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Volume preserving mean curvature flow in asymptotically flat spaces

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This Thesis is based on the following works:

- Carlo Sinestrari and Jacopo Tenan, *Volume preserving mean curvature flow of round surfaces in asymptotically flat spaces*, [ST25];
- Jacopo Tenan, *Volume preserving spacetime mean curvature flow in initial data sets and applications to General Relativity*, [Ten25].

Contents

Chapter 1

Introduction

> In the general theory of relativity the doctrine of space and time, or kinematics, no longer figures as a fundamental independent of the rest of physics. The geometrical behaviour of bodies and the motion of clocks rather depend on gravitational fields, which in their turn are produced by matter.
>
> A. Einstein

With these words, in his non-scientific book "*The World As I See It*" [Ein35], Albert Einstein effectively resumes one of his most famous equations. Mathematically, this translates into having a complete Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ satisfying the equation

$$\mathbf{Ric} - \left(\frac{\mathbf{S}}{2}\right)\mathbf{g} = \mathbf{T}, \tag{1.1}$$

where $\mathbf{Ric} = \mathbf{Ric}(\cdot, \cdot)$ is the Ricci tensor of $(\mathcal{M}, \mathbf{g})$, $\mathbf{S} = \mathrm{tr}_{\mathbf{g}}\mathbf{Ric}$ and $\mathbf{T}$ is a given symmetric smooth (0,2)-type tensor field called *energy-momentum tensor*. Even if in the present Thesis the Einstein equation (1.1) plays a marginal role, the author believes appropriate to review it here since it describes the "world" where the protagonists of the next Chapters live.

We will focus our attention on *isolated gravitational systems*. These physical models are well-described by 3-dimensional submanifolds of $(\mathcal{M}, \mathbf{g})$ with suitable asymptotics. In the following, we will assume that there exists a smooth immersion $\mathbf{j} : M \hookrightarrow \mathcal{M}$ and a smooth vector field $\boldsymbol{e}_0$ on $\mathcal{M}$ such that $\boldsymbol{e}_0$ is timelike and the restriction of $\mathbf{g}$ to M , i.e. $\overline{\mathrm{g}} := \mathbf{j}^*\mathbf{g}$, is a Riemannian metric. We additionally suppose that M contains a compact subset $\mathrm{C} \subset M$ such that $M \setminus \mathrm{C}$, called *end* of the manifold M, is diffeomorphic to $\mathbb{R}^3 \setminus \overline{\mathbb{B}}_1(\vec{0})$ through a chart $\vec{x} : M \setminus \mathrm{C} \to \mathbb{R}^3 \setminus \overline{\mathbb{B}}_1(\vec{0})$ which induces a Euclidean metric $\overline{\mathrm{g}}^e := \left(\vec{x}^{-1}\right)^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3}$. Furthermore, we assume that $\overline{\mathrm{g}} - \overline{\mathrm{g}}^e$, and its derivatives, decay suitably at infinity. At the moment we do not specify which decay we need, see Definition 1.1.1 for details. We finally set

$$\overline{\mathrm{K}}(\cdot, \cdot) := \langle \mathbf{A}(\cdot, \cdot), -\boldsymbol{e}_0 \rangle, \qquad \overline{\mu} := \mathbf{T}(\boldsymbol{e}_0, \boldsymbol{e}_0), \qquad \overline{\mathrm{J}}(\cdot) := \mathbf{T}(\boldsymbol{e}_0, \cdot), \tag{1.2}$$

where $\mathbf{A}$ is the second fundamental form of the immersion $\mathbf{j}$. $\overline{\mathrm{K}}$, $\overline{\mu}$ and $\overline{\mathrm{J}}$ are, respectively, the (scalar) *spacetime second fundamental form* of M, the *energy density* and the *momentum density*. In term of these new quantities, the Einstein equations (1.1) can be written, through the Gauss-Mainardi-Codazzi equations [Lee18, Thm. 8.3], as

$$\begin{cases} \overline{\mathrm{S}} - |\overline{\mathrm{K}}|^2 + (\mathrm{tr}_{\overline{\mathrm{g}}}\overline{\mathrm{K}})^2 = 2\overline{\mu} \\ \overline{\nabla} \cdot (\overline{\mathrm{K}}) - \overline{\mathrm{d}}\left(\mathrm{tr}_{\overline{\mathrm{g}}}\overline{\mathrm{K}}\right) = \overline{\mathrm{J}}. \end{cases} \tag{1.3}$$

Choquet-Bruhat proved that the validity of the system (1.3) for a tuple $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}}, \overline{\mu}, \overline{\mathrm{J}})$ implies the existence of a spacetime associated to this tuple in the sense of (1.2), see [CB09], [Lee19]. This remark allows us to define an initial data set as a notion which is independent from the one of spacetime manifold (at least formally), but including the whole information encoded.

For isolated gravitational systems, a notion of energy can be given, provided they satisfy suitable asymptotics. More precisely, if the scalar curvature of $(M, \overline{\mathrm{g}})$ is integrable, i.e. $\overline{\mathrm{S}} \in L^1(M, \overline{\mathrm{g}})$, then it is possible to define the so called ADM-*energy*, named after Arnowitt, Deser and Misner [ADM61], given by the following limit of flux integrals

$$E_{\mathrm{ADM}} := \lim_{r\to\infty} (16\pi)^{-1} \sum_{\alpha,\beta} \int_{\vec{x}^{-1}(\mathbb{S}_r(\vec{0}))} \left(\partial_\beta \overline{\mathrm{g}}_{\alpha\beta} - \partial_\alpha \overline{\mathrm{g}}_{\beta\beta}\right) \nu_r^\alpha \, d\mu_r, \tag{1.4}$$

where $\vec{x}^{-1}(\mathbb{S}_r(\vec{0}))$ is the Euclidean sphere immersed in $(M, \overline{\mathrm{g}})$, ν_r and $d\mu_r$ are, respectively, its normal vector and its volume form in $(M, \overline{\mathrm{g}})$ and ∂_α is the derivative in local coordinates in the chart $\vec{x}$. In a similar way, Beig-Ó Murchadha [BOM87] defined the so called ADM-*center of mass*. It is a vector of $\mathbb{R}^3$ which is given as a limit of flux integrals on Euclidean spheres, similarly to the definition of ADM-energy. Explicitly,

