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Cohomological rank functions and Seshadri constants on abelian varieties

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Notations and conventions

We work over an algebraically closed field k of characteristic zero. Other conditions on the field, if necessary, will be oportunely noted. By an abelian variety A defined over k we mean a complete smooth projective algebraic group (in particular, commutative) over k . Usually g will indicate the dimension of A. Given a point $a \in A$, we consider the translation map

$$t_a : A \to A \quad \text{given by} \quad x \mapsto x + a,$$

where "+" stands for the group law. For a positive integer b we write

$$b_A : A \to A, \quad x \mapsto b \cdot x = \underbrace{x + \cdots + x}_{b\text{-times}},$$

for the multiplication by b isogeny. We write $A[b]$ for the subgroup of the b-torsion points of A, that is, the kernel of b_A.

$\mathrm{Pic}^0(A)$ is the subgroup of $\mathrm{Pic}(A)$ given by translation-invariant line bundles (i.e: $L \in \mathrm{Pic}(A)$ such that $t_a^* L \simeq L$ for every $a \in A$). By a polarization on A we mean an ample class in $\mathrm{NS}(A) = \mathrm{Pic}(A)/\mathrm{Pic}^0(A)$. Usually we will use capital letters for line bundles and small letters for their class in $\mathrm{NS}(A)$. For example, ℓ usually will denote the class of $L \in \mathrm{Pic}(A)$ in $\mathrm{NS}(A)$.

Given an abelian variety A defined over k we denote by $\hat{A}$ its dual abelian variety. That is, $\hat{A}$ is the k-abelian variety representing the functor $\mathscr{P}ic^0_A : \mathbf{Sch}_k \to \mathbf{Sets}$ given by

$$\mathscr{P}ic^0_A(T) = \left\{ \mathcal{L} \in \mathrm{Pic}(A \times_k T) : \mathcal{L}|_{A \times \{t\}} \in \mathrm{Pic}^0(A) \text{ for every } t \in T \right\} / \sim$$

where $\mathcal{L} \sim \mathcal{M}$ if there exists $M \in \mathrm{Pic}(T)$ such that $\mathcal{L} \otimes p^* M \simeq \mathcal{M}$, $p : A \times_k T \to T$ being the natural projection.

In the above setting, given a closed point $\alpha \in \hat{A}$ we write $P_\alpha \in \mathrm{Pic}^0(A)$ for the corresponding line bundle. We write $\mathcal{P}$ for a line bundle on $A \times \hat{A}$ which has the property that

$$\mathcal{P}|_{A \times \alpha} = P_\alpha \;\text{ and }\; \mathcal{P}|_{0 \times \hat{A}} = \mathcal{O}_{\hat{A}},$$

such $\mathcal{P}$ is called *a normalized Poincaré bundle*. In the notation of the above paragraph we have that $\mathcal{P} \in \mathscr{P}ic^0_A(\hat{A})$ corresponds to id_A under the natural isomorphism $\mathscr{P}ic^0_A \simeq \mathrm{Hom}(-, \hat{A})$.

For a polarization ℓ with representant L we will write φ_ℓ or φ_L for the isogeny $A \to \hat{A}$ which, under the identification $\hat{A}(k) \simeq \mathrm{Pic}^0(A)$, is given by

$$\varphi_L(a) = t_a^* L \otimes L^\vee,$$

where $L^\vee$ stands for the line bundle $\mathscr{H}om(L, \mathcal{O}_A)$.

For a k-projective variety X we will write $\mathrm{Coh}(X)$ for the category of coherent sheaves on X and $D^b(X)$ for $D^b(\mathrm{Coh}(X)$, the derived category of bounded complexes in $\mathrm{Coh}(X)$. For $x \in X$ $k(x) \in \mathrm{Coh}(X)$ denotes the skycraper sheaf on supported on x. For an object $\mathcal{G} \in \mathrm{Coh}(X)$ we will also write $\mathcal{G} \in D^b(X)$ for the complex whose unique non-zero term is in degree zero and it is equal to $\mathcal{G}$. For $\mathcal{F} \in D^b(A)$, $\mathcal{F}^\vee$ will stand for the **derived** dual of $\mathcal{F}$, that is

$$\mathcal{F}^\vee := R\mathscr{H}om(\mathcal{F}, \mathcal{O}_A)$$

(note that if $\mathcal{F}$ is a line bundle then there is no harm in using the same notation as in the precedent paragraph). Given $n \in \mathbb{Z}$, $[n] : D^b(X) \to D^b(X)$ denote the shift which at the level of complexes is defined by $B^\bullet[n]_r = B^{n+r}$. By $\underline{\otimes}$ we mean the **derived** tensor product. Given $f : X \to Y$ morphism of schemes we will write $Rf_* : D^b(X) \to D^b(Y)$ for the derived pushforward. As most of the morphism involved in this work are flat, usually we will not make distinction between the usual pullback f^* and the derived one Lf^*. For $\mathcal{F} \in D^b(X)$ we write $H^i(X, \mathcal{F})$ (or $H^i(\mathcal{F})$ if X is clear from the context) for the sheaf (hyper)cohomology of $\mathcal{F}$ (that is, the i-th cohomology of the derived functor of the global sections functor).

For a sheaf $\mathcal{F}$ contained in a line bundle over a variety we write $|\mathcal{F}|$ for the linear system $\mathbb{P}H^0(\mathcal{F})$. Moreover, sometimes we will (somehow abusevily) write "$D \in |\mathcal{F}|$", meaning that D is the zero locus of a section $s \in H^0(\mathcal{F})$ and say that "D belongs to the linear system $|\mathcal{F}|$". For instance, given a line bundle L on X and a subvariety $Y \subset X$ we write $|L|_Y|$ for the linear system consisting of the zero loci of global sections of $L|_Y$. Similarly, $\left|I_x^k \otimes L\right|$ is the linear system consisting of sections of L passing with multiplicity at least k through the point $x \in X$.

More symbols, notations and conventions will be introduced and clarified during the text as needed.

Introduction

In this thesis we study abelian varieties using the machinery of the Fourier-Mukai-Poincaré functor (FMP transform). Since Mukai's remarkable paper [Muk81] it has become clear that this functor is a very useful tool to study such kind of varieties because it defines an equivalence between the bounded derived category of an abelian variety A with the one of its dual $\hat{A}$ and, as in general an abelian variety is not isomorphic to its dual, this equivalence gives a non-trivial symmetry between objects in A and objects on $\hat{A}$, which turns out to be very fruitful.

More precisely, in the first two parts of this thesis, which are the content of the preprints [Alv24] and [AP24], we introduce and study certain numerical invariants that one can associate to a polarization on an abelian variety by means of some *cohomological vanishing conditions* or, equivalently, by means of the dimension of the support of the cohomology sheaves of the Fourier transform of an opportune object. Finally, in the third part of this work we propose an approach, based in the use of the FMP transform, to characterize a special type of principally polarized abelian varieties, namely, hyperelliptic jacobians, by means of a classical numerical invariant, namely the Seshadri constant of the theta divisor.

Turning to a more detailed exposition, we start by giving an idea of what are the invariants that we introduce and study. The general principle is that we would like to quantify "how positive" is an ample line bundle. Roughly speaking, the idea is that a high multiple of an ample line bundle is "very positive" because it has many good properties. Moreover, for abelian varieties, we can usually be very explicit about what "high multiple" means. For instance, if ℓ is a polarization on an abelian variety, then it is well known that 2ℓ is already globally generated ([BL04, Proposition 4.1.5]), $(p+2)\ell$ is p-ample ([BS97] and also [PP11b]) and $(p+3)\ell$ satisfy the property N_p ([Laz89, Conjecture 1.5.1], [Par00, Theorem]). However, for primitive line bundles the situation is very different: the only intrinsic numerical invariant that we have a priori, namely the h^0, does not completely encode the properties that we mentioned above since, for instance, for any $d \in \mathbb{Z}_{\geq 1}$ there exist polarizations with $h^0 = 2d$ that are not globally generated ([BLR93, Example 4.2]). In this context it is interesting to find finer numerical invariants guaranteeing effective results regarding properties such as global generation, projective normality or, more generally, property N_p.

An interesting invariant was recently introduced by Pareschi and Jiang in [JP20] by means of $\mathbb{Q}$-twisted vanishing conditions. Intuitively, their invariant, called the basepoint-freeness threshold $\beta(\ell)$, is defined as the minimal t such that $t\ell$ "is generically globally generated" and thus $\beta(\ell) \leq 1$ with equality if and only if ℓ has base points. To present a precise definition, let us recall that a coherent sheaf $\mathcal{F}$ is said to be IT(0) (short for "satisfy the index theorem with index zero") if

$$H^i(\mathcal{F} \otimes P_\alpha) = 0 \quad \text{for every } i > 0 \text{ and } \alpha \in \hat{A}.$$

In *loc.cit* the cited authors extended this notion to the setting of $\mathbb{Q}$-twisted sheaves $\mathcal{F}\langle t\ell\rangle$, that is, "sheaves twisted by a rational multiple of a polarization" (see Section 1.3 for a precise definition of these objects). In this context, we may define the *vanishing threshold of $\mathcal{F}$ with respect to ℓ* as the real number

$$\nu_\ell(\mathcal{F}) = \inf \{t \in \mathbb{Q} : \mathcal{F}\langle t\ell\rangle \text{ is IT}(0)\}. \tag{1}$$

When we take $\mathcal{F}$ to be the ideal I_x of a (any) closed point $x \in A$ we obtain a number which is sensible to be called the *basepoint-freeness threshold of ℓ*. Indeed, if L is an line bundle on an abelian variety then L is globally generated at x if and only if $H^1(I_x \otimes L) = 0$. On the other hand, for $\alpha \in \hat{A}$ the line bundle $L \otimes P_\alpha$ are isomorphic to $t_y^* L$ for some $y \in A$ and hence we have the following equivalences:

$$\begin{aligned} H^1(I_x \otimes L \otimes P_\alpha) = 0 \ \text{ for every } \alpha \in \hat{A} &\iff H^1(I_x \otimes t_y^* L) = 0 \ \text{ for every } y \in A \\ &\iff t_y^* L \ \text{ is globally generated at } x \text{ for every } y \in A \\ &\iff L \ \text{ is globally generated} \end{aligned}$$

and, as the higher cohomology groups automatically vanish, the above conditions are equivalent to IT(0) (for the sheaf $I_x \otimes L$). In practice, a polarization with small basepoint-freenes threshold is a polarization that godes properties similar to the ones that high-multiplies of polarizations have. For instance, in [Cau20, Corollary 1.2] Caucci shows that if $\beta(\ell) < (p+2)^{-1}$, which holds for example in the case that $\ell = (p+3)\ell'$, then ℓ satisfy N_p.

In the first part of this work, in a slightly more general way, we consider the vanishing thresholds, with respect to a polarization, associated to the ideal of a closed subscheme of an abelian variety. In symbols: for a closed subscheme $Z \subset A$ we write

$$\epsilon(Z, \ell) = \nu_\ell(I_Z).$$

As a particular case, we introduce the p-jets-separation thresholds which is nothing else than the vanishing threshold of p-th infinitesimal neighborhood $x^{(p)}$ of a closed point, that is, the closed subscheme defined by the ideal I_x^{p+1} and hence they are higher-order analogues of the basepoint freeness threshold. In symbols:

$$\epsilon_p(\ell) := \epsilon(x^{(p)}, \ell) = \nu_\ell(I_x^{p+1}).$$

Here we prove two results regarding these numbers, the first of them is that the sequence of jets-separation thresholds, suitably normalized, converges to the Seshadri constant $\varepsilon(\ell)$ of the polarization. This is no surprise since, by a result of Demailly ([Dem92, Theorem 6.4]), it is well known that the Seshadri constant of a line bundle at a point in a smooth projective variety is related to the jets-separation. However, unlike the numbers $s(kL, x)$

considered by Demailly (see [Laz04a, Proposition 5.1.6] for their definition), which are integers and depend on the point, our jets-separation thresholds depends just on the numerical class of L and are allowed to be real numbers. In this sense, our result can be thought as a finer version, valid for abelian varieties, of Demailly's result. Our result is the following:

Theorem 0.1. *Let A be a g-dimensional abelian variety and $Z \subset A$ a closed subscheme with ideal I_Z. For $p \in \mathbb{Z}_{\geq 0}$ write $Z^{(p)}$ for the closed subscheme of A defined by the ideal I_Z^{p+1}. Then:*

1. *The following inequalities hold:*

$$\sup \frac{p+1}{\epsilon(Z^{(p)}, \ell)} \geq \varepsilon(I_Z, \ell) \geq \limsup \frac{p+1}{\epsilon(Z^{(p)}, \ell)},$$

where $\varepsilon(I_Z, \ell)$ is the Seshadri constant of the ideal I_Z with respect to ℓ (see Section 2.2 for its definition).

2. *If $\dim Z \leq 1$ then*

   (a) For $p, r \in \mathbb{Z}_{\geq 0}$ with $p < r$ we have

$$0 \leq \epsilon(Z^{(r)}, \ell) \leq \epsilon(Z^{(r-p-1)}, \ell) + \epsilon(Z^{(p)}, \ell)$$

   (b) The sequence $\{(p+1)^{-1}\epsilon(Z^{(p)}, \ell)\}_{p \in \mathbb{Z}_{\geq 0}}$ converges and

$$\varepsilon(I_Z, L) = \lim_{p \to \infty} \frac{p+1}{\epsilon(Z^{(p)}, \ell)} = \sup_p \frac{p+1}{\epsilon(Z^{(p)}, \ell)}.$$

3. *For a complex abelian variety the following inequalities of jets separation thresholds hold:*

$$\frac{p+1}{\varepsilon(\ell)} \leq \epsilon_p(\ell) \leq \frac{g+p}{\varepsilon(\ell)}.$$

Our second result says that jets-separation thresholds are related, via the FMP transform, in a very explicit way to the surjectivity of certain Gauss-Wahl maps. To state this result, we first give an informal introduction of what are these maps (in Section 2.3 the reader can find a more detailed exposition). They were introduced by Wahl in [Wah87] and they are a hierarchy of linear maps $\{\gamma^p_{L,M}\}_{p \in \mathbb{Z}_{\geq 0}}$ associated to a pair of line bundles L, M, where $\gamma^0_{L,M}$ is no other than the multiplication map of global sections. In this context, our result is a higher order version of the fact, proved in [JP20], that basepoint-freeness thresholds are related to a (suitably defined) surjectivity threshold for the multiplication map of global sections.

Briefly, given two line bundles L, M on a projective variety A, the associated p-th Gauss-Wahl map is a linear map

$$\gamma^p_{L,M} : H^0(R_L^{(p-1)} \otimes M) \longrightarrow H^0(\operatorname{Sym}^p \Omega_A \otimes L \otimes M),$$

where $R_L^{(p-1)}$ is the kernel of the natural *evaluation map*

$$H^0(L) \otimes \mathcal{O}_A \to P^{p-1}(L), \tag{2}$$

where $P^{p-1}(L)$ is the $p-1$-*principal parts bundle* (we refer to Section 2.3 for the precise definition of this sheaf). In the case that L is very ample, $R_L^{(p-1)}$ is nothing else than the $p-1$-order conormal bundle associated to the embedding of A into $\mathbb{P}H^0(L)$ (for instance, $R_L^{(1)}$ can be identified with $N^\vee_{A/\mathbb{P}H^0(L)} \otimes L$).

When L separates p-jets at every point, the surjectivity of $\gamma^p_{L,M}$ is detected by the vanishing of $H^1(R_L^{(p)} \otimes M)$ (we refer again to section 2.3 for details). In this context, informally speaking, the vanishing threshold $\nu_\ell(R_L^{(p)})$ (see (1)) of the sheaf $R_L^{(p)}$ with respect to ℓ may be seen as the minimal "rational power" of L such that the "rational" Gauss-Wahl map $\gamma^p_{L,\nu L}$ is surjective. Extending [JP20, Proposition 8.1], our second result establish a relation between the numbers

$$\mu_p(\ell) := \nu_\ell\left(R_L^{(p)}\right)$$

and the jets-separations thresholds. Concretely, we have:

Theorem 0.2. *Let L be an ample line bundle on an abelian variety A. Let $\ell \in \mathrm{NS}(A)$ be the class of L and suppose that L separates p-jets at every point of A (thus $\epsilon_p(\ell) < 1$). Then the following equality holds:*

$$\mu_p(\ell) = \frac{\epsilon_p(\ell)}{1-\epsilon_p(\ell)}.$$

As an application, we can use Theorem 0.2 above to quantify, in terms of the jets-separations thresholds and hence also in terms of the Seshadri constant, the surjectivity of Gauss-Wahl maps:

Theorem 0.3. *Let L, M ample and algebraically equivalent ample line bundles on an abelian variety A. Let c, d be positive integers and write ℓ for the class of L and M in* $\mathrm{NS}(A)$. *Assume that*

$$\epsilon_p(\ell) < \frac{cd}{c+d}$$

for some $p \in \mathbb{Z}_{\geq 0}$. Then the Gauss-Wahl map $\gamma^p_{cL,dM}$ is surjective.

For instance, the above result says that if $\epsilon_p(\ell) < 1$, that is, if ℓ separates p-jets at every point, then $\gamma^p_{2L,2M}$ is surjective, while $\gamma^p_{L,M}$ is already surjective whenever $\epsilon_p(\ell) < 1/2$. Now, combining with Theorem 0.1 3) we obtain a condition ensuring the surjectivity of certain Gauss-Wahl maps in terms of the Seshadri constant which, at least asymptotically, improves the results present in the literature ([Par95, Theorem 2.2], for instance):

Corollary 0.4. *Let ℓ be a polarization on an abelian variery. Consider a positive integer c such that $c \cdot \varepsilon(\ell) > g+p$, where $\varepsilon(\ell)$ is the Seshadri constant of ℓ. Then*

$$\mu_p(c\ell) < \frac{g+p}{c\varepsilon(\ell)-(g+p)}.$$

In particular, for L, M ample line bundles representing ℓ, the Gauss-Wahl map $\gamma^p_{cL,dM}$ is surjective as soon as

$$d \;\geq\; \frac{c(g+p)}{c\varepsilon(L)-(g+p)}.$$

On the other hand, a combination of Theorem 0.1 and Theorem 0.2 gives then an interesting expression of the Seshadri constant in terms of the asymptotic surjectivity of higher Gauss-Wahl maps:

Corollary 0.5. *Let $\ell \in \mathrm{NS}(A)$ be a polarization on an abelian variety A. Then:*

$$\varepsilon(\ell) = 1 + \lim_{p \to \infty} \frac{1}{\mu_p((p+2)\ell)}$$

Surprisingly, this result also says that certain special types of polarized abelian varieties can be characterized in terms of the surjectivity of certain Gauss-Wahl maps.

1. Decomposable abelian varieties with a curve factor: In [Nak96] Nakamaye proved that if (A, ℓ) is a polarized abelian variety, then $\varepsilon(\ell) \geq 1$ and, moreover, $\varepsilon(\ell) = 1$ if and only if

$$(A, \ell) \simeq (E, \theta) \boxtimes (B, m), \tag{3}$$

where (E, θ) is a principally polarized elliptic curve and (B, m) is another polarized abelian variety. In this context, Corollary 0.5 says that varieties of the form (3) are also characterized by the following equivalent conditions:

a) $\epsilon_p(\ell) = p + 1$ for every $p \in \mathbb{Z}_{\geq 0}$

b) the sequence $\{\mu_p((p+2)\ell)\}_{p \in \mathbb{Z}_{\geq 0}}$ is unbounded

2. Hyperelliptic jacobians: In [Deb04, Theorem 7] it is shown that if C is a curve of genus $g \geq 5$ and C is not hyperelliptic then $\varepsilon(\operatorname{Jac} C, \theta) > 2$ while $\varepsilon(\operatorname{Jac} C, \theta) = 2g/(g+1) < 2$ in the hyperelliptic case. Moreover, it is also conjectured (see Conjecture 0.14 below) that the condition $\varepsilon(\theta) < 2$ should characterize jacobians of hyperelliptic curves among indecomposable principally polarized abelian varieties (i.p.p.a.v) of dimension $g \geq 5$.

In this context, Corollary 0.5 says that hyperelliptic curves are characterized among smooth curves by the fact that $\mu_p((p+2)\theta) > 1$ for $p \gg 0$, that is, for the failure of surjectivity of the Gauss-Wahl map $\gamma^p_{(p+2)\theta,(p+2)\theta}$ for $p \gg 0$ and, moreover, conjecturally this fact characterize hyperelliptic jacobians among all i.p.p.a.v.

Based on this, a natural question that arise is wether we can render the above characterizations effective. That is, we may ask if there exists an specific Gauss-Wahl map whose non-surjectivity characterizes p.p.a.v. with small Seshadri constant. More generally, we can ask the following:

Question 0.6. *Given a positive real number u. Does there exist a positive real number $t(u)$ and a positive integer $p(u)$ such that for any g-dimensional principally polarized abelian variety (A, θ) we have*

$$\varepsilon(\theta) > u \iff \epsilon_{p(u)}(\theta) < t(u) \quad ?$$

As an example, we answer the above question in the case that $u = 1$. More precisely, we prove the following:

Proposition 0.7. *Let (A, θ) be a g-dimensional principally polarized abelian variety. Then the following are equivalent:*

a) (A, θ) *is not as* (3)

b) $\varepsilon(\theta) > 1$

c) $\epsilon_2(\theta) < 3$.

As a consequence, we have that the Gauss-Wahl map $\gamma^2_{6\Theta,d\Theta}$ *is surjective for* $d \geq 7$ *and if* $\gamma^2_{6\Theta,6\Theta}$ *fails to be surjective then* (A, θ) *is of the form* (3).

In the second part of this thesis we revisit some cohomological aspects of $\mathbb{Q}$-twisted objects. More precisely, in [JP20] Jiang and Pareschi introduce the *cohomological rank functions* which are a sensible way to define the dimension of the (generic) cohomology of a $\mathbb{Q}$-twisted object. Here we revisit this notion, providing an interpretation of them in terms of certain Mukai's semi-homogeneous vector bundles.

More precisely, due to [Muk78, Theorem 7], given a polarization ℓ and a rational number λ there exists a, unique up to twist by an element of $\mathrm{Pic}^0(A)$, simple and semi-homogeneous vector bundle $E = E_{\lambda,\ell}$ such that

$$\frac{\det E}{\operatorname{rk} E} = \lambda \cdot \ell \in \mathrm{NS}(A)_{\mathbb{Q}}.$$

If $\lambda = a/b$, this vector bundle satisfy that $b_A^* E \simeq (L^{ab})^{\oplus \operatorname{rk} E}$, where L is a representant of ℓ. This means that it is possible to compute cohomological rank functions (and hence, the various thresholds that we define) using the vector bundles $E_{\lambda,\ell}$, which plays the role of the "rational power" $\frac{a}{b}\ell$ with the advantage that now we are dealing with concrete sheaves (and sheaf maps) rather than with a certain somehow elusive equivalence class.

Using the above interpretation, we focalize in the study of the cohomological rank functions of the ideal of a point and its relation to the surjectivity of certain multiplication maps of global sections. More precisely, we first observe that for $t = a/b$ the cohomological rank function $h^1(I_0 \langle t\ell \rangle)$ can be computed as the generic (normalized) dimension of the cokernel of the restriction map

$$H^0(E_{\lambda,\ell}) \longrightarrow E_{\lambda,\ell} \otimes k(x),$$

which means that we have the following vector bundle interpretation of the basepoint-freeness threshold:

- $\epsilon_0(\ell) = \beta^1_A(\ell) < a/b$ if and only if the vector bundle $E_{a,b,\ell}$ is globally generated
- $\epsilon_0(\ell) = \beta^1_A(\ell) = a/b$ if and only if $E_{a,b,\ell}$ is generically globally generated but not globally generated.

We can also introduce the following number:

$$\beta^0_A(\ell) := \sup\{\lambda \in \mathbb{Q} : I_0 \langle \lambda\ell \rangle \text{ is IT(1)}\}$$

and, as before, this has a corresponding vector bundle interpretation:

- $\beta^0_A(\ell) > a/b$ if every non-zero section of $E_{a,b,\ell}$ is nowhere zero

- $\beta^0_A(\ell) = a/b$ if there are sections of $E_{a,b,\ell}$ vanishing at some points but the zero loci of those sections do not cover the whole variety.

For instance, using this observation we can prove the following:

Proposition 0.8. *Let ℓ be a polarization on an abelian variety A. Then the following inequalities hold:*

$$\beta^1_A(\ell) \ \geq \ \chi(\ell)^{-1/g} \ \geq \ \beta^0_A(\ell).$$

Now, in this context, the relation between the cohomological rank function of the ideal of a point and the surjectivity of multiplication maps ([JP20, §8], which also corresponds to the $p = 0$ case in Theorem 0.2) translates as a relation between the restriction map of the global sections of a vector bundle and the surjectivity of the multiplication of global sections of vector bundles. More precisely, for $\lambda, t \in \mathbb{Q}$ and $\alpha \in \hat{A}$ we can consider the multiplication maps

$$m^t_{A,\lambda\ell,\alpha} : H^0(E_{\lambda,\ell}) \otimes H^0(E_{t\lambda,\ell} \otimes P_\alpha) \longrightarrow H^0(E_{\lambda,\ell} \otimes E_{\lambda t,\ell} \otimes P_\alpha)$$

and define the following numbers:

$$s^0_A(\lambda \cdot \ell) := \sup\{t \in \mathbb{Q} : m^t_{A,\lambda\ell,\alpha} \text{ is injective } \forall \alpha \in \hat{A}\}$$

and

$$s^1_A(\lambda \cdot \ell) := \inf\{t \in \mathbb{Q} : m^t_{A,\lambda\ell,\alpha} \text{ is surjective } \forall \alpha \in \hat{A}\}.$$

Here we prove the followings relations:

Proposition 0.9. *For every $\lambda \in \mathbb{Q}_{>0}$ we have*

$$s^i_A(\lambda \cdot \ell) = \frac{\beta^i_A(\ell)}{\lambda - \beta^i_A(\ell)}.$$

On the other hand, we show that there exists a duality between the cohomological rank functions of the ideal of a point in A with respect to a polarization ℓ and the ones of the ideal of a point in $\hat{A}$ with respect to the dual polarization ℓ_δ ([BL04, Proposition 14.4.1]). Concretely, we prove the following:

Proposition 0.10. *Let ℓ be a polarization of type $(d_1, ..., d_g)$ on an abelian variety A with dual $(\hat{A}, \ell_\delta)$. Then*

$$\beta^i_A(\ell) = \frac{1}{d_1 d_g \cdot \beta^{1-i}_{\hat{A}}(\ell_\delta)}.$$

Combining with Proposition 0.9 above we obtain relations between thresholds on A and thresholds on $\hat{A}$. Concretely:

Corollary 0.11. *For $i = 0, 1$ and $\lambda \in \mathbb{Q}$ we have*

$$\beta^i_A(\ell) = \frac{1}{\lambda \cdot d_1 d_g} \cdot \left(1 + \frac{1}{s^{1-i}_{\hat{A}}(\lambda \cdot \ell_\delta)}\right)$$

In particular, this means that the basepoint-freeness thresholds can be controlled by studying the (non) **injectivity** of certain multiplications maps on the dual abelian variety $\hat{A}$. As a consequence, the basepoint-freeness threshold can be bounded by below whenever we are able to find non-trivial elements in the kernel of an appropiate multiplication map. Indeed, we shall see that if $E = E_{\lambda \ell_\delta}$ is generically globally generated of rank r, then the image of $\wedge^{r+1} H^0(E)$ in $H^0(E) \otimes H^0(\det E)$ is a non-zero subspace contained in the kernel of the multiplication map. Ellaborating on this approach we reach to find a lower bound for the basepoint-freeness threshold. The precise inequality is the following:

Theorem 0.12. *We have that*

$$\beta_A^1(\ell) \geq \sup_{\lambda > \beta_A^0(\ell_\delta)} \frac{1}{\lambda \cdot d_1 d_g} \cdot \left(1 + \frac{1}{\operatorname{rk} E_{\lambda \ell_\delta}}\right).$$

As we pointed out before, a low basepoint-freeness threshold ensures good properties for the polarizations and hence, a lower bound for such invariant represent an obstruction to have such good properties. Up to the knowledge of the author, this is the first result that goes into this direction, improving the picture that we have of the numerical behaviour of this invariant and complementing the results present in the literature (see [Ito22a] or [Jia23], for example), which give obstructions to upper bounds for β_A^1 in terms of the presence of low degree abelian subvarieties.

As a particular example, as we know that a polarization with $\beta_A^1(\ell) > 1/2$ can not be projectively normal, we can apply Theorem 0.12 above in order to find obstructions for a polarization to satisfy such property in terms of its type. Interestingly enough, our inequality contains many of the results on the subject and cover new cases that were out of reach by means of more classical methods. To illustrate this, we consider the case of $\lambda = \frac{1}{d_1}$. In this situation we have that $\lambda \ell_\delta$ is a line bundle and hence we obtain that, if $d_1 < d_g$ (we need this to apply Theorem 0.12) we obtain the following universal bound:

$$\beta_A^1(\ell) \ \geq \ \frac{2}{d_g}.$$

In particular, we obtain that polarizations of type $(1, ..., 2)$ has $\beta_A^1 \geq 1$ and hence are never globally generated (a previously known result, [NR95]) while polarizations of type $(1, ..., 3)$ has $\beta_A^1 \geq 2/3 > 1/2$ and hence are never projectively normal, which, up to the knowledge of the author, was a previously unknown case. On the other hand, we obtain that polarizations of type $(1, ..., 4)$ has $\beta_A^1 \geq 1/2$, which means that there exists a point $\alpha \in \hat{A}$ such that the multiplication map

$$H^0(L) \otimes H^0(L \otimes P_\alpha) \longrightarrow H^0(L^{\otimes 2} \otimes P_\alpha) \tag{4}$$

is not surjective. In this situation we have that L is not projectively normal if and only if we can take α to be the origin. Now, at the end of Chapter 3 we study the cases $d_1 d_g = 4$ and $d_1 d_g = 6$, where projective normality is not clear (even with computational methods, [FG05]), in a bit more detail. Here we prove that although in general is not true that $\hat{0}$ belongs to the jump-locus, it is always the case that there exist a finite order point in such subvariety. Concretely:

Proposition 0.13. *Let L be an ample line bundle of type $(d_1, ..., d_g)$ with $d_1 d_g \in \{4, 6\}$. Then there exists a point $\alpha \in \hat{A}[d_1 d_g]$ such that the multiplication map* (4) *is not surjective.*

Finally, in the last chapter of this thesis we report a work in progress with G. Pareschi regarding an approach to to a question of Debarre that we now present.

One of the most important problems in the study of abelian varieties ask for characteriza -tions of (polarized) jacobians of curves among all the indecomposable principally polarized abelian varieties (i.p.p.a.v) (A, θ) (see [Gru12] for an excellent account on this problem). Examples of such characterizations are the Matsusaka's criterion ([Mat59]) which states that an i.p.p.a.v is a jacobian if and only if there exists a curve with minimal cohomology class $\theta^{g-1}/(g-1)! \in H^{2g-2}(A, \mathbb{Z})$ or Schreieder's theorem ([Sch16, Corollary 3]) which states that jacobians are characterized by the fact that the theta divisor is dominated by a product of curves.

In this context, we may ask if it is possible to characterize jacobians by means of the Seshadri constant of θ. Now, althought it is known that jacobians tend to have small Seshadri constant (for instance, in [Laz96, Proposition (i)] Lazarsfeld proves that $\varepsilon(\operatorname{Jac} C, \theta_C) \leq \sqrt{g}$), there exist i.p.p.a.v's with $\varepsilon(\theta) = 2$ that are not jacobians while for a non-hyperelliptic curve it is known that $\varepsilon(\operatorname{Jac} C, \theta_C) > 2$ (see [Deb04, Theorem 7]). However, there is the following:

Conjecture 0.14 ([Deb04])**.** *If (A, θ) is an i.p.p.a.v with $\varepsilon(\theta) < 2$ then $(A, \theta) \simeq (\operatorname{Jac} C, \theta_C)$ for an hyperelliptic curve C.*

Our approach is based on Debarre's observation which says that the Seshadri constant can be bounded by below by studying the base locus of the *continuous linear systems*

$$\{k\theta\}_{0,m} := \bigcup_{\Theta \in \theta} |I_0^m(k\Theta)|\,,$$

where the union runs over all the line bundles with class θ. More precisely, note that we always have that 0 belongs to $\operatorname{Bs}(\{k\theta\}_{0,m})$ and Debarre's observation ([Deb04, Lemma 1]) says that $\varepsilon(\theta) \geq k/m$ whenever 0 is an isolated point of such base locus. In particular, a possible approach to study Conjecture 0.14 above is based on the study of the base locus of the continuous linear system $\{2\theta\}_{0,4}$. Now, of course, such system can also be rewritten as

$$\{2\theta\}_{0,4} = \bigcup_{\alpha \in \hat{A}} |I_0^m \otimes \mathcal{O}_A(k\Theta) \otimes P_\alpha|\,,$$

where Θ is a (any) divisor representing the polarization θ, which suggests that we may study its base locus using the Fourier-Mukai transform. Concretely, we note that for $x \neq 0$ we have that x lies in $\operatorname{Bs}(\{2\Theta\}_{0,4})$ if and only if the evaluation map

$$\bigoplus_{\alpha \in \hat{A}} H^0(I_0^4(2\Theta) \otimes P_\alpha) \otimes P_\alpha^\vee \longrightarrow I_0^4(2\Theta) \otimes k(x) \simeq k(x)$$

fails to be surjective, that is, that $I_0^4(2\Theta)$ is not continuously globally generated at x. Now, we can study the surjectivity of this map using the methods developed in [Par23] (where also a finer version of *generation* is introduced), which allow us to prove the following strange implication:

Theorem 0.15. *Let (A, θ) be a principally polarized abelian variety and let Θ be a symmetric divisor representing θ. Let Q be the cokernel of the dual of the natural map* (2)

$$\phi : P^3(2\Theta)^\vee \to H^0(\mathcal{O}_A(2\Theta))^\vee \otimes \mathcal{O}_A.$$

Then, if $\varepsilon(\theta) < 2$ then there exists a torsion subsheaf of Q with non-reduced support.

