases, while the other components remain constant. In particular $c_x(n) = (1, \dots, 1)$. For an example see Figure 4.4. Note that the cactus map c_x uniquely determines the cactus x.

Now we are ready to define $\theta_{n_1,\dots,n_k} : \mathcal{C}_k \times \mathcal{C}_{n_1} \times \cdots \times \mathcal{C}_{n_k} \to \mathcal{C}_{n_1+\cdots+n_k}$.

Definition 4.3. For $k, n_1, \dots, n_k \in N$ we define embeddings

$$\theta_{n_1,\dots,n_k} : \mathcal{C}_k \times \mathcal{C}_{n_1} \times \cdots \times \mathcal{C}_{n_k} \to \mathcal{C}_{n_1+\cdots+n_k}$$

as follows: fix $x \in \mathcal{C}_k$ and $x_j \in \mathcal{C}_{n_j}$ for $j = 1, \dots, k$. $\theta_{n_1,\dots,n_k}(x, x_1, \dots, x_k)$ will be described by the corresponding cactus map: let $n \coloneqq n_1 + \cdots + n_k$ and consider the product of dilations

$$\begin{aligned} D : [0,1]^k &\to \prod_{j=1}^k [0, n_j] \\ (t_1, \dots, t_k) &\mapsto (t_1 n_1, \dots, t_k n_k) \end{aligned}$$

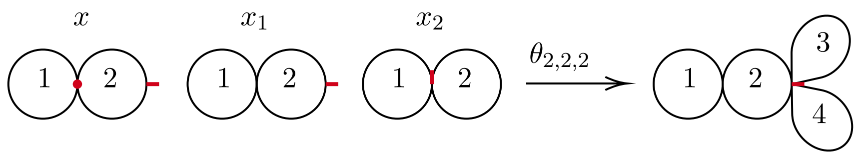


Figure 4.5: On the left we see three cacti with two lobes. The red bullet on the leftmost cactus denotes the local base point of lobe 1. On the right we see a picture of the composite $\theta_{2,2,2}(x, x_1, x_2)$.

There is a unique piecewise linear homomorphism $\alpha : [0, k] \to [0, n]$ and a unique piecewise oriented isometry onto its image $c : [0, n] \to \prod_{j=1}^{k}[0, n_j]$ such that the following square is commutative:

$$\begin{tikzcd}
{[0,k]} \arrow[r, "c_x"] \arrow[d, "\alpha"] & {[0,1]^k} \arrow[d, "D"] \\
{[0,n]} \arrow[r, "c"] & \prod_{j=1}^{k}[0,n_j]
\end{tikzcd}$$

Then $\theta_{n_1,\ldots,n_k}(x, x_1, \ldots, x_k)$ is uniquely determined by the cactus map

$$\left(\prod_{j=1}^{k} c_{x_j}\right) \circ c : [0, n] \to [0, 1]^n$$

Intuitively the composition of cacti works as follows: fix $x \in \mathcal{C}_k$ and $x_j \in \mathcal{C}_{n_j}$ for $j = 1, \ldots, k$. Observe that each lobe l of x has a *local base point*: if l contains the base point then the local base point coincides with it. Otherwise the local base point is the intersection point of l with the connected component of $x - l$ containing the base point. $\theta_{n_1,\ldots,n_k}(x, x_1, \ldots, x_k)$ is obtained by inserting each cactus x_i into the i-th lobe of x so that the base point of x_i coincides with the local base point of the i-th lobe of x. For a picture see Figure 4.5.

Theorem 4.5 (Salvatore, [47])**.** *The maps $\theta_{n_1,\ldots,n_k}$ induce an operad structure on the sequence of chain complexes $Cact := \{C_*^{cell}(\mathcal{C}_n)\}_{n\in\mathbb{N}}$, where $C_*^{cell}(\mathcal{C}_n)$ are the cellular chains. Therefore, $Cact$ is a chain model for the little two disk operad $\mathcal{D}_2$.*

Remark 4.6. The operad $Cact$ is isomorphic to $\mathcal{S}_2$, a natural E_2 suboperad of the E_∞ surjection operad $\mathcal{S}$ by McClure and Smith [40]. The reader can find some details on the comparison between $Cact$ and $\mathcal{S}_2$ in [47] and [49].

Remark 4.7. There are many variants of the space of cacti:

1. One can do the same construction as before but with the additional data of a base point (also called a *spine*) for each lobe. We call the resulting space $f\mathcal{C}_n$ the **space of cacti with spines** (or *framed cacti*). From the configuration space point of view this corresponds to assign an element of S^1 to each point

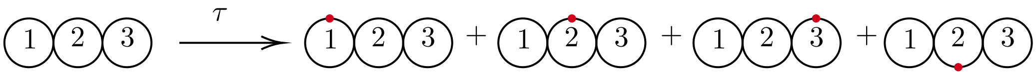


Figure 4.6: An example of how the transfer τ works (with $\mathbb{F}_2$ coefficients).

of the configuration. Therefore $f\mathcal{C}_n$ will be homotopy equivalent to the space of framed configurations $fF_n(\mathbb{C}) := F_n(\mathbb{C}) \times (S^1)^n$ and the family of cellular chains $fCact := \{C_*^{cell}(f\mathcal{C}_n)\}_{n\in\mathbb{N}}$ will be a chain model for the framed little two disks operad $f\mathcal{D}_2$.

2. The quotient $\mathcal{C}_n/S^1$ is still a regular CW-complex, and its cells are described by the same combinatorics of $\mathcal{C}_n$ except that we do not have a base point on our cacti (see Figure 4.8 for an example). We call $\mathcal{C}_n/S^1$ the **space of unbased cacti**. By Theorem 4.4 $\mathcal{C}_n/S^1$ is homotopy equivalent to $F_n(\mathbb{C})/S^1 \simeq \mathcal{M}_{0,n+1}$. The operadic structure of $Cact$ induces an operad structure on the cellular chains $C_*^{cell}(\mathcal{C}_n/S^1)$, giving a chain model for the Gravity operad. More precisely, the collection of chain complexes $grav := \{sC_*^{cell}(\mathcal{C}_n/S^1)\}$ is an operad, with partial compositions defined by the following diagram:

$$\begin{tikzcd}
C_*^{cell}(\mathcal{C}_n) \otimes C_*^{cell}(\mathcal{C}_m) \arrow[r, "\circ_i"] & C_*^{cell}(\mathcal{C}_{n+m-1}) \\
sC_*^{cell}(\mathcal{C}_n/S^1) \otimes sC_*^{cell}(\mathcal{C}_m/S^1) \arrow[u, "\tau\otimes\tau"] \arrow[r, dashed, "\circ_i"] & sC_*^{cell}(\mathcal{C}_{n+m-1}/S^1) \arrow[u, "\tau"']
\end{tikzcd}$$

Here $\tau : sC_*^{cell}(\mathcal{C}_n/S^1) \to C_*^{cell}(\mathcal{C}_n)$ is a chain model for the transfer

$$H_*(F_n(\mathbb{C})/S^1) \to H_{*+1}(F_n(\mathbb{C}))$$

and takes an (unbased) cactus to the sum (with signs, depending on the orientations) of all cacti one can obtain by adjoining a base point in one of the lobes (see Figure 4.6). It is possible to see that if $C_1 \in sC_*^{cell}(\mathcal{C}_n/S^1)$ and $C_2 \in sC_*^{cell}(\mathcal{C}_m/S^1)$, then $\tau(C_1) \circ_i \tau(C_2) = \tau(C)$ for a unique $C \in sC_*^{cell}(\mathcal{C}_{n+m-1}/S^1)$, so we set $C_1 \circ_i C_2 := C$. To be more explicit, the composition $C_1 \circ_i C_2$ is the sum of all (unbased) cacti which one obtains by inserting C_2 into the i-th lobe of C_1 in all possible ways. For an explicit example see Figure 4.7.

We summarize these observations in the next table:

Space	$F_n(\mathbb{C})/S^1$	$F_n(\mathbb{C})$	$fF_n(\mathbb{C})$
Chain model	$\mathcal{C}_n/S^1$	$\mathcal{C}_n$	$f\mathcal{C}_n$
Operad in Top		$\mathcal{D}_2$	$f\mathcal{D}_2$
Operad in $Ch(\mathbb{Z})$	$grav$	$Cact$	$fCact$

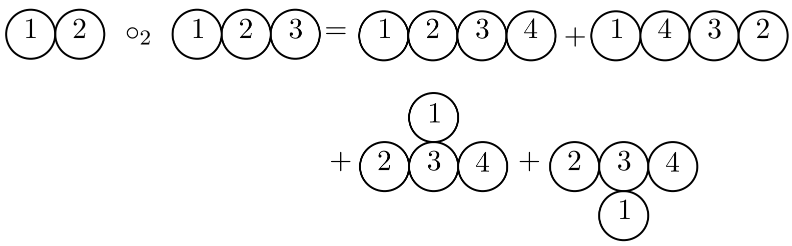


Figure 4.7: An example of how the partial composition of grav works (with $\mathbb{F}_2$ coefficients).

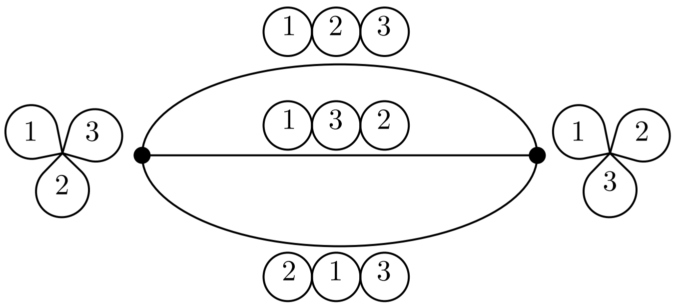


Figure 4.8: This picture shows the CW-complex $\mathcal{C}_3/S^1 \simeq \mathcal{M}_{0,4}$. There are two zero cells and three edges.

4.2 Nested trees

In this section we introduce some combinatorial notions that will be useful for the rest of the chapter.

Definition 4.4. Let R be a finite set. A **nested tree with leaves labelled in** R is a collection $\mathcal{S}$ of subsets of R of cardinality at least 2, called vertices, such that:

- $R \in \mathcal{S}$. This vertex is called the **root**.
- If $S_1, S_2 \in \mathcal{S}$ then either $S_1 \cap S_2 = \emptyset$ or $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

We denote by N_R the set of all nested trees with leaves labelled in R. A pair $(S_1, S_2) \in \mathcal{S} \times \mathcal{S}$ is called an **internal edge** if $S_1 \subsetneq S_2$ and there is no vertex $S_3 \in \mathcal{S}$ such that $S_1 \subsetneq S_3 \subsetneq S_2$. We say that the internal edge (S_1, S_2) goes out of S_1 and into S_2. For each $i \in R$ let S_i be the minimal element of $\mathcal{S}$ containing i. We say that i is an **open edge** going into S_i. The **valence** $|S|$ of a vertex $S \in \mathcal{S}$ is the number of edges (either internal or open) going into it. For an example see Figure 4.9. Finally, observe that N_R is partially ordered by inclusion. Geometrically, if $\mathcal{S} \subseteq \mathcal{T}$ then $\mathcal{S}$ can be obtained from $\mathcal{T}$ by collapsing some edges.

Remark 4.8. When $R = \{1, \dots, n\}$ we will use the notation N_n instead of $N_{\{1,\dots,n\}}$ and we will call an element of N_n a **nested tree with n-leaves**. If R is a finite set with n elements, the data of a total order on R induces a bijection between N_R and N_n.

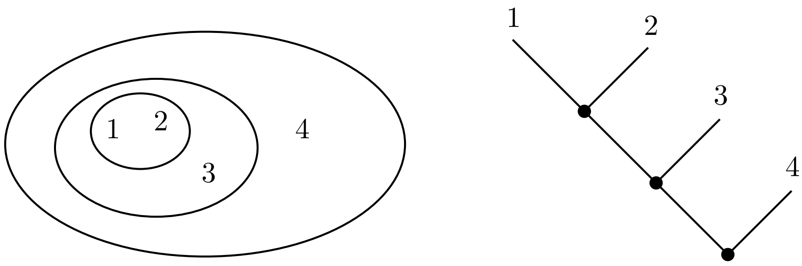
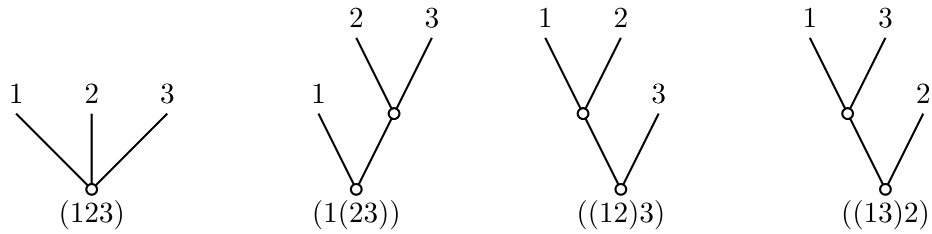


Figure 4.9: In this picture we see two graphical representations of the nested tree $\mathcal{S} = \{\{1,2,3,4\},\{1,2,3\},\{1,2\}\}$. In this example there are two internal edges, three vertices and each vertex has valence two.

Example. There are four nested trees on three leaves: the parenthesis indicates the elements of $\{1,2,3\}$ which are contained in the same $S \in \mathcal{S}$.



Definition 4.5. Let $\mathcal{S}$ be a nested tree with n-leaves. The **composition of cacti** associated to $\mathcal{S}$ is the map

$$\theta_{\mathcal{S}} : \prod_{S \in \mathcal{S}} \mathcal{C}_{|S|} \to \mathcal{C}_n$$

defined as follows: the collapse of any internal edge (S,T) of $\mathcal{S}$ corresponds to a composition $\circ_i : \mathcal{C}_{|T|} \times \mathcal{C}_{|S|} \to \mathcal{C}_{|T|+|S|-1}$. $\theta_{\mathcal{S}}$ is then defined as the composition of such $\circ_i$, where we first collapse the internal edges which are closer to the leaves ad then proceed iteratively towards the root.

Definition 4.6. For any $k \geq 2$ let $\sigma_k : \mathcal{C}_k/S^1 \to \mathcal{C}_k$ be a fixed section of the trivial bundle $p : \mathcal{C}_k \to \mathcal{C}_k/S^1$ and $\mathcal{S}$ be a nested tree with n-leaves. We define a map

$$\gamma^{\sigma}_{\mathcal{S}} : \mathcal{C}_{|R|}/S^1 \times \prod_{S \in \mathcal{S} - R} \left(S^1 \times \mathcal{C}_{|S|}/S^1\right) \to \mathcal{C}_n$$

iteratively on n as follows:

$n = 2$**:** The only nested tree with two leaves is the corolla $\mathcal{S} = (12)$, and in this case we set $\gamma^{\sigma}_{\mathcal{S}} := \sigma_2$.

$n \geq 3$**:** If $\mathcal{S}$ is the corolla with n-leaves, put $\gamma^{\sigma}_{\mathcal{S}} := \sigma_n$. Now suppose that $\mathcal{S}$ is not the corolla, and denote by $(S_1, R), \ldots, (S_m, R)$ be the internal edges going into the root $R := \{1, \ldots, n\}$. If we delete the root R, we obtain a collection of nested trees $\mathcal{S}_1 \ldots, \mathcal{S}_m$ with leaves labelled (respectively) by $S_1, \ldots, S_m$. We will denote by n_i the cardinality of S_i and by $\mathcal{T}$ the nested tree given by $\{S_1, \dots, S_m, R\}$. Then $\gamma^g_{\mathcal{S}}$ is defined to be the following composition:

$$\begin{tikzcd}
\mathcal{C}_{|R|}/S^1 \times \prod_{\substack{S\in\mathcal{S}\\ S\neq R}} \left(S^1 \times \mathcal{C}_{|S|}/S^1\right) \arrow[rr, dashed, "\gamma^g_{\mathcal{S}}"] \arrow[d] & & \mathcal{C}_n \\
\mathcal{C}_{|R|}/S^1 \times \prod_{i=1}^m \prod_{S\in\mathcal{S}_i} \left(S^1 \times \mathcal{C}_{|S|}/S^1\right) \arrow[d, "\sigma_{|R|}\times\gamma^g_{\mathcal{S}_1}\times\cdots\times\gamma^g_{\mathcal{S}_m}"] & & \\
\mathcal{C}_{|R|} \times \prod_{i=1}^m S^1 \times \mathcal{C}_{n_i} \arrow[rr, "1\times\rho^m"] & & \mathcal{C}_{|R|} \times \prod_{i=1}^m \mathcal{C}_{n_i} \arrow[uu, "\theta_{\mathcal{T}}"']
\end{tikzcd} \tag{4.1}$$

In this diagram the first vertical arrow is just a permutation of the factors in the cartesian product, $\rho : S^1 \times \mathcal{C}_k \to \mathcal{C}_k$ is the circle action and $\theta_{\mathcal{T}}$ is the composition of cacti associated to the nested tree $\mathcal{T}$.

Example. Let $\mathcal{S} \in N_5$ given by $R = \{1,2,3,4,5\}$, $S = \{2,3,4,5\}$, $T = \{4,5\}$. Then

$$\gamma^g_{\mathcal{S}} : \mathcal{C}_{|R|}/S^1 \times (S^1 \times \mathcal{C}_{|S|}/S^1) \times (S^1 \times \mathcal{C}_{|T|}/S^1) \to \mathcal{C}_5$$

sends $(x_R, \alpha_S, x_S, \alpha_T, x_T)$ to $\sigma_2(x_R) \circ_2 (\alpha_S \cdot (\sigma_3(x_S) \circ_3 (\alpha_T \cdot \sigma_2(x_T))))$. For a picture see Figure 4.10.

Lemma 4.9. *For any nested tree $\mathcal{S} \in N_n$ the composite*

$$\mathcal{C}_{|R|}/S^1 \times \prod_{S\in\mathcal{S}-R} \left(S^1 \times \mathcal{C}_{|S|}/S^1\right) \xrightarrow{\gamma^g_{\mathcal{S}}} \mathcal{C}_n \xrightarrow{p} \mathcal{C}_n/S^1$$

is an embedding.

Proof. Since the domain is compact and the target is Hausdorff it suffices to show the injectivity. Let us proceed by induction on the number of leaves n:

$n = 2$: the only nested tree with two leaves is the corolla $\mathcal{S} = (12)$. By definition $\gamma^g_{\mathcal{S}} \coloneqq \sigma_2$, so $p \circ \gamma^g_{\mathcal{S}} = Id$.

$n \geq 3$: if $\mathcal{S}$ is the corolla then by the same argument as before $p \circ \gamma^g_{\mathcal{S}} = Id$, so it is injective. Now suppose $\mathcal{S}$ is not the corolla: if we have

$$p \circ \gamma^g_{\mathcal{S}}(x_R, (\alpha_S, x_S)_{S\in\mathcal{S}-R}) = p \circ \gamma^g_{\mathcal{S}}(y_R, (\beta_S, y_S)_{S\in\mathcal{S}-R})$$

then $\gamma^g_{\mathcal{S}}(x_R, (\alpha_S, x_S)_{S\in\mathcal{S}-R})$ and $\gamma^g_{\mathcal{S}}(y_R, (\beta_S, y_S)_{S\in\mathcal{S}-R})$ differ by a rotation. Let us denote by $(S_1, R), \dots, (S_m, R)$ be the internal edges going into the root $R \coloneqq \{1, \dots, n\}$. If we delete the root R, we obtain a collection of nested trees $\mathcal{S}_1, \dots, \mathcal{S}_m$ with leaves labelled (respectively) by $S_1, \dots, S_m$. We will denote

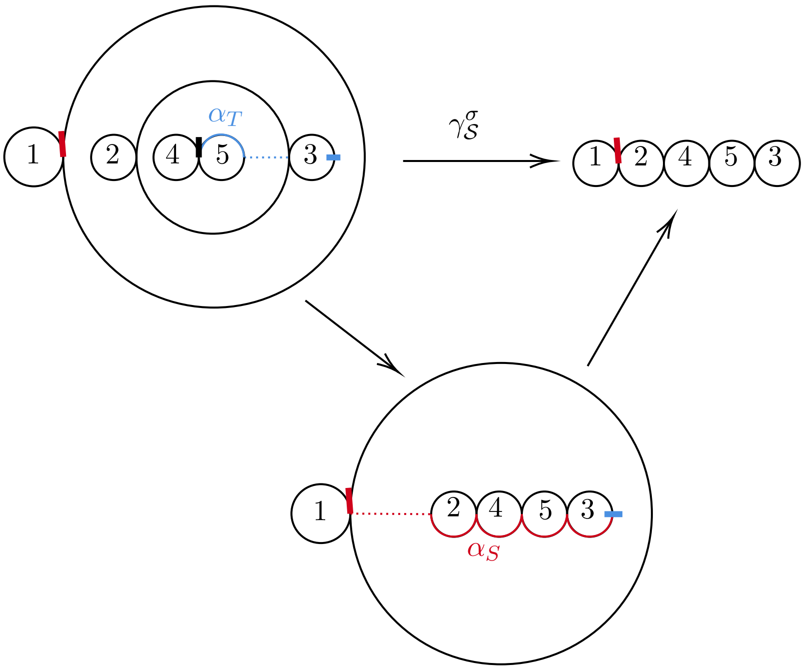


Figure 4.10: In this picture the element $(x_R, \alpha_S, x_S, \alpha_T, x_T)$ is depicted on the left: x_R is the bigger cactus, x_S is the middle one and x_T is the smaller. They are nested one into the other according to the nested tree $\mathcal{S} = (1(23(45)))$. The base points depicted on the cacti are those obtained using the sections $\sigma_k : \mathcal{C}_k/S^1 \to \mathcal{C}_k$. To compute $\gamma^\sigma_{\mathcal{S}}$ we proceed as follows: start from the cactus x_T, use the section $\sigma_2 : \mathcal{C}_2/S^1 \to \mathcal{C}_2$ to get a base point (the black one) and then rotate by α_T: the result is $\alpha_T \cdot \sigma_2(x_T)$. Then glue this cactus into the third lobe of $\sigma_3(x_S)$, obtaining the lower nested cactus. Now use α_S to rotate the base point and glue the resulting cactus in the second lobe of $\sigma_2(x_R)$. The dotted lines in this picture shows how to glue a smaller cactus into a bigger one.

by n_i the cardinality of S_i and by $\mathcal{T}$ the nested tree given by $\{S_1, \dots, S_m, R\}$. The map $\gamma^{\sigma}_{\mathcal{S}}$ is defined to be the following composition:

$$\begin{tikzcd}
\mathcal{C}_{|R|}/S^1 \times \prod_{\substack{S\in\mathcal{S}\\ S\neq R}} (S^1 \times \mathcal{C}_{|S|}/S^1) \arrow[rr, "\gamma^{\sigma}_{\mathcal{S}}"] \arrow[dr, "f"'] & & \mathcal{C}_n \\
& \mathcal{C}_{|R|} \times \prod_{i=1}^m \mathcal{C}_{n_i} \arrow[ur, "\theta_{\mathcal{T}}"'] &
\end{tikzcd}$$

where f is the composite of the first three arrows of diagram 4.1 and $\theta_{\mathcal{T}}$ is the composition of cacti associated to the nested tree $\mathcal{T}$. By definition

$$\begin{cases} f(x_R, (\alpha_S, x_S)_{S\in\mathcal{S}-R}) = (\sigma_m(x_R), a_1, \dots, a_m) \\ f(y_R, (\beta_S, y_S)_{S\in\mathcal{S}-R}) = (\sigma_m(y_R), b_1, \dots, b_m) \end{cases}$$

for some elements $a_i, b_i \in \mathcal{C}_{n_i}$, $i = 1, \dots, m$. Since $\theta_{\mathcal{T}}(\sigma_m(y_R), b_1, \dots, b_m)$ and $\theta_{\mathcal{T}}(\sigma_m(x_R), a_1, \dots, a_m)$ differ by a rotation, we have that

$$x_R = p(\sigma_m(x_R)) = p(\sigma_m(y_R)) = y_R$$

Therefore $\sigma_m(x_R) = \sigma_m(y_R)$ and we get that

$$\theta_{\mathcal{T}}(\sigma_m(y_R), b_1, \dots, b_m) = \theta_{\mathcal{T}}(\sigma_m(x_R), a_1, \dots, a_m)$$

But $\theta_{\mathcal{T}}$ is an embedding, so $a_i = b_i$ for any $i = 1, \dots, m$. Now observe that

$$\begin{aligned} p \circ \gamma^{\sigma}_{\mathcal{S}_i}(x_{S_i}, (\alpha_S, x_S)_{S\in\mathcal{S}_i - S_i}) &= p(a_i) \\ &= p(b_i) \\ &= p \circ \gamma^{\sigma}_{\mathcal{S}_i}(y_{S_i}, (\beta_S, y_S)_{S\in\mathcal{S}_i - S_i}) \end{aligned}$$

and use the inductive hypothesis to conclude that for any $i = 1, \dots, m$

$$\begin{cases} x_S = y_S \text{ for all } S \in \mathcal{S}_i \\ \alpha_S = \beta_S \text{ for all } S \in \mathcal{S}_i - S_i \end{cases}$$

Finally, observe that

$$\alpha_{S_i} \cdot \gamma^{\sigma}_{\mathcal{S}_i}(x_{S_i}, (\alpha_S, x_S)_{S\in\mathcal{S}_i - S_i}) = a_i = b_i = \beta_{S_i} \cdot \gamma^{\sigma}_{\mathcal{S}_i}(y_{S_i}, (\beta_S, y_S)_{S\in\mathcal{S}_i - S_i})$$

and since the circle acts freely on the space of cacti we conclude that $\alpha_{S_i} = \beta_{S_i}$, concluding the proof.

$\square$

4.3 The space of nested cacti

In this section we introduce a regular CW-complex $N_n^\sigma(\mathcal{C}/S^1)$, called the *space of nested cacti*. This space turns out to be homeomorphic to $\overline{\mathcal{M}}_{0,n+1}$, giving a CW-decomposition of $\overline{\mathcal{M}}_{0,n+1}$.

Definition 4.7. Let $\mathcal{S}$ be a nested tree with n-leaves and root R. We define

$$St(\mathcal{S}) \coloneqq \mathcal{C}_{|R|}/S^1 \times \prod_{S \in \mathcal{S}-R} (D_2 \times \mathcal{C}_{|S|}/S^1)$$

We will call $St(\mathcal{S})$ the **stratum** associated to $\mathcal{S}$. In what follows we will think $D_2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ as the quotient of the cylinder $[0,1] \times S^1$ by collapsing to a point the base $\{0\} \times S^1$. So we will identify a point in D_2 by a pair $(t, \alpha) \in [0,1] \times S^1$.

Definition 4.8. For any $k \geq 2$ fix a section $\sigma_k : \mathcal{C}_k/S^1 \to \mathcal{C}_k$. The **space of nested cacti** (with n leaves) is defined as the quotient

$$N_n^\sigma(\mathcal{C}/S^1) \coloneqq \frac{\coprod_{\mathcal{S} \in N_n} \mathcal{C}_{|R|}/S^1 \times \prod_{S \in \mathcal{S}-R} (D_2 \times \mathcal{C}_{|S|}/S^1)}{\sim}$$

The equivalence relation $\sim$ is defined as follows: fix a point $(x_R, (t_S, \alpha_S, x_S)_{S \in \mathcal{S}-R})$, with $t_S \in [0,1]$, $\alpha_S \in S^1$, $x_S \in \mathcal{C}_{|S|}$. Suppose $t_{S_0} = 1$, and let $\mathcal{T} \subseteq \mathcal{S}$ be the maximal nested tree containing S_0 such that $t_S \neq 0$ for any $S \in \mathcal{T} - U$, where U is the root of $\mathcal{T}$. Then $(x_R, (t_S, \alpha_S, x_S)_{S \in \mathcal{S}-R})) \sim (x'_R, (t'_S, \alpha'_S, x'_S)_{S \in \mathcal{S}-\{R, S_0\}})$ if and only if

$$\begin{cases} x_S = x'_S \text{ for all } S \in \mathcal{S} - \mathcal{T} \\ t_S = t'_S \text{ for all } S \in \mathcal{S} - S_0 \\ \alpha_S = \alpha'_S \text{ for all } S \in \mathcal{S} - \mathcal{T} \\ p \circ \gamma^\sigma_{\mathcal{T}}(x_U, (\alpha_S, x_S)_{S \in \mathcal{T}-U}) = p \circ \gamma^\sigma_{\mathcal{T}-S_0}(x'_U, (\alpha'_S, x'_S)_{S \in \mathcal{T}-\{U, S_0\}}) \end{cases}$$

For an example of nested cacti that are identified under this relation see Figure 4.11. To sum up, a point in $N_n^\sigma(\mathcal{C}/S^1)$ is specified by the following data:

- A nested tree $\mathcal{S}$ with n-leaves.
- For any vertex $S \in \mathcal{S}$ we have an unbased cactus $x_S \in \mathcal{C}_{|S|}/S^1$.
- For any vertex $S \in \mathcal{S} - R$ we have a parameter $t_S \in [0,1]$. We will call it the **radial parameter** and we think of it as a decoration of the unique edge going out the vertex S.
- For any vertex $S \in \mathcal{S} - R$ we have a parameter $\alpha_S \in S^1$. We will call it the **angular parameter**. Note that if $t_S = 0$ the angular parameter α_S is superfluous, so we will omit it.

We will call a point in $N_n^\sigma(\mathcal{C}/S^1)$ a **nested cactus**.

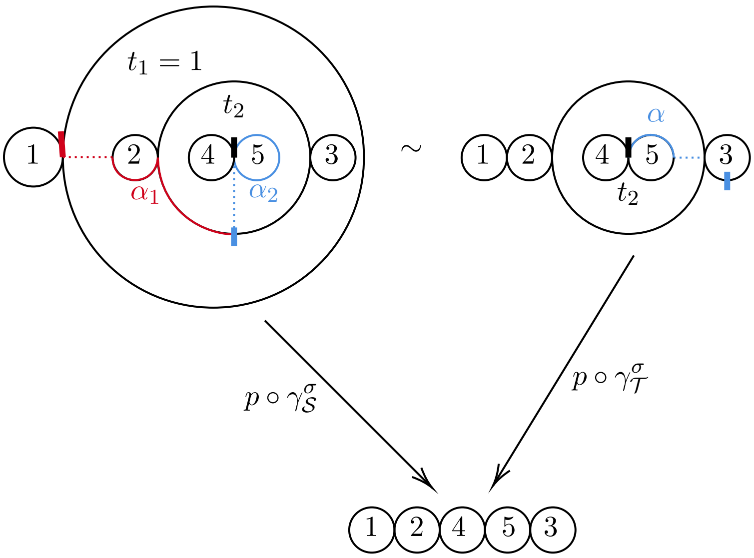


Figure 4.11: In this picture we see two nested cactus that are identified in $N_5^{\sigma}(\mathcal{C}/S^1)$: the nested tree underlying the leftmost (resp. rightmost) nested cactus is $\mathcal{S} = (1(23(45)))$ (resp. $\mathcal{T} = (123(45))$). $t_1, t_2 \in (0,1]$ are radial parameters, $\alpha, \alpha_1, \alpha_2 \in S^1$ are angular parameters. The base points depicted on the cacti are those obtained using the sections $\sigma_k : \mathcal{C}_k/S^1 \to \mathcal{C}_k$. In the leftmost cactus t_1, α_1 and t_2, α_2 are the parameters associated respectively to $\{2,3,4,5\}$ and $\{4,5\}$. Similarly, in the right cactus t_2 and α are the parameters associated to the vertex $\{4,5\}$. If we compute $p \circ \gamma^{\sigma}_{\mathcal{S}}$ on the left nested cactus and $p \circ \gamma^{\sigma}_{\mathcal{T}}$ on the right nested cactus we end up with the same unbased cactus, which is depicted below.

4.3.1 Cell decomposition

The space of nested cacti $N_n^\sigma(\mathcal{C}/S^1)$ has a natural cell structure that we describe next. A cell τ is determined by:

1. A nested tree $\mathcal{S}$ with n-leaves.

2. A cell $C_S \subseteq \mathcal{C}_{|S|}/S^1$ for any vertex $S \in \mathcal{S}$.

and contains all the nested cacti $(x_R, (t_S, \alpha_S, x_S)_{S\in\mathcal{S}-R}))$ such that $x_S \in C_S$ and $t_S < 1$ for all $S \in \mathcal{S}$. This subspace of $N_n^\sigma(\mathcal{C}/S^1)$ is homeomorphic to

$$C_R \times \prod_{S\in\mathcal{S}-R} (\mathring{D}_2 \times C_S)$$

The dimension of such a cell is

$$dim(\tau) = 2E + \sum_{S\in\mathcal{S}} dim(C_S)$$

where E is the number of internal edges of $\mathcal{S}$. The boundary of τ is the union of cells with a similar description that are obtained by an iteration of the following moves (for a concrete example see Figure 4.12):

Move 1 (cacti boundary): for a fixed vertex $S \in \mathcal{S}$ replace the cell C_S with a cell $C'_S \subseteq \partial C_S$.

Move 2 (edge contraction): replace an edge (S, T) of the nested tree $\mathcal{S}$ with a vertex (i.e. contract the edge) labelled by a cell of $\mathcal{C}_{|S|+|T|-1}/S^1$ which is obtained by inserting the cactus C_S in the lobe of C_T corresponding to the edge (S, T).

When we want to highlight the combinatorial data defining a cell τ we will write $\tau = (\mathcal{S}, (C_S)_{S\in\mathcal{S}})$.

Proposition 4.10. *The space $N_n^\sigma(\mathcal{C}/S^1)$ is a regular CW-complex.*

Proof. The cell decomposition just described is regular, because $\mathcal{C}_k/S^1$ is a regular CW-complex and the maps $p \circ \gamma_S^\sigma$ are embeddings. □

4.4 Nested cacti vs labelled trees

In this section we show that the space of labelled treed Tr_n is homeomorphic to a subspace of $N_n^\sigma(\mathcal{C}/S^1)$. First we remind a result from [47]:

Proposition 4.11 (Salvatore, [47])**.** *There is a deformation retraction $\beta_n : Tr_n \to \mathcal{C}_n/S^1$.*

Lemma 4.12. *Consider the subspace of $N_n^\sigma(\mathcal{C}/S^1)$ defined as follows:*

$$N_n^\sigma(\mathcal{C}/S^1)^{>0} := \{(x_R, (t_S, \alpha_S, x_S)_{S\in\mathcal{S}-R}) \in N_n^\sigma(\mathcal{C}/S^1) \mid t_S > 0 \textit{ for any } S \in \mathcal{S} - R\}$$

Then the space of labelled trees Tr_n is homeomorphic to $N_n^\sigma(\mathcal{C}/S^1)^{>0}$.

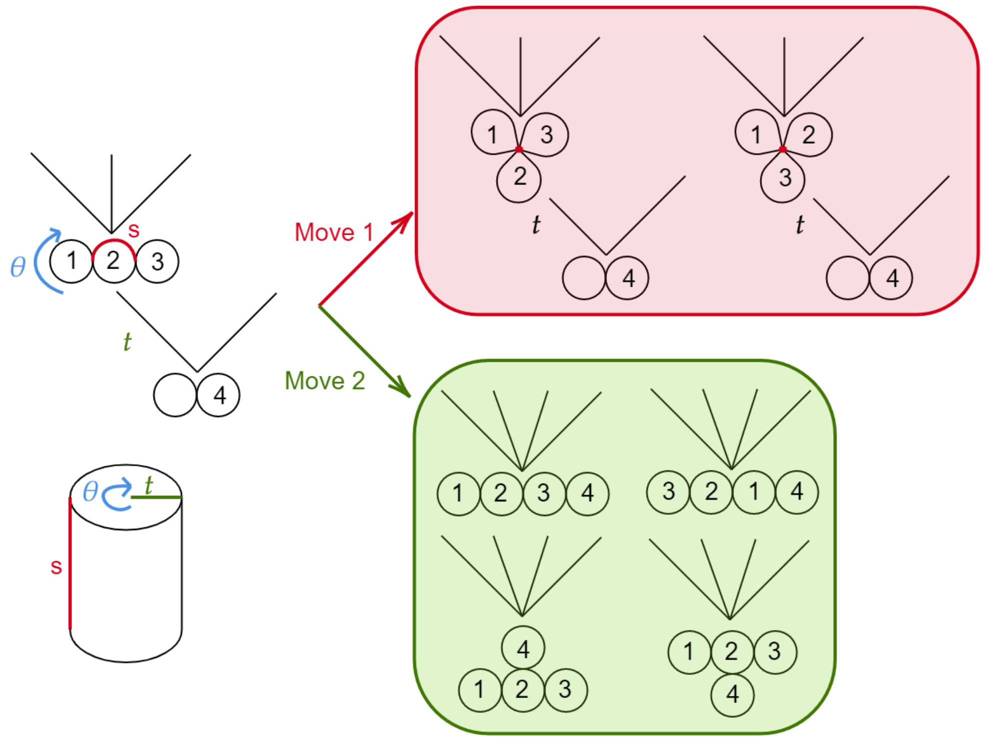


Figure 4.12: The cell in this picture has dimension three (it is a cylinder). If we compute the boundary of the cactus with three lobes (move 1) we obtain the cells in the red rectangle, corresponding to the base and the top of the cylinder. If we contract the internal edge (move 2) we obtain the cells in the green rectangle.

Proof. Fix a point $x := (T, f : B \to (0,1], \{g_w : E_w \to (0,1)\}_{w\in W})$ in Tr_n. Here T is an admissible tree with set of black (resp. white) vertices B (resp. W), f are the parameters of the black vertices, g_w are the parameters associated to a white vertex $w \in W$ (for more details see Section 3.1). We want to associate to x a point $f(x) \in N_n^\sigma(\mathcal{C}/S^1)^{>0}$, so we need to specify a nested tree $\mathcal{S}$ with n-leaves, a cactus $x_S \in \mathcal{C}_{|S|}/S^1$ for any $S \in \mathcal{S}$ and a point $(t_S, \alpha_S) \in D_2$ for any $S \in \mathcal{S} - R$. Let us do this step by step:

Nested tree: we obtain the nested tree $\mathcal{S}$ by the following procedure:

1. Remove from T all black vertices with label 1 together with the edges that are incident to them. In this way we obtain a forest. If T_i is a connected component of this forest, let S_i be the set of labels of the white vertices of T_i.
2. For each connected component T_i of this forest which contains at least one black vertex, let λ_i the maximum label of a black vertex in that component. Now divide by λ_i the labels of the black vertices in T_i and we can iterate the procedure in each component. The procedure stops when there are no black vertices left.

At the end of this procedure we get a bunch of subsets of $\{1, \dots, n\}$ (the S_i defined at the first step of the algorithm) which form a nested tree $\mathcal{S}$ with n-leaves. See Figure 4.13 for an example.

Radial parameters: any $S \in \mathcal{S}$ corresponds to one of the trees T_i obtained from T by the previous procedure. Then we define $t_S \in (0,1]$ to be λ_i, the maximum label of a black vertex of T_i. For an explicit example see Figure 4.13.

Decorative cacti and angular parameters: consider the deformation retraction $\beta_n : Tr_n \to \mathcal{C}_n/S^1$ and the embedding

$$p \circ \gamma_{\mathcal{S}}^{\sigma} : \mathcal{C}_{|R|}/S^1 \times \prod_{S\in\mathcal{S}-R} \left(S^1 \times \mathcal{C}_{|S|}/S^1\right) \to \mathcal{C}_n/S^1$$

We claim that $\beta_n(x) \in Im(p \circ \gamma_{\mathcal{S}}^{\sigma})$: this is true because for any $S \in \mathcal{S}$ the union of the lobes of $\beta_n(x)$ labelled by S is again a cactus. Therefore we take $(p \circ \gamma_{\mathcal{S}}^{\sigma})^{-1}(\beta_n(x))$ and we get a decorative cactus $x_S \in \mathcal{C}_{|S|}/S^1$ for any $S \in \mathcal{S}$, together with an angular parameter $\alpha_S \in S^1$ when S is not the root.