$$\begin{aligned}\left(\vec{\mathcal{C}}_{\mathrm{ADM}}\right)_\gamma := \frac{1}{16\pi E_{\mathrm{ADM}}} \lim_{r\to\infty} \Bigg[&\int_{\vec{x}^{-1}(\mathbb{S}_r(\vec{0}))} \sum_{\alpha,\beta} \vec{x}_\gamma \left(\partial_\alpha \overline{\mathrm{g}}_{\alpha\beta} - \partial_\beta \overline{\mathrm{g}}_{\alpha\alpha}\right) \frac{\vec{x}_\beta}{r} \, d\mu_r \\ &- \int_{\vec{x}^{-1}(\mathbb{S}_r(\vec{0}))} \sum_{\alpha} \left(\overline{\mathrm{g}}_{\alpha\gamma} \frac{\vec{x}_\alpha}{r} - \overline{\mathrm{g}}_{\alpha\alpha} \frac{\vec{x}_\gamma}{r} \right) d\mu_r \Bigg],\end{aligned} \tag{1.5}$$

for $\gamma \in \{1,2,3\}$. Note that in these definitions the Euclidean foliation $\left\{\vec{x}^{-1}\left(\mathbb{S}_r(\vec{0})\right)\right\}_{r\geq r_0}$ has the important role of being used in order to encode physically relevant information of the system. With few computations, one can show that $\vec{x}^{-1}\left(\mathbb{S}_r(\vec{0})\right)$ in (1.4) can be replaced with other families of surfaces, provided they are sufficiently *round*, in a precise sense (see Lemma 3.1.3).

Round surfaces such as the Euclidean spheres, however, do not represent in general a good coordinate system for $(M, \overline{\mathrm{g}})$. This is essentially due to the fact that the round surfaces do not necessarily have constant mean curvature (CMC). In the context of Mathematical General Relativity, in the late '80s Christodoulou and Yau employed CMC surfaces in the study of the quasi-local mass.

In 1996, the seminal work of Huisken and Yau [HY96] showed that, in an asymptotically Schwarzschildean setting of positive mass, i.e. a metric $\overline{\mathrm{g}}^S$ on $M \setminus \mathrm{C}$ which satisfies, for some $m > 0$,

$$\sum_{N=0}^{4} |\vec{x}|^N \left| \partial^{|N|} \left(\overline{\mathrm{g}}^S_{\alpha\beta} - \left(1 + \frac{m}{2|\vec{x}|}\right)^4 \delta_{\alpha\beta} \right) \right| \leq \frac{\overline{c}}{|\vec{x}|^2}, \tag{1.6}$$

one can construct a family of CMC-surfaces $\{\Sigma^\sigma\}_{\sigma\geq\sigma_0}$ which exhaust the end of the manifold. Moreover they do not self-intersect and $H^{\Sigma^\sigma} \to 0$ as $\sigma \to \infty$. Setting $s_\sigma := \frac{2}{H^{\Sigma_\sigma}}$, since Σ^σ is diffeomorphic to $\mathbb{S}_1$, one can define a bijective map

$$\Phi : (s_0, \infty) \times \mathbb{S}_1 \to M \setminus \mathrm{C}. \tag{1.7}$$

such that, for every $s > s_0$, $\Phi(s, \cdot)$ maps $\mathbb{S}_1$ in the unique CMC round surface Σ^s of mean curvature $2/s$. The definition of this map goes through the use of the inverse function, see e.g. [Hua12, Sect. 5.3]. Such a family of *leaves* is called CMC-*foliation*. From a physical point of view, a CMC-foliation is a sort of *abstract* center of mass, which allows to describe the manifold $(M, \overline{\mathrm{g}})$ through the polar representation (1.7).

In the construction of Huisken-Yau, each CMC-leaf is obtained trough the deformation of a Euclidean sphere, performed by the so called *volume preserving mean curvature flow*. Once the large time existence of this flow is proved, each CMC-leaf is obtained as a long time limit of the flow. This flow has no longer been studied in the context of asymptotically flat manifolds, except for the work of Corvino and Wu [CW08].

After this seminal result, the foliation described above has been constructed by different methods and under different hypotheses in various papers. Ye, and later Huang, based their construction on the implicit function theorem, see [Ye97],[Hua10], while the work of Metzger [Met07] gave rise to a branch of the field of study in which the foliation is constructed through a continuity method. This culminated with the work of Nerz [Ner15], which obtained the foliation under the decay assumption we will describe in Definition 1.1.1. The optimality of these hypotheses has been highlighted in [Ner18]. Nerz's result was later modified by Cederbaum and Sakovich in order to construct a different type of foliation, which we will explain in more detail below. Recently, Eichmair and Koerber presented a new construction of the foliation trough a Lyapunov-Schmidt reduction [EK24]. The advantage of the method employed by Eichmair and Koerber is that this works in every dimension $n \geq 3$.

The foliation-approach started by Huisken and Yau allows to introduce another notion of center of mass, whose definition formally coincides with the one we would use in a Euclidean context, i.e. the limit of the integral mean of barycenters. This new center of mass, sometimes called *geometric center of mass*, is given by

$$\vec{\mathcal{C}}_{\mathrm{CMC}} := \lim_{\sigma\to\infty} \frac{1}{|\Sigma^\sigma|} \int_{\Sigma^\sigma} \vec{x}\, d\mu^\sigma, \tag{1.8}$$

where $d\mu^\sigma$ is the measure induced by $\overline{\mathrm{g}}$ on Σ^σ and $|\Sigma^\sigma|$ is its area. In other words, passing to the CMC-coordinate system flattens the manifold without losing the non-Euclidean information.

Under suitable symmetry assumptions, known as *weak Regge-Teitelboim conditions*, it was proved that the CMC-center of mass exists if and only if the Beig-Ó Murchadha center of mass exists. In the case in which both exist, they coincide. Moreover, under the so called *strong Regge-Teitelboim conditions*, the Beig-Ó Murchadha center of mass exists, and thus also the CMC-center of mass. On the other hand, on an asymptotically flat manifold where the Regge-Teitelboim condition are not satisfied, these centers of mass may not be well-defined. In particular, Cederbaum and Nerz [CN15] constructed explicit examples where both these objects do not converge.