Finally, we revisit a well-known characterization of hyperelliptic jacobians in terms of the non-generation of the sheaves $I_\tau(\Theta)$, where τ is a length-two subscheme of A supported on the origin. More precisely, a theorem by Marini ([Mar93]) says that hyperelliptic jacobians are characterized among i.p.p.a.v by the fact that there exists a divisor D in Θ belonging to the linear system $\mathcal{O}_\Theta(\Theta)$ such that all the components of D are not reduced. On the other hand, we can prove the following:

Theorem 0.16. *Let τ be a length two subscheme of A supported on the origin. Then:*

a) The scheme $V(\tau) := \operatorname{Supp} R^g\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee)$ (which is called the Θ-dual of τ) belongs to the linear system $|\mathcal{O}_\Theta(\Theta)|$. Moreover, any divisor in such linear system arise in this way.

b) If the sheaf $I_\tau(\Theta)$ is not generated (see Definition 4.5) then $V(\tau)$ is not reduced.

Here we note that in item b) of the above theorem we are not claiming that **all** of the components are not reduced, just that there exists a non reduced component, and hence we do not obtain an hyperelliptic jacobian. However, due to a result proved by Beauville and Debarre ([BD86, Théorème 2.1]), the existence of a non integral (i.e not reduced or not irreducible) divisor $D \in |\mathcal{O}_\Theta(\Theta)|$ implies that either $\dim \operatorname{Sing} \Theta \geq g-4$ or there exists an elliptic curve $E \subset A$ with $(\theta \cdot E) = 2$. Now, for a very general abelian variety the latter option can not occur while the locus in $\mathcal{A}_g$ (the moduli space of p.p.a.v) where the former holds contains the jacobian locus as an irreducible component ([AM67]) and, even more, the locus where $\dim \operatorname{Sing} \Theta \geq g-3$ contains the hyperelliptic locus as a component. In this context, these last two theorems leave a couple of open questions:

1. Is there a better description of the sheaf Q from Theorem 0.15? More precisely, what is its torsion filtration (see Definition 4.10) ?

2. Suppose that $I_\tau(\Theta)$ is generated for every length two subscheme τ supported on the origin. Does it follow that $I_0^4(2\Theta)$ is generated (and hence $\varepsilon(\theta) \geq 2$)?

3. Let (A, θ) be an i.p.p.a.v. and suppose that there exists a non-reduced divisor D belonging to the linear system $|\mathcal{O}_\Theta(\Theta)|$. Does it follow that either there exist an elliptic curve E with $(\theta \cdot E) = 2$ or $\dim \operatorname{Sing} \Theta \geq g-3$?

This thesis is structured as follows: in Chapter 1 we review the preliminar material regarding the FMP transform and generic vanishing conditions that we need throughout this work; in Chapter 2 we study the relation between jets-separation thresholds, the Seshadri constant and the surjectivity of the Gauss-Wahl maps, proving Theorem 0.1 and 0.2, which are the content of the author's pre-print [Alv24]; in Chapter 3 we review the theory of Mukai's semi-homogeneous vector bundles and prove Theorem 0.12, which is the content of the pre-print [AP24], written by the author together with G. Pareschi. Finally, in Chapter 4 we present our approach to Debarre's conjecture, proving Theorem 0.15 and 0.16, which is a work in progress with G. Pareschi.

Chapter 1

Generic vanishing theory

In this chapter we survey the general background material about generic vanishing theory, as it will needed throughout this work. At the beginning of each chapter the reader will find more specific preliminaries concerning the particular subject treated at the moment.

1.1 Fourier-Mukai functors

One of the most important tecnical tool employed throughout our work is the Fourier-Mukai-Poincaré functor, which we now proceed to review.

Definition 1.1. Let A be an abelian variety defined over an algebraically closed field k and let $\hat{A}$ be its dual. Let $\mathcal{P} \in \mathrm{Coh}(A \times_k \hat{A})$ be a normalized Poincaré bundle. Then *the Fourier-Mukai-Poincaré (FMP) functor with kernel* $\mathcal{P}$ is the functor

$$\Phi_{\mathcal{P}} : D^b(A) \to D^b(\hat{A}) : F \to Rp_{2*}\left(p_1^* F \otimes \mathcal{P}\right),$$

where p_1, p_2 are the product projections.

Remark 1.2. *Of course, $\mathcal{P}$ define also a functor $D^b(\hat{A}) \to D^b(A)$ by means of the identification $\widehat{\hat{A}} \simeq A$. When confussion may arise we will explicitly mention the source and target of our functors.*

This functor is particulary interesting in view of the following remarkable theorem due to Mukai:

Theorem 1.3 (Theorem 2.2 in [Muk81])**.** *$\Phi_{\mathcal{P}}$ is an equivalence of categories. A quasi-inverse is given by the composition*

$$D^b(\hat{A}) \xrightarrow{(-1_A)^*\Phi_{\mathcal{P}}} D^b(A) \xrightarrow{[g]} D^b(A)$$

An important feature of the equivalence $\Phi_{\mathcal{P}}$ is the fact that interchanges skycraper sheaves of a point with the line bundle parametrized by such a point:

Example 1.4. *We have that $\Phi_{\mathcal{P}}(k(0)) = \mathcal{O}_{\hat{A}}$. From Theorem 1.3 and the identification $A \simeq A^{\wedge\wedge}$ we see that $\Phi_{\mathcal{P}}(\mathcal{O}_A) = k(\hat{0})[-g]$, where $\hat{0}$ is the zero of the abelian variety $\hat{A}$.*

We now list some of the main properties of Fourier-Mukai transforms.

- *Exchange of translations and tensors* ([Muk81, (3.1)]) Let $\alpha \in \hat{A}$ and $P_\alpha \in \mathrm{Pic}^0(A)$ the corresponding line bundle. Then we have the following isomorphism of functors $D^b(A) \to D^b(\hat{A})$

$$\Phi_{\mathcal{P}} \circ (P_\alpha \otimes -) \simeq t_\alpha^* \circ \Phi_{\mathcal{P}}. \tag{1.1}$$

Similarly, for $a \in A$ we have a natural ismorphism

$$(P_a \otimes -) \circ \Phi_{\mathcal{P}} \simeq \Phi_{\mathcal{P}} \circ t_{-a}^* \tag{1.2}$$

-*Grothendieck-Verdier duality* ([Muk81, (3.8)] and [PP11a, Lemma 2.2]) For $\mathcal{F} \in D^b(A)$ we have that

$$\Phi_{\mathcal{P}}(\mathcal{F})^\vee \simeq \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)[g] \simeq (-1_{\hat{A}})^* \Phi_{\mathcal{P}}(\mathcal{F}^\vee)[g]. \tag{1.3}$$

-*Fourier transforms of ample line bundles* ([Muk81, Proposition 3.11(1)])

For an ample line bundle $L \in \mathrm{Pic}(A)$ we have that $\Phi_{\mathcal{P}}(L)$ is a vector bundle on $\hat{A}$, denoted by $\widehat{L}$. Moreover, we have that

$$\varphi_L^* \widehat{L} \simeq H^0(A, L) \otimes L^\vee, \tag{1.4}$$

where $\varphi_L : A \to \hat{A}$ is the isogeny associated to L. In particular, if $L = \mathcal{O}_A(\Theta)$ is a principal polarization and we use φ_L to identify A with $\hat{A}$ then we have that

$$\widehat{\mathcal{O}_A(\Theta)} = \mathcal{O}_A(-\Theta). \tag{1.5}$$

-*Exchange of pull-back and pushforward under isogenies* ([Muk81, (3.4)])

Let $f : B \to A$ be an isogeny between abelian varieties. Then for every $\mathcal{F} \in D^b(A)$ and $\mathcal{G} \in D^b(B)$ we have that

$$\Phi_{\mathcal{P}_B}(f^*\mathcal{F}) \simeq \hat{f}_* \Phi_{\mathcal{P}_A}(\mathcal{G}) \quad \text{and} \quad \Phi_{\mathcal{P}_A}(f_*\mathcal{G}) \simeq \hat{f}^* \Phi_{\mathcal{P}_B}(\mathcal{G}), \tag{1.6}$$

where $\hat{f} : \hat{A} \to \hat{B}$ is the dual isogeny of f.

-*Hypercohomology* ([PP11a, Lemma 2.1]) Given $\mathcal{F} \in D^b(A)$ and $\mathcal{G} \in D^b(\hat{A})$ we have that

$$H^i(\hat{A}, \Phi_{\mathcal{P}}(\mathcal{F}) \underline{\otimes} \mathcal{G}) \simeq H^i\left(A, \mathcal{F} \underline{\otimes} \Phi_{\mathcal{P}}(\mathcal{G})\right), \tag{1.7}$$

where $\Phi_{\mathcal{P}}$ denotes both the functors $D^b(A) \to D^b(\hat{A})$ and $D^b(\hat{A}) \to D^b(A)$.

Now, for the next and last properties that we survey here, we recall the following definition:

Definition 1.5. Let A be an abelian variety with group law $m : A \times A \to A$. Given $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(A)$, the Pontryagin product of $\mathcal{F}$ and $\mathcal{G}$ is the sheaf

$$\mathcal{F} * \mathcal{G} = m_* (\mathcal{F} \boxtimes \mathcal{G}).$$

Given $\mathcal{F}, \mathcal{G} \in D^b(A)$, the derived Pontryagin product of $\mathcal{F}$ and $\mathcal{G}$ is

$$\mathcal{F} \underline{*} \mathcal{G} = Rm_* (\mathcal{F} \boxtimes \mathcal{G}).$$

One of the most important properties of $\Phi_{\mathcal{P}}$ is the fact that it interchanges tensor and Pontryagin products ([Muk81, (3.7)]). That is, for $\mathcal{F}, \mathcal{G} \in D^b(A)$ we have that

$$\Phi_{\mathcal{P}}(\mathcal{F} \underline{\otimes} \mathcal{G}) \simeq \Phi_{\mathcal{P}}(\mathcal{F}) \underline{*} \Phi_{\mathcal{P}}(\mathcal{G})[g] \tag{1.8}$$

and

$$\Phi_{\mathcal{P}}\left(\mathcal{F} \underline{*} \mathcal{G}\right) \simeq \Phi_{\mathcal{P}}(\mathcal{F}) \underline{\otimes} \Phi_{\mathcal{P}}(\mathcal{G}). \tag{1.9}$$

Finally, for an ample line bundle L, [Muk81, (3.10)] gives the following isomorphism of functors:

$$L \otimes \varphi_L^* \Phi_{\mathcal{P}}(L \otimes -) \simeq L \underline{*} \; - . \tag{1.10}$$

1.2 Cohomological support loci

"Cohomology and base change" theorem ([Har77, III, Theorem 12.11]) states that the higher direct images of a sheaf under a flat morphism are related to the cohomology of the fibers of such sheaf. More precisely, from this theorem it follows that if for $\mathcal{F} \in \mathrm{Coh}(A)$ we consider the sheaves

$$R^i\Phi_{\mathcal{P}}(\mathcal{F}) := R^i p_{\hat{A}*}\left(p_A^*\mathcal{F} \otimes \mathcal{P}\right) \in \mathrm{Coh}(\hat{A})$$

then

$$\mathrm{Supp}(R^i\Phi_{\mathcal{P}}(\mathcal{F})) \subseteq \left\{ \alpha \in \hat{A} \middle| H^i(A, \mathcal{F} \otimes P_\alpha) \neq 0 \right\}. \tag{1.11}$$

In particular it follows that $R^i\Phi_{\mathcal{P}}(\mathcal{F}) = 0$, whenever the vanishing $H^i(A, \mathcal{F} \otimes P_\alpha) = 0$ holds for every α. More generally, [Gro63, Théorème (7.7.5) II] implies that the contention (1.11) still holds for an *object* $\mathcal{F} \in D^b(A)$.

In this context, is natural to study the subsets of $\hat{A}$ at the right-hand side of (1.11).

Definition 1.6. Let $\mathcal{F} \in D^b(A)$ and $i \in \mathbb{Z}_{\geq 0}$. Then the i-th cohomological support loci of $\mathcal{F}$ is the set

$$V^i(A, \mathcal{F}) = \left\{ \alpha \in \hat{A} \middle| H^i(A, \mathcal{F} \otimes P_\alpha) \neq 0 \right\}.$$

When A is clear from the context we may simply write $V^i(\mathcal{F})$ for $V^i(A, \mathcal{F})$.

Remark 1.7. *From the semi-continuity theorem [Gro63, Théorème (7.7.5) I] ([Har77, Theorem 12.8] for sheaves) it follows that the sets $V^i(\mathcal{F})$ are closed in the Zariski topology of $\hat{A}$.*

In this context, in [PP11a] there were introduced the following notions of generic vanishing for a sheaf on an abelian variety

Definition 1.8. An object $\mathcal{F} \in D^b(A)$ is said to

1. satisfy the index theorem of index j (I.T(j), from now) if $V^i(\mathcal{F}) \neq \emptyset$ if and only if $i = j$

2. satisfy the weak index theorem of index j (W.I.T(j), from now) if $R^i\Phi_{\mathcal{P}}(\mathcal{F}) \neq 0$ if and only if $i = j$

3. be a GV-object if $\operatorname{codim}_{\hat{A}}(V^i(\mathcal{F})) \geq i$ for every $i \geq 0$

4. be a M-regular object if $\operatorname{codim}_{\hat{A}}(V^i(\mathcal{F})) > i$ for every $i > 0$

Example 1.9. *It may happen that $V^i(\mathcal{F}) \neq \emptyset$ but $R^i\Phi_{\mathcal{P}}(\mathcal{F}) = 0$. Indeed: From [Mum08, pag. 76, (vii)] we know that $V^i(\mathcal{O}_A) = \{\hat{0}\} \neq \emptyset$ for evey $i \geq 0$ but $R^i\Phi_{\mathcal{P}}(\mathcal{O}_A) = 0$ for $i \neq g$ (see Example 1.4). However, the condition of being GV can be controlled either by $\operatorname{codim}_{\hat{A}} V^i(\mathcal{F})$ or by $\operatorname{codim}_{\hat{A}} \operatorname{Supp} R^i\Phi_{\mathcal{P}}(\mathcal{F})$ (see [PP11a, Lemma 3.6]).*

Remark 1.10. *Of course, from (1.11) it follows that if $\mathcal{F}$ satisfies I.T(j) then it also satisfies W.I.T(j). Moreover, from Grauert's theorem ([Har77, Corollary 12.9]) and the fact that the Euler characteristic is constant in families ([Gro63, Théorème (7.9.4)]) it follows that in that case $R^j\Phi_{\mathcal{P}}(\mathcal{F})$ is locally free.*

We now give some examples which is good to have in mind for the next chapters.

Example 1.11. *If $Z \subset A$ is a zero-dimensional subscheme then obviously $\mathcal{O}_Z$ satisfies I.T(0).*

Example 1.12. *Example 1.9 says that $\mathcal{O}_A$ is GV and satisfies W.I.T(g).*

Example 1.13. *If L is an ample line bundle, then we know that $H^i(L \otimes P_\alpha) = 0$ for every $i > 0$ ([Mum08, pag.150]). In other words, $V^i(L) = \emptyset$ for every $i > 0$, that is, L satisfies I.T(0). By Serre-duality this implies that $L^\vee$ satisfies I.T(g).*

Example 1.14. *Let L be an ample line bundle, $0 \in A$ the zero of the group law and I_0 its ideal sheaf. From the exact sequence*

$$0 \to I_0 \otimes L \to L \to k(0) \to 0$$

we see that $H^i(I_0 \otimes L \otimes P_\alpha) = 0$ for every $i \geq 2$ and that $H^1(I_0 \otimes L \otimes P_\alpha) \neq 0$ if and only if the evaluation map

$$H^0(L \otimes P_\alpha) \to k(0)$$

fails to be surjective, that is, if and only if 0 is a base point of the complete linear system associated to $L \otimes P_\alpha$. This means that $V^i(I_0 \otimes L) = \emptyset$ for $i \geq 2$ and

$$V^1(I_0 \otimes L) = -\varphi_L(\operatorname{Bs}(L))$$

because $L \otimes P_\alpha \simeq t_a^*L$ for every $a \in \varphi_L^{-1}(\{\alpha\})$. This means that $I_0 \otimes L$ is always GV, it is M-regular if and only if $\operatorname{Bs}(L)$ does not have divisorial part and it is I.T(0) if and only if L does not have base points.

Sometimes, instead of using $\Phi_{\mathcal{P}}$ we shall use the 'dual FM-transform'

$$\Phi_{\mathcal{P}^\vee}((-)^\vee) : D^b(A) \to D^b(\hat{A}).$$

The reason is the following characterization of GV-objects:

Theorem 1.15 (Theorem 3.7 in [PP11a])**.** *An object $\mathcal{F} \in D^b(A)$ is GV if and only if*

$$\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) = R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)[-g],$$

that is, if and only if $\mathcal{F}^\vee$ satisfies W.I.T(g).

We conclude this section mentioning a result of preservation of vanishing under (derived) tensor products, which is proved in [PP11b, Proposition 3.1 and Theorem 3.2] when $\mathcal{E}$ is a locally free sheaf. In Remark 3.3 of such reference it is observed that the more general result stated below is true but no proof is given. Since in Chapter 2 we will need this more general version, we prove it here for the sake of completeness, although it follows essentially the same lines of the cited reference.

Theorem 1.16. *Let $\mathcal{E} \in \mathrm{Coh}(A)$ and $\mathcal{F} \in D^b(A)$. Then*

a) If $\mathcal{E}$ satisfies I.T(0) and $\mathcal{F}$ is GV then $\mathcal{E} \underline{\otimes} \mathcal{F}$ satisfies I.T(0)

b) If $\mathcal{E}$ and $\mathcal{F}$ are both GV then $\mathcal{E} \underline{\otimes} \mathcal{F} \in D^b(A)$ is a GV-object.

Proof: a) Let $\alpha \in \hat{A}$ and we need to show that $H^i(\mathcal{F} \underline{\otimes} \mathcal{E} \otimes P_\alpha) = 0$. Now, since $\mathcal{E}$ is IT(0) it follows that $\mathcal{E} \otimes P_\alpha$ is also IT(0) and thus the sheaf

$$N_\alpha := R^0\Phi_{\mathcal{P}}(\mathcal{E} \otimes P_\alpha) = \Phi_{\mathcal{P}}(\mathcal{E} \otimes P_\alpha)$$

is locally free. On the other hand, by (1.3) we have that

$$\mathcal{E} \otimes P_\alpha \simeq \Phi_{\mathcal{P}}((-1_A)^* N_\alpha)[g]$$

and thus by (1.7) we have the following isomorphisms:

$$\begin{aligned} H^i(\mathcal{F} \underline{\otimes} \mathcal{E} \otimes P_\alpha) &= H^i(\mathcal{F} \underline{\otimes} \Phi_{\mathcal{P}}((-1_A)^* N_\alpha)[g]) \\ &\simeq H^i(\Phi_{\mathcal{P}}(\mathcal{F}) \underline{\otimes} (-1_A)^* N_\alpha[g]) \\ &= H^{i+g}(\Phi_{\mathcal{P}}(\mathcal{F}) \otimes (-1_A)^* N_\alpha). \end{aligned}$$

Now, the latter group can be computed from the Leray spectral sequence ([Huy06, p.74 (3.5)] combined with the projection formula):

$$E_2^{p,q} = H^p(R^q\Phi_{\mathcal{P}}(\mathcal{F}) \otimes (-1_A)^* N_\alpha) \implies H^{p+q}(\Phi_{\mathcal{P}}(\mathcal{F}) \otimes (-1_A)^* N_\alpha).$$

Since $\mathcal{F}$ is GV we have that $\dim \mathrm{Supp}(R^q\Phi_{\mathcal{P}}(\mathcal{F})) \leq g - q$ and thus $E_2^{p,q} = 0$ whenever $p + q > g$. The spectral sequence then gives

$$H^{g+i}(\Phi_{\mathcal{P}}(\mathcal{F}) \otimes (-1_A)^* N_\alpha) = 0 \text{ for every } \alpha \in \hat{A},$$

as we wanted to see.

b) Let L be an ample and symmetric line bundle and write $\widehat{L}$ for the vector bundle $R^g\Phi_{\mathcal{P}^\vee}(L^\vee) = \Phi_{\mathcal{P}^\vee}(L^\vee)[g]$. First, we claim that $\mathcal{E} \otimes \widehat{L^\vee}$ is IT(0). To prove the claim, note that we have the following sequence of isomorphisms:

$$\begin{aligned} H^i(\mathcal{E} \otimes \widehat{L^\vee} \otimes P_\alpha) &= H^{g+i}(\mathcal{E} \otimes P_\alpha \otimes (-1_A)^* \Phi_{\mathcal{P}}(L^\vee)) \\ &\simeq H^{g+i}(\Phi_{\mathcal{P}}(\mathcal{E} \otimes P_\alpha) \otimes L^\vee) && (1.7) \\ &\simeq H^{-i}(\Phi_{\mathcal{P}}(\mathcal{E} \otimes P_\alpha)^\vee \otimes L) && \text{Grothendieck- Serre duality} \\ &\simeq H^{g-i}(\Phi_{\mathcal{P}}((\mathcal{E} \otimes P_\alpha)^\vee) \otimes L) && (1.3) \end{aligned}$$

Now, as $\mathcal{E}$ is GV, $\mathcal{E} \otimes P_\alpha$ is also GV and thus by 1.15 we have that $\Phi_{\mathcal{P}^\vee}((\mathcal{E} \otimes P_\alpha)^\vee) = R^g\Phi_{\mathcal{P}^\vee}((\mathcal{E} \otimes P_\alpha)^\vee)[-g]$ and thus

$$H^i(\mathcal{E} \otimes \widehat{L^\vee} \otimes P_\alpha) \simeq H^{-i}(R^g\Phi_{\mathcal{P}^\vee}((\mathcal{E} \otimes P_\alpha)^\vee) \otimes L),$$

in particular, all such groups vanish for $i > 0$ and the claim is proved.

As $\mathcal{E} \otimes \widehat{L^\vee}$ is an IT(0) sheaf, from part a) it follows that $\mathcal{F} \underline{\otimes} \mathcal{E} \otimes \widehat{L^\vee}$ is IT(0), in particular this means that

$$H^i(\mathcal{F} \underline{\otimes} \mathcal{E} \otimes \widehat{L^\vee}) = 0 \text{ for every ample line bundle } L \text{ and } i > 0.$$

In particular, reasoning as before, we obtain that for any ample line bundle L we have

$$0 = H^i(\mathcal{F} \underline{\otimes} \mathcal{E} \otimes \widehat{L^\vee}) \simeq H^{g-i}(\Phi_{\mathcal{P}}(\mathcal{F}^\vee \underline{\otimes} \mathcal{E}^\vee) \otimes L) \text{ for } i > 0. \qquad (*)$$

We will apply the previous observation to the case in which $L = kM$, where M is an ample line bundle and k is a sufficiently big positive integer such that:

i) $H^p(R^q\Phi_{\mathcal{P}}(\mathcal{F}^\vee \underline{\otimes} \mathcal{E}^\vee) \otimes kM) = 0$ for every $p > 0$ and

ii) $R^q\Phi_{\mathcal{P}}(\mathcal{F}^\vee \underline{\otimes} \mathcal{E}^\vee) \otimes kM$ is globally generated for every $q \geq 0$.

(such k exist by [Laz04a, Theorem 1.2.6]). Now, the group at the right side in $(*)$ can be computed using the Leray spectral sequence, which, from condition i) degenerates to give isomorphisms

$$0 = H^{g-i}(\Phi_{\mathcal{P}}(\mathcal{F}^\vee \underline{\otimes} \mathcal{E}^\vee) \otimes kM) \simeq H^0(R^{g-i}\Phi_{\mathcal{P}}(\mathcal{F}^\vee \underline{\otimes} \mathcal{E}^\vee) \otimes kM) \text{ for } i > 0.$$

At this point, condition ii) implies that $R^{g-i}\Phi_{\mathcal{P}}(\mathcal{F}^\vee \underline{\otimes} \mathcal{E}^\vee) = 0$ for $i \neq 0$ and therefore

$$\Phi_{\mathcal{P}}((\mathcal{E} \underline{\otimes} \mathcal{F})^\vee) = R^g\Phi_{\mathcal{P}}(\mathcal{F}^\vee \underline{\otimes} \mathcal{E}^\vee)[-g],$$

which by 1.15 means that $\mathcal{E} \underline{\otimes} \mathcal{F}$ is a GV object, as we wanted to see.

□

1.3 $\mathbb{Q}$-twisted objects and cohomological rank functions

In this work we will need to deal not just with *objects*, but also with $\mathbb{Q}$-twisted objects, which informally speaking are twists of objects with "rational powers" of line bundles. More precisely, we consider pairs $(\mathcal{F}, t\ell)$, where $\mathcal{F}$ is an object in $D^b(A)$, t is a rational number and $\ell \in \mathrm{NS}(A)$. On these pairs we impose the relation given by identifying $(\mathcal{F} \otimes L^{\otimes m}, t\ell)$ with the pair $(\mathcal{F}, (t+m)\ell)$ for every integer m and for every L representing ℓ. Note that, for example, $(L \otimes P_\alpha, 0 \cdot \ell)$ is identified with $(L, 0 \cdot \ell)$ because both of these pairs are equivalent to $(\mathcal{O}_A, \ell)$; this causes no harm since we will deal with properties that depends just on the algebraic class of a line bundle (and not on the line bundle itself).

Definition 1.17. A $\mathbb{Q}$-twisted object, written $\mathcal{F}\langle t\ell \rangle$, is the class of the pair $(\mathcal{F}, t\ell)$ under the relation described in the previous paragraph.

In [JP20, Section 5] the *generic vanishing* conditions presented in the previous section are extended to the $\mathbb{Q}$-twisted setting. Indeed, for a $\mathbb{Q}$-twisted object $\mathcal{F}\langle t\ell\rangle$ it is possible to define its cohomological support loci $V^i(\mathcal{F}\langle t\ell\rangle)$ as follows. Let L be a line bundle representing ℓ and consider $a, b \in \mathbb{Z}$ with $b > 0$ and $t = a/b$. Write

$$V^i(\mathcal{F}, L, a, b) := \left\{\alpha \in \hat{A} : H^i(b_A^*\mathcal{F} \otimes L^{ab} \otimes P_\alpha) \neq 0\right\},$$

where H^i stands for the sheaf (hyper)cohomology. Now, if N is another representant of ℓ then $V^i(\mathcal{F}, N, a, b)$ is just a traslation of $V^i(\mathcal{F}, L, a, b)$. On the other hand we note that if we change a, b for ac, bc with c a positive integer and we consider a symmetric L, then:

$$\begin{aligned}
V^i(\mathcal{F}, L, ac, bc) &= \left\{\alpha \in \hat{A} : H^i((bc)_A^*\mathcal{F} \otimes L^{c^2ab} \otimes P_\alpha) \neq 0\right\} \\
&= \left\{\alpha \in \hat{A} : H^i(c_A^*(b_A^*\mathcal{F} \otimes L^{ab} \otimes P_\gamma)) \neq 0 \ \text{ for a (any) } \gamma \in c_A^{-1}(\alpha)\right\} \\
&= \left\{\alpha \in \hat{A} : \bigoplus_{\gamma \in c_A^{-1}(\alpha)} H^i(b_A^*\mathcal{F} \otimes L^{ab} \otimes P_\gamma) \neq 0\right\}, \\
&= c_A\left(V^i(\mathcal{F}, L, a, b)\right).
\end{aligned}$$

where, to get the direct sum we used [Mum08, p.72] and the projection formula.

In this context, the next definition naturally arise:

Definition 1.18. For a $\mathbb{Q}$-twisted object $\mathcal{F}\langle t\ell\rangle$ its i-th cohomological support loci $V^i(\mathcal{F}\langle t\ell\rangle)$ is the class of $V^i(\mathcal{F}, L, a, b)$, where L is a representant of ℓ and $a/b = t$, under the equivalence relation generated by translations and by taking direct images of multiplication isogenies.

In particular, although $V^i(\mathcal{F}\langle t\ell\rangle)$ is not a set, its dimension and codimension are well defined and are denoted by $\dim V^i(\mathcal{F}\langle t\ell\rangle)$ and $\operatorname{codim}_{\hat{A}} V^i(\mathcal{F}\langle t\ell\rangle)$, respectively. In [JP20, Section 5] the cited authors use these loci in order to define generic vanishing notions for $\mathbb{Q}$-twisted objects:

Definition 1.19. We say that a $\mathbb{Q}$-twisted object $\mathcal{F}\langle t\ell\rangle$ is

1. IT(0) if a (any) representant of $V^i(\mathcal{F}\langle t\ell\rangle)$ is empty for $i \neq 0$
2. GV if $\operatorname{codim}_{\hat{A}} V^i(\mathcal{F}\langle t\ell\rangle) \geq i$ for every $i \geq 0$

Example 1.20. *By Kodaira-vanishing theorem, it follows that $\mathcal{O}_A\langle t\ell\rangle$ is IT(0) for every $t > 0$.*

We have the following characterization of the behaviour of the generic vanishing conditions of $\mathcal{F}\langle t\ell\rangle$ for varying t :

Theorem 1.21 ([JP20], Theorem 5.2)**.**

1. *A $\mathbb{Q}$-twisted object $\mathcal{F}\langle t_0\ell\rangle$ is GV if and only if $\mathcal{F}\langle t\ell\rangle$ is IT(0) for every $t > t_0$.*

2. *If* $\mathcal{F}\langle t_0\ell\rangle$ *is GV but not IT(0) then for every* $t < t_0$ *the* $\mathbb{Q}$*-twisted object* $\mathcal{F}\langle t\ell\rangle$ *is not GV*

3. $\mathcal{F}\langle t_0\ell\rangle$ *is IT(0) if and only if* $\mathcal{F}\langle (t_0-\eta)\ell\rangle$ *is IT(0) for every* $\eta > 0$ *small enough*

The rational analogues of the theorems of the previous sections are the following:

Theorem 1.22.

1. *A* $\mathbb{Q}$*-twisted object* $\mathcal{F}\langle t\ell\rangle$ *is GV if and only if*

$$\operatorname{codim}_{\hat{A}} \operatorname{Supp}\left(R^i\Phi_{\mathcal{P}}(b_A^*\mathcal{F}\otimes L^{ab})\right) \geq i$$

for a (any) representation $t = a/b$ *and for a (any) representant* L *of* ℓ*.*

2. *Let* $\mathcal{F}\langle t\ell\rangle$ *be a* $\mathbb{Q}$*-twisted object and* $\mathcal{G}\langle s\ell\rangle$ *a* $\mathbb{Q}$*-twisted sheaf. If both of them are GV then* $(\mathcal{F}\underline{\otimes}\mathcal{G})\langle (t+s)\ell\rangle$ *is also GV.*

We conclude this chapter by reviewing the theory of *cohomological rank functions*, introduced in [JP20], which are a sensible way of defining the (dimension of the generic) cohomology of a $\mathbb{Q}$-twisted object, which are also a convenient way of expressing wether $V^i(\mathcal{F}\langle t\ell\rangle) = \hat{A}$ or not, when we let t vary.

Definition 1.23. Let A be a g-dimensional abelian variety, $\ell \in \mathrm{NS}(A)$ an ample class and $\mathcal{F} \in D^b(A)$. Given $i \in \mathbb{Z}$, the i-th cohomological rank function of $\mathcal{F}$ with respect to ℓ is the function $\mathbb{Q} \to \mathbb{Q}$ given by

$$\frac{a}{b} \longrightarrow h^i\left(\mathcal{F}\left\langle \frac{a}{b}\ell\right\rangle\right) := \frac{1}{b^{2g}} \cdot \min\left\{h^i(b_A^*\mathcal{F}\otimes L^{ab}\otimes P_\alpha) : \alpha \in \hat{A}\right\},$$

where L is a (any) line bundle representing ℓ.

Here a couple of observations are in place in order to make precise the paragraph preceding the definition:

- By semicontinuity, the minimum involved in the definition is actually the generic value of $h^i(b_A^*\mathcal{F}\otimes L^{ab}\otimes P_\alpha)$ for varying α. In other words, $h^i(\mathcal{F}\langle t\ell\rangle) = 0$ if and only if $V^i(\mathcal{F}\langle t\ell\rangle) \neq \hat{A}$. In particular, from Example 1.14 we see that $I_0\langle t\ell\rangle$ is GV if and only if $h^1(I_0\langle t\ell\rangle) = 0$.
- If we take another representant M of ℓ, then $M = L\otimes P_\alpha$ for some $\alpha \in \hat{A}$ and thus obviously $h^i(\mathcal{F}\langle t\ell\rangle)$ does not depend on that choice. In particular, we can always choose a symmetric representant (i.e L such that $(-1_A)^*L \simeq L$), for which we have $b_A^*L \simeq L^{b^2}$.
- The definition depends just on t and not on a particular representation as a/b. Indeed, in characteristic zero we can argue as follows: let c be a positive integer, we need to show that for a generic $\alpha \in \hat{A}$ and a representant L of ℓ we have

$$h^i((bc)_A^*(\mathcal{F})\otimes L^{abc^2}\otimes P_\alpha) = c^{2g}\cdot h^i(b_A^*(\mathcal{F})\otimes L^{ab}\otimes P_\alpha).$$

Without loss of generality we may assume that L is symmetric and thus $c_A^* L \simeq L^{c^2}$. In this setting we have:

$$\begin{aligned} H^i((bc)_A^*(\mathcal{F}) \otimes L^{abc^2} \otimes P_\alpha) &\simeq H^i(c_A^*(b_A^*(\mathcal{F}) \otimes L^{ab} \otimes P_{\alpha/c})) \qquad \text{for } \alpha/c \in c_{\hat{A}}^{-1}(\alpha) \\ &\simeq \bigoplus_{\gamma \in \hat{A}[c]} H^i(b_A^*(\mathcal{F}) \otimes L^{ab} \otimes P_{\alpha/c} \otimes P_\gamma) \qquad [\text{Mum08}, p.72], \end{aligned}$$

now, for generic α all of the direct summands at the righ side have the generic dimension and thus the claim follows.