To sum up, we have constructed a continuous map $f : Tr_n \to N_n^\sigma(\mathcal{C}/S^1)^{>0}$. We show that it is a homeomorphism by exhibiting an explicit continuous inverse. If $x := (x_R, (t_S, \alpha_S, x_S)_{S\in\mathcal{S}-R}) \in N_n^\sigma(\mathcal{C}/S^1)^{>0}$, we want to construct $f^{-1}(x)$, so we need to specify an admissible tree T, a function $f : B \to (0,1]$ (parameters of the black vertices) and for any white vertex $w \in W$ a map $g_w : E_w \to (0,1)$ (parameters of the white vertices): the admissible tree T will have n white vertices and a black vertex for each intersection point of the lobes of a cactus x_S, $S \in \mathcal{S}$. If b is a black vertex corresponding to an intersection point of the lobes of some cactus x_S, we put

$$f(b) := \prod_{\substack{T\in\mathcal{S}-R \\ T\supseteq S}} t_T$$

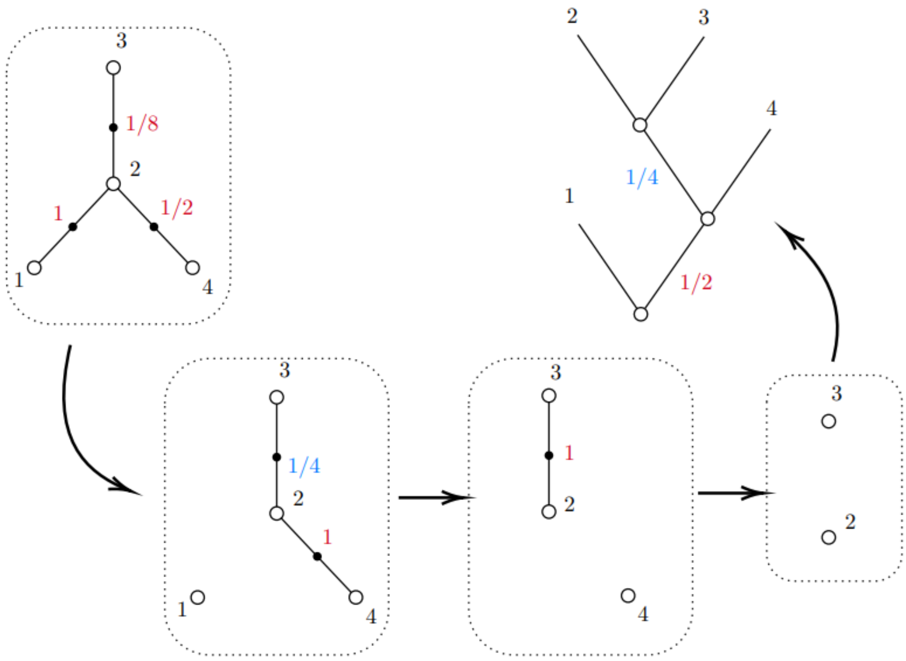


Figure 4.13: An explicit example of the procedure that assigns a nested tree to a labelled tree.

Now we define the edges of T and the parameters of the white vertices inductively: start form the vertices $S \in \mathcal{S}$ that are furthest from the root. Take the decorative cactus $x_S \in \mathcal{C}_{|S|}/S^1$ and consider the pointed cactus $\alpha_S \sigma_{|S|}(x_S) \in \mathcal{C}_{|S|}$. Passing to the dual graph we obtain a labelled tree (with a base point). Now let us move towards the root: suppose $S \in \mathcal{S}$ is a fixed vertex and that all the labelled trees corresponding to edges into S have been constructed. If x_S is the decorative cactus associated to S, then consider $y_S := \sigma_{|S|}(x_S) \in \mathcal{C}_{|S|}$. This is a pointed cactus, so each lobe of y_S has a canonical base point. Let b be a black vertex associated to the intersection of some lobes of y_S. We will denote this lobes as $l_1, \dots, l_m$. Each lobe corresponds to an edge into S (either internal or open). If l_j is associated to an open edge (labelled by $k \in \{1, \dots, n\}$), then draw an edge from b to a white vertex labelled by k. If l_j corresponds to an internal edge (S_j, S), then we will draw an edge connecting b to the labelled tree (with base point) T_i associated to S_i as follows: using the deformation retraction of Proposition 4.11 we can associate to T_i a cactus (with base point) C_i. Now we can use the composition of based cacti to identify C_i with the corresponding lobe l_i of y_S. Therefore our black vertex b will correspond to a point p in the cactus C_i. If this point is an intersection of lobes, then connect b to the black vertex of T_i associated to p which has the largest label. If p is not an intersection of lobes, then connect b to the corresponding white vertex of T_i. After we have done this for any black vertex associated to x_S we get a labelled tree with a base point; then we rotate the base point by the angular parameter α_S and we get a labelled tree with base point T_S. Now we iterate this procedure until we get to the

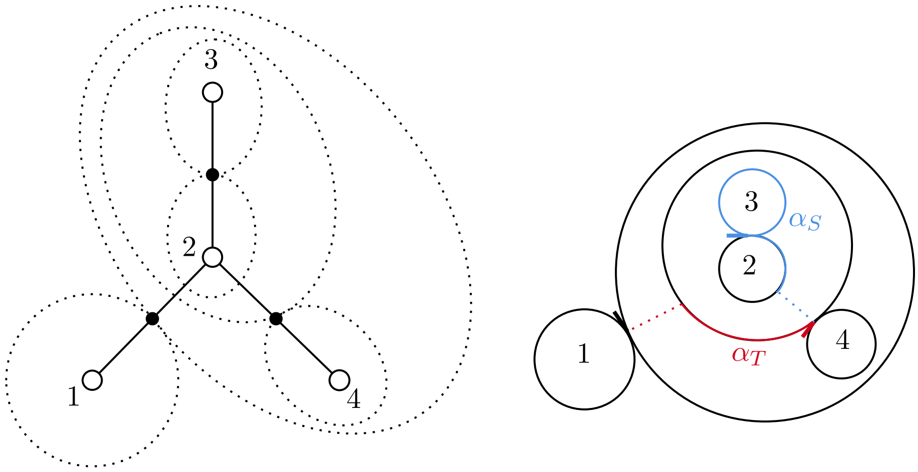


Figure 4.14: In this picture we see the nested cactus associated to the labelled tree on the left. We assume that the label of the black vertices are the same as in Figure 4.13. The base point on the cacti are those obtained using the sections $\sigma_k : \mathcal{C}_k/S^1 \to \mathcal{C}_k$. The angular parameters α_S, α_T are represented as distances between the base point given by the section and the base point induced by the bigger cactus (dotted lines). The angular parameters are omitted.

root. At the end we obtain a labelled tree with base point; forgetting this base point we get $f^{-1}(x)$. See Figure 4.14 for an example. □

4.5 A CW-decomposition of $\overline{\mathcal{M}}_{0,n+1}$

Combining Theorem 3.4 and Lemma 4.12, for each choice of weights $a_1, \dots, a_n$ we get an embedding

$$\phi_{a_1 \dots, a_n} : \mathcal{M}_{0,n+1} \to Tr_n \cong N_n^\sigma(\mathcal{C}/S^1)^{>0} \subseteq N_n^\sigma(\mathcal{C}/S^1) \tag{4.2}$$

We will see that this embedding extends to a homeomorphism between the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n+1}$ and $N_n^\sigma(\mathcal{C}/S^1)$, giving a regular CW decomposition of $\overline{\mathcal{M}}_{0,n+1}$. In what follows it is convenient to use a coordinate free approach: for any finite set R with r elements we define $\mathcal{M}_{0,R+1}$ to be the space of injective maps from R to $\mathbb{C}$, modulo the action of $\mathbb{C} \rtimes \mathbb{C}^*$ on the target. A total order on R induces a homeomorphism between $\mathcal{M}_{0,R+1}$ and $\mathcal{M}_{0,r+1}$. Similarly it is possible to define $N_R^\sigma(\mathcal{C}/S^1) \cong N_r^\sigma(\mathcal{C}/S^1)$, the space of nested cacti with leaves labelled by R. The choice of a function $a : R \to \mathbb{R}^{>0}$ gives an embedding $\phi_a : \mathcal{M}_{0,R+1} \to N_R^\sigma(\mathcal{C}/S^1)$ analogous to 4.2. When $R = \{y_1 < y_2 < \dots < y_r\}$ is equipped with a total order ϕ_a corresponds to $\phi_{a(y_1)/A, \dots, a(y_r)/A}$, with $A = \sum_{i=1}^r a(y_i)$.

Definition 4.9. Let $\mathcal{S}$ be a nested tree with n-leaves. For any $S \in \mathcal{S}$, $\hat{S}$ will denote the quotient of S by the equivalence relation given by

$$s_1 \sim s_2 \iff \exists T \in \mathcal{S}, T \subseteq S \mid s_1, s_2 \in T$$

We will denote by $\pi_S : S \to \hat{S}$ the quotient map.

Recall that $\overline{\mathcal{M}}_{0,n+1}$ is a stratified space: for each nested tree $\mathcal{S} \in N_n$ we have $\overline{\mathcal{M}}_{0,n+1} = \bigcup_{\mathcal{S}\in N_n} \mathcal{M}(\mathcal{S})$, where $\mathcal{M}(\mathcal{S})$ contains all the stable curves whose dual graph is $\mathcal{S}$. For example, if $\mathcal{S}$ is the corolla, then $\mathcal{M}(\mathcal{S}) = \mathcal{M}_{0,n+1}$. In general we have a homeomorphism

$$\beta_{\mathcal{S}} : \prod_{S\in\mathcal{S}} \mathcal{M}_{0,\hat{S}+1} \to \mathcal{M}(\mathcal{S})$$

Definition 4.10. Let $a_1, \dots, a_n \in \mathbb{R}^{>0}$ such that $\sum_{i=1}^n a_i = 1$. We will define a function

$$\overline{\phi}_{a_1,\dots,a_n} : \overline{\mathcal{M}}_{0,n+1} \to N_n(\mathcal{C}/S^1)$$

stratum-wise as follows: given a nested tree $\mathcal{S} \in N_n$, take $(y_S)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \mathcal{M}_{0,\hat{S}+1} \cong \mathcal{M}(\mathcal{S})$. In order to define

$$\overline{\phi}_{a_1,\dots,a_n}(\beta_{\mathcal{S}}((y_S)_{S\in\mathcal{S}}))$$

we have to specify a nested tree $\mathcal{T}$ with n-leaves, an unbased cactus for any vertex and radial/angular parameters for any vertex which is not the root. For any $S \in \mathcal{S}$, consider the labelled nested tree

$$\phi_{a_S}(y_S) \in N^{\sigma}_{\hat{S}}(\mathcal{C}/S^1)$$

where $a_S : \hat{S} \to \mathbb{R}^{>0}$ is defined as $a_S(x) := \sum_{i\in\pi_S^{-1}(x)} a_i$. Let $\mathcal{T}_S$ be the underlying nested tree. Then $\mathcal{T} \in N_n$ is defined to be the collection of subsets of the form $\pi_S^{-1}(U) \subseteq \{1, \dots, n\}$, with $U \in \mathcal{T}_S$ and $S \in \mathcal{S}$. Intuitively $\mathcal{T}$ is obtained by grafting together the nested trees $\{T_S\}_{S\in\mathcal{S}}$. Now we specify the unbased cacti and the radial/angular parameters decorating each vertex: given $\pi_S^{-1}(U) \in \mathcal{T}$, with $U \in \mathcal{T}_S$ and $S \in \mathcal{S}$ we put:

1. The unbased cactus associated to $\pi_S^{-1}(U)$ will be the one decorating $U \in \mathcal{T}_S$ in $\phi_{a_S}(y_S)$.
2. If $U \neq \hat{S}$ the radial parameter associated to $\pi_S^{-1}(U)$ will be the one associated to $U \in \mathcal{T}_S$ in $\phi_{a_S}(y_S)$. It will be zero if $U = \hat{S}$.
3. If $U \neq \hat{S}$ the angular parameter associated to $\pi_S^{-1}(U)$ will be the one associated to $U \in \mathcal{T}_S$ in $\phi_{a_S}(y_S)$ if $U \neq \hat{S}$. If $U = \hat{S}$ we do not have to specify it since the radial parameter is zero.

Example. Consider the nested tree $\mathcal{S} := ((1, 2, 3), 4)$ and let us denote by R the root and by S the other vertex. Fix some weights a_1, a_2, a_3, a_4 and take a point $(y_R, y_S) \in \mathcal{M}_{0,\hat{R}+1} \times \mathcal{M}_{0,\hat{S}+1} \cong \mathcal{M}(\mathcal{S})$. Then $\overline{\phi}_{a_1,a_2,a_3,a_4}(\beta_{\mathcal{S}}(y_R, y_S))$ is obtained by grafting $\phi_{a_1,a_2,a_3}(y_S)$ to the first leaf of $\phi_{a_1+a_2+a_3,a_4}(y_R)$ and setting the radial parameter $t_S = 0$.

Theorem 4.13. *For each choice of weights $a_1, \dots, a_n > 0$, $\sum_{i=1}^n a_i = 1$, the map*

$$\overline{\phi}_{a_1\dots,a_n} : \overline{\mathcal{M}}_{0,n+1} \to N^{\sigma}_n(\mathcal{C}/S^1)$$

is a homeomorphism.

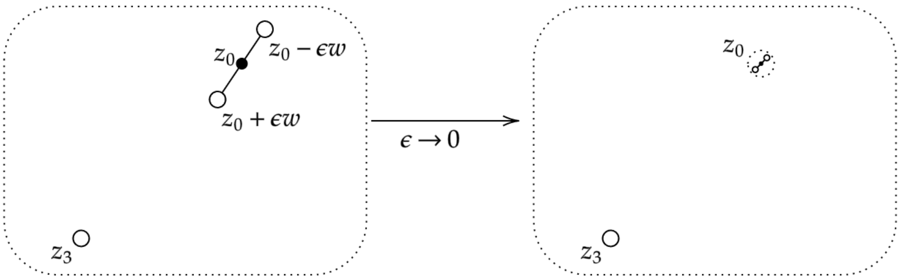


Figure 4.15: This picture represent the collision of two charges.

Proof. $\overline{\phi}_{a_1 \ldots, a_n}$ is bijective, the domain is compact and the target is Hausdorff. So it is enough to prove the continuity of $\overline{\phi}_{a_1 \ldots, a_n}$. By definition $\overline{\phi}_{a_1 \ldots, a_n}$ restricted to $\mathcal{M}_{0,n+1} \subseteq \overline{\mathcal{M}}_{0,n+1}$ is the map $\phi_{a_1,\ldots,a_n}$ of 4.2, which is continuous. Thus we only need to understand what happens when two or more points collide. Let us consider for example the configuration $[z_0 - \epsilon w, z_0 + \epsilon w, z_3] \in \mathcal{M}_{0,4}$ for $\epsilon > 0$ (see Figure 4.15) and fix some weights a_1, a_2, a_3 (again the physical intuition is useful, so think the weights as the value of some charges placed in $z_0 - \epsilon w, z_0 + \epsilon w, z_3$). When $\epsilon \to 0$ the configuration $[z_0 - \epsilon w, z_0 + \epsilon w, z_3]$ tends to the following infinitesimal configuration: the third charge remains in z_3, with value a_3. The other two charges collide and became infinitesimal at the point z_0. So, looking at this configuration from "far away" it seems that there are only two charges: one in z_0 with value $a_1 + a_2$, and the other in z_3 with value a_3; but when we zoom in on z_0 we realize that in fact there are two charges of values a_1 and a_2 which are very close to each other. In other words, the limit of this configuration when $\epsilon \to 0$ is a stable curve in $\mathcal{M}(\mathcal{S})$, where $\mathcal{S} := \{R, S\}$ is the nested tree given by $R = \{1, 2, 3\}$ and $S = \{1, 2\}$. More precisely, the limit is $\beta_{\mathcal{S}}(y_R, y_S)$ where $y_R = [z_0, z_3]$ and $y_S = [-w, w]$. The black vertices of the labelled tree which represents $[z_0 - \epsilon w, z_0 + \epsilon w, z_3]$ are the two (possibly coinciding) critical points w_1^ϵ, w_2^ϵ of

$$h(z) = (z - z_0 + \epsilon w)^{a_1}(z - z_0 - \epsilon w)^{a_2}(z - z_3)^{a_3}$$

that are the zeros of

$$p_\epsilon(z) = a_1(z - z_0 - \epsilon w)(z - z_3) + a_2(z - z_0 + \epsilon w)(z - z_3) + a_3((z - z_0)^2 - \epsilon^2 w^2)$$

These zeros depend continuously on ϵ, and in the limit tend to the zeros of

$$p_0(z) = (z - z_0)((a_1 + a_2)(z - z_3) + a_3(z - z_0))$$

that are $w_1^0 = z_0$ and w_2^0, which is the critical point associated to the configuration $[z_0, z_3]$ with weights $a_1 + a_2, a_3$. For $\epsilon \to 0$ we have $|h(w_1^\epsilon)| \to 0$ and $|h(w_2^\epsilon)| \to |h(w_2^0)| \neq 0$, so the ratio of their values tends to 0. This implies that the radial parameter associated to $S \in \mathcal{S}$ in the nested cactus $\phi_{a_1,a_2,a_3}([z_0 - \epsilon w, z_0 + \epsilon w, z_3])$ tends to zero. Moreover, the cactus labelling the root of $\phi_{a_1,a_2,a_3}([z_0 - \epsilon w, z_0 + \epsilon w, z_3])$ tends to the cactus $\phi_{a_1 + a_2, a_3}([z_0, z_3])$ and the one labelling $S = \{1, 2\}$ tends to $\phi_{a_1,a_2}([-w, w])$. In the general case we can do a similar computation to show the continuity of $\overline{\phi}_{a_1,\ldots,a_n}$. ∎

Remark 4.14. By definition the space $N_n^\sigma(\mathcal{C}/S^1)$ depends on the choice of sections $\sigma_k : \mathcal{C}/S^1 \to \mathcal{C}$. Theorem 4.13 tells us that this choice is not so important, since in any case we obtain a CW-complex which is homeomorphic to $\overline{\mathcal{M}}_{0,n+1}$.

Corollary 4.15. *Each choice of weights $a_1, \dots, a_n > 0$, $\sum_{i=1}^n a_i = 1$ and sections $\sigma_k : \mathcal{C}_k/S^1 \to \mathcal{C}_k$ determines a regular CW-decomposition of $\overline{\mathcal{M}}_{0,n+1}$. A cell is determined by*

1. *A nested tree $\mathcal{S}$ with n-leaves.*
2. *A cell $C_S \subseteq \mathcal{C}_{|S|}/S^1$ for any vertex $S \in \mathcal{S}$.*

and its closure is homeomorphic to

$$\overline{C}_R \times \prod_{S \in \mathcal{S}-R} D_2 \times \overline{C}_S$$

Proof. Just use Theorem 4.13 to transfer the regular cell structure of $N_n^\sigma(\mathcal{C}/S^1)$ to $\overline{\mathcal{M}}_{0,n+1}$. □

Example. Let us describe explicitly the cell structure of $\overline{\mathcal{M}}_{0,4}$: there are two 0-cells C_1, C_2, three 1-cells B_1, B_2, B_3 and three 2-cells A_1, A_2, A_3. The cells of dimension 0 and 1 are described by a corolla with three leaves decorated by an unbased cactus with three lobes. The two dimensional cells are associated to the three nested trees ((12)3)), ((13)2) and (1(23). Topologically A_1, A_2 and A_3 are homeomorphic to D_2. The center of each such a disk represents the stable curve whose dual graph is the nested tree representing the cell. See Figure 4.16.

Remark 4.16. As we know $Tr_n \cong \mathcal{M}_{0,n+1}$. If $n = 3$ we get the (open) cell decomposition of $\mathcal{M}_{0,4}$ depicted in Figure 3.4. Adding the three missing points (i.e. the three stable curves of $\overline{\mathcal{M}}_{0,4}$) we get a CW-decomposition of $\overline{\mathcal{M}}_{0,4}$. However it is different to the one we obtained in this paragraph: in this decomposition the stable curves lie in the boundary of the 2-cells (see Figure 3.4), while in the decomposition given by nested cacti they are the center of the 2-cells.

Remark 4.17. The regular cell decomposition of $\overline{\mathcal{M}}_{0,n+1}$ obtained in this way is not compatible with the operad structure. Indeed the operadic composition maps

$$\circ_i : \overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,m+n}$$

are not cellular maps (in the domain we put the product cell structure). For example consider $\circ_1 : \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{0,4}$: its image is the stable curve associated to the nested tree ((12)3) which is not a 0-cell of $\overline{\mathcal{M}}_{0,4}$ (actually it is the center of the 2-cell A_1 of Figure 4.16).

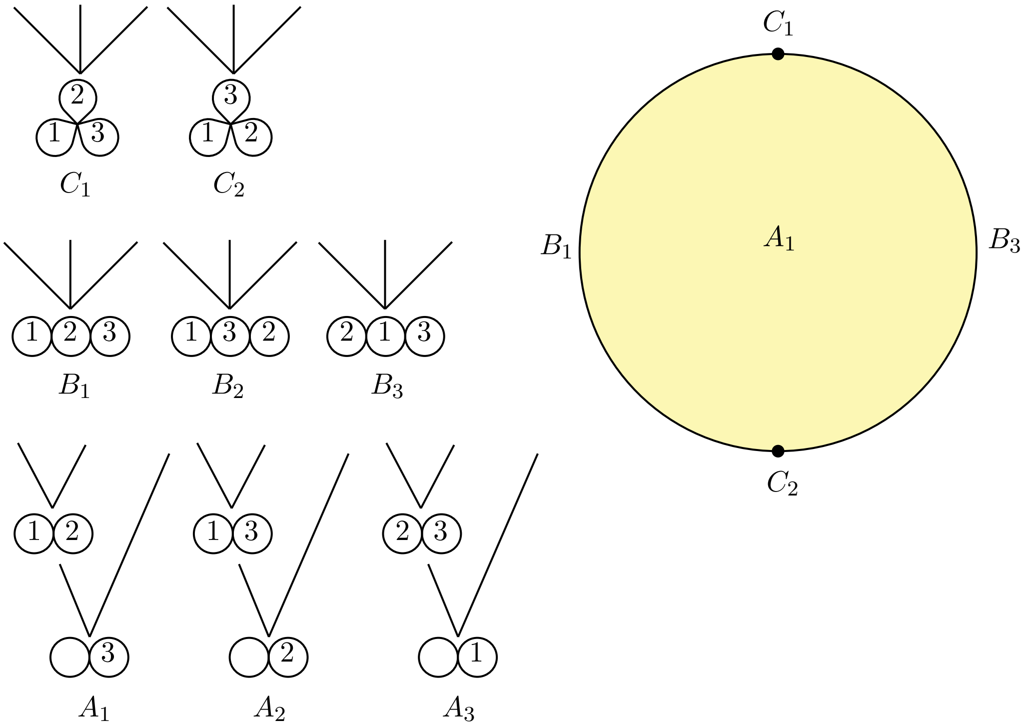


Figure 4.16: In this picture we see the cell decomposition of $\overline{\mathcal{M}}_{0,4}$. On the left there are the cells. On the right it is depicted the cell A_1, together with its boundary.

Chapter 5

A chain model for the Hypercommutative operad

In this Chapter we present a new chain model for the Hypercommutative operad (Theorem 5.10). The idea is the following: in Section 5.1 we identify (up to a shift in degrees) the operadic bar construction $B(grav)$ with the collection of chain complexes

$$C_*^{cell}(\overline{\mathcal{M}}) := \{C_*^{cell}(\overline{\mathcal{M}}_{0,n+1})\}_{n\geq 2}$$

where $C_*^{cell}(\overline{\mathcal{M}}_{0,n+1})$ is the complex of cellular chains associated to the CW decomposition of Section 4.5. $B(grav)$ is a co-operad by definition, so we get a co-operad structure on $C_*^{cell}(\overline{\mathcal{M}})$. Dualizing, i.e. replacing chains with cochains we get an operad in cochain complexes

$$C^*_{cell}(\overline{\mathcal{M}}) := \{C^*_{cell}(\overline{\mathcal{M}}_{0,n+1})\}_{n\geq 2}$$

Since the Gravity and Hypercommutative operad are Koszul dual to each other ([21], [26]) this will be a (co)chain model for the Hypercommutative operad. In this chapter we will expand a bit this observation, giving a more down to earth explanation of what we just said using Poincaré duality instead of Koszul duality: in the eyes of Poincaré duality, passing to cochains is the same as taking the dual cell decomposition. So, if $C_*^{dual}(\overline{\mathcal{M}}_{0,n+1})$ denotes the cellular chains respect to the dual cell decomposition, we expect that

$$C_*^{dual}(\overline{\mathcal{M}}) := \{C_*^{dual}(\overline{\mathcal{M}}_{0,n+1})\}_{n\geq 2}$$

has an operad structure, giving a chain model for the Hypercommutative operad. Here is an overview of the chapter:

Section 5.1 contains the proof that $B(grav)$ can be identified with $C_*^{cell}(\overline{\mathcal{M}})$ up to a shift in degrees.

Section 5.2 explains how to construct the dual cell decomposition.

Section 5.3 shows that the collection of chain complexes $C_*^{dual}(\overline{\mathcal{M}})$ has a natural operad structure and that the homology of this operad is exactly the Hypercommutative operad.

Section 5.4 contains some speculations about an open problem I was not able to solve.

5.1 Relation with the bar construction

In the previous chapter we described a cell decomposition of $\overline{\mathcal{M}}_{0,n+1}$ where the cells are described by the following combinatorics:

- A nested cactus $\mathcal{S}$ with n-leaves.
- For each vertex $S \in \mathcal{S}$ we have a cell $C_S \subseteq \mathcal{C}_{|S|}/S^1$.

If we want to highlight this combinatorial data we will write $(\mathcal{S}, (C_S)_{S\in\mathcal{S}})$.

Remark 5.1. These cells resembles a bar construction: the elements of $C_*^{cell}(\overline{\mathcal{M}}_{0,n+1})$ are linear combinations of nested trees with n-leaves decorated by unbased cacti, and this is exactly what we would obtain after performing a bar construction to an *operad of unbased cacti*. But we already encountered such an operad! Indeed, in Section 4.1 we saw that the

$$grav := \{sC_*^{cell}(\mathcal{C}_n/S^1)\}_{n\geq 2}$$

is a chain model for the Gravity operad.

After this observation, the next Proposition seems to be more plausible:

Proposition 5.2. *$B(grav)(n) = s^2C_*^{cell}(\overline{\mathcal{M}}_{0,n+1})$ for any $n \geq 2$, where B is the operadic bar construction.*

Proof. By definition $B(grav) = (T^c(sgrav), d)$, where T^c denotes the free co-operad on the Σ-sequence $sgrav := \{s^2C_*^{cell}(\mathcal{C}_n/S^1)\}_{n\geq 2}$. Forgetting the differential for a moment,

$$B(grav)(n) = \bigoplus_{\mathcal{S}\in N_n}\bigotimes_{S\in\mathcal{S}} sgrav(|S|) = \bigoplus_{\mathcal{S}\in N_n}\bigotimes_{S\in\mathcal{S}} s^2C_*^{cell}(\mathcal{C}_{|S|}/S^1)$$

so we can think of an element in the chain complex $B(grav)(n)$ as a sum of nested trees decorated by unbased cacti whose dimension is raised by two. Now consider the following isomorphism of graded abelian groups:

$$\begin{aligned} f : C_*^{cell}(\overline{\mathcal{M}}_{0,n+1}) &\to B(grav)(n) \\ (\mathcal{S}, (C_S)_{S\in\mathcal{S}}) &\mapsto \otimes_{S\in\mathcal{S}} s^2 C_S \end{aligned}$$

If $\tau = (\mathcal{S}, (C_S)_{S\in\mathcal{S}})$ is a cell in $C_*^{cell}(\overline{\mathcal{M}}_{0,n+1})$, then

$$dim(\tau) = 2(V-1) + \sum_{S\in\mathcal{S}} dim(C_S)$$

where V is the number of vertices of $\mathcal{S}$. The corresponding element $f(\tau)$ has degree

$$deg(f(\tau)) = 2V + \sum_{S\in\mathcal{S}} dim(C_S)$$

therefore f is a degree two map and induces an isomorphism

$$B(grav)(n) \cong s^2 C_*^{cell}(\overline{\mathcal{M}}_{0,n+1})$$

of graded abelian groups. Finally, let us prove that this identification is indeed an isomorphism of chain complexes: the differential of $B(grav)(n)$ splits in two parts $d = d_1 + d_2$: d_1 acts on an element $f(\tau) \in B(grav)(n)$ by performing the differential of $grav$ one vertex at the time, and then summing all the results. d_2 acts on $f(\tau)$ by contracting edges and composing the corresponding cacti. But this is exactly how we compute the boundary of the cell τ (see Paragraph 4.3.1), and this proves the claim. □

5.2 The Dual cell decomposition

Suppose from now on that we have fixed some weights $(a_1, \dots a_n) \in \mathring{\Delta}^{n-1}$. In this way we can identify $\overline{\mathcal{M}}_{0,n+1}$ with the space of nested cacti $N_n^\sigma(\mathcal{C}/S^1)$ (Theorem 4.13) and get a regular CW-decomposition of $\overline{\mathcal{M}}_{0,n+1}$. In this section we give an explicit construction of the dual cell decomposition. We will work on $N_n^\sigma(\mathcal{C}/S^1)$, and then use Theorem 4.13 to define the dual cells on $\overline{\mathcal{M}}_{0,n+1}$.

Construction (Dual cell decomposition)**.** Let X be a regular cell decomposition of a compact oriented n-dimensional manifold M. The dual cell decomposition X^* of M is constructed as follows:

1. For each cell $\tau \in X$ choose a point $B\tau$ in its interior, which we will call the **barycenter**.

2. For any strictly increasing chain $\tau_0 \subset \tau_1 \subset \cdots \subset \tau_k$ of cells, we define inductively $(\tau_0, \dots, \tau_k) \subseteq \tau_k$ as follows: $(\tau_0) := B\tau_0$. Suppose that $(\tau_0, \dots, \tau_{k-1}) \subseteq \tau_{k-1}$ is defined, then $(\tau_0, \dots, \tau_k)$ is will be the cone on $(\tau_0, \dots, \tau_{k-1})$ with vertex $B\tau_k$. Observe that $(\tau_0, \dots, \tau_k)$ is homeomorphic to the k-simplex.

3. Given a cell τ, the dual cell τ^* is defined as the union of all simplices $(\tau_0, \dots, \tau_k)$ with $\tau_0 = \tau$, $k \in \mathbb{N}$.

In the case of $N_n^\sigma(\mathcal{C}/S^1)$ we have a natural choice for the barycenters:

Definition 5.1. Let $\tau = (\mathcal{S}, (C_S)_{S \in \mathcal{S}})$ be a cell of $N_n^\sigma(\mathcal{C}/S^1)$. The **barycenter** $B\tau = (x_R, (t_S, \alpha_S, x_S)_{S \in \mathcal{S}-R})$ is the point specified by the following parameters:

- For any $S \in \mathcal{S} - R$ put $t_S = 0$. Note that since the radial parameter is zero we do not have to specify any angular parameter.

- For any $S \in \mathcal{S}$, x_S will be the barycenter of C_S (C_S is a product of simplices, so its barycenter is just the product of the barycenters of each simplex).

Now that we have choosen the barycenters we can construct the dual cells as just described. Let us prove some easy properties of the dual cells:

Definition 5.2. Fix a nested tree $\mathcal{S}$ with n-leaves. We will indicate with $\overline{N_n(\mathcal{S})}$ the following subspace of $N_n^\sigma(\mathcal{C}/S^1)$:

$$\overline{N_n(\mathcal{S})} := \{(x_R, (t_T, \alpha_T, x_T)_{T\in\mathcal{T}-R}) \in N_n^\sigma(\mathcal{C}/S^1)) \mid \mathcal{S} \subseteq \mathcal{T} \text{ and } t_S = 0 \text{ for all } S \in \mathcal{S}\}$$

Remark 5.3. Under the homeomorphism of Theorem 4.13 the subspace $\overline{N_n(\mathcal{S})}$ corresponds to the closure of the stratum $\mathcal{M}(\mathcal{S})$.

Lemma 5.4. *Let $\tau = (\mathcal{S}, (C_S)_{S\in\mathcal{S}})$ be a cell of $N_n^\sigma(\mathcal{C}/S^1)$. Then $\tau^* \subseteq \overline{N_n(\mathcal{S})}$.*

Proof. By the construction of τ^* it suffices to show that any simplex of the form $(\tau, \tau_1, \ldots, \tau_k)$ is contained in $\overline{N_n(\mathcal{S})}$. We proceed by induction on k:

$k = 0$**:** in this case we have the zero simplex $(\tau) := B\tau$, which is a point of $\overline{N_n(\mathcal{S})}$ by construction.

$k \geq 1$**:** by definition $(\tau, \tau_1, \ldots, \tau_k)$ is the cone on $(\tau, \tau_1, \ldots, \tau_{k-1})$ with vertex $B\tau_k$. Since $\tau \subseteq \tau_k$, the cell τ_k will be of the form $\tau_k = (\mathcal{T}_k, (C'_T)_{T\in\mathcal{T}_k})$ for some nested tree $\mathcal{T}_k \supseteq \mathcal{S}$. The barycenter of τ_k is then $B\tau_k = (b_R, (0, b_T)_{T\in\mathcal{T}_k-R})$, where b_T is the barycenter of C'_T. Now take a point $p \in (\tau, \ldots, \tau_{k-1})$, where $(\tau, \ldots, \tau_{k-1})$ is seen as a subspace of $\partial\tau_k$. Then p will be of the form

$$p = (x_R, (t_T, \alpha_T, x_T)_{T\in\mathcal{T}_k-R})$$

By induction $(\tau, \tau_1, \ldots, \tau_{k-1}) \subseteq \overline{N_n(\mathcal{S})}$, so $t_S = 0$ for all $S \in \mathcal{S}$. The segment connecting $B\tau_k$ and p interpolates between the coordinates of $B\tau_k$ and p which are different from each other. Since also $B\tau_k$ has $t_S = 0$ for all $S \in \mathcal{S}$, any point on such a segment will have $t_S = 0$ for any $S \in \mathcal{S}$. Finally we observe that $(\tau, \tau_1, \ldots, \tau_k)$ is the union of all such segments, so we get the statement.

□

A converse statement also holds:

Lemma 5.5. *Fix a nested tree $\mathcal{S}$ and let $x \in \overline{N_n(\mathcal{S})}$. If τ^* is the unique dual cell containing x, then $\mathcal{S} \subseteq tree(\tau)$, where $tree(\tau)$ is the nested tree underlying the cell τ.*

Proof. Since $x \in \overline{N_n(\mathcal{S})}$ it has the form $x = (x_R, (t_T, \alpha_T, x_T)_{T\in\mathcal{T}-R})$ for some $\mathcal{T} \supseteq \mathcal{S}$, and $t_S = 0$ for any $S \in \mathcal{S}$. Without loss of generality we can suppose that all the radial parameters are different from 1. Let τ_x be the unique (open) cell containing x. Then $\tau_x = (\mathcal{T}, (C_T)_{T\in\mathcal{T}})$, with $x_T \in C_T$ for any $T \in \mathcal{T}$. We prove the statement by induction on $k := \dim(\tau_x)$:

$k = 0$**:** in this case τ_x is just the corolla with n-leaves decorated by a cactus C_R of dimension zero. Since $\mathcal{S} \subseteq \mathcal{T}$, we have that also $\mathcal{S}$ is the corolla and the statement becomes trivial.

$k \geq 1$**:** If $x = B\tau_x$ then $x \in \tau_x^*$ and the statement follows. Now suppose $x \neq B\tau_x$: in this case there must be a radial parameter $t_{T_0} \neq 0$, for some $T_0 \in \mathcal{T}$, $T_0 \notin \mathcal{S}$. Consider the point $y \in \partial\tau_x$ which has the same coordinates as x except for the radial parameter of T_0, which we set to be 1. Let $\tau_y \subseteq \partial\tau_x$ be the unique (open) cell containing y and σ^* be the unique (open) dual cell containing y. By induction the nested tree $tree(\sigma)$ contains $\mathcal{S}$. Now let $(\sigma, \sigma_1, \dots, \sigma_m)$ be the unique simplex containing y in its interior. Then $y \in \sigma_m$, but since there is a unique (open) cell containing y we conclude that $\sigma_m = \tau_y \subseteq \partial\tau_x$. Therefore $x \in (\sigma, \sigma_1, \dots, \sigma_m, \tau_x)$, so $x \in \sigma^*$ as well concluding the proof.

□

Proposition 5.6. *Fix a nested tree $\mathcal{S}$. Then*

$$\overline{N_n(\mathcal{S})} = \bigcup_{\substack{\tau \text{ cell} \\ tree(\tau) \supseteq \mathcal{S}}} \tau^*$$

In other words this Proposition tells us that if we put the dual cell decomposition on $N_n^\sigma(\mathcal{C}/S^1)$, then $\overline{N_n(\mathcal{S})}$ becomes a subcomplex.

Proof. Fix a cell τ such that $\mathcal{T} := tree(\tau) \supseteq \mathcal{S}$. By Lemma 5.4 we have that $\tau^* \subseteq \overline{N_n(\mathcal{T})}$. This space is contained in $\overline{N_n(\mathcal{S})}$ because $\mathcal{T} \supseteq \mathcal{S}$. Therefore

$$\bigcup_{\substack{\tau \text{ cell} \\ tree(\tau) \supseteq \mathcal{S}}} \tau^* \subseteq \overline{N_n(\mathcal{S})}$$

The other inclusion is given by Lemma 5.5. □

Example (Dual cell decomposition of $\overline{\mathcal{M}}_{0,4}$)**.** In Section 4.5 we described in detail the CW-decomposition of $\overline{\mathcal{M}}_{0,4}$ given by nested cacti: we had two 0-cells C_1, C_2, three 1-cells B_1, B_2, B_3 and three 2-cells A_1, A_2, A_3 (see Figure 4.16). The dual cell decomposition will have three 0-cells A_1^*, A_2^*, A_3^*, (corresponding to the three stable curves), three 1-cells B_1^*, B_2^*, B_3^* and two 2-cells C_1^*, C_2^*. See Figure 5.1 for a picture.

5.3 Operad stucture on the dual cells

In the prevoius section we constructed explicitly the dual cell decomposition of $\overline{\mathcal{M}}_{0,n+1}$ (remember that this decomposition depends on the initial choice of weights $a_1, \dots, a_n$). We will denote by $C_*^{dual}(\overline{\mathcal{M}}_{0,n+1})$ the cellular chains respect to the dual cell decomposition.