For this reason, Cederbaum and Sakovich [CS21] introduced a new foliation, based on a modified curvature, whose leaves are not CMC but satisfy a prescribed mean curvature equation. As said before, their construction is based on the method of continuation of Nerz. In order to understand their foliation, it is necessary to introduce the notion of *spacetime* mean curvature of a surface Σ, which is given by

$$\mathcal{H}^2 = H^2 - P^2, \tag{1.9}$$

where $P := \mathrm{tr}_g(\overline{\mathrm{K}})$ and g is the metric on Σ induced by $\overline{\mathrm{g}}$. See Section 2.1 for more details. In the present Thesis, we will also work with a generalization of (1.9), see equation (2.12). In [CS21], Cederbaum and Sakovich prove the existence of a constant spacetime mean curvature (CSTMC) foliation of the outer part of M, and define a corresponding CSTMC-center of mass as the limit of the barycenters of the leaves. It is proved that the CSTMC-center of mass exists also in some cases in which the previous one does not. Moreover, the new center of mass has a physical relevance. From a spacelike point of view, the equation satisfied by each CSTMC surface looks like a prescribed mean curvature equation. A similar equation was present in the work of Metzger [Met07], who constructed surfaces satisfying the so called *constant expansion equation* $\Theta_\pm := H \pm P \equiv \text{const}$, where $\Theta^\pm$ are the *null curvatures* of Σ.

Instead, each leaf in the foliation constructed by Cederbaum and Sakovich in [CS21] satisfies the equation

$$\mathrm{const} \equiv \mathcal{H} = \sqrt{H^2 - P^2} = \sqrt{H+P}\sqrt{H-P} =: \sqrt{\Theta_+}\sqrt{\Theta_-}.$$

The two equations are different if the right hand side is strictly positive, as in the case of our interest. On the other hand, it is interesting to notice that they coincide if the right-hand side is zero, in which case we recover a well-known class called *trapped surfaces*, or MOTS, which has been studied by various authors (for example [AEM11] or [EHLS15]).

1.1 Results of the present Thesis

This Thesis has two main goals. The first is recovering the CMC-foliation constructed by Nerz in [Ner15], through a *volume preserving mean curvature flow*, i.e. generalizing the work of Huisken-Yau [HY96]. Secondly, we also aim to recover the spacetime CMC-foliation of Cederbaum-Sakovich through a volume preserving flow. In order to do this, we have to define a non-linear version of the volume preserving mean curvature flow, which we call *volume preserving spacetime mean curvature flow.*

Volume preserving mean curvature flow (Chapter 4). Similarly to Huisken-Yau, we study a volume preserving mean curvature flow starting from Euclidean spheres, namely we consider a smooth family of immersions $\{F_t\}_{t\in[0,T)}$, $0 \le T \le \infty$ which evolves according to

$$\begin{cases} \frac{\partial F_t}{\partial t}(\cdot) = -\left(H(t,\cdot) - h(t)\right)\nu(t,\cdot) \\ F_0 = \iota \end{cases}, \tag{1.10}$$

where ι is the immersion of the Euclidean sphere $\mathbb{S}_r(\vec{0})$ in M. Moreover, from now on, we are supposing M to be an *asymptotically flat* manifold, see Definition 1.1.1. We remark that our hypotheses on the ambient manifold M are more general and differs from those of Huisken-Yau because of two main reasons, as we will explain after the following Definition.

Definition 1.1.1. *Let $\delta \in (0, \frac{1}{2}]$. A Riemannian 3-manifold $(M, \overline{\mathrm{g}})$ is said to be $C^2_{\frac{1}{2}+\delta}$-asymptotically flat if there exist a compact subset $\mathrm{C} \subset M$, a constant $\overline{c} > 0$ and a diffeomorphism $\vec{x} : M \setminus \mathrm{C} \to \mathbb{R}^3 \setminus \overline{\mathbb{B}}_1(\vec{0})$ such that*

$$|\overline{\mathrm{g}}_{\alpha\beta} - \delta_{\alpha\beta}| + |\vec{x}|\left|\partial_\gamma \overline{\mathrm{g}}_{\alpha\beta}\right| + |\vec{x}|^2\left|\partial_\gamma\partial_\omega \overline{\mathrm{g}}_{\alpha\beta}\right| \le \overline{c}|\vec{x}|^{-\frac{1}{2}-\delta}, \tag{1.11}$$

where $\overline{\mathrm{g}}_{\alpha\beta} := (\vec{x}^*\overline{\mathrm{g}})_{\alpha\beta}$ and ∂_γ is the local derivative in the chart. We moreover assume that the scalar curvature $\overline{\mathrm{S}} = \mathrm{tr}_{\overline{\mathrm{g}}}\left(\overline{\mathrm{Ric}}\right)$ satisfies $|\overline{\mathrm{S}}| \le \overline{c}|\vec{x}|^{-3-\delta}$. We will often refer to this hypothesis as the mass condition.

In the following, we will always assume that $E_{\mathrm{ADM}} > 0$, and we refer to this condition as the *positive mass condition*. The integrability of the scalar curvature is a sort of reminiscence of the scalar flatness of the Schwarzschild space. Some comments concerning the negative mass case can be found in Section 4.3.2.

Note that in Definition 1.1.1 we require a suitable decay only for two derivatives of the metrics, and not for four derivatives as in [HY96], see (1.6). Note also that in the asymptotically Schwarzschildean case of Huisken-Yau we have $\delta = \frac{1}{2}$. For this reason, our methods to prove long time existence and asymptotic convergence of the flow use different techniques from those of Huisken-Yau. In particular, the lack of control on the derivatives of the Riemann tensor of our case does not allow to control the roundness of the evolving surfaces by the usual maximum principle arguments employed in the literature on mean curvature flow. The core of our proof is a suitable definition of the class of round surfaces, which involves integral norms of the traceless second fundamental form and of the oscillation of the mean curvature. Since we are interested in the analysis of these integral quantities, a crucial role in the proof is played by integration by parts, in the spirit of Metzger, see for example [Met07, Lemma 3.3].

We are then able to prove invariance of this class by a careful analysis of the time evolution of our integral quantities, combined with powerful recent results from the literature [DLM05], and estimates on the barycenter of the evolving surface. In order to carry out a fruitful analysis on the evolution of the barycenter, we need to additionally suppose that $(M, \overline{\mathrm{g}})$ satisfies a weak Regge-Teitelboim condition, see for example Definition 3.1.19. This is again a reminiscence of the Schwarzschildean setting of Huisken-Yau. However, being asymptotically Schwarzschildean is a requirement stronger than the Regge-Teitelboim condition, since it includes an asymptotic radial symmetry assumption. A crucial ingredient in the proof is a slight generalization of the spectral theory of the stability operator as presented in [Ner15] and [CS21]. We present an analysis of this operator which is similar to the one carried out by Nerz and Cederbaum-Sakovich. Since our surfaces are just round and not CMC, the presence of some additional terms which does not allow us to deduce the positivity of the stability operator. However, in our dynamical analysis, we only apply the stability operator to the deformation $H - h$, and in this case also the additional terms will give good contributions to our purposes. We end the paragraph stating the first main theorem of the Thesis.