- Similarly, for an isogeny $f : A \to B$ between abelian varieties, we have that

$$h^i(A, (f^*\mathcal{F}) \langle t \cdot f^*\ell \rangle) = \deg(f) \cdot h^i(B, \mathcal{F} \langle t\ell \rangle) \tag{1.12}$$

- For any $m \in \mathbb{Z}$, $t = a/b$ and generic $\alpha \in \hat{A}$ we have

$$\begin{aligned} h^i((\mathcal{F} \otimes L^m) \langle t\ell \rangle) &:= \frac{1}{b^{2g}} \cdot h^i(b_A^*(\mathcal{F} \otimes L^m) \otimes L^{ab} \otimes P_\alpha) \\ &= \frac{1}{b^{2g}} \cdot h^i(b_A^*(\mathcal{F}) \otimes L^{mb^2+ab} \otimes P_\alpha) \\ &= \frac{1}{b^{2g}} \cdot h^i(b_A^*(\mathcal{F}) \otimes L^{b(a+mb)} \otimes P_\alpha) \\ &= h^i\left(\mathcal{F} \left\langle \frac{a+mb}{b}\ell \right\rangle\right) = h^i(\mathcal{F} \langle (t+m)\ell \rangle), \end{aligned}$$

so the cohomological rank functions respect the equivalence defining $\mathbb{Q}$-twisted sheaves.

One of the main properties of the cohomological rank functions is the following fundamental transformation formula with respect to the FMP functor:

Proposition 1.24 ([JP20], Proposition 2.3)**.** *For $\mathcal{F}$ and ℓ as in Definition 1.23 above and $t > 0$, the following equalities hold:*

a) $h^i(\mathcal{F} \langle -t\ell \rangle) = \frac{t^g}{\chi(\ell)} \cdot h^i\left((\varphi_\ell^* \Phi_{\mathcal{P}}(\mathcal{F})) \left\langle \frac{1}{t}\ell \right\rangle\right)$

b) $h^i(\mathcal{F} \langle t\ell \rangle) = \frac{t^g}{\chi(\ell)} \cdot h^{g-i}\left((\varphi_\ell^* \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)) \left\langle \frac{1}{t}\ell \right\rangle\right)$, *where $\mathcal{F}^\vee$ stands for the* derived *dual $R\mathscr{H}om(\mathcal{F}, \mathcal{O}_A)$.*

Chapter 2

Jets separations, Seshadri constants and higher Gauss-Wahl maps

2.1 Vanishing thresholds

In this section we introduce the notion of vanishing thresholds for coherent sheaves. Afterwards, we give a some examples and at the end we prove Theorem from the introduction.

Definition 2.1.

1. Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A and ℓ be a polarization on A. The vanishing threshold of $\mathcal{F}$ with respect to ℓ is the real number
$$\nu_\ell(\mathcal{F}) := \inf\left\{t \in \mathbb{Q} : \mathcal{F}\langle t\ell\rangle \text{ satisfies IT}(0)\right\}.$$
2. Given a closed subscheme Z of A and a non-negative integer p we write
$$\epsilon_p(Z, \ell) = \nu_\ell\left(I_Z^{p+1}\right),$$
where I_Z is the ideal sheaf of Z.

We note that, by definition, we have that $\nu_{n\ell}(\mathcal{F}) = n^{-1}\nu_\ell(\mathcal{F})$.

The main example for our work regards the notion of separation of jets that we now recall:

Definition 2.2. Let X be a projective variety, and x a closed point in X with ideal I_x. Given a line bundle L on X and a non-negative integer p, we say that L separates p-jets at x if the natural map

$$H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_X/I_x^{p+1})$$

is surjective.

In particular, L separates 0-jets at x if and only if x is not a basepoint of L, while L separates 1-jets at x if and only if x is not a basepoint and L separates tangent vectors at x (i.e the map between tangent spaces induced by the rational map $X - - > \mathbb{P}H^0(X, L)$ is injective). Arguing as in Example 1.14 we see that L separates p-jets at every point if and only if $L \otimes I_0^{p+1}$ is IT(0). In particular, it is natural to consider the following:

Definition 2.3. The p-jets separation threshold of ℓ is the number

$$\epsilon_p(\ell) := \epsilon_p(\{0\}, \ell),$$

where we consider the reduced scheme structure on the set $\{0\}$.

Example 2.4. *We have that $\epsilon_0(\ell)$ is the basepoint-freeness threshold $\beta(\ell)$ considered in [JP20, §8]. In particular $\epsilon_0(\ell) \leq 1$ and $\epsilon_0(\ell) = 1$ if and only if ℓ has base points. More generally, from Proposition 1.21 we see that*

1. *$\epsilon_p(\ell) \leq 1$ if and only if there exist a representant L of ℓ such that L separates p-jets at 0. Equivalently, for any representant L of ℓ there exists a point such that L separates p-jets at such a point.*

2. *$\epsilon_p(\ell) < 1$ if and only if for every representant L of ℓ, L separates p-jets at 0. Equivalently, any representant L of ℓ separates p-jets at every point of A.*

Example 2.5. *Consider a principal polarization θ and Θ a symmetric divisor representing it. If θ is indecomposable, then $\mathcal{O}_A(2\Theta)$ fails to separate 1-jets (i.e tangent vectors) just in the 2-torsion points of A. That is, $I_0^2 \langle 2\theta \rangle$ is GV but not IT(0) which, by Proposition 1.21, means that $\epsilon_1(\theta) = 2$. From [BL04, p.99], the same conclusion holds if θ is decomposable.*

Example 2.6. *Let C be a smooth curve of genus g and $u : C \hookrightarrow \operatorname{Jac} C$ an Abel-Jacobi map. By [JP20, Theorem 7.5 and Proposition 7.6] we know that*

$$h^1\left(\mathcal{O}_{u(C)} \langle t\theta \rangle\right) = t^g - gt + (g-1) > 0 \quad \textit{for } t \in [0, 1).$$

Write $A = \operatorname{Jac} C$. From the exact sequence

$$0 \to b_A^* I_{u(C)} \otimes \mathcal{O}_A(ab\Theta) \to \mathcal{O}_A(ab\Theta) \to b_A^* \mathcal{O}_{u(C)}(ab\Theta) \to 0$$

it follows that

$$h^2\left(I_{u(C)} \langle t\theta \rangle\right) = h^1\left(\mathcal{O}_{u(C)} \langle t\theta \rangle\right) > 0 \ \ \textit{for } t \in [0, 1).$$

This means that

$$H^2\left(b_A^* I_{u(C)} \otimes \mathcal{O}_A(ab\Theta) \otimes P_\alpha\right) \neq 0 \quad \textit{for every } \alpha \in \hat{A} \textit{ and } a < b$$

and hence $I_{u(C)} \langle t\theta \rangle$ is not GV for every $t < 1$. On the other hand, in [PP03, Proposition 4.4] it is shown that $I_{u(C)} \langle \theta \rangle$ is GV and thus $\epsilon_0(u(C), \theta) = 1$.

By [PP08, Theorem 6.1] this is esentially the only example of a curve inside an abelian variety whose vanishing threshold is at most one. More precisely, if C is a smooth curve that generates an abelian variety A and $I_C \langle \theta \rangle$ is GV for a principal polarization θ, then C has minimal cohomology class $\theta^{g-1}/(g-1)!$ and thus by Matsusaka's criterion [Mat59, Theorem 3] it follows that $(A, \theta) \simeq (\operatorname{Jac} C, \theta_C)$.

Example 2.7. *Let (A, θ) be a g-dimensional principally polarized abelian variety and τ a length two closed subscheme supported in the origin. Again in this case, we have that*

$$\epsilon_0(\tau, \theta) = \inf \left\{ t \in \mathbb{Q} : h^1(I_\tau \langle t\theta \rangle) = 0 \right\}.$$

Let Θ be a symmetric divisor representing θ. We now compute the dual Fourier transform of $I_\tau(\Theta)$: first, we note that $I_{0/\tau} \simeq k(0)$ and thus we have an exact sequence

$$0 \to I_\tau(\Theta) \to I_0(\Theta) \to k(0) \to 0.$$

Now, $\Phi_{\mathcal{P}^\vee}(k(0)^\vee) \simeq \mathcal{O}_A[-g]$ and $\Phi_{\mathcal{P}^\vee}(I_0(\Theta)^\vee) \simeq \mathcal{O}_\Theta(\Theta)[-g]$, whence we identify A with $\hat{A}$ via θ (see Examples 1.4 and 1.14). It follows that

$$R^j\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee) = 0 \quad \textit{for } j \leq g-2$$

and that we have an exact sequence

$$0 \to R^{g-1}\Phi_{\mathcal{P}^\vee}\left(I_\tau(\Theta)^\vee\right) \to \mathcal{O}_A \to \mathcal{O}_\Theta(\Theta) \to R^g\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee) \to 0. \tag{2.1}$$

For this example the value of R^g it does not matter (see, however, Section 4.4) and thus we focus on R^{g-1}. From (2.1) *we have that*

$$R^{g-1}\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee) \simeq \ker\left[\mathcal{O}_A \to \mathcal{O}_\Theta(\Theta)\right] \tag{2.2}$$

and thus we need to understand such map. From the chain of contentions $0 \subset \tau \subset 0^2$, where 0^2 is the subscheme defined by I_0^2, we have a commutative diagram

$$\begin{array}{ccc} I_0(\Theta) & \longrightarrow & I_{0/\tau}(\Theta) \simeq k(0) \\ & \searrow \quad \nearrow & \\ & I_0(\Theta)/I_0^2(\Theta) & \end{array}$$

Now, $I_0(\Theta)/I_0^2(\Theta) \simeq T_{A,0} \otimes k(0)$ (the tangent space at the origin) and the induced morphism

$$\mathcal{O}_A \simeq R^g\Phi_{\mathcal{P}^\vee}(I_{0/\tau}^\vee) \to R^g\Phi_{\mathcal{P}^\vee}(T_{A,0} \otimes k(0)) \simeq T_{A,0} \otimes \mathcal{O}_A,$$

is no other than the morphism corresponding to τ. On the other hand, we have that

$$T_{A,0} \otimes k(0) \simeq H^1(\mathcal{O}_A) \simeq H^0(\mathcal{O}_\Theta(\Theta)),$$

*where the first isomorphism comes from [**?**] while the second one comes from taking cohomology to the exact sequence*

$$0 \to \mathcal{O}_A \to \mathcal{O}_A(\Theta) \to \mathcal{O}_\Theta(\Theta) \to 0.$$

Summarizing, the map in (2.2) *is the composition*

$$\mathcal{O}_A \xrightarrow{\tau} H^0(\mathcal{O}_\Theta(\Theta)) \otimes \mathcal{O}_A \xrightarrow{ev} \mathcal{O}_\Theta(\Theta) ,$$

which means that

$$R^{g-1}\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee) \simeq \ker\left[V\otimes\mathcal{O}_A \to \mathcal{O}_\Theta(\Theta)\right],$$

where $V \subset H^0(\mathcal{O}_\Theta(\Theta))$ is a one-dimensional subspace. Since this last map factors through the evaluation map

$$V\otimes\mathcal{O}_\Theta \to \mathcal{O}_\Theta(\Theta),$$

it follows that its image is isomorphic to $\mathcal{O}_\Theta$ and thus

$$R^{g-1}\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee) \simeq I_{\Theta/A} \simeq \mathcal{O}_A(-\Theta).$$

We want to compute the cohomological rank function $h^1(I_\tau\langle t\theta\rangle)$. From the transformation formula (Proposition 1.24b)) for $t = a/b > 0$ we have

$$h^1(I_\tau\langle(1+t)\theta\rangle) = h^1(I_\tau(\Theta)\langle t\theta\rangle) = t^g\cdot h^{g-1}\left(\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee)\left\langle\frac{1}{t}\theta\right\rangle\right), \qquad (2.3)$$

which means that we need to compute the hypercohomology

$$H^{g-1}(a_A^*\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee)\otimes\mathcal{O}_A(ab\Theta)\otimes P_\alpha) \quad \text{for } a,b>0 \text{ and } \alpha\in\hat{A}.$$

From our previous calculation we see that the spectral sequence

$$E_2^{p,q} = H^p(a_A^*R^q\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta))\otimes\mathcal{O}_A(ab\Theta)\otimes P_\alpha) \implies H^{p+q}(a_A^*\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta))\otimes\mathcal{O}_A(ab\Theta)\otimes P_\alpha)$$

for $t\le 1$ collapse to give:

$$h^1\left(\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee)\left\langle\frac{1}{t}\theta\right\rangle\right) = h^0\left(\mathcal{O}_A(-\Theta)\left\langle\frac{1}{t}\theta\right\rangle\right) = \left(\frac{1}{t}-1\right)^g.$$

From (2.3) *it follows that*

$$h^1(I_\tau\langle(1+t)\theta\rangle) = (1-t)^g \quad \text{for } t\in[0,1].$$

Since it is clear that $h^1(I_\tau\langle u\theta\rangle)\neq 0$ for $u\in(0,1)$ we conclude that

$$\epsilon_0(\tau,\theta) = 2.$$

A couple of formal properties are the following:

Proposition 2.8. *a) If Z is a smooth closed subscheme of an abelian variety A then*

$$\epsilon_p(Z,\ell) \le \max\left\{\epsilon_{p+1}(Z,\ell), \nu_\ell\left(\mathrm{Sym}^{p+1} N_{Z/A}^\vee\right)\right\},$$

where $N_{Z/A}^\vee$ is the conormal sheaf of Z in A.

b) The sequence $\{\epsilon_p(\ell)\}_{p\in\mathbb{Z}_{\ge 0}}$ of jets-separation thresholds is unbounded and not decreasing

Proof: a) We want to prove that $I_Z^p \langle t\ell \rangle$ is GV whenever both $I_Z^{p+1} \langle t\ell \rangle$ and $(\operatorname{Sym}^p N_{Z/A}^\vee) \langle t\ell \rangle$ are GV. As Z is smooth, we have that

$$I_Z^p / I_Z^{p+1} \simeq \operatorname{Sym}^p \left(I_Z / I_Z^2 \right) = \operatorname{Sym}^p \ N_{Z/A}^\vee$$

and thus we get exact sequences

$$0 \to I_Z^{p+1} \to I_Z^p \to \operatorname{Sym}^p \ N_{Z/A}^\vee \to 0$$

and thus the claim follows from [Ito22a, Lemma 2.8 (1)].

b) The fact that the sequence is not decreasing follows from the previous part considering $Z = \{0\}$ (with reduced scheme structure). Now, the sequence $\{h^0(\mathcal{O}_A/I_0^{p+1})\}_{p \in \mathbb{Z}_{\geq 0}}$ is unbounded and for every $p \in \mathbb{Z}$ and for every line bundle M we have

$$M \otimes (\mathcal{O}_A/I_0^{p+1}) \simeq \mathcal{O}_A/I_0^{p+1}.$$

It follows that, for any fixed $N \in \mathbb{Z}$ there exists $p \gg 0$ such that for every $\alpha \in \hat{A}$ the restriction map

$$H^0(L^N \otimes P_\alpha) \to H^0 \left(L^N \otimes P_\alpha \otimes (\mathcal{O}_A/I_0^{p+1}) \right)$$

is not surjective and thus $I_0^{p+1} \langle N\ell \rangle$ can not be GV. In other words, for any N there exists p such that $\epsilon_p(\ell) \geq N$ and hence the sequence $\{\epsilon_p(\ell)\}_p$ is unbounded.

□

We also have the following:

Theorem 2.9. *Let ℓ be a polarization on an abelian variety A and $Z \subset A$ a closed subscheme of dimension at most one. Then for $p, r \in \mathbb{Z}$ positive integers with $p < r$, the following inequalities hold:*

$$0 \leq \epsilon_r(Z, \ell) \leq \epsilon_{r-p-1}(Z, \ell) + \epsilon_p(Z, \ell).$$

In particular, the sequence $\{(p+1)^{-1}\epsilon_p(Z, \ell)\}_{p \in \mathbb{Z}_{\geq 1}}$ converges and

$$\lim_{p \to \infty} \frac{\epsilon_p(Z, \ell)}{p+1} = \inf_p \frac{\epsilon_p(Z, \ell)}{p+1}.$$

In order to prove the theorem, the following subadditivity lemma is fundamental:

Lemma 2.10. *Let I, J ideals sheaves of subschemes of dimension at most one. Let M be a line bundle such that $I \underline{\otimes} J \otimes M$ is GV. Then both $I \otimes J \otimes M$ and $IJ \otimes M$ are GV.*

Proof of the Lemma: For an object $\mathcal{F}$ we write $V^i(\mathcal{F}) = \{\alpha \in \hat{A} : H^i(\mathcal{F} \otimes P_\alpha) \neq 0\}$. In this context, the hypothesis means that

$$\operatorname{codim}_{\hat{A}} V^i(I \underline{\otimes} J \otimes M) \geq i \ \text{ for all } i$$

and we need to prove

$$\operatorname{codim}_{\hat{A}} V^i(IJ \otimes M) \geq i. \tag{2.4}$$

Now, we claim that

$$V^i(IJ \otimes M) \subseteq V^i(I \otimes J \otimes M) \subseteq V^i(I \underline{\otimes} J \otimes M). \tag{2.5}$$

It is clear that, once the claim is proved, the result follows. Now, to prove the claim, we prove the opposite contentions between the complements, that is, we show that if for $M_\alpha := M \otimes P_\alpha$ we have $H^i(I \underline{\otimes} J \otimes M_\alpha) = 0$ then $H^i(I \otimes J \otimes M_\alpha) = 0$ and if such vanishing holds then $H^i(IJ \otimes M_\alpha) = 0$. To do this we use the (fourth-quadrant) spectral sequence ([**?**, (3.5)])

$$E_2^{p,q} = H^p(\mathscr{T}or_{-q}(I,J) \otimes M_\alpha) \implies H^{p+q}(I \underline{\otimes} J \otimes M_\alpha).$$

Tensoring the exact sequence $0 \to I \to \mathcal{O}_A \to \mathcal{O}_A/I \to 0$ by J we see that the support of the sheaves $\mathscr{T}or_i(I,J)$ for $i > 0$ is contained in the intersection of the cosupports of I and J. As such scheme has dimension at most one, it follows that

$$H^p(\mathscr{T}or_i(I,J) \otimes M_\alpha) = 0 \text{ for } i > 0 \text{ and } p \geq 2$$

and spectral sequence already degenerates at the second page. It follows that $E_2^{i,0} = H^i(I \otimes J \otimes M_\alpha)$ is a subquotient of $H^i(I \underline{\otimes} J \otimes M)$ and thus

$$V^i(I \otimes J \otimes M) \subseteq V^i(I \underline{\otimes} J \otimes M),$$

as we wanted to see.

Now, to prove that $IJ \otimes M$ is also GV, we note that we have the exact sequence

$$0 \to \mathscr{T}or_1(I,J) \otimes M_\alpha \to I \otimes J \otimes M_\alpha \to IJ \otimes M_\alpha \to 0,$$

and thus taking cohomology we get an exact sequence

$$H^i(I \otimes J \otimes M_\alpha) \to H^i(IJ \otimes M_\alpha) \to H^{i+1}(\mathscr{T}or_1(I,J) \otimes M_\alpha).$$

As $\dim \operatorname{Supp} \mathscr{T}or_1(I,J) \leq 1$ the H^{i+1} at the right is zero for $i \geq 1$ and hence we get the contention

$$V^i(IJ \otimes M) \subseteq V^i(I \otimes J \otimes M)$$

that we wanted to prove.

□

Proof: Let $s,t \in \mathbb{Q}$ such that both $I_Z^{r-k}\langle t\ell\rangle$ and $I_Z^{k+1}\langle s\ell\rangle$ are GV. By Theorem 1.16 we have that $(I_Z^{r-k} \underline{\otimes} I_Z^{k+1})\langle (s+t)\ell\rangle$ is GV. If we write $s = a/b$ and $t = c/e$, the latter condition means that $\lambda_{be}^*(I_Z^{r-k} \underline{\otimes} I_Z^{k+1}) \otimes L^{be(ae+bc)}$ is GV. Now, as pullbacks commutes with derived tensor products and λ_{be} is a flat morphism (so $\lambda_{be}^* I_Z^{k+1}$ and $\lambda_{be}^* I_Z^{r-k}$ are also ideals), the Lemma implies that $I_Z^{r+1}\langle (s+t)\ell\rangle$ is GV, as we wanted to see. The result then follows from the well known Lemma 2.11 below.

□

Lemma 2.11. *Let $\{x_k\}_{k\in\mathbb{Z}_{>0}}$ be a sequence of real numbers such that*

$$x_{k+r} \leq x_k + x_r \quad \text{for every } k,r \in \mathbb{Z}_{>0}.$$

Then the sequence $\{x_k/k\}_{k\in\mathbb{Z}_{>0}}$ converges and

$$\lim_{k\to\infty} \frac{x_k}{k} = \inf_{k\in\mathbb{Z}_{>0}} \frac{x_k}{k}.$$

Proof of the Lemma: We will show that for every $k \in \mathbb{Z}_{>0}$ we have

$$\limsup \frac{x_n}{n} \leq \frac{x_k}{k}, \tag{$*$}$$

which implies

$$\limsup \frac{x_n}{n} \leq \inf_k \frac{x_k}{k} \leq \liminf \frac{x_n}{n},$$

from which the result would follow. It is enough then to show $(*)$. To do this, fix $k \in \mathbb{Z}_{\geq 0}$ and for $n \in \mathbb{Z}_{>0}$ write $n = q_n k + r_n$ with $r_n \in \{0, ..., k-1\}$. We then have

$$\frac{x_n}{n} \leq \frac{q_n x_k}{q_n k + r_n} + \frac{x_r}{n} \leq \frac{x_k}{k} + \frac{1}{n} \cdot \max\{x_0, ..., x_{k-1}\}.$$

Letting $n \to \infty$ we obtain the desired inequality.

□

An immediate consequence of Theorem 2.9 above is the following important particular case:

Corollary 2.12. *For every $p \in \mathbb{Z}_{\geq 0}$ the following inequalites of jets-separation thresholds hold:*

$$\epsilon_p(\ell) \leq (p+1)\epsilon_0(\ell) \leq p+1$$

and if $\epsilon_p(\ell) = p+1$ then $\epsilon_r(\ell) = r+1$ for every $r \leq p$.

Remark 2.13. *Note that using the above proposition we see that if $\epsilon_0(\ell) < (p+1)^{-1}$ then $\epsilon_p(\ell) < 1$ and thus ℓ separates p-jets at every point. In particular we recover the well known fact ([BS97, Theorem 1],[PP11b, Theorem (4)]) that for every $p \in \mathbb{Z}_{\geq 0}$ we have that $(p+2)\ell$ separates p-jets at every point.*

2.2 Seshadri constants

In the previous section we proved that for a closed subscheme Z of dimension at most one, the sequence $\{(p+1)^{-1}\epsilon_p(Z, \ell)\}_p$ converges. In this section we establish that the limit is actually the inverse of the Seshadri constant of the ideal I_Z with respect to ℓ.

First, recall that for an ample line bundle L and an ideal sheaf I, the Seshadri constant of I with respect to L is the real number

$$\varepsilon(I, L) = \sup\left\{t \in \mathbb{Q} : \sigma^* L - tE \text{ is nef}\right\},$$

where $\sigma : \mathrm{Bl}_I A \to A$ is the blow-up along I with exceptional divisor E.

In particular, the "classical Seshadri constant" $\varepsilon(L, x)$ of a line bundle L at a point x is no other than $\varepsilon(I_x, L)$. By [Laz04a, Proposition 5.1.5] we have

$$\varepsilon(L, x) = \inf_{C \ni x} \frac{(L \cdot C)}{\mathrm{mult}_x(C)},$$

where the infimum runs through all the irreducible curves containing x. In particular, it follows that $\varepsilon(L,x)$ depends just on the numerical class of L and, in the case of abelian varieties, it does not depend on the particular point x. In this context, we may write

$$\varepsilon(\ell) = \varepsilon(L,x) = \varepsilon(I_x, L),$$

where x is a (any) closed point and ℓ is the class of L.

Theorem 2.14. *Let L be an ample line bundle on an abelian variety A and ℓ the corresponding polarization. Let Z be a closed subscheme of A. Then:*

$$\sup \frac{p+1}{\epsilon_p(Z,\ell)} \geq \varepsilon(I_Z, L) \geq \limsup \frac{p+1}{\epsilon_p(Z,\ell)}.$$

If $\dim Z \leq 1$ *then*

$$\varepsilon(I_Z, L) = \lim_{p\to\infty} \frac{p+1}{\epsilon_p(Z,\ell)} = \sup_p \frac{p+1}{\epsilon_p(Z,\ell)}.$$

In particular, in this case we have that $\epsilon_p(Z,\ell) \geq (p+1)\varepsilon(I_Z,L)^{-1}$ for every $p \in \mathbb{Z}_{\geq 0}$.

Proof: The proof of this statement closely follows [CEL01, Theorem 3.2]. First we prove that $\varepsilon(I_Z, L) \geq \sup_p (p+1)\epsilon_p(Z,\ell)^{-1}$. In order to do this we consider a rational number $t = a/b < \varepsilon(I_Z, L)$ and we need to show that there exists k (possibly very big) such that $\epsilon_k(Z,\ell) \leq t^{-1}(k+1)$, for which is enough to find $k \gg 0$ such that

$$H^j(a_A^* I_Z^{k+1} \otimes L^{ab(k+1)}) = 0 \ \text{ for } j > 0. \tag{2.6}$$

To do this, we start by noticing that by [CEL01, Lemma 3.3] there exists r_0 such that

$$H^j(I_Z^r \otimes M) \simeq H^j(\sigma^* M \otimes \mathcal{O}_{X'}(-rE)) \tag{2.7}$$

for all $j \geq 0$, all $r \geq r_0$ and all line bundles M, where $X' = \mathrm{Bl}_Z X$ and $\sigma : X' \to X$ is the corresponding projection. Now, let u such that $au \geq r_0$. For all such u consider $k_u = au - 1$ and thus we have

$$H^j(a_A^* I_Z^{k_u+1} \otimes L^{ab(k_u+1)}) = H^j(a_A^* I_Z^{au} \otimes L^{a^2bu}) \simeq \bigoplus_{\alpha\in\hat{A}[a]} H^j(I_Z^{au} \otimes L^{bu} \otimes P_\alpha),$$

so, in order to get the vanishing (2.6), we need to ensure the vanishing of each direct summand. Now, by (2.7) and the way we choose u we have that

$$\begin{aligned} H^j(I_Z^{au} \otimes L^{bu} \otimes P_\alpha) &\simeq H^j(\sigma^*(L^{bu} \otimes P_\alpha) \otimes \mathcal{O}_{X'}(-auE)) \\ &\simeq H^j((\sigma^* P_\alpha) \otimes \mathcal{O}_{X'}(b\sigma^* L - aE)^{\otimes u}). \end{aligned}$$

Now, as $\varepsilon(I_Z, L) > a/b$ we have that $b\sigma^*L - aE$ is ample and thus, as we just need to consider finite α's, by Serre's vanishing we can take $u \gg 0$ such that the desired vanishings (2.6) hold. Summarizing, we have established that $I_Z^{k_u+1}\left\langle \frac{b(k_u+1)}{a}\ell\right\rangle$ is GV for for $u \gg 0$ and thus

$$\sup_k \frac{k+1}{\epsilon_k(Z,\ell)} \geq \frac{k_u+1}{\epsilon_{k_u}(Z,\ell)} \geq \frac{a}{b} = t.$$

As t can be arbitrarily close to $\varepsilon(I_Z, L)$ we conclude that

$$\sup_k \frac{k+1}{\epsilon_k(Z,\ell)} \geq \varepsilon(I_Z, L).$$

To prove the opposite inequality, we first note that by [Laz04a, Proposition 5.4.5] we have that

$$(k+1)\varepsilon(I_Z, L)^{-1} \leq d_L(I_Z^{k+1}) := \min\{d \in \mathbb{Z}_{\geq 0} : I_Z^{k+1} \otimes L^{\otimes d} \text{ is globally generated}\},$$

while Mumford's theorem ([Laz04a, Theorem 1.8.5]) says that

$$d_L(I_Z^{k+1}) \leq \operatorname{reg}_L(I_Z^{k+1}) := \min\{m \in \mathbb{Z}_{\geq 0} : I_Z^{k+1} \text{ is } m\text{-regular with respect to } L\},$$

where being m-regular with respect to L means that $H^j(I_Z^{k+1} \otimes (m-j)L) = 0$ for $j > 0$. Now, by definition, we have that

$$H^j(I_Z^{k+1} \otimes (1 + \lceil \epsilon_k(Z,L) \rceil)L) = 0 \text{ for all } j > 0,$$

and hence

$$\operatorname{reg}_L(I_Z^{k+1}) \leq 1 + g + \lceil \epsilon_k(Z,L) \rceil \leq 2 + g + \epsilon_k(Z,L).$$

It follows that

$$\varepsilon(I_Z, L)^{-1} \leq \frac{2 + g + \epsilon_k(Z,L)}{k+1} \quad \text{for all } k$$

and therefore, passing to the limit we get

$$\varepsilon(I_Z, L) \geq \limsup \frac{k+1}{\epsilon_k(Z,L)},$$

as we wanted to see. If $\dim Z \leq 1$, Theorem 2.9 above implies that we have equalities. □

2.3 Principal parts bundles and Gauss-Wahl maps

In this section we recall the definition of the Gauss-Wahl maps and in particular we introduce some thresholds that may be thought as the surjectivity thresholds of such maps. Then we relate these thresholds with the jets-separation thresholds.

To start, consider a smooth projective variety X. We write $p_1, p_2 : X \times X \to X$ for the projections, $\Delta \subset X \times X$ for the diagonal and $I_\Delta \subset \mathcal{O}_{X \times X}$ for the corresponding ideal sheaf. For $n \in \mathbb{Z}_{\geq 0}$ we write $n\Delta$ for the closed subscheme of $X \times X$ defined by the ideal I_Δ^n, (we consider $I_\Delta^0 = \mathcal{O}_{X \times X}$) thus

$$\mathcal{O}_{n\Delta} = \mathcal{O}_{X \times X} / I_\Delta^n.$$

Definition 2.15. Let L be a line bundle on X. Given $n \in \mathbb{Z}_{\geq -1}$, the sheaf of n-principal parts of L over X (or the sheaf of n-jets of L over X) is the sheaf

$$P^n(L) := p_{1*}\left(p_2^* L \otimes \mathcal{O}_{(n+1)\Delta}\right).$$

We also consider the sheaf of n-jets-relations

$$R_L^{(n)} = p_{1*}\left(p_2^* L \otimes I_\Delta^{n+1}\right).$$

From the definition it follows that $P^{-1}(L) = 0$ and $P^0(L) = L$. On the other hand, when L is very ample, we can identify $R_L^{(1)}$ with the conormal sheaf associated to the embedding of X in $\mathbb{P}H^0(L)$.

Applying $p_{1*}(p_2^* L \otimes -)$ to the exact sequence $0 \to I_\Delta^{n+1} \to \mathcal{O}_{X\times X} \to \mathcal{O}_{(n+1)\Delta} \to 0$ we get an exact sequence

$$0 \longrightarrow R_L^{(n)} \longrightarrow H^0(L) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} P^n(L), \tag{2.8}$$

where the fiber over x of the last arrow is given by the restriction

$$H^0(L) \to H^0\left(L \otimes \mathcal{O}_X / I_x^{n+1}\right).$$

In particular, the right arrow is surjective if and only if L generates n-jets at every point of X. In this way we also see that $R_L^{(-1)} = H^0(L) \otimes \mathcal{O}_X$ and $R_L^{(0)}$ is the kernel of the evaluation map (which in [JP20] is denoted by M_L). On the other hand, applying $p_{1*}(p_2^* L \otimes -)$ to the exact sequence

$$0 \to I_\Delta^{n+1} \to I_\Delta^n \to I_\Delta^n / I_\Delta^{n+1} \to 0$$

we get an exact sequence

$$0 \longrightarrow R_L^{(n)} \longrightarrow R_L^{(n-1)} \xrightarrow{u} L \otimes \operatorname{Sym}^n \Omega_X, \tag{2.9}$$

fitting in the following commutative and exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & R_L^{(n)} & = & R_L^{(n)} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & R_L^{(n-1)} & \longrightarrow & H^0(L) \otimes \mathcal{O}_X & \xrightarrow{\text{ev}_{n-1}} & P^{n-1}(L) & & \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \text{ev}_n} & & \| & & \\
0 & \longrightarrow & L \otimes \operatorname{Sym}^n \Omega_X & \longrightarrow & P^n(L) & \longrightarrow & P^{n-1}(L) & \longrightarrow & 0
\end{array} \tag{2.10}$$

where the surjectivity at the bottom follows from the fact that the sheaf $I_\Delta^n / I_\Delta^{n+1}$ is supported on the diagonal (for example, see [GR20, Proposition 1.3] for details). In particular, from the 5-lemma it follows that u is surjective as soon as the evaluation map ev_n is surjective, that is, as soon as L generates n-jets at every point of X.