Remark 5.7. The dual cells in $\overline{\mathcal{M}}_{0,n+1}$ depend on the choice of weights, but the chain complex $C_*^{dual}(\overline{\mathcal{M}}_{0,n+1})$ depends only on the combinatorics of the cells: a (dual) cell τ^* of $\overline{\mathcal{M}}_{0,n+1}$ will be determined by:

1. A nested tree $\mathcal{S}$ on n-leaves.

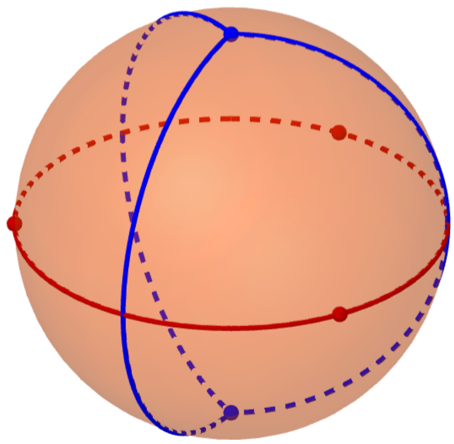


Figure 5.1: In blue we see the CW-decomposition of $\overline{\mathcal{M}}_{0,4}$ given by nested cacti. In red we see the dual cells.

2. A cell $C_S \subseteq \mathcal{C}_{|S|}/S^1$ for any $S \in \mathcal{S}$.

In the case we want to highlight this data we will write $\tau^* = (\mathcal{S}, (C_S)_{S\in\mathcal{S}})$. Such a cell has dimension

$$dim(\tau^*) = 2(n-2) - 2E(\mathcal{S}) - \sum_{S\in\mathcal{S}} dim(C_S)$$

where $E(\mathcal{S})$ is the number of internal edges of $\mathcal{S}$. The boundary of $\tau^* = (\mathcal{S}, (C_S)_{S\in\mathcal{S}})$ is the sum (with signs depending on the orientations) of all the codimension one cells obtained from τ^* by one of the following moves:

Move 1 (cacti co-boundary): for a fixed vertex $S \in \mathcal{S}$ replace C_S with a cell C'_S such that $C_S \subseteq \partial C'_S$.

Move 2 (vertex expansion): for a fixed vertex $S_0 \in \mathcal{S}$, chose a subset $S_1 \subseteq S_0$ with at least two elements such that $\mathcal{S} \cup S_1$ is still a nested tree. Then consider following cell $\sigma^* = (\mathcal{T}, (C'_T)_{T\in\mathcal{T}})$:

- As underlying nested tree we take $\mathcal{T} := \mathcal{S} \cup S_1$.
- As decorative cacti we put $C'_T = C_T$ for $T \neq S_0, S_1$. C'_{S_0} and C'_{S_1} will be two cacti such that C_{S_0} is obtained from C'_{S_0} gluing C'_{S_1} in the lobe corresponding to the edge (S_1, S_0).

Definition 5.3. Let $\mathcal{S} \in N_n$ and $\mathcal{T} \in N_m$. For any $i = 1, \ldots, n$ we define $\mathcal{S} \circ_i \mathcal{T} \in N_{n+m-1}$ as follows:

- For any $T \in \mathcal{T}$, $\mathcal{S} \circ_i \mathcal{T}$ contains $T + i := \{t + i - 1 \mid t \in T\}$.
- If $S \in \mathcal{S}$ does not contain i, then $\mathcal{S} \circ_i \mathcal{T}$ contains

$$\overline{S} := \{s \in S \mid s < i\} \cup \{s + m - 1 \mid s \in S, s \geq i + 1\}$$

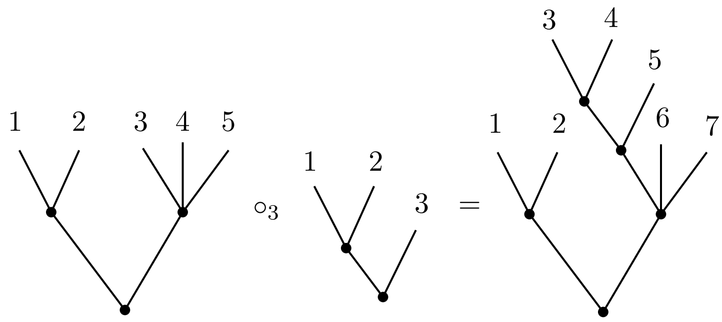


Figure 5.2: In this picture we see the grafting $\mathcal{S} \circ_3 \mathcal{T}$, where $\mathcal{S} = ((12)(345))$ and $\mathcal{T} = ((12)3)$.

- If $S \in \mathcal{S}$ is a vertex containing i, then $\mathcal{S} \circ_i \mathcal{T}$ contains

$$\overline{S} \coloneqq \{s \in S \mid s < i\} \cup \{i, \ldots, i+m-1\} \cup \{s+m-1 \mid s \in S, s \geq i+1\}$$

Intuitively $\mathcal{S} \circ_i \mathcal{T}$ is obtained by grafting $\mathcal{T}$ to the i-th leaf of $\mathcal{S}$. See Figure 5.2 for an example.

We are now ready to define an operad structure on the collection of graded abelian groups $\{C_*^{dual}(\overline{\mathcal{M}}_{0,n+1}; \mathbb{Z})\}_{n \geq 2}$.

Definition 5.4. Let $\sigma^* \in C_*^{dual}(\overline{\mathcal{M}}_{0,n+1})$, $\tau^* \in C_*^{dual}(\overline{\mathcal{M}}_{0,m+1})$ two dual cells, with $\sigma^* = (\mathcal{S}, (C_S)_{S \in \mathcal{S}})$ and $\tau^* = (\mathcal{T}, (C'_T)_{T \in \mathcal{T}})$. For any $i \in \{1, \ldots, n\}$, the **i-th composition** $\sigma^* \circ_i \tau^*$ is the cell of $C_*^{dual}(\overline{\mathcal{M}}_{0,n+m})$ specified by the following combinatorial data:

1. As nested tree we put $\mathcal{S} \circ_i \mathcal{T}$.
2. Given $U \in \mathcal{S} \circ_i \mathcal{T}$, the decorative cacti of this vertex is given by:
   - C'_T if $U = T + i$ for some $T \in \mathcal{T}$.
   - C_S if $U = \overline{S}$ for some $S \in \mathcal{S}$.

Since the dual cells are the generators of $C_*^{dual}(\overline{\mathcal{M}}_{0,n+1})$, we can extend $\circ_i$ by linearity and obtain a morphism

$$\circ_i : C_*^{dual}(\overline{\mathcal{M}}_{0,n+1}) \otimes C_*^{dual}(\overline{\mathcal{M}}_{0,m+1}) \to C_*^{dual}(\overline{\mathcal{M}}_{0,m+n})$$

Lemma 5.8. *The map $\circ_i : C_*^{dual}(\overline{\mathcal{M}}_{0,n+1}) \otimes C_*^{dual}(\overline{\mathcal{M}}_{0,m+1}) \to C_*^{dual}(\overline{\mathcal{M}}_{0,m+n})$ is a degree zero morphism of chain complexes.*

Proof. To show the statement it suffices to verify it on the generators; so take $\sigma^* \in C_*^{dual}(\overline{\mathcal{M}}_{0,n+1})$, $\tau^* \in C_*^{dual}(\overline{\mathcal{M}}_{0,m+1})$ two dual cells, with $\sigma^* = (\mathcal{S}, (C_S)_{S \in \mathcal{S}})$ and $\tau^* = (\mathcal{T}, (C'_T)_{T \in \mathcal{T}})$. We know that

$$dim(\sigma^* \circ_i \tau^*) = 2(n+m-1-2) - 2E(\mathcal{S} \circ_i \mathcal{T}) - \sum_{S \in \mathcal{S}} dim(C_S) - \sum_{T \in \mathcal{T}} dim(C'_T)$$

By construction $E(\mathcal{S} \circ_i \mathcal{T}) = E(\mathcal{S}) + E(\mathcal{T}) + 1$, so

$$\begin{aligned} dim(\sigma^* \circ_i \tau^*) &= 2(n+m-1-2) - 2E(\mathcal{S}) - 2E(\mathcal{T}) - 2 - \sum_{S\in\mathcal{S}} dim(C_S) - \sum_{T\in\mathcal{T}} dim(C'_T) \\ &= 2(n-2) - 2E(\mathcal{S}) - \sum_{S\in\mathcal{S}} dim(C_S) + 2(m-2) - 2E(\mathcal{T}) - \sum_{T\in\mathcal{T}} dim(C'_T) \\ &= dim(\sigma^*) + dim(\tau^*) \end{aligned}$$

proving that $\circ_i$ is a degree zero map. For the compatibility with the differential observe that $d(\sigma^* \otimes \tau^*) = d(\sigma^*) \otimes \tau^* + \pm\sigma^* \otimes d(\tau^*)$. The term $d(\sigma^*) \otimes \tau^*$ is a sum of elements of the form $\omega^* \otimes \tau^*$, where ω^* is obtained from σ^* by Move 1 (cacti coboundary) or Move 2 (vertex expansion). Similarly we can describe the other term $\sigma^* \otimes d(\tau^*)$. Now it is clear that if we apply $\circ_i$ to $d(\sigma^* \otimes \tau^*)$ we obtain $d(\sigma^* \circ_i \tau^*)$, concluding the proof. □

Definition 5.5 (Chain model for the hypercommutative operad)**.** Let $hycom(n) := C_*^{dual}(\overline{\mathcal{M}}_{0,n+1}; \mathbb{Z})$. With this notation, the collection of chain complexes

$$hycom := \{hycom(n)\}_{n\geq 2}$$

together with the maps $\circ_i : hycom(n) \otimes hycom(m) \to hycom(n+m-1)$ defined before forms a symmetric operad in $Ch(\mathbb{Z})$.

We now prove that the homology of this operad is precisely the Hypercommutative operad. First we need a preliminary result:

Proposition 5.9. *Let $\mathcal{S}$ be a nested tree on n-leaves and $\overline{\mathcal{M}}(\mathcal{S})$ be the closure of the corresponding stratum $\mathcal{M}(\mathcal{S}) \subseteq \overline{\mathcal{M}}_{0,n+1}$. In $C_*^{dual}(\overline{\mathcal{M}}_{0,n+1})$ consider the following element:*

$$c(\mathcal{S}) := \sum_{\substack{\tau^*=(\mathcal{S},(C_S)_{S\in\mathcal{S}}) \\ dim(C_S)=0, S\in\mathcal{S}}} \pm\tau^*$$

where the sign depends on the chosen orientation of the cells. Then c is a cycle and

$$[c(\mathcal{S})] = [\overline{\mathcal{M}}(\mathcal{S})] \in H_*(\overline{\mathcal{M}}_{0,n+1})$$

Proof. For any choice of weights the homeomorphism of Theorem 4.13 sends $\overline{\mathcal{M}}(\mathcal{S})$ to $\overline{N_n(\mathcal{S})}$. By Proposition 5.6 $\overline{N_n(\mathcal{S})}$ is a subcomplex of $N_n^\sigma(\mathcal{C}/S^1)$ (with the dual cell structure): it is the union of the dual cells $\tau^* = (\mathcal{T}, (C_T)_{T\in\mathcal{T}})$ whose underlying tree $\mathcal{T} \supseteq \mathcal{S}$. Therefore $[\overline{\mathcal{M}}(\mathcal{S})]$ will be represented at the chain level by the sum of the cells cells $\tau^* = (\mathcal{T}, (C_T)_{T\in\mathcal{T}})$ such that $\mathcal{T} \supseteq \mathcal{S}$ and

$$dim(\overline{\mathcal{M}}(\mathcal{S})) = dim(\tau^*) = 2(n-2) - 2E(\mathcal{T}) - \sum_{T\in\mathcal{T}} dim(C_T)$$

The dimension of $\overline{\mathcal{M}}(\mathcal{S})$ is $2(n-2) - 2E(\mathcal{S})$, so the above equality holds if and only if

$$2(E(\mathcal{S}) - E(\mathcal{T})) = \sum_{T\in\mathcal{T}} dim(C_T) \tag{5.1}$$

The left term is less or equal than zero since $\mathcal{S} \subseteq \mathcal{T}$, so $\sum_{T\in\mathcal{T}} dim(C_T) \leq 0$. But this can happen if and only if C_T has dimension zero for each $T \in \mathcal{T}$. Therefore Equation 5.1 became $E(\mathcal{T}) - E(\mathcal{S}) = 0$, and we conclude that $\mathcal{T} = \mathcal{S}$. To sum up, we have proven that τ^* must be of the form $\tau^* = (\mathcal{S}, (C_S)_{S\in\mathcal{S}})$, with C_S a zero dimensional cell and this concludes the proof. □

Theorem 5.10. *hycom is a chain model for the Hypercommutative operad, i.e*

$$H_*(hycom) = Hycom$$

Proof. Since $hycom(n) = C_*^{dual}(\overline{\mathcal{M}}_{0,n+1}; \mathbb{Z})$, it is clear that

$$H_*(hycom(n)) = H_*(\overline{\mathcal{M}}_{0,n+1}) = Hycom(n)$$

It remains to show that the operad composition of $H_*(hycom)$ is that of $Hycom$. As an operad $Hycom$ is generated by the fundamental classes $[\overline{\mathcal{M}}_{0,n+1}]$: in particular

$$[\overline{\mathcal{M}}_{0,n+1}] \circ_i [\overline{\mathcal{M}}_{0,m+1}] = [\overline{\mathcal{M}}(\mathcal{S})]$$

where $\mathcal{S}$ is the nested tree obtained by grafting the m-corolla to the i-th leaf of the n-corolla. $\mathcal{S}$ has two vertices, which we will call R and S. By Proposition 5.9 $[\overline{\mathcal{M}}_{0,n+1}]$ is represented at the chain level by the sum of cells whose underlying nested tree is the n-corolla, and whose decorative cactus at the root is a zero dimensional cell of $\mathcal{C}_n/S^1$. The same holds for $[\overline{\mathcal{M}}_{0,m+1}]$. Let us denote by $a \in hycom(n)$ and $b \in hycom(m)$ these chain level representatives. By definition $a \circ_i b \in hycom(n + m - 1)$ is the sum over all the dual cells τ^* whose underlying nested tree is $\mathcal{S}$ and such that the decorative cacti C_S and C_R are zero dimensional cells. Proposition 5.9 tells us that $a \circ_i b$ is precisely a chain level representative for $[\overline{\mathcal{M}}(\mathcal{S})]$, concluding the proof. □

5.4 An open problem

To finish this Chapter I would like to present a problem I have been working on for quite a long time, but without getting to the solution. In [47] Salvatore constructed a cellular decomposition of the two dimensional Fulton MacPherson operad FM_2. More precisely, he defined a CW-decomposition for each space $FM_2(n)$ and proved that the operad compositions $\circ_i : FM_2(n) \times FM_2(m) \to FM_2(m+n-1)$ are cellular maps. So we might try to do the same for the Deligne-Mumford operad $\overline{\mathcal{M}}$:

Problem: For any n find a cell decomposition of $\overline{\mathcal{M}}_{0,n+1}$ such that the operad compositions $\circ_i : \overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,n+m}$ are cellular maps.

The first step is to construct a CW-decomposition of $\overline{\mathcal{M}}_{0,n+1}$ for any $n \geq 2$: for example we can choose the cell structures described in Corollary 4.15. However these cells are not a a good candidate to solve our problem, because they are not compatible with the operad structure (see Remark 4.17). A better choice would be to put on $\overline{\mathcal{M}}_{0,n+1}$ the dual cell decomposition. Indeed, at least at the level of chains we get the right operad structure (Theorem 5.10). However even in this case we encounter some issues, basically because the dual cell decompositions depends on the choice of some weights $(a_1, \ldots, a_n)$. Let me explain this last sentence: in order to solve our problem one could try to work inductively. $\overline{\mathcal{M}}_{0,3}$ is just a point, and we put on it the obvious cell structure. Now consider the operadic composition

$$\circ_1 : \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{0,4}$$

It is an embedding whose image is the stable curve associated to the nested tree $((12)3)$. Similarly, the image of $\circ_2$ is the stable curve associated to $(1(23))$. If we put on $\overline{\mathcal{M}}_{0,4}$ the dual cell structure (in this case any choice of weights is fine) we get that $\circ_1, \circ_2$ are cellular maps (see Figure 5.1). By induction suppose we have constructed CW-decompositions of $\overline{\mathcal{M}}_{0,n}$ for $n \leq k-1$ which are compatible with the operad compositions. We would like to define a CW-decomposition of $\overline{\mathcal{M}}_{0,k}$ compatible with the operad structure. Let $n, m \in \mathbb{N}$ such that $n + m = k$. Then for $i = 1, \ldots, n$ consider

$$\circ_i : \overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,k}$$

We would like that $\circ_i$ is a cellular map. Since $\circ_i$ is an embedding we can put on $Im(\circ_i)$ the product cell structure of $\overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1}$. In this way we obtain a cell decomposition of the closed strata $\overline{\mathcal{M}}(\mathcal{S}) \subseteq \overline{\mathcal{M}}_{0,k}$, with $\mathcal{S}$ a nested tree on $(k-1)$-leaves which is not the corolla. The challenge is to extend this CW-decomposition to the whole $\overline{\mathcal{M}}_{0,k}$. The problem is that if we take the dual cell decomposition of $\overline{\mathcal{M}}_{0,k}$ associated to some weights $a_1, \ldots, a_k$, in general it does not match the cell decomposition of the strata described above. Paolo Salvatore suggested that maybe we can deform the dual cells of $\overline{\mathcal{M}}_{0,k}$ and forcing them to intersect properly the CW-decomposition of the strata defined inductively. Figure 5.3 illustrate this idea.

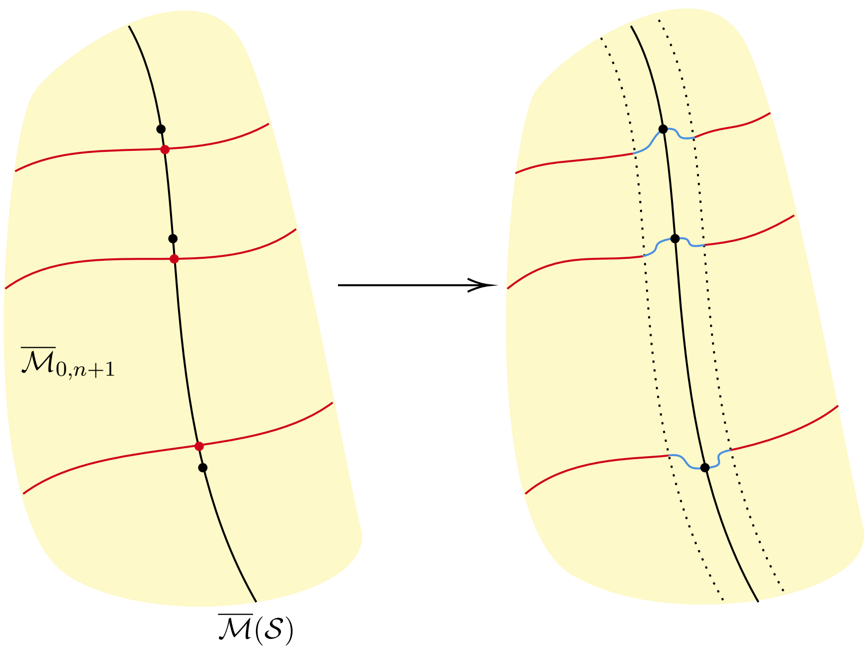


Figure 5.3: On the left we see that the red CW-decomposition of $\overline{\mathcal{M}}_{0,n+1}$ does not match the CW-decompostion of the stratum $\overline{\mathcal{M}}(\mathcal{S})$. The idea is to deform the red 1-cells to make them intersect properly the cells of $\overline{\mathcal{M}}(\mathcal{S})$ defined inductively, obtaining the cell decomposition on the right.

Part II

Homology operations for gravity algebras

Chapter 6

Homology operations for gravity algebras

This chapter is devoted to study homology operations for gravity algebras. To introduce the reader to the problem I believe it is useful to think about this parallelism: in the theory of $H_*(\mathcal{D}_2)$-algebras a key role is played by the Dyer-Lashof operations, which correspond to classes in $H_*^{\Sigma_n}(\mathcal{D}_2(n);\mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{D}_2(n);\mathbb{F}_p(\pm 1))$ (here $\mathbb{F}_p(\pm 1)$ denotes the sign representation). Similarly, classes in $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p(\pm 1))$ give rise to operations for Gravity algebras. In Section 6.1 we define explicitly these operations. In Section 6.2 we prepare the ground for computing $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$. The key observation is the following: the homotopy quotients $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ and $C_n(\mathbb{C})_{S^1}$ are both models for the classifying space of $B_n/Z(B_n)$, the quotient of the braid group by its center. This allows us to do the computation of $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$ by looking at the Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \longrightarrow BS^1 \tag{6.1}$$

which is much simpler than

$$\mathcal{M}_{0,n+1} \hookrightarrow (\mathcal{M}_{0,n+1})_{\Sigma_n} \longrightarrow B\Sigma_n$$

because in the first case there is not any monodromy. Moreover, in the (homological) Serre spectral sequence associated to (6.1) the homology of the fiber is well known, and the differential of the second page is given by the BV-operator Δ. So everything is now quite explicit and one can hope to calculate $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$. This computation will be the topic of the Chapter 7. Here is the outline of the chapter:

Section 6.1 contains a chain level definition of the homology operations for gravity algebras.

Section 6.2 contains the proof that $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ and $C_n(\mathbb{C})_{S^1}$ are both models for the classifying space of $B_n/Z(B_n)$. We also discuss some results about the Serre spectral sequence associated to the fibration $X \to X_{S^1} \to BS^1$ for X any space with a circle action.

6.1 Equivariant operations

In this section we explain how to define homology operations for gravity algebras. This will serve as a motivation for the main results of Chapters 7 and 8, i.e. a complete calculation of $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$ and $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p(\pm 1))$ for any $n \in \mathbb{N}$ and p a prime number.

Let p be a prime and use $\mathbb{F}_p$-coefficients for (co)homology from now on. Fix *grav* a chain model for the Gravity operad (for example the one described in Section 4.1). Explicitly this means that

$$H_*(grav(n)) = sH_*(\mathcal{M}_{0,n+1};\mathbb{F}_p)$$

Remark 6.1. Without loss of generality we can suppose that

(A) For any $n \in \mathbb{N}$ the symmetric group Σ_n acts freely on $grav(n)$

Indeed if we consider the tensor product $grav \otimes \mathbb{E}_\infty$ we get a chain model for the gravity operad with this property. From now on we will suppose that our chain model *grav* satisfy property (A), therefore $grav(n)$ is made of free (hence flat) $\mathbb{F}[\Sigma_n]$-modules.

Remark 6.2. The chain model described in Section 4.1 do not satisfy property (A). Indeed in arity two we have the cellular chains of $\mathcal{C}_2/S^1 = *$ and Σ_2 acts trivially.

Now let (A, d_A) be a Gravity algebra in the category $Ch(\mathbb{F}_p)$. Since the maps $grav(n) \otimes A^{\otimes n} \to A$ which give the gravity algebra structure are Σ_n-equivariant, they factor through the coinvariants:

$$\begin{tikzcd} grav(n) \otimes A^{\otimes n} \arrow[r] \arrow[d] & A \\ grav(n) \otimes_{\Sigma_n} A^{\otimes n} \arrow[ur, "\gamma"'] & \end{tikzcd}$$

Passing to homology we get

$$\gamma_* : H_*(grav(n) \otimes_{\Sigma_n} A^{\otimes n}) \to H_*(A)$$

Remark 6.3. Since we are working with coefficients in a field, we can define a quasi-isomorphism $H_*(A) \to A$ by choosing a basis for $H_*(A)$ and assigning to each element of it a representative cycle in $Z_*(A) \subseteq A$. Σ_n acts freely on $grav(n)$ and therefore $grav(n)$ is made of flat $\mathbb{F}[\Sigma_n]$-modules. Thus we get an isomorphism between $H_*(grav(n) \otimes_{\Sigma_n} H_*(A)^{\otimes n})$ and $H_*(grav(n) \otimes_{\Sigma_n} A^{\otimes n})$.

Remark 6.4. Since Σ_n acts freely on $grav(n)$, we have that the coinvariants $grav(n)_{\Sigma_n}$ compute (up to a degree shift) the Σ_n-equivariant homology of $\mathcal{M}_{0,n+1}$. More explicitly,

$$H_*(grav(n)_{\Sigma_n}) \cong sH_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$$

It is not hard to see that the homotopy quotient $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ is the classifying space of $B_n/Z(B_n)$ (see Section 6.2 for details), the quotient of the braid group on n-strands by its center. So $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$ can be interpreted in purely algebraic terms as the group homology of $B_n/Z(B_n)$ with trivial coefficients. Besides $H_*(B_n/Z(B_n);\mathbb{F}_p)$, in what follows $H_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$ will also be important (where $\mathbb{F}_p(\pm 1)$ denotes the sign representation). Essentially by definition, this homology is computed (up to a shift) by the quotient of $grav(n)$ by the subspace $< x-(-1)^\sigma \sigma\cdot x \mid x \in grav(n), \sigma \in \Sigma_n >$. So,

$$H_*\left(\frac{grav(n)}{< x-(-1)^\sigma \sigma \cdot x >}\right) \cong sH_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$$

We are now ready to define what is an equivariant homology operation for a gravity algebra, and to explain in detail the connection with $H_*(B_n/Z(B_n);\mathbb{F}_p)$ and $H_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$.

Definition 6.1 (Equivariant operations for even classes)**.** Let $Q \in H_*(B_n/Z(B_n);\mathbb{F}_p)$ and let $q \in grav(n)$ be an element such that $[q] \in grav(n)_{\Sigma_n}$ is a representative for the class $sQ \in sH_*(B_n/Z(B_n);\mathbb{F}_p)$. Since there is a shift of degree, $|q| = |Q|+1$. If $[a] \in H_*(A)$ is an even degree class, then it is not hard to verify that $q \otimes a^{\otimes n}$ is a cycle in $grav(n) \otimes_{\Sigma_n} A^{\otimes n}$. Then we define

$$Q(a) \coloneqq \gamma_*(q \otimes a^{\otimes n}) \in H_*(A)$$

It is not hard to see that $Q(a)$ does not depend neither on the choice of q, nor on the choice of a representative cycle for $[a]$. So the definition is well posed. To sum up, any class $Q \in H_*(B_n/Z(B_n);\mathbb{F}_p) = H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$ gives rise to a homology operation (defined only for classes of even degrees)

$$Q(-) : H_{2m}(A) \to H_{2mn+|Q|+1}(A)$$

Definition 6.2 (Equivariant operations for odd classes)**.** Consider a class $Q \in H_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$ and let $q \in grav(n)$ be an element such that $[q] \in grav(n)/ < x-(-1)^\sigma \sigma \cdot x \mid x \in grav(n), \sigma \in \Sigma_n >$ is a representative for the class $sQ \in sH_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$. Since there is a shift of degree, $|q| = |Q|+1$. If $[a] \in H_*(A)$ is an odd degree class, then it is not hard to verify that $q \otimes a^{\otimes n}$ is a cycle in $grav(n) \otimes_{\Sigma_n} A^{\otimes n}$. Then we define

$$Q(a) \coloneqq \gamma_*(q \otimes a^{\otimes n}) \in H_*(A)$$

It is not hard to see that $Q(a)$ does not depend neither on the the choice of q, nor on the choice of a representative cycle for $[a]$. So the definition is well posed. To sum up, any class $Q \in H_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$ gives rise to a homology operation (defined only for classes of odd degrees)

$$Q(-) : H_{2m+1}(A) \to H_{(2m+1)n+|Q|+1}(A)$$

To conclude, if one wants to understand the homology operations for gravity algebras, the first step is to compute $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p) = H_*(B_n/Z(B_n);\mathbb{F}_p)$ and $H_*(B_n/Z(B_n);\mathbb{F}_p(\pm 1))$. This will be the main achievement of Chapter 7 and Chapter 8. Then one can ask if there are any relations between the composite of two such operations (e.g. Adem relations). This topic is discussed in Chapter 9, even if a complete answer to this question is far from being clear (at least to the author).

6.2 Homotopy models for $(\mathcal{M}_{0,n+1})_{\Sigma_n}$

As explained in the previous section, to understand the equivariant operations on gravity algebras one has to compute $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1})$. The first thing one might try to do is to study the Serre spectral sequence associated to the fibration

$$\mathcal{M}_{0,n+1} \hookrightarrow (\mathcal{M}_{0,n+1})_{\Sigma_n} \to B\Sigma_n \tag{6.2}$$

However the action of Σ_n on the homology of $\mathcal{M}_{0,n+1}$ is not trivial and this complicates the whole computation. To overcome this problem the key observation is the following:

Lemma 6.5. *$(\mathcal{M}_{0,n+1})_{\Sigma_n}$ is homotopy equivalent to $C_n(\mathbb{C})_{S^1}$.*

Proof. Recall that $\mathcal{M}_{0,n+1}$ is Σ_n-homotopy equivalent to $F_n(\mathbb{C})/S^1$, so the homotopy quotients $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ and $(F_n(\mathbb{C})/S^1)_{\Sigma_n}$ are homotopy equivalent. S^1 acts freely on $F_n(\mathbb{C})$ so $(F_n(\mathbb{C})/S^1)_{\Sigma_n}$ is homotopy equivalent to $(F_n(\mathbb{C})_{S^1})_{\Sigma_n}$. The action of Σ_n on $F_n(\mathbb{C})$ and that of S^1 commute so we get a homotopy equivalence between $(F_n(\mathbb{C})_{S^1})_{\Sigma_n}$ and $(F_n(\mathbb{C})_{\Sigma_n})_{S^1}$, which is homotopy equivalent to $C_n(\mathbb{C})_{S^1}$ since Σ_n acts freely on $F_n(\mathbb{C})$. $\square$

Thanks to this Lemma we have an isomorphism $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1}) \cong H_*^{S^1}(C_n(\mathbb{C}))$. Chapter 7 will be dedicated to the computation of $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ for any $n \in \mathbb{N}$ and any prime number p. This is obtained by the Serre spectral sequence associated to

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \to BS^1 \tag{6.3}$$

Remark 6.6. The spectral sequence of the fibration 6.3 is much easier than the one associated to 6.2. First of all in this case we have trivial monodromy since BS^1 is simply connected. If we fix a field of coefficients $\mathbb{F}$ for (co)homology, the E_2 page of the spectral sequence is given by

$$E^2_{p,q} = H_p(C_n(\mathbb{C})) \otimes H_q(BS^1)$$

In addition, the differential d^2 of the E^2 page is well known in this case, as we will explain in the rest of this section.

Proposition 6.7. *Let X be any topological space such that $H_i(X;\mathbb{Z})$ is a finitely generated abelian group for each $i \in \mathbb{N}$. Consider $S^1 \times X$ with the natural action of S^1 by multiplication on the left. Fix a field $\mathbb{F}$ of coefficients for (co)homology. Then the homological spectral sequence associated to the fibration $S^1 \times X \to (S^1 \times X)_{S^1} \to BS^1$ has the following form:*

1. *$E^2 = H_*(S^1) \otimes H_*(X) \otimes H_*(BS^1)$.*
2. *Let y_{2i} be a generator of $H_{2i}(BS^1;\mathbb{F})$, e_0 a generator of $H_0(S^1)$ and $x \in H_*(X;\mathbb{F})$. Then the differential d^2 of the second page is given by*

$$d^2(e_0 \otimes x \otimes y_{2i}) = \begin{cases} 0 \text{ if } i = 0 \\ [S^1] \otimes x \otimes y_{2i-2} \text{ otherwise} \end{cases} \qquad d^2([S^1] \otimes x \otimes y_{2i}) = 0$$

3. *The spectral sequence degenerates at the third page, which is given by:*

$$E^3_{i,j} = \begin{cases} e_0 \otimes H_j(X) \otimes y_0 \text{ if } i = 0 \\ 0 \text{ if } i > 0 \end{cases}$$

Proof. Point (1) is clear, (3) follows from (2). So the only thing to prove is the statement of point (2). Since S^1 acts freely on $S^1 \times X$, the homotopy quotient $(S^1 \times X)_{S^1}$ is homotopy equivalent to the strict quotient $(S^1 \times X)/S^1 = X$. The original fiber sequence can be rewritten as $S^1 \times X \to X \to BS^1$, where the first map $p : S^1 \times X \to X$ is the projection on the second factor. We prove the dual statement in order to exploit the multiplicativity of the cohomological spectral sequence. The second page looks as follows:

$$E_2 = \frac{\mathbb{F}[a]}{(a^2)} \otimes H^*(X;\mathbb{F}) \otimes \mathbb{F}[c]$$

where a is a generator of $H^1(S^1)$ and c is a generator of $H^2(BS^1)$. The classes $y \in H^*(X) \subseteq E_2^{0,*}$ are infinite cycles because they belong to the image of $p^* : H^*(X) \to H^*(S^1 \times X)$. This observation implies that $E_3 = E_\infty$, because the only multiplicative generator which can have non zero differentials is a, which is a class in $E_2^{0,1}$, and therefore $d_n(a) = 0$ for $n \geq 3$. Now we claim that $d_2(a)$ is a generator of $E_2^{2,0} = \mathbb{F}c$: consider the projection $p : S^1 \times X \to S^1$ on the first factor. This map is S^1-equivariant, so we get a map of fibrations

$$\begin{tikzcd} S^1 \times X \arrow[r] \arrow[d, hook] & S^1 \arrow[d, hook] \\ X \arrow[r] \arrow[d] & * \arrow[d] \\ BS^1 \arrow[r] & BS^1 \end{tikzcd}$$

The claim now follows by comparing the spectral sequences of the right and left fibration. ☐

Now let X be a S^1-space. The action $\theta : S^1 \times X \to X$ induces an operator $\Delta : H_*(X;\mathbb{Z}) \to H_{*+1}(X;\mathbb{Z})$ by the composition

$$H_*(X) \longrightarrow H_*(S^1) \otimes H_*(X) \xrightarrow{\times} H_*(S^1 \times X) \xrightarrow{\theta_*} H_*(X)$$

where the first map take a class $x \in H_*(X)$ and send it to $[S^1] \otimes x$. See the paper of E. Getzler [20] or Section 7.5 for further details about Δ.

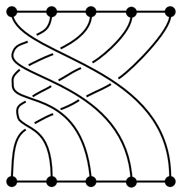
$\delta =$ 

Figure 6.1: The braid δ.

Proposition 6.8. *Let X be a topological space of finite type equipped with an S^1 action. Fix $\mathbb{F}$ a field of coefficients for (co)homology. Then the differential d^2 of the second page of the homological spectral sequence associated to $X \hookrightarrow X_{S^1} \to BS^1$ is given by*

$$d^2(x \otimes y_{2i}) = \begin{cases} 0 \text{ if } i = 0 \\ \Delta(x) \otimes y_{2i-2} \text{ otherwise} \end{cases}$$

where y_{2i} is the generator of $H_{2i}(BS^1; \mathbb{F})$.

Proof. Consider the map of fibrations

$$\begin{tikzcd} S^1 \times X \arrow[r, "\theta"] \arrow[d, hook] & X \arrow[d, hook] \\ (S^1 \times X)_{S^1} \arrow[r] \arrow[d] & X_{S^1} \arrow[d] \\ BS^1 \arrow[r] & BS^1 \end{tikzcd}$$

The statement follows combining the definition of Δ and the formula for d^2 given in Proposition 6.7. $\square$

We end this Section observing that $(\mathcal{M}_{0,n+1})_{\Sigma_n} \simeq C_n(\mathbb{C})_{S^1}$ is the classifying space for the quotient of the braid group by its center: let B_n be the Braid group on n-strands and let us denote by σ_i the i-th generator of B_n. It is well known that the center of B_n is an infinite cyclic group generated by δ^2, where $\delta \in B_n$ is the element

$$\delta := \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1)\dots(\sigma_{n-1}\sigma_{n-2}\dots\sigma_1)$$

See Figure 6.1 for a picture.

Remark 6.9. The center of the Pure braid group PB_n is an infinite cyclic group as well, generated by δ^2.

Proposition 6.10. *The homotopy quotient $C_n(\mathbb{C})_{S^1}$ is the classifying space for the group $B_n/Z(B_n)$. Similarly, $F_n(\mathbb{C})_{S^1}$ is the classifying space for $PB_n/Z(PB_n)$.*

Proof. Consider the long exact sequence for homotopy groups associated to the fibration $S^1 \hookrightarrow ES^1 \times C_n(\mathbb{C}) \to C_n(\mathbb{C})_{S^1}$. To get the result just observe that the map $i_* : \pi_1(S^1) \to \pi_1(ES^1 \times C_n(\mathbb{C})) \cong \pi_1(C_n(\mathbb{C}))$ induced by the inclusion of a fiber sends the generator of $\pi_1(S^1)$ to δ^2. The case of the ordered configurations is completely analogous. $\square$

Remark 6.11. If $n > 1$ the S^1-action on $F_n(\mathbb{C})$ is free, therefore the homotopy quotient is homotopy equivalent to the strict quotient $F_n(\mathbb{C})/S^1$. This space is in turn homotopy equivalent to $\mathcal{M}_{0,n+1}$, as we already observed.

We can summarize these observations in the following table:

Group	Models for the classifying space
PB_n	$F_n(\mathbb{C})$
B_n	$C_n(\mathbb{C})$
$PB_n/Z(PB_n)$	$F_n(\mathbb{C})_{S^1}$ $\quad$ $\mathcal{M}_{0,n+1}$
$B_n/Z(B_n)$	$C_n(\mathbb{C})_{S^1}$ $\quad$ $(\mathcal{M}_{0,n+1})_{\Sigma_n}$

Chapter 7

Operations for even degree classes

In this chapter we compute $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$ for any $n\in\mathbb{N}$ and any prime number p. As explained in Section 6.2 this computation can be done by looking at the Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \longrightarrow BS^1$$

instead of

$$\mathcal{M}_{0,n+1} \hookrightarrow (\mathcal{M}_{0,n+1})_{\Sigma_n} \longrightarrow B\Sigma_n$$

Indeed the homotopy quotients $(\mathcal{M}_{0,n+1})_{\Sigma_n}$ and $C_n(\mathbb{C})_{S^1}$ are homotopy equivalent, so their homology is the same. The main results are the following (Theorem 7.16 and Theorem 7.19):

- Let $n\in\mathbb{N}$ and p be a prime. If $n=0,1 \mod p$ we have $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p) = H_*(C_n(\mathbb{C});\mathbb{F}_p)\otimes H_*(BS^1;\mathbb{F}_p)$.
- Otherwise $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ is isomorphic to the quotient of $H_*(C_n(\mathbb{C});\mathbb{F}_p)$ by the image of Δ.

These results will be proved using techniques from equivariant cohomology, such as the Localization Theorem. Here is the outline of this chapter:

Section 7.1 contains some classical results from equivariant cohomology and configuration spaces.

Section 7.2 contains the computation of $H_*^{\mathbb{Z}/p}(C_n(\mathbb{C});\mathbb{F}_p)$ when $n=0,1 \mod p$ (Theorem 7.8).

Section 7.3 contains the computation of $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ when $n=0,1 \mod p$ (Theorem 7.16).

Section 7.4 contains the computation of $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ when $n\neq 0,1 \mod p$ (Theorem 7.19).

Section 7.5 recollects some properties of the BV-operator Δ that are used in the prevoius sections.

7.1 Preliminares

In this section we recall some facts that will be useful to compute $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$.

7.1.1 Equivariant cohomology

In what follows we recall some basic theorems about equivariant cohomology. We refer to [16] and [8] for further details. Let $G = \mathbb{Z}/n$ or S^1 and M be an abelian group which we use as coefficients for (co)homology. We also suppose that X is a finite dimensional G-complex of finite orbit type. Let $c \in H^2(BG;\mathbb{Z})$ be a generator, and consider the multiplicative subset $S := \{1, c, c^2, \dots\}$. Consider the following subspace:

$$FX := \{x \in X \mid \tilde{H}^*(BG_x; M) \neq 0\}$$

where G_x denotes the stabilizer of a point $x \in X$. A crucial result is the so called *Localization Theorem*:

Theorem 7.1 ([16], p.198). *The inclusion $i : FX \to X$ induces an isomorphism*

$$i^* : S^{-1}H_G^*(X;M) \to S^{-1}H_G^*(FX;M)$$

where $S^{-1}H_G^*(X;M)$ is the localization of the $H^*(BG;M)$-module $H_G^*(X;M)$ to the subset S.