Theorem 1.1.2 (Sinestrari-T.). *Let $(M, \overline{\mathrm{g}})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat 3-manifold with $E_{\mathrm{ADM}} > 0$ and satisfying the weak Regge-Teitelboim conditions. Let Σ_t be the solution of the volume preserving mean curvature flow starting from the Euclidean coordinate sphere $\vec{x}^{-1}\left(\mathbb{S}_r(\vec{0})\right)$, for a large enough radius $r > 0$. Then Σ_t exists for all $t \in [0, \infty)$ and exponentially converges to a CMC-surface as $t \to \infty$.*

Volume preserving spacetime mean curvature flow (Chapter 5). The aim of Chapter 5 is to consider again the flow approach of Huisken-Yau and extend it to the context of spacetime mean curvature. We study here a volume preserving flow where the mean curvature is replaced by the spacetime mean curvature. In particular, we consider an initial data set $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ which satisfies (1.1.1) and the following asymptotic flatness.

Definition 1.1.3. *Let $\delta \in (0, \frac{1}{2}]$. An asymptotically flat initial data set is a triple $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ such that the pair $(M, \overline{\mathrm{g}})$ is a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat (with respect to the chart $\vec{x}$) and moreover it holds*

$$|\overline{\mathrm{K}}_{\alpha\beta}| + |\vec{x}||\partial_\gamma \overline{\mathrm{K}}_{\alpha\beta}| \leq \overline{c}|\vec{x}|^{-\frac{3}{2}-\delta}, \tag{1.12}$$

where $\overline{\mathrm{K}}_{\alpha\beta} = (\vec{x}^* \overline{\mathrm{K}})_{\alpha\beta}$. On the other hand, we say that $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ is constrained by the pair $(\overline{\mu}, \overline{\mathrm{J}})$ if (1.3) holds together with

$$|\overline{\mu}| + |\overline{\mathrm{J}}| \leq \overline{c}|\vec{x}|^{-3-\delta}. \tag{1.13}$$

Observe that in the definition of initial data set the decay of the scalar curvature is a consequence of equation (1.3). In this context, for an arbitrary but fixed power $q \geq 2$, we set

$$\mathcal{H} = \sqrt[q]{H^q - |P|^q}. \tag{1.14}$$

We denote with $\hbar$ the integral mean of $\mathcal{H}$. We then aim to show long time existence and convergence of solutions of the system

$$\begin{cases} \frac{\partial F_t}{\partial t}(\cdot) = -\left[\mathcal{H}(t,\cdot) - \hbar(t)\right]\nu(t,\cdot) \\ F_0 = \iota \end{cases} . \tag{1.15}$$

This time we choose as initial data the immersion $\iota : \Sigma \hookrightarrow M$ of a surface of the spacelike CMC-foliation, which satisfies better estimates than a Euclidean coordinate sphere. A similar choice is made in [CS21], where the continuity method is implemented taking the CMC-leaves as starting surfaces. As for the flow (1.10) above, we show that with our choice of initial data the solution of (1.15) exists for all times and converges to a limit that is CSTMC. The CMC-property of the initial surface allows us to carry out the spectral analysis on a more restricted class of surfaces, on which the stability operator has a better behaviour than in the previous case. On the other hand, there are new difficulties because the speed of the flow is nonlinear in H and moreover the flow is no longer area-decreasing. In particular, the crucial step of our analysis (Theorem 5.0.14), where the invariance of the roundness class under the flow is obtained, requires a different argument from the space-like case, which combines the estimates of different integral norms of the oscillation of the space time mean curvature. Our convergence result provides an alternative construction of the foliation obtained in [CS21], and has an independent interest in the analysis of the behaviour of curvature flows in asymptotically flat spaces. A further motivation for our study comes from the recent work of Huisken and Wolff, who study a spacetime version of the inverse mean curvature flow, see [HW22].

We finally highlight that the hypotheses of the results described in Chapter 5 could be generalized, for example modifying $\mathcal{H}$ to be a nonlinear function of H with suitable asymptotics. However, we do not focus our interest on these technicalities in order to maintain a point of contact between the flow and the physical setting. We end stating the main theorem of this Chapter.

Theorem 1.1.4 (T.). *Let $(M, \overline{g}, \overline{K})$ be a constrained $C^2_{\frac{1}{2}+\delta}$-asymptotically flat initial data set in the sense of Definition 1.1.3, and suppose that $E_{\mathrm{ADM}} > 0$. Fix $q \geq 2$. Let $\iota : \Sigma \hookrightarrow M$ be a* closed CMC-surface immersed *in $(M, \overline{g})$ and, setting $\sigma = \sigma_\Sigma := \sqrt{|\Sigma|/4\pi}$, suppose that there exists $C_0 > 0$ such that*

$$\|\overset{\circ}{A}\|_{L^4(\Sigma)} \leq C_0 \sigma^{-1-\delta}, \qquad |\vec{z}_\Sigma| \leq C_0 \sigma^{1-\delta}, \qquad \frac{\sigma}{r_\Sigma} \leq 1 + C_0^{-1}, \tag{1.16}$$

where $r_\Sigma := \min_{x\in\Sigma} |\vec{x}(x)|$. Then, there exists $\sigma_0 = \sigma_0(C_0, \overline{c}, \delta, q) > 1$ such that if $\sigma > \sigma_0$, the solution Σ_t to the spacetime mean curvature flow (1.15) exists for every $t \in [0, \infty)$ and converges exponentially fast to a surface $\Sigma^{\mathrm{st}}_\infty$ satisfying the prescribed mean curvature equation

$$H^q_{\Sigma^{\mathrm{st}}_\infty} = |P|^q_{\Sigma^{\mathrm{st}}_\infty} + \hbar^q_{\Sigma^{\mathrm{st}}_\infty} \tag{1.17}$$

for some constant $\hbar_{\Sigma^{\mathrm{st}}_\infty} > 0$.

Chapter 2

Surfaces in asymptotically flat spaces

2.0.1 Definitions and basic properties

In this introductory Section we briefly review some definitions, notations and well-known results in Riemannian geometry.