Definition 2.16. Given L, M line bundles on a smooth projective variety X and $n \in \mathbb{Z}_{\geq 0}$.

a) The n-th Gauss-Wahl map associated to L, M is the map

$$\gamma_{L,M}^n = H^0(u \otimes M) : H^0(R_L^{(n-1)} \otimes M) \to H^0(L \otimes M \otimes \operatorname{Sym}^n \Omega_X),$$

where u is the morphism in (2.9).

b) The n-th multiplication map associated to L, M is the map

$$m_{L,M}^n = H^0(\mathrm{ev}_n \otimes M) : H^0(L) \otimes H^0(M) \to H^0(P^n(L) \otimes M).$$

From the diagram (2.10) we see that the surjectivity of $m_{L,M}^n$ implies the surjectivity of $\gamma_{L,M}^n$. Moreover, from (2.9) and the remark below it, we see that if L generate n-jets at every point then the surjectivity of $m_{L,M}^n$ is equivalent to the vanishing of $H^1(R_L^{(n)} \otimes M)$ and hence such vanishing implies the surjectivity of $\gamma_{L,M}^n$. In the case that $M = L^k$ and X is an abelian variety, this suggests to study the cohomological rank function $h^1(R_L \langle -\ell \rangle)$, where ℓ is the class of L in $\mathrm{NS}(X)$.

Lemma 2.17. *Let L be an ample line bundle on an abelian variety of dimension $g \geq 2$. Then:*

a) For $t > -1$ $P^n(L)\langle t\ell \rangle$ is IT(0) and $h^0(P^n(L)\langle t\ell\rangle) = (1+t)^g h^0(L)\binom{n+g}{g}$

b) $h^1(R_L^{(n)}\langle u\ell\rangle) = 0$ for $u \leq -1$.

c) If L separate n-jets at every point then $h^1(R_L^{(n)}\langle v\ell\rangle) = (1+v)^g h^0(L)\binom{n+g}{g}$ for $v \in (-1, 0)$.

d) If L separates n-jets at every point then we have $h^j(R_L^{(n)}\langle s\ell\rangle) = 0$ for every $s > 0$ and $j \geq 2$.

Proof: a) Write $t = a/b$ with $0 > a > -b$. Without loss of generality we may assume that L is symmetric and hence $b_A^* L \simeq L^{b^2}$. Applying $b_A^*(-) \otimes L^{ab} \otimes P_\alpha$ to the exact sequences

$$0 \to L \otimes \mathrm{Sym}^n \Omega_A \to P^n(L) \to P^{n-1}(L) \to 0 \tag{2.11}$$

we get that for every $j \geq 1$ we have

$$H^j(b_A^* P^n(L) \otimes L^{ab} \otimes P_\alpha) \simeq H^j(b_A^* P^{n-1}(L) \otimes L^{ab} \otimes P_\alpha)$$

and thus, inductively, we see that

$$H^j(b_A^* P^n(L) \otimes L^{ab} \otimes P_\alpha) \simeq H^j(b_A^* P^0(L) \otimes L^{ab} \otimes P_\alpha) = H^j(L^{b(a+b)} \otimes P_\alpha) = 0,$$

as we wanted to prove. It also follows, inductively, that

$$h^0(b_A^* P^n(L) \otimes L^{ab} \otimes P_\alpha) = h^0(L_\alpha^{b(a+b)}) \sum_{k=0}^{n} \binom{g+k-1}{g-1} = b^g(b+a)^g h^0(L)\binom{n+g}{g}.$$

b) Write $u = -r/q$ with $r, q > 0$ and $L_\alpha^{-rq} = L^{-rq} \otimes P_\alpha$. Consider the exact sequence

$$0 \to R_L^{(n)} \to H^0(L) \otimes \mathcal{O}_A \to \mathcal{Q} \to 0,$$

where

$$\mathcal{Q} = \mathrm{Im}\left[H^0(L) \otimes \mathcal{O}_A \to P^n(L)\right] \subset P^n(L).$$

As $H^0(L_\alpha^{-rq}) = H^1(L_\alpha^{-rq}) = 0$ (because $g \geq 2$), applying $q_A^*(-) \otimes L_\alpha^{-rq}$ to the above exact sequence and taking cohomology we get

$$H^1(q_A^* R_L^{(n)} \otimes L_\alpha^{-rq}) \simeq H^0(q_A^* \mathcal{Q} \otimes L_\alpha^{-rq}) \subset H^0(q_A^* P^n(L) \otimes L_\alpha^{-rq}), \tag{2.12}$$

with equality whenever L separates n-jets at every point. In particular, to prove the first part of b) it is enough to show that the group at the right vanishes when $r > q$. Now, again by (2.11), inductively we get that

$$H^0(q_A^* P^n(L) \otimes L_\alpha^{-rq}) = H^0(q_A^* P^0(L) \otimes L_\alpha^{-rq}) = H^0(L^{q(q-r)}) = 0$$

as we wanted to see.

c) In the case that L separates n-jets, in (2.12) there is equality and thus the result follows from item a).

d) Write $s = c/d$ with c, d positive integers. As L separates n-jets at every point we get that

$$H^j(d_A^* R_L^{(n)} \otimes L_\alpha^{cd}) \simeq H^{j-1}(d_A^* P^n(L) \otimes L_\alpha^{cd}) \text{ for every } j \geq 2.$$

The result then follows from part a).

□

In the spirit of [JP20, §8] we introduce the following numbers:

Definition 2.18. Let L be an ample line bundle on an abelian variety A. For $p \in \mathbb{Z}_{\geq 0}$ we write

$$\mu_p(L) = \inf\left\{t \in (-1, \infty) \cap \mathbb{Q} : h^1(R_L^{(p)} \langle t\ell \rangle) = 0\right\},$$

where ℓ is the class of L in $\mathrm{NS}(A)$.

Note that, when L separates p-jets at every point, the above lemma tell us that $\mu_p(L)$ is the vanishing threshold $\nu_\ell(R_L^{(p)})$ (see Definition 2.1). Note also that the invariant $\mu_0(L)$ is already considered in [JP20], where it is denoted by $s(L)$. In the cited reference it is shown that there is a relation between such number and the base-point freeness threshold, when L is globally generated. In the following we compute $\Phi_{\mathcal{P}}(I_0^{p+1} \otimes L)$ to show that there is an analagous relation between μ_p and ϵ_p.

Before stating our result, we need to introduce further notation. Let L be an ample line bundle on an abelian variety A and $\alpha \in \hat{A}$. For $a, b \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$ we write

$$m_{L,\alpha}^n(a,b) = H^0(b^* \mathrm{ev}_n \otimes L^{\otimes ab} \otimes P_\alpha) : H^0(L) \otimes H^0(L^{ab} \otimes P_\alpha) \to H^0(b^* P^n(L) \otimes L^{\otimes ab} \otimes P_\alpha),$$

and for $t = a/b \in \mathbb{Q}$ we write

$$\operatorname{gcorank} m_L^n(t) = \frac{1}{b^{2g}} \cdot \min_{\alpha \in \hat{A}} \operatorname{corank} m_{L,\alpha}^n(a,b).$$

Here two observations are in place:

- The number gcorank $m_L^n(t)$ does not depend on the representation $t = a/b$. To see this, note that

$$\begin{aligned} \operatorname{gcorank} m_L^n(t) &= h^0\left(P^n(L)\langle t\ell\rangle\right) - \frac{h^0(L)\cdot h^0(L^{ab})}{b^{2g}} + h^0\left(R_L^{(n)}\langle t\ell\rangle\right) \\ &= h^0\left(P^n(L)\langle t\ell\rangle\right) - t^g h^0(L)^2 + h^0\left(R_L^{(n)}\langle t\ell\rangle\right) \end{aligned}$$

  and the right hand side does not depend on the representation.

- When L separates n-jets at every point, gcorank $m_L^n(t)$ is no other than $h^1(R_L^{(n)}\langle t\ell\rangle)$. In the proof of Theorem 2.19 below we will see that, in general, gcorank $m_L^n(t)$ can be seen as the cohomological rank function of an appropiate *object*.

Theorem 2.19. *Let A be an abelian variety. Let L be an ample line bundle on A with class $\ell \in \mathrm{NS}(A)$. Then for every $y \in (0,1)$ the following equality holds:*

$$h^1\left(I_0^{p+1}\langle y\ell\rangle\right) = \frac{(1-y)^g}{h^0(L)}\cdot \operatorname{gcorank} m_L^n\left(\frac{y}{1-y}\right).$$

In particular, if L separates p-jets at every point, the following equality holds:

$$\mu_p(L) = \frac{\epsilon_p(\ell)}{1-\epsilon_p(\ell)}.$$

and if L and L' are numerically equivalent then $\mu_p(L) = \mu_p(L')$.

The main tool to prove this is the following lemma:

Lemma 2.20. *Let A be an abelian variety and L an ample line bundle over A. We write 0^n for the closed subscheme of A defined by the ideal I_0^n. Then we have*

$$\varphi_L^*\Phi_{\mathcal{P}}\left[\ L \xrightarrow{\text{res}} L\otimes\mathcal{O}_{0^n}\ \right] \simeq \left[\ H^0(L)\otimes\mathcal{O}_A \xrightarrow{\text{ev}} P^{n-1}(L)\ \right]\otimes L^\vee.$$

Proof: Let $m : A\times A \to A$ be the multiplication map and $p_1, p_2 : A\times A\to A$ be the projections.

By (1.10) we have that

$$\varphi_L^*\Phi_{\mathcal{P}}\left[L\otimes\left(\ \mathcal{O}_A \xrightarrow{res} \mathcal{O}_{0^n}\ \right)\right]\otimes L \simeq m_*\left[p_1^*\left(\ \mathcal{O}_A \xrightarrow{res} \mathcal{O}_{0^n}\ \right)\otimes p_2^*L\right].$$

Now, let $\mu : A\times A\to A\times A$ be the map given by $\mu(x,y) = (m(x,y),y)$, whose inverse is given by $\nu(x,y) = (\delta(x,y),y)$, where $\delta : A\times A\to A$ is the difference map. Then we have:

$$\begin{aligned} m_*\left[p_1^*\left(\ \mathcal{O}_A \xrightarrow{res} \mathcal{O}_{0^n}\ \right)\otimes p_2^*L\right] &\simeq p_{1*}\mu_*\left[p_1^*\left(\ \mathcal{O}_A \xrightarrow{res} \mathcal{O}_{0^n}\ \right)\otimes p_2^*L\right] \\ &\simeq p_{1*}\left[(p_1\circ\nu)^*\left(\ \mathcal{O}_A \xrightarrow{res} \mathcal{O}_{0^n}\ \right)\otimes (p_2\circ\nu)^*L\right] \\ &\simeq p_{1*}\left[\delta^*\left(\ \mathcal{O}_A \xrightarrow{res} \mathcal{O}_{0^n}\ \right)\otimes p_2^*L\right] \\ &\simeq\ H^0(L)\otimes\mathcal{O}_A \xrightarrow{ev} P^{n-1}(L)\ , \end{aligned}$$

where in the last line we used the fact that $\delta^*\mathcal{O}_{0^n} = \mathcal{O}_{n\Delta}$ and the definition of the natural map in (2.8).

□

Proof: We have that $I_0^{p+1} \otimes L$ is isomorphic in $D^b(A)$ to the complex $L \to L \otimes \mathcal{O}_{0^{p+1}}$ (concentrated in degrees 0 and 1). By Lemma 2.20 above, this implies that in $D^b(A)$ we have the following isomorphism

$$\varphi_L^* \Phi_{\mathcal{P}}(I_0^{p+1} \otimes L) \simeq \mathrm{EV}_p^\bullet \otimes L^\vee,$$

where $\mathrm{EV}_p^\bullet$ is the complex (concentrated in degrees 0 and 1) given by

$$H^0(L) \otimes \mathcal{O}_A \xrightarrow{\mathrm{ev}} P^p(L) .$$

By the transormation formula (Proposition 1.24) it follows that for $s \in \mathbb{Q}^-$ we have

$$h^1\left((I_0^{p+1} \otimes L)\left<s\ell\right>\right) = \frac{(-s)^g}{h^0(L)} \cdot h^1\left((\mathrm{EV}_p^\bullet \otimes L^\vee)\left<-\frac{1}{s}\ell\right>\right).$$

Now, by Lemma 2.17 above, we have that both $(P^p(L) \otimes L^\vee)\left<-\frac{1}{s}\ell\right>$ and $L^\vee\left<-\frac{1}{s}\ell\right>$ are IT(0) for $s \in (-1, 0)$. In this context, writing $s = -a/b$ with $b > a > 0$, the spectral sequence ([**?**, Remark 2.67])

$$E_1^{r,q} = H^q(a_A^*(\mathrm{EV}_p^r \otimes L^\vee) \otimes L^{ab} \otimes P_\alpha) \implies H^{r+q}(a_A^*(\mathrm{EV}_p^\bullet \otimes L^\vee) \otimes L^{ab} \otimes P_\alpha)$$

says that

$$H^1(a_A^*(\mathrm{EV}_p^\bullet \otimes L^\vee) \otimes L^{ab} \otimes P_\alpha) \simeq \operatorname{Coker} m_{L,\alpha}^p(b-a, a)$$

and thus

$$h^1\left((\mathrm{EV}_p^\bullet \otimes L^\vee)\left<-\frac{1}{s}\ell\right>\right) = \operatorname{gcorank} m_L^p\left(-\left(1 + \frac{1}{s}\right)\right).$$

Finally, as $h^1((I_0^{p+1} \otimes L)\left<s\ell\right>) = h^1(I_0^{p+1}\left<(1+s)\ell\right>)$, setting $y = 1+s$, the result follows. □

An important consequence of Theorem 2.19 is that we can ensure the surjectivity of certain Gauss-Wahl maps when we know that ϵ_p is small. We discuss this application in detail in the next section. For the moment, we limit ourselves to discuss formal consequences of the above theorem and its relation with the Seshadri constant.

First, we note that, unlike the jets-separation thresholds, a priori it is not clear how $\mu_p(L)$ changes when we replace L by a multiple of it. However, using the previous theorem, we are able to give such a formula:

Corollary 2.21. *Let L be an ample line bundle and suppose that L separates p-jets at every point. Then for every $n \in \mathbb{Z}_{\geq 1}$ we have that*

$$\mu_p(nL) = \frac{\mu_p(L)}{n + (n-1)\mu_p(L)}.$$

Proof: As L separates p-jets, it follows that also nL separates p-jets for every positive n and hence, applying the previous theorem we get that the following equalities hold:

$$\begin{aligned}\mu_p(nL) &= \frac{\epsilon_p(n\ell)}{1-\epsilon_p(n\ell)} = \frac{n^{-1}\epsilon_p(\ell)}{1-n^{-1}\epsilon_p(\ell)} \\ &= \frac{\epsilon_p(\ell)}{n-\epsilon_p(\ell)} = \frac{\frac{\mu_p(L)}{1+\mu_p(L)}}{n-\frac{\mu_p(L)}{1+\mu_p(L)}} \\ &= \frac{\mu_p(L)}{n+(n-1)\mu_p(L)}.\end{aligned}$$

□

Combining with the results from previous section we get an expression of the Seshadri constant in terms of the thresholds μ_p.

Corollary 2.22. *Let L be an ample line bundle on an abelian variety, then*

$$\varepsilon(L) = \varepsilon(I_x, L) = 1 + \lim_{p\to\infty} \frac{1}{\mu_p((p+2)L)},$$

where μ_p is the threshold given in Definition 2.18.

Proof: From Remark 2.13 we see that if L is an ample line bundle then $L^{\otimes(p+2)}$ generates p-jets at every point. Therefore, by Theorem 2.19 we have

$$\mu_p\left((p+2)L\right) = \frac{\epsilon_p\left((p+2)\ell\right)}{1-\epsilon_p\left((p+2)\ell\right)}.$$

On the other hand, by definition we know that $\epsilon_p((p+2)\ell) = (p+2)^{-1}\epsilon_p(\ell)$. Therefore we have

$$\mu_p((p+2)L) = \frac{\epsilon_p(\ell)}{2+p-\epsilon_p(\ell)} = \frac{\frac{\epsilon_p(\ell)}{p}}{\frac{2+p}{p}-\frac{\epsilon_p(\ell)}{p}}.$$

In particular the limit $\lim_{p\to\infty}\mu_p((p+2)L)$ exist as an element of $\mathbb{R}\cup\{\infty\}$ and by Proposition 2.14 it follows that

$$\lim_{p\to\infty}\mu_p((p+2)L) = \frac{\frac{1}{\varepsilon(\ell)}}{1-\frac{1}{\varepsilon(\ell)}} = \frac{1}{\varepsilon(\ell)-1}$$

or, in other words, we have

$$\varepsilon(\ell) = 1 + \lim_{p\to\infty}\frac{1}{\mu_p((p+2)L)},$$

as we wanted to see.

□

Using the results of this section, we get the following:

Corollary 2.23. *Let A be an abelian variety of dimension g and a polarization $\ell \in \mathrm{NS}(A)$. Then the following are equivalent:*

a) $\varepsilon(\ell) = 1$

b) $\epsilon_p(\ell) = p + 1$ *for every* $p \in \mathbb{Z}_{\geq 0}$

c) For L a line bundle representing ℓ the sequence $\{\mu_p((p+2)L)\}_{p \in \mathbb{Z}_{\geq 0}}$ *is unbounded*

d) There exists a principally polarized elliptic curve (E, θ) and a $g-1$-dimensional polarized abelian variety $(B, \underline{m})$ such that

$$(A, \ell) \simeq (E, \theta) \boxtimes (B, \underline{m}) \tag{2.13}$$

Proof: By 2.14 and 2.12

$$\varepsilon(\ell) = \sup_p \frac{p+1}{\epsilon_p(\ell)} \;\geq\; \frac{p+1}{\epsilon_p(\ell)} \;\geq 1.$$

The equivalence a) $\iff$ b) is then clear.

From Corollary 2.22 above, the equivalence a) $\iff$ c) is immediate. Finally, the equivalence a) $\iff$ d) is the content of Nakamaye's theorem [Nak96, Theorem 1.1]. □

2.4 Effective surjectivity of higher Gauss-Wahl maps

In this section we establish Theorem 0.3 from the introduction.

Theorem 2.24. *Let L and M be ample and algebraically equivalent line bundles on an abelian variety A. Let $c, d \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{\geq 0}$. Then the Gauss-Wahl map*

$$\gamma^p_{cL,dM} : H^0(R^{(p-1)}_{cL} \otimes M^{\otimes d}) \to H^0(L^{\otimes c} \otimes M^{\otimes d} \otimes \mathrm{Sym}^p\, \Omega_A) \tag{2.14}$$

is surjective whenever

$$\epsilon_p(\ell) < \frac{cd}{c+d}.$$

In particular, if L separates p-jets at every point then $\gamma^p_{2L,2M}$ is surjective.

Proof: The surjectivity of (2.14) follows as soon $\mu_p(cL) < d/c$. Indeed: this inequality means that the $\mathbb{Q}$-twisted sheaf

$$R^{(p)}_{cL}\left\langle c^{-1}d \cdot c\ell \right\rangle = R^{(p)}_{cL}\left\langle d\ell \right\rangle$$

is IT(0), where ℓ is the class of L and M in $\mathrm{NS}(A)$. This means that $H^1\left(R^{(p)}_{cL} \otimes M^{\otimes d}\right) = 0$ and thus that $\gamma^p_{cL,dM}$ is surjective.

Now, we note that by hypothesis we have

$$\epsilon_p(\ell) < \frac{dc}{c+d} < c,$$

which means that $\epsilon_p(c\ell) < 1$, that is, cL separates p-jets at every point of A. From Theorem (2.19) it follows that

$$\mu_p(cL) = \frac{\epsilon_p(cL)}{1-\epsilon_p(cL)} = \frac{\epsilon_p(L)}{c-\epsilon_p(L)}.$$

Now, as the function $[0,c) \to \mathbb{R} : t \mapsto \frac{t}{c-t}$ is strictly increasing, we have

$$\mu_p(cL) < \frac{\frac{dc}{c+d}}{c - \frac{dc}{c+d}} = \frac{d}{c},$$

and hence the result follows.

Regarding the last part, we note that hypothesis means that $\epsilon_p(\ell) < 1$. On the other hand, if $c, d \geq 2$ then $cd/(c+d) \geq 1$ and thus the result follows.

□

Finally, we use multiplier ideals in order to bound the jets-separation thresholds $\epsilon_p(\ell)$ in terms of the Seshadri constant of ℓ. To do this, we will need the following:

Lemma 2.25. *Let A be an abelian variety and L an ample line bundle on it. Fix a rational number $t_0 = u_0/v_0$ with $t_0 < \varepsilon(L)$. Then for every positive integer $m \gg 0$ there exists a divisor D_0 such that*

i) $D_0 \in |mv_0 L|$

ii) $\mathrm{mult}_0 D_0 = mu_0$

iii) $\mu : \mathrm{Bl}_0 A \to A$ *is a log-resolution of* D_0 *and the strict transform* $\mu_*^{-1} D_0$ *is reduced*

Proof: By definition of the Seshadri constant we have that $v_0\mu^*L - u_0 E$ is ample, where E is the exceptional divisor. It follows that $mv_0\mu^*L - mu_0E$ is globally generated for $m \gg 0$. By Bertini's theorem, a general member D of the linear system $|mv_0\mu^*L - mu_0E|$ is smooth and thus it is enough to put $D_0 = \mu_*(D)$.

□

Now, if we fix t_0 and $m \gg 0$ as in the previous lemma then considering $\Delta = \frac{g+p}{mu_0} D_0$ and $r \in \mathbb{Q}$ with $r > \frac{g+p}{t_0}$, we obtain:

$$\begin{aligned}\mu^*\Delta &= \frac{g+p}{mu_0}\mu_*^{-1}D_0 + \left(\frac{g+p}{mu_0} \cdot \mathrm{mult}_0 D_0\right) E \\ &= \frac{g+p}{mu_0}\mu_*^{-1}D_0 + (g+p)E.\end{aligned}$$

Now, as μ is a log-resolution of D_0 we have that $\mu_*^{-1}D_0 + E$ is snc and hence, as $m \gg 0$, it follows that

$$\lfloor \mu^*\Delta \rfloor = (g+p)E.$$

In this way we can compute the multiplier ideal

$$\mathcal{J}(A,\Delta) = \mu_*\mathcal{O}_{\mathrm{Bl}_0 X}\left((g-1)E - (g+p)E\right) = \mu_*\mathcal{O}_{\mathrm{Bl}_0 X}(-(p+1)E) = I_0^{p+1}.$$

On the other hand:

$$rL - \Delta \equiv \left(r - \frac{g+p}{t_0}\right) L,$$

is ample because of the way we chose r. Therefore, the following inequality directly follows by letting $t_0 \to \varepsilon(L)$ and Nadel's vanishing ([Laz04b, Theorem 9.4.8]):

Proposition 2.26. *For a polarization ℓ on an abelian variety the following inequality holds*

$$\epsilon_p(\ell) \ \leq \ r_p(L) \ \leq \ \frac{g+p}{\varepsilon(L)},$$

where

$$r_p(L) := \inf \left\{ r \in \mathbb{Q} \,\middle|\, \begin{array}{c} \exists \text{ an effective } \mathbb{Q}\text{-divisor } \Delta \text{ such that } \mathcal{J}(X,\Delta) = I_0^{p+1} \\ \text{and } rL - \Delta \text{ is ample} \end{array} \right\},$$

and L is a line bundle representing ℓ.

Now, from Proposition 2.26 and Theorem 2.24 we are able to prove the following:

Corollary 2.27. *Let L, M be ample and algebraically equivalent line bundles and $p \in \mathbb{Z}_{\geq 0}$. Consider a positive integer c such that $\varepsilon(cL) > g+p$. Then the Gauss-Wahl map $\gamma^p_{cL,dM}$ is surjective as soon as*

$$d \ > \ \frac{c(g+p)}{c\varepsilon(L) - (g+p)}. \tag{2.15}$$

Proof: From Theorem 2.24 and Proposition 2.26, the desired surjectivity follows as soon as

$$\frac{g+p}{\varepsilon(\ell)} < \frac{cd}{d+c},$$

where ℓ is the class of L and M. Now, as $\varepsilon(c\ell) > g+p$ the above inequality becomes (2.15). The inequality of $\mu_p(cL)$ follows as in the proof of Theorem 2.24.

□

To illustrate the usage of this result, let L, M be algebraically equivalent ample line bundles as in the theorem. Consider $f, h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ two functions and suppose that we want to ensure the surjectivity of $\gamma^p_{cL,dM}$ for all $c \geq f(p)$ and $d \geq h(p)$. In order to apply the corollary we would just need

$$\varepsilon(\ell) \ > \ \frac{g+p}{f(p)} \ \text{ and } \ h(p) > \frac{(g+p)f(p)}{f(p)\varepsilon(\ell) - (g+p)}$$

or, equivalently:

$$\varepsilon(\ell) > \frac{(f(p)+h(p))(g+p)}{f(p)h(p)}.$$

As an example, we contrast with [Par95, Theorem 2.2]. In such reference it is proved that $\gamma^p_{cL,dM}$ is surjective for all $c, d \geq 2(p+1)$ with $c+d \geq 4p+5$. Now, if we consider

$f(p) = h(p) = 2p+1$ we get that $\gamma^p_{cL,dM}$ is surjective for all $c, d \geq 2(p+1)$ (no matter $c+d$) whenever

$$\varepsilon(\ell) > \frac{g+p}{p+1/2} \tag{2.16}$$

As $(g+p)/(p+1/2) \to 1$ as $p \to \infty$, from Nakamaye's theorem we get that if (A, ℓ) is not as in (2.13) then (2.16) holds for $p \gg 0$. In this sense, Corollary 2.27 above asymptotically improves the cited result.

Using the above result, we can also find conditions for the surjectivity of Gauss-Wahl maps on (non necessary abelian) subvarieties of abelian varieties. More precisely, we prove the following:

Proposition 2.28. *Let L, M be two ample and algebraically equivalent line bundles on an abelian variety A. Let Y be a smooth (but not necessarily abelian) subvariety of A. Suppose that*

1. $\epsilon_1(\ell) < 1/2$, *where ℓ is the class of L (and M) in* $\mathrm{NS}(A)$
2. $H^1(A, I_Y \otimes L \otimes M) = 0$, *where I_Y is the ideal of Y in A*
3. $H^1(Y, N^\vee_{Y/A} \otimes (L \otimes M)|_Y) = 0$, *where $N_{Y/A}$ is the normal sheaf of Y in A.*

Then the Gauss-Wahl map (on Y)

$$\gamma^1_{L|_Y, M|_Y} : H^0\left(Y, R^{(0)}_{L|_Y} \otimes M|_Y\right) \to H^0\left(Y, (L \otimes M)|_Y \otimes \Omega_Y\right)$$

is surjective.

If in addition we suppose that Y is a divisor and $\epsilon_p(\ell) < 1/2$ then $\gamma^p_{L|_Y, M|_Y}$ is surjective.

Proof: We note that we have the following commutative diagram

$$\begin{array}{ccc}
R^{(n-1)}_L \otimes M & \xrightarrow{p_{1*}(p_2^* L \otimes \pi) \otimes M} & L \otimes M \otimes \mathrm{Sym}^n \Omega_X \\
\big\downarrow & & \big\downarrow \\
 & & L|_Y \otimes M|_Y \otimes \mathrm{Sym}^n\, \Omega_X|_Y \\
 & & \big\downarrow \\
R^{(n-1)}_{L|_Y} \otimes M|_Y & \xrightarrow[q_{1*}(q_2^* L|_Y \otimes \pi') \otimes M|_Y]{} & L|_Y \otimes M|_Y \otimes \mathrm{Sym}^n \Omega_Y
\end{array}$$

where $\pi : I^n_\Delta \to I^n_\Delta / I^{n+1}_\Delta$ and $\pi' : I^n_{\Delta_Y/Y\times Y} \to I^n_{\Delta_Y/Y\times Y}/I^{n+1}_{\Delta_Y/Y\times Y}$ are the projections; while $\Delta_Y \subset Y \times Y \subset X \times X$ is the diagonal of Y and $q_1, q_2 : Y \times Y \to Y$ are the product-projections.

Taking cohomology to the commutative diagram above, the lower horizontal map is no other than the Gauss-Wahl map $\gamma^p_{L|_Y, M|_Y}$ and therefore, we conclude that such map is surjective as soon as the composition:

$$H^0(R_L^{(p-1)} \otimes M) \xrightarrow{\gamma^k_{L,M}} H^0(L \otimes M \otimes \mathrm{Sym}^p\,\Omega_A) \xrightarrow{u} H^0\left((L\otimes M)|_Y \otimes \mathrm{Sym}^p\,\Omega_A|_Y\right) \xrightarrow{v} H^0((L\otimes M)|_Y \otimes \mathrm{Sym}^p\,\Omega_Y)$$

is surjective. For this to hold is enough to ensure the surjectivity of each step. By Theorem 2.24, the condition 1) ensures the surjectivity of the first map. Now, taking cohomology to the exact sequence

$$(L \otimes M \otimes \mathrm{Sym}^p\,\Omega_X) \otimes [0 \to I_Y \to \mathcal{O}_A \to \mathcal{O}_Y \to 0]\,,$$

we see that the surjectivity of u follows from hypothesis 2). To handle the the third map, we note that from the conormal exact sequence we get an exact sequence

$$\bigwedge^2 N^\vee_{Y/A} \otimes \mathrm{Sym}^{p-2}\,\Omega_A|_Y \to N^\vee_{Y/A} \otimes \mathrm{Sym}^{p-1}\,\Omega_A|_Y \to \mathrm{Sym}^p\,\Omega_A|_Y \to \mathrm{Sym}^p\,\Omega_Y \to 0.$$

When Y is a divisor, the left term in the above sequence vanish. The result then follows by taking cohomology since Ω_A is trivial.

□

A particular situation where we can apply the previous result is in the case of curves contained in abelian surfaces. It is well known (see [CFP12, Theorem A]) that if C is a smooth curve contained in an abelian surface then the second Gauss-Wahl $\gamma^2_{\omega_C,\omega_C}$ is never surjective while the maps $\gamma^p_{\omega_C,\omega_C\otimes\eta}$ for $\eta \in \mathrm{Pic}^0(C)$ with $\eta \neq 0$ "tend to be surjective" ([CF13], [BF14] and [CV93]). In this context, the following result gives a quantification of this phenomenon in terms of the jets-separation thresholds (using Corollary 2.27, a similar statement can be done using the Seshadri constant):

Corollary 2.29. *Let C be a smooth curve of genus $g \geq 2$ contained in an abelian surface A. Let $\eta \in \mathrm{Im}\left[\mathrm{Pic}^0(A) \to \mathrm{Pic}^0(C)\right]$ with $\eta \neq 0$. If $\epsilon_p\left(\mathcal{O}_A(C)\right) < 1/2$ then $\gamma^p_{\omega_C,\omega_C\otimes\eta}$ is surjective.*

Proof: Write $L = \mathcal{O}_A(C)$ and $M = L \otimes P_\alpha$ for $\alpha \in \hat{A}$ with $\alpha \neq 0$. In this case $N_{C/A} \simeq \omega_C \simeq L|_C$ and $I_{C/A} \simeq L^\vee$. Now, as $g(C) \geq 2$ it follows that C is an ample divisor [1] and thus

$$H^1(I_{C/A} \otimes L \otimes M) = H^1(L \otimes P_\alpha) = 0,$$

while

$$H^1(N^\vee_{C/A} \otimes (L \otimes M)|_C) = H^1((L\otimes P_\alpha)|_C).$$

Now, taking cohomology to

$$0 \to P_\alpha \to L \otimes P_\alpha \to (L \otimes P_\alpha)|_C \to 0$$

[1] On an abelian variety any effective divisor is nef and any big divisor is ample. It follows that a curve inside an abelian surface is ample if and only if it has positive self-intersection. By Riemann-Roch this is equivalent to $g(C) \geq 2$.

we get that $H^1(C, (L \otimes P_\alpha)|_C) \simeq H^2(A, P_\alpha) = 0$, since we took $\alpha \neq 0$. The result then follows from Proposition 2.28 and the fact that the restriction map $\mathrm{Pic}^0(A) \to \mathrm{Pic}^0(C)$ is injective by [Fuj80, Theorem p.155].

□

2.5 Further questions

We conclude this chapter by stating a couple of questions that might be interesting for future research:

2.5.1 Effective convergence of jets-separation thresholds:

First, we may ask for a more detailed study of the convergence $\varepsilon(\ell) = \lim_{p\to\infty} \frac{p+1}{\epsilon_p(\ell)}$. More precisely, we may ask the following:

Question 2.30. *Given a positive real number u. Does there exist a positive real number $t(u)$ and a positive integer $p(u)$ such that for any g-dimensional principally polarized abelian variety (A, θ) we have*

$$\varepsilon(\theta) > u \iff \epsilon_{p(u)}(\theta) < t(u) \quad ?$$

Here it is worth to point out that, in view of the results of this chapter, an affirmative answer to the above question would also give a characterization of p.p.a.v with small Seshadri constant in terms of the failure of the surjectivity of an specific Gauss-Wahl map. As an example, here we answer the above question in the case $u = 1$:

Proposition 2.31. *Let (A, θ) be a g-dimensional principally polarized abelian variety. Then the following are equivalent:*

a) $(A, \theta) \simeq (E, \theta_E) \boxtimes (B, \theta_B)$, *where (E, θ_E) is a principally polarized elliptic curve and* $\dim B = g - 1$

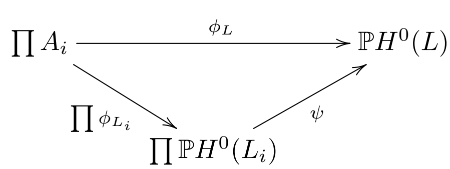
b) The sheaf $I_0^2(2\Theta)$ is not M-regular

c) $\epsilon_2(\theta) = 3$

Proof: Write

$$(A, \theta) \simeq (A_1, \theta_1) \boxtimes \cdots \boxtimes (A_r, \theta_r)$$

for the decomposition of (A, θ) in indecomposable principally polarized abelian varieties ([BL04, Theorem 4.3]). Write L, respectively L_i, for a symmetric line bundle representing the polarization 2θ, respectively $2\theta_i$. By [BL04, p.99] we have the following commutative diagram:



where ψ is the Segre embedding. At the level of tangent spaces we have then that for $a = (a_1, ..., a_r) \in A$, the differential $d_a\phi_L : T_{A,a} \to T_{\mathbb{P}H^0(L),\phi_L(a)}$ is injective if and only if all the differentials $d_{a_i}\phi_{L_i} : T_{A_i,a_i} \to T_{\mathbb{P}H^0(L_i),\phi_{L_i}(a_i)}$ are injective. As (A_i, θ_i) is indecomposable we have that $d_{a_i}\phi_{L_i}$ fails to be injective just in $A_i[2]$. It follows that

$$L \text{ does not separate 1-jets at } a \iff a \in \bigcup A_1 \times \cdots \times A_i[2] \times \cdots A_r$$

and hence

$$\operatorname{codim}_{\hat{A}} V^1(I_0^2(2\Theta)) = \begin{cases} \min_i \dim A_i & \text{if } r \geq 2 \\ g & \text{if } r = 1 \end{cases}.$$

We conclude that that $I_0^2(2\Theta)$ is M-regular if and only if (A, θ) is not as in a).