A consequence of this Theorem is the following:

Theorem 7.2 ([16], p.199). *Suppose $H^i(X;M) = 0$ for $i > n$. Then the inclusion $i : FX \to X$ induces an isomorphism*

$$H_G^i(X;M) \to H_G^i(FX;M)$$

for any $i > n - dim(G)$ and an epimorphism for $i = n - dim(G)$.

Now let us restrict to the case $G = \mathbb{Z}/p$, with p a prime number. In the following we will use $\mathbb{F}_p$ coefficients for the $\mathbb{Z}/p$-equivariant (co)homology of X.

Theorem 7.3 ([16] p. 200). *Suppose $dim_{\mathbb{F}_p} \bigoplus_{k\in\mathbb{N}} H^k(X)$ is finite and $H^k(X) = 0$ for any $k > n$. Then we have*

$$dim_{\mathbb{F}_p} \bigoplus_{k\in\mathbb{N}} H^k(X^{\mathbb{Z}/p}) \leq dim_{\mathbb{F}_p} \bigoplus_{k\in\mathbb{N}} H^k(X) \tag{7.1}$$

Moreover, the following assertions are equivalent:

1. *Equality holds in (7.1).*
2. *The map induced by the inclusion $i^* : H^*_{\mathbb{Z}/p}(X) \to H^*(X)$ is surjective.*
3. *$dim_{\mathbb{F}_p} H^k_{\mathbb{Z}/p}(X) = dim_{\mathbb{F}_p} \bigoplus_{i\in\mathbb{N}} H^i(X)$ for $k > n$.*
4. *$\mathbb{Z}/p$ acts trivially on $H^*(X)$ and the Serre spectral sequence of $X \hookrightarrow X_{\mathbb{Z}/p} \to B\mathbb{Z}/p$ degenerates at the E_2 page.*

Remark 7.4. The previous Theorem holds as well if we replace $\mathbb{Z}/p$ with S^1 and take $\mathbb{Q}$ as field of coefficients. There is also a relative version of the Theorem, for more details see [16].

7.1.2 Labelled configuration spaces

Definition 7.1 (Bödigheimer, [6]). Let M be a manifold and $(X, *)$ be a based CW-complex, not necessarily connected. The space of configurations in M with labels in X is defined as

$$C(M; X) := \bigsqcup_{n \in \mathbb{N}} F_n(M) \times_{\Sigma_n} X^n / \sim$$

where $(p_1, \ldots, p_n; x_1, \ldots, x_n) \sim (p_1, \ldots, \hat{p}_i, \ldots, p_n; x_1, \ldots, \hat{x}_i, \ldots, x_n)$ if $x_i = *$.

Example. $C(\mathbb{C}; S^0)$ is just the disjoint union $\bigsqcup_{n \in \mathbb{N}} C_n(\mathbb{C})$. To ease the notation we sometimes abbreviate $C(\mathbb{C}; S^0)$ by $C(\mathbb{C})$.

When $M = \mathbb{R}^n$ the homology of $C(\mathbb{R}^n; X)$ is known, thanks to the work of Cohen [12]. The idea is the following: $C(\mathbb{R}^n; X)$ is homotopy equivalent to the free $\mathcal{D}_n$-algebra on X. Therefore $H_*(C(\mathbb{R}^n; X); \mathbb{F}_p)$ can be described as a functor of $H_*(X; \mathbb{F}_p)$. For the purpose of this work it is enough to recall the results in the case $n = 2$.

Definition 7.2. Let p be a prime number. Fix a basis $\mathcal{B}$ of $H_*(X; \mathbb{F}_p)/[*]$, where $[*] \in H_0(X; \mathbb{F}_p)$ is the class of the base point. We define a **basic bracket of weight k** inductively as follows:

- A basic bracket of weight 1 is just an element $a \in \mathcal{B}$. Its degree is by definition the homological degree of a. Observe that any class of $H_*(X; \mathbb{F}_p)$ can be seen as a class in $H_*(C(\mathbb{C}; X); \mathbb{F}_p)$.

- By induction assume that the basic brackets of weight j have been defined and equipped with a total ordering compatible with weight for $j < k$. Then a basic bracket of weight k is a homology class $[a, b] \in H_*(C(\mathbb{C}; X); \mathbb{F}_p)$, where $[-, -]$ is the Browder bracket and a, b are basic brackets such that:

  1. $weight(a) + weight(b) = k$.
  2. $a < b$ and if $b = [c, d]$ then $c \leq a$.

  The degree of $[a, b]$ is by definition $deg(a) + deg(b) + 1$.

In the case $p \neq 2$ we also include as basic brackets classes of the form $[a, a]$, where a is a basic bracket of even degree.

Theorem 7.5 (Cohen, [12]). *Let p be any prime, and $Q : H_q(C(\mathbb{C}; X); \mathbb{F}_p) \to H_{pq+p-1}(C(\mathbb{C}; X); \mathbb{F}_p)$ be the first Dyer-Lashof operation (when p is odd it acts only on classes of odd degree q). Then $H_*(C(\mathbb{C}; X); \mathbb{F}_p)$ has the following form:*

$p = 2$: *$H_*(C(\mathbb{C}; X); \mathbb{F}_2)$ is the free graded commutative algebra on classes $Q^i(x)$, where Q^i denotes the i-th iteration of Q and x is a basic bracket.*

$p \neq 2$: *$H_*(C(\mathbb{C}; X); \mathbb{F}_p)$ is the free graded commutative algebra on classes $Q^i(x)$ and $\beta Q^i(x)$, where β is the Bockstein operator, Q^i denotes the i-th iteration of Q and x is a basic bracket of odd degree.*

Corollary 7.6. *Let $C(\mathbb{C}) := \coprod_{n\in\mathbb{N}} C_n(\mathbb{C})$ be the disjoint union of all unordered configuration spaces of points in the complex plane. Then if p is an odd prime*

$$H_*(C(\mathbb{C});\mathbb{F}_p) = \mathbb{F}_p[\iota, \beta Q[\iota,\iota], \beta Q^2[\iota,\iota], \dots] \otimes \Lambda[[\iota,\iota], Q[\iota,\iota], Q^2[\iota,\iota], \dots]$$

where ι is the generator of $H_0(C_1(\mathbb{C}))$. When $p = 2$ we have

$$H_*(C(\mathbb{C});\mathbb{F}_2) = \mathbb{F}_2[\iota, Q\iota, Q^2\iota, \dots]$$

Remark 7.7. In what follows we will adopt the following notation:

$$u := [\iota,\iota]$$
$$\beta_i := \beta Q^i[\iota,\iota]$$
$$\alpha_i := Q^i[\iota,\iota]$$

Moreover, the following table will be useful in the next section:

Homology class	Number of points	Degree
ι	1	0
u	2	1
α_i	$2p^i$	$2p^i - 1$
β_i	$2p^i$	$2p^i - 2$

7.2 Computation of $H_*^{\mathbb{Z}/p}(C_n(\mathbb{C});\mathbb{F}_p)$ when $n = 0, 1 \bmod p$

In this section we compute $H_*^{\mathbb{Z}/p}(C_n(\mathbb{C});\mathbb{F}_p)$ when p is a prime that divides n or $n-1$. Here we are considering $\mathbb{Z}/p$ as the subgroup of p-th roots of unity inside S^1, so its generator acts on $C_n(\mathbb{C})$ by the rotation of $2\pi/p$. Some of the statements of this section are stated with the assumption that p is an odd prime, but similar statements holds for $p = 2$ with minor modifications. A different proof for the case $p = 2$ will be included in Chapter 8 (see Corollary 8.10).

Theorem 7.8. *Let p be a prime, $n \in \mathbb{N}$ such that $p|n$ or $p|n-1$. Then*

$$H_*^{\mathbb{Z}/p}(C_n(\mathbb{C});\mathbb{F}_p) \cong H_*(C_n(\mathbb{C});\mathbb{F}_p) \otimes H_*(B\mathbb{Z}/p;\mathbb{F}_p)$$

Proof. Consider the homological spectral sequence associated to the fibration

$$C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{\mathbb{Z}/p} \to B\mathbb{Z}/p$$

Since $\mathbb{Z}/p$ acts by rotations on $C_n(\mathbb{C})$, the monodromy action is trivial. Therefore

$$E^2_{p,q} \cong H_p(C_n(\mathbb{C});\mathbb{F}_p) \otimes H_q(B\mathbb{Z}/p;\mathbb{F}_p)$$

We will see that

$$dim_{\mathbb{F}_p} \bigoplus_{k\in\mathbb{N}} H^k(C_n(\mathbb{C})^{\mathbb{Z}/p}) = dim_{\mathbb{F}_p} \bigoplus_{k\in\mathbb{N}} H^k(C_n(\mathbb{C})) \tag{7.2}$$

So the result follows applying Theorem 7.3. ☐

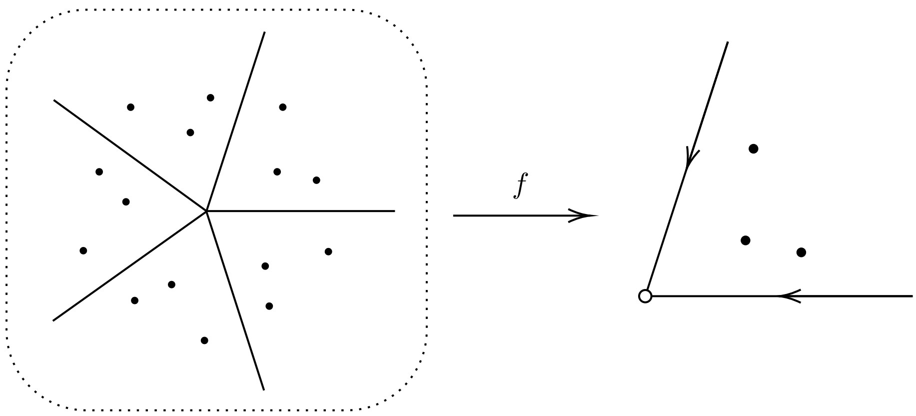


Figure 7.1: This picture shows how the homeomorphism $f : C_{15}(\mathbb{C})^{\mathbb{Z}/5} \to C_3(\mathbb{C}^*)$ works.

Now we focus in proving the equality 7.2. For the moment we restrict to the case $n = pq$, then we will extend the result to the case $n = pq + 1$.

Lemma 7.9. *Let $n = pq$ or $n = pq + 1$. Then the fixed points $C_n(\mathbb{C})^{\mathbb{Z}/p}$ are homeomorphic to $C_q(\mathbb{C}^*)$.*

Proof. Let us prove the statement when $n = pq$, the other case in similar. Let us denote by $\zeta := e^{i2\pi/p}$ the generator of $\mathbb{Z}/p$. Consider the quotient space

$$H := \{z \in \mathbb{C}^* \mid arg(z) \in [0, 2\pi/p]\}/\sim$$

where $\sim$ identifies a point $z \in \{z \in \mathbb{C}^* \mid arg(z) = 0\}$ with ζz. So H is homeomorphic to $\mathbb{C}^*$. Now observe that any configuration in $C_n(\mathbb{C})^{\mathbb{Z}/p}$ is of the form $\{z_1, \zeta z_1, \dots \zeta^{p-1} z_1, \dots, z_q, \zeta z_q, \dots \zeta^{p-1} z_q\}$, where $z_1, \dots, z_q$ are distinct points in $\{z \in \mathbb{C}^* \mid arg(z) \in [0, 2\pi/p)\}$. The association $\{z_1, \zeta z_1, \dots \zeta^{p-1} z_1, \dots, z_q, \zeta z_q, \dots \zeta^{p-1} z_q\} \mapsto \{z_1, \dots, z_q\}$ defines a continuous map

$$f : C_n(\mathbb{C})^{\mathbb{Z}/p} \to C_q(H)$$

Conversely, if we have a configuration $\{z_1, \dots, z_q\} \in C_q(H)$, we can produce a configuration of $C_n(\mathbb{C})^{\mathbb{Z}/p}$ by taking the $\mathbb{Z}/p$-orbits of every point. More precisely, the association $\{z_1, \dots, z_q\} \mapsto \{z_1, \zeta z_1, \dots \zeta^{p-1} z_1, \dots, z_q, \zeta z_q, \dots \zeta^{p-1} z_q\}$ defines a continuous function $C_q(H) \to C_n(\mathbb{C})^{\mathbb{Z}/p}$, which is the inverse of f. See Figure 7.1 for a pictorial description of f. □

Remark 7.10. $C_q(\mathbb{C}^*)$ is homotopy equivalent to the configuration space of q black particles and one white particle in the plane. Therefore $H_*(C_q(\mathbb{C}^*);\mathbb{F}_p)$ will be the subspace of $H_*(C(\mathbb{C}, S^0 \vee S^0);\mathbb{F}_p)$ spanned by those classes that involve only q black particles and one white particle. Let us denote by a (resp. b) the class in $H_0(S^0 \vee S^0)$ which represent a black particle (resp. a white particle). Then $H_*(C(\mathbb{C}, S^0 \vee S^0);\mathbb{F}_p)$ can be computed using Theorem 7.5. The Lemma stated below just identifies explicitly $H_*(C_q(\mathbb{C}^*);\mathbb{F}_p)$ as a subspace of $H_*(C(\mathbb{C}, S^0 \vee S^0);\mathbb{F}_p)$.

Lemma 7.11. *Let p be any prime and use $\mathbb{F}_p$ coefficients for homology. Consider the space $C(\mathbb{C}^*) := \coprod_{n\in\mathbb{N}} C_n(\mathbb{C}^*)$. Then we have:*

$$H_*(C(\mathbb{C}^*)) = b \cdot H_*(C(\mathbb{C})) + [a,b] \cdot H_*(C(\mathbb{C})) + [a,[a,b]] \cdot H_*(C(\mathbb{C})) + \dots$$

Proof. We use Theorem 7.5 to compute $H_*(C(\mathbb{C}; S^0 \vee S^0); \mathbb{F}_p)$ and then we identify $H_*(C(\mathbb{C}^*); \mathbb{F}_p)$ as the subspace spanned by classes involving exactly one white particle. Let us denote by a (resp. b) the class in $H_0(S^0 \vee S^0)$ which represent a black particle (resp. a white particle). The basic brackets involving only one white particle turns out to be b, $[a,b]$, $[a,[a,b]]$, $[a,[a,[a,b]]]$ etc. Now if x is one of these brackets, $Q(x)$ will be a class containing p white particles. So the classes of $H_*(C(\mathbb{C}^*); \mathbb{F}_p)$ are of the form $x \cdot y$, where $y \in H_*(C(\mathbb{C}); \mathbb{F}_p)$ and x is one of the basic bracket listed above, and this proves the statement. $\square$

Corollary 7.12. *Let q be any natural number and take $\mathbb{F}_p$-coefficients for homology, for p a fixed prime. Then*

$$H_*(C_q(\mathbb{C}^*)) = b \cdot H_*(C_q(\mathbb{C})) + [a,b] \cdot H_{*-1}(C_{q-1}(\mathbb{C})) + \dots + [a,[a,\dots[a,b]]] \cdot H_{*-q}(C_0(\mathbb{C}))$$

Definition 7.3. Let us denote by $d(q)$ the dimension of $H_*(C_q(\mathbb{C}); \mathbb{F}_p)$ as $\mathbb{F}_p$-vector space.

Lemma 7.9 and Corollary 7.12 allows us to rewrite equation (7.2) in the following way:

$$d(pq) = d(q) + d(q-1) + \dots + d(1) + d(0) \tag{7.3}$$

Before going into the details of the proof of this equation, let us look at an example. The general proof will be a generalization of the methods we are going to use in this specific case.

Example. Let $p = q = 3$. To prove equation 7.3 one can proceed by induction on q, so by inductive hypothesis it suffices to show that $d(pq) = d(q) + d(p(q-1))$. Let us verify this equality in this specific case. $H_*(C_3(\mathbb{C}); \mathbb{F}_3)$ is generated by ι^3 and ιu, so $d(3) = 2$. The generators of $H_*(C_9(\mathbb{C}); \mathbb{F}_3)$ are listed in the left table, while those of $H_*(C_6(\mathbb{C}); \mathbb{F}_3)$ are in the right one:

Homology class	Degree
ι^9	0
$\iota^7 u$	1
$\iota^3 \beta_1$	4
$\iota u \beta_1$	5
$\iota^3 \alpha_1$	5
$\iota u \alpha_1$	6

Homology class	Degree
ι^6	0
$\iota^4 u$	1
β_1	4
α_1	5

Therefore $d(9) = 6 = d(6) + d(3)$ and the equality holds. A more conceptual proof of this equality is the following: observe that four classes of $H_*(C_9(\mathbb{C}); \mathbb{F}_3)$ are obtained just multiplying by ι^3 the generators of $H_*(C_6(\mathbb{C}); \mathbb{F}_3)$. The remaining generators are $\iota u\beta_1$ and $\iota u\alpha_1$ and they can be obtained from those of $H_*(C_3(\mathbb{C});\mathbb{F}_3)$ by the following change of variables:

$$u \mapsto \alpha_1$$
$$\iota^{2l+1} \mapsto \iota^{p-2}u\beta_1^l$$

This procedure can be useed to prove Equality 7.3 in general, as we will see in Proposition 7.13.

Proposition 7.13. *For any p prime and $q \in \mathbb{N}$ we have $d(pq) = d(q) + d(q-1) + \cdots + d(1) + d(0)$.*

Proof. We proceed by induction on q. If $q = 0$ there is nothing to prove. Let us suppose that the equation holds until q, let us prove it for $q + 1$: by induction we have

$$\sum_{i=0}^{q+1} d(i) = d(pq) + d(q+1) \tag{7.4}$$

Therefore it suffices to show that $d(pq) + d(q+1) = d(p(q+1))$. We will do this by constructing an explicit isomorphism of vector spaces between $H_*(C_{pq}(\mathbb{C})) \oplus H_*(C_{q+1}(\mathbb{C}))$ and $H_*(C_{p(q+1)}(\mathbb{C}))$. Let us suppose p is an odd prime (the case $p = 2$ is similar). We refer to Remark 7.7 for the notation we are going to use. Consider the linear map

$$f : H_*(C_{pq}(\mathbb{C})) \oplus H_*(C_{q+1}(\mathbb{C})) \to H_*(C_{p(q+1)}(\mathbb{C}))$$

defined on the basis monomials as follows:

- If x is a monomial of $H_*(C_{pq}(\mathbb{C}))$, then $f(x) = \iota^p x$.
- If $x = \iota^k u^\epsilon \alpha_{i_1} \cdots \alpha_{i_m} \beta_{j_1}^{a_1} \cdots \beta_{j_n}^{a_n}$ is a monomial of $H_*(C_{q+1}(\mathbb{C}))$, with $\epsilon = 0, 1$, then

  $$f(x) := \begin{cases} \beta_1^l \alpha_1^\epsilon \alpha_{i_1+1} \cdots \alpha_{i_m+1} \beta_{j_1+1}^{a_1} \cdots \beta_{j_n+1}^{a_n} & \text{if } k = 2l \\ (\beta_1^l u \iota^{p-2}) \alpha_1^\epsilon \alpha_{i_1+1} \cdots \alpha_{i_m+1} \beta_{j_1+1}^{a_1} \cdots \beta_{j_n+1}^{a_n} & \text{if } k = 2l+1 \end{cases}$$

  In other words, $f(x)$ is the monomial of $H_*(C_{p(q+1)}(\mathbb{C}))$ obtained from x by the following substitution of variables:

  (a) $\alpha_i \mapsto \alpha_{i+1}$
  (b) $\beta_i \mapsto \beta_{i+1}$
  (c) $u \mapsto \alpha_1$
  (d) $\iota^k \mapsto \begin{cases} \beta_1^l \text{ if } k = 2l \\ \beta_1^l u \iota^{p-2} \text{ if } k = 2l+1 \end{cases}$

We claim that f is an isomorphim of vector spaces:

- f is well defined: clearly if we multiply by ι^p a monomial of $H_*(C_{pq}(\mathbb{C}))$ we obtain a monomial of $H_*(C_{p(q+1)}(\mathbb{C}))$. So let us pick an element
$$x = \iota^k u^\epsilon \alpha_{i_1} \cdots \alpha_{i_m} \beta_{j_1}^{a_1} \cdots \beta_{j_n}^{a_n} \in H_*(C_{q+1}(\mathbb{C}))$$
We prove that $f(x) \in H_*(C_{p(q+1)}(\mathbb{C}))$: by hypothesis we know that
$$q+1 = k + 2\epsilon + \sum_{r=1}^{m} 2p^{i_r} + \sum_{s=1}^{n} a_s 2p^{j_s}$$
If $k = 2l$, then $f(x)$ is a class which involves the following number of points:
$$2lp + 2p\epsilon + \sum_{r=1}^{m} 2p^{i_r+1} + \sum_{s=1}^{n} a_s 2p^{j_s+1} = p(q+1)$$
proving our claim. If $k = 2l+1$ the computation is analogous.

- f is injective: f restricted to the subspaces $H_*(C_{pq}(\mathbb{C}))$ and $H_*(C_{q+1}(\mathbb{C}))$ is injective by definition. Moreover the intersection of $f(H_*(C_{pq}(\mathbb{C})))$ and $f(H_*(C_{q+1}(\mathbb{C})))$ contains only the zero element: the elements of $f(H_*(C_{pq}(\mathbb{C})))$ are sum of monomials which contain ι^p, while $f(H_*(C_{pq}(\mathbb{C})))$ is spacced by monomials not containing ι^p.

- f is surjective: to achieve this we need to prove that if x is a basic monomial in $H_*(C_{p(q+1)}(\mathbb{C}))$, then it is of the following forms:
  1. $x = \iota^p y$ for some $y \in H_*(C_{pq}(\mathbb{C}))$
  2. x contains only the letters from $\{\alpha_i, \beta_i\}_{i \geq 1}$. In other words, x does not contain ι and u.
  3. $x = \iota^{p-2} u y$ with y a monomial in $\{\alpha_i, \beta_i\}_{i \geq 1}$.

  But this is exactly the content of the following Proposition 7.14, therefore f is surjective.

□

Proposition 7.14. *Let p be an odd prime, $n \in \mathbb{N}$ and $n = k$ mod p. If $x \in H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ is a basic monomial, then it has one of the following forms:*

1. $x = \iota^p y$ *for some* $y \in H_*(C_{n-p}(\mathbb{C}))$
2. $x = \iota^k \alpha_{i_1} \cdots \alpha_{i_m} \beta_{j_1}^{a_1} \cdots \beta_{j_s}^{a_s}$
3. $x = (\iota^{k-2} u) \alpha_{i_1} \cdots \alpha_{i_m} \beta_{j_1}^{a_1} \cdots \beta_{j_s}^{a_s}$

If $k = 0$ (resp. $k = 1$) replace the exponent $k-2$ in point (c) with the corresponding class in $\mathbb{Z}/p$, i.e with $p-2$ (resp. $p-1$) to get the correct statement.

Proof. We proceed by induction on n:

- If $n = 1$ the only class in $H_*(C_1(\mathbb{C});\mathbb{F}_p)$ is ι, so the statement is true. If $n \in \{2,\ldots,p\}$ then $H_*(C_n(\mathbb{C});\mathbb{F}_p)$ contains only two classes, ι^n and $\iota^{n-2}u$, and the statement follows.

- Assume the result is true until n, let us prove it for $n+p$: let
$$x = \iota^l u^\epsilon \alpha_{i_1}\cdots\alpha_{i_m}\beta_{j_1}^{a_1}\cdots\beta_{j_s}^{a_s} \in H_*(C_{n+p}(\mathbb{C}))$$
If $l \geq p$ then we are in the first case. So let us restrict to the case where $l \leq p-1$. If x contains some β_i then $x = \beta_i \cdot x'$ with $x' \in H_*(C_{n+p-2p^i}(\mathbb{C}))$. Since $i \geq 1$ we have $n+p-2p^i \leq n$. Moreover $n+p-2p^i = k$ mod p therefore we can use the inductive hypothesis for x' and get the result. We can proceed in the same way when x contains one of the variables $\{\alpha_i\}_{i\geq 1}$. The only case which is not yet considered is when $x = \iota^l u^\epsilon$. In this case $n+p$ must be equal to $l+2\epsilon$. Therefore $l = n+p-2\epsilon \leq p-1$ if and only if $n \leq 2\epsilon - 1 \leq 1$ which is not the case we are considering.

□

This concludes the proof of Theorem 7.8 in the case $p|n$. It remains to prove the statement for $n = pq+1$. This is equivalent to show that
$$d(pq+1) = d(q) + d(q-1) + \cdots + d(0) = d(pq)$$
Where the last equality holds for Proposition 7.13. But this is an easy consequence of Proposition 7.14:

Corollary 7.15. *If p divides n, then $H_*(C_n(\mathbb{C});\mathbb{F}_p)$ and $H_*(C_{n+1}(\mathbb{C});\mathbb{F}_p)$ have the same dimension as $\mathbb{F}_p$-vector spaces.*

Proof. We observe that multiplication by ι is an isomorphism between $H_*(C_n(\mathbb{C});\mathbb{F}_p)$ and $H_*(C_{n+1}(\mathbb{C});\mathbb{F}_p)$. Clearly it is injective. By Proposition 7.14 we have that each monomial in $H_*(C_{n+1}(\mathbb{C});\mathbb{F}_p)$ contains at least one ι, and this proves the surjectivity. □

7.3 Computation of $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ when $n = 0,1$ mod p

In this section we compute $H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ when $n = 0,1$ mod p.

Theorem 7.16. *Let p be a prime, $n \in \mathbb{N}$ such that $n = 0,1$ mod p. Then*
$$H_*^{S^1}(C_n(\mathbb{C});\mathbb{F}_p) \cong H_*(C_n(\mathbb{C});\mathbb{F}_p) \otimes H_*(BS^1;\mathbb{F}_p)$$

Proof. Consider the map of fibrations
$$\begin{tikzcd}
C_n(\mathbb{C}) \arrow[r] \arrow[d, hook] & C_n(\mathbb{C}) \arrow[d, hook] \\
C_n(\mathbb{C})_{\mathbb{Z}/p} \arrow[r] \arrow[d] & C_n(\mathbb{C})_{S^1} \arrow[d] \\
B\mathbb{Z}/p \arrow[r] & BS^1
\end{tikzcd}$$

Then observe that with coefficients in $\mathbb{F}_p$ this map induces a surjection between the E^2 pages of the homological spectral sequences. The result now follows from Theorem 7.8. □

Remark 7.17. It would be interesting to understand $H^*_{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ as a ring. However there are some non-trivial extension problems to solve. For example, if we put $n = p = 2$ the homotopy quotient $C_2(\mathbb{C})_{S^1}$ is a model for $B(\mathbb{Z}/2)$, therefore

$$H^*_{S^1}(C_2(\mathbb{C});\mathbb{F}_2) = \mathbb{F}_2[x]$$

where x is a variable of degree one. Theorem 7.16 tells us that

$$H^*_{S^1}(C_2(\mathbb{C});\mathbb{F}_2) \cong \frac{\mathbb{F}_2[x]}{(x^2)} \otimes \mathbb{F}_2[c]$$

as $\mathbb{F}_2[c]$-module, where x (respectively c) is a generator of $H^1(C_2(\mathbb{C});\mathbb{F}_2)$ (respectively of $H^2(BS^1;\mathbb{F}_2)$). To get the correct ring structure we need to impose the relation $x^2 = c$.

Remark 7.18. As we already said, $H^{S^1}_*(C_n(\mathbb{C});\mathbb{F}_p) \cong H^{\Sigma_n}_*(\mathcal{M}_{0,n+1};\mathbb{F}_p)$. Since the inclusion $\mathcal{M}_{0,n+1} \to \overline{\mathcal{M}}_{0,n+1}$ is Σ_n-equivariant we get a map of fibrations

$$\begin{tikzcd}
\mathcal{M}_{0,n+1} \arrow[r] \arrow[d, hook] & \overline{\mathcal{M}}_{0,n+1} \arrow[d, hook] \\
(\mathcal{M}_{0,n+1})_{\Sigma_n} \arrow[r] \arrow[d] & (\overline{\mathcal{M}}_{0,n+1})_{\Sigma_n} \arrow[d] \\
B\Sigma_n \arrow[r] & B\Sigma_n
\end{tikzcd}$$

It would be interesting if this map of fibrations can be used to understand the Σ_n-equivariant (co)homology of $\overline{\mathcal{M}}_{0,n+1}$ with $\mathbb{F}_p$ coefficients. Such a computation would refine the results of D. Kim and N. Wilkins presented in [31].

7.4 Computation of $H^{S^1}_*(C_n(\mathbb{C});\mathbb{F}_p)$ when $n \neq 0,1 \bmod p$

In this section we compute $H^{S^1}_*(C_n(\mathbb{C});\mathbb{F}_p)$ when $n \neq 0,1 \bmod p$. Of course this request is not empty only if p is an odd prime. The main result is the following:

Theorem 7.19. *Let p be an odd prime, $n \in \mathbb{N}$ such that $n \neq 0,1$ mod p. Then*

$$H^{S^1}_*(C_n(\mathbb{C});\mathbb{F}_p) \cong coker(\Delta)$$

where $\Delta : H_*(C_n(\mathbb{C});\mathbb{F}_p) \to H_{*+1}(C_n(\mathbb{C});\mathbb{F}_p)$ is the BV-operator.

Proof. Consider the homological Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}) \to C_n(\mathbb{C})_{S^1} \to BS^1$$

By Proposition 6.8 the second page is given by

$$E^2_{i,j} = H_i(C_n(\mathbb{C})) \otimes H_j(BS^1) \qquad d^2(x \otimes y_{2j}) = \begin{cases} \Delta(x) \otimes y_{2j-2} \text{ if } j \geq 1 \\ 0 \text{ if } j = 0 \end{cases}$$

where y_{2j} is the generator of $H_{2j}(BS^1)$. By Theorem 7.5 any class of $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ is a product of the variables ι, $[\iota, \iota]$, $Q^i[\iota, \iota]$ and $\beta Q^i[\iota, \iota]$, $i \geq 1$. In particular any class is of the form $\iota^k[\iota, \iota]^l x$, where $k \in \mathbb{N}$, $l = 0, 1$ and x is a monomial which contains only the letters $\{Q^i[\iota, \iota], \beta Q^i[\iota, \iota]\}_{i \geq 1}$. We claim that the operator Δ acts as follows:

$$\Delta(\iota^k x) = k(k-1)\iota^{k-2}[\iota, \iota]x \qquad \Delta(\iota^k[\iota, \iota]x) = 0 \tag{7.5}$$

A detailed proof of these formulas will be given in Section 7.5 (Proposition 7.25). Observe that if we have a monomial of the form $\iota^k x$, $x \in H_*(C_m(\mathbb{C}); \mathbb{F}_p)$, then $n = k + m = k \bmod p$ (x contains only letters from $\{Q^i[\iota, \iota], \beta Q^i[\iota, \iota]\}_{i \geq 1}$, so the number of points m is divisible by p). Therefore $k \neq 0, 1 \bmod p$ and the term $k(k-1)\iota^{k-2}[\iota, \iota]x$ is never zero. Now we can conclude: the formula 7.5 shows that the third page of the Serre spectral sequence looks as follows: $E^3_{i,j} = 0$ for any $j \geq 1$, while the first column $E^3_{0,*}$ may contain some non zero elements. To be more precise, $E^3_{0,*}$ is the quotient of $E^2_{0,*} = H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ by the image of the differential. Since the differential of the second page is given by Δ, we get that

$$E^3_{0,*} \cong coker(\Delta)$$

The third page contains only the first column, so the spectral sequence degenerates and we get the statement. □

Remark 7.20. We can explicitly describe $coker(\Delta)$: a basis of this vector space is given by (the image of) classes in $H_*(C_n(\mathbb{C}); \mathbb{F}_p)$ which do not contain the bracket $[\iota, \iota]$ (Equation 7.5).

7.5 Auxiliary computations

In this section we recollect some easy algebraic computations which are relevant for Section 7.4. In particular we prove the formulas 7.5 in Proposition 7.25. In what follows p will be an odd prime. We begin by recalling some basic properties of the bracket and Δ; to ease the notation we will write $(-1)^x$ instead of $(-1)^{deg(x)}$, where x is an element in some graded vector space. We refer to [12] and [20] for further details.

1. Graded anticommutativity: $[x, y] = (-1)^{xy+x+y}[y, x]$.
2. Jacobi relation: $[x, [y, z]] = [[x, y], z] - (-1)^{y+x+xy}[y, [x, z]]$.
3. The bracket is a derivation: $[x, yz] = [x, y]z + (-1)^{y+yx}y[x, z]$.
4. $\Delta(xy) = \Delta(x)y + (-1)^x x\Delta(y) + (-1)^x[x, y]$.

5. $\Delta[x,y] = [\Delta x, y] + (-1)^{x+1}[x,\Delta y]$.
6. $[x, Qy] = ad^p(y)(x)$, where $ad(y)(x) := [x,y]$ and for any $n \in \mathbb{N}$ we define $ad^n(y)(x) := ad(y)(ad^{n-1}(y)(x))$. For example, $ad^2(y)(x) = [[x,y],y]$, $ad^3(y)(x) = [[[x,y],y],y]$ and so on.
7. $[x, \beta Qy] = [x, ad^{p-1}(y)(\beta y)]$

Lemma 7.21. *Let p be an odd prime, $k \in \mathbb{N}$. Then $\Delta(\iota^k) = k(k-1)\iota^{k-2}[\iota,\iota]$.*

Proof. It suffices to proceed by induction: $\Delta(\iota) = 0$ since ι is the top class of $H_*(C_1(\mathbb{C}));\mathbb{F}_p)$. $\Delta(\iota^2) = [\iota,\iota]$ by equation (4) at the beginning of this section. The general formula follows easily using equation (4). □

Lemma 7.22. *Let $x \in H_*(C(\mathbb{C});\mathbb{F}_p)$ be a monomial containing only the letters $\{Q^i[\iota,\iota], \beta Q^i[\iota,\iota]\}_{i\geq 1}$. Then $\Delta(x) = 0$ and $\Delta(\iota x) = 0$.*

Proof. We prove that $\Delta(x) = 0$, the other case is analogous. Since x contains only the letters $\{Q^i[\iota,\iota], \beta Q^i[\iota,\iota]\}_{i\geq 1}$, it is a class of $H_*(C_n(\mathbb{C});\mathbb{F}_p)$ for some n divisible by p. Theorem 7.16 tells us that the homological Serre spectral sequence associated to

$$C_n(\mathbb{C}) \to C_n(\mathbb{C})_{S^1} \to BS^1$$

degenerates at the second page. But the differential of the second page is given by Δ (Proposition 6.8) so $\Delta(x) = 0$. □

Lemma 7.23. *Let $x \in H_*(C(\mathbb{C});\mathbb{F}_p)$ be a monomial containing only the letters $\{Q^i[\iota,\iota], \beta Q^i[\iota,\iota]\}_{i\geq 1}$. Then $[\iota^k, x] = 0$.*

Proof. Since the bracket is a derivation it is enough to prove that $[\iota, x] = 0$. By Lemma 7.22 $\Delta(\iota x) = 0$, therefore

$$0 = \Delta(\iota x) = \Delta(\iota)x \pm \iota\Delta(x) \pm [\iota, x] = \pm[\iota, x]$$

The last equality holds because $\Delta(\iota) = 0$ (ι is the top class of $H_*(C_1(\mathbb{C});\mathbb{F}_p)$) and $\Delta(x) = 0$ (Lemma 7.22). □

Lemma 7.24. *Let $x \in H_*(C(\mathbb{C}),\mathbb{F}_p)$ be a monomial which contains only the letters $\{Q^i[\iota,\iota], \beta Q^i[\iota,\iota]\}_{i\geq 1}$. Then $[[\iota,\iota],x] = 0$.*

Proof. Since the bracket is a derivation it is enough to show that $[[\iota,\iota], Q^i[\iota,\iota]] = 0$ and that $[[\iota,\iota], \beta Q^i[\iota,\iota]] = 0$. In the first case we proceed by induction: if $i = 1$ we get $[[\iota,\iota], Q[\iota,\iota]] = ad^p([\iota,\iota])([\iota,\iota])$ by property (6), and the latter term is zero by Jacobi. In general we use again property (6) and we get:

$$[[\iota,\iota], Q^i[\iota,\iota]] = ad^p(Q^{i-1}[\iota,\iota])([\iota,\iota]) = 0$$

where the last equality holds by induction. The other formula can be proved as follows: by equation (7) we get

$$[[\iota,\iota], \beta Q^i[\iota,\iota]] = [[\iota,\iota], ad^{p-1}(Q^{i-1}[\iota,\iota])(\beta Q^{i-1}[\iota,\iota])]$$

One of the brackets contained in $ad^{p-1}(Q^{i-1}[\iota,\iota])(\beta Q^{i-1}[\iota,\iota])$ is $[\beta Q^{i-1}[\iota,\iota],Q^{i-1}[\iota,\iota]]$ which we claim is zero:

$$\begin{aligned}0 &= \Delta(\beta Q^{i-1}[\iota,\iota]Q^{i-1}[\iota,\iota])\\ &= \Delta(\beta Q^{i-1}[\iota,\iota])Q^{i-1}[\iota,\iota] \pm \beta Q^{i-1}[\iota,\iota]\Delta(Q^{i-1}[\iota,\iota]) \pm [\beta Q^{i-1}[\iota,\iota],Q^{i-1}[\iota,\iota]]\\ &= \pm[\beta Q^{i-1}[\iota,\iota],Q^{i-1}[\iota,\iota]]\end{aligned}$$

where in the first and third equality we used Lemma 7.22. ☐

Proposition 7.25. *Let $n \in \mathbb{N}$ and suppose $n \neq 0,1$ mod p. Let $\iota^k x$ and $\iota^k[\iota,\iota]x$ be classes is $H_*(C_n(\mathbb{C});\mathbb{F}_p)$, where x be a monomial which contains only the variables $\{Q^i[\iota,\iota],\beta Q^i[\iota,\iota]\}_{i\geq 1}$. Then*

$$\begin{cases}\Delta(\iota^k x) = k(k-1)\iota^{k-2}[\iota,\iota]x\\ \Delta(\iota^k[\iota,\iota]x) = 0\end{cases}$$

Proof. By Lemma 7.21 we have $\Delta(\iota^k) = k(k-1)\iota^{k-2}[\iota,\iota]$. Using property (4) of Δ together with Lemma 7.22 and Lemma 7.23 we get

$$\Delta(\iota^k x) = \Delta(\iota^k)x \pm \iota^k\Delta(x) \pm [\iota^k,x] = k(k-1)\iota^{k-2}[\iota,\iota]$$

and this proves the first part of the statement. Similarly,

$$\Delta(\iota^k[\iota,\iota]x) = \Delta(\iota^k[\iota,\iota])x \pm \iota^k[\iota,\iota]\Delta(x) \pm [\iota^k[\iota,\iota],x]$$

The last term is zero since the bracket is a derivation and $[\iota^k,x] = 0 = [[\iota,\iota],x]$ (Lemma 7.23 and Lemma 7.24). The middle term is zero by Lemma 7.22. The first term is zero as well for the following reason:

$$\Delta(\iota^k[\iota,\iota]) = k(k-1)\iota^{k-2}[\iota,\iota][\iota,\iota] \pm \iota^k\Delta[\iota,\iota] \pm [\iota^k,[\iota,\iota]]$$

The first term is zero since $[\iota,\iota]$ is a variable of odd degree, so it squares to zero. The second term is zero since $[\iota,\iota]$ is the top class of $H_*(C_2(\mathbb{C});\mathbb{F}_p)$. To see that the last term is zero just use the fact that the bracket is a derivation and that $[\iota,[\iota,\iota]] = 0$ (if $p \neq 3$ the Jacobi relation imply $[\iota,[\iota,\iota]] = 0$, while for $p = 3$ this iterated bracket is zero by definition). ☐

Chapter 8

Operations for odd degree classes

As we saw in Section 6.1 the homology operations for odd degree elements in a gravity algebra are governed by $H_*(B_n/Z(B_n); \mathbb{F}_p(\pm 1))$, where p is any fixed prime. In this Chapter we compute this homology (Theorem 8.7). Observe that if $p = 2$ the sign representation is actually the trivial representation, therefore we get an alternative computation of $H_*(B_n/Z(B_n); \mathbb{F}_2)$. The techniques involved comes from the theory of fiberwise configurations spaces, which we will quickly review in Section 8.1. The outline of this chapter is the following:

Section 8.1 is a recollection of known facts about fiberwise configuration spaces.