In the following we will always indicate with $(M, \overline{g})$ a 3-dimensional complete Riemannian manifold. Local coordinates on M will be indicated with Greek letters, such as $\alpha, \beta, \gamma, \omega$, etc. Moreover, Σ or $\iota : \Sigma \hookrightarrow M$ will always be a surface immersed in M, with induced metric $g := \iota^* \overline{g}$. Local coordinates on Σ will be indicated with Latin letters, such as i, j, k, l, etc. Similarly to the notation for the metrics, the overlined geometric quantity will refer to $(\Sigma, \overline{g})$; otherwise, they refer to (Σ, g). Sometimes, for example when writing the Ricci tensor of Σ, we will put the focus on Σ writing Ric^{Σ}.

We will indicate with TM the tangent space to M, and with $T\Sigma$ the tangent (phase) plane to Σ. The outer unit normal to Σ in TM will be indicated with ν. We highlight that the field ν is defined on Σ but for each $x \in \Sigma$ we have $\nu_x \in T_{\iota(x)}M$. In general, to indicate that V is a smooth vector field, we write $V \in \mathcal{F}^\infty$.

As we will review in Section 2.1, we represent with $A = \{h_{ij}\}$ the *second fundamental form* of Σ, and with $H = g^{ij} h_{ij} = \mathrm{tr}_g(A)$ the *mean curvature* of Σ. Moreover, $d\mu = d\mu_g$ will be the volume form of Σ induced by g.

We now recall the main well-known identities. The Gauss equation and the Gauss-Codazzi equation say, respectively, that

$$\mathrm{Rm}_{kilm} = \overline{\mathrm{Rm}}_{kilm} + h_{kl} h_{im} - h_{km} h_{il}, \tag{2.1}$$

$$\nabla_i h_{jk} - \nabla_k h_{ij} = \overline{\mathrm{Rm}}_{\omega jki} \nu^\omega. \tag{2.2}$$

Tracing (2.2) with respect to the indexes i and j, we get

$$\nabla_i h_{ik} - \nabla_k H = \overline{\mathrm{Ric}}_{\omega k} \nu^\omega - \overline{\mathrm{Rm}}_{\omega\alpha k\beta} \nu^\omega \nu^\alpha \nu^\beta = \overline{\mathrm{Ric}}_{\omega k} \nu^\omega. \tag{2.3}$$

This allows us to deduce the Simons' identity. We briefly prove it as stated in [Met07]. This identity relates the Hessian of the mean curvature to the laplacian of the second fundamental form.

Lemma 2.0.1 (Simons' identity)**.**

$$\begin{aligned}\Delta h_{ij} =& \nabla_i \nabla_j H + H h_i^l h_{lj} - |A|^2 h_{ij} + h_i^l \overline{\mathrm{Rm}}_{kjkl} + h^{lk} \overline{\mathrm{Rm}}_{lijk} \\ &+ \nabla_j \left(\overline{\mathrm{Ric}}_{i\omega} \nu^\omega \right) + \nabla^l \left(\overline{\mathrm{Rm}}_{\omega ijl} \nu^\omega \right)\end{aligned} \tag{2.4}$$

Proof. We use normal coordinates on Σ. Taking the derivative ∇_k of (2.2) we get

$$\nabla_k \nabla_l h_{ij} = \nabla_k \nabla_i h_{jl} + \nabla_k \left(\overline{\mathrm{Rm}}_{\omega jil} \nu^\omega \right). \tag{2.5}$$

By the commutation of the derivatives and the symmetry of the second fundamental form, it holds

$$\nabla_k \nabla_i h_{lj} = \nabla_i \nabla_k h_{lj} + \mathrm{Rm}^{\Sigma}_{kilm} h_{mj} + \mathrm{Rm}^{\Sigma}_{kijm} h_{ml}.$$

Combining this with (2.2) and the Gauss equation (2.1), we have

$$\begin{aligned} \nabla_k \nabla_l h_{ij} =& \nabla_i \nabla_k h_{lj} + \overline{\mathrm{Rm}}_{kilm} h_{mj} + \left(h_{kl} h_{im} - h_{km} h_{il}\right) h_{mj} \\ &+ \overline{\mathrm{Rm}}_{kijm} h_{ml} + \left(h_{kj} h_{im} - h_{km} h_{ij}\right) h_{ml} + \nabla_k \left(\overline{\mathrm{Rm}}_{\omega jil} \nu^{\omega}\right) \end{aligned} \tag{2.6}$$

Using again (2.2), we have

$$\nabla_i \nabla_k h_{lj} = \nabla_i \left(\nabla_j h_{lk} + \overline{\mathrm{Rm}}_{\omega lkj} \nu^{\omega}\right). \tag{2.7}$$

The thesis follows combining (2.6) and (2.7), summing over $k = l$ and noting that

$$h_{kj} h_{im} h_{mk} - h_{km} h_{ik} h_{mj} = h_{jk} h_{km} h_{mi} - h_{jm} h_{mk} h_{ki} = 0.$$

□

Lemma 2.0.2. *Let M and Σ be as above, and consider a smooth bilinear form $\overline{\mathrm{B}}_{\alpha\beta}$ on M. Then*

$$\nabla_k \overline{\mathrm{B}}_{ij} = \left(\overline{\nabla}\left(\overline{\mathrm{B}}\right)\right)_{ijk} - h_{ki} \overline{\mathrm{B}}_{\omega j} \nu^{\omega} - h_{kj} \overline{\mathrm{B}}_{i\gamma} \nu^{\gamma}, \tag{2.8}$$

where $\overline{\nabla}\left(\overline{\mathrm{B}}\right)$ is the covariant derivative of $\overline{\mathrm{B}}$.

Proof. Consider local normal coordinates $\{e_i\}_i$ on Σ, completed by ν to a local frame of M. Since $\overline{\mathrm{B}}(e_i, e_j)$ is a scalar function on Σ,

$$\nabla_k(\overline{\mathrm{B}}(e_i, e_j)) = \left(\overline{\nabla}\left(\overline{\mathrm{B}}\right)\right)(e_i, e_j, e_k) - \overline{\mathrm{B}}(\overline{\nabla}_k e_i, e_j) - \overline{\mathrm{B}}(e_i, \overline{\nabla}_k e_j).$$

We used that ∇_k and $\overline{\nabla}_k$, on scalar functions, coincide with the derivative with respect to e_k. Note that, by definition of second fundamental form and the choice of normal coordinates we have

$$\overline{\nabla}_k e_i = \nabla_k e_i + h_{ki} \nu = h_{ki} \nu,$$

and thus the thesis follows. □

2.0.2 Energy and center of mass

The mass condition introduced in Definition 1.1.1 can be generalized by the request $\overline{\mathrm{S}} \in L^1(M, \overline{\mathrm{g}})$. This allows to define the ADM-*energy*, named after Arnowitt, Deser and Misner [ADM61] recalled in (1.4). However, for our aims, we will use the following equivalent characterization of the energy of the system, proved in [MT16] but already well-known in literature (see, for example, [Chr86], [Sch88]).