Now, if $I_0^2(2\Theta)$ is M-regular then by [Ito22b, Proposition 3.1 (ii)] we have that $I_0^3(3\Theta)$ is IT(0) and thus $\epsilon_2(\theta) < 3$. On the other hand, if $\epsilon_2(\theta) < 3$ then by Corollary 2.23 we have that (A, θ) is not as in a). This completes the proof.

□

Corollary 2.32. *Let (A, θ) be a principally polarized abelian variety and let Θ be a line bundle representing it. Consider the second Gauss-Wahl map*

$$\gamma^2_{6\Theta,d\Theta} : H^0\left(R^{(1)}_{6\Theta} \otimes \Theta^6\right) \to H^0(\Theta^{d+6} \otimes S^2\Omega_A).$$

We have that $\gamma^2_{6\Theta,d\Theta}$ is surjective for $d \geq 7$ and if $\gamma^2_{6\Theta,6\Theta}$ fails to be surjective then

$$(A, \theta) \simeq (E, \theta_E) \boxtimes (B, \theta_B), \tag{2.17}$$

where (E, θ_E) is a principally polarized elliptic curve and $\dim B = g - 1$

Proof: For the first point we note that as we always have $\epsilon_2(\theta) \leq 3$. From Theorem 2.24 it follows then that $\gamma^2_{6\Theta,d\Theta}$ is surjective whenever $3 < 6d/(d+6)$, that is, as soon as $d > 6$.

Regarding the second statement, we note that by Theorem 2.19 we have that

$$\mu_2(6\theta) = \frac{\epsilon_2(6\theta)}{1 - \epsilon_2(6\theta)} = \frac{\epsilon_2(\theta)}{6 - \epsilon_2(\theta)}.$$

If (A, θ) is not as in (2.17), Proposition 2.31 above implies that $\epsilon_2(\theta) < 3$ and hence

$$\mu_2(6\theta) < \frac{3}{6-3} = 1$$

which means that $\gamma^2_{6\Theta,6\Theta}$ is surjective.

□

2.5.2 Seshadri constant in positive characteristic:

The theory of the Fourier-Mukai functors, generic vanishing and cohomological rank functions work also over algebraically closed fields of positive characteristic. In this context, most of the results of this work also hold in that setting. However, in positive characteristic Nadel's vanishing theorem does not hold and hence we are not able to prove Proposition 2.26. We may then ask the following:

Question 2.33. *Let (A, ℓ) be a polarized abelian variety over a field of positive characteristic. Does the inequality $\epsilon_p(\ell) \le (g+p)\varepsilon(\ell)^{-1}$ hold?*

On the other hand, in [MS14] and in [Mur18], there are introduced the *Frobenius-Seshadri constants* $\varepsilon_F^k(L, x)$ of a line bundle L at a point x of a smooth variety X over an algebraically closed field of positive characteristic and it is shown that the following inequalities hold:

$$\frac{k+1}{\varepsilon(L,x)} \le \frac{k+1}{\varepsilon_F^k(L,x)} \le \frac{k+\dim X}{\varepsilon(L,x)},$$

where $\varepsilon(L, x)$ is the usual Seshadri constant at x. This, together with our Theorem 0.1 3), suggests to study the following:

Question 2.34. *Let L be an ample line bundle on an abelian variety defined over an algebraically closed field of positive characteristic. Compare the number $(k+1)/\varepsilon_F^k(L, x)$ and the k-jets separation threshold $\epsilon_k(\ell)$, where ℓ is the class of L.*

For instance, if we are able to prove that

$$\epsilon_k(\ell) \le \frac{k+1}{\varepsilon_F^k(L,x)}, \tag{2.18}$$

then Question 2.33 would have positive answer.

Chapter 3

Semi-homogeneous vector bundles and projective normality

3.1 Simple and semi-homogeneous vector bundles

Definition 3.1. A vector bundle E on an abelian variety A is said to be semi-homogeneous if for every $x \in A$ there exists $\alpha \in \hat{A}$ (wich depends on x) such that

$$t_x^* E \;\simeq\; E \otimes P_\alpha.$$

Definition 3.2. A vector bundle E on a k-variety X is said to be simple if its only endomorphisms are scalar multiples of the identity. In symbols:

$$\mathrm{Hom}_{\mathcal{O}_X}(E, E) = k \cdot \mathrm{id}.$$

In Mukai's fundamental paper [Muk78], the following result is proved:

Theorem 3.3 (Theorem 7.11 in [Muk78])**.** *Let ℓ be a polarization on an abelian variety A and $\lambda \in \mathbb{Q}$. Then there exists a simple and semi-homogeneous vector bundle E such that*

$$\frac{[\det E]}{\mathrm{rk}\, E} = \lambda\ell \in \mathrm{NS}(A) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{3.1}$$

Moreover, this vector bundle is unique up to tensorization by P_α for $\alpha \in \hat{A}$.

In what follows we write $E_{A,\lambda\ell}$ for a simple and semi-homogeneous vector bundle on an abelian variety A which satisfies (3.1). The full subcategory of $\mathrm{Coh}(A)$ whose objects are the (not necessarily simple) semi-homogeneous vector bundles satisfying (3.1) will be denoted by $\mathbb{S}_{A,\lambda\ell}$. Sometimes we will omit A from the notation.

Here we review some of their main properties of $E_{\lambda\ell}$ and the category $\mathbb{S}_{A,\lambda\ell}$:

(a) Write $\lambda = \frac{a}{b}$ with $b > 0$ and denote $A[b]$ the group of n-torsion points in A. Let

$$u^2_{A,l}(a,b) := ord(A[b] \cap K(al)).$$

Then, by [Muk78, Theorem 7.11(5)], we have that

$$r_{A,l}(\lambda) := \operatorname{rk} E_{A,\lambda l} = \frac{b^g}{u_{A,l}(a,b)} \quad , \qquad \chi(E_{A,\lambda l}) = \frac{\chi(al)}{u_{A,l}(a,b)} = \frac{a^g \chi(l)}{u_{A,l}(a,b)} \tag{3.2}$$

(b) ([Muk78, Proposition 7.3]) There exists an isogeny $\pi : B \to A$ and a line bundle M on B such that

$$\pi^* E_{A,\lambda l} \simeq M^{\oplus r}.$$

Moreover, writing $\lambda = a/b$, by Proposition 7.6(2) and Theorem 7.11(3) in [Muk78], all the isogenies with this property are the ones factoring through the projection

$$\Phi(E_{\lambda\ell}) := \operatorname{Im}(b_A, \varphi_{a\ell}) \subset A \times \hat{A} \xrightarrow{p} A$$

(c) ([Muk78, Proposition 6.18, Theorem 7.11(2)]) Any $F \in \mathbb{S}_{A,\lambda\ell}$, not necessarily simple, is of the form

$$F \ \simeq \ \bigoplus_i U_i \otimes E_{\lambda\ell} \otimes P_{\alpha_i} \quad \text{for some } \alpha_i \in \hat{A},$$

where U_i is a unipotent vector bundle, i.e. it has a filtration $0 \subset G_1 \subset \cdots \subset G_{k-1} \subset G_k = U_i$ with $G_j/G_{j-1} \simeq \mathcal{O}_A$ (in particular, $U_i \otimes E_{\lambda\ell} \otimes P_{\alpha_i} \in \mathbb{S}_{\lambda\ell}$).

3.2 Cohomological rank functions and semi-homogeneous vector bundles

In this section we formalize the fact, mentioned in the introduction, that we can compute the cohomological rank functions of an object $\mathcal{F}$ with respect to a polarization ℓ using the vector bundles $E_{\lambda\ell}$ introduced above. The main point is the following precisation of point (b) from the previous section.

Proposition 3.4. *Let $\lambda = \frac{a}{b} \in \mathbb{Q}$, let $E_{A,\lambda\ell}$ be a simple bundle in $\mathbb{S}_{A,\lambda\ell}$ and let $r_{A,l}(\lambda) = \operatorname{rk} E_{A,\lambda\ell}$. Then*

$$b_A^* E_{A,\lambda\ell} \simeq (L^{\otimes ab})^{\oplus r_{A,\ell}(\lambda)}, \tag{3.3}$$

for a line bundle L representing ℓ.

Proof: It follows from (b) of the previous section that

$$b_A^* E_{A,\lambda\ell} \simeq M^{\oplus r_{A,\ell}(\lambda)}$$

for some line bundle M on A. Taking determinant we get the following equalities

$$r[M] = b^2[\det W] = abr\ell \in \operatorname{NS}(A)$$

and hence (as $\operatorname{NS}(A)$ is torsion-free) we get that $[M] = ab\ell$, and thus the result follows. $\square$

Remark 3.5. *It is interesting to note that, conversely, given any line bundle L representing ℓ there exists a simple and semi-homogeneous vector bundle $E \in \mathbb{S}_{\lambda\ell}$ with $b_A^* E \simeq (L^{ab})^{\oplus r}$, where, as usual, $\lambda = a/b$. This directly follows from the proposition by twisting with an appropiate element of* $\mathrm{Pic}^0(A)$.

As announced in the introduction, it turns out that the $\mathbb{Q}$-twisted sheaf $\mathcal{O}_A^{\oplus r_{A,l}(\lambda)} \langle \lambda\ell \rangle$ behaves cohomologically as the bundle $E_{A,\lambda l}$, in the following sense:

Proposition 3.6. *Keeping the notation of Proposition 3.4 and of Section 3.1. For all object $\mathcal{F}$ in $D^b(A)$ and for all $\lambda, t \in \mathbb{Q}$ and for all simple bundle $E_{A,\lambda\ell} \in \mathbb{S}_{A,\lambda\ell}$ we have*

$$h^i\left(A, (\mathcal{F} \otimes E_{A,\lambda\ell}) \langle t\ell \rangle\right) = r_{A,\ell}(\lambda) \cdot h^i\left(A, \mathcal{F}\langle (t+\lambda)\ell \rangle\right).$$

Moreover a $\mathbb{Q}$-twisted object $\mathcal{F}\langle \lambda\ell \rangle$ is IT(i) if and only if the object $\mathcal{F} \otimes E_{A,\lambda\ell}$ is so.

Proof: Let $\lambda = \frac{a}{b}$ and $t = \frac{c}{d}$. For general $\alpha \in \hat{A}$ and L a representant of ℓ satisfying (3.3) we have:

$$\begin{aligned}
h^i\left((\mathcal{F} \otimes E_{\lambda\ell}) \langle t\ell \rangle\right) &= \frac{1}{(bd)^{2g}} \cdot h^i\left((b_A d_A)^*(\mathcal{F} \otimes E_{\lambda\ell}) \otimes L^{b^2cd} \otimes P_\alpha\right) \\
&= \frac{1}{(bd)^{2g}} \cdot h^i\left((b_A d_A)^*(\mathcal{F}) \otimes d_A^*(L^{ab})^{\oplus r_{A,\ell}(\lambda)} \otimes L^{b^2cd} \otimes P_\alpha\right) \\
&= \frac{r_{A,\ell}(\lambda)}{(bd)^{2g}} \cdot h^i\left((b_A d_A)^*(\mathcal{F}) \otimes L^{bd(ad+bc)} \otimes P_\alpha\right) \\
&= \frac{r_{A,\ell}(\lambda)}{(bd)^{2g}} \cdot (bd)^{2g} \cdot h^i\left(\mathcal{F}\left\langle \frac{ad+bc}{bd}\ell \right\rangle\right) \\
&= r_{A,\ell}(\lambda) \cdot h^i\left(\mathcal{F}\langle (t+\lambda)\ell \rangle\right),
\end{aligned}$$

as we wanted to see.

□

We now study the behaviour of semi-homogeneous vector bundles under the Fourier-Mukai equivalence. To do this we recall the notion of the *dual* ℓ_δ of a polarization ℓ.

Proposition 3.7 (/Definition; Proposition 14.4.1 in [BL04])**.** *Let A be an abelian variety and $\ell \in \mathrm{NS}(A)$ be a polarization of type $(d_1, ..., d_g)$. Then there exist a unique polarization ℓ_δ on $\widehat{A}$ satisfying the following equivalent conditions:*

a) $\varphi_\ell^* \ell_\delta = d_1 d_g \ell$

b) $\varphi_{\ell_\delta} \circ \varphi_\ell = (d_1 d_g)_A$.

A couple of important properties of ℓ_δ are the following.

- ℓ_δ has type $\left(d_1, \frac{d_1 d_g}{d_{g-1}}, \cdots, \frac{d_1 d_g}{d_2}, d_g\right)$ and thus

$$\chi(\ell) \cdot \chi(\ell_\delta) = (d_1 d_g)^g \tag{3.4}$$

- ([BL04, Proposition 14.4.3]) ℓ_δ is related to the Fourier transform as follows: let L be a representant of ℓ, then in $\mathrm{NS}(A)$ the following equality holds

$$[\det \Phi_{\mathcal{P}}(L)] = -\frac{\chi(\ell)}{d_1 d_g} \cdot \ell_\delta \tag{3.5}$$

We have the following:

Proposition 3.8. *Let ℓ be a polarization of type $(d_1, \ldots, d_g)$ on an abelian variety A of dimension g. Let $E \in \mathbb{S}_{A,\lambda\ell}$. If $\lambda \in \mathbb{Q}_{>0}$ then E is IT(0) and $\widehat{E} := \Phi_{\mathcal{P}}(E) \in \mathbb{S}_{\widehat{A},-\frac{1}{d_1 d_g \lambda}\ell_\delta}$.*

Proof: By item (c) from Section 3.1 above, we know that E is a direct sum of $E_{A\lambda l} \otimes P_{\alpha_i}$-potent vector bundles, where $E_{A\lambda\ell}$ is, as usual, a reference simple vector bundle in $\mathbb{S}_{A,\lambda\ell}$ and thus it is enough to prove the result assuming that E is simple. In this context, the assertion regarding the vanishing conditions directly follows from Proposition 3.4.

Concerning the Fourier-Mukai transform we first note that $\widehat{E}$ is simple, since $\Phi_{\mathcal{P}}$ is an equivalence. We claim that $\widehat{E}$ is also semi-homogeneous. From (1.1) and (1.2) we see that in order to prove the claim it is enough to prove that the projection

$$q : \Phi^0(E) := \{(x, \alpha) \in A \times \widehat{A} : t_x^* E \simeq E \otimes P_\alpha\} \longrightarrow \widehat{A}$$

is surjective. Now, the source of this map is a subgroup of $A \times \widehat{A}$ and q is a group homomorphism whose kernel is

$$\{(x, \hat{0}) \in A \times \widehat{A} : t_x^* E \simeq E\} \hookrightarrow K(\det E),$$

and thus is finite, since $\det E$ is ample (resp. antiample). It follows that

$$\dim \operatorname{Im} q = \dim \Phi^0(E).$$

On the other hand, since E is semi-homogeneous, we have that the first projection $\Phi^0(E) \to A$ is an isogeny and thus $\dim \Phi^0(E) = g$, which means that q is surjective, completing the proof of the claim.

The rest of the statement is proved as follows. In the first place, we note that from (1.6) and Proposition 3.4 we have the following equalities in $\mathrm{NS}(\widehat{A})$:

$$\left[\det\left(b_{\widehat{A}*}\widehat{E}\right)\right] = \left[\det\left(\widehat{b_A^* E}\right)\right] = \left[\det\left(\widehat{L^{ab}}^{\oplus \operatorname{rk}(E)}\right)\right] = -\frac{(ab)^{g-1}\chi(\ell)}{d_1 d_g} \cdot \operatorname{rk}(E) \cdot \ell_\delta, \tag{3.6}$$

where the last equality follows from (3.5). On the other hand we have that

$$b^2 \left[\det(b_{\widehat{A}*}\widehat{E})\right] = \left[b_{\widehat{A}}^* \det(b_{\widehat{A}*} \det \widehat{E})\right] = b^{2g} \left[\det \widehat{E}\right], \tag{3.7}$$

where the second equality comes from Lemma 3.9 below. Combining (3.6) and (3.7) we obtain

$$\frac{[\det \widehat{E}]}{\operatorname{rk}(\widehat{E})} = -\frac{(ab)^{g-1}\chi(\ell)}{b^{2g-2} d_1 d_g} \cdot \frac{\operatorname{rk}(E)}{\operatorname{rk}(\widehat{E})} \cdot \ell_\delta = -\frac{\lambda^{g-1}\chi(\ell)}{d_1 d_g} \cdot \frac{\operatorname{rk}(E)}{\operatorname{rk}(\widehat{E})} \cdot \ell_\delta.$$

Now, by duality one can assume that $\lambda \in \mathbb{Q}_{>0}$. If this is the case the Fourier-Mukai transform exchanges χ with the rank, more precisely, the rank of $\widehat{E}$ is $h^0(E) = \chi(E)$ (by Grauert's theorem). Now, by 3.1(b), $\frac{\chi(E)}{\mathrm{rk}(E)} = \lambda^g \cdot \chi(\ell)$ and hence

$$\frac{[\det \widehat{E}]}{\mathrm{rk}(\widehat{E})} = -\frac{1}{\lambda d_1 d_g} \cdot \ell_\delta,$$

as we wanted to see.

□

Lemma 3.9. *Let $\pi : A \to B$ be an isogeny between abelian varieties. Let E be a vecor bundle on A. Then*

$$\pi^* \pi_* E \simeq \bigoplus_{a \in \ker \pi} t_a^* E.$$

In particular, in $\mathrm{NS}(A)$ *the following equality holds:*

$$\pi^*[\det(\pi_* E)] = \deg(\pi) \cdot [\det E].$$

Proof: Since $\Phi_{\mathcal{P}}$ is an equivalence, it is enough to prove that the Fourier transform of the left side is isomorphic to the Fourier transform of the right hand. Now, by (1.6) we have:

$$\Phi_{\mathcal{P}_A}(\pi^* \pi_* E) \simeq \widehat{\pi}_* \widehat{\pi}^* \Phi_{\mathcal{P}_A}(E).$$

On the other hand, by [Mum08, p.72] we have

$$\widehat{\pi}_* \widehat{\pi}^* \Phi_{\mathcal{P}_A}(E) \simeq \bigoplus_{a \in \ker \pi} \Phi_{\mathcal{P}_A}(E) \otimes P_a \simeq \Phi_{\mathcal{P}_A}\left(\bigoplus_{a \in \ker \pi} t_a^* E \right),$$

where the last equality comes from (1.2).

□

As a consequence we record here the following relation between the cohomological rank functions of an object $\mathcal{F}$ with respect to ℓ and the ones of $\Phi_{\mathcal{P}}(\mathcal{F})$ with respect to ℓ_δ, which turn to be equivalent to the formulas given by Proposition 1.24:

Proposition 3.10. *Let A an abelian variety and let $\mathcal{F} \in D(A)$. For $\lambda \in \mathbb{Q}_{<0}$*

$$h^i(A, \mathcal{F}\langle \lambda \ell \rangle) = (-\lambda)^g \chi(\ell) \cdot h^i\left(\widehat{A}, \Phi_{\mathcal{P}}(\mathcal{F}) \left\langle -\frac{1}{d_1 d_g \lambda} \ell_\delta \right\rangle \right). \tag{3.8}$$

For $\lambda \in \mathbb{Q}_{>0}$

$$h^i(A, \mathcal{F}\langle \lambda \ell \rangle) = \chi(\ell) \lambda^g \cdot h^{g-i}\left(\widehat{A}, \Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) \left\langle \frac{1}{d_1 d_g \lambda} \ell_\delta \right\rangle \right). \tag{3.9}$$

Proof: Let $\lambda \in \mathbb{Q}_{<0}$ and let $E := E_{\widehat{A}, -\frac{1}{d_1 d_g \lambda}\ell_\delta} \in \mathbb{S}_{\widehat{A}, -\frac{1}{d_1 d_g \lambda}\ell_\delta}$ be a simple vector bundle. By the previous proposition E is an IT(0) vector bundle i.e. $\Phi_{\mathcal{P}}(E) = \widehat{E}$ (concentrated in degree zero). By (1.2) and (1.7) we have

$$H^i(A, \mathcal{F} \otimes \widehat{E} \otimes P_\alpha) \simeq H^i(\widehat{A}, \Phi_{\mathcal{P}}(\mathcal{F}) \otimes t_\alpha^* E)$$

By Proposition 3.8 above, we have $\widehat{E} \otimes P_\alpha \in \mathbb{S}_{A,\lambda\ell}$. Hence, by Proposition 3.6 (for $t = 0$), we have that

$$\begin{aligned} r_{\widehat{A},\ell}(\lambda) \cdot h^i(A, \mathcal{F}\langle \lambda\ell \rangle) &= h^i(A, (\mathcal{F} \otimes \widehat{E})\langle 0 \cdot \ell \rangle) \\ &= h^i(\widehat{A}, (\Phi_{\mathcal{P}}(\mathcal{F}) \otimes E)\langle 0 \cdot \ell_\delta \rangle) \\ &= r_{\widehat{A},\ell_\delta}\left(-\frac{1}{d_1 d_g \lambda}\right) \cdot h^i\left(\widehat{A}, \Phi_{\mathcal{P}}(\mathcal{F})\left\langle -\frac{1}{d_1 d_g \lambda}\ell_\delta \right\rangle\right). \end{aligned} \tag{3.10}$$

Now, by the rank-formula in 3.1(a) and (3.4), for $\lambda = -a/b$ we have that

$$r_{\widehat{A},\ell_\delta}\left(-\frac{1}{d_1 d_g \lambda}\right) = \frac{a^g (d_1 d_g)^g}{u_{\widehat{A},\ell_\delta}(b, a d_1 d_g)} = \frac{a^g \chi(\ell)\chi(\ell_\delta)}{u_{\widehat{A},\ell_\delta}(b, a d_1 d_g)} = \frac{a^g \chi(\ell)\chi(E)}{b^g} = (-\lambda)^g \chi(\ell)\chi(E).$$

On the other hand, from the proof of Proposition 3.8 we have that $\chi(E) = \operatorname{rk} \widehat{E} = r_{A,\ell}(\lambda)$ and thus from (3.10) we obtain the equality (3.8).

The proof for $\lambda \in \mathbb{Q}_{>0}$ is similar. Let $E \in \mathbb{S}_{\widehat{A},\frac{1}{d_1 d_g \lambda}\ell_\delta}$ be a simple vector bundle. In the same way, by (1.2) and (1.7) we have

$$H^i(A, \mathcal{F} \otimes \widehat{E} \otimes P_{-\alpha}) \simeq H^i(\widehat{A}, \Phi_{\mathcal{P}}(\mathcal{F}) \otimes t_\alpha^* E) \tag{3.11}$$

By Grothendieck-Serre duality we have that

$$H^i(A, \mathcal{F} \otimes \widehat{E} \otimes P_{-\alpha}) \simeq H^{g-i}(A, \mathcal{F}^\vee \otimes (\widehat{E})^\vee \otimes P_\alpha)^\vee \tag{3.12}$$

The second equality follows from Proposition 3.6 in the same way, because by Proposition 3.8 we have that $(\widehat{E})^\vee \otimes P_\alpha$ is a simple bundle in $\mathbb{S}_{A,\lambda\ell}$.

□

Now, regarding the equivalence of the above formulas to Proposition 1.24, for $\lambda \in \mathbb{Q}_{<0}$ we have

$$\begin{aligned} h^i\left(A, (\varphi_\ell^* \Phi_{\mathcal{P}}(\mathcal{F}))\left\langle \frac{1}{\lambda}\ell \right\rangle\right) &= h^i\left(A, (\varphi_\ell^* \Phi_{\mathcal{P}}(\mathcal{F}))\left\langle \frac{1}{\lambda d_1 d_g} \cdot d_1 d_g \ell \right\rangle\right) \\ &= h^i\left(A, (\varphi_\ell^* \Phi_{\mathcal{P}}(\mathcal{F}))\left\langle \frac{1}{\lambda d_1 d_g} \cdot \varphi_\ell^* \ell_\delta \right\rangle\right) && \text{Definition of } \ell_\delta \\ &= \chi(\ell)^2 \cdot h^i\left(\hat{A}, \Phi_{\mathcal{P}}(\mathcal{F})\left\langle \frac{1}{\lambda d_1 d_g} \cdot \ell_\delta \right\rangle\right) && (1.12) \\ &= \frac{\chi(\ell)}{(-\lambda)^g} \cdot h^i(A, \mathcal{F}\langle \lambda\ell \rangle) && (3.8), \end{aligned}$$

and thus we obtain the claimed transformation formula. The case $\lambda \in \mathbb{Q}_{>0}$ follows similarly.

3.3 Basepoint freeness threshold revisited

In the following two sections we give an interpretation in terms of semi-homogeneous vector bundles of the well known relation between the cohomological rank functions of the ideal of a point and multiplication maps of global sections (see [JP20, §8] and Chapter 2). More precisely, we use the duality given by Proposition 3.10 above to bound the basepoint-freeness threshold (the 0-jets separation threshold from Chapter 2) and study the multiplication maps of global sections. The key point is the following simple technical lemma (recall that in this work, $\mathcal{F}^\vee$ stands for the *derived* dual $R\mathscr{H}om(\mathcal{F}, \mathcal{O}_A)$):

Lemma 3.11. *Let $0 \in A$ (resp. $\hat{0}$) be the origin of the group law of A (resp. $\widehat{A}$) and write I_0 (resp. $I_{\hat{0}}$) for the corresponding ideal sheaves. Then we have:*

$$\Phi_{\mathcal{P}}(I_0^\vee) = I_{\hat{0}}[-(g-1)].$$

Proof: By Rees' theorem ([BH93, Theorem 1.2.5 and Lemma 1.2.4]), dualizing the trivial exact sequence $0 \to I_0 \to \mathcal{O}_A \to k(0) \to 0$ we get that

$$\mathscr{E}xt^i(I_0, \mathcal{O}_A) = \begin{cases} \mathcal{O}_A & \text{for } i = 0 \\ k(0) & \text{for } i = g-1 \\ 0 & \text{otherwise} \end{cases}.$$

Now, it is well known that $\Phi_{\mathcal{P}}(\mathcal{O}_A) = k(\hat{0})[-g]$ ([Mum08, Proof of Theorem in §13]), and it is clear that $\Phi_{\mathcal{P}}(k(0)) = \mathcal{O}_{\widehat{A}}$. Therefore the spectral sequence

$$E_2^{p,q} = R^p\Phi_{\mathcal{P}}(\mathscr{E}xt^q(I_0, \mathcal{O}_A)) \implies R^{p+q}\Phi_{\mathcal{P}}(I_0^\vee)$$

collapses giving the exact sequence

$$0 \to R^{g-1}\Phi_{\mathcal{P}}(I_0^\vee) \to \mathcal{O}_{\widehat{A}} \to k(\hat{0}) \to 0$$

and the vanishings $R^i\Phi_{\mathcal{P}}(I_0^\vee) = 0$ for $i \neq g-1$.

□

Plugging Lemma 3.11 into the second formula of Proposition 3.10 we get the following basic symmetry satisfied by the cohomological rank functions of the ideal of one point.

Corollary 3.12. *For $\lambda \in \mathbb{Q}_{>0}$ and $i = 0, 1$ we have*

$$h^i(A, I_0 \langle \lambda\ell \rangle) = \chi(\ell)\lambda^g \cdot h^{1-i}\left(\widehat{A}, I_{\hat{0}}\left\langle \frac{1}{d_1 d_g \lambda}\ell_\delta \right\rangle\right).$$

As we saw in Chapter 2, for $\lambda \in \mathbb{Q}_{>0}$ the cohomological rank functions $h^i(A, I_0\langle\lambda\ell\rangle)$ are identically zero for $i \neq 0, 1$. Now, from the tautological exact sequence $0 \to I_0 \to \mathcal{O}_A \to k(0) \to 0$ we have

$$h^0(A, I_0\langle\lambda\ell\rangle) - h^1(A, I_0\langle\lambda\ell\rangle) =: \chi(A, I_0\langle\lambda\ell\rangle) = \chi(\ell)\lambda^g - 1.$$

Now, it is easy to see that if $I_0\langle\lambda\ell\rangle$ is IT(0) (respectively IT(1)) then the same holds for all $\lambda' \geq \lambda$ (resp. $\lambda' \leq \lambda$). Therefore it is natural to consider the following two thresholds:

Definition 3.13. For a polarized abelian variety (A, ℓ) we write

$$\beta^0_A(\ell) := \sup\{\lambda \in \mathbb{Q} : I_0 \langle \lambda\ell \rangle \text{ is IT}(1)\} = \sup\{\lambda \in \mathbb{Q} : h^0(I_0 \langle \lambda\ell \rangle) = 0\}$$

and

$$\beta^1_A(\ell) := \epsilon_0(\ell) = \inf\{\lambda \in \mathbb{Q} : I_0 \langle \lambda\ell \rangle \text{ is IT}(0)\} = \inf\{\lambda \in \mathbb{Q} : h^1(I_0 \langle \lambda\ell \rangle) = 0\}.$$

The following property is a direct consequence of Corollary 3.12 above:

Corollary 3.14. *Let (A, ℓ) be a polarized abelian variety of type $(d_1, ..., d_g)$ with dual $(\widehat{A}, \ell_\delta)$. Then the following equality holds:*

$$\beta^i_A(\ell) = \frac{1}{d_1 d_g \beta^{1-i}_{\widehat{A}}(\ell_\delta)}.$$

Note that the threshold $\beta^1_A(\ell)$ is not new, but it is worth to point out that now we can give an interpretation of it in terms of semi-homogeneous vector bundles. Concretely, from Proposition 3.6 we have that

$$h^i(I_0 \langle \lambda\ell \rangle) = \frac{1}{r_{A,\ell}(\lambda)} \cdot h^i((I_0 \otimes E_{A,\lambda\ell}) \langle 0 \cdot \ell \rangle).$$

Taking cohomology to the exact sequence

$$[0 \to I_0 \to \mathcal{O}_A \to k(0) \to 0] \otimes E_{A,\lambda\ell} \otimes P_\alpha,$$

we see that $h^0(I_0 \langle \lambda\ell \rangle)$ is the generic dimension of the kernel of the restriction map

$$H^0(E_{A,\lambda\ell} \otimes P_\alpha) \to H^0(E_{A,\lambda\ell} \otimes P_\alpha \otimes k(0)) \qquad (*)$$

normalized by the rank of $E_{A,\lambda\ell}$, while $h^1(I_0 \langle \lambda\ell \rangle)$ is the (normalized) generic dimension of the cokernel of such map. On the other hand, as we saw in the proof of Proposition 3.8, the vector bundles $E_{A,\lambda\ell} \otimes P_\alpha$ are all traslates of $E_{A,\lambda\ell}$ and thus the maps $(*)$ correspond to the maps

$$H^0(E_{A,\lambda\ell}) \to H^0(E_{A,\lambda\ell} \otimes k(x)) \quad \text{for } x \in A. \qquad (**)$$

The announced interpretation is then the following:

a) $\beta^1_A(\ell) < \lambda$ if $E_{A,\lambda\ell}$ is globally generated

b) $\beta^1_A(\ell) = \lambda$ if $E_{A,\lambda\ell}$ is generically globally generated but not globally generated

c) $\beta^0_A(\ell) > \lambda$ if every section of $E_{A,\lambda\ell}$ is nowhere zero

d) $\beta^0_A(\ell) = \lambda$ means that the restriction map $(**)$ is injective for generic x but not all x. This implies that the morphism

$$\mathrm{ev}_{A,\lambda\ell} : H^0(E_{A,\lambda\ell}) \otimes \mathcal{O}_A \to E_{A,\lambda\ell}$$

is injective as a morphism of sheaves (i.e injective at the stalks) but not as a morphism of vector bundles (i.e at the level of fibers).

Using the above interpretation we directly obtain the following:

Proposition 3.15. *Let ℓ be a polarization on an abelian variety A. Then the following inequalities hold:*

$$\beta^1_A(\ell) \geq \chi(\ell)^{-1/g} \geq \beta^0_A(\ell).$$

Proof: The first inequality is well known and it is proved in [Ito22a, Lemma 3.4]. For the second one, we have that if $\lambda > \chi(\ell)^{-1/g}$ then from 3.1(a) we have that $h^0(E_{\lambda\ell}) > \mathrm{rk}(E_{\lambda\ell})$ and hence the evaluation map can not be injective.

□

3.4 Duality and multiplication maps

Here we will establish a relation between the cohomological rank functions of the ideal of one point and the ones of the evaluation complex of a suitable simple semihomogeneous vector bundle. This generalize the content of [JP20, §8], allowing us to give a vector-bundle interpretation of such relation and allowing us to work with *fractionally polarized abelian varieties* in a simpler way.