Section 8.2 contains the proof that $B(B_n/Z(B_n))$ is homotopy equivalent to a fiberwise configuration space (Proposition 8.4).

Section 8.3 is about the computation of $H_*(B_n/Z(B_n); \mathbb{F}_p(\pm 1))$. This will be a consequence of Theorem 8.7.

8.1 Fiberwise configuration spaces

In this section we give an overview of the theory of fiberwise configuration spaces. For further details we refer to [11] and [7]. Let $\lambda : E \to B$ be a fiber bundle with fiber Y. We consider the following space of **(ordered) fiberwise configurations** of points

$$E(\lambda, n) \coloneqq \{(e_1, \ldots, e_n) \in E^n \mid e_i \neq e_j \text{ and } \lambda(e_i) = \lambda(e_j) \text{ if } i \neq j\}$$

The symmetric group Σ_n acts on $E(\lambda, n)$ by permuting the coordinates, so we can also define the space of **unordered fiberwise configurations** as the quotient $E(\lambda, n)/\Sigma_n$. In particular there are fiber bundles

$$F_n(Y) \hookrightarrow E(\lambda, n) \to B \qquad\qquad C_n(Y) \hookrightarrow E(\lambda, n)/\Sigma_n \to B$$

Now let X be a connected CW-complex with base point $*$. We consider the following space of **fiberwise configurations with label in** X

$$E(\lambda; X) \coloneqq \bigsqcup_{n=0}^{\infty} E(\lambda, n) \times_{\Sigma_n} X^n / \sim$$

where $\sim$ is the equivalence relation determined by

$$(e_1, \ldots, e_n) \times (x_1, \ldots, x_n) \sim (e_1, \ldots, \hat{e}_i, \ldots, e_n) \times (x_1, \ldots, \hat{x}_i, \ldots, x_n)$$

when $x_i = *$.

Remark 8.1. If the fiber bundle is given by a constant map $\lambda : E \to \{*\}$ the space $E(\lambda; X)$ is usually denoted by $C(E; X)$. These are just configurations of points on E with label in X.

The spaces $E(\lambda; X)$ are equipped with a natural filtration by the number of points

$$\{*\} \subseteq E_1(\lambda; X) \subseteq E_2(\lambda; X) \subseteq \cdots \subseteq E(\lambda; X)$$

where $E_k(\lambda; X)$ is the subspace

$$E_k(\lambda; X) \coloneqq \bigsqcup_{n=0}^{k} E(\lambda, n) \times_{\Sigma_n} X^n / \sim$$

In the literature it is standard to denote the quotient $E_k(\lambda; X)/E_{k-1}(\lambda; X)$ by $D_k(\lambda; X)$.

Theorem 8.2 ([7])**.** *Let X be a connected CW-complex and $Y \hookrightarrow E \xrightarrow{\lambda} B$ be a fiber bundle. Then $E(\lambda; X)$ is stably equivalent to $\bigvee_{k\in\mathbb{N}} D_k(\lambda; X)$. In particular we have an homology isomorphism*

$$\tilde{H}_*(E(\lambda; X); \mathbb{Z}) \cong \bigoplus_{k=1}^{\infty} \tilde{H}_*(D_k(\lambda; X); \mathbb{Z})$$

Proposition 8.3 ([11], p. 8)**.** *Let X be a connected CW complex with base point $* \in X$ and $\mathbb{F}$ be any field. Then we have a quasi-isomorphism*

$$C_*(E(\lambda, k)) \otimes_{\Sigma_k} \tilde{H}_*(X)^{\otimes k} \to C_*(D_k(\lambda; X))$$

where $C_*(-)$ are the singular chains with $\mathbb{F}$-coefficients and Σ_k acts on the graded $\mathbb{F}$-vector space $\tilde{H}_*(X)^{\otimes k}$ by permutation of variables with the usual sign convention.

Proof. Since we are working with coefficients on a field $\mathbb{F}$ one can always find a quasi-isomorphism $j : H_*(X) \to C_*(X)$. Now observe that $C_*(E(\lambda, k))$ is a chain complex of free (hence flat) $\mathbb{F}[\Sigma_k]$-modules because Σ_k acts freely on $E(\lambda, k)$. Therefore we have an induced quasi-isomorphism

$$C_*(E(\lambda, k)) \otimes_{\Sigma_k} H_*(X)^{\otimes k} \to C_*(E(\lambda, k)) \otimes_{\Sigma_k} C_*(X)^{\otimes k}$$

Using the freeness of the Σ_k-action on $E(\lambda, k) \times X^k$ together with the Künneth map we get a quasi-isomorphism

$$C_*(E(\lambda, k)) \otimes_{\Sigma_k} C_*(X)^k \to C_*(E(\lambda, k) \times_{\Sigma_k} X^k)$$

For more details we refer to [35, Ch. XI Sec. 7]. Combining these two we finally get a quasi-isomorphism

$$f : C_*(E(\lambda, k)) \otimes_{\Sigma_k} H_*(X)^{\otimes k} \to C_*(E(\lambda, k) \times_{\Sigma_k} X^k)$$

Since X is a path-connected space with non-degenerate base point we have a cofibration

$$E(\lambda, k) \times_{\Sigma_k} X_k \hookrightarrow E(\lambda, k) \times_{\Sigma_k} X^k \to D_k(\lambda; X)$$

where $X_k \coloneqq \{(x_1, \ldots, x_k) \in X^k \mid x_i = * \text{ for some } i\}$. Therefore f induces a quasi isomorphism

$$C_*(E(\lambda, k)) \otimes_{\Sigma_k} \tilde{H}_*(X)^{\otimes k} \to C_* \left(\frac{E(\lambda, k) \times_{\Sigma_k} X^k}{E(\lambda, k) \times_{\Sigma_k} X_k} \right) = C_*(D_k(\lambda; X))$$

and we get the statement. □

8.2 A model for $B(B_n/Z(B_n))$ with fiberwise configuration spaces

Let $\mathbb{C} \hookrightarrow E \xrightarrow{\lambda} \mathbb{C}P^\infty$ be the tautological line bundle. Explicitly, the total space is

$$E \coloneqq \{(v, l) \in \mathbb{C}^\infty \times \mathbb{C}P^\infty \mid v \in l\}$$

and $\lambda : E \to \mathbb{C}P^\infty$ is the projection on the second coordinate. The most important observation of this paragraph is the following proposition:

Proposition 8.4. *The unorderd fiberwise configurations $E(\lambda, n)/\Sigma_n$ is a model for the classifying space $B(B_n/Z(B_n))$. Similarly, $E(\lambda, n)$ is a model for the classifying space $B(PB_n/Z(PB_n))$.*

Proof. We prove the statement for $E(\lambda, n)/\Sigma_n$, the other case is analogous. Consider the fibration $C_n(\mathbb{C}) \hookrightarrow E(\lambda, n)/\Sigma_n \to \mathbb{C}P^\infty$. The long exact sequence for homotopy groups shows that $\pi_i(E(\lambda, n)/\Sigma_n) = 0$ for all $i \geq 3$. Moreover, we get the following exact sequence:

$$0 \longrightarrow \pi_2(E(\lambda, n)/\Sigma_n) \longrightarrow \mathbb{Z} \xrightarrow{\partial} B_n \longrightarrow \pi_1(E(\lambda, n)/\Sigma_n) \longrightarrow 0$$

Now we claim that the connecting homomorphism ∂ includes $\mathbb{Z}$ as the center of B_n. To prove this, let us consider the map

$$\begin{aligned} f : S^\infty &\to E(\lambda, n)/\Sigma_n \\ v &\mapsto \{(v, l_v), (\zeta \cdot v, l_v), \ldots, (\zeta^{n-1} \cdot v, l_v)\} \end{aligned}$$

where $\zeta := e^{2\pi i/n}$ acts on $S^\infty \subseteq \mathbb{C}^\infty$ by multiplication and l_v denotes the line spanned by v. This is a map of fibrations

$$\begin{tikzcd}
C_n(\mathbb{C}) \arrow[d, hook] & S^1 \arrow[l] \arrow[d, hook] \\
E(\lambda, n)/\Sigma_n \arrow[d] & S^\infty \arrow[l, "f"'] \arrow[d] \\
\mathbb{C}P^\infty & \mathbb{C}P^\infty \arrow[l]
\end{tikzcd}$$

so we get the following commutative diagram, whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & \pi_2(E(\lambda,n)/\Sigma_n) \arrow[r] & \mathbb{Z} \arrow[r, "\partial"] & B_n \arrow[r] & \pi_1(E(\lambda,n)/\Sigma_n) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r, "\cong"] \arrow[u, "id"] & \mathbb{Z} \arrow[r] \arrow[u, "f_*"'] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Finally observe that f_* includes $\mathbb{Z}$ as the center of B_n, so the same holds for ∂. □

Remark 8.5. The Braid group B_n is equipped by a natural morphism to Σ_n, whose kernel is the pure braid group PB_n. Since this morphism sends the generator of the center δ^2 to the identity permutation, there is a factorization

$$\begin{tikzcd}
1 \arrow[r] & PB_n \arrow[r, hook] \arrow[d] & B_n \arrow[r] \arrow[d] & \Sigma_n \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & PB_n/Z(PB_n) \arrow[r, hook] & B_n/Z(B_n) \arrow[r] & \Sigma_n \arrow[r] & 1
\end{tikzcd}$$

Therefore we can regard any Σ_n-module as a $B_n/Z(B_n)$ module.

Now let V be a graded vector space over some field $\mathbb{F}$, and assume that V in concentrated in degree greater than 1. In this case we can always find a bouquet of spheres S_V such that

$$V \cong \tilde{H}_*(S_V; \mathbb{F})$$

We assume that the symmetric group Σ_n acts on $V^{\otimes n}$ with the usual sign conventions. By the previous remark we can see $V^{\otimes n}$ as a $B_n/Z(B_n)$-module.

Proposition 8.6. *Let $\mathbb{F}$ be any field, $q \in \mathbb{N}$. Then we have an isomorphism*

$$H_*(B_n/Z(B_n); V^{\otimes n}) \cong \tilde{H}_*(D_n(\lambda; S_V); \mathbb{F})$$

In particular, if we choose V to be a copy of $\mathbb{F}$ concentrated in degree $2q+1$ (resp. $2q$) we get

$$H_*(B_n/Z(B_n); \mathbb{F}(\pm 1)) \cong H_{*+(2q+1)n}(D_n(\lambda; S^{2q+1}); \mathbb{F}) \tag{8.1}$$

$$H_*(B_n/Z(B_n); \mathbb{F}) \cong H_{*+2qn}(D_n(\lambda; S^{2q}); \mathbb{F}) \tag{8.2}$$

Proof. To ease the notation, let us call $G_n := B_n/Z(B_n)$ and $H_n := PB_n/Z(PB_n)$. By definition $H_*(G_n; V^{\otimes n})$ is computed by $C_*^{cell}(EG_n) \otimes_{G_n} V^{\otimes n}$, where if G is a discrete group $C_*^{cell}(EG)$ denotes the standard resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$. Since H_n is a subgroup of G_n, we can take EG_n as a model for EH_n. Therefore:

$$C_*^{cell}(EG_n) \otimes_{G_n} V^{\otimes n} \cong \frac{C_*^{cell}(EG_n) \otimes_{H_n} V^{\otimes n}}{\Sigma_n} \cong C_*^{cell}(EG_n)_{H_n} \otimes_{\Sigma_n} V^{\otimes n}$$

where the last isomorphism holds because H_n acts trivially on $V^{\otimes n}$. Note that $C_*^{cell}(EG_n)_{H_n}$ computes the homology of H_n. Proposition 8.4 tells us that $E(\lambda, n)$ is a model for the classifying space of $PB_n/Z(PB_n)$ and Proposition 8.3 gives us the quasi-isomorphism

$$C_*(E(\lambda, n)) \otimes_{\Sigma_n} \tilde{H}_*(S_V)^{\otimes n} \to C_*(D_n(\lambda; S_V))$$

which allow us to conclude the proof. □

8.3 Computations

By Proposition 8.6 and Theorem 8.2 the computation of $H_*(B_n/Z(B_n); \mathbb{F}_p(\pm 1))$ is reduced to the computation of $H_*(E(\lambda; S^{2q+1}); \mathbb{F}_p)$.

Theorem 8.7. *Let p be an odd prime. Then*

$$H_*(E(\lambda; S^{2q+1}); \mathbb{F}_p) \cong H_*(C(\mathbb{C}; S^{2q+1}); \mathbb{F}_p) \otimes H_*(BS^1; \mathbb{F}_p)$$

Proof. We get the statement by proving that the Serre spectral sequence with $\mathbb{F}_p$-coefficients associated to the fibration $C(\mathbb{C}; S^{2q+1}) \hookrightarrow E(\lambda; S^{2q+1}) \to BS^1$ degenerates at page E_2. In particular, it suffices to show that the map induced by the inclusion $i_* : H_*(C(\mathbb{C}; S^{2q+1}); \mathbb{F}_p) \to H_*(E(\lambda; S^{2q+1}); \mathbb{F}_p)$ is injective. Consider the following map

$$\begin{gathered} \psi : E(\lambda; S^{2q+1}) \to C(\mathbb{C}^\infty; S^{2q+1}) \\ [((v_1, l), \ldots, (v_n, l)) \times (p_1, \ldots, p_n)] \mapsto [(v_1, \ldots, v_n) \times (p_1, \ldots, p_n)] \end{gathered}$$

where (v_i, l) are points in the total space of the tautological line bundle $\mathbb{C} \hookrightarrow E \to \mathbb{C}P^\infty$, and $p_i \in S^{2q+1}$ are the corresponding labels. Now observe that the inclusion $j : C(\mathbb{C}; S^{2q+1}) \to C(\mathbb{C}^\infty; S^{2q+1})$ factors through $E(\lambda; S^{2q+1})$:

$$\begin{tikzcd} C(\mathbb{C}; S^{2q+1}) \arrow[r, "j"] \arrow[d, "i"] & C(\mathbb{C}^\infty; S^{2q+1}) \\ E(\lambda; S^{2q+1}) \arrow[ur, "\psi"] & \end{tikzcd}$$

Since j induces an injective map in mod p homology (see [12] or [48]), the above commutative diagram shows that i_* is injective as well. □

Remark 8.8. If we replace S^{2q+1} with S^{2q} the proof written above does not work, indeed the map $i_* : H_*(C(\mathbb{C}; S^{2q}); \mathbb{F}_p) \to H_*(C(\mathbb{C}^\infty; S^{2q}); \mathbb{F}_p)$ is not injective anymore, except when $p = 2$. For details about the homology of labelled configuration spaces we refer to the work of F. Cohen [12].

Theorem 8.9. *Let $q \geq 1$. Then*

$$H_*(E(\lambda; S^{2q}); \mathbb{F}_2) \cong H_*(C(\mathbb{C}; S^{2q}); \mathbb{F}_2) \otimes H_*(BS^1; \mathbb{F}_2)$$

Proof. Follow the proof of Theorem 8.7, just replace $2q+1$ with $2q$ and p with 2. □

We end this section with a proof of Theorem 7.16 based on labelled configuration spaces:

Corollary 8.10. *For any $n \in \mathbb{N}$ we have*

$$H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_2) \cong H_*(C_n(\mathbb{C}); \mathbb{F}_2) \otimes H_*(BS^1; \mathbb{F}_2)$$

Proof. By Proposition 8.6 we have

$$H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_2) = H_{*+2qn}(D_n(\lambda; S^{2q}); \mathbb{F}_2)$$

By Theorem 8.2 we get

$$\tilde{H}_*(E(\lambda; S^{2q}); \mathbb{F}_2) = \bigoplus_{n=1}^{\infty} H_*(D_n(\lambda; S^{2q}); \mathbb{F}_2)$$

so $H_*^{S^1}(C_n(\mathbb{C}); \mathbb{F}_2)$ can be seen as a subspace of $H_*(E(\lambda; S^{2q}); \mathbb{F}_2)$. Finally, the Theorem 8.9 tells us that

$$H_*(E(\lambda; S^{2q}); \mathbb{F}_2) = \mathbb{F}_2[\iota, Q\iota, Q^2\iota, \dots] \otimes H_*(BS^1; \mathbb{F}_2)$$

where $\iota \in H_{2q}(S^{2q}); \mathbb{F}_2)$ is the fundamental class, and this is enough to get the statement. □

Chapter 9

Composition of equivariant operations

Let (A, d_A) be a gravity algebra over the field $\mathbb{F}_p$. For the rest of this Chapter we will denote by M either the trivial representation $\mathbb{F}_p$ or the sign representation $\mathbb{F}_p(\pm 1)$. As we already observed in Section 6.1, classes in $H_*(B_n/Z(B_n); M)$ give us homology operations for gravity algebras, which we will denote with the same letter. More precisely, any class $Q \in H_*(B_n/Z(B_n); \mathbb{F}_p)$ (resp. $Q \in H_*(B_n/Z(B_n); \mathbb{F}_p(\pm 1))$) gives rise to an operation

$$Q : H_*(A) \to H_{*n+|Q|+1}(A)$$

which acts on even (resp. odd) degree classes. These operations can be naturally composed one after the other, provided that some parity condition are satisfied. For example, if $Q_1 \in H_*(B_n/Z(B_n); \mathbb{F}_p)$, $Q_2 \in H_*(B_m/Z(B_m); \mathbb{F}_p)$ and $a \in H_*(A)$ is an even degree class, then $Q_1Q_2(a)$ is defined if and only if $|Q_2(a)| = |Q_2| + 1 + m|a|$ is even, i.e. if Q_2 has odd degree. Similar conditions hold in the other cases. These are summarized in the next table, where $+$ (resp. $-$) indicates if a equivariant operation acts on even (resp. odd) classes:

Operation	Arity				
Q_1	n	$+$	$+$	$-$	$-$
Q_2	m	$+$	$-$	$+$	$-$
Q_1Q_2	nm	$\lvert Q_2\rvert = 1$	$\lvert Q_2\rvert = 1 + m$	$\lvert Q_2\rvert = 0$	$\lvert Q_2\rvert = m$

The last row contains the parity condition that must be satisfied in order to define Q_1Q_2. All the equalities should be intended in $\mathbb{Z}/2$. In any case, the composition Q_1Q_2 will be represented by a class (which we indicate with the same symbol) $Q_1Q_2 \in H_*(B_{nm}/Z(B_{nm}); M)$. In this Chapter we give some details about how to properly define the class Q_1Q_2 associated to the composition of two equivariant operations. Here is the outline:

Section 9.1 contains a slightly different description of the gravity operad respect to that of Section 2.2.

Section 9.2 contains a geometric definition of the composition of equivariant operations.

Section 9.3 describes the composition of equivariant operations in terms of group homology.

Section 9.4 provides some geometric models for the classifying spaces of the groups involved in Section 9.3.

9.1 The gravity operad revisited

Let $\gamma_1 : \mathcal{D}_2(n) \times \mathcal{D}_2(m)^n \to \mathcal{D}_2(nm)$ be the composition of the little two discs operad. $(S^1)^{n+1}$ acts on $\mathcal{D}_2(n) \times \mathcal{D}_2(m)^n$ by rotations in each component. Consider the following diagram

$$\begin{tikzcd}
\mathcal{D}_2(n) \times \mathcal{D}_2(m)^n \arrow[r, "\gamma_1"] \arrow[d, shift left, "p_1"] & \mathcal{D}_2(nm) \arrow[d, shift left, "p_3"] \\
\frac{\mathcal{D}_2(n)\times\mathcal{D}_2(m)^n}{S^1} \arrow[r, "\gamma_2"] \arrow[u, dashed, shift left, "\tau_1"] \arrow[d, shift left, "p_2"] & \frac{\mathcal{D}_2(nm)}{S^1} \arrow[u, dashed, shift left, "\tau_3"] \\
\frac{\mathcal{D}_2(n)}{S^1} \times \left(\frac{\mathcal{D}_2(m)}{S^1}\right)^n \arrow[u, dashed, shift left, "\tau_2"] \arrow[ur, dashed, "\gamma"] \arrow[uu, dashed, bend left=60, "\tau"] &
\end{tikzcd} \tag{9.1}$$

Here $\mathcal{D}_2(n)\times\mathcal{D}_2(m)^n/S^1$ is the quotient respect to the diagonal action of S^1, p_1, p_2, p_3 are the obvious quotient maps and τ_1, τ_2, τ_3 are the (homological) transfers (we refer to [27] for details about transfer maps). The map $p_2 \circ p_1$ is just the quotient by the $(S^1)^{n+1}$ action and the associated transfer is $\tau = \tau_1 \circ \tau_2$. γ_2 is induced by γ_1 and

$$\gamma : H_*\left(\frac{\mathcal{D}_2(n)}{S^1} \times \left(\frac{\mathcal{D}_2(m)}{S^1}\right)^n\right) \to H_{*+n}\left(\frac{\mathcal{D}_2(nm)}{S^1}\right)$$

is defined as the composition $(\tau_3)^{-1} \circ (\gamma_1)_* \circ \tau$ (further details about the construction of this map can be found in Section 2.2). If we precompose γ with the Künneth isomorphism we get the composition of the Gravity operad (see also Section 2.2). The next Lemma shows that we can define the composition of the Gravity operad only by looking at the lower triangle of diagram 9.1:

Lemma 9.1. $\gamma = (\gamma_2)_* \circ \tau_2$

Proof. The upper square of diagram 9.1 is a pullback, so by the naturality of transfer maps we find

$$(\gamma_1)_* \circ \tau_1 = \tau_3 \circ (\gamma_2)_*$$

Combining this with the equality $\tau = \tau_1 \circ \tau_2$ we get that $\gamma = (\gamma_2)_* \circ \tau_2$. □

9.2 Composition of equivariant operations

In the previous paragraph we saw that the composition of the gravity operad factorizes as

$$\begin{tikzcd} H_{*+n}\left(\frac{\mathcal{D}_2(n)\times\mathcal{D}_2(m)^n}{S^1}\right) \arrow[r, "(\gamma_2)_*"] & H_{*+n}\left(\frac{\mathcal{D}_2(nm)}{S^1}\right) \\ H_*\left(\frac{\mathcal{D}_2(n)}{S^1}\times\left(\frac{\mathcal{D}_2(m)}{S^1}\right)^n\right) \arrow[u, "\tau_2"] \arrow[ur, "\gamma"'] & \end{tikzcd} \tag{9.2}$$

If we want to compose equivariant operations, we need to replace singular homology with equivariant homology in the diagram above. Look at diagram 9.1: the spaces $(\mathcal{D}_2(n)\times\mathcal{D}_2(m)^n)/S^1$ and $\mathcal{D}_2(n)/S^1\times(\mathcal{D}_2(m)/S^1)^n$ have a natural action of $\Sigma_m\wr\Sigma_n$, and the projection p_2 is $\Sigma_m\wr\Sigma_n$-equivariant so we get a fiber sequence

$$(S^1)^n \xrightarrow{i} \left(\frac{\mathcal{D}_2(n)\times\mathcal{D}_2(m)^n}{S^1}\right)_{\Sigma_m\wr\Sigma_n} \longrightarrow \left(\frac{\mathcal{D}_2(n)}{S^1}\times\left(\frac{\mathcal{D}_2(m)}{S^1}\right)^n\right)_{\Sigma_m\wr\Sigma_n} \tag{9.3}$$

We will denote by τ the (homological) transfer associated to this fiber sequence. To be completely explicit, let M be either the trivial or the sign representation of Σ_{nm} (over $\mathbb{F}_p$). Note that the standard inclusion $\Sigma_m\wr\Sigma_n\to\Sigma_{nm}$ makes M a $\Sigma_m\wr\Sigma_n$-module. The Leray-Serre spectral sequence associated to this fiber sequence is given by

$$H_p^{\Sigma_m\wr\Sigma_n}\left(\frac{\mathcal{D}_2(n)}{S^1}\times\left(\frac{\mathcal{D}_2(m)}{S^1}\right)^n;H_q((S^1)^n;i^*M)\right) \Longrightarrow H_{p+q}^{\Sigma_m\wr\Sigma_n}\left(\frac{\mathcal{D}_2(n)\times\mathcal{D}_2(m)^n}{S^1};M\right)$$

It is not hard to see that i^*M is (in any case) the trivial representation $\mathbb{F}_p$ (see the next section), so $H_n((S^1)^n;i^*M)=\mathbb{F}_p$ and there are no homology classes of higher degree. Therefore we can define a transfer map in the spirit of [27] as follows:

$$\begin{tikzcd} H_p^{\Sigma_m\wr\Sigma_n}\left(\frac{\mathcal{D}_2(n)}{S^1}\times\left(\frac{\mathcal{D}_2(m)}{S^1}\right)^n;H_n((S^1)^n;i^*M)\right) \arrow[rr, dashed, "\tau"] \arrow[dr] & & H_{p+n}^{\Sigma_m\wr\Sigma_n}\left(\frac{\mathcal{D}_2(n)\times\mathcal{D}_2(m)^n}{S^1};M\right) \\ & E^\infty_{p,n} \arrow[ur] & \end{tikzcd}$$

Moreover Σ_{nm} acts on $\mathcal{D}_2(nm)/S^1$ and the map γ_2 is equivariant. Therefore we get an induced map in equivariant homology and we can write the equivariant version of diagram 9.2:

$$\begin{tikzcd} H_{*+n}^{\Sigma_m\wr\Sigma_n}\left(\frac{\mathcal{D}_2(n)\times\mathcal{D}_2(m)^n}{S^1};M\right) \arrow[r, "(\gamma_2)_*"] & H_{*+n}^{\Sigma_{nm}}\left(\frac{\mathcal{D}_2(nm)}{S^1};M\right) \\ H_*^{\Sigma_n\wr\Sigma_m}\left(\frac{\mathcal{D}_2(n)}{S^1}\times\left(\frac{\mathcal{D}_2(m)}{S^1}\right)^n;H_n((S^1)^n;i^*M)\right) \arrow[u, "\tau"] \arrow[ur, "\gamma"'] & \end{tikzcd} \tag{9.4}$$

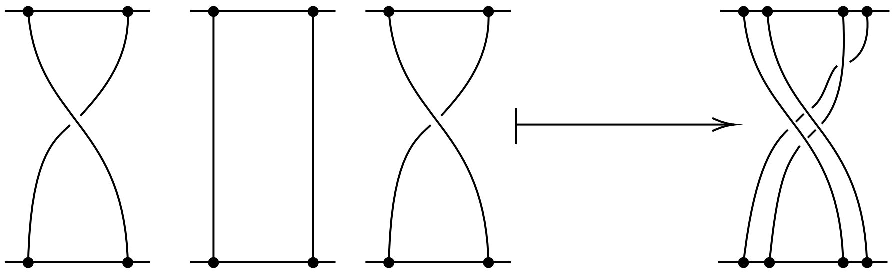


Figure 9.1: In this picture we see an example of how $B_2 \ltimes (B_2)^2 \to B_4$ acts.

Remark 9.2. All the spaces involved in the previous diagram are classifying spaces for some groups. Consider the semi-direct product $B_n \ltimes (B_m)^n$, where B_n acts on $(B_m)^n$ by permuting the factors via the morphism $B_n \to \Sigma_n$. Let δ^2 be the generator of $Z(B_n)$ and δ_i^2 be the generator of the center of the i-th copy of B_m inside $B_n \ltimes (B_m)^n$. Then

- $(\mathcal{D}_2(nm)/S^1)_{\Sigma_{nm}} \simeq B(B_{nm}/Z(B_{nm}))$
- $\left(\frac{\mathcal{D}_2(n)\times\mathcal{D}_2(m)^n}{S^1}\right)_{\Sigma_m\wr\Sigma_n} \simeq B\left(\frac{B_n\ltimes(B_m)^n}{<\delta^2\delta_1^2\cdots\delta_n^2>}\right)$
- $\left(\frac{\mathcal{D}_2(n)}{S^1} \times \left(\frac{\mathcal{D}_2(m)}{S^1}\right)^n\right)_{\Sigma_m\wr\Sigma_n} \simeq B\left(\frac{B_n\ltimes(B_m)^n}{<\delta^2,\delta_1^2,\ldots,\delta_n^2>}\right)$

Thanks to this Remark we will be able to rephrase all this construction in terms of group homology. The advantage of this perspective is that it is more combinatorial, so it is easier to define the composition of equivariant operations using this model.

9.3 Composition of equivariant operations via group homology

In this section we rewrite in terms of group homology the diagram 9.4. In this way we will be able to give a chain level definition of the composition of equivariant operations.

We start by describing the map $(\gamma_2)_*$ of diagram 9.4 in terms of group homology. The composition of the little two disk operad $\mathcal{D}_2(n) \times \mathcal{D}_2(m) \to \mathcal{D}_2(nm)$ is equivariant so it induces a map between the homotopy quotients

$$(\mathcal{D}_2(n) \times \mathcal{D}_2(m))_{\Sigma_m\wr\Sigma_n} \to \mathcal{D}_2(nm)_{\Sigma_{nm}} \tag{9.5}$$

Passing to the fundamental groups we obtain a morphism of groups

$$B_n \ltimes (B_m)^n \to B_{nm}$$

which takes an element $\sigma\sigma_1\cdots\sigma_n \in B_n \ltimes (B_m)^n$ to the braid obtained from σ by replacing the i-th strand with σ_i. See Figure 9.1 for a picture. This morphism factors as

$$\begin{tikzcd}
B_n \ltimes (B_m)^n \arrow[r] \arrow[d] & B_{nm} \arrow[d] \\
\frac{B_n \ltimes (B_m)^n}{<\delta^2\delta_1^2\cdots\delta_n^2>} \arrow[r, "\gamma_2"] & \frac{B_{nm}}{Z(B_{nm})}
\end{tikzcd}$$

where the vertical arrows are the quotient maps. The map induced in homology by γ_2 is precisely the map $(\gamma_2)_*$ of diagram 9.4. Now let us give an algebraic description of the transfer τ of diagram 9.4: consider the exact sequence of groups

$$1 \longrightarrow \mathbb{Z}^n \hookrightarrow \frac{B_n \ltimes (B_m)^n}{<\delta^2\delta_1^2\cdots\delta_n^2>} \longrightarrow \frac{B_n \ltimes (B_m)^n}{<\delta^2,\delta_1^2,\cdots,\delta_n^2>} \longrightarrow 1 \tag{9.6}$$

where $\mathbb{Z}^n$ is embedded in $B_n \ltimes (B_m)^n / < \delta^2\delta_1^2\cdots\delta_n^2 >$ as the subgroup generated by $\{\delta_1^2, \ldots, \delta_n^2\}$. Note that if we apply the classifying space functor to 9.6 we obtain the fiber sequence 9.3. As before we will denote by M either the trivial or the sign representation of Σ_{nm}. The Lyndon-Hochschild-Serre spectral sequence associated to 9.6 is given by

$$E^2_{p,q} := H_p\left(\frac{B_n \ltimes (B_m)^n}{< \delta^2, \delta_1^2, \ldots, \delta_n^2 >}; H_q(\mathbb{Z}^n; M)\right) \implies H_{p+q}\left(\frac{B_n \ltimes (B_m)^n}{< \delta^2\delta_1^2\cdots\delta_n^2 >}; M\right)$$

Note that the subgroup $\mathbb{Z}^n$ contains only pure braids, so it acts trivially on M. The transfer map τ we are interested in is defined as follows:

$$\begin{tikzcd}
E^2_{i,n} = H_i(\frac{B_n \ltimes (B_m)^n}{<\delta^2,\delta_1^2,\ldots,\delta_n^2>}; H_n(\mathbb{Z}^n; M)) \arrow[rr, dashed, "\tau"] \arrow[dr] & & H_{i+n}(\frac{B_n \ltimes (B_m)^n}{<\delta^2\delta_1^2\cdots\delta_n^2>}; M) \\
& E^\infty_{i,n} \arrow[ur] &
\end{tikzcd}$$

Therefore diagram 9.4 can be written in terms of group homology as follows:

$$\begin{tikzcd}
H_{*+n}\left(\frac{B_n \ltimes (B_m)^n}{<\delta^2\delta_1^2\cdots\delta_n^2>}; M)\right) \arrow[r, "(\gamma_2)_*"] & H_{*+n}\left(\frac{B_{nm}}{Z(B_{nm})}; M\right) \\
H_*\left(\frac{B_n \ltimes (B_m)^n}{<\delta^2,\delta_1^2,\ldots,\delta_n^2>}; H_n(\mathbb{Z}^n; M))\right) \arrow[u, "\tau"] \arrow[ur, "\gamma"] &
\end{tikzcd}$$

Remark 9.3. Let us analyse the action of $B_n \ltimes (B_m)^n / < \delta^2, \delta_1^2, \ldots, \delta_n^2 >$ on $H_n(\mathbb{Z}^n; M)$. First we set up some notation: $\sigma_1, \ldots, \sigma_{n-1}$ will be the generators of B_n, while $\sigma_1^i, \ldots, \sigma_{m-1}^i$ will be the generators of the i-th copy of B_m inside $B_n \ltimes (B_m)^n$. By an abuse of notation we use the same letters to denote the images of these generators in the groups $B_n \ltimes (B_m)^n / < \delta^2, \delta_1^2, \ldots, \delta_n^2 >$ and $B_n \ltimes (B_m)^n / < \delta^2\delta_1^2\cdots\delta_n^2 >$. Now look at the exact sequence 9.6: for any $i = 1, \ldots, n$ and any $j = 1, \ldots, m-1$ the element σ_j^i acts trivially by conjugation on $\mathbb{Z}^n$, so the induced map in homology is the identity. Now consider σ_j, $j = 1, \dots, n-1$: this element acts on $H_1(\mathbb{Z}^n; \mathbb{F}_p) \cong \mathbb{F}_p x_1 \oplus \cdots \oplus \mathbb{F}_p x_n$ by permuting the factors. Therefore it acts on the generator of $H_n(\mathbb{Z}^n; \mathbb{F}_p)$ by the sign representation. Combining these information with the knowledge of the action of $B_n \ltimes (B_m)^n / < \delta^2, \delta_1^2, \dots, \delta_n^2 >$ on the coefficients M we get an explicit description of the action.

Now we can define the composition of two equivariant classes at the chain level: we will denote by $W(n)$ any $B_n/Z(B_n)$-free resolution of $\mathbb{F}_p$ and by G the group $B_n \ltimes (B_m)^n / < \delta^2, \delta_1^2, \dots, \delta_n^2 >$. Then $W(n) \otimes W(m)^{\otimes n}$ will be a G-free resolution of $\mathbb{F}_p$.

Definition 9.1 (Composition of equivariant classes)**.** Fix Q_1 and Q_2 two composable operations of arity n and m respectively. In particular suppose $Q_2 \in H_*(B_m/Z(B_m); M)$, where M is either the trivial or the sign representation of Σ_m. The G-module $H_n(\mathbb{Z}^n; M)$ is a one dimensional representation of G and we will denote by x a fixed generator of it. If $q_1 \in W(n)$ and $q_2 \in W(m)$ are two chain level representatives for Q_1, Q_2, then $q_1 \otimes q_2^{\otimes n} \otimes x$ is a cycle in $(W(n) \otimes W(m)^{\otimes n}) \otimes_G H_n(\mathbb{Z}^n; M))$. Therefore it makes sense to define

$$Q_1 Q_2 := (\gamma_2)_* \tau (q_1 \otimes q_2^{\otimes n} \otimes x)$$

Remark 9.4. By the previous discussion the operation associated to $Q_1 Q_2$ is precisely the composition of the operation associated to Q_1 with that of Q_2.

9.4 Some geometric models

As we saw in the previous section the groups $(B_n \ltimes (B_m)^n)/ < \delta^2, \delta_1^2, \dots, \delta_n^2 >$ and $(B_n \ltimes (B_m)^n)/ < \delta^2 \delta_1^2 \cdots \delta_n^2 >$ play a crucial role to define the composition of equivariant operations. In this section we just point out some nice geometric models for the classifying space of these groups, based on labelled configuration spaces.

Definition 9.2. Let X, Y be two topological spaces. The **configuration space of** n **points in** X **with labels in** Y is defined as

$$C_n(X; Y) := F_n(X) \times_{\Sigma_n} Y^n$$

where Σ_n acts diagonally on $F_n(X) \times Y^n$.

The most relevant cases for us are $X = \mathbb{C}$ and $Y = C_m(\mathbb{C})$ or $Y = C_m(\mathbb{C})_{S^1}$.

Proposition 9.5. *Consider the space $C_n(\mathbb{C}; C_m(\mathbb{C}))$ with the diagonal action of S^1. Then*

$$C_n(\mathbb{C}; C_m(\mathbb{C}))_{S^1} \simeq B\left(\frac{B_n \ltimes (B_m)^n}{< \delta^2 \delta_1^2 \cdots \delta_n^2 >}\right)$$

Proof. Just observe that $C_n(\mathbb{C}; C_m(\mathbb{C}))$ is a model for the classifying space of $B_n \ltimes (B_m)^n$. Then use the long exact sequence for homotopy groups associated to the fibration

$$C_n(\mathbb{C}; C_m(\mathbb{C})) \to C_n(\mathbb{C}; C_m(\mathbb{C}))_{S^1} \to BS^1$$

to prove that

$$\pi_i(C_n(\mathbb{C};C_m(\mathbb{C}))_{S^1}) = \begin{cases} 0 \text{ if } i \geq 2 \\ \frac{B_n \ltimes (B_m)^n}{<\delta^2 \delta_1^2 \cdots \delta_n^2>} \text{ if } i = 1 \end{cases}$$

This concludes the proof. □

Proposition 9.6. *Consider the space $C_n(\mathbb{C};C_m(\mathbb{C})_{S^1})$ with the action of S^1 induced by the one on $F_n(\mathbb{C})$. Then*

$$C_n(\mathbb{C};C_m(\mathbb{C})_{S^1})_{S^1} \simeq B\left(\frac{B_n \ltimes (B_m)^n}{< \delta^2, \delta_1^2, \ldots, \delta_n^2 >}\right)$$

Proof. Use the same argument as in the previous Proposition. □

Chapter 10

Some computations

As we already observed in the introduction of Chapter 6 the equivariant operations we defined in Section 6.1 are the analogue for Gravity algebras of the Dyer-Lashof operations. Therefore it makes sense the following question: *is there an analogue of the Adem relation for the composite of our equivariant operations?* This Chapter is devoted to answer this question in a very special case, i.e. when we use $\mathbb{F}_2$ coefficients and we compose two binary equivariant operations. The main result is Theorem 10.1: it tells us that the composition of any two equivariant operations (with $\mathbb{F}_2$ coefficients) is zero. The outline of the chapter is the following:

Section 10.1 contains the proof of Theorem 10.1.