Definition 2.0.3 (ADM-energy)**.** *Let $(M, \overline{\mathrm{g}})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat 3-manifold that satisfies the mass condition. The ADM-energy is defined as*

$$E_{\mathrm{ADM}} := -\lim_{R\to\infty} \frac{R}{8\pi} \int_{\vec{x}^{-1}(\mathbb{S}_R(\vec{0}))} \overline{\mathrm{G}}(\nu_R, \nu_R)\, d\mu_R, \tag{2.9}$$

where $\overline{\mathrm{G}} := \overline{\mathrm{Ric}} - \left(\frac{\overline{\mathrm{S}}}{2}\right) \overline{\mathrm{g}}$ is the (spacelike) Einstein tensor.

In the time-symmetric case (i.e. $\overline{\mathrm{K}} \equiv 0$) the ADM-energy is also called ADM-mass. However, when $\overline{\mathrm{K}} \not\equiv 0$, the two definitions differ. Thus, avoiding ambiguities, during this

Thesis we will refer to the limit in (2.9) always as ADM-energy. We remark also that, in the case of the asymptotically Schwarzschildean manifolds of Huisken-Yau [HY96], the ADM-energy coincides with the parameter $m > 0$. We now introduce the notion of ADM-mass when $\overline{\mathrm{K}} \not\equiv 0$.

Definition 2.0.4 (ADM-mass)**.** *Let* $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ *be a* $C^2_{1/2+\delta}$ *asymptotically flat initial data set and define* $\overline{\varrho} := \left(\mathrm{tr}_{\overline{\mathrm{g}}}\overline{\mathrm{K}}\right)\overline{\mathrm{g}} - \overline{\mathrm{K}}$*, the so called conjugate momentum tensor. The ADM-linear momentum is a vector* $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in \mathbb{R}^3$ *defined by*

$$\pi_\beta := \frac{1}{8\pi} \lim_{R\to\infty} \int_{|\vec{x}|=R} \sum_{\alpha=1}^{3} \varrho_{\alpha\beta} \frac{\vec{x}_\alpha}{R}\, d\mu^e_R,$$

where $d\mu^e_R$ *is the Euclidean volume form on* $\{|\vec{x}| = R\}$*. Then the ADM-mass of* $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ *is defined as*

$$\overline{m}_{\mathrm{ADM}} := \sqrt{E^2_{\mathrm{ADM}} - |\vec{\pi}|^2}.$$

Using the divergence theorem, it can be easily proved that the mass condition implies that the ADM-energy is well defined. The notion of mass also clarifies why we require the decay exponent $\frac{1}{2} + \delta$, with $\delta \in (0, \frac{1}{2}]$. In fact, in the case $\delta \leq 0$ it has been proved that $\mathbb{R}^3$ can be equipped with a chart which does not have zero energy, as one would expect from the Euclidean space, see [DS83]. On the other hand, if $\delta > \frac{1}{2}$, it is easy to see, from the definition of E_{ADM}, that $E_{\mathrm{ADM}} = 0$. This is not desirable when working with centers of mass (as can be seen in the definition we will give in a moment); moreover, we will work just in the case of positive ADM-energy, as we will see in the next Chapters and as we underlined in the introduction.

In [RT74] and [BOM87], Regge-Teitelboim and Beig-Ó Murchadha introduced the so called ADM-*center of mass*, named again after Arnowitt, Deser and Misner and recalled in (1.5). Similarly to the case of the ADM-energy, we use here an equivalent definition introduced by Miao and Tam in [MT16].

Definition 2.0.5 (ADM-center of mass)**.** *Let* $(M, \overline{\mathrm{g}}, \vec{x})$ *be a* $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold *with* $E_{\mathrm{ADM}} \neq 0$*. We say that* $(M, \overline{\mathrm{g}}, \vec{x})$ *admits the* ADM-center of mass *if the limits*

$$\omega_\alpha := \lim_{R\to\infty} \int_{\{|\vec{x}|=R\}} \overline{\mathrm{G}}(Y_\alpha, \nu_R)\, d\mu_R \tag{2.10}$$

exist finite, where $Y_\alpha(\overline{x}) = \left(R^2\delta^{\alpha\beta} - 2\overline{x}^\alpha\overline{x}^\beta\right)\frac{\partial}{\partial \overline{x}_\beta}$*. In this case, we set* $\vec{\omega} := (\omega_1, \omega_2, \omega_3)$ *and* $\vec{\mathcal{C}} := \frac{\vec{\omega}}{8\pi E_{\mathrm{ADM}}}$.

The mass condition is not enough to assure the existence of the vector $\vec{\mathcal{C}}$. However, an asymptotic symmetry condition which guarantees the existence of the center of mass is the so called (strong-)Regge-Teitelboim condition. Even if the original references for this condition goes back to [RT74], we mainly refer to [Hua10], [Ner15].

Definition 2.0.6 (Strong Regge-Teitelboim condition)**.** *Let* $(M, \overline{\mathrm{g}}, \vec{x})$ *a* $C^2_{\frac{1}{2}+\delta}$*-asymptotically flat manifold. We say that this manifold satisfies the strong or* $C^2_{\frac{3}{2}+\delta}$*-Regge-Teitelboim condition if there exists a positive constant* $\overline{c} > 0$ *such that*

$$\left|\left(\overline{\mathrm{g}}_{\alpha\beta}\right)_{\overline{x}} - \left(\overline{\mathrm{g}}_{\alpha\beta}\right)_{-\overline{x}}\right| + |\overline{x}|\left|\overline{\Gamma}^\gamma_{\alpha\beta}(\overline{x}) + \overline{\Gamma}^\gamma_{\alpha\beta}(-\overline{x})\right| + |\overline{x}|^2\left|\overline{\mathrm{Ric}}_{\alpha\beta}(\overline{x}) - \overline{\mathrm{Ric}}_{\alpha\beta}(-\overline{x})\right| \\ + |\overline{x}|^{\frac{5}{2}}\left|\overline{\mathrm{S}}(\overline{x}) - \overline{\mathrm{S}}(-\overline{x})\right| \leq \frac{\overline{c}}{|\overline{x}|^{\frac{3}{2}+\delta}},$$

for every $\overline{x} \in M \setminus \mathrm{C}$.