We start by noticing that we can formally define a fractional polarization on an abelian variety to be a symbol $\nu \cdot \ell$, where $\nu \in \mathbb{Q}$ and $\ell \in \mathrm{NS}(A)$ is an ample class. In this setting, the cohomological rank functions of an object $\mathcal{F}$ with respect to a rationally polarized abelian variety $(A, \nu \cdot \ell)$ can be formally defined as

$$h^i((A, \nu \cdot \ell), \mathcal{F}; t) := h^i(A, \mathcal{F}\langle \nu t \cdot \ell\rangle) \tag{3.13}$$

and the formulas from the previous section remain valid. More precisely, for a fractional polarization $\nu \cdot \ell$ we formally define its type to be $(\nu d_1, ..., \nu d_g)$, where $(d_1, ..., d_g)$ is the type of ℓ and its dual $(\nu \cdot \ell)_\delta := \nu \cdot \ell_\delta$. With this notation, the transformation formula (3.8) for $\lambda \in \mathbb{Q}_{<0}$ becomes:

$$h^i((A, \nu \cdot \ell), \mathcal{F}; \lambda) = \chi(\nu \cdot \ell) \cdot (-\lambda)^g \cdot h^i\left(\left(\hat{A}, (\nu \cdot \ell)_\delta\right), \Phi_{\mathcal{P}}(\mathcal{F}); -\frac{1}{(\nu d_1) \cdot (\nu d_2) \cdot \lambda}\right),$$

where $\chi(\nu \cdot \ell) := \nu^g \chi(\ell)$. Indeed:

$$\begin{aligned}
h^i((A, \nu \cdot \ell), \mathcal{F}); \lambda) &= h^i(A, \mathcal{F}\langle \nu\lambda \cdot \ell\rangle) \\
&= (-\lambda\nu)^g \chi(\ell) \cdot h^i\left(\hat{A}, \Phi_{\mathcal{P}}(\mathcal{F})\left\langle -\frac{1}{d_1 d_g \lambda\nu} \cdot \ell_\delta\right\rangle\right) && \text{by (3.8)} \\
&= (-\lambda\nu)^g \chi(\ell) \cdot h^i\left(\hat{A}, \Phi_{\mathcal{P}}(\mathcal{F})\left\langle -\frac{\nu}{d_1 d_g \lambda\nu^2} \cdot \ell_\delta\right\rangle\right) \\
&= \chi(\nu \cdot \ell) \cdot (-\lambda)^g \cdot h^i\left(\left(\widehat{A}, \nu \cdot \ell_\delta\right), \Phi_{\mathcal{P}}(\mathcal{F})); -\frac{1}{\nu^2 d_1 d_g \lambda}\right)
\end{aligned}$$

We have the following fundamental lemma:

Lemma 3.16. *Let $E_{A,\nu\ell}$ be a simple semihomogeneous vector bundle in $\mathbb{S}_{A,\nu\ell}$, where $\nu = \frac{a}{b}$, with $a, b > 0$. Applying the functor $\varphi^*_{a\ell}\Phi_{\mathcal{P}}$ to the restriction map*

$$E_{A,\nu\ell} \longrightarrow E_{A,\nu\ell} \otimes k(0)$$

one gets the twisted evaluation map

$$b_A^* H^0(A, E_{A,\nu\ell}) \otimes L^{\otimes -ab} \longrightarrow \mathcal{O}_A^{\oplus r_{A,\ell}(\nu)} \simeq (b_A^* E_{A,\nu\ell}) \otimes L^{\otimes -ab},$$

where $r_{A,\ell}(\nu) = \operatorname{rk} E_{A,\nu\ell}$ *and* L *is a line bundle representing* ℓ *such that* $b_A^* E \simeq (L^{\otimes ab})^{\oplus r_{A,\ell}(\nu)}$ *(See Proposition 3.4).*

Proof: We know that $E_{A,\nu\ell}$ is an IT(0) vector bundle. We claim that

$$\varphi_{a\ell}^* \widehat{E_{A,\nu l}} \simeq (L'^{\otimes -ab})^{\oplus h^0(A, E_{A,\nu\ell})} \tag{3.14}$$

where L' is a line bundle representing ℓ. To prove this we argue as in the proof of Proposition 3.4. We have that $\operatorname{rk} \widehat{E_{A,\nu\ell}} = h^0(A, E_{A,\lambda\ell})$. Moreover $\widehat{E_{A,\nu\ell}}$ is simple and belongs to $\mathbb{S}_{\widehat{A}, -\frac{1}{d_1 d_g \nu}\ell_\delta}$ (Proposition 3.8). Therefore, according to Subsection 3.1(b), we have that the subgroup $\Phi(\widehat{E_{A,\nu\ell}}) \subset \widehat{A} \times A$ is the image of the morphism $((ad_1 d_g)_{\widehat{A}}, \varphi_{b\ell_\delta}) : \widehat{A} \to \widehat{A} \times A$, and we have that

$$\varphi_{a\ell}^* \widehat{E_{A,\nu l}} \simeq M^{\oplus h^0(A, E_{A,\nu l})} \tag{$*$}$$

if and only if $\varphi_{a\ell}$ factors through the projection $\Phi(\widehat{E_{A,\nu\ell}}) \to \widehat{A}$. We claim that $\varphi_{a\ell}$ satisfies such condition and therefore $\varphi_{a\ell}^* \widehat{E_{A,\lambda\ell}}$ splits. Indeed, since $(ad_1 d_g)_{\widehat{A}} = \varphi_{a\ell}\varphi_{\ell_\delta}$ (Proposition 3.7), we have that for any $\alpha \in \ker \varphi_{\ell_\delta}$ we have

$$(ad_1 d_g)_{\hat{A}}(\alpha) = \varphi_{a\ell}\varphi_{\ell_\delta}(\alpha) = 0$$

and thus we can define an isogeny

$$A \longrightarrow \Phi(\widehat{E_{A,\nu\ell}}) : x \longrightarrow ad_1 d_g \cdot \beta,$$

where β is a (any) element in the preimage of x under the isogeny φ_{ℓ_δ} and it is clear that $\varphi_{a\ell}$ factors through this isogeny. Taking determinants and recalling that $\varphi_\ell^* \ell_\delta = d_1 d_g \cdot \ell$ (Proposition 3.7)), the claimed (3.14) follows.

Now, we need to understand the induced map

$$\varphi_{a\ell}^*(\widehat{E_{A,\nu\ell}}) \longrightarrow \varphi_{a\ell}^*(\widehat{E_{A,\nu\ell} \otimes k}(0)). \tag{3.15}$$

Concerning the target, we have the isomorphism $\varphi_{a\ell}^*(\widehat{E_{A,\nu\ell} \otimes k}(0)) = \mathcal{O}_A^{\oplus r_{A,\ell}(\nu)}$, where L is a line bundle representing ℓ such that the isomorphism of Proposition 3.4 holds. By (3.14), (3.15) is identified to a morphism $(L'^{\otimes -ab})^{\oplus h^0(A, E_{A,\nu\ell})} \to \mathcal{O}_A^{\oplus r_{A,\ell}(\nu)}$. From the definition of the Fourier-Mukai functor, it is not difficult to see that (3.15), at the level of fibers, is an evaluation map of global sections of $E_{A,\nu\ell}$ tensored by $L'^{\otimes -ab}$. Since we know by Proposition 3.4 that $b_A^* E_{A,\nu\ell} \otimes L^{\otimes -ab}$ is trivial (for a suitable line bundle L representing ℓ), it follows that 3.15 is identified to a twisted evaluation map from $b_A^* H^0(A, E_{A,\nu\ell}) \otimes L'^{\otimes -ab}$ to $b_A^* E_{A,\nu\ell} \otimes L^{\otimes -ab}$. It follows that $L^{-\otimes ab} = L'^{\otimes -ab}$ (otherwise there would not be a non-zero morphism) and the proof is complete.

□

Let us write $\mathrm{EV}^\bullet_{A,\nu\ell}$ for the complex

$$0 \to H^0(A, E_{A,\nu\ell}) \otimes \mathcal{O}_A \to E_{A,\nu\ell} \to 0. \tag{3.16}$$

We observe that there is a slight abuse of notation, in the sense that the complex above does not depend just on ν and ℓ but also on the choice of a simple vector bundle in $\mathbb{S}_{A,\nu\ell}$. However, the cohomological rank functions built on it does not depend on this choice.

The basic result we are interested in is the following relation between the cohomological rank functions of the ideal sheaf of the origin and the ones of the above complex.

Proposition 3.17. *Let $\nu \in \mathbb{Q}_{>0}$ and $\lambda \in (0,1) \cap \mathbb{Q}$. Then, in the notation of* (3.13)

$$h^i((A, \nu \cdot \ell), I_0; \lambda) = \frac{(1-\lambda)^g}{\chi(\nu\ell) r_{A,\ell}(\nu)} \cdot h^i\left((A, \nu\ell), \mathrm{EV}^\bullet_{A,\nu\ell}; \frac{\lambda}{1-\lambda}\right) \tag{3.17}$$

Proof: Let $t \in \mathbb{Q} \cap (-1, 0)$ and write $\nu = \frac{a}{b}$. We have that

$$\begin{aligned} h^i((A,\nu\ell), I_0, 1+t) = h^i(A, I_0 \langle \nu(1+t)\ell \rangle) &= \frac{1}{r_{A,\ell}(\nu)} \cdot h^i(A, (I_0 \otimes E_{A,\nu\ell}) \langle \nu t \ell \rangle) \\ &= \frac{1}{r_{A,\ell}(\nu)} \cdot h^i\left(A, (I_0 \otimes E_{A,\nu\ell}) \left\langle \frac{t}{b} \cdot a\ell \right\rangle\right) \end{aligned} \tag{3.18}$$

where the second equality follows from Proposition 3.6. Since $I_0 \otimes E_{A,\nu\ell}$ is isomorphic (in $D^b(A)$) to the complex $0 \to E_{A,\nu\ell} \stackrel{\mathrm{res}}{\to} E_{A,\nu\ell} \otimes k(e) \to 0$, it follows from Lemma 3.16 that $\varphi^*_{a\ell}\Phi_{\mathcal{P}}(I_0 \otimes E_{A,\nu\ell})$ is isomorphic, in $D(A)$, to the complex $(b^*_A \mathrm{EV}^\bullet_{A,\nu\ell}) \otimes L^{-ab}$. Thus the transformation formula (Proposition 1.24 yields that

$$h^i\left(A, (I_0 \otimes E_{A,\nu\ell}) \left\langle \frac{t}{b} \cdot a\ell \right\rangle\right) = \frac{(-\frac{t}{b})^g}{\chi(a\ell)} \cdot h^i\left(A, (b^*_A \mathrm{EV}^\bullet_{A,\nu\ell} \otimes L^{\otimes -ab}) \left\langle -\frac{b}{t} \cdot a\ell \right\rangle\right) \tag{3.19}$$

Manipulating the right hand side we get

$$\begin{aligned} \frac{(-\frac{t}{b})^g}{\chi(a\ell)} \cdot h^i\left(A, (b^*_A \mathrm{EV}^\bullet_{A,\nu\ell} \otimes L^{\otimes -ab}) \left\langle -\frac{b}{t} \cdot a\ell \right\rangle\right) &= \frac{(-\frac{t}{b})^g}{\chi(a\ell)} \cdot h^i\left(A, b^*_A \mathrm{EV}^\bullet_{A,\nu\ell} \left\langle -ab\left(1+\frac{1}{t}\right) \cdot \ell \right\rangle\right) \\ &= \frac{(-\frac{t}{b})^g}{\chi(a\ell)} \cdot b^{2g} \cdot h^i\left(A, \mathrm{EV}^\bullet_{A,\nu\ell} \left\langle -\frac{a}{b}\left(1+\frac{1}{t}\right)\ell \right\rangle\right) \\ &= \frac{(-t)^g}{\chi(\nu\ell)} \cdot h^i\left(A, \mathrm{EV}^\bullet_{A,\nu\ell} \left\langle -\frac{\nu(1+t)}{t}\ell \right\rangle\right) \end{aligned}$$

Combining with (3.18) we get

$$h^i\left((A,\nu\ell), I_0; 1+t\right) = \frac{(-t)^g}{\chi(\nu\ell) r_{A,\ell}(\nu)} \cdot h^i\left((A,\nu\ell), \mathrm{EV}^\bullet_{A,\nu\ell}; -\frac{1+t}{t}\right).$$

The Proposition then follows letting $\lambda = 1+t$.

□

Plugging $\nu = 1$ and combining with Corollary 3.12, we get the following duality:

Corollary 3.18. *For $\lambda \in (0,1) \cap \mathbb{Q}$ the following equality holds:*

$$\frac{(1-\lambda)^g}{\chi(\ell) \cdot r_{A,\ell}\left(\frac{\lambda}{1-\lambda}\right)} \cdot h^i\left(A, \mathrm{EV}^{\bullet}_{A,\ell} \otimes E_{A,\frac{\lambda}{1-\lambda}\ell}\langle 0 \cdot \ell\rangle\right)$$
$$= \lambda^g \chi(\ell) \cdot \frac{\left(1 - \frac{1}{d_1 d_g \lambda}\right)^g}{\chi(\ell_\delta) \cdot r_{\widehat{A},\ell_\delta}\left(\frac{1}{d_1 d_g \lambda - 1}\right)} \cdot h^{1-i}\left(\widehat{A}, \mathrm{EV}^{\bullet}_{\widehat{A},\hat{\ell}} \otimes E_{\widehat{A},\frac{1}{d_1 d_g \lambda - 1}\ell_\delta}\langle 0 \cdot \ell_\delta\rangle\right).$$

□

As before, we can give an interpretation of the formula (3.17) above in terms of semi-homogeneous vector bundles. Indeed: using Proposition 3.4, the afore mentioned formula can be equivalently stated as follows:

$$h^i\left((A,\nu\ell), I_0, \lambda\right) = \frac{(1-\lambda)^g}{\chi(\nu\ell) \cdot r_{A,\ell}(\nu) \cdot r_{A,\ell}\left(\frac{\lambda\nu}{1-\lambda}\right)} \cdot h^i\left(A, (\mathrm{EV}^{\bullet}_{A,\nu\ell} \otimes E^{\bullet}_{\frac{\lambda\nu}{1-\lambda}\ell})\langle 0 \cdot \ell\rangle\right). \quad (3.20)$$

for $\nu \in \mathbb{Q}_{>0}$ and $\lambda \in (0,1) \cap \mathbb{Q}$. Now, for those values of λ and ν we have that $E_{A,\nu\ell}$ and $E_{A,\frac{\lambda\nu}{1-\lambda}\ell}$ are both IT(0) and thus the E_1-spectral sequence ([Huy06, Remark 2.67]) computing the hypercohomology of $\mathrm{EV}^{\bullet}_{A,\nu\ell} \otimes E_{\frac{\lambda\nu}{1-\lambda}\ell}$ degenerates to the complex

$$0 \to H^0(A, E_{A,\nu\ell}) \otimes H^0(A, E_{A,\frac{\lambda\nu}{1-\lambda}\ell} \otimes P_\alpha) \xrightarrow{m^{1,\frac{\lambda}{1-\lambda}}_{A,\nu\ell,\alpha}} H^0(A, E_{A,\nu\ell} \otimes E_{A,\frac{\lambda\nu}{1-\lambda}\ell} \otimes P_\alpha) \to 0. \quad (3.21)$$

where we wrote $m^{1,\frac{\lambda}{1-\lambda}}_{A,\nu\ell,\alpha}$ for the corresponding multiplication map of global sections. In particular, we see that $h^0\left(A, (\mathrm{EV}^{\bullet}_{A,\nu\ell} \otimes E^{\bullet}_{\frac{\lambda\nu}{1-\lambda}\ell})\langle 0 \cdot \ell\rangle\right)$ is the generic dimension of the kernel of the multiplication map while the h^1 is no other than the generic corank of those maps.

Again, we point out that writing $m^{1,\frac{\lambda}{1-\lambda}}_{A,\nu\ell,\alpha}$ is already a slight notational abuse since these maps depends on the choice of a simple vector bundle, but the cohomological rank functions do not depend on this choice. In particular, the following thresholds are independent of this choice:

Definition 3.19.

$$s^0_A(\nu \cdot \ell) = \sup\{y \in \mathbb{Q} : m^{1,y}_{A,\nu\ell,\alpha} \quad \text{is injective } \forall \alpha \in \widehat{A}\}$$

and

$$s^1_A(\nu \cdot \ell) = \inf\{y \in \mathbb{Q} : m^{1,y}_{A,\nu\ell,\alpha} \quad \text{is surjective } \forall \alpha \in \widehat{A}\}.$$

We can directly obtain the following relations:

Corollary 3.20.

$$s^i_A(\nu \cdot \ell) = \frac{\beta^i_A(\ell)}{\nu - \beta^i_A(\ell)}. \quad (3.22)$$

Proof: We can formally introduce the numbers

$$\beta^0_A(\nu\cdot\ell) := \sup\{\lambda\in\mathbb{Q} : h^0((A,\nu\cdot\ell),I_0,\lambda)\neq 0\}$$

and

$$\beta^1_A(\nu\cdot\ell) := \inf\{\lambda\in\mathbb{Q} : h^1((A,\nu\cdot\ell),I_0,\lambda) = 0\}.$$

We have that $\beta^i_A(\nu\cdot\ell) = \nu^{-1}\cdot\beta^i_A(\ell)$, where $\beta^i_A(\ell)$ is the threshold introduced in Definition 3.13. From (3.20) and (3.21) we obtain that

$$s^i_1(\nu\cdot\ell) = \frac{\beta^i_A(\nu\cdot\ell)}{1-\beta^i_A(\nu\cdot\ell)} = \frac{\beta^i_A(\ell)}{\nu-\beta^i_A(\ell)}.$$

□

Corollary 3.21. *For $\nu,\mu\in\mathbb{Q}$ we have*

$$\beta^i_A(\ell) = \frac{\nu\cdot s^i_A(\nu\cdot\ell)}{1+s^i_A(\nu\cdot\ell)} = \frac{\mu\cdot s^i_A(\mu\cdot\ell)}{1+s^i_A(\mu\cdot\ell)}.$$

□

Finally, combining with Corollary 3.14 we obtain the following relation that will be fundamental in the following section:

Corollary 3.22. *For $i=0,1$ and $\nu\in\mathbb{Q}$ we have*

$$\beta^i_A(\ell) = \frac{1}{\nu\cdot d_1 d_g}\cdot\left(1+\frac{1}{s^{1-i}_{\hat{A}}(\nu\cdot\ell_\delta)}\right).$$

□

To conclude this section we mention that, even though until we have used only **simple** semi-homogeneous vector bundles in $\mathbb{S}_{A,\nu\ell}$, to compute the thresholds $s^i_A(\nu\cdot\ell)$ we may omit the simplicity hypothesis. The reason is the following

Lemma 3.23. *Let E be a (not necessarily simple) semi-homogenenous vector bundle in $\mathbb{S}_{\lambda\ell}$. Then*

$$h^i\left((\mathrm{EV}^\bullet_{\nu\ell}\otimes E)\langle 0\cdot\ell\rangle\right) = \frac{\mathrm{rk}\,E}{r_{A,\ell}(\lambda)}\cdot h^i\left((\mathrm{EV}^\bullet_{\nu\ell}\otimes E_{\lambda\ell})\langle 0\cdot\ell\rangle\right),$$

where, as usual, $E_{\lambda\ell}$ is a simple vector bundle in $\mathbb{S}_{\lambda\ell}$.

Proof: By 3.1 we know that every vector bundle in $\mathbb{S}_{\lambda\ell}$ is direct sum of vector bundles of the form $U\otimes E_{\lambda\ell}\otimes P_\alpha$, where U is unipotent. Thus, in order to prove the statement it is enough to prove that if U is unipotent and $F\in\mathbb{S}_{\lambda\ell}$ then

$$h^i(\mathrm{EV}^\bullet_{\nu\ell}\otimes F\otimes U) = \mathrm{rk}(U)\cdot h^i(\mathrm{EV}^\bullet_{\nu\ell}\otimes F).$$

To start we note that by [Muk78, Theorem 4.12] $U=\Phi_{\mathcal{P}}(\mathcal{O}_Z)$ for a zero-dimensional subscheme $Z\subset\hat{A}$ with length $Z=\mathrm{rk}\,U$. On the other hand, by (1.7) we have

$$H^i(\mathrm{EV}^\bullet_{\nu\ell}\otimes F\otimes U)\simeq H^i(\Phi_{\mathcal{P}}(\mathrm{EV}^\bullet_{\nu\ell}\otimes F)\,\underline{\otimes}\,\mathcal{O}_Z).$$

Now, as $\dim Z = 0$, the spectral sequence

$$E_2^{p,q} = H^p\left(\mathscr{T}or_{-q}(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F), \mathcal{O}_Z)\right) \implies H^{p+q}\left(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F) \underline{\otimes} \mathcal{O}_Z\right)$$

degenerates to give isomorphisms

$$H^i(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F) \underline{\otimes} \mathcal{O}_Z) \simeq H^0\left(\mathscr{T}or_i(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F), \mathcal{O}_Z)\right).$$

Now, it is easy to see that $\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F)$ is represented in $D^b(\widehat{A})$ by the complex of locally free sheaves

$$H^0(E_{\nu\ell}) \otimes \widehat{F} \longrightarrow \widehat{E_{\nu\ell} \otimes F}.$$

It follows then that

$$\begin{aligned}
\mathscr{T}or_0(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F), \mathcal{O}_Z) &= \ker\left[\left(H^0(E_{\nu\ell}) \otimes \widehat{F} \to \widehat{E \otimes F}\right) \otimes \mathcal{O}_Z\right] \\
&\simeq \ker\left[\left(H^0(E_{\nu\ell}) \otimes \widehat{F} \to \widehat{E \otimes F}\right) \otimes k(\hat{0})\right] \otimes \mathcal{O}_Z \\
&\simeq \mathscr{T}or_0(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F), k(\hat{0})) \otimes \mathcal{O}_Z
\end{aligned}$$

Similarly,

$$\mathscr{T}or_1(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F), \mathcal{O}_Z) \simeq \mathscr{T}or_1(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F), k(\hat{0})) \otimes \mathcal{O}_Z.$$

Finally, as both $\mathscr{T}or_1$ and $\mathscr{T}or_0$ are supported on a point, we have that their h^0 is nothing else than their rank and thus

$$h^0(\mathscr{T}or_i(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F), \mathcal{O}_Z)) = \mathrm{rk}(\mathscr{T}or_i(\Phi_{\mathcal{P}}(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F), k(\hat{0}))) \cdot \mathrm{length}\, Z = h^i(\mathrm{EV}^{\bullet}_{\nu\ell} \otimes F) \cdot \mathrm{rk}\, U,$$

where the last equality comes from the previous calculations (for $Z = \{\hat{0}\}$). The proof is then complete.

□

3.5 Lower bounds for the basepoint-freeness threshold: obstructions to projective normality

In this section we use the results from the previous section to prove a lower bound for the base-point freeness threshold. The approach is based on Corollary 3.22 above which says that a lower bound for $\beta^1_A(\ell)$ is equivalent to an upper bound for $s^0_{\hat{A}}(\nu \cdot \ell_\delta)$, and an upper bound this number can be found by proving that certain multiplication maps can not be injective.

Proposition 3.24. *If $\nu > \beta^0_{\hat{A}}(\ell_\delta)$ then*

$$s^0_{\hat{A}}(\nu \cdot \ell_\delta) \leq r_{\hat{A},\ell_\delta}(\nu).$$

To heart of the proposition is the following lemma:

Lemma 3.25. *Let E be a generically globally generated vector bundle with* $\operatorname{rk} E < h^0(E)$. *Then the multiplication map*

$$H^0(E) \otimes H^0(\det E) \longrightarrow H^0(E \otimes \det E)$$

is not injective.

Proof: Let $r = \operatorname{rk} E$. Let ψ be the composition

$$\bigwedge^{r+1} H^0(E) \longrightarrow H^0(E) \otimes \bigwedge^{r} H^0(E) \longrightarrow H^0(E) \otimes H^0(\det E),$$

where the first map is given by

$$s_1 \wedge \cdots \wedge s_{r+1} \longrightarrow \frac{1}{r!} \cdot \sum_{\sigma \in S_{r+1}} \operatorname{sgn}(\sigma) s_{\sigma(1)} \otimes \left[s_{\sigma(2)} \wedge \cdots \wedge s_{\sigma(r+1)}\right]$$

(for example, for $r = 2$ this map is given by

$$s_1 \wedge s_2 \wedge s_3 \longrightarrow s_1 \otimes (s_2 \wedge s_3) - s_2 \otimes (s_1 \wedge s_3) + s_3 \otimes (s_1 \wedge s_2)).$$

Clearly we have that

$$\operatorname{Im}\psi \subseteq \ker\left[H^0(E) \otimes H^0(\det E) \to H^0(E \otimes \det E)\right]$$

and hence the lemma follows once we prove that $\operatorname{Im}\psi \neq 0$. To do this, we note that the fact that E is generically globally generated implies that the natural map

$$\varepsilon : \wedge^r H^0(E) \to H^0(\det E)$$

is not zero. Let $s_2, ..., s_r \in H^0(E)$ such that $s_2 \wedge \cdots s_r \neq 0 \in H^0(\det E)$ and consider $s_1 \in H^0(E)$ such that $\{s_1, ..., s_{r+1}\}$ is a linearly independent subset of $H^0(E)$. We claim that

$$\psi(s_1 \wedge s_2 \wedge \cdots \wedge s_{r+1}) \neq 0.$$

To justify this, let $V = \operatorname{span}_{\mathbb{C}}(s_1, ..., s_{r+1}) \subset H^0(E)$ and suppose that the image (via ε) of $\wedge^r V$ has rank k. Let $\tau_1, ..., \tau_k \in S_{r+1}$ such that

$$\left\{s_{\tau_1(2)} \wedge \cdots \wedge s_{\tau_1(r+1)}, ..., s_{\tau_k(2)} \wedge \cdots \wedge s_{\tau_k(r+1)}\right\}$$

is a base of $\varepsilon\left(\bigwedge^k V\right) \subset H^0(\det E)$. If we write

$$s_{\tau_l} := s_{\tau_l(2)} \wedge \cdots \wedge s_{\tau_l(r+1)}$$

then for $\sigma \in S_{r+1}$ we have that

$$s_{\sigma(2)} \wedge \cdots \wedge s_{\sigma(r+1)} = \sum_{l=1}^{k} \lambda_{\sigma,l} s_{\tau_l} \quad \text{for certains } \lambda_{\sigma,l} \in \mathbb{C}.$$

We have then

$$\begin{aligned}
\psi\left(s_1 \wedge s_2 \wedge \cdots \wedge s_{r+1}\right) &= \frac{1}{r!} \cdot \sum_{\sigma \in S_{r+1}} \sum_{l=1}^{k} \operatorname{sgn}(\sigma) \lambda_{\sigma,l} s_{\sigma(1)} \otimes s_{\tau_l} \\
&= \frac{1}{r!} \cdot \sum_{l=1}^{k} \sum_{j=1}^{r+1} \sum_{\sigma \mid \sigma(1)=j} \operatorname{sgn}(\sigma) \lambda_{\sigma,l} s_j \otimes s_{\tau_l} \\
&= \frac{1}{r!} \cdot \sum_{l=1}^{k} \sum_{j=1}^{r+1} \left[\sum_{\sigma \mid \sigma(1)=j} \operatorname{sgn}(\sigma) \lambda_{\sigma,l} \right] s_j \otimes s_{\tau_l} \\
&= \sum_{l,j} \operatorname{sgn}(\sigma_j) \lambda_{\sigma_j,l} s_j \otimes s_{\tau_l}
\end{aligned}$$

where in the last line we chose a $\sigma_j \in S_{r+1}$ with $\sigma(1) = j$ and we used the fact that if $\sigma, \sigma' \in S_{r+1}$ both satisfy $\sigma(1) = \sigma'(1)$ then $\lambda_{\sigma,l} = \operatorname{sgn}(\sigma'\sigma^{-1})\lambda_{\sigma',l}$ (and that there are $r!$ of such σ's). Finally, as $\{s_j \otimes s_{\tau_l} : j, l\}$ is a linearly independent subset of $H^0(E) \otimes H^0(\det E)$ and $\lambda_{\mathrm{id},l} \neq 0$ for some l, we conclude that $\psi(s_1 \wedge s_2 \wedge \cdots \wedge s_{r+1}) \neq 0$, as we wanted to prove.

□

Proof of the Proposition: By Proposition 3.15 we have that $\beta^1_{\hat{A}}(\ell_\delta) \geq \beta^0_{\hat{A}}(\ell_\delta)$. To prove the proposition we will distinguish two cases, namely:

$$\nu \geq \beta^1_{\hat{A}}(\ell_\delta) \tag{a}$$

and

$$\beta^1_{\hat{A}}(\ell_\delta) > \nu > \beta^0_{\hat{A}}(\ell_\delta). \tag{b}$$

In case (a) we have that $E_{\nu\ell_\delta}$ is generically globally generated and hence the lemma implies that the multiplication map

$$H^0(E_{\nu\ell_\delta}) \otimes H^0(\det E_{\nu\ell_\delta}) \longrightarrow H^0(E_{\nu\ell_\delta} \otimes \det E_{\nu\ell_\delta}) \tag{3.23}$$

is not injective unless $h^0(E_{\nu\ell}) = \operatorname{rk} E_{\nu\ell}$. As shown in Proposition 3.15, if the latter happens then $\nu^g \cdot \chi(\ell_\delta) = 1$ and $\beta^1_{\hat{A}}(\ell_\delta) = \beta^0_{\hat{A}}(\ell_\delta) = \nu$, contradicting the hypothesis $\nu > \beta^0_{A}(\ell_\delta)$. We have then that (3.23) is not injective. Now, by definition we have that $\det E_{\nu\ell_\delta}$ is a simple (line) vector bundle in $\mathbb{S}_{\hat{A}, r_{\hat{A},\ell_\delta}(\nu)\nu\ell_\delta}$ and hence it follows that

$$s^0_{\hat{A}}(\nu \cdot \ell_\delta) \leq r_{\hat{A},\ell_\delta}(\nu),$$

as we wanted to see.

Now, in case (b), we have anyways that the evaluation map $H^0(E_{\nu\ell_\delta}) \otimes \mathcal{O}_{\hat{A}} \to E_{\nu\ell_\delta}$ is not injective and thus the rank of its the image, say u, is strictly less than the rank of $E_{\nu\ell_\delta}$. A similar computation as in the lemma then shows that the multiplication map

$$H^0(E_{\nu\ell_\delta}) \otimes H^0\left(\wedge^u E_{\nu\ell_\delta}\right) \longrightarrow H^0\left(E_{\nu\ell_\delta} \otimes \wedge^u E_{\nu\ell_\delta}\right)$$

is not injective. Now, although $\wedge^u E_{\nu\ell_\delta}$ is not necessarily simple, it is a direct summand of $E_{\nu\ell_\delta}^{\otimes u}$ and thus by Lemma 3.23 we have that

$$s^0_{\hat{A}}(\nu\cdot\ell)\leq u<\operatorname{rk}E_{\nu\ell_\delta},$$

as we wanted to prove.

□

Corollary 3.26. *We have that*

$$\beta^1_A(\ell)\geq\sup_{\nu>\beta^0_{\hat{A}}(\ell_\delta)}\left\{\frac{1}{\nu\cdot d_1d_g}\cdot\left(1+\frac{1}{r_{\hat{A},\ell_\delta}(\nu)}\right)\right\}.$$

In particular:

$$\beta^1_A(\ell)\geq\sup_{\nu>\frac{1}{\chi(\ell_\delta)^{1/g}}}\left\{\frac{1}{\nu\cdot d_1d_g}\cdot\left(1+\frac{1}{r_{\hat{A},\ell_\delta}(\nu)}\right)\right\}.$$

Proof: The first inequality is a direct combination of Corollary 3.22 and Proposition 3.24. The "in particular" statement follows from Propostion 3.15.

□

Example 3.27. *If the type of ℓ satisfy $d_1<d_g$ then in the above corollary we can consider $\nu=1/d_1$. In this case $E_{\nu\ell_\delta}$ is nothing else than a line bundle representing $d_1^{-1}\ell_\delta$ and hence we get that*

$$\beta^1_A(\ell)\geq\frac{2}{d_g}.$$

Concretely:

- *If $d_g=2$ we get $\beta^1_A(\ell)\geq 1$ and hence $\beta^1_A(\ell)=1$, which means that polarizations of type $(1,\ldots,2)$ are never globally generated. This was an already known example ([NR95]).*
- *If $d_g=3$ we get $\beta^1_A(\ell)\geq 2/3>1/2$ and hence polarizations of type $(1,\ldots,3)$ are never normally generated. Note that for $g\geq 5$ we have that $3^{g-1}>2^{g+1}-1$ and hence this was not dimensionally obvious and at the best knowledge of the author this case was not covered in the literature before.*
- *If $d_g=4$ then we get that $\beta^1_A(\ell)\geq 1/2$. This means that there is a non-empty subvariety Z of $\widehat{A}$ such that for every $\alpha\in Z$ the multiplication map*

  $$H^0(L)\otimes H^0(L\otimes P_\alpha)\to H^0(L^{\otimes 2}\otimes P_\alpha)$$

  is not surjective. Here we point out that L fails to be normally generated if and only if $\hat{0}$ belongs to Z and it is worth to mention that, by means of computational methods ([FG05]) it is known that there are polarizations with $d_1d_g=4$ such that L is normally generated (so $\hat{0}\notin Z$) but there also exist polarizations of that type that are not normally generated. In Corollary 3.33 below we give a slighlty more detailed result concerning this "jump locus".