Section 10.2 contains the computation of the homology (with $\mathbb{F}_2$-coefficients) of the group $< a, b, c \mid ac = ba, ab = ca, bc = cb, a^2b^2c^2 = 1 >$. This was needed in Section 10.1.

10.1 Adem relations (in a very special case)

We use $\mathbb{F}_2$ coefficients for (co)homology from now on. The goal of this section is to prove that the composition of any equivariant binary operations is zero. More precisely:

Theorem 10.1. *Given $Q_1, Q_2 \in H_*(B_2/Z(B_2); \mathbb{F}_2)$ equivariant operations, then Q_1Q_2 is the zero class in $H_*(B_4/Z(B_4); \mathbb{F}_2)$.*

Remark 10.2. $B_2/Z(B_2)$ is just the cyclic group of order two. Therefore the binary equivariant operations are classes in $H_*(C_2; \mathbb{F}_2)$.

Before going into the details of the proof of this Theorem, let us fix some notation. If we let $n = m = 2$ the exact sequence of groups 9.6 becomes

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow (\mathbb{Z} \ltimes \mathbb{Z}^2)/ < a^2b^2c^2 > \longrightarrow (\mathbb{Z} \ltimes \mathbb{Z}^2)/ < a^2, b^2, c^2 > \longrightarrow 0 \quad (10.1)$$

where we have denoted by a, b, c the generators of the three copies of $\mathbb{Z}$ in $\mathbb{Z} \ltimes \mathbb{Z}^2$. The injective map embeds $\mathbb{Z}^2$ in $(\mathbb{Z} \ltimes \mathbb{Z}^2)/ < a^2b^2c^2 >$ as the subgroup generated by b^2 and c^2.

Remark 10.3. The quotient $(\mathbb{Z} \ltimes \mathbb{Z}^2)/ < a^2, b^2, c^2 >$ is just the wreath product $C_2 \wr C_2$ of the cyclic group with two elements C_2 with itself. The cohomology of this group with $\mathbb{F}_2$ coefficients has the following form:

$$H^*(C_2 \wr C_2; \mathbb{F}_2) = \mathbb{F}_2[u, v, w]/(u \cdot w)$$

where u, w are classes of degree one, while v has degree two. Moreover, the map $i^* : H^*(C_2 \wr C_2; \mathbb{F}_2) \to H^*(C_2 \times C_2; \mathbb{F}_2) = \mathbb{F}_2[x, y]$ induced by the inclusion $C_2 \times C_2 \to C_2 \wr C_2$ sends u (resp. v) to the symmetric polynomial $x + y$ (resp. xy). Combining this together with the compatibility of i^* with the cup product we easily get that $i^*(w) = 0$. For a proof of these facts one can see the book [1] or the classical paper by M. Nakaoka [42]

To get the composition of two binary equivariant operations we have to study the composite

$$H_*(C_2 \wr C_2; \mathbb{F}_2) \xrightarrow{\tau} H_{*+2}(\tfrac{\mathbb{Z} \ltimes \mathbb{Z}^2}{<a^2 b^2 c^2>}; \mathbb{F}_2) \xrightarrow{\gamma_*} H_{*+2}(B_4/Z(B_4); \mathbb{F}_2)$$

Theorem 10.1 follows immediately from

Proposition 10.4. *The transfer*

$$\tau : H_*(C_2 \wr C_2; \mathbb{F}_2) \to H_{*+2}(\frac{\mathbb{Z} \ltimes \mathbb{Z}^2}{< a^2 b^2 c^2 >}; \mathbb{F}_2)$$

is zero.

The proof of Proposition 10.4 is a bit long, so we subdivide it in several steps. We begin by writing explicitly the second page of the (co)homological Serre spectral sequence associated to the fibration 10.1 written above.

The E_2 page: we will work with the cohomological spectral sequence, in order to exploit the multiplicative structure. The E_2 page is

$$E_2^{p,q} = H^p(C_2 \wr C_2; H^q(\mathbb{Z}^2))$$

If $q = 0, 2$ $H^q(\mathbb{Z}^2; \mathbb{F}_2) = \mathbb{F}_2$ and the monodromy action is trivial (for any group G the only one dimensional G-representation over $\mathbb{F}_2$ is the trivial one). So $E_2^{*,0} = E_2^{*,2} = H^*(C_2 \wr C_2)$, which is well known (see Remark 10.3). Let us denote by x and y the two generators of $H^1(\mathbb{Z}^2; \mathbb{F}_2) = \mathbb{F}_2 \oplus \mathbb{F}_2$ and by a, b, c the generators of each copy of C_2 in $C_2 \wr C_2$. It is not hard so see that $C_2 \wr C_2$ acts on $H^1(\mathbb{Z}^2; \mathbb{F}_2)$ as follows: b and c act as the identity, while a acts by swapping x and y. Thus the $(C_2 \wr C_2)$-module $H^1(\mathbb{Z}^2; \mathbb{F}_2)$ is isomorphic to the coinduction $Coind_{C_2 \times C_2}^{C_2 \wr C_2}(\mathbb{F}_2)$, where $C_2 \times C_2$ acts trivially on $\mathbb{F}_2$. Now we use the Shapiro's Lemma to get

$$E_2^{*,1} := H^*(C_2 \wr C_2; Coind_{C_2 \times C_2}^{C_2 \wr C_2}(\mathbb{F}_2)) \cong H^*(C_2 \times C_2; \mathbb{F}_2)$$

Therefore the E_2 page of the Lyndon-Hochschild-Serre spectral sequence associated to the short exact sequence 10.1 is (the numbers represent the ranks of $E_2^{p,q}$ as $\mathbb{F}_2$-vector spaces):

$$
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\hline
 & & & & & & & &
\end{array}
$$

E_2 page

Multiplicative structure of the second page: let $i : C_2 \times C_2 \to C_2 \wr C_2$ be the natural inclusion of $C_2 \times C_2$ as normal subgroup of $C_2 \wr C_2$ and

$$s : H^*(C_2 \wr C_2; Coind_{C_2 \times C_2}^{C_2 \wr C_2}(\mathbb{F}_2)) \to H^*(C_2 \times C_2; \mathbb{F}_2)$$

be the isomorphism of the Shapiro's Lemma. The following diagram is commutative because of the naturality of the cup product with respect to morphisms of groups and morphisms of the coefficient module:

$$
\begin{tikzcd}
H^p(C_2 \wr C_2; H^1(\mathbb{Z}^2)) \otimes H^q(C_2 \wr C_2; H^0(\mathbb{Z}^2)) \arrow[r, "\cdot"] \arrow[d, "s\otimes 1"] & H^{p+q}(C_2 \wr C_2; H^1(\mathbb{Z}^2)) \arrow[d, "s"] \\
H^p(C_2 \times C_2; \mathbb{F}_2) \otimes H^q(C_2 \wr C_2; H^0(\mathbb{Z}^2)) \arrow[r, "-\cup i^*(-)"] & H^{p+q}(C_2 \times C_2; \mathbb{F}_2)
\end{tikzcd}
$$

Here the top horizontal arrow is the product of the second page of the spectral sequence, the lower horizontal arrow maps an element $a \otimes b$ to $a \cup i^*(b)$. Therefore, if we identify $E_2^{*,1}$ with $H^*(C_2 \times C_2; \mathbb{F}_2)$, we know exactly how to multiply a class $a \in E_2^{*,1}$ with a class $b \in E_2^{*,0}$:

$$a \cdot b = a \cup i^*(b) \in E_2^{*,1} \tag{10.2}$$

The E_∞ page : we know that the spectral sequence we are interested in converges to

$$H^*\left(\frac{\mathbb{Z} \ltimes \mathbb{Z}^2}{< a^2b^2c^2 >}; \mathbb{F}_2\right)$$

The computation of this cohomology is not so hard, but is quite long so for the moment we only state the result:

$$H^i\left(\frac{\mathbb{Z} \ltimes \mathbb{Z}^2}{< a^2b^2c^2 >}; \mathbb{F}_2\right) = \begin{cases} \mathbb{F}_2 \oplus \mathbb{F}_2 \text{ if } i \geq 1 \\ \mathbb{F}_2 \text{ if } i = 0 \end{cases} \tag{10.3}$$

The details about this computation can be found in Section 10.2.

The differentials: we now determine the differentials of page E_2 and E_3 by comparing them with E_∞ and using the multiplicative structure to propagate non trivial differentials. The pages E_2 and E_∞ of the spectral sequence are reported below:

$$\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\hline
 & & & & & & & &
\end{array}
\qquad
\begin{array}{c|cccccccc}
 & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \cdots \\
 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\
 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \cdots \\
\hline
 & & & & & & & &
\end{array}$$

E_2 page $\qquad$ E_∞ page

By Equation 10.3 we know that

$$\begin{cases} c_i + b_{i+1} + a_{i+2} = 2 \text{ if } i \geq 0 \\ b_0 + a_1 = 2 \\ a_0 = 1 \end{cases}$$

In particular $a_i, b_i, c_i \leq 2$ for all i . This information will be crucial in the rest of this section.

Lemma 10.5. $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ *is injective.*

Proof. Suppose by contradiction that $d_2^{0,2}$ is not injective: since $E_2^{0,2}$ has dimension one, then $d_2^{0,2} = 0$. By the multiplicative structure of the spectral sequence it follows that all the differentials $d_2^{*,2}$ are zero. Then the third page of the spectral sequence looks as follows:

$$\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & \cdots \\
 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \cdots \\
\hline
 & & & & & & & &
\end{array}$$

E_3 page

In particular $x_{i+3} = i+4-rk(d_2^{i+1,1})$ and y_{i+1} is the dimension of $Ker(d_2^{i+1,1})$. The elements of $E_3^{*,1}$ are infinite cycles, so $y_i = b_i \leq 2$. Therefore we have an upper bound for x_{i+3}:

$$\begin{aligned} x_{i+3} &= i + 4 - rk(d_2^{i+1,1}) \\ &= i + 4 - (i + 2 - y_{i+1}) = 2 + y_{i+1} \leq 4 \end{aligned}$$

From this inequality we get

$$dim(E_\infty^{i,2}) = dim(Ker(d_3^{i,2})) = i + 1 - rk(d_3^{i,2}) \geq i + 1 - x_{i+3} \geq i - 3$$

This is a contradiction when $i \geq 6$, indeed we know that the dimension of $E_\infty^{p,q}$ must be less or equal to 2 for any p and q. $\square$

Corollary 10.6. *The differentials* $d_2^{2i,2}$ *and* $d_2^{2i+1,2}$ *have both rank* $i+1$.

Proof. Let 1 be the unit of the ring $H^*(C_2 \wr C_2; \mathbb{F}_2)$ and u be the generator of $H^2(\mathbb{Z}^2; \mathbb{F}_2)$. With this notation, any class in $E_2^{p,2}$ is of the form $\alpha \otimes u$, where $\alpha \in H^p(C_2 \wr C_2; \mathbb{F}_2)$. The previous Lemma tells us that $d_2(1 \otimes u) \neq 0$ in $E_2^{2,1} = H^2(C_2 \times C_2; \mathbb{F}_2)$. The multiplicative structure of the Serre spectral sequence combined with Equation 10.2 imply that

$$d_2(\alpha \otimes u) = d_2(1 \otimes u) \cup i^*(\alpha)$$

Since $H^*(C_2 \times C_2; \mathbb{F}_2)$ is a polynomial ring, $d_2(\alpha \otimes u) = 0$ if and only if $i^*(\alpha) = 0$. But Remark 10.3 tells us exactly what classes are sent to zero by i^*. An easy computation shows that for $k = 2i$ or $k = 2i + 1$ the rank of

$$i^* : H^k(C_2 \wr C_2; \mathbb{F}_2) \to H^k(C_2 \times C_2; \mathbb{F}_2)$$

is exactly $i + 1$, and this concludes the proof. □

We are now ready to prove Proposition 10.4:

Proof. (of Proposition 10.4) We prove the dual statement, i.e that the cohomological transfer

$$\tau : H^*(\frac{\mathbb{Z} \ltimes \mathbb{Z}^2}{< a^2b^2c^2 >}; \mathbb{F}_2) \to H^{*-2}(C_2 \wr C_2; \mathbb{F}_2)$$

is zero. Remember that this transfer is given by the composite

$$H^*(\frac{\mathbb{Z} \ltimes \mathbb{Z}^2}{< a^2b^2c^2 >}; \mathbb{F}_2) \to E_\infty^{*-2,2} \to E_2^{*-2,2} = H^{*-2}(C_2 \wr C_2; \mathbb{F}_2)$$

We will see that the row $E_\infty^{*,2}$ is zero, so the transfer will be zero as well. We start by comparing the pages E_2, E_3 and $E_4 = E_\infty$ of the spectral sequence: first of all note that Corollary 10.6 tells us how many classes of the top row survive at the E_3 page. Since the dimension of $E_\infty^{1,0} = E_2^{1,0}$ is 2, Equation 10.3 tells us that $d_2^{0,1}$ is injective. Therefore $E_3^{2,0} = E_\infty^{2,0}$ has dimension two; using again Equation 10.3 we get that $d_2^{1,1}$ is also injective. Summarizing we have:

$$\begin{tikzcd}
1 \arrow[drr] & 2 \arrow[drr] & 3 \arrow[drr] & 4 \arrow[drr] & 5 \arrow[drr] & 6 & 7 & \cdots \\
1 \arrow[drr] & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots
\end{tikzcd}$$

E_2 page

$$\begin{array}{cccccccc}
0 & 1 & 1 & 2 & 2 & 3 & 3 & \cdots \\
0 & 0 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\
1 & 2 & 2 & x_3 & x_4 & x_5 & x_6 & \cdots
\end{array}$$

E_3 page

$$\begin{array}{cccccccc}
0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \cdots \\
0 & 0 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\
1 & 2 & 2 & a_3 & a_4 & a_5 & a_6 & \cdots
\end{array}$$

E_∞ page

Now we can proceed inductively and determine the whole E_∞ page: suppose by induction that $c_i = 0$, $b_{i+1} = 0$ and $a_{i+2} = 2$. we prove that

$$\begin{cases} c_{i+1} = 0 \\ b_{i+2} = 0 \\ a_{i+3} = 2 \end{cases}$$

When $i = 0$ this is true, as we just verified. Let us see how the induction step works: since $c_i = 0$ we have that $d_3^{i,2}$ is injective. So its rank is $\lceil i/2 \rceil$ and

$$\begin{aligned} a_{i+3} &= x_{i+3} - \lceil i/2 \rceil \\ &= i + 4 - rk(d_2^{i+1,1}) - \lceil i/2 \rceil \\ &= i + 4 - (i + 2 - dimKer(d_2^{i+1,1})) - \lceil i/2 \rceil \\ &= 2 + rk(d_2^{i-1,2})) - \lceil i/2 \rceil \end{aligned}$$

where the last equality holds because by inductive hypothesis $b_{i+1} = 0$. By Corollary 10.6 we know that $d_2^{i-1,2}$ has rank $\lfloor (i-1)/2 \rfloor + 1 = \lceil i/2 \rceil$. So we conclude that $a_{i+3} = 2$, and Equation 10.3 implies that $b_{i+2} = c_{i+1} = 0$ as we claimed. $\square$

Remark 10.7. In the preceding proof we determined completely all the differentials of the spectral sequence we where interested in. So it worth writing explicitly the pages:

$$\begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ \hline & & & & & & & & \end{array}$$

E_2 page

$$\begin{array}{c|cccccccc} & 0 & 1 & 1 & 2 & 2 & 3 & 3 & \cdots \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 1 & 2 & 2 & 2 & 3 & 3 & 4 & \cdots \\ \hline & & & & & & & & \end{array}$$

E_3 page

$$\begin{array}{c|cccccccc} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 1 & 2 & 2 & 2 & 2 & 2 & 2 & \cdots \\ \hline & & & & & & & & \end{array}$$

E_∞ page

10.2 Auxiliary computations

In this section we compute the cohomology of $\mathbb{Z} \ltimes \mathbb{Z}^2 / < a^2b^2c^2 >$ with coefficients in $\mathbb{F}_2$. This result was needed in Section 10.1 to prove that the composition of any two equivariant binary operations is zero. Since we have a nice geometric model for the classifying space of $\mathbb{Z} \ltimes \mathbb{Z}^2 / < a^2b^2c^2 >$, i.e. $C_2(\mathbb{C}; C_2(\mathbb{C}))_{S^1}$ (see Proposition 9.5), one can try to compute this cohomology by looking at the Serre spectral sequence of the fibration

$$C_2(\mathbb{C}; C_2(\mathbb{C})) \hookrightarrow C_2(\mathbb{C}; C_2(\mathbb{C}))_{S^1} \to BS^1$$

Let us do this computation: in high degrees we can use some techniques from equivariant cohomology, as the following Proposition shows:

Proposition 10.8. *For any $i > 2$ we have $H^i_{S^1}(C_2(\mathbb{C}; C_2(\mathbb{C})); \mathbb{F}_2) = \mathbb{F}_2 \oplus \mathbb{F}_2$. Moreover $H^2_{S^1}(C_2(\mathbb{C}; C_2(\mathbb{C})); \mathbb{F}_2)$ has dimension greater than two.*

Proof. Let $j : C_2(\mathbb{C}; C_2(\mathbb{C}))^{\mathbb{Z}/2} \hookrightarrow C_2(\mathbb{C}; C_2(\mathbb{C}))$ be the inclusion of the $\mathbb{Z}/2$ fixed points. By Theorem 7.2 this map induces an isomorphism (resp. epimorphism)

$$j^* : H^i_{S^1}(C_2(\mathbb{C}; C_2(\mathbb{C})); \mathbb{F}_2) \to H^i_{S^1}(C_2(\mathbb{C}; C_2(\mathbb{C}))^{\mathbb{Z}/2}; \mathbb{F}_2)$$

in degrees $i > 2$ (resp. $i = 2$). The right hand term can be computed by comparing the Serre spectral sequences associated to the fibrations:

$$\begin{tikzcd}
C_2(\mathbb{C}; C_2(\mathbb{C}))^{\mathbb{Z}/2} \arrow[r] \arrow[d, hook] & C_2(\mathbb{C}; C_2(\mathbb{C}))^{\mathbb{Z}/2} \arrow[d, hook] \\
\left(C_2(\mathbb{C}; C_2(\mathbb{C}))^{\mathbb{Z}/2}\right)_{\mathbb{Z}/2} \arrow[r] \arrow[d] & \left(C_2(\mathbb{C}; C_2(\mathbb{C}))^{\mathbb{Z}/2}\right)_{S^1} \arrow[d] \\
B\mathbb{Z}/2 \arrow[r] & BS^1
\end{tikzcd}$$

The cohomological spectral sequence of the left fibration degenerates at the second page since $\mathbb{Z}/2$ acts trivially. Moreover the map of spectral sequences induced by the map of fibrations above is injective at the second page, so the spectral sequence of the right fibration degenerates at the second page as well. Finally observe that the $\mathbb{Z}/2$-fixed points $C_2(\mathbb{C}; C_2(\mathbb{C}))^{\mathbb{Z}/2}$ are homotopy equivalent to $S^1 \times S^1$, and the statement follows.

□

So it remains to compute $H^i_{S^1}(C_2(\mathbb{C}; C_2(\mathbb{C})); \mathbb{F}_2)$ for $i = 1, 2$. We determine these vector spaces by examining the Serre spectral sequence associated to the fibration

$$C_2(\mathbb{C}; C_2(\mathbb{C})) \hookrightarrow C_2(\mathbb{C}; C_2(\mathbb{C}))_{S^1} \to BS^1$$

We start by recalling how to compute the homology of the fiber.

The (co)homology of the fiber: The homology of the fiber can be computed by Theorem 7.5: $H_*(C_2(\mathbb{C}; C_2(\mathbb{C})); \mathbb{F}_2)$ can be seen as a subspace of $H_*(C(\mathbb{C}; C_2(\mathbb{C})_+); \mathbb{F}_2)$, where $C(\mathbb{C}; C_2(\mathbb{C})_+)$ is the space of configurations of points in the plane with labels in $C_2(\mathbb{C})_+ \coloneqq C_2(\mathbb{C}) \sqcup \{*\}$. Theorem 7.5 explains exactly how to compute the homology of this larger space. The final result is the following: let us denote by ι (resp. u)

the class of degree 0 (resp. 1) in $H_*(C_2(\mathbb{C}); \mathbb{F}_2)$ and let Q be the first Dyer-Lashof operation. Then $H_*(C_2(\mathbb{C}; C_2(\mathbb{C}); \mathbb{F}_2)$ is spanned by the following classes:

Homology class	Degree
ι^2	0
$u \cdot \iota$	1
$Q\iota$	1
$[\iota, u]$	2
u^2	2
Qu	3

Another way to get the same result is to observe that $C_2(\mathbb{C}; C_2(\mathbb{C}))$ is the classifying space of the wreath product $\mathbb{Z} \ltimes \mathbb{Z}^2$, where the generator of $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by swapping the coordinates. Thus its homology can be determined by group homology techniques.

Analysis of the spectral sequence: Now that we know the additive structure of the cohomology of the fiber we have that the E_2 page of the cohomological Serre spectral sequence looks as follows:

$$\begin{tikzcd}
1 \arrow[drr] & 0 & 1 & 0 & 1 & \cdots \\
2 \arrow[drr] & 0 & 2 & 0 & 2 & \cdots \\
2 \arrow[drr] & 0 & 2 & 0 & 2 & \cdots \\
1 & 0 & 1 & 0 & 1 & \cdots
\end{tikzcd}$$

where the numbers represent the ranks of the vector spaces that appear at page E_2. We now state a few Lemmas about the ranks of the differentials of the second page of the Serre spectral sequence associated to the fibration 10.2.

Lemma 10.9. *$E_2^{p,3} = E_\infty^{p,3}$ for any $p \in \mathbb{N}$.*

Proof. By the multiplicativity of the Serre spectral sequence it suffices to show that all the differentials starting at $E_2^{0,3}$ are zero. To prove this claim consider the map of fibrations

$$\begin{tikzcd}
C_2(\mathbb{C}; C_2(\mathbb{C})) \arrow[r, "f"] \arrow[d, hook] & C_4(\mathbb{C}) \arrow[d, hook] \\
C_2(\mathbb{C}; C_2(\mathbb{C}))_{S^1} \arrow[r] \arrow[d] & C_4(\mathbb{C})_{S^1} \arrow[d] \\
BS^1 \arrow[r] & BS^1
\end{tikzcd}$$

where f is the map induced by the group morphism $B_2 \ltimes (B_2)^2 \to B_4$ after applying the classifying space functor. First of all note that this morphism induces an isomorphism between $H_4(C_2(\mathbb{C}; C_2(\mathbb{C}); \mathbb{F}_2)$ and $H_4(C_4(\mathbb{C}); \mathbb{F}_2)$. Dualizing we get that the generator of $H^*(C_4(\mathbb{C}); \mathbb{F}_2)$ is pulled back to the generator of $H^4(C_2(\mathbb{C}; C_2(\mathbb{C}); \mathbb{F}_2)$. By Corollary 8.10 the spectral sequence of the right fibration degenerates at the second page. Therefore the generator of $H^4(C_2(\mathbb{C}; C_2(\mathbb{C}); \mathbb{F}_2)$ is an infinite cycle, and this proves the statement. □

Lemma 10.10. *$E_2^{p,0} = E_\infty^{p,0}$ for any $p \in \mathbb{N}$.*

Proof. We want to prove that all the differentials with target the classes in the 0-th row of the cohomological spectral sequence are zero. Consider the map of fibrations

$$\begin{tikzcd}
C_2(\mathbb{C}; C_2(\mathbb{C})) \arrow[r] \arrow[d, hook] & C_2(\mathbb{C}; C_2(\mathbb{C})) \arrow[d, hook] \\
C_2(\mathbb{C}; C_2(\mathbb{C}))_{\mathbb{Z}/2} \arrow[r] \arrow[d] & C_2(\mathbb{C}; C_2(\mathbb{C}))_{S^1} \arrow[d] \\
B\mathbb{Z}/2 \arrow[r] & BS^1
\end{tikzcd}$$

Since $C_2(\mathbb{C}; C_2(\mathbb{C}))^{\mathbb{Z}/2}$ is non empty the left fibration has a section, and this implies that the classes of the 0-th row are infinite cycles. The map of cohomological spectral sequences induced by the map of fibrations above is injective at the second page, thus all the differentials $d_2^{*,1}$ of the right spectral sequence are zero by comparison. The statement now follow since the classes of $E_2^{p,0} = E_3^{p,0}$ can be killed only by $d_4^{p-4,3}$, which is zero by the preceding Lemma. □

Lemma 10.11. *The differential $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ has rank one.*

Proof. By the previous Lemmas we know that the E_3 page of our spectral sequence has the following form:

$$\begin{array}{c|cccccc}
 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & a & 0 & a & 0 & a & \cdots \\
 & 2 & 0 & b & 0 & b & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\hline
 & & & & & &
\end{array}$$

where a is the dimension of $Ker(d_2^{0,2})$ and $b = 2 - rk(d_2^{0,2})$. By the previous Lemmas we know that the 0-th and top rows consist of infinite cycles, therefore $E_3 = E_\infty$. Proposition 10.8 tells us that

$$\bigoplus_{p+q=i} E_\infty^{p,q} = \mathbb{F}_2 \oplus \mathbb{F}_2$$

provided that i is big enough. Therefore $a = b = 1$, as claimed. □

Now that we understand properly the spectral sequence, we are able to determine $H^i_{S^1}(C_2(\mathbb{C}; C_2(\mathbb{C})); \mathbb{F}_2)$ also for $i = 1, 2$. We sum up the computations of this section in the next statement:

Corollary 10.12. $H^i_{S^1}(C_2(\mathbb{C}; C_2(\mathbb{C})); \mathbb{F}_2) = \mathbb{F}_2 \oplus \mathbb{F}_2$ *for all* $i > 0$.

Part III

On the topology of some (strict) quotients

Chapter 11

Topology of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$

In this chapter we focus on the topology of the strict quotients $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$. The main results are listed below:

- $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$ are not topological manifolds for any $n \geq 4$ (Theorem 11.8).
- $\mathcal{M}_{0,n+1}/\Sigma_n$ can be realized as the complement of a hypersurface inside the weighted projective space $\mathbb{P}(n, n-1, \dots, 2)$ (Proposition 11.9).
- $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$ are simply connected (Theorems 11.16 and 11.17).
- $\mathcal{M}_{0,n+1}/\Sigma_n$ has the same rational homology of the point (Theorem 11.19).

Here is the outline of the chapter:

Section 11.1 presents an embedding of $\mathcal{M}_{0,n+1}/\Sigma_n$ into the weighted projective space $\mathbb{P}(n, n-1, \dots, 2)$ as an open dense subset. This is crucial to prove that $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ are not topological manifolds for $n \geq 4$ (Theorem 11.8).

Section 11.2 contains a description of $\mathcal{M}_{0,n+1}/\Sigma_n$ as the complement of an explicit hypersurface of $\mathbb{P}(n, n-1, \dots, 2)$.

Section 11.3 contains a cambinatorial model for $\mathcal{M}_{0,n+1}/\Sigma_n$ based on cacti. This is useful to do computations when n is small.

Section 11.4 deals with the proof that $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ are simply connected (Theorem 11.17 and Theorem 11.16).

Section 11.5 is about the computation of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Q})$.

11.1 Orbifold structure

In this section we show that $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ are not topological manifolds for $n \geq 4$. For completeness we also discuss the case $n = 3$.

Proposition 11.1. *$\overline{\mathcal{M}}_{0,4}/\Sigma_3$ is homeomorphic to S^2. $\mathcal{M}_{0,4}/\Sigma_3$ is homeomorphic to $S^2 - \{0\}$.*

Proof. $\overline{\mathcal{M}}_{0,4}$ is a compact Riemann surface, which is well known to be homeomorphic to S^2. Since Σ_3 acts faithfully and by biholomorphisms we get that the quotient $\overline{\mathcal{M}}_{0,4}/\Sigma_3$ is again a Riemann surface. By the Riemann-Hurwitz formula we can compute the genus of $\overline{\mathcal{M}}_{0,4}/\Sigma_3$, which turns out to be zero giving the first part of the statement. For $\mathcal{M}_{0,4}/\Sigma_3$ just observe that $\mathcal{M}_{0,4}$ is $\overline{\mathcal{M}}_{0,4}$ minus three points (corresponding to the three stable curves). These points are all identified in $\overline{\mathcal{M}}_{0,4}/\Sigma_3$, so $\mathcal{M}_{0,4}/\Sigma_3$ is $\overline{\mathcal{M}}_{0,4}/\Sigma_3$ minus a point, as claimed. □

We now focus on $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ when $n \geq 4$.

Proposition 11.2. *There is a homeomorphism*

$$\phi : \frac{\mathcal{M}_{0,n+1}}{\Sigma_n} \to \frac{C_n(\mathbb{C})}{\mathbb{C} \rtimes \mathbb{C}^*}$$
$$[[z_1, \ldots, z_n]] \mapsto [\{z_1, \ldots, z_n\}]$$

where $[[z_1 \ldots, z_n]]$ is the class associated to $[(z_1, \ldots, z_n)] \in \mathcal{M}_{0,n+1} \cong F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$.

Proof. $F_n(\mathbb{C})$ is equipped with an action of $\mathbb{C} \rtimes \mathbb{C}^*$ by translations, dilations and rotations. The quotient by this action is exactly $\mathcal{M}_{0,n+1}$. Σ_n acts on $F_n(\mathbb{C})$ by permuting the coordinates, and this action commutes with that of $\mathbb{C} \rtimes \mathbb{C}^*$. Therefore the quotients $(F_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*)/\Sigma_n = \mathcal{M}_{0,n+1}/\Sigma_n$ and $(F_n(\mathbb{C})/\Sigma_n)/\mathbb{C} \rtimes \mathbb{C}^* = C_n(\mathbb{C})/\Sigma_n$ are homeomorphic.

□

Remark 11.3. The unordered configuration space is naturally a subspace of the symmetric power $SP^n(\mathbb{C})$ which is homeomorphic to $\mathbb{C}^n$. The homeomorphism maps an unordered n-uple $\{z_1, \ldots, z_n\}$ to the coefficients $(a_0, \ldots, a_{n-1})$ of the unique monic polynomial $a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n = (z - z_1) \cdots (z - z_n)$ which has $\{z_1, \ldots, z_n\}$ as roots. If we denote by $s_i(z_1, \ldots, z_n)$ the i-th symmetric polynomial in n variables, the coefficient a_i is $s_i(-z_1, \ldots, -z_n)$. Explicitly, we have:

$$\begin{aligned}
a_0 &= (-1)^n z_1 \cdots z_n \\
a_1 &= (-1)^{n-1} \sum_{i=1}^{n} z_1 \cdots \hat{z}_i \cdots z_n \\
&\cdots \\
a_{n-2} &= \sum_{i<j} z_i z_j \\
a_{n-1} &= -(z_1 + \cdots + z_n)
\end{aligned}$$

Definition 11.1. If $\mathbf{z} := (z_1, \dots, z_n)$ is a n-uple of complex numbers, its **barycenter** is

$$B(\mathbf{z}) := \frac{z_1 + \cdots + z_n}{n}$$

Clearly, the barycenter does not depend on the order of the points $z_1, \dots, z_n$.

Before we get into the topology of $\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ we recall the definitions of weighted projective space and lens complex, just to fix the notation.

Definition 11.2. Let $(b_0, \dots, b_n)$ be a $(n+1)$-tuple of positive integers. The **weighted projective space** (of weights $(b_0, \dots, b_n)$) is defined as

$$\mathbb{P}(b_0, \dots, b_n) := \mathbb{C}^{n+1} - \{0\}/\sim$$

where $(z_0, \dots, z_n) \sim (t^{b_0} z_0, \dots, t^{b_n} z_n)$ for any $t \in \mathbb{C}^*$.

Definition 11.3. Let $(b_0, \dots, b_n)$ be a $(n+1)$-tuple of positive integers and $\zeta := e^{2\pi i/b_n}$. The **lens complex** is defined as

$$L(b_n; b_0, \dots, b_{n-1}) := S^{2n-1}/\sim$$

where $(z_0, \dots, z_{n-1}) \sim (\zeta^{b_0} z_0, \dots, \zeta^{b_{n-1}} z_{n-1})$.

Remark 11.4. The topology of $\mathbb{P}(b_0, \dots, b_n)$ and of $L(b_n; b_0, \dots, b_{n-1})$ is well known. For example, the cohomology rings of these spaces were computed by Kawasaki in [29].

Proposition 11.5. *There is a homeomorphism*

$$\begin{aligned} \psi : \frac{SP^n(\mathbb{C}) - \Delta}{\mathbb{C} \rtimes \mathbb{C}^*} &\to \mathbb{P}(n, n-1, \dots, 2) \\ [\{z_1, \dots, z_n\}] &\mapsto [a_0 : \cdots : a_{n-2}] \end{aligned}$$

where $(a_0, \dots, a_{n-2})$ *are the coefficients of the monic polynomial* $(z - z_1 + B(\mathbf{z})) \cdots (z - z_n + B(\mathbf{z}))$ *and* $\Delta := \{\{z, \dots, z\} \in SP^n(\mathbb{C}) \mid z \in \mathbb{C}\}$ *is the diagonal of* $SP^n(\mathbb{C})$.

Remark 11.6. The coefficient a_{n-1} of the monic polynomial $(z - z_1 + B(\mathbf{z})) \cdots (z - z_n + B(\mathbf{z}))$ is 0, indeed

$$a_{n-1} = -z_1 + B(\mathbf{z}) + \cdots - z_n + B(\mathbf{z}) = -(z_1 + \cdots + z_n) + nB(\mathbf{z}) = 0$$

Proof. (of Proposition 11.5) Consider the map

$$\begin{aligned} f : SP^n(\mathbb{C}) &\to SP^n(\mathbb{C}) \\ \{z_1, \dots, z_n\} &\mapsto \{z_1 - B(\mathbf{z}), \dots, z_n - B(\mathbf{z})\} \end{aligned}$$

whose image consists of the unordered n-uples whose barycenter is the origin. By Remark 11.3 there is a homeomorphism $g : SP^n(\mathbb{C}) \to \mathbb{C}^n$ which sends a set of n complex numbers $\{z_1, \dots, z_n\}$ to the coefficients $(a_0, \dots, a_{n-1})$ of the monic polynomial $(z - z_1) \cdots (z - z_n)$. The composite $g \circ f$ maps $\{z_1, \dots, z_n\}$ to the coefficients of the monic polynomial $(z - z_1 + B(\mathbf{z})) \cdots (z - z_n + B(\mathbf{z}))$. By Remark 11.6 the image of $g \circ f$ is $\{(a_0, \dots, a_{n-2}, 0) \mid a_i \in \mathbb{C}\} \cong \mathbb{C}^{n-1}$ therefore we get a map

$$SP^n(\mathbb{C}) - \Delta \xrightarrow{g \circ f} \mathbb{C}^{n-1} - \{0\} \to \mathbb{P}(n, n-1, \dots, 2)$$

where the last map is the natural projection $\mathbb{C}^{n-1} - \{0\} \to \mathbb{P}(n, n-1, \dots, 2)$. It is easy to show that this map is constant on the $\mathbb{C} \rtimes \mathbb{C}^*$ orbits, so it induces a bijective map

$$\psi : \frac{SP^n(\mathbb{C}) - \Delta}{\mathbb{C} \rtimes \mathbb{C}^*} \to \mathbb{P}(n, n-1, \dots, 2)$$

which is a homeomorphism because the source is compact and the target is Hausdorff. $\square$

Remark 11.7. By Proposition 11.2 $\mathcal{M}_{0,n+1}/\Sigma_n$ is homeomorphic to $C_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$, and the latter is naturally an open dense subspace of $SP^n(\mathbb{C}) - \Delta/\mathbb{C} \rtimes \mathbb{C}^* \cong \mathbb{P}(n, n-1, \dots, 2)$. From now on we will see $\mathcal{M}_{0,n+1}/\Sigma_n$ as an open subset of $\mathbb{P}(n, n-1 \dots, 2)$ without no more referring to the embedding just described.

Theorem 11.8. *$\mathcal{M}_{0,n+1}/\Sigma_n$ and $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ are not topological manifolds for any $n \geq 4$.*

Proof. Since $\mathcal{M}_{0,n+1}/\Sigma_n$ is an open subset of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$, it suffices to show that there exist a point in $\mathcal{M}_{0,n+1}/\Sigma_n$ which does not have an Euclidean neighbourhood. Let us consider the point $p \in \mathcal{M}_{0,n+1}/\Sigma_n$ consisting of the roots of the polynomial $z^n + 1$. By the identification of Remark 11.7, this corresponds to the point $[1 : 0 : \cdots : 0]$ of $\mathbb{P}(n, n-1, \dots, 2)$. Since $\mathcal{M}_{0,n+1}/\Sigma_n$ is an open subset of $\mathbb{P}(n, n-1, \dots, 2)$ our claim is equivalent to show that p does not have an Euclidean neighbourhood contained in $\mathbb{P}(n, n-1, \dots, 2)$. Consider the open set

$$\begin{aligned} U_0 &\coloneqq \{[a_0 : \cdots : a_{n-2}] \in \mathbb{P}(n, n-1 \dots, 2) \mid a_0 \neq 0\} \\ &= \{[1 : a_1 : \cdots : a_{n-2}] \in \mathbb{P}(n, n-1 \dots, 2)\} \end{aligned}$$

If $[1 : a_1 : \cdots : a_{n-2}] = [1 : b_1 : \cdots : b_{n-2}]$ then exists $\lambda \in \mathbb{C}^*$ such that

$$\begin{cases} \lambda^n = 1 \\ \lambda^{n-k} a_k = b_k \text{ for } k = 1, \dots, n-2 \end{cases}$$

Therefore $U_0 \cong \mathbb{C}^{n-2}/\sim$ where $(a_1, \dots, a_{n-2}) \sim (b_1, \dots, b_{n-2})$ if and only if there is a n-th root of unity λ such that $\lambda^{n-k} a_k = b_k$ for all $k = 1, \dots, n-2$. Therefore U_0 is homeomorphic to a cone on the lens complex $L(n; n-1, \dots, 2)$. The point $p = [1, 0, \dots, 0]$ is precisely the vertex of this cone, therefore (by excision) we have

$$\begin{aligned} H_k(\mathbb{P}(n, n-1, \dots, 2), \mathbb{P}(n, n-1, \dots, 2) - \{p\}) &= H_k(U_0, U_0 - \{p\}) \\ &= H_{k-1}(U_0 - \{p\}) \\ &= H_{k-1}(L(n; n-1, \dots, 2)) \end{aligned}$$

Suppose by contradiction that p has a neighbourhood U which is homeomorphic to the open unit disk. Then, by excision

$$\begin{aligned} H_k(\mathbb{P}(n, n-1, \ldots, 2), \mathbb{P}(n, n-1, \ldots, 2) - \{p\}) &= H_k(U, U - \{p\}) \\ &= \begin{cases} \mathbb{Z} \text{ if } k = 0, 2(n-2) - 1 \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

But this is a contradiction, because the lens complex $L(n; n-1, \ldots, 2)$ is not a homology spere if $n \geq 4$. For example, using the computations of Kawasaki [29], it is easy to see that $H_{2n-6}(L(n; n-1, \ldots, 2)) = \mathbb{Z}/n\mathbb{Z}$. □

11.2 An algebro-geometric description

Probably an algebraic-geometry minded reader already noticed that the embedding

$$\mathcal{M}_{0,n+1}/\Sigma_n \hookrightarrow \mathbb{P}(n, n-1, \ldots, 2) \tag{11.1}$$

we described in Section 11.1 exhibit $\mathcal{M}_{0,n+1}/\Sigma_n$ as the complement of an algebraic subvariety of $\mathbb{P}(n, n-1, \ldots, 2)$. In this short section we simply make this description explicit. We first recall the definition and the main properties of the discriminant of a polynomial.