The following Lemma shows how the divergence theorem and the Regge-Teitelboim condition assure the existence of $\vec{\mathcal{C}}$.

Lemma 2.0.7. *Let* $(M, \overline{\mathrm{g}}, \vec{x})$ *be a* $C^2_{\frac{1}{2}+\delta}$*-asymptotically flat manifold. If it satisfies the* $C^2_{\frac{3}{2}+\delta}$*-Regge-Teitelboim conditions, then the* ADM-center of mass, *i.e. the vector* $\vec{\mathcal{C}} = \frac{\vec{\omega}}{8\pi E}$*, is well-defined.*

Proof. We prove the existence of the limits in (2.10). Identifying, for sake of simplicity, $\mathbb{S}_R(\vec{0})$ with its preimage through $\vec{x}$, and using the divergence theorem, we compute, for S larger than R,

$$\left| \int_{\mathbb{S}_S(\vec{0})} \overline{\mathrm{G}}\left(Y_\alpha, \nu\right)\, d\mu - \int_{\mathbb{S}_R(\vec{0})} \overline{\mathrm{G}}\left(Y_\alpha, \nu\right)\, d\mu \right| = \left| \int_{\mathbb{B}_S(\vec{0})\setminus\mathbb{B}_R(\vec{0})} \overline{\mathrm{G}}_{\beta\gamma} \overline{\partial}_{\overline{x}_\beta} \left(Y_\alpha\right)^\gamma \, \mathrm{d}\overline{\mathrm{x}} \right|, \tag{2.11}$$

where $\mathrm{d}\overline{\mathrm{x}}$ is the volume form of $(M, \overline{\mathrm{g}})$. Since $\overline{\partial}_{\overline{x}_\beta}(Y_\alpha)^\gamma = 2\overline{x}^\beta \delta^{\alpha\gamma} - 2\delta_{\alpha\beta}\overline{x}^\gamma - 2\overline{x}^\alpha \delta_{\beta\gamma}$, we find that equation (2.11) equals

$$\begin{aligned} &\left| 2\int_{\mathbb{B}_S(\vec{0})\setminus\mathbb{B}_R(\vec{0})} \overline{\mathrm{G}}_{\beta\alpha}\overline{x}^\beta \, \mathrm{d}\overline{\mathrm{x}} - 2\int_{\mathbb{B}_S(\vec{0})\setminus\mathbb{B}_R(\vec{0})} \overline{\mathrm{G}}_{\alpha\gamma}\overline{x}^\gamma \, \mathrm{d}\overline{\mathrm{x}} + 2\int_{\mathbb{B}_S(\vec{0})\setminus\mathbb{B}_R(\vec{0})} \overline{\mathrm{G}}_{\beta\beta}\overline{x}^\alpha \, \mathrm{d}\overline{\mathrm{x}} \right| \\ &\qquad = 2\left| \int_{\mathbb{B}_S(\vec{0})\setminus\mathbb{B}_R(\vec{0})} \overline{\mathrm{G}}_{\beta\beta}\overline{x}^\alpha \, \mathrm{d}\overline{\mathrm{x}} \right|, \end{aligned}$$

using that $\overline{\mathrm{G}}$ is symmetric. Since $\overline{\mathrm{G}}_{\beta\beta} = \frac{\overline{\mathrm{S}}}{2}$, we proceed as follows. We introduce the antipodal map on M, given by $p : M \to M$ that sends $\overline{x} \mapsto (\vec{x})^{-1}(-\vec{x}(\overline{x})) =: -\overline{x}$. Then, there exists $\mathrm{U} \subset M$ such that $\mathbb{B}_S(\vec{0}) \setminus \mathbb{B}_R(\vec{0}) = \mathrm{U} \cup p(\mathrm{U})$. Then

$$\int_{\mathbb{B}_S(\vec{0})\setminus\mathbb{B}_R(\vec{0})} \overline{\mathrm{S}}_{\overline{x}}\overline{x}^\alpha \, \mathrm{d}\overline{\mathrm{x}} = \int_{\mathrm{U}} \overline{\mathrm{S}}_{\overline{x}}\overline{x}^\alpha \, \mathrm{d}\overline{\mathrm{x}} + \int_{p(\mathrm{U})} \overline{\mathrm{S}}_{\overline{x}}\overline{x}^\alpha \, \mathrm{d}\overline{\mathrm{x}} = \int_{\mathrm{U}} \overline{\mathrm{S}}_{\overline{x}}\overline{x}^\alpha \, \mathrm{d}\overline{\mathrm{x}} - \int_{\mathrm{U}} \overline{\mathrm{S}}_{-\overline{x}}\overline{x}^\alpha \, \mathrm{d}\overline{\mathrm{x}}.$$

Then

$$\left| \int_{\mathbb{B}_S(\vec{0})\setminus\mathbb{B}_R(\vec{0})} \overline{\mathrm{S}}_{\overline{x}}\overline{x}^\alpha \, \mathrm{d}\overline{\mathrm{x}} \right| \le \int_{\mathrm{U}} |\overline{\mathrm{S}}_{\overline{x}} - \overline{\mathrm{S}}_{-\overline{x}}||\overline{x}^\alpha| \, \mathrm{d}\overline{\mathrm{x}} \le \overline{c} \int_{\mathbb{B}_S(\vec{0})\setminus\mathbb{B}_R(\vec{0})} |\overline{x}^\alpha||\overline{x}|^{-4-\delta} \, \mathrm{d}\overline{\mathrm{x}},$$

using the Regge-Teitelboim condition. Using polar coordinates, we conclude that

$$\lim_{R,S\to\infty} \left| \int_{\mathbb{S}_S(\vec{0})} \overline{\mathrm{G}}\left(Y_\alpha, \nu\right)\, d\mu - \int_{\mathbb{S}_R(\vec{0})} \overline{\mathrm{G}}\left(Y_\alpha, \nu\right)\, d\mu \right| = 0.$$

This proves the claim. □

Weak Regge-Teitelboim condition. As we have already seen in the hypothesis of Theorem 1.1.2 in Chapter 1, we will assume a weak version of the Regge-Teitelboim condition, which differs from Definition 1.1.2 for a decay $1 + \delta$ instead of $\frac{3}{2} + \delta$. This change of the decay assumptions does not assure anymore the existence of $\vec{\mathcal{C}}$. However, it is well-known that if the weak Regge-Teitelboim condition holds and $\vec{\mathcal{C}}$ exists, then the foliation constructed in Theorem 1.1.2 admits a barycenter which coincides with the ADM-center of mass. See [Ner15, Thm. 6.3].