Example 3.28. *Consider a polarization ℓ of type $(1, a, ..., a, ab)$ with $a > b$. In this case the dual polarization ℓ_δ has type $(1, b, ..., b, ab)$ and thus $\chi(\ell_\delta) = b^{g-1}a > b^g$. We can then consider $\nu = 1/b$. We have*

$$K(L_\delta) \simeq (\mathbb{Z}/b\mathbb{Z})^{2(g-2)} \times (\mathbb{Z}/ab\mathbb{Z})^2$$

and thus

$$K(L_\delta)[b] = K(L_\delta) \cap \hat{A}[b] \simeq (\mathbb{Z}/b\mathbb{Z})^{2(g-1)}.$$

Therefore, we have $u(1, b, \ell_\delta) = b^{g-1}$ and thus $r_{\hat{A},\ell_\delta} = b$. We get then

$$\beta^1_A(\ell) \geq \frac{1+b}{ab}.$$

For instance, if $a = 3$ and $b = 2$ we obtain $\beta^1_A(\ell) \geq 1/2$, which again means that there exists a non-empty jump - locus. This case is particularly interesting because it was out of reach even using computational methods in [FG05]. In Corollary 3.34 below we study this case in more detail.

3.6 On the jump - locus for the multiplication map

We conclude this chapter by studying wether the origin $\hat{0}$ belongs to the jump-locus, that is, wether the multiplication map

$$H^0(L) \otimes H^0(L) \to H^0(L^2) \tag{3.24}$$

is surjective or not, giving a slightly better picture of the situation in the above examples. To start, we recall that if L is globally generated, then the map (3.24) above is surjective if and only if

$$H^1(M_L \otimes L) = 0$$

where M_L is the kernel of the evaluation map $H^0(L) \otimes \mathcal{O}_A \to L$. It is then natural to seek for a more detailed description of such a group. The key point is that, by Lemma 2.20 we know that

$$M_L \otimes L \simeq \varphi_L^* \Phi_{\mathcal{P}}(I_0 \otimes L) \otimes L^2.$$

We have then the following isomorphisms

$$\begin{aligned} H^i(M_L \otimes L) &\simeq H^i(\varphi_L^* \Phi_{\mathcal{P}}(I_0 \otimes L) \otimes L^2) \\ &\simeq H^i(\Phi_{\mathcal{P}}(I_0 \otimes L) \otimes \varphi_{L*}(L^2)) \\ &\simeq H^i(I_0 \otimes L \otimes \Phi_{\mathcal{P}}(\varphi_{L*}(L^2))) \end{aligned} \tag{3.25}$$

where the last isomorphism comes from (1.7). This suggests to study the sheaf $\varphi_{L*}(L^2)$. We know that this is a semi-homogeneous vector bundle and hence admits a direct sum decomposition as in 3.1(c). A more explicit description of such decomposition is given by the following:

Proposition 3.29. *Let L be an ample line bundle on an abelian variety A with class $\ell \in \mathrm{NS}(A)$. Fix a simple vector bundle $W \in \mathbb{S}_{A,\frac{1}{2}\ell}$ such that $2_A^* W \simeq (L^{\otimes 2})^{\oplus \mathrm{rk}(W)}$ and assume that L^2 is symmetric. We have*

$$\varphi_{L*}(L^{\otimes 2}) \simeq \bigoplus_{\alpha} \left(\Phi_{\mathcal{P}}(W \otimes P_\alpha)^\vee\right)^{\oplus \chi(W)}, \tag{3.26}$$

and α runs over the quotient $\hat{A}[2]/\Sigma(W)$, where

$$\Sigma(W) = \left\{ \alpha \in \hat{A} : W \otimes P_\alpha \simeq W \right\}.$$

Before proving the proposition, we point out a couple of things:

1. From the relation $2^* W \simeq (L^{\otimes 2})^{\oplus \mathrm{rk}\, W}$ we see that $\Sigma(W) \subset \hat{A}[2]$. It follows that the quotient $\hat{A}[2]/\Sigma(W)$ makes sense and, moreover, the isomorphism class $W \otimes P_\alpha$ is determined by the class of α in such quotient. Also, by [Muk78, Proposition 6.1] we have that $|\Sigma(W)| = \mathrm{rank}(W)^2$.

2. Recall that $\mathrm{rk}\, \Phi_{\mathcal{P}}(W) = \chi(W)$. Concreteley, this means that in the decomposition (3.26) there are $2^{2g}/\,\mathrm{rk}(W)^2$ different simple vector bundles, each one of those has rank $\chi(W)$ and appears $\chi(W)$ times (note thas this make sense since $\chi(L) \cdot \mathrm{rk}(W) = 2^g \cdot \chi(W)$ and $\varphi_{L*}(2L)$ has rank $\chi(L)^2$).

Example 3.30. *In the $(1, 4, 4)$ case we have $K(L) \simeq (\mathbb{Z}/4\mathbb{Z})^4$ and hence $K(L)[2] \simeq (\mathbb{Z}/2\mathbb{Z})^4$. This means that $R = 2^3/2^2 = 2$, $N = 2^4/2^2 = 4$ and*

$$\left| \hat{A}[2]/\Sigma(W) \right| = 2^6/2^2 = 2^4.$$

That is, $\varphi_{L*}(2L)$ (which is a vector bundle of rank 2^8), is the direct sum of 4 copies of a direct sum of 2^4 vector bundles of rank 4.

Now we prove the proposition:

Proof of the Proposition: The main point is to write the sheaf $\varphi_{L*}\varphi_{2L}^* \widehat{2L}^\vee$ in two different ways. For one part, by (1.4) we have

$$\varphi_{2L}^* \widehat{2L} \simeq H^0(2L) \otimes L^{-\otimes 2}.$$

Applying φ_{L*} we get

$$\varphi_{L*}\varphi_{2L}^* \widehat{2L}^\vee \simeq H^g(2L^\vee) \otimes \varphi_{L*}(L^{\otimes 2}).$$

On the other hand, by projection formula we have:

$$\begin{aligned} \varphi_{L*}\varphi_{2L}^* \widehat{2L}^\vee &\simeq 2_{\hat{A}}^* \widehat{2L}^\vee \otimes \varphi_{L*}\mathcal{O}_A \\ &\simeq \bigoplus_{a \in K(L)} 2^* \widehat{2L}^\vee \otimes P_a. \end{aligned}$$

Now, we claim that $2^*_{\hat{A}}\widehat{2L}^\vee \otimes P_a \simeq 2^*\widehat{2L}^\vee$. To see this, we note that for any b with $2b = a$ we have

$$\begin{aligned} 2^*_{\hat{A}}\widehat{2L}^\vee \otimes P_a &\simeq \Phi_{\mathcal{P}}\left(t^*_{-a}2_{A*}L^{-2}\right)[g] \\ &\simeq \Phi_{\mathcal{P}}\left(2_{A*}t^*_{-b}L^{-2}\right)[g] \\ &\simeq 2^*\widehat{t^*_{-b}(2L)}^\vee \simeq 2^*\widehat{2L}^\vee, \end{aligned}$$

where we used (1.3) (and the fact that $2L$ is symmetric) and the cartesian diagram

$$\begin{array}{ccc} A & \xrightarrow{t_{-b}} & A \\ {\scriptstyle t_{-a}}\downarrow & & \downarrow{\scriptstyle 2_A} \\ A & \xrightarrow[2_A]{} & A \end{array} .$$

Summarizing, we have:

$$\left(2^*_{\hat{A}}\widehat{2L}^\vee\right)^{\oplus\chi(L)^2} \simeq \varphi_L\varphi^*_{2L}\widehat{2L}^\vee \simeq \varphi_{L*}(2L)^{\oplus 2^g\chi(L)}. \tag{3.27}$$

Now we use Mukai's semi-homogeneous vector bundles to decompose the left side of the above equation. More precisely, we have that

$$2^*_{\hat{A}}\widehat{2L}^\vee \simeq \Phi_{\mathcal{P}}(2_{A*}(L^{\otimes 2}))^\vee, \tag{3.28}$$

and thus we need to compute $2_{A*}(L^{\otimes 2})$. We have

$$\bigoplus_{\alpha\in\hat{A}[2]} W \otimes P_\alpha \simeq 2_{A*}2^*_A W \simeq (2_{A*}L^{\otimes 2})^{\oplus \operatorname{rk} W}.$$

Now, as W is simple, it is in particular indecomposable and hence by [Ati56, Theorem 3] it follows that $2_{A*}L^{\otimes 2}$ is the direct sum of vector bundles $W \otimes P_\alpha$ for $\alpha \in \hat{A}[2]$ and, moreover, all the possible isomorphism classes $W \otimes P_\alpha$ with $\alpha \in \hat{A}[2]$ must appear in such decomposition (and with the same "multiplicity"), that is:

$$2_{A*}(L^{\otimes 2}) \simeq \bigoplus_{\alpha\in\frac{\hat{A}[2]}{\Sigma(W)}} (W \otimes P_\alpha)^{\oplus n},$$

for some integer n. Now, n must satisfy the equation

$$2^{2g} = \frac{2^{2g}}{\operatorname{rk}(W)^2} \cdot \operatorname{rk}(W) \cdot n,$$

and thus $n = \operatorname{rk}(W)$. Substituing in (3.28) and (3.27) we get

$$\bigoplus_{\alpha\in\frac{\hat{A}[2]}{\Sigma(W)}} (\Phi_{\mathcal{P}}(W \otimes P_\alpha)^\vee)^{\oplus \operatorname{rk}(W)\cdot\chi(L)^2} \simeq \varphi_{L*}(L^{\otimes 2})^{\oplus 2^g\chi(L)}.$$

Again by [Ati56] it follows that

$$\varphi_{L*}(L^{\otimes 2}) \simeq \bigoplus_{\alpha \in \frac{\hat{A}[2]}{\Sigma(W)}} \left(\Phi_{\mathcal{P}}(W \otimes P_\alpha)^\vee\right)^{\oplus m},$$

where

$$m = \frac{\mathrm{rk}(W) \cdot \chi(L)^2}{2^g \chi(L)} = \frac{\chi(L)}{\sqrt{|K(L)[2]|}} = \chi(W),$$

as we wanted to see.

□

Combining with (3.25) we obtain:

Corollary 3.31. *Let L be a symmetric and ample line bundle on an abelian variety A. Let $\ell \in \mathrm{NS}(A)$ the class of L and fix a simple vector bundle $W \in \mathbb{S}_{\frac{1}{2}\ell}$ with $2_A^* W \simeq (L^{\otimes 2})^{\oplus r}$. Then we have:*

$$H^1(M_L \otimes L) \simeq \bigoplus_{\alpha \in \hat{A}[2]/\Sigma(W)} H^1\left(I_0 \otimes W \otimes P_\alpha\right).$$

In particular, if L is globally generated then it is projectively normal if and only if

$$\{\alpha \in \hat{A}[2] : W \otimes P_\alpha \textit{ not globally generated in } 0\} = \emptyset.$$

Proof: First note that, as L is symmetric it follows that $L^{\otimes 2}$ is also symmetric and thus we are able to apply the previous proposition. From (3.25) and (1.3) it follows then that

$$H^1(M_L \otimes L) \simeq \bigoplus H^1(I_0 \otimes L \otimes (-1_A)^* W^\vee \otimes P_\alpha),$$

where α and the number of copies are as in the proposition. It remains to show that $L \otimes (-1_A)^* W^\vee \simeq W$. To do this we note that $L \otimes (-1_A)^* W^\vee$ is a simple vector bundle in $\mathbb{S}_{\frac{1}{2}\ell}$ and thus $L \otimes W^\vee \simeq W \otimes P_\beta$ for some $\beta \in \hat{A}$. Applying 2_A^* and the fact that L is symmetric (thus $2_A^* L \simeq L^{\otimes 4}$) we conclude that $\beta \in \hat{A}[2]$.

The final part follows since $W \otimes P_\alpha$ is IT(0) and thus $W \otimes P_\alpha$ fails to be globally generated at 0 if and only if $H^1(I_0 \otimes W \otimes P_\alpha) \neq 0$.

□

For instance, if $L = N^{\otimes 2}$ for a symmetric N, then in the above setting we have $W = N$ and we get Ohbuchi's theorem ([Ohb88, Theorem]), that is, L is normally generated if and only if

$$0 \notin \bigcup_{\alpha \in \hat{A}[2]} \mathrm{Bs}(N \otimes P_\alpha).$$

A new example regards the cases $d_1 d_g = 4$ and $d_1 d_g = 6$, which we study a bit more in detail in the remaining of this chapter. More precisely, in Corollary 3.33 below we will prove that in these cases the jump-locus

$$V^1(M_L, L) = \{\alpha \in \hat{A} : H^1(M_L \otimes L) \neq 0\}$$

contains points of order $d_1 d_g$. To do so we will need the following statement regarding such jump-locus:

Lemma 3.32. *Let L be an ample and symmetric line bundle (of any type) on an abelian variety A. Suppose that there exists $\beta \in \widehat{A}$ such that*

$$H^1(2_A^* I_0 \otimes L^2 \otimes P_\beta) \neq 0.$$

Then $\beta \in V^1(M_L, L)$.

Proof: We have the following sequence of isomorphisms:

$$\begin{aligned} H^1(2_A^* I_0 \otimes L^2 \otimes P_\beta) &\simeq \mathrm{Ext}^1(2^* I_0^\vee, L^2 \otimes P_\beta) \simeq \mathrm{Ext}^1(2_A^*(I_0^\vee \otimes L^\vee), L^{-2} \otimes P_\beta) \\ &\simeq \mathrm{Ext}^1(I_0^\vee \otimes L^\vee, 2_{A*}(L^{-2} \otimes P_\beta)) \\ &\simeq \mathrm{Ext}^1(\Phi_{\mathcal{P}}(I_0^\vee \otimes L^\vee), 2_{\hat{A}}^* \Phi_{\mathcal{P}}(L^{-2} \otimes P_\beta)) \end{aligned}$$

now, the latter group is a direct summand of

$$\begin{aligned} \mathrm{Ext}^1(\varphi_L^* \Phi_{\mathcal{P}}(I_0^\vee \otimes L^\vee), \varphi_{2L}^* \Phi_{\mathcal{P}}(L^{-2} \otimes P_\beta)) &\simeq \mathrm{Ext}^1(M_L^\vee \otimes L, L^2 \otimes P_\beta)^{\oplus h^0(L^2 \otimes P_\beta)} \\ &\simeq H^1(M_L \otimes L \otimes P_\beta)^{\oplus h^0(L^2)}, \end{aligned}$$

concluding the proof.

□

Corollary 3.33. *Let L be a symmetric and ample line bundle of type $(d_1, ..., d_g)$. If $d_1 d_g = 4$ then*

$$V^1(M_L, L) \cap \hat{A}[4] \neq \emptyset.$$

Proof: Consider a symmetric representant L_δ of ℓ_δ. From Lemma 3.25 we have that the multiplication map $H^0(L_\delta)^{\otimes 2} \to H^0(L_\delta^{\otimes 2})$ is not injective and thus $H^0(M_{L_\delta} \otimes L_\delta) \neq 0$. Now, fix a simple vector bundle $E \in \mathbb{S}_{\hat{A}, \frac{1}{2}\ell_\delta}$ with $2_{\hat{A}}^* E \simeq (L_\delta^2)^{\oplus \operatorname{rk} E}$ (which exist by Remark 3.5). From Corollary 3.31 we get then that

$$H^0(I_{\hat{0}} \otimes E \otimes P_a) \neq 0 \quad \text{for some } a \in A[2].$$

Now, by Serre-duality, (1.7) and Lemma 3.11, the above non-vanishing can alternatively be described as

$$H^1(I_0 \otimes \Phi_{\mathcal{P}}(E \otimes P_a)^\vee) \neq 0 \quad \text{for some } a \in A[2].$$

The idea is to apply Lemma 3.32 above, so we just need to establish that

$$2_A^* \Phi_{\mathcal{P}}(E \otimes P_a)^\vee \simeq (L^2 \otimes P_\alpha)^{\oplus \chi(E)} \quad \text{for some } \alpha \in \hat{A}[4]. \tag{3.29}$$

To do this we first claim that $E \simeq E_0 \otimes P_b$ with $b \in A[4]$ and $E_0 \in \mathbb{S}_{\hat{A}, \frac{1}{2}\ell_\delta}$ simple and symmetric. Indeed: we have that $(-1_A)^* E \in \mathbb{S}_{\frac{1}{2}\ell_\delta}$ is simple and hence

$$(-1_A)^* E \simeq E \otimes P_c \ \text{ for some } c \in A.$$

Applying $2_{\hat{A}}^*$ we get that $c \in A[2]$. It follows that for $b \in A$ with $2b = c$ we have that $E_0 := E \otimes P_{-b} \in \mathbb{S}_{\lambda\ell}$ is simple and symmetric, as we claimed. We get then that

$F_0 := E_0 \otimes P_a$ is a simple and symmetric vector bundle in $\mathbb{S}_{\hat{A},\frac{1}{2}\ell_\delta}$. In this setting, writting $y = a + b \in A[4]$, the left side of (3.29) above can be developed as:

$$\begin{aligned} 2_A^* \Phi_{\mathcal{P}}(F_0 \otimes P_y)^\vee &\simeq 2_A^* t_y^* \Phi_{\mathcal{P}}(F_0)^\vee \\ &\simeq t_{y/2}^* 2_A^* \Phi_{\mathcal{P}}(F_0)^\vee \\ &\simeq t_{y/2}^* (M^{\otimes 2})^{\oplus \chi(E)}, \end{aligned}$$

where in the last line M is line bundle algebraically equivalent to L (and the isomorphism comes from Proposition 3.4). Now, as F_0 is symmetric it follows that $M^{\otimes 2}$ is also symmetric and hence $M^{\otimes 2} \simeq L^{\otimes 2} \otimes P_\beta$ for some $\beta \in \hat{A}[2]$. Finally:

$$t_{y/2}^*(L^{\otimes 2} \otimes P_\beta) \simeq L^{\otimes 2} \otimes P_{\beta + \varphi_L(y)}$$

and we get the desired isomorphism (3.29) for $\alpha = \beta + \varphi_L(y)$.

□

A similar proof shows that if L has type $(1, ..., 2)$ then $0 \in \mathrm{Bs}(L \otimes P_\alpha)$ for some $\alpha \in \hat{A}[2]$. It follows from Corollary 3.31 that polarizations of type $(2, ..., 4)$ are not normally generated, an already known fact ([Rub98]).

With a bit more of work we can study the case from Example 3.28:

Corollary 3.34. *If L has type $(1, 3, ..., 3, 6)$ then $V^1(M_L, L) \cap \hat{A}[6] \neq \emptyset$.*

Proof: Let ℓ be a polarization as in the statement. Let L_δ be a symmetric representant of ℓ_δ and fix a simple vector bundle (necessarily of rank two, see Example 3.28) $E \in \mathbb{S}_{\frac{1}{2}\ell_\delta}$ with $2_{\hat{A}}^* E \simeq (L_\delta^{\otimes 2})^{\oplus 2}$. It follows that $\det E \simeq L_\delta \otimes P_a$ for some $a \in A[2]$. Now, from Lemma 3.25 we have that the multiplication map $H^0(E) \otimes H^0(\det E) \to H^0(E \otimes \det E)$ is not injective and thus

$$H^0(\mathrm{EV}^\bullet \otimes L_\delta \otimes P_a) \neq 0 \quad \text{for some } a \in A[2]$$

(see (3.16) for the notation) and thus $H^0(2_{\hat{A}}^* \mathrm{EV}^\bullet \otimes L_\delta^4) \neq 0$. On the other hand, both $\varphi_{L_\delta}^* L$ and $L_\delta^{\otimes 6}$ are symmetric algebraically equivalent line bundles and thus they differ by an element $x \in A[2]$. Write $x = \varphi_{L_\delta}(\alpha)$. We have then the following isomorphisms:

$$\begin{aligned} 0 \neq H^0(2_{\hat{A}}^* \mathrm{EV}^\bullet \otimes L_\delta^4) &\simeq H^0(2_{\hat{A}}^* \mathrm{EV}^\bullet \otimes L_\delta^{-2} \otimes L_\delta^6) \\ &\simeq H^0\left(\varphi_{L_\delta}^*(\Phi_{\mathcal{P}}(I_{\hat{0}} \otimes E) \otimes L \otimes P_\alpha)\right) \\ &\simeq \bigoplus_{\beta \in K(L_\delta)} H^0\left(\Phi_{\mathcal{P}}(I_{\hat{0}} \otimes E) \otimes L \otimes P_{\alpha+\beta}\right) \\ &\simeq \bigoplus_{\beta \in K(L_\delta)} H^0\left(I_{\hat{0}} \otimes E \otimes \Phi_{\mathcal{P}}(L \otimes P_{\alpha+\beta})\right). \end{aligned}$$

Now, we note that $E \otimes \Phi_{\mathcal{P}}(L \otimes P_{\alpha+\beta})$ is a (not simple) element of $\mathbb{S}_{\hat{A},\frac{1}{3}\ell_\delta}$, which is a direct summand of the following sheaf:

$$\begin{aligned} 6_{\hat{A}*} 6_{\hat{A}}^* \left(E \otimes \Phi_{\mathcal{P}}(L \otimes P_{\alpha+\beta})\right) &\simeq 6_{\hat{A}*}\left(3_{\hat{A}}^* 2_{\hat{A}}^* E \otimes \varphi_{L_\delta}^* \varphi_L^* \Phi_{\mathcal{P}}(L \otimes P_{\alpha+\beta})\right) \\ &\simeq 6_{\hat{A}*}\left(L_\delta^{18} \otimes \varphi_{L_\delta}^*(L^\vee \otimes P_{-\alpha-\beta})\right)^{\oplus N} \\ &\simeq 6_{\hat{A}*}\left(L_\delta^{18} \otimes L_\delta^{-6} \otimes P_{-x} \otimes P_{-\varphi_{L_\delta}(\alpha+\beta)}\right) \simeq 6_{\hat{A}*}(L_\delta^{12})^{\oplus N} \end{aligned}$$

where the value of N does not matter for this argument and the last line is justified since $\beta \in K(L_\delta)$ and $\varphi_{L_\delta}(\alpha) = x \in A[2]$. Now, continuing with the above computation, since $\varphi_{L_\delta}^* L \simeq L_\delta^6 \otimes P_x$ with $x \in A[2]$, using the definition of the dual polarization we have:

$$\begin{aligned} 6_{\hat{A}*}(L_\delta^{12})^{\oplus N} &\simeq \varphi_{L*}\varphi_{L_\delta *}\varphi_{L_\delta}^*(L^2)^{\oplus N} \\ &\simeq \bigoplus_{\gamma \in K(L_\delta)} (\varphi_{L*}(L^{\otimes 2}) \otimes P_{c_\gamma})^{\oplus N} \end{aligned}$$

where in the last line $c_\gamma \in A$ is a (any) preimage of γ via φ_L. Finally, from Proposition 3.29 we conclude that

$$6_{\hat{A}*}(L_\delta^{12})^{\oplus N} \simeq \bigoplus_{\gamma \in K(L_\delta)} \bigoplus_{\nu \in \hat{A}[2]} \left(\Phi_{\mathcal{P}}(W \otimes P_\nu)^\vee \otimes P_{c_\gamma}\right)^{\oplus N},$$

where $W \in \mathbb{S}_{A,\frac{1}{2}\ell}$ with $2_A^* W \simeq (L^{\otimes 2})^{\oplus \operatorname{rk} W}$. It follows then there exists $\nu \in \hat{A}[2]$ and $\gamma \in K(L_\delta)$ such that

$$\begin{aligned} 0 \neq H^0(I_{\hat{0}} \otimes \Phi_{\mathcal{P}}(W \otimes P_\nu)^\vee \otimes P_{c_\gamma}) &\simeq H^0(I_{\hat{0}} \otimes \Phi_{\mathcal{P}}(t_c^* W \otimes P_\nu)^\vee) \\ &\simeq H^1(I_0 \otimes t_c^* W \otimes P_\nu) \end{aligned}$$

which implies that

$$\begin{aligned} 0 \neq H^1(2_A^* I_0 \otimes t_{c/2}^* 2_A^* W) &\simeq H^1(2_A^* I_0 \otimes t_{c/2}^*(L^{\otimes 2}))^{\oplus \operatorname{rk} W} \\ &\simeq H^1(I_0 \otimes L^{\otimes 2} \otimes P_\gamma). \end{aligned}$$

Finally, as $\gamma \in K(L_\delta)$ it follows that $\gamma \in \hat{A}[6]$ and hence the result follows from Lemma 3.32 above.

□

Chapter 4

Generation of some twisted ideals

In this chapter we report on a work in progress with G. Pareschi which propose a generic vanishing approach to characterize hyperelliptic jacobians and, in particular, to study van Geemen-van der Geer's conjecture. We start by reviewing the statement of such conjecture and its relation with a conjecture regarding the Seshadri constant of the theta divisor. Afterwards, we review the notion of *generation* of sheaves and see how the generation of some special twisted ideals is related to these conjectures.

4.1 The Γ_{00}-conjecture

Let (A, θ) be a principally polarized abelian variety and Θ a symmetric divisor representing θ. The Van Geemen-van der Geer conjecture, also known as Γ_{00}-conjecture, concerns the base locus of the linear system $|2\Theta|_{0,4} := \left|I_0^4(2\Theta)\right|$ (which in [Gru10] and [BD89] is denoted by Γ_{00}). More precisely, based in the work of Welters ([Wel86]), who proved that when $A = \operatorname{Jac} C$ such base locus is the surface $C - C$, in [vGvdG86] van Geemen and van der Geer proposed the following [1]:

Conjecture 4.1. *Let (A, θ) be an i.p.p.a.v. Then*

$$(A, \theta) \text{ is a polarized jacobian if and only if } \operatorname{Bs}(|2\Theta|_{0,4}) \neq \{0\}.$$

On the other hand, Debarre also noted that it is useful to consider the *continuous* linear system:

$$\{2\Theta\}_{0,4} := \bigcup_{\alpha \in \hat{A}} \mathbb{P}H^0(I_0^4(2\Theta) \otimes P_\alpha) = \{D \equiv_{num} 2\Theta : \operatorname{mult}_0 D \geq 4\}.$$

Indeed, in [Deb04] is proved that the geometry of the base locus of this continuous linear system influences the size of the Seshadri constant. More precisely, in such reference the following is shown:

[1] In [BD89] it is explained how this conjecture relates with other approaches to characterize jacobians such as the trisecant conjecture

Proposition 4.2 (Lemma 1 in [Deb04])**.** *Let Y be a subvariety of A. If either:*

- *Y is a curve and*
$$\frac{(\theta \cdot Y)}{\mathrm{mult}_0 Y} \;<\; 2$$
or
- $\dim Y \geq 2$ *and Y*
$$\left(\frac{(\theta^r \cdot Y)}{\mathrm{mult}_0 Y}\right)^{1/r} \;=\; \varepsilon(\theta) \leq 2 \tag{4.1}$$

then $Y \subseteq \mathrm{Bs}(\{2\Theta\}_{0,4})$.

The relation with the Seshadri constant is as follows: by definition of Seshadri constant, writing $\sigma : \widetilde{A} = \mathrm{Bl}_0 A \to A$ for the blow-up map and E for the exceptional divisor, we have that the $\mathbb{R}$-divisor $D = \sigma^*\theta - \varepsilon(\theta) \cdot E$ is nef but not ample. By Camapana-Peternell' theorem (Theorem 2.3.18 in [Laz04a]) it follows that there exists a positive-dimensional subvariety $\widetilde{Y} \subset \widetilde{A}$ such that $(D \cdot \widetilde{Y}) = 0$ or, equivalently, a subvariety $Y \subset A$ satisfying the equality in (4.1). This means that if $\mathrm{Bs}(\{2\Theta\}_{0,4})$ is zero-dimensional then $\varepsilon(\theta) \geq 2$. Combining with Conjecture 4.1 the following weaker conjecture arise:

Conjecture 4.3 ((7) in [Deb04], Remark 5.3.13 [Laz04a])**.**

Let (A, θ) be an i.p.p.a.v. If $\varepsilon(\theta) < 2$ then (A, θ) is a polarized jacobian.

Moreover, in Theorem 7 of *loc.cit* Debarre also computed the Seshadri constant of the theta divisor of a jacobian, proving that for $g \geq 4$ we have that $\varepsilon(\theta_C) < 2$ just in the hyperelliptic case. Summarizing, we have the following:

Conjecture 4.4.
Let (A, θ) be an i.p.p.a.v of dimension $g \geq 4$.

If $\varepsilon(\theta) < 2$ then (A, θ) is the polarized jacobian of an hyperelliptic curve.

In the following section we review the notion of *generated sheaves* and in the subsequent one we study the generation of the sheaf $I_0^4(2\Theta)$, which we will see is intimately related to the base locus of the continous linear system $\{2\Theta\}_{0,4}$ and hence to the Seshadri constant of Θ.

4.2 Generated sheaves and the Fourier transform

In this section, as usual, for an object $\mathcal{F} \in D^b(A)$ we write $\mathcal{F}^\vee$ for the *derived* dual $R\mathscr{H}om(\mathcal{F}, \mathcal{O}_A)$.

Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A. For an open subset U of $\widehat{A} = \mathrm{Pic}^0(A)$ and a point $x \in A$ we can consider the *continuous evaluation map*

$$\mathrm{ev}_U(x) : \bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes P_\alpha) \otimes P_\alpha^\vee \longrightarrow \mathcal{F} \otimes k(x).$$

In [Par23] the cited author introduce the following notion:

Definition 4.5. Let $\mathcal{F}$ and A as above. Let $\mathcal{Z} = \{Z_1, ..., Z_r\}$ be a finite collection of irreducible subvarieties of $\widehat{A}$.

1. We say that $\mathcal{F}$ is **generated** by $\mathcal{Z}$ at a point $x \in A$ if the evaluation map $\mathrm{ev}_U(x)$ is surjective whenever $U \cap Z_i \neq \emptyset$ for $i = 1, ..., r$.

2. We say that $\mathcal{F}$ is generated by $\mathcal{Z}$ if it is generated at x for all x

Example 4.6. *A globally generated sheaf is generated by* $\{\widehat{0}\}$.

A sheaf generated by the whole $\widehat{A}$ *(i.e when we can take* $\mathcal{Z} = \{\widehat{A}\}$*) is said to be* continuously globally generated *(CGG). In [Deb06, Corollary 3.2] it is shown that a CGG sheaf is ample (in the sense of [Kub70]). On the other hand, in [PP03, Proposition 2.13] it is shown that an M-regular sheaf is CGG.*

In order to work with this definition, a criterion for the surjectivity of the maps $\mathrm{ev}_U(x)$ is needed. Such a criterion is provided by [Par23, Corollary 3.2.1]. To state such a result we need to recall the definition of the *edge map* of a bounded spectral sequence.

Definition / Construction 4.7. *Let* $E_2^{p,q} \implies E^{p+q}$ *be a first-quadrant spectral sequence. For a positive integer* r *the edge map*

$$\mathrm{ed}^r : E^r \longrightarrow E_2^{0,r}$$

is the map constructed as follows.

First, as $E_2^{p,r} = 0$ *for* $p < 0$ *we have that the stable value* $E_\infty^{0,r}$ *injects into* $E_2^{0,r}$.

On the other hand, by definition of spectral sequence, there is a decreasing filtration $\{F^p E^r\}_{p \in \mathbb{Z}}$ *of* E^r *such that*

a) $\bigcup F^p E^r = E^r$ *and* $\bigcap E^r = 0$

b) $F^p E^r / F^{p+1} E^r \simeq E_\infty^{p, r-p}$.

In particular, it follows that $E_\infty^{0,r} = F^0 E^r / F^1 E^r$. *Now, as* $E_\infty^{p,r} = 0$ *for* $p < 0$ *(since* $E_2^{p,r}$ *is already zero in those cases), from a) and b) it easily follows that*

$$F^0 E^r = F^{-1} E^r = \cdots = F^p E^r = \cdots = E^r \quad \forall p < 0$$

and hence we have a quotient map

$$E^r \twoheadrightarrow E_\infty^{0,r}.$$

The edge map is then the composition

$$\mathrm{ed}^r : E^r \twoheadrightarrow E_\infty^{0,r} = \frac{E^r}{F^1 E^r} \hookrightarrow E_2^{0,r}.$$

Moreover, the following easy observation directly follows from the above construction:

Remark 4.8. *Let* $E_2^{p,q} \implies E^{p+q}$ *be a first quadrant spectral sequence. Then the edge map* ed^r *is injective if and only if* $F^1 E^r = 0$.

In particular, we may define an edge map for the Leray spectral sequence associated to the Fourier-Mukai functor. Concretely, for $x \in A$ and a coherent sheaf $\mathcal{F} \in \mathrm{Coh}(A)$ we have the first-quadrant spectral sequence

$$E_2^{p,q} = H^p(R^q\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) \otimes P_x) \implies H^{p+q}(\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) \otimes P_x)$$

and hence we obtain an edge map

$$\mathrm{ed}_x : H^g(\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) \otimes P_x) \longrightarrow H^0(R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) \otimes P_x).$$

With these language we can state the announced criterion for the surjectivity of $\mathrm{ev}_U(x)$:

Proposition 4.9 (Corollary 3.2.1 in [Par23])**.** *Let $\mathcal{F} \in \mathrm{Coh}(A)$ and $x \in A$ as above. For a non-empty open set $U \subset \widehat{A}$ the evaluation map $\mathrm{ev}_U(x)$ is surjective if and only if the following two conditions hold:*

a) the edge map ed_x is injective

b) the simultaneous evaluation map

$$\mathrm{ev}_x(U) : H^0(R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) \otimes P_x) \longrightarrow \prod_{\alpha \in U} R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee) \otimes P_x \otimes k(\alpha)$$

restricted to the image of ed_x is injective (here, as usual, $k(\alpha)$ denotes the skycraper sheaf supported on the point $\alpha \in \widehat{A}$).

First, note that condition a) above does not depend on the open set U. In particular, this condition holds once we know that $\mathrm{ev}_U(x)$ is surjective for *some* open set U.