Definition 11.4. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial with complex coefficients. The **discriminant** of $p(z)$ is defined as

$$Disc(p(z)) := \frac{(-1)^{n(n-1)/2}}{a_n} \cdot Res(p(z), p'(z))$$

where $Res(p(z), p'(z))$ is the resultant of $p(z)$ and its derivative $p'(z)$, i.e. it is the determinant of the Sylvester matrix of $p(z)$ and $p'(z)$. In particular, $Disc(p(z))$ is a homogeneus polynomial in $a_0, \ldots, a_n$ of degree $2n-2$.

Two important properties of the discriminant are the following:

1. $Disc(p(z)) = 0$ if an only if $p(z)$ has a multiple root.
2. If $a_0^{i_0} a_1^{i_1} \ldots a_n^{i_n}$ is a monomial of $Disc(p(z))$, then $i_0, \ldots, i_n$ satisfy the equation

$$n i_0 + (n-1) i_1 + \cdots + i_{n-1} = n(n-1) \tag{11.2}$$

   In particular, if $a_n = 1$ we get that $Disc(p(z)) \in \mathbb{C}[a_0, \ldots, a_{n-1}]$ is a weighted homogeneus polynomial (with a_i variable of weight $n-i$) and it makes sense to talk about the zero locus of $Disc(p(z))$ inside $\mathbb{P}(n, n-1, \ldots, 1)$.

Proposition 11.9. *Let $p(z) = a_0 + a_1 z + \cdots + a_{n-2} z^{n-2} + z^n$ be a generic polynomial. The embedding*

$$\mathcal{M}_{0,n+1}/\Sigma_n \hookrightarrow \mathbb{P}(n, n-1, \ldots, 2)$$

maps the space $\mathcal{M}_{0,n+1}/\Sigma_n$ to the complement of $\{Disc(p(z)) = 0\} \subseteq \mathbb{P}(n, n-1, \ldots, 2)$.

Proof. If we identify $\mathcal{M}_{0,n+1}/\Sigma_n$ with $C_n(\mathbb{C})/\mathbb{C} \rtimes \mathbb{C}^*$, then the embedding maps a configuration $[\{z_1, \dots, z_n\}]$ to the coefficients $[a_0 : \cdots : a_{n-2}]$ of the polynomial $p(z) = (z - z_1 + B(\mathbf{z})) \cdots (z - z_n + B(\mathbf{z}))$. The complement of the image of this embedding consists exactly of the points $[a_0 : \cdots : a_{n-2}] \in \mathbb{P}(n, n-1, \dots, 2)$ such that the polynomial $a_0 + a_1 z + \dots a_{n-2} z^{n-2} + z^n$ has a multiple root. But this is equivalent to say that the discriminant of such a polynomial is zero. Equation 11.2 tells us that $Disc(p(z))$ is a weighted polynomial with variables a_i of weight $n-i$, thus its zero locus is well defined. □

11.3 A combinatorial model using cacti

In this section we will see how to produce a small combinatorial model for $\mathcal{M}_{0,n+1}/\Sigma_n$ which could be useful to do explicit computations for n small. This model is based on cacti, and we refer to Section 4.1 for more details about the combinatorics of the cacti complex.

Proposition 11.10. *$\mathcal{M}_{0,n+1}/\Sigma_n$ is homotopy equivalent to $\mathcal{C}_n/(S^1 \times \Sigma_n)$.*

Proof. By Proposition 4.4 $F_n(\mathbb{C})$ is $(S^1 \times \Sigma_n)$-equivariantly homotopy equivalent to $\mathcal{C}_n$, so we have a homotopy equivalence between the quotients

$$F_n(\mathbb{C})/(S^1 \times \Sigma_n) \simeq \mathcal{C}_n/(S^1 \times \Sigma_n)$$

The left hand space is homotopy equivalent to $\mathcal{M}_{0,n+1}/\Sigma_n$, and this proves the statement. □

Now let us describe in details the space $\mathcal{C}_n/(S^1 \times \Sigma_n)$: recall that the space of cacti $\mathcal{C}_n$ is a regular CW-complex whose cells are described by cacti (with numbered lobes and a base point). The boundary of a cell is described by collapsing arcs. The $(S^1 \times \Sigma_n)$-action is encoded as follows:

- S^1 acts by rotating the base point.
- Σ_n acts by relabelling the lobes.

To better understand the quotient $\mathcal{C}_n/(S^1 \times \Sigma_n)$ it is useful to first consider the space of unbased cactus $\mathcal{C}_n/S^1 \simeq \mathcal{M}_{0,n+1}$ and then quotient by the Σ_n-action: $\mathcal{C}_n/S^1$ is a CW-complex where a cell is given by an unbased cactus, and the boundary of each cell is obtained by collapsing arcs as in $\mathcal{C}_k$. Now we quotient by the Σ_n-action: this corresponds to relabelling the lobes. This time we have some cells with non-trivial stabilizer and we need to take this into account (some examples of these cells are depicted in Figure 11.1). To sum up, $\mathcal{C}_n/(S^1 \times \Sigma_n)$ is built up by two types of cells, those with trivial stabilizer ("not symmetric cells") and those with non-trivial stabilizer ("symmetric cells"):

1. **Not symmetric cells:** take a cactus c (without base point and labelling of the lobes) and fixes an arbitrary labelling of the lobes. Being a not symmetric

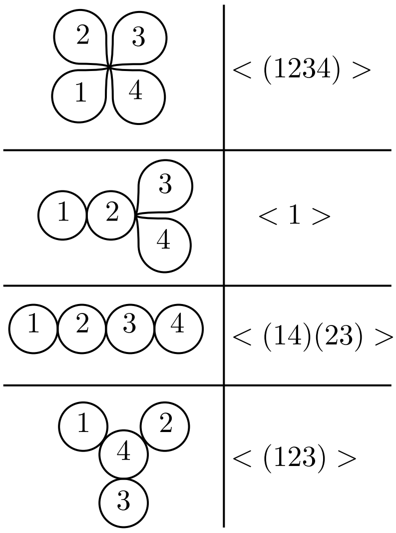


Figure 11.1: On the left there are some cells of $\mathcal{C}_4/S^1$. On the right is represented the stabilizer of the cell respect to the Σ_4 action by relabelling the lobes. In the first row we see a 0-dimensional cell, whose stabilizer is the cyclic group of order four generated by $(1234) \in \Sigma_4$. In the second row there is a 1-cell, which has trivial stabilizer. The last two cells are two dimensional with stabilizer respectively a cyclic group of order two and three.

cell means that the only permutation of Σ_n which fix c as a labelled cactus is the identity. The cell $\sigma(c)$ associated to c is given by:

$$\sigma(c) := \prod_{i=1}^{n} \Delta^{n_i - 1}$$

where n_i is the number of intersection points of the i-th lobe with the other lobes. As for the space of cacti $\mathcal{C}_n$ the parameters of Δ^{n_i-1} represent the length of the arcs between two intersection points. The boundary of $\sigma(c)$ is given by sending to zero some parameter, i.e. collapsing some arc.

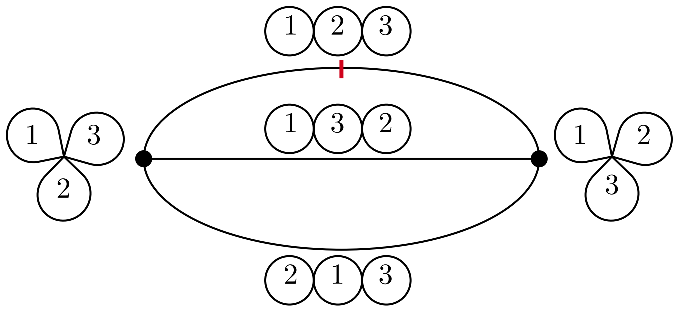
2. **Symmetric cells:** let c be a cactus (without base point and labelling of the lobes) and fix an arbitrary labelling of the lobes. Being a symmetric cell means that the isotropy group of c as a labelled cactus is a non trivial subroup $G_c \leq \Sigma_n$. This cactus gives us a cell

$$\sigma(c) := \frac{\prod_{i=1}^{n} \Delta^{n_i-1}}{G_c}$$

In other words, $\sigma(c)$ is the quotient of the cell associated to c as a labelled cactus (with an arbitrary fixed labelling) by the action of its isotropy group. The boundary of $\sigma(c)$ is given by obtained by sending to zero some parameter, i.e. collapsing some arc.

We conclude this section by discussing some explicit examples.

$\mathcal{C}_3/S^1 =$



$\mathcal{C}_3/\left(S^1 \times \Sigma_3\right) =$

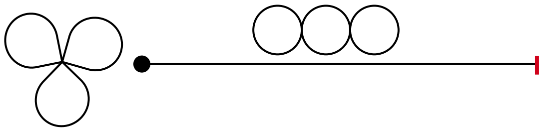


Figure 11.2: On top of this picture we see the CW-complex $\mathcal{C}_3/S^1$; the red segment indicate the middle of the 1-cell. $\mathcal{C}_3/(S^1 \times \Sigma_3)$ is depicted below and it is obtained from $\mathcal{C}_3/S^1$ by quotienting the Σ_3-action: the two 0-cells of $\mathcal{C}_3/S^1$ are identified, and the same happens for the three 1-cells. Since the one cells have as stabilizer $\mathbb{Z}/2$, we have an additional identification: we need to glue together the two halves of any 1-cell.

Example (n=3)**.** There are only two cacti (unlabelled, without base point) with three lobes, let us call them c_0 and c_1 (see Figure 11.2). c_0 is fixed by a rotation of $2\pi/3$, so its stabilizer is $\mathbb{Z}/3$. The corresponding cell is a point. c_1 is fixed by a rotation of π, so its stabilizer is $\mathbb{Z}/2$. The corresponding cell $\sigma(c_1)$ is obtained from $\Delta^1 = \{(t_0, t_1) \in [0,1]^2 \mid t_0 + t_1 = 1\}$ quotienting by the relation $(t_0, t_1) \sim (t_1, t_0)$. Therefore $\mathcal{C}_3/(S^1 \times \Sigma_3)$ is contractible.

Example (n=4)**.** There are four cacti (unlabelled, without base point) with four lobes, let us call them c_0, c_1, c_2 and c_3 (see Figure 11.1 where such cacti are depicted with an arbitrary labelling of the lobes). c_0 is fixed by a rotation of $2\pi/4$, so its stabilizer is $\mathbb{Z}/4$. The corresponding cell is a point. c_1 is not fixed by any rotation, so its corresponding cell $\sigma(c_1)$ is a copy of Δ^1. c_2 is fixed by a rotation of π, so its stabilizer is $\mathbb{Z}/2$. The corresponding cell $\sigma(c_2)$ is obtained from $\Delta^1 \times \Delta^1$ by imposing the relation

$$(t_0, t_1) \times (s_0, s_1) \sim (s_1, s_0) \times (t_1, t_0)$$

Finally, c_3 is fixed by a rotation of $2\pi/3$, so its stabilizer is $\mathbb{Z}/3$. The corresponding cell $\sigma(c_3)$ is obtained from Δ^2 by quotienting the $\mathbb{Z}/3$-action. To be explicit, the generator of $\mathbb{Z}/3$ acts on Δ^2 by permuting cyclically the coordinates (t_0, t_1, t_2).

11.4 Fundamental group

This section contains the computation of $\pi_1(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n)$ and $\pi_1(\mathcal{M}_{0,n+1}/\Sigma_n)$. It turns out that both this spaces are simply connected. We begin with a quick overview of the theory needed to do these computations.

11.4.1 On the fundamental group of orbit spaces

The following results can be found in the paper of B. Noohi [43], which extend a previous work by M.A. Armstrong [2]. The goal of this paragraph is to explain the relation between $\pi_1(X)$ and $\pi_1(X/G)$, when G is a topological group acting on a nice topological space X. In order to compute $\pi_1(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n)$ and $\pi_1(\mathcal{M}_{0,n+1}/\Sigma_n)$ we will only need Theorem 11.12; however I decided to include some more details to give a better description of the theory.

By nice topological space we mean that X is connected, locally path-connected, semilocally simply-connected. We also assume that the orbit space X/G is semilocally simply-connected. From now on we will always assume that we are in this situation.

Remark 11.11. The statements of the next two Theorems contain the notion of *action with slice property*, which is discussed in [45] and [43]. Instead of giving the precise definition we give some important examples where this property is satisfied:

1. If G is a finite group then any action of G on X has the slice property.

2. G Lie group (not necessarily compact), X locally compact. If the map

$$\begin{aligned} G \times X &\to X \times X \\ (g, x) &\mapsto (gx, x) \end{aligned}$$

   is proper, then the action has the slice property.

3. If G is compact Lie group acting on a completely regular space, then the action of G on X has the slice property.

Theorem 11.12 ([43], p. 23)**.** *Let G be a topological group acting on a topological space X which is connected, locally path-connected and semilocally simply-connected. We also assume that X/G is semilocally simply connected. Fix a base point $x_0 \in X$ and let $[x_0]$ be its image in X/G. Suppose that the action has the slice property. If all stabilizer groups G_x are locally path-connected then we have an exact sequence*

$$\pi_1(X, x_0) \longrightarrow \pi_1(X/G, [x_0]) \longrightarrow \pi_0(G)/I \longrightarrow 1$$

where $I \subset \pi_0(G)$ is the subgroup generated by the path components of G containing an element g which fix some point of X.

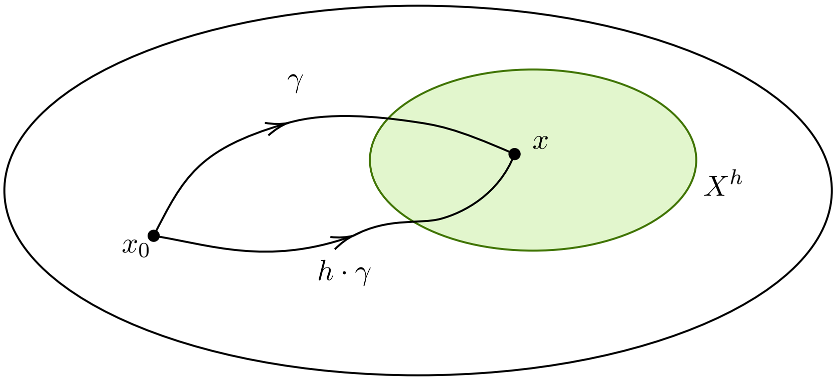


Figure 11.3: In this picture we see the construction of the loop $\alpha_{h,x,\gamma}$. The green area represent X^h.

Remark 11.13. The first map in the exact sequence above is the one induced by the projection $p : X \to X/G$. The second map is defined as follows: given a loop α in X/G we can lift it to a path $\tilde{\alpha}$ in X beginning at the basepont x_0 and ending at $\tilde{\alpha}(1)$. Since α is a loop in X/G we have that $\tilde{\alpha}(1) = g_\alpha x_0$ for some $g_\alpha \in G$. Let us denote by $[g_\alpha]$ the element in $\pi_0(G)/I$ which corresponds to the path component of G containing g_α. It is easy to verify that the assignment

$$\begin{aligned} \pi_1(X/G, [x_0]) &\to \pi_0(G)/I \\ [\alpha] &\mapsto [g_\alpha] \end{aligned}$$

is a well defined function.

Now suppose that the base point $x_0 \in X$ is a fixed point for the action. Under this assumption we have $\pi_0(G)/I = 0$ and therefore $p_* : \pi_1(X, x_0) \to \pi_1(X/G, [x_0])$ is an epimorphism by Theorem 11.12. The next Theorem describes quite explicitly the kernel of this map.

Construction. For every triple (h, x, γ) with $h \in G$, $x \in X^h$ and γ a path from x_0 to x, define $\alpha_{h,x,\gamma}$ to be the loop $(h\gamma)^{-1}\gamma$. Let K be the subgroup of $\pi_1(X, x_0)$ generated by all such loops. This is easily seen to be a normal subgroup. See Figure 11.3 for a picture.

Theorem 11.14 ([43], p.25)**.** *Let G, X be as in Theorem 11.12. Suppose that the base point $x_0 \in X$ is a fixed point for the action. Then the projection $p : X \to X/G$ induces an isomorphism*

$$p_* : \frac{\pi_1(X, x_0)}{K} \to \pi_1(X/G, [x_0])$$

Remark 11.15. The hypothesis that the action have a fixed point is not too restrictive. Indeed we can always find a G-space $C(i)$ such that it has a fixed point and $C(i)/G$ is homotopy equivalent to X/G. The construction of $C(i)$ is the following:

fix a point $x \in X$ and consider the orbit $[x] = \{g \cdot x \mid g \in G\}$. Then define $C(i)$ as the mapping cone associated to the inclusion $i : [x] \hookrightarrow X$ and observe that the action of G on X can be extended to $C(i)$. Therefore $C(i)$ is a G-space and the tip of the cone is a fixed point for the entire action. The quotient $C(i)/G$ is homotopy equivalent to X/G because it is obtained from X/G by gluing $[0,1]$ according to the rule $1 \sim [x]$.

11.4.2 Computations

We now apply the results of the previous paragraph to compute the fundamental group of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ and $\mathcal{M}_{0,n+1}/\Sigma_n$.

Theorem 11.16. *The quotient $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ is simply connected.*

Proof. Since $\overline{\mathcal{M}}_{0,n+1}$ is simply connected, Theorem 11.12 gives us an isomorphism between $\pi_1(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n)$ and Σ_n/I. Let $(i,j) \in \Sigma_n$ be a transposition. It has at least one fixed point since it fixes a configuration whose clustering is of the form $((i,j), 1 \dots, \hat{i}, \dots, \hat{j}, \dots, n)$. Therefore the subgroup I contains all the transpositions, so it must be Σ_n itself. This concludes the proof. $\square$

Theorem 11.17. *The quotient $\mathcal{M}_{0,n+1}/\Sigma_n$ is simply connected.*

Proof. The space $\mathcal{M}_{0,n+1}/\Sigma_n$ is homotopy equivalent to $C_n(\mathbb{C})/S^1$, therefore it suffices to prove statement for this last space. Let us denote by $p : C_n(\mathbb{C}) \to C_n(\mathbb{C})/S^1$ the quotient map. The proof follows several steps:

1. $p_* : B_n \to \pi_1(C_n(\mathbb{C})/S^1)$ is surjective: this follows from Theorem 11.12 with $G = S^1$ because S^1 is connected. Therefore we have a short exact sequence

$$1 \longrightarrow Ker(p_*) \hookrightarrow B_n \xrightarrow{p_*} \pi_1(C_n(\mathbb{C})/S^1) \longrightarrow 1$$

2. $\pi_1(C_n(\mathbb{C})/S^1)$ is abelian: the previous step tells us that $\pi_1(C_n(\mathbb{C})/S^1)$ is a quotient of the braid group, in particular it is generated by elements $\sigma_1, \dots, \sigma_{n-1}$ and in addition to the relations of B_n there are some extra relations coming from $Ker(p_*)$. Let us make explicit some of these relations: the braid $\Delta \coloneqq \sigma_1(\sigma_2\sigma_1)\dots(\sigma_{n-1}\sigma_{n-2}\dots\sigma_1)$ belongs to $Ker(p_*)$ since it is the given by a rotation of π. Therefore we have the relation

$$\sigma_i = \sigma_i\Delta = \Delta\sigma_{n-i} = \sigma_{n-i}$$

Now suppose n is odd. The previous relation enable us to prove that σ_i and σ_{i+1} commute in $\pi_1(C_n(\mathbb{C})/S^1)$:

$$\sigma_i\sigma_{i+1} = \sigma_{n-i}\sigma_{i+1} = \sigma_{i+1}\sigma_{n-i} = \sigma_{i+1}\sigma_i$$

where the middle equality holds because n is odd and therefore $n - i \neq i$ and $n - i \neq i + 2$ for each $i = 1, \dots, n-1$. To get the statement in the case $n = 2k$ we can do the same procedure to show that σ_i and σ_{i+1} commute for each $i \neq k$ and $i \neq k-1$. So it remains to prove the equality $\sigma_k\sigma_{k+1} = \sigma_{k+1}\sigma_k$: combining $\sigma_{k+1}\sigma_{k+2}\sigma_{k+1} = \sigma_{k+2}\sigma_{k+1}\sigma_{k+2}$ and $\sigma_{k+2}\sigma_{k+1} = \sigma_{k+1}\sigma_{k+2}$ we get that $\sigma_{k+2} = \sigma_{k+1}$. Therefore $\sigma_k\sigma_{k+1} = \sigma_k\sigma_{k+2} = \sigma_{k+2}\sigma_k = \sigma_{k+1}\sigma_k$.

3. $\pi_1(C_n(\mathbb{C})/S^1)$ is generated by σ_1: by point (2) we know that $\pi_1(C_n(\mathbb{C})/S^1)$ is abelian. Combining this fact with the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ we get $\sigma_i = \sigma_{i+1}$ for all $i = 1, \dots, n-2$. Thus $\pi_1(C_n(\mathbb{C})/S^1)$ is generated by σ_1.

4. $\pi_1(C_n(\mathbb{C})/S^1)$ is the trivial group: by the previous discussion we know that $\pi_1(C_n(\mathbb{C})/S^1)$ is an abelian group, so it is isomorphic to $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$ by Hurewicz. Moreover $\pi_1(C_n(\mathbb{C})/S^1)$ it is generated by σ_1. So it suffices to show that $\sigma_1 = 0$. We proceed as follows: consider the n-th roots of unity $\{\zeta_1, \dots, \zeta_n\} \in C_n(\mathbb{C})$. Now take the loop
$$\begin{aligned}\alpha : [0,1] &\to C_n(\mathbb{C})\\ t &\mapsto \{e^{2t\pi i/n}\zeta_1, \dots, e^{2t\pi i/n}\zeta_n\}\end{aligned}$$
In plain words α rotates the n-agon $\{\zeta_1, \dots, \zeta_n\}$ counter-clockwise between 0 and $2\pi/n$ degrees. It is easy to see that this loop represent the class $(n-1)\sigma_1 \in H_1(C_n(\mathbb{C});\mathbb{Z})$ (see figure 11.4 for a picture). Therefore $p_*(\alpha)$ is $(n-1)$ times the generator of $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$. But $p_*(\alpha)$ is a constant loop, so it is the zero class in homology. Therefore we get the equation $(n-1)\sigma_1 = 0$ in $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$. Similarly, let $\{\zeta_1, \dots, \zeta_{n-1}\}$ be the set of $(n-1)$-th roots of unity. Consider the loop
$$\begin{aligned}\beta : [0,1] &\to C_n(\mathbb{C})\\ t &\mapsto \{e^{2t\pi i/(n-1)}\zeta_1, \dots, e^{2t\pi i/(n-1)}\zeta_{n-1}, 0\}\end{aligned}$$
In plain words α rotates the configuration $\{\zeta_1, \dots, \zeta_{n-1}, 0\}$ counter-clockwise between 0 and $2\pi/(n-1)$ degrees. It is easy to see that this loop represent the class $n\sigma_1 \in H_1(C_n(\mathbb{C});\mathbb{Z})$ (see Figure 11.4). Therefore $p_*(\alpha)$ is n times the generator of $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$. But $p_*(\alpha)$ is a constant loop, so it is the zero class in homology. Therefore we get the equation $n\sigma_1 = 0$ in $H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$. Finally we can conclude: we know that $\pi_1(C_n(\mathbb{C})/S^1) \cong H_1(C_n(\mathbb{C})/S^1;\mathbb{Z})$ is a cyclic group with generator σ_1. However $n\sigma_1 = 0 = (n-1)\sigma_1$, therefore $\sigma_1 = 0$.

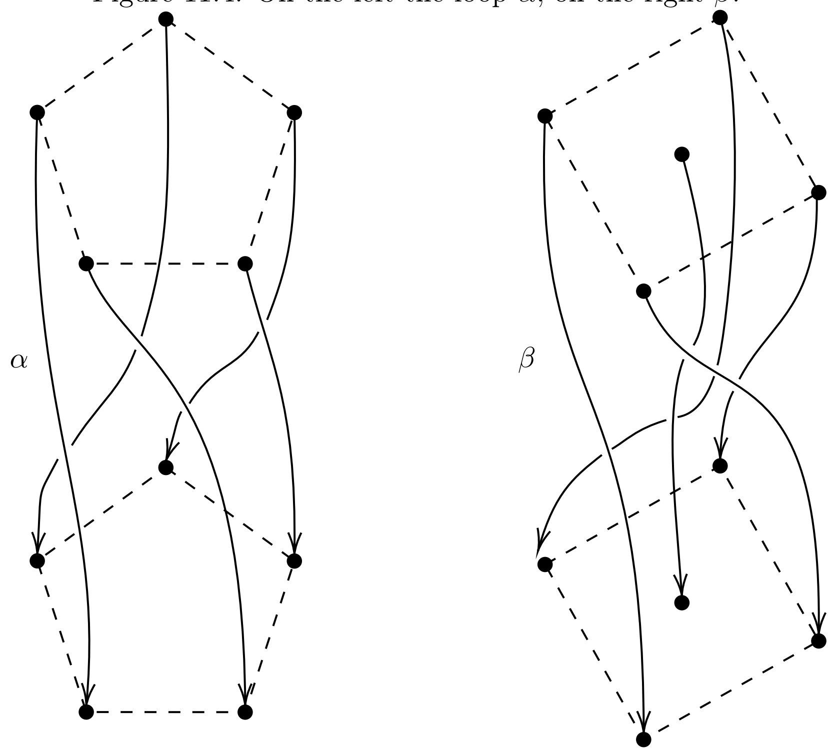
□

11.5 Rational homology

In this section we compute the rational homology of $\mathcal{M}_{0,n+1}/\Sigma_n$ by means of some operad theory. We also discuss some results about the rational homology of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$.

Remark 11.18. Σ_n acts on $\mathcal{M}_{0,n+1}$ with finite stabilizers therefore the map
$$(\mathcal{M}_{0,n+1})_{\Sigma_n} \to \mathcal{M}_{0,n+1}/\Sigma_n$$
induces an isomorphism in rational homology
$$H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{Q})) \xrightarrow{\cong} H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Q})$$
Therefore the classes of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Q})$ may be interpreted as equivariant homology operations for gravity algebras (over the field $\mathbb{Q}$).

Figure 11.4: On the left the loop α, on the right β.



Theorem 11.19. $H_k(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Q}) = \begin{cases} \mathbb{Q} \ \textit{if } k=0 \\ 0 \ \textit{otherwise} \end{cases}$

To proof of this Theorem is elementary but a bit long, so we will subdivide it in several steps. The key observation is that up to a shift in degrees the vector space

$$\bigoplus_{n=2}^{\infty} H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Q})$$

carries a very nice algebraic structure:

Proposition 11.20. *Let us denote by $Grav(x)$ the free gravity algebra on a generator x of degree zero. Then*

$$Grav(x) = \bigoplus_{n=2}^{\infty} sH_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Q})$$

Proof. Let $\mathbb{Q}x$ be the graded vector space of dimension 1 spanned by x. By definition

$$Grav(x) \coloneqq \bigoplus_{n=2}^{\infty} Grav(n) \otimes_{\Sigma_n} (\mathbb{Q}x)^{\otimes n}$$

where $Grav(n) = sH_*(\mathcal{M}_{0,n+1},\mathbb{Q})$. Explicitly, if $p \in H_*(\mathcal{M}_{0,n+1},\mathbb{Q})$ and $\sigma \in \Sigma_n$ we have the identification $(\sigma \cdot p) \otimes x \otimes \cdots \otimes x = p \otimes \sigma \cdot (x \otimes \cdots \otimes x)$. Since x is a variable of degree zero Σ_n acts trivially on $(\mathbb{Q}x)^{\otimes n}$, so we get the identification

$$(\sigma p) \otimes x \otimes \cdots \otimes x \sim p \otimes x \otimes \cdots \otimes x$$

Therefore $Grav(x) = \bigoplus_{n=2}^{\infty} sH_*(\mathcal{M}_{0,n+1}; \mathbb{Q})_{\Sigma_n} = \bigoplus_{n=2}^{\infty} sH_*(\mathcal{M}_{0,n+1}/\Sigma_n; \mathbb{Q})$. $\square$

Lemma 11.21. *In $Grav(x)$, we have $\{\{x, x\}, x, \ldots, x\} = 0$.*

Proof. Let us consider generalized Jacobi relation (Equation 2.2) with $l = 0$ and $n \geq 3$:

$$\sum_{1 \leq i < j \leq n} \pm\{\{a_i, a_j\}, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_n\} = 0 \tag{11.3}$$

The relevant case for us is when $x_k = x$ for all $k = 1, \ldots, n$ and x is a variable of degree zero. In this case all the signs are positive and we get

$$\binom{n}{2}\{\{x, x\}, x, \ldots, x\} = 0$$

and the statement follows.

$\square$

Corollary 11.22. *The only non trivial elements in $Grav(x)$ are of the form $\{x, \ldots, x\}$.*

Proof. Let us consider the generalized Jacobi relation associated to any $k \in \mathbb{N}$ and $l > 0$:

$$\sum_{\leq i < j \leq k} \pm\{\{a_i, a_j\}, a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k, b_1, \ldots, b_l\} = \{\{a_1, \ldots, a_k\}, b_1, \ldots, b_l\}$$

If all the variables a_i, b_i are equal to a variable x of degree zero, the left hand side of the equation written above is 0 because of the previous Lemma. It follows that for all $k \geq 3$, $l > 0$ we have

$$\{\underbrace{\{x, \ldots, x\}}_{k}, x, \ldots, x\} = 0$$

Therefore all the iterated brackets are 0 and the only non trivial elements of $Grav(x)$ are of the form $\{x, \ldots, x\}$. $\square$

Now we are ready to prove Theorem 11.19:

Proof. (of Theorem 11.19) By the previous Corollary, in $Grav(x)$ the only non trivial elements are of the arity k brackets $\{x, \ldots, x\}$. Since x has degree 0 and the arity k bracket $\{-, \ldots, -\}$ has degree one, the element $\{x, \ldots, x\}$ has degree one. By Proposition 11.20 we conclude that the arity k bracket $\{x, \ldots, x\}$ is the suspension of the generator of $H_0(\mathcal{M}_{0,n+1}/\Sigma_n, \mathbb{Q})$. Since these are the only non zero elements of $Grav(x)$, the statement of Theorem 11.19 follows. $\square$

A similar approach can be used to compute $H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$, as the following Proposition suggests:

Proposition 11.23. *Let us denote by $HyCom(x)$ the free hypercommutative algebra on one generator x of degree zero. Then we have*

$$HyCom(x)=\bigoplus_{n=2}^{\infty}H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$$

Proof. Let $\mathbb{Q}x$ be the graded vector space of dimension 1 spanned by x. By definition

$$HyCom(x):=\bigoplus_{n=2}^{\infty}HyCom(n)\otimes_{\Sigma_n}(\mathbb{Q}x)^{\otimes n}$$

where $HyCom(n)=H_*(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})$. Explicitly, if $p\in H_*(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})$ and $\sigma\in\Sigma_n$ we have the identification $(\sigma\cdot p)\otimes x\otimes\cdots\otimes x=p\otimes\sigma\cdot(x\otimes\cdots\otimes x)$. The right hand side is equal to $p\otimes x\otimes\cdots\otimes x$ since x is a degree zero variable. It follows that

$$(\sigma p)\otimes x\otimes\cdots\otimes x\sim p\otimes x\otimes\cdots\otimes x$$

Finally, $HyCom(x)=\bigoplus_{n=2}^{\infty}H_*(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})_{\Sigma_n}=\bigoplus_{n=2}^{\infty}H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$. □

Remark 11.24. This Proposition is the *Koszul dual* (in the sense of Ginzburg-Kapranov [26], see also [21]) of Proposition 11.20. While $\mathcal{M}_{0,n+1}/\Sigma_n$ has the rational homology of a point, the case of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ is highly non trivial. I started thinking about this problem in the Fall 2022 and some computations I did at that time are reported below. However, in March 2023 the problem was solved in the paper [10]: $\overline{\mathcal{M}}_{0,n+1}$ can be constructed by iterated blow-ups from $\mathbb{C}P^{n-2}$. Since the centers of these blow-ups are Σ_n-equivariant, the blow-up formula for cohomology allows the authors to compute $H^*(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})$ as Σ_n-representation. By taking the Σ_n-invariant part they get a formula for the Poincarè polynomial of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$, even if the combinatorics involved looks difficult. So, if you want to know the ranks of the homology of $\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$ you have to count some combinatorial objects, but there is not any closed formula/recursion that may help you in this task. It would be interesting to see if our operadic approach can bring to a better formula for these ranks. I thank Professor Vladimir Dotsenko for pointing out to me the paper [10].

Remark 11.25. In principle Proposition 11.23 gives us a complete computation of $H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$: indeed the Hypercommutative operad has a nice presentation in terms of generators and relations (see Section 2.1); however there are infinitely many of them, making explicit computations a bit involved.

Remark 11.26. Σ_n acts on $\overline{\mathcal{M}}_{0,n+1}$ with finite stabilizers therefore the map

$$(\overline{\mathcal{M}}_{0,n+1})_{\Sigma_n}\to\overline{\mathcal{M}}_{0,n+1}/\Sigma_n$$

induces an isomorphism in rational homology

$$H_*^{\Sigma_n}(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q}))\xrightarrow{\cong}H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$$

We end this section by stating some results about the ranks of $H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$.

Proposition 11.27. *$H_i(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ and $H_{2(n-2)-i}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ have the same dimension as $\mathbb{Q}$-vector spaces.*

Proof. By Poincarè duality we have an isomorphism

$$\begin{aligned}\varphi : H^i(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q}) &\to H_{2(n-2)-i}(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})\\ \psi &\mapsto [\overline{\mathcal{M}}_{0,n+1}]\cap\psi\end{aligned}$$

If $g\in\Sigma_n$, then

$$\begin{aligned}\varphi(g\cdot\psi) &= [\overline{\mathcal{M}}_{0,n+1}]\cap g^*\psi\\ &= (g_*)^{-1}g_*([\overline{\mathcal{M}}_{0,n+1}]\cap g^*\psi)\\ &= (g_*)^{-1}(g_*([\overline{\mathcal{M}}_{0,n+1}])\cap\psi)\\ &= (g_*)^{-1}([\overline{\mathcal{M}}_{0,n+1}]\cap\psi) = g^{-1}\cdot\varphi(\psi)\end{aligned}$$

where the fourth equality holds because Σ_n acts on $\overline{\mathcal{M}}_{0,n+1}$ by complex automorphism, so each $g\in\Sigma_n$ preserves the orientation and, as a consequence, the fundamental class. Therefore the Poincarè duality isomorphism induce an isomorphism between the coinvariants

$$H^i(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})_{\Sigma_n}\xrightarrow{\cong}H_{2(n-2)-i}(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})_{\Sigma_n}$$

The right hand term is isomorphic to $H_{2(n-2)-i}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ Since we are working rationally. The left hand term is isomorphic (by the norm map) to the invariants $H^i(\overline{\mathcal{M}}_{0,n+1};\mathbb{Q})^{\Sigma_n}\cong H^i(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$. Now the statement follows by the Universal Coefficient Theorem which allow us to identify $H^i(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ with $H_i(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$. $\square$

A simple application of Proposition 11.27 and Proposition 11.23 is the following computation:

Corollary 11.28. *$H_2(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ has dimension $n-2$.*

Proof. $H_2(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ is isomorphic to $H_{2(n-2)-2}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ by Proposition 11.27, so it suffices to compute the dimension of this last vector space. By the presentation of the Hypercommutative operad, $H_{2(n-2)-2}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ is generated by classes of the form

$$(\underbrace{(x,\dots,x)}_{k\text{ times}},\underbrace{x,\dots,x}_{n-k\text{ times}})\qquad k=2,\dots,n-1$$

Now we claim the these classes are actually a basis for $H_{2(n-2)-2}(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$: indeed in this case the generalized associativity relations are of the form

$$\sum_{k=2}^{n-1}\binom{n-3}{k-2}(\underbrace{(x,\dots,x)}_{k\text{ times}},\underbrace{x,\dots,x}_{n-k\text{ times}}) = \sum_{k=2}^{n-1}\binom{n-3}{k-2}(x,\underbrace{(x,\dots,x)}_{k\text{ times}},\underbrace{x,\dots,x}_{n-k-1\text{ times}})$$

But x and the symmetric products $(x,\dots,x)$ are all variables of even degree, so we can switch them without producing signs. Therefore the right hand term of the equality above is really the same as the left hand term, so the generalized associativity is actually the trivial relation $0 = 0$. The claim is now proved since there are no other relations. $\square$

More generally, one can combine Proposition 11.23 and Proposition 11.27 to compute $H_*(\overline{\mathcal{M}}_{0,n+1}/\Sigma_n;\mathbb{Q})$ for small values of n. We summarize the results in the following table:

n	Poincarè polynomial
2	1
3	$1+t^2$
4	$1+2t^2+t^4$
5	$1+3t^2+3t^4+t^6$
6	$1+4t^2+7t^4+4t^6+t^8$

Chapter 12

Homotopy quotients vs strict quotients

The aim of this Chapter is to study $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p)$. We do not get a complete answer but in some cases it is possible to compute explicitly this homology. The main result is the computation $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p)$ for any $n \neq 0,1 \bmod p$ (Theorem 12.7 and the proof that $\tilde{H}_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p) = 0$ when $n = p, p+1$ (Theorem 12.10). These computations turns out to be useful to prove that $\mathcal{M}_{0,n+1}/\Sigma_n$ is contractible for $n \leq 5$. Here is the outline of the chapter:

Section 12.1 contains an upper bound for the order of a class in $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z})$ (Theorem 12.1).

Section 12.2 deals with the construction of a long exact sequence (Proposition 12.5) that expresses $H^*(X/S^1;\mathbb{F}_p)$ in terms of $H^*_{S^1}(X;\mathbb{F}_p)$, $H^*_{S^1}(X^{\mathbb{Z}/p};\mathbb{F}_p)$ and $H^*(X^{\mathbb{Z}/p}/S^1;\mathbb{F}_p)$, where X is any suitably nice S^1-space. This sequence is an easy consequence of classical facts about transformation groups, but I was not able to find a reference for it. So I decided to include a detailed construction.

Section 12.3 contains the computation of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p)$ for any $n \neq 0,1 \bmod p$ (Theorem 12.7) and the proof that $\tilde{H}_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p) = 0$ when $n = p, p+1$ (Theorem 12.10). These computations are an easy application of the long exact sequence constructed before.

Section 12.4 is about some explicit computations of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z})$ for n small. In particular we prove that $\mathcal{M}_{0,n+1}/\Sigma_n$ is contractible for any $n \leq 5$, while for $n = 6$ we get a space with non trivial homology.

Section 12.5 is about the computation of $H^{S^1}_*(C_n(\mathbb{C}^*);\mathbb{F}_p)$ when p does not divide n (Theorem 12.12). This result was used in Section 12.3 and Section 12.4.

12.1 On the torsion of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z})$

Theorem 11.19 tells us that rationally $\mathcal{M}_{0,n+1}/\Sigma_n$ has the same homology of a point. In particular each class in the integral homology is a torsion class. This section contains some partial answers to the question "what kind of torsion appears in $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z})$?".