ADM-CoM in initial data sets. Cederbaum-Nerz [CN15] showed the existence of explicit examples of initial data sets with non-converging ADM-center of mass. These examples arise as 3-dimensional submanifolds of the Schwarzschild spacetime, solution to the vacuum field equations, and thus they are expected to have $\vec{0}$ as ADM-center of mass. Recently, Cederbaum-Sakovich [CS21] have given a characterization for the existence of a modified center of mass that takes into account the spacetime nature of the examples constructed in [CN15]. In particular, the ADM-center of mass is modified by a correction term which compensates the non-converging behavior of the limit in the definition of center of mass, giving the expected center of mass. In the second part of the Thesis (Chapter 5) we will define a spacetime version of the volume preserving mean curvature flow with the aim of recovering the modified ADM-center of mass introduced in [CS21].

2.1 Surfaces in asymptotically flat manifolds

We dedicate this Section to surfaces in asymptotically flat manifolds, both in the time-symmetric and the non time-symmetric case. Even if in the Introduction we fixed the notation for describing Riemannian 3-manifolds, we saved until the present Section the moment to fix the basic definitions about surfaces, since they are the main object of the whole Thesis, and we want to imprint their notion in the context of the asymptotically flat manifolds as presented up to this point in the Chapter.

From now on, with 2-*surface* we mean an immersion $\iota : \Sigma \hookrightarrow M \setminus \mathrm{C}$, with $\dim \Sigma = 2$, which is closed, connected and 2-faced. Since $M \setminus \mathrm{C}$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{\mathbb{B}}_1(\vec{0})$, the surface Σ inherits two Riemannian metrics: a *physical* metric $g := \iota^* \overline{\mathrm{g}}$ and a *Euclidean* metric $g^e := \iota^* \overline{\mathrm{g}}^e$, where $\overline{\mathrm{g}}^e$ is the Euclidean metric on M. From now on, we will use the apex e each time a quantity is computed with respect to the Euclidean metric, and we will omit the apex if it is computed using the physical metric. Then, fixed an outer unit normal $\nu : \Sigma \to TM$, we represent with $A = \{h_{ij}\}$, H and $d\mu$, respectively, the second fundamental form, the mean curvature and the volume form of Σ with respect to g. Moreover we write $\overset{\circ}{A} = A - \frac{H}{2} g$. On the other hand, if $\nu^e : \Sigma \to TM$ is the outer normal field of Σ with respect to $\overline{\mathrm{g}}^e$, we represent the same quantities with $A^e = \{h^e_{ij}\}$, H^e, $d\mu^e$ and $\overset{\circ}{A}{}^e$. Observe that, when we are on a hypsersurface Σ, we use the latin indexes i, j, k, l, etc, to distinguish from the ambiental coordinates, which are indicated with the greek indexes $\alpha, \beta, \gamma, \epsilon$, etc. Finally, we define

$$h := \frac{1}{|\Sigma|} \int_\Sigma H \, d\mu, \qquad h^e := \frac{1}{|\Sigma^e|} \int_\Sigma H^e \, d\mu^e,$$

which are, respectively, the mean of the mean curvature computed with respect to the physical and the Euclidean metric. Here $|\Sigma| = \int_\Sigma d\mu$ and $|\Sigma^e| = \int_\Sigma d\mu^e$.

In order to estimate the Euclidean position of an immersed surface and its area, we introduce now some useful definitions.

Definition 2.1.1. *Let $(M, \overline{\mathrm{g}})$ be a 3-manifold, and consider an immersed surface $\iota : \Sigma \to M \setminus \mathrm{C}$ with induced metric $g = \iota^* \overline{\mathrm{g}}$. Then we set*

$$r_\Sigma := \min_{x \in \Sigma} |\vec{x}(\iota(x))|, \qquad R_\Sigma := \max_{x \in \Sigma} |\vec{x}(\iota(x))|, \qquad \sigma_\Sigma := \sqrt{\frac{|\Sigma|_g}{4\pi}}.$$

These radii are called Euclidean radius, Euclidean diameter and area radius, respectively.

Surfaces of codimension 2. If $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ is an asymptotically flat initial data set and $\Sigma \hookrightarrow M$, the surface Σ can be seen as a codimension 2 submanifold of the spacetime associated to $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$. As a result, together with the spacelike mean curvature, which we continue to call simply *mean curvature*, we have an additional scalar mean curvature called *timelike mean curvature* and a *spacetime mean curvature* which takes into account each extrinsic curvature.

Definition 2.1.2. *Fix $q \geq 2$. Let $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ be $C^2_{\frac{1}{2}+\delta}$-asymptotically flat initial data set. Let $\iota : \Sigma \hookrightarrow M$ be a surface of M, with induced metric $g := \iota^* \overline{\mathrm{g}}$. We define the timelike mean curvature of Σ as $P := \mathrm{tr}_g(\overline{\mathrm{K}}) := g^{ij}\overline{\mathrm{K}}_{ij}$. Let moreover H be the (spacelike) mean curvature of Σ. Then, we define the spacetime (q-)mean curvature of Σ, if it exists, as*

$$\mathcal{H}_q := \sqrt[q]{H^q - |P|^q}, \tag{2.12}$$

This is essentially the Minkowski q-length of the vector $(\vec{H}, P)$, where $\vec{H}$ is the vector mean curvature of Σ. In the case (2.12) is globally defined on Σ, we furthermore set

$$\hbar_q := \fint_\Sigma \mathcal{H}_q \, d\mu. \tag{2.13}$$

Since q will be arbitrary but fixed, in the following we will simply write $\mathcal{H}$ and $\hbar$, without ambiguities.

Lemma 2.1.3. *Let $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat initial data set. Then there exist constants $C = C(\overline{c}) > 0$ and $\mathrm{c_{in}} = \mathrm{c_{in}}(\overline{c}) > 0$, also depending on the choice of q, such that if $\iota : \Sigma \hookrightarrow M \setminus \mathrm{C}$ is a* surface *with induced metric $g := \iota^* \overline{\mathrm{g}}$ and there exists $\sigma > 1$ such that*

$$2\sigma \geq r_\Sigma \geq \frac{\sigma}{2}, \qquad \frac{1}{\sigma} \leq H_x \leq \frac{\sqrt{5}}{\sigma} \quad \forall x \in \Sigma. \tag{2.14}$$

then the following properties hold.

(i) $\|P\|_{L^\infty(\Sigma