Next, note that the failure of condition b) means that there exist a non-zero section which vanishes at all fibers of an open subset of $\hat{A}$, which is, a priori, a weird condition. Note however that there are two very natural situations where this can happen:

- Consider a coherent sheaf $\mathcal{F}$ and a global section $s : \mathcal{O}_{\hat{A}} \to \mathcal{F}$ and let $\mathrm{Zeroes}(s)$ be the closure of the set $\{\alpha \in \hat{A} : s \otimes k(\alpha) = 0\}$. If we take the open set $U = \hat{A} \backslash \mathrm{Zeroes}(s)$ and assume that this is not empty, then we tautologically see that $s \in \ker \mathrm{ev}_x(U)$.
- For a non-reduced scheme it is perfectly possible for a non-zero section to vanish at all fibers. For instance, for $X = \mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)$ we have that the non-zero section of $\mathcal{O}_X$ given by $\cdot\varepsilon : k[\varepsilon]/(\varepsilon^2) \to k[\varepsilon]/(\varepsilon^2)$ vanish at every (i.e the unique) fiber of X.

Moreover, in Corollary 4.12 below [2] we shall see that these are esentially the only two cases where b) may fail. Before stating such corollary, we recall the following:

Definition 4.10 (Definition 1.1.4 in [HL10])**.** Let $\mathcal{F}$ be a coherent sheaf on a noetherian scheme X.

[2] whose proof, already present in [Par23], is an excercise in scheme theory which we include just to highlight how the non-reducedness arise

1. The *torsion filtration* of $\mathcal{F}$ is the filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_d = \mathcal{F},$$

   where $\mathcal{F}_k$ is the maximal subsheaf of $\mathcal{F}$ whose support has dimension at most k and $d = \dim \operatorname{Supp} \mathcal{F}$. Each $\mathcal{F}_k$ will be called a *torsion component* of $\mathcal{F}$.

2. The support of $\mathcal{F}$, denoted $\operatorname{Supp}(\mathcal{F})$, is the closed set $\{x \in X : \mathcal{F}_x \neq 0\}$ endowed with the scheme structure determined by the ideal

$$\operatorname{Ann}(\mathcal{F}) := \ker \left[\mathcal{O}_X \to \mathscr{H}om(\mathcal{F}, \mathcal{F})\right].$$

Remark 4.11. *It is worth to point out that the noetherianity of X ensures the existence of such a filtration. Also, there is no problem to have $\mathcal{F} = \mathcal{F}_{k+1}$ for some k. For example, for a line bundle L and a closed subscheme Z we have that the torsion filtration of $L \oplus \mathcal{O}_Z$ is given by*

$$\mathcal{F}_k = \begin{cases} 0 & \text{if } k < \dim Z \\ \mathcal{O}_Z & \text{if } \dim Z \leq k < \dim X \\ L \oplus \mathcal{O}_Z & \text{if } k = \dim X \end{cases}.$$

We then have:

Corollary 4.12 (Corollary 3.2.4 in [Par23])**.** *Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A. Suppose that*

i) The edge map ed_x is injective

ii) All torsion components $\mathcal{T}_k$ of $R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)$ have reduced support.

Then $\mathcal{F}$ is generated at x by the set of irreducible components of the support of the sheaves $\mathcal{T}_k$.

Proof: The only thing to prove is that (ii) ensures condition b) from Proposition 4.9 above for all open sets intersecting the support of the sheaves $\mathcal{T}_k$. To do this, suppose that we have a non-zero section $s \in \ker \mathrm{ev}_x(U)$, that is $s \otimes k(\alpha) = 0$ for all $\alpha \in U$. Let $s : \mathcal{O}_{\hat{A}} \to \widehat{\mathcal{F}}$ be the corresponding morphism where, for this proof, we write $\widehat{\mathcal{F}}$ for $R^g\Phi_{\mathcal{P}^\vee}(\mathcal{F}^\vee)$.

First, we claim that the image sheaf $\mathcal{I}m(s)$ is a torsion subsheaf of $\widehat{\mathcal{F}}$. To see this, let $\eta \in \hat{A}$ be the generic point. As U is open we have that $\eta \in U$ and hence $s \otimes k(\eta) = 0 \in \widehat{\mathcal{F}} \otimes k(\eta) = \widehat{\mathcal{F}}_\eta$, where in the last equality we use the fact that $\hat{A}$ is reduced [3]. This means that the composition

$$\mathcal{O}_{\hat{A},\eta} \twoheadrightarrow \mathcal{I}m(s)_\eta \hookrightarrow \widehat{\mathcal{F}}_\eta$$

is zero and hence $\mathcal{I}m(s)_\eta = 0$. It follows that the sheaf $\mathcal{I}m(s)$ is generically zero and therefore of torsion.

Being a torsion sheaf, it is contained in $\mathcal{T}_d$ for $d = \dim \operatorname{Supp} \mathcal{I}m(s)$. By the definition of the torsion filtration it follows that each irreducible component of $\operatorname{Supp} \mathcal{I}m(s)$ is an irreducible component of $\mathcal{T}_d$.

[3] If B is a reduced noetherian ring and $\mathfrak{p}$ is a minimal prime ideal then $\mathfrak{p}B_{\mathfrak{p}} = 0$

We claim that if $S := \operatorname{Supp} \mathcal{I}m(s)$ is reduced, then the fact that $s \neq 0$ belongs to $\ker \mathrm{ev}_x(U)$ implies that U is contained in the complement of such support. To prove the claim it is enough to show that the set $V := \{y \in \hat{A} : s \otimes k(y) \neq 0\}$ is not empty. To do this note that s factors through a morphism $t : \mathcal{O}_S \to \mathcal{I}m(s)$ of $\mathcal{O}_S$-modules. Moreover, $\mathcal{I}m(s)$ is torsion free as $\mathcal{O}_S$-module and if $s \otimes k(x) = 0$ for all x then t has the same property. We conclude that we just need to prove the following statement:

Let X be a noetherian and reduced scheme. Let t be a section of a coherent and torsion free sheaf $\mathcal{G}$ on X. Suppose that $t \otimes k(x) = 0$ for all $x \in X$. Then $t = 0$.

Now, this statement follows easily since $\mathcal{G}$ is torsion free and hence t is either zero or injective, but, as in the second paragraph, the hypothesis implies that t is generically zero and thus the latter can not hold.

□

4.3 Generation of $I_0^4(2\Theta)$

This section is devoted to the proof of the following result:

Theorem 4.13. *Let (A, θ) be a principally polarized abelian variety and Θ a symmetric line bundle representing θ. Let Q be the cokernel of the natural morphism (see Chapter 2, (2.8))*

$$\phi : P^3 \left(\mathcal{O}_A(2\Theta)\right)^\vee \to H^0 \left(\mathcal{O}_A(2\Theta)\right)^\vee \otimes \mathcal{O}_A.$$

Then, if $\varepsilon(\theta) < 2$ then there exist a torsion component of Q (Definition 4.10) with non-reduced support.

The skeleton of the proof has the following steps:

1. In Lemma 4.14 below we compute the Fourier transform of $I_0^4(2\Theta)^\vee$, proving that

$$\varphi_{2\theta}^* R^g \Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) = Q \otimes \mathcal{O}_A(2\Theta)$$

   while

$$\varphi_{2\theta}^* R^{g-1} \Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) = K \otimes \mathcal{O}_A(2\Theta),$$

   where $K = \ker \phi$.

2. In Lemma 4.15 we prove that if all the irreducible components of the supports of the torsion components of Q are reduced then for $x \neq 0$ the fact that x belongs to $\mathrm{Bs}\{2\Theta\}_{0,4}$ is detected by the non-injectivity of the edge map

$$\mathrm{ed}_x : H^g(\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x) \to H^0(R^g \Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x),$$

   and this turns out to be equivalent to the non-vanishing of

$$H^1(R^{g-1} \Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x).$$

3. From the previous point we deduce that if all the components of Q are reduced then we have the following contentions:

$$\mathrm{Bs}(\{2\Theta\}_{0,4}) \setminus A[2] \subset \varphi_{2\theta}^{-1}(V^1(K \otimes \mathcal{O}_A(2\Theta)) \setminus \{\hat{0}\}) \subset \mathrm{Bs}(\{2\Theta\}_{0,4}) + A[2].$$

   In particular, we have that $V^1(K \otimes \mathcal{O}_A(2\Theta)) \neq \hat{A}$.

4. Finally, in Lemma 4.16 we compute the cohomology of $K \otimes \mathcal{O}_A(2\Theta) \otimes P_\alpha$, proving that for $i \neq 1$ we have $V^i(K \otimes \mathcal{O}_A(2\Theta)) \subseteq \{\hat{0}\}$. Using the previous step we deduce that $K \otimes \mathcal{O}_A(2\Theta)$ is a GV sheaf. From Hacon's lemma (Lemma 1.7 in [Par12]) we deduce that $V^1(K \otimes \mathcal{O}_A(2\Theta)) \subseteq V^0(K \otimes \mathcal{O}_A(2\Theta)) = \{\hat{0}\}$.

5. The previous point implies then that

$$\mathrm{Bs}(\{2\Theta\}_{0,4}) \subset \ker \varphi_{2\theta} = A[2].$$

   and the result then follows from Proposition 4.2

We start by the first step:

Lemma 4.14. *We have that $R^k\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) = 0$ for $k \leq g-2$ while*

$$\varphi_{2\theta}^* R^g \Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) = Q \otimes \mathcal{O}_A(2\Theta)$$

and

$$\varphi_{2\theta}^* R^{g-1} \Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) = K \otimes \mathcal{O}_A(2\Theta),$$

where $K = \ker \phi$.

Proof: First, by Grothendieck-Verdier duality (1.3) and the fact that Θ is symmetric we have

$$\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) = \Phi_{\mathcal{P}}(I_0^4(2\Theta))^\vee[-g]. \tag{$*$}$$

By Lemma 2.20 we have that

$$\varphi_{2\theta}^* \Phi_{\mathcal{P}}(I_0^4(2\Theta)) = \mathrm{EV}_3^\bullet \otimes \mathcal{O}_A(-2\Theta),$$

where

$$\mathrm{EV}_3^\bullet = \left[H^0(2\Theta) \otimes \mathcal{O}_A \to P^3(2\Theta)\right].$$

To compute the dual, we use then the spectral sequence (Remark 2.67 and Example 2.70(ii) in [Huy06]):

$$E_1^{p,q} = \mathscr{E}xt^q(\mathrm{EV}_3^{-p}, \mathcal{O}_A(2\Theta)) \implies \mathscr{E}xt^{p+q}(\Phi_{\mathcal{P}}(I_0^4(2\Theta)), \mathcal{O}_{\hat{A}}).$$

As EV_3^{-q} is locally free for all q, the above spectral sequence gives

$$\varphi_{2\theta}^* \mathscr{E}xt^0(\Phi_{\mathcal{P}}(I_0^4(2\Theta)), \mathcal{O}_{\hat{A}}) = \operatorname{coker}\left[P^3(2\Theta)^\vee \to H^0(L)^\vee \otimes \mathcal{O}_A\right] \otimes \mathcal{O}_A(2\Theta)$$

and

$$\varphi_{2\theta}^* \mathscr{E}xt^{-1}(\Phi_{\mathcal{P}}(I_0^4(2\Theta)), \mathcal{O}_{\hat{A}}) = \ker\left[P^3(2\Theta)^\vee \to H^0(L)^\vee \otimes \mathcal{O}_A\right] \otimes \mathcal{O}_A(2\Theta) = K \otimes \mathcal{O}_A(2\Theta).$$

From $(*)$ it follows then that

$$\varphi_{2\theta}^* R^{g-1} \Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) = K \otimes \mathcal{O}_A(2\Theta) \tag{4.2}$$

and similarly for R^g.

□

Now we proceed to the second step:

Lemma 4.15. *If $x \neq 0$ and all the torsion components of Q have reduced support then the following conditions are equivalent:*

a) $x \in \mathrm{Bs}(\{2\Theta\}_{0,4})$

b) the edge map

$$\mathrm{ed}_x : H^g(\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x) \to H^0(R^g\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x)$$

is not injective

c) $H^1(R^{g-1}\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x) \neq 0$, *i.e* $x \in V^1(R^{g-1}\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee))$

In particular, the following contentions hold:

$$\mathrm{Bs}(\{2\Theta\}_{0,4}) \setminus A[2] \subset \varphi_{2\theta}^{-1}(V^1(K \otimes \mathcal{O}_A(2\Theta)) \setminus \{\hat{0}\}) \subset \mathrm{Bs}(\{2\Theta\}_{0,4}) + A[2] \subsetneq A.$$

Proof: As $R^k\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) = 0$ for $k \leq g-2$, the equivalence b) $\iff$ c) follows from Remark 4.8.

Now, we note that for $x \neq 0$ the condition a) is equivalent to the nullity, for all α, of the map

$$H^0(I_0^4(2\Theta) \otimes P_\alpha) \otimes P_\alpha^\vee \longrightarrow I_0^4(2\Theta) \otimes k(x)$$

or, equivalently, to the non-surjectivity, for all open sets $U \subset \hat{A}$, of the map

$$\mathrm{ev}_U(x) : \bigoplus_{\alpha \in U} H^0(I_0^4(2\Theta) \otimes P_\alpha) \otimes P_\alpha^\vee \longrightarrow I_0^4(2\Theta) \otimes k(x).$$

By Proposition 4.9 this means that either ed_x is not injective or the simultaneous evaluation

$$\mathrm{ev}_x(U) : H^0(R^g\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x) \longrightarrow \prod_{\alpha \in U} R^g\Phi_{\mathcal{P}^\vee} \otimes P_x \otimes k(\alpha)$$

fails to be injective for all U. Now, if $\{s_1, ..., s_r\}$ is a basis of $H^0(R^g\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x)$ then the reduced hypothesis and the previous lemma imply that the open set $U := \hat{A} \setminus \cup \mathrm{Zeroes}(s_i)$ is not empty and $\mathrm{ev}_x(U)$ is injective and hence the only possibility is that ed_x is not injective.

Now, regarding the "as a consequence part" we proceed as follows. As usual, applying $\varphi_{2\theta *}$, tensoring with P_x and taking cohomology to (4.2) we obtain that

$$\bigoplus_{y \in A[2]} H^1(R^{g-1}\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_{x+y}) = H^1(K \otimes \mathcal{O}_A(2\Theta) \otimes P_{\varphi_{2\theta}(x)}).$$

In particular, we directly obtain that

$$\mathrm{Bs}(\{2\Theta\}_{0,4}) \setminus \{0\} \subset \varphi_{2\theta}^{-1}(V^1(K \otimes \mathcal{O}_A(2\Theta)).$$

In a similar fashion, if $\varphi_{2\theta}(x)$ lies in $V^1(K \otimes \mathcal{O}_A(2\Theta))$ then there exist $y \in A[2]$ such that $H^1(R^{g-1} \otimes P_{x+y}) \neq 0$ and therefore, if $x + y \neq 0$, then $x + y \in \mathrm{Bs}(\{2\Theta\}_{0,4})$ and hence $x \in \mathrm{Bs}(\{2\Theta\}_{0,4}) + A[2]$ whenever $x \notin A[2]$. Summarizing, we have proved that

$$\varphi_{2\theta}^{-1}\left(V^1(K \otimes \mathcal{O}_A(2\Theta)) \setminus \{\hat{0}\}\right) \subset \mathrm{Bs}(\{2\Theta\}_{0,4}) + A[2],$$

as we wanted to see.

□

Finally, we proceed to prove the last step which concludes the proof of the theorem:

Lemma 4.16. *Let $K = \ker\phi$ as above. Then we have that the sheaf $K \otimes \mathcal{O}_A(2\Theta)$ is GV and $V^0(K \otimes \mathcal{O}_A(2\Theta)) \subset \{\hat{0}\}$. In particular, $V^1(K \otimes \mathcal{O}_A(2\Theta) \subset \{\hat{0}\}$.*

Proof: We need to stimate the codimensions of the jump loci $V^i(K \otimes \mathcal{O}_A(2\Theta))$. To start, as $K \otimes \mathcal{O}_A(2\Theta) \otimes P_\alpha$ is a subsheaf of the locally free sheaf $P^3(2\Theta)^\vee \otimes \mathcal{O}_A(2\Theta) \otimes P_\alpha$ and thus

$$H^0(K \otimes \mathcal{O}_A(2\Theta) \otimes P_\alpha) \subset H^0(P^3(2\Theta)^\vee \otimes \mathcal{O}_A(2\Theta) \otimes P_\alpha) \simeq H^g(P^3(2\Theta) \otimes \mathcal{O}_A(-2\Theta) \otimes P_{-\alpha})^\vee$$

and, as we see in the proof of Lemma 2.17(a), the latter group vanishes for $\alpha \neq 0$. In other words, $V^0(K \otimes \mathcal{O}_A(2\Theta)) \subset \{\hat{0}\}$.

Now, from the previous lemma we know that V^1 is proper closed subset of $\hat{A}$ and hence, in any case, $\operatorname{codim}_{\hat{A}} V^1 \geq 1$. It remains then to compute V^i for $i \geq 2$. We claim that $V^i(K \otimes \mathcal{O}_A(2\Theta)) \subset \{\hat{0}\}$. To do this, we use again the Leray spectral sequence:

$$E_2^{p,q} = H^p(R^q\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x) \implies H^{p+q}(\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x).$$

From (1.7) we have

$$H^k(\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee) \otimes P_x) \simeq H^{k-g}(I_0^4(2\Theta) \underline{\otimes} k(x)).$$

We claim that for $x \neq 0$ the derived tensor at the right is nothing else than the usual one. Indeed: we have the exact sequence

$$0 \to I_0^4 \to \mathcal{O}_A \to \mathcal{O}_{0^4} \to 0.$$

Tensoring with $k(x)$ for $x \neq 0$ we get that for all $k \geq 1$ we have

$$\mathscr{T}or_k(I_0^4, k(x)) \simeq \mathscr{T}or_{k+1}(\mathcal{O}_{0^4}, k(x)) = 0,$$

which proves the claim. We conclude that

$$H^k(I_0^4(2\Theta) \underline{\otimes} k(x)) \simeq H^k(I_0^4(2\Theta) \otimes k(x)) = 0 \text{ for all } k \geq 1.$$

From the spectral sequence we obtain then

$$V^k(R^{g-1}\Phi_{\mathcal{P}^\vee}(I_0^4(2\Theta)^\vee)) \subseteq \{0\} \text{ for all } k \geq 2$$

and thus from (4.2) we see that $V^k(K \otimes \mathcal{O}_A(2\Theta))$ is zero-dimensional for all $k \geq 2$, which is more than enough to conclude that $K \otimes \mathcal{O}_A(2\Theta)$ is GV. As mentioned before, the "in particular" assertion follows then from Hacon's lemma.

□

4.4 Θ-duals and Marini's theorem revisited

A well known characterization of hyperelliptic jacobians is given by the following theorem due to Marini:

Theorem 4.17 (Theorem in [Mar93])**.** *Let (A, θ) be an i.p.p.a.v. and Θ a symmetric representant of θ. Then (A, θ) is the polarized jacobian of an hyperelliptic curve if and only if there exists a divisor $D \in |\mathcal{O}_\Theta(\Theta)|$ such that all the irreducible components of D are not reduced.*

In this section we give an interpretation of the above result in terms of Θ*-duals* of length two subschemes (i.e tangent directions) of A. To do so we first briefly recall what Θ-duals are.

Definition 4.18. Let Z be a closed subscheme of a p.p.a.v (A, θ) and let Θ be a symmetric representant of θ. The Θ-dual $V(Z)$ of Z is the scheme

$$V(Z) := \operatorname{Supp}(R^g\Phi_{\mathcal{P}^\vee}(I_Z(\Theta)^\vee)) \subset \hat{A}.$$

Here it is worth to point out that, using the identification $\varphi_\theta : A \to \hat{A}$, we have that $V(Z)$ is supported on the set

$$V^0(I_Z(\Theta)) = \{\alpha \in \hat{A} : H^0(I_Z(\Theta) \otimes P_\alpha) \neq 0\} \simeq \{a \in A : Z \subset \Theta - a\}.$$

As an example, when we identify A with $\hat{A}$ via Θ, we have that $\Phi_{\mathcal{P}^\vee}(I_0(\Theta)^\vee) \simeq \mathcal{O}_\Theta(\Theta)[-g]$ and hence $V(\{0\}) = \Theta$. More generally, we have the following fundamental relation:

Lemma 4.19 (Corollary 4.3 in [PP08])**.**

$$R^g\Phi_{\mathcal{P}^\vee}(I_Z(\Theta)^\vee) \simeq (-1_{\hat{A}})^*\mathcal{O}_{V(Z)}(\Theta),$$

where in the right side we identified Θ with its image in $\hat{A}$.

We now state and prove the main results of this section:

Theorem 4.20. *Formation of Θ-duals gives a bijection*

$$\left\{\begin{matrix}\textit{length two subschemes}\\ \textit{supported on the origin}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{Divisors (in } \Theta\textit{) in the}\\ \textit{linear system } \mathcal{O}_\Theta(\Theta)\end{matrix}\right\}.$$

This has the following consequence:

Corollary 4.21. *Assume that* $\dim A \geq 2$*. If there exists a length two subscheme τ such that $I_\tau(\Theta)$ is not generated then there exist a non-reduced divisor in the linear system $|\mathcal{O}_\Theta(\Theta)|$.*

Proof of the Corollary: The idea is to use Corollary 4.12 above. As in the case of $I_0^4(2\Theta)$, for $x \in A$ the injectivity of the edge map

$$H^g(\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee) \otimes P_x) \to H^0(\mathcal{O}_{V(\tau)}(\Theta) \otimes P_x)$$

is equivalent to the vanishing of $H^1(R^{g-1}\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee)\otimes P_x)$. Now, in Example 2.7 we computed that $R^{g-1}\Phi_{\mathcal{P}^\vee}(I_\tau(\Theta)^\vee) = \mathcal{O}_A(-\Theta)$ and hence such vanishing (and hence the injectivity) is automatic. It follows that, if $V(\tau)$ is reduced, then $I_\tau(\Theta)$ is generated and therefore the result follows from the theorem.

□

Proof of the Theorem: We first prove that if τ is a length two subscheme supported on the origin then $V(\tau)$ belongs to $|\mathcal{O}_\Theta(\Theta)|$. To do this we start by noticing that from Example 2.7 we have the following exact sequence:

$$0 \to \mathcal{O}_A(-\Theta) \to \mathcal{O}_A \to \mathcal{O}_\Theta(\Theta) \to \mathcal{O}_{V(\tau)}(\Theta) \to 0,$$

which implies that $V(\tau)$ is a closed subscheme in Θ and, moreover, we can compute its ideal $I_{V(\tau)/\Theta}$:

$$\begin{aligned} I_{V(\tau)/\Theta}(\Theta) &= \operatorname{Im}\Big[\ \mathcal{O}_A \longrightarrow \mathcal{O}_\Theta(\Theta)\ \Big] \\ &= \operatorname{Im}\Big[\ \mathcal{O}_A \xrightarrow{\cdot\tau} H^0(\mathcal{O}_\Theta(\Theta))\otimes\mathcal{O}_A \longrightarrow \mathcal{O}_\Theta(\Theta)\ \Big] \\ &\simeq \mathcal{O}_\Theta, \end{aligned}$$

that is, $I_{V(\tau)/\Theta} \simeq \mathcal{O}_\Theta(-\Theta)$. In other words, we have that $V(\tau)$ is a Cartier divisor in Θ and belongs to the linear system $|\mathcal{O}_\Theta(\Theta)|$, as we wanted to see.

Now, we proceed to prove that if $D \in |\mathcal{O}_\Theta(\Theta)|$ then its theta dual $V(D)$ is a length two subscheme supported on the origin. To do this, the strategy is the following: first we will compute $\Phi_{\mathcal{P}}(I_D(\Theta))$ and then, dualizing, we compute $\Phi_{\mathcal{P}^\vee}(I_D(\Theta)^\vee)$, proving that we have an exact sequence

$$0 \to k(0) \to R^g\Phi_{\mathcal{P}^\vee}(I_D(\Theta)^\vee) \to k(0) \to 0. \qquad (*)$$

At this point, the result will follow from Lemma 4.19 above.

We proceed then to compute $\Phi_{\mathcal{P}}(I_D(\Theta))$. We claim that it is concentrated in degree $g-1$. To prove this assertion start by noticing that, as $\dim D = g-2$ we we automatically have that $R^g\Phi_{\mathcal{P}}(I_D(\Theta)) = 0$. Now, we have the exact sequence

$$0 \to \mathcal{O}_{\hat{A}} \to I_D(\hat{\Theta}) \to \mathcal{O}_{\hat{\Theta}} \to 0$$

obtained by twisting $0 \to I_\Theta \to I_D \to I_{D/\Theta} \to 0$ (and using the fact that $D \in |\mathcal{O}_\Theta(\Theta)|$). Applying $\Phi_{\mathcal{P}}$ and taking the corresponding long exact sequence we deduce that

$$R^i\Phi_{\mathcal{P}}(I_D(\hat{\Theta})) \simeq R^i\Phi_{\mathcal{P}}(\mathcal{O}_{\hat{\Theta}}) \qquad \text{for } i \leq g-2. \qquad (4.3)$$

and that we have an exact sequence

$$0 \to R^{g-1}\Phi_{\mathcal{P}}(I_D(\hat{\Theta})) \to R^{g-1}\Phi_{\mathcal{P}}(\mathcal{O}_{\hat{\Theta}}) \to k(0) \to 0. \qquad (4.4)$$

Therefore, from (4.3) and (4.4) we see that in order to prove our claim it is enough to compute $\Phi_{\mathcal{P}}(\mathcal{O}_\Theta)$. Now, applying $\Phi_{\mathcal{P}}$ to $0 \to \mathcal{O}_A(-\Theta) \to \mathcal{O}_A \to \mathcal{O}_\Theta \to 0$ it is easy to see that $\Phi_{\mathcal{P}}(\mathcal{O}_\Theta) = I_0[-(g-1)]$ and hence we conclude that $\Phi_{\mathcal{P}}(I_D(\Theta)^\vee)$ is concentrated in degree $g-1$, proving our claim. Moreover, we obtain an exact sequence

$$0 \to R^{g-1}\Phi_{\mathcal{P}}(I_D(\hat{\Theta})) \to I_0(\Theta) \to k(0) \to 0. \tag{4.5}$$

Now, we need to dualize (that is, use Grothendieck-Verdier (1.3)) to compute the sheaf $R^g\Phi_{\mathcal{P}^\vee}(I_D(\Theta)^\vee) \simeq \mathscr{E}xt^0(\Phi_{\mathcal{P}}(I_D(\hat{\Theta})), \mathcal{O}_A)$ to obtain an exact sequence of the form $(*)$. As we have a fourth quadrant spectral sequence (see e.g [Huy06, (3.8)])

$$E_2^{p,q} = \mathscr{E}xt^p\left(R^{-q}\Phi_{\mathcal{P}}(I_D(\hat{\Theta})), \mathcal{O}_A\right) \implies \mathscr{E}xt^{p+q}\left(\Phi_{\mathcal{P}}(I_D(\hat{\Theta})), \mathcal{O}_A\right) \tag{4.6}$$

we shall compute the sheaves

$$\mathscr{E}xt^p\left(R^{g-1}\Phi_{\mathcal{P}}(I_D(\hat{\Theta})), \mathcal{O}_A\right) \qquad \text{for } p = 0, 1, ..., g.$$

Now, from the exact sequence (4.5) and Lemma 4.22 below it follows that

$$\mathscr{E}xt^p\left(R^{g-1}\Phi_{\mathcal{P}}(I_D(\hat{\Theta})), \mathcal{O}_A\right) = 0 \qquad \text{for } p \neq 0, g-1,$$

and that we have the exact sequence

$$0 \to k(0) \to \mathscr{E}xt^{g-1}\left(R^{g-1}\Phi_{\mathcal{P}}(I_D(\hat{\Theta})), \mathcal{O}_A\right) \to k(0) \to 0.$$

From these observations the result follows because, by our previous calculations, in the espectral sequence (4.6) $E_2^{p,q} = 0$ except for $(p,q) = (g-1, -(g-1))$ and $(p,q) = (0, -(g-1))$ and hence it degenerates to give an isomorphism

$$\mathscr{E}xt^{g-1}\left(R^{g-1}\Phi_{\mathcal{P}}(I_D(\hat{\Theta})), \mathcal{O}_A\right) \simeq \mathscr{E}xt^0(\Phi_{\mathcal{P}}(I_D(\Theta), \mathcal{O}_A)) \simeq R^g\Phi_{\mathcal{P}^\vee}(I_D(\Theta)^\vee).$$

Until now we have proved that formation of Θ-dual gives maps between the sets of the statement of the theorem. It remains to show that these maps are inverse to each other. More precisely, we need to show that if τ is a length two subscheme then $V(V(\tau)) = \tau$ and that if $D \in |\mathcal{O}_\Theta(\Theta)|$ then $V(V(D)) = D$. Now, by [GL11, Remark 2.7] we have an inclusion $\tau \subset V(V(\tau))$ and as both τ and $V(V(\tau))$ have length two and are supported on the same point they must be equal. Similarly, we have an inclusion $D \subset V(V(D))$, and as D is linearly equivalent to $V(V(D))$ they must coincide. This completes the proof of the theorem.

□

Lemma 4.22. *We have that*

a) $\mathscr{E}xt^p(k(0), \mathcal{O}_A) = 0$ *for* $p < g$ *and* $\mathscr{E}xt^g(k(0), \mathcal{O}_A) = k(0)$

b) $\mathscr{E}xt^0(I_0(\Theta), \mathcal{O}_A) = \mathcal{O}_A(-\Theta)$, $\mathscr{E}xt^{g-1}(I_0(\Theta), \mathcal{O}_A) = k(0)$ *and* $\mathscr{E}xt^p(I_0, \mathcal{O}_A) = 0$ *otherwise*

Proof of the Lemma: a) We know that the sheaves $\mathscr{E}xt^p(k(0), \mathcal{O}_A)$ are all supported on $\{0\}$ and, moreover

$$\mathscr{E}xt^p(k(0), \mathcal{O}_A)_0 = \mathrm{Ext}^p_{\mathcal{O}_{A,0}}(k(0), \mathcal{O}_{A,0})$$

(here we are seeing $k(0)$ as the residual field of the local ring $\mathcal{O}_{A,0}$). Now, as A is smooth, $\mathcal{O}_{A,0}$ is in particular Cohen-Macaulay and hence the Rees' theorem ([BH93, Theorem 1.2.5]) tell us that $\mathscr{E}xt^p(k(0), \mathcal{O}_A) = 0$ for $p < g$. On the other hand by [BH93, Lemma 1.2.4] we have that

$$\mathrm{Ext}^g_{\mathcal{O}_{A,0}}(k(0), \mathcal{O}_{A,0}) = \mathrm{Hom}_{\mathcal{O}_{A,0}}(k(0), k(0)) = k(0)$$

and hence $\mathscr{E}xt^g(k(0), \mathcal{O}_A) = k(0)$.

b) This is the content of Lemma 3.11.

□

4.5 Further questions

We conclude this chapter by stating a couple of pending questions arising from Theorem 4.13 and 4.20.

1. First at all, it would be interesting to have a better geometric understanding of the sheaf Q from Theorem 4.13, at least in the jacobian case. More precisely, we may ask

Question 4.23. *What is the torsion filtration of Q?*

2. In the proof of Theorem 4.13 we esentially studied the generation of the sheaf $I_0^4(2\Theta)$ outside the origin and hence it is natural to ask the following:

Question 4.24. *What can we say about the generation of $I_0^4(2\Theta)$ at the origin?*

3. We saw that the non-generation of a length two subscheme implies that a divisor $D \in \mathcal{O}_\Theta(\Theta)$ is not reduced. Now, to actually obtain an hyperelliptic jacobian we need to ensure that **all** the components of D are not reduced. For instance, if we know a priori that (A, θ) is a jacobian then we can show that if there exist such non-reduced D, then it must have just one irreducible component with a multiplicity two scheme structure and (A, θ) must be an hyperelliptic jacobian. In this context, it is natural to ask:

Question 4.25. *Can it happen that a divisor $D \in |\mathcal{O}_\Theta(\Theta)|$ has both reduced and non-reduced components? Can we characterize the situations where this happens?*

4. In [BD86] it is proved that if there exists a non-integral divisor $D \in |\mathcal{O}_\Theta(\Theta)|$ then $\dim \mathrm{Sing}\, \Theta \geq g - 4$ unless there exists an elliptic curve E with $(\theta \cdot E) = 2$. We may then ask:

Question 4.26. *Assuming that there exists a non-reduced divisor $D \in |\mathcal{O}_\Theta(\Theta)|$. Does it follow that $\dim \mathrm{Sing}\, \Theta \geq g - 3$ unless there exists an elliptic curve E with $(\theta \cdot E) = 2$?*

5. Finally, we may ask also a more precise question regarding the generation of the twisted-ideals studied in this chapter. Concretely:

Question 4.27. *Suppose that $I_\tau(\Theta)$ is generated for every length two subscheme τ supported on the origin. Does it follow that $I_0^4(2\Theta)$ is also generated (at least outside of the origin)?*

The above question is sensible since the tensor product of generated sheaves is generated and quotients of generated sheaves are generated; in particular, the supposition in the question implies that $(I_\tau I_\mu)(2\Theta)$ is generated for every couple τ, μ of length two subschemes and, on the other hand, we have that $I_0^4(2\Theta) = \bigcap_{\tau,\mu}(I_\tau I_\mu)(2\Theta)$.

Here it is worth to point out that affirmative answers to questions 4.26 and 4.27, combined with Theorem 4.20 would give the following implication:

$$\text{If } \varepsilon(\theta) < 2 \text{ then either exist an elliptic curve } E \text{ with } (\theta \cdot E) = 2 \text{ or } \dim \operatorname{Sing} \Theta \geq g - 3$$

Where it is worth to highlight the fact that the hyperelliptic locus in $\mathcal{A}_g$ is a component of the locus where the latter condition holds.

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