Theorem 12.1. *Let x be a class in $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z})$. Then the order of x divides $n!$.*

Proof. The quotient map $p : \mathcal{M}_{0,n+1} \to \mathcal{M}_{0,n+1}/\Sigma_n$ is a $n!$ fold ramified covering in the sense of [51]. Therefore we have a transfer map $\tau : H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z}) \to H_*(\mathcal{M}_{0,n+1};Z)$ such that the diagram

$$\begin{tikzcd} H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z}) \arrow[rr, "\cdot n!"] \arrow[dr, "\tau"'] & & H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z}) \\ & H_*(\mathcal{M}_{0,n+1};\mathbb{Z}) \arrow[ur, "p_*"'] & \end{tikzcd}$$

commutes. Given $x \in H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z})$ we know that it is a torsion class; however $H_*(\mathcal{M}_{0,n+1};\mathbb{Z})$ is torsion free, so τ is the zero map and we get the statement. □

In order to get more information on the torsion of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{Z})$ we will follow this approach:

- First of all we focus on the calculation of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p)$, with p be any prime number. The advantage is that now we have coefficients in a field and we can use linear algebra.
- Secondly we note that $\mathcal{M}_{0,n+1}/\Sigma_n$ is homotopy equivalent to $C_n(\mathbb{C})/S^1$, which is a bit easier to handle.
- To compute $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p)$ the main idea is that we can compare it to $H_*^{\Sigma_n}(\mathcal{M}_{0,n+1};\mathbb{F}_p)$, which we already know from Section 7.3 and Section 7.4 . I thank my advisor Paolo Salvatore for suggesting this approach to me.

We will get a complete calculation of $H_*(\mathcal{M}_{0,n+1}/\Sigma_n;\mathbb{F}_p)$ when $n \neq 0, 1 \bmod p$ and $n = p, p+1$. The other cases seems more complicated. Before going into the details of the computation we discuss a bit the relation between homotopy quotients and strict quotients.

12.2 Homotopy quotients vs strict quotients

Let X be an Hausdorff space equipped with an S^1-action. In particular, being Hausdorff ensures that the isotropy groups G_x are closed subgroups, i.e. $G_x = S^1$ or it is a finite group. In this paragraph we discuss the relation between the homotopy quotient X_{S^1} and the strict quotient X/S^1. Most of the results presented are easy consequences of the theory of transformation groups, which is developed in the books [8] and [16]. I believe that what follows is well known to the experts in the field, but I was not able to find it in the classical references cited above.

Proposition 12.2 ([8], p. 371). *Let X be a S^1-space, $A \subseteq X$ be a closed invariant subspace and F be an abelian group of coefficients. Suppose that for any $x \in X - (A \cup X^{S^1})$ we have $H^i(BG_x; F) = 0$ for all $i > 0$. Then for any $i \in \mathbb{N}$ the map $f : X_{S^1} \to X/S^1$ induces an isomorphism*

$$f^* : H^i(X/S^1, A/S^1 \cup X^{S^1}; F) \to H^i_{S^1}(X, A \cup X^{S^1}; F)$$

If we fix $\mathbb{F}_p$ as coefficients and $A = X^{\mathbb{Z}/p}$ we get:

Corollary 12.3. *Let X be an S^1-space. Then for any $i \in \mathbb{N}$ the map $f : X_{S^1} \to X/S^1$ induces an isomorphism*

$$f^* : H^i(X/S^1, X^{\mathbb{Z}/p}/S^1; \mathbb{F}_p) \to H^i_{S^1}(X, X^{\mathbb{Z}/p}; \mathbb{F}_p)$$

Remark 12.4. If there are not $\mathbb{Z}/p$-fixed points, then we have

$$H^i(X/S^1; \mathbb{F}_p) \cong H^i_{S^1}(X; \mathbb{F}_p)$$

Intuitively we can justify this statement like this: the order of the stabilizer G_x of any point $x \in X$ is prime to p, so $\tilde{H}_*(G_x; \mathbb{F}_p) = 0$ for any $x \in X$. Therefore the S^1-action on X behave as it would be free, and there is no difference (in (co)homology mod p) between the homotopy quotient and the strict quotient.

We can use this Corollary to get a *Mayer-Vietoris* sequence as follows: the map $f : X_{S^1} \to X/S^1$ sends $(X^{\mathbb{Z}/p})_{S^1}$ to $X^{\mathbb{Z}/p}/S^1$ therefore we get a map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(\frac{X}{S^1}, \frac{X^{\mathbb{Z}/p}}{S^1}) \arrow[r] \arrow[d, red] & H^i(\frac{X}{S^1}) \arrow[r] \arrow[d] & H^i(\frac{X^{\mathbb{Z}/p}}{S^1}) \arrow[r] \arrow[d] & H^{i+1}(\frac{X}{S^1}, \frac{X^{\mathbb{Z}/p}}{S^1}) \arrow[r] \arrow[d, red] & \cdots \\
\cdots \arrow[r] & H^i_{S^1}(X, X^{\mathbb{Z}/p}) \arrow[r] & H^i_{S^1}(X) \arrow[r] & H^i_{S^1}(X^{\mathbb{Z}/p}) \arrow[r] & H^i_{S^1}(X, X^{\mathbb{Z}/p}) \arrow[r] & \cdots
\end{tikzcd}$$

where the red vertical arrows are isomorphisms by the previous Corollary. Therefore we get:

Proposition 12.5 (Mayer-Vietoris). *Fix p a prime and use $\mathbb{F}_p$ as field of coefficients for cohomology. Then we have a long exact sequence*

$$\cdots \longrightarrow H^i(\tfrac{X}{S^1}) \longrightarrow H^i(\tfrac{X^{\mathbb{Z}/p}}{S^1}) \oplus H^i_{S^1}(X) \longrightarrow H^i_{S^1}(X^{\mathbb{Z}/p}) \longrightarrow H^{i+1}(\tfrac{X}{S^1}) \longrightarrow \cdots$$

Remark 12.6. Proposition 12.5 continues to hold if we replace $X^{\mathbb{Z}/p}$ with X^{S^1} and use rational coefficients for cohomology.

Example. An interesting case where we can apply the Mayer-Vietoris sequence is the following: fix $\mathbb{Q}$ as coefficients for cohomology and let Y be any topological space. The free loop space LY carries a natural S^1-action by rotating the loops and the S^1-fixed points are the constant loops, so $LY^{S^1} \cong Y$. In this case the Mayer-Vietoris sequence becomes

$$\cdots \longrightarrow H^i(LY/S^1) \longrightarrow H^i(Y) \oplus H^i_{S^1}(LY) \longrightarrow H^i_{S^1}(Y) \longrightarrow H^{i+1}(LY/S^1) \longrightarrow \cdots$$

12.3 Computations

From now on we fix $\mathbb{F}_p$ as field of coefficients for homology, where p is any prime number. We are interested in the quotient $\mathcal{M}_{0,n+1}/\Sigma_n$, which is homotopy equivalent to $C_n(\mathbb{C})/S^1$. Corollary 12.3 allow us to do a very nice computation:

Theorem 12.7. *If $n \neq 0, 1$ mod p the map $f : C_n(\mathbb{C})_{S^1} \to C_n(\mathbb{C})/S^1$ induces an isomorphism*

$$f^* : H^i(C_n(\mathbb{C})/S^1; \mathbb{F}_p) \to H^i_{S^1}(C_n(\mathbb{C}); \mathbb{F}_p)$$

Proof. Just observe that if $n \neq 0, 1$ mod p there are no $\mathbb{Z}/p$-fixed points. ☐

Now suppose that p divides n (or $n-1$). We would like to apply the Mayer-Vietoris sequence of Proposition 12.5

$$\cdots \longrightarrow H^i(\tfrac{C_n(\mathbb{C})}{S^1}) \longrightarrow H^i(\tfrac{C_n(\mathbb{C})^{\mathbb{Z}/p}}{S^1}) \oplus H^i_{S^1}(C_n(\mathbb{C})) \longrightarrow H^i_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}) \longrightarrow \cdots \tag{12.1}$$

to compute of $H^i(C_n(\mathbb{C})/S^1; \mathbb{F}_p)$, but in this generality the problem is hard. Indeed we do not know $H^*(C_n(\mathbb{C})^{\mathbb{Z}/p}/S^1; \mathbb{F}_p)$, which could be as difficult to compute as $H^*(C_n(\mathbb{C})/S^1; \mathbb{F}_p)$. The only advantage is that $C_n(\mathbb{C})^{\mathbb{Z}/p}/S^1$ is a much smaller space with respect to $C_n(\mathbb{C})/S^1$. Moreover, even if we would be able to do such computation it remains to understand the maps that fit into the Mayer-Vietoris sequence, which can be an even harder problem.

Remark 12.8. In the Mayer-Vietoris sequence above the terms $H^i_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}; \mathbb{F}_p)$ can be computed easily: consider the map of fibrations

$$\begin{tikzcd}
C_n(\mathbb{C})^{\mathbb{Z}/p} \arrow[r] \arrow[d, hook] & C_n(\mathbb{C})^{\mathbb{Z}/p} \arrow[d, hook] \\
\left(C_n(\mathbb{C})^{\mathbb{Z}/p}\right)_{\mathbb{Z}/p} \arrow[r] \arrow[d] & \left(C_n(\mathbb{C})^{\mathbb{Z}/p}\right)_{S^1} \arrow[d] \\
B\mathbb{Z}/p \arrow[r] & BS^1
\end{tikzcd}$$

and observe that the mod p Serre spectral sequence of the left fibration degenerates at the second page. The map of cohomological spectral sequences induced by the map of fibrations above is surjective at the E_2 page so the Serre spectral sequence of the right fibration degenerates at the second page as well. Therefore $H^*_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}; \mathbb{F}_p) = H^*(BS^1; \mathbb{F}_p) \otimes H^*(C_n(\mathbb{C})^{\mathbb{Z}/p}; \mathbb{F}_p)$ as $\mathbb{F}_p$-vector space. Finally we note that $H^*_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}; \mathbb{F}_p)$ is known: $C_n(\mathbb{C})^{\mathbb{Z}/p}$ is homeomorphic to the space of unordered configurations of points in $\mathbb{C}^*$ (Lemma 7.9) and the homology of this space is given by Corollary 7.12.

Remark 12.9. In some cases we can compute all the terms in the Mayer-Vietoris sequence 12.1. For example take $n = pq$ (or $n = pq + 1$) with $(p, q) = 1$. The terms $H^*(C_n(\mathbb{C})^{\mathbb{Z}/p}/S^1; \mathbb{F}_p)$ can be computed as follows: first recall that $C_n(\mathbb{C})^{\mathbb{Z}/p}$ is homeomorphic to $C_q(\mathbb{C}^*)$ (Lemma 7.9), therefore $C_n(\mathbb{C})^{\mathbb{Z}/p}/S^1 \cong C_q(\mathbb{C}^*)/S^1$. Since p does not divide q the subspace $C_q(\mathbb{C}^*)^{\mathbb{Z}/p}$ is empty and by Remark 12.4 we deduce that

$$H^*(C_q(\mathbb{C}^*)/S^1;\mathbb{F}_p) \cong H^*_{S^1}(C_q(\mathbb{C}^*);\mathbb{F}_p)$$

The advantage is that $H^*_{S^1}(C_q(\mathbb{C}^*);\mathbb{F}_p)$ can be computed by the Serre spectral sequence associated to the fibration

$$C_q(\mathbb{C}^*) \hookrightarrow C_q(\mathbb{C}^*)_{S^1} \to BS^1$$

We will postpone this computation to Section 12.5 (Theorem 12.12).

We end this section with an acyclicity result:

Theorem 12.10. *Let p be a prime. Then*

$$H^*(C_p(\mathbb{C})/S^1;\mathbb{F}_p) \cong H^*(C_{p+1}(\mathbb{C})/S^1;\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p & \textit{if } * = 0 \\ 0 & \textit{otherwise} \end{cases}$$

In order to prove this Theorem we need a preliminary Lemma:

Lemma 12.11. *Let $n \in \mathbb{N}$ and p be a prime. Suppose $n = 0, 1$ mod p. If $i : C_n(\mathbb{C})^{\mathbb{Z}/p} \hookrightarrow C_n(\mathbb{C})$ is the inclusion then*

$$i^* : H^k_{S^1}(C_n(\mathbb{C});\mathbb{F}_p) \to H^k_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)$$

is a monomorphism.

Proof. Consider the commutative diagram

$$\begin{tikzcd} H^k_{S^1}(C_n(\mathbb{C})) \arrow[r, "i^*"] \arrow[d] & H^k_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p}) \arrow[d] \\ \left(S^{-1}H^*_{S^1}(C_n(\mathbb{C}))\right)^k \arrow[r, "i^*"] & \left(S^{-1}H^*_{S^1}(C_n(\mathbb{C})^{\mathbb{Z}/p})\right)^k \end{tikzcd}$$

where S is the multiplicatively closed subset $\{1, c, c^2, c^3, \dots\} \subseteq H^*(BS^1;\mathbb{F}_p) = \mathbb{F}_p[c]$, with c be a variable of degree two. By the Localization Theorem 7.1 the bottom horizontal arrow is an isomorphism. The cohomological version of Theorem 7.16 tells us that the mod p spectral sequence associated to the fibration $C_n(\mathbb{C}) \hookrightarrow C_n(\mathbb{C})_{S^1} \to BS^1$ degenerates at the E_2 page, so $H^*_{S^1}(C_n(\mathbb{C});\mathbb{F}_p)$ is a free $H^*(BS^1)$-module. Therefore the left vertical arrow is a monomorphism, and this proves the statement. □

Proof. (of Theorem 12.10) We do the case of $C_p(\mathbb{C})$, the other one is completely analogous. First of all note that $C_p(\mathbb{C})^{\mathbb{Z}/p} \cong C_1(\mathbb{C}^*) = \mathbb{C}^*$. Therefore

$$C_p(\mathbb{C})^{\mathbb{Z}/p}/S^1 \cong \mathbb{C}^*/S^1$$

and we conclude that $C_p(\mathbb{C})^{\mathbb{Z}/p}/S^1$ is contractible. If $i \geq 1$ the Mayer-Vietoris sequence 12.1 becomes

$$\cdots \longrightarrow H^i(\tfrac{C_p(\mathbb{C})}{S^1}) \longrightarrow H^i_{S^1}(C_p(\mathbb{C})) \longrightarrow H^i_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p}) \longrightarrow \cdots$$

By Lemma 12.11 the map $H^k_{S^1}(C_p(\mathbb{C});\mathbb{F}_p) \to H^k_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)$ is a monomorphism, therefore the Mayer-Vietoris sequence splits (for $i \geq 1$) as:

$$0 \longrightarrow H^i_{S^1}(C_p(\mathbb{C})) \longrightarrow H^i_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p}) \longrightarrow H^{i+1}(\tfrac{C_p(\mathbb{C})}{S^1}) \longrightarrow 0$$

To conclude the proof we show that $H^i_{S^1}(C_p(\mathbb{C});\mathbb{F}_p)$ and $H^i_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)$ are vector spaces of the same dimension: by Theorem 7.16 we have $H^*_{S^1}(C_p(\mathbb{C});\mathbb{F}_p) \cong H^*(C_p(\mathbb{C});\mathbb{F}_p)\otimes H^*(BS^1;\mathbb{F}_p)$ as $H^*(BS^1;\mathbb{F}_p)$-module. Similarly $H^*_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p) \cong H^*(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)\otimes H^*(BS^1;\mathbb{F}_p)$ by Remark 12.8. Finally note that $H_*(C_p(\mathbb{C});\mathbb{F}_p)$ is generated by a class of degree zero and one of degree one (i.e. ι^p and $[\iota,\iota]\iota^{p-2}$) therefore $H^i(C_p(\mathbb{C});\mathbb{F}_p)$ and $H^i(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p) \cong H^i(S^1;\mathbb{F}_p)$ have the same dimension for each $i \in \mathbb{N}$. As a consequence we get that $H^i_{S^1}(C_p(\mathbb{C});\mathbb{F}_p)$ and $H^i_{S^1}(C_p(\mathbb{C})^{\mathbb{Z}/p};\mathbb{F}_p)$ have the same dimension, as claimed. □

12.4 Examples

In what follows we use the results of the previous section to do some explicit computations of $H_*(C_n(\mathbb{C})/S^1;\mathbb{F}_p)$. In particular, we will see that the quotients $C_n(\mathbb{C})/S^1$ are contractible until $n = 6$, which is the first non contractible space.

n=1: $C_1(\mathbb{C})/S^1$ is homeomorphic to a half line, so it is contractible.

n=2: by the combinatorial model explained in Section 11.3 $C_2(\mathbb{C})/S^1$ is homotopy equivalent to a CW-complex with only one cell of dimension zero, so it is contractible.

n=3: by the combinatorial model explained in Section 11.3 $C_3(\mathbb{C})/S^1$ is homotopy equivalent to a CW-complex of dimension one (see also Figure 11.2), so there are no homology classes of degrees strictly greater than 1. But $C_3(\mathbb{C})/S^1$ is simply connected, so $H_1(C_3(\mathbb{C})/S^1;\mathbb{Z}) = 0$ and this implies that $C_3(\mathbb{C})/S^1$ is contractible.

n=4: We prove that $C_4(\mathbb{C})/S^1$ is contractible by showing that $H_i(C_4(\mathbb{C})/S^1;\mathbb{Z}) = 0$ for any $i \geq 1$. This is enough to prove the claim since $C_4(\mathbb{C})/S^1$ is homotopic to a CW complex of dimension two, and it is simply connected. By the cellular model we know that there are no homology classes of degrees strictly greater than 2. By Theorem 11.19 we know that the Euler characteristic must be one, so

$$1 = b_0 - b_1 + b_2$$

where b_i is i-th Betti number. But $C_4(\mathbb{C})/S^1$ is simply connected, so $b_1 = 0$ and combining this fact with the equation above we get that $b_2 = 0$. Therefore $C_4(\mathbb{C})/S^1$ is contractible.

n=5: We will prove that $C_5(\mathbb{C})/S^1$ is contractible by showing that for any prime p and any $i \geq 1$ $H_i(C_5(\mathbb{C})/S^1; \mathbb{F}_p) = 0$. The cellular model tells us that $C_5(\mathbb{C})/S^1$ has no homology classes of degrees strictly greater than three. By Theorem 12.1 the order of any class in $H_*(C_5(\mathbb{C}); \mathbb{Z})$ divides 5!. So the only interesting cases are when we take $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_5$ as coefficients for homology.

- $\mathbb{F}_2$-coefficients: by Proposition 12.5 we have a Mayer-Vietoris sequence

$$\cdots \longrightarrow H^i\left(\tfrac{C_5(\mathbb{C})}{S^1}\right) \longrightarrow H^i\left(\tfrac{C_5(\mathbb{C})^{\mathbb{Z}/2}}{S^1}\right) \oplus H^i_{S^1}(C_5(\mathbb{C})) \longrightarrow H^i_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2}) \longrightarrow \cdots$$

Now observe that $C_5(\mathbb{C})^{\mathbb{Z}/2}$ is homeomorphic $C_2(\mathbb{C}^*)$, whose homology with $\mathbb{F}_2$-coefficients is generated by the classes listed below:

Homology class	Degree
a^2b	0
$a[a,b]$	1
$b \cdot Qa$	1
$[a,[a,b]]$	2

For the notation and more details about $H_*(C_n(\mathbb{C}^*); \mathbb{F}_p)$ see Corollary 7.12. Moreover, $C_2(\mathbb{C}^*)/S^1$ is homotopy equivalent to a circle (see Figure 12.1 and its caption for an explanation). The second pages of the Serre spectral sequences that compute $H^*_{S^1}(C_5(\mathbb{C}); \mathbb{F}_2)$ and $H^*_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2}; \mathbb{F}_2)$ are displayed below:

$$\begin{array}{c|cccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\hline
 & \multicolumn{8}{c}{H^*_{S^1}(C_5(\mathbb{C}); \mathbb{F}_2)}
\end{array}
\qquad
\begin{array}{c|cccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\hline
 & \multicolumn{8}{c}{H^*_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2}; \mathbb{F}_2)}
\end{array}$$

As we know, both the spectral sequences degenerate at the second page (Theorem 7.16 and Remark 12.8) and summing over the diagonals we get the ranks of $H^i_{S^1}(C_5(\mathbb{C}))$ and $H^i_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2})$. By Lemma 12.11 the inclusion $i : C_5(\mathbb{C})^{\mathbb{Z}/2} \hookrightarrow C_5(\mathbb{C})$ induces a monomorphism $i^* : H^i_{S^1}(C_5(\mathbb{C}); \mathbb{F}_2) \to H^i_{S^1}(C_5(\mathbb{C})^{\mathbb{Z}/2}; \mathbb{F}_2)$ in each degree, so we can conclude that it is an isomorphism for any $i \geq 2$ by looking at the ranks. If we put all these information in the Mayer-Vietoris sequence we conclude immediately that

$$H^i(C_5(\mathbb{C})/S^1; \mathbb{F}_2) = 0 \quad \text{for all } i \geq 3$$

Now $C_5(\mathbb{C})/S^1$ is simply connected, so the first (co)homology group is zero. By the same argument as before using the Euler characteristic we can conclude that $H^2(C_5(\mathbb{C})/S^1; \mathbb{F}_2) = 0$ as well. So there is no 2-torsion in $H_*(C_5(\mathbb{C})/S^1; \mathbb{Z})$.

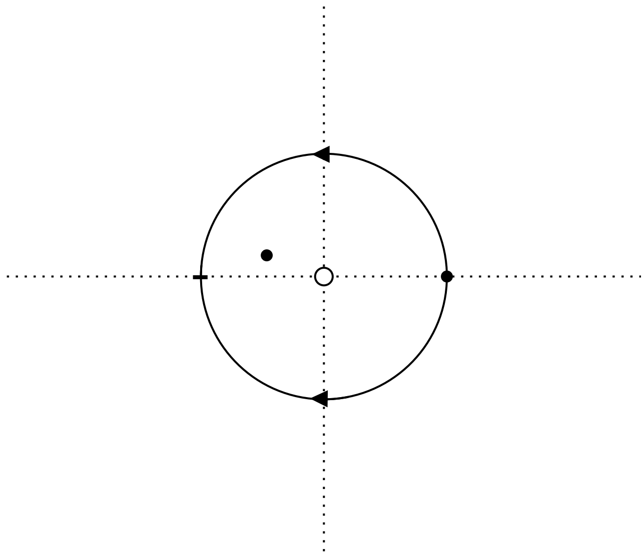


Figure 12.1: This picture explains the homotopy equivalence between $C_2(\mathbb{C}^*)/S^1$ and S^1: first of all note that $C_2(\mathbb{C}^*)/S^1$ is homotopy equivalent to $C_2(\mathbb{C}^*)/\mathbb{C}^*$. Now observe that any configuration $\{z_1, z_2\} \in C_2(\mathbb{C}^*)$ is equivalent (up to rotations and dilations) to a configuration of the form $\{1, z\}$, where $z \in D^2 - \{0, 1\}$. If $|z_1| \neq |z_2|$ there is a unique representative $\{1, z\}$ of the class $[z_1, z_2] \in C_2(\mathbb{C}^*)/\mathbb{C}^*$. In the case $|z_1| = |z_2|$ there are two representatives of $[z_1, z_2]$: $\{1, z_1 z_2^{-1}\}$ and $\{1, z_2 z_1^{-1}\}$. Therefore $C_2(\mathbb{C}^*)/\mathbb{C}^*$ is homeomorphic to the space obtained from $D^2 - \{0, 1\}$ by gluing the boundary of the disk according to the relation $z \sim z^{-1}$. This space is homeomorphic to S^2 minus two points, so we can conclude that $C_2(\mathbb{C}^*)/\mathbb{C}^*$ is homotopy equivalent to a circle.

- $\mathbb{F}_3$-coefficients: by Theorem 12.7 $H_*(C_5(\mathbb{C})/S^1;\mathbb{F}_3) \cong H_*^{S^1}(C_5(\mathbb{C});\mathbb{F}_3)$. Moreover $H_*^{S^1}(C_5(\mathbb{C});\mathbb{F}_3)$ is isomorphic to the subspace of $H_*(C_5(\mathbb{C});\mathbb{F}_3)$ spanned by the classes which do not contain the bracket $[\iota,\iota]$ (Theorem 7.19). In this case $H_*(C_5(\mathbb{C});\mathbb{F}_3)$ has only two classes: ι^5 of degree zero and $\iota^3[\iota,\iota]$ of degree one, so we can conclude that $H_*^{S^1}(C_5(\mathbb{C});\mathbb{F}_3)$ is trivial. Hence there is no 3-torsion in $H_*(C_5(\mathbb{C})/S^1;\mathbb{Z})$.

- $\mathbb{F}_5$-coefficients: By Theorem 12.10 $H^i(C_5(\mathbb{C})/S^1;\mathbb{F}_5) = 0$ for any $i \geq 1$. Thus there is no 5-torsion in $H_*(C_5(\mathbb{C})/S^1;\mathbb{Z})$.

n=6: as we said at the beginning of this section, $C_6(\mathbb{C})/S^1$ is the first non contractible space. In particular its homology with $\mathbb{F}_3$-coefficients will be not trivial. By Theorem 12.1 the order of any class in $H_*(C_6(\mathbb{C})/S^1;\mathbb{Z})$ divides 6!. So the only interesting coefficients for homology are $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_5$. From the cellular model described in Section 11.3 we know that $C_6(\mathbb{C})/S^1$ is homotopy equivalent to a CW-complex of dimension 4, therefore $H^i(C_6(\mathbb{C})/S^1;\mathbb{Z}) = 0$ for any $i \geq 5$. Moreover $C_6(\mathbb{C})/S^1$ is simply connected and its Euler characteristic is equal to one, so we get $0 = b_2 - b_3 + b_4$.

- $\mathbb{F}_2$-coefficients: this case is a bit complicated and I was not able to fully compute $H_*(C_6(\mathbb{C})/S^1;\mathbb{F}_2)$ using the Mayer-Vietoris sequence. To complete the computation one should get some information about the ranks of the maps that fits into the Mayer-Vietoris sequence, but at the moment I do not know how to get these informaiton. I expect that there is no 2-torsion in $H_*(C_6(\mathbb{C})/S^1;\mathbb{F}_2)$ because of some brute force computation I have done with the cellular model explained in Section 11.3.

- $\mathbb{F}_3$-coefficients: consider the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^i(\tfrac{C_6(\mathbb{C})}{S^1}) \longrightarrow H^i(\tfrac{C_6(\mathbb{C})^{\mathbb{Z}/3}}{S^1}) \oplus H^i_{S^1}(C_6(\mathbb{C})) \longrightarrow H^i_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3}) \longrightarrow \cdots$$

Now observe that $C_6(\mathbb{C})^{\mathbb{Z}/3}$ is homeomorphic $C_2(\mathbb{C}^*)$, whose homology with $\mathbb{F}_3$-coefficients is generated by the classes listed below:

Homology class	Degree
a^2b	0
$b[a,a]$	1
$a[a,b]$	1
$[a,[a,b]]$	2

As we already know $C_2(\mathbb{C}^*)/S^1$ is homotopy equivalent to a circle, so it has no classes of degrees greater that two. The second pages of the Serre spectral sequences that compute $H^*_{S^1}(C_6(\mathbb{C});\mathbb{F}_3)$ and $H^*_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3};\mathbb{F}_3)$ are displayed below:

$$\begin{array}{c|cccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\hline
 & \multicolumn{8}{c}{H^*_{S^1}(C_6(\mathbb{C});\mathbb{F}_3)}
\end{array}
\qquad
\begin{array}{c|cccccccc}
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & \cdots \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
\hline
 & \multicolumn{8}{c}{H^*_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3};\mathbb{F}_3)}
\end{array}$$

As we know, both the spectral sequences degenerate at the second page and summing over the diagonals we get the ranks of $H^i_{S^1}(C_6(\mathbb{C}))$ and $H^i_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3})$. By Lemma 12.11 the inclusion $i : C_6(\mathbb{C})^{\mathbb{Z}/3} \hookrightarrow C_6(\mathbb{C})$ induces a monomorphism $i^* : H^i_{S^1}(C_6(\mathbb{C});\mathbb{F}_3) \to H^i_{S^1}(C_6(\mathbb{C})^{\mathbb{Z}/3};\mathbb{F}_3)$ in each degree, so we can conclude that it is an isomorphism for any $i \geq 4$ by looking at the ranks. If we put all these information in the Mayer-Vietoris sequence we conclude immediately that

$$H^i(C_6(\mathbb{C})/S^1;\mathbb{F}_3) = \begin{cases} 0 & \text{for all } i \geq 5 \text{ and } i = 1,2 \\ \mathbb{F}_3 & \text{for } i = 0,3,4 \end{cases}$$

Thus $C_6(\mathbb{C})/S^1$ is not contractible, since there are 3-torsion classes in $H_*(C_6(\mathbb{C})/S^1;\mathbb{Z})$.

- $\mathbb{F}_5$-coefficients: by Theorem 12.10 $H^i(C_6(\mathbb{C})/S^1;\mathbb{F}_5) = 0$ for any $i \geq 1$, so there is no 5-torsion in $H_*(C_5(\mathbb{C})/S^1;\mathbb{Z})$.

12.5 Extra: computation of $H^{S^1}_*(C_n(\mathbb{C}^*);\mathbb{F}_p)$ when $n \neq 0$ mod p

This section collects some results about the additive structure of $H^*_{S^1}(C_n(\mathbb{C}^*);\mathbb{F}_p)$. In particular we present an explicit computation in the case $p \nmid n$. The case when n is divisible by p remains open. The main result is the following:

Theorem 12.12. *Let p be any prime and $n \in \mathbb{N}$ be a natural number not divisible by p. Then*

$$H^{S^1}_*(C_n(\mathbb{C}^*);\mathbb{F}_p) \cong coker(\Delta)$$

where $\Delta : H_*(C_n(\mathbb{C}^*)) \to H_{*+1}(C_n(\mathbb{C}^*))$ is the BV-operator.

Remark 12.13. This Theorem gives us an explicit description of $H^{S^1}_*(C_n(\mathbb{C}^*);\mathbb{F}_p)$: by Corollary 7.12 we get that any class in $H_*(C_n(\mathbb{C}^*);\mathbb{F}_p)$ is a linear combination of monomials of the form $b_k \cdot x$ where $x \in H_*(C_{n-k}(\mathbb{C});\mathbb{F}_p)$ and $b_k := ad^k(a)(b)$ (for this notation see Section 7.5). To be explicit

$$\begin{aligned}
b_0 &= b\\
b_1 &= [b,a]\\
b_2 &= [[b,a],a]\\
b_3 &= [[[b,a],a],a]\\
&\dots
\end{aligned}$$

Therefore if we want to compute Δ on $H_*(C_n(\mathbb{C}^*);\mathbb{F}_p)$ it suffices to know its value on these generators. Since

$$\Delta(xy) = \Delta(x)y + (-1)^x x\Delta(y) + (-1)^x[x,y]$$

it suffices to know how Δ acts on the brackets $\{b_k\}_{k\in\mathbb{N}}$ and on the classes $x \in H_*(C_{n-k}(\mathbb{C}^*);\mathbb{F}_p)$. b_k is a top dimensional class in $H_*(C_k(\mathbb{C}^*);\mathbb{F}_p)$ therefore

$$\Delta(b_k) = 0 \quad k \in \mathbb{N}$$

On the other hand, $\Delta(x) = 0$ if $n = 0,1$ mod p (Theorem 7.16), while if $n \neq 0,1$ mod p $\Delta(x)$ is given by the formulas 7.5. For example, let us suppose $n = 3$ and $p = 2$. The generators of $H_*(C_3(\mathbb{C}^*);\mathbb{F}_2)$ are listed in the first column of this table:

Homology class x	Degree	$\Delta(x)$
a^3b	0	$a^2[a,b]$
$a^2[a,b]$	1	0
$abQa$	1	$[a,b]Qa + a[a,[a,b]]$
$[a,b]Qa$	2	$[a,[a,[a,b]]]$
$[a,[a,b]]a$	2	$[a,[a,[a,b]]]$
$[a,[a,[a,b]]]$	3	0

In the last column we see the image of each generator through Δ. Therefore

$$H_i^{S^1}(C_3(\mathbb{C}^*);\mathbb{F}_2) = \begin{cases} \mathbb{F}_2 \text{ if } i = 0,1,2\\ 0 \text{ otherwise}\end{cases}$$

The idea behind the proof of Theorem 12.12 is to analyse the Serre spectral sequence associated to the fibration

$$C_n(\mathbb{C}^*) \hookrightarrow C_n(\mathbb{C}^*)_{S^1} \to BS^1$$

As usual it is useful to compare the spectral sequence to one that is better understood. To define the comparison spectral sequence we need to introduce some notation:

Definition 12.1. Let us denote by $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)$ the space $F_n(\mathbb{C}^*)/\Sigma_{n-1}$, where Σ_{n-1} is embedded in Σ_n as the subgroup that fix n. As the notation suggests, we can think of a point of $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)$ as a configuration in $\mathbb{C}^*$ of $n-1$ indistinguishable black particles and one white particle.

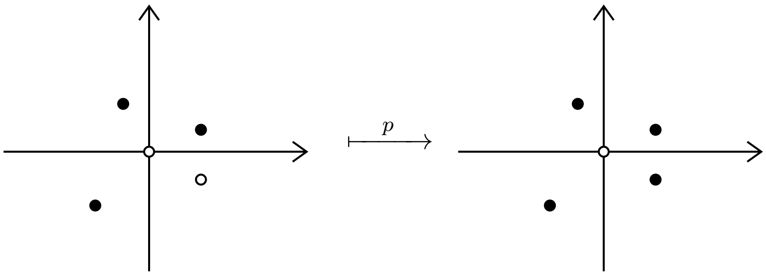


Figure 12.2: The n-fold covering $p : C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \to C_n(\mathbb{C}^*)$

S^1 acts by rotations on $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)$ and we have a n-fold covering

$$p : C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \to C_n(\mathbb{C}^*)$$

which colors black the white particle (see Figure 12.2). Since p is S^1-equivariant we get a map of fibrations which will be the key to prove Theorem 12.12:

$$\begin{tikzcd}
C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \arrow[r] \arrow[d, hook] & C_n(\mathbb{C}^*) \arrow[d, hook] \\
C_{(n-1)\bullet+\circ}(\mathbb{C}^*)_{S^1} \arrow[r] \arrow[d] & C_n(\mathbb{C}^*)_{S^1} \arrow[d] \\
BS^1 \arrow[r] & BS^1
\end{tikzcd}$$

Lemma 12.14. *Fix a field of coefficients $\mathbb{F}$ for (co)homology. Then the (co)homological spectral sequence associated to the fibration*

$$C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \hookrightarrow C_{(n-1)\bullet+\circ}(\mathbb{C}^*)_{S^1} \to BS^1$$

degenerates at page E^3. More precisely, we have

$$E^3_{i,j} = \begin{cases} H_j(C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1) \text{ if } i = 0 \\ 0 \text{ if } i > 0 \end{cases}$$

Proof. We first observe a few elementary facts:

1. S^1 acts freely on $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)$, therefore the homotopy quotient $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)_{S^1}$ and the strict quotient $C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1$ are homotopy equivalent.

2. The continuous map

$$s : F_n(\mathbb{C}^*) \to F_n(\mathbb{C}^*)$$
$$(z_1, \ldots, z_n) \mapsto \left(\frac{z_n}{|z_n|}\right)^{-1} \cdot (z_1, \ldots, z_n)$$

is $(\Sigma_{n-1} \times S^1)$-equivariant, so it induces a map

$$\begin{tikzcd} F_n(\mathbb{C}^*) \arrow[r, "s"] \arrow[d] & F_n(\mathbb{C}^*) \arrow[r] & C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \\ C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1 \arrow[urr, dashed] & & \end{tikzcd}$$

which is a section to the S^1-principal bundle

$$\pi : C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \to C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1$$

Therefore $C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \cong S^1 \times C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1$.

From the second point we can rewrite the fibration of the statement as

$$S^1 \times C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1 \hookrightarrow C_{(n-1)\bullet+\circ}(\mathbb{C}^*)/S^1 \to BS^1$$

Now apply Proposition 6.7 and conclude. $\square$

Now we are ready to prove Theorem 12.12:

Proof. (of Theorem 12.12) Consider the Serre spectral sequence with $\mathbb{F}_p$-coefficients associated to the fibration

$$C_n(\mathbb{C}^*) \hookrightarrow C_n(\mathbb{C}^*)_{S^1} \to BS^1$$

We claim that $E^3_{p,q} = 0$ for any $p > 0$, $q \in \mathbb{N}$, therefore the spectral sequence degenerates at the third page and

$$H_*^{S^1}(C_n(\mathbb{C}^*);\mathbb{F}_p) \cong E^3_{0,*} \cong coker(\Delta)$$

where the last isomorphisms follows directly from Proposition 6.8. Consider the map of fibrations

$$\begin{tikzcd} C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \arrow[r, "p"] \arrow[d, hook] & C_n(\mathbb{C}^*) \arrow[d, hook] \\ C_{(n-1)\bullet+\circ}(\mathbb{C}^*)_{S^1} \arrow[r] \arrow[d] & C_n(\mathbb{C}^*)_{S^1} \arrow[d] \\ BS^1 \arrow[r] & BS^1 \end{tikzcd}$$

Let us denote by (E_*, d_*) (resp. (E'_*, d'_*)) the homological spectral sequence associated to the right (resp. left) fibration . Fix a cycle $x \in E_2^{i,j}$ with $i > 0$, and let $\tau : H_*(C_n(\mathbb{C}^*);\mathbb{F}_p) \to H_*(C_{(n-1)\bullet+\circ}(\mathbb{C}^*);\mathbb{F}_p)$ be the transfer map associated to the n-fold covering p. We are going to show that x does not survive at page E_3, i.e. page E_3 contains elements only in the first column and this proves the claim. The key property we need to prove the Theorem is:

$$(\tau \otimes 1) \circ d_2 = d'_2 \circ (\tau \otimes 1) \tag{12.2}$$

We refer to Corollary 12.15 for a proof of this equation. If we assume this result, then we have that $d_2' \circ (\tau \otimes 1)(x) = (\tau \otimes 1) \circ d_2(x) = 0$ because x is a cycle. By the previous Lemma $(\tau \otimes 1)(x)$ does not survive at E_3', so there exists $y \in E_2'$ such that $d_2'(y) = (\tau \otimes 1)(x)$. Applying $p_* \otimes 1$ we obtain

$$(p_* \otimes 1)(d_2'(y)) = (p_* \otimes 1) \circ (\tau \otimes 1)(x) = n \cdot x$$

But n is not zero in $\mathbb{F}_p$ so we can divide by n and get

$$d_2 \circ (p_* \otimes 1)(y/n) = (p_* \otimes 1) \circ d_2'(y/n) = x$$

which is exactly what we claimed. □

We end this section justifying Equation 12.2:

Corollary 12.15. *The equality 12.2 holds.*

Proof. By Proposition 6.8 d_2 and d_2' are given by the operator Δ, so it suffices to prove the equality

$$\tau\Delta = \Delta\tau$$

Consider the pullback square

$$\begin{tikzcd} S^1 \times C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \arrow[r, "\theta'"] \arrow[d, "1\times p"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_{(n-1)\bullet+\circ}(\mathbb{C}^*) \arrow[d, "p"] \\ S^1 \times C_n(\mathbb{C}^*) \arrow[r, "\theta"] & C_n(\mathbb{C}^*) \end{tikzcd}$$

where the horizontal arrows are the S^1-actions. Let us denote by τ (resp. τ') the homological transfer associated to right (resp. left) vertical arrow. Fix a class $y \in H_*(C_n(\mathbb{C}^*))$ and consider its cross product $[S^1] \times y \in H_*(S^1 \times C_n(\mathbb{C}^*))$; by the naturality of the transfer under pullbacks we get

$$\begin{aligned} \tau\Delta(y) &= \tau\theta_*([S^1] \times y) \\ &= \theta'_*\tau'([S^1] \times y) \\ &= \theta'_*([S^1] \times \tau(y)) = \Delta\tau(y) \end{aligned}$$

□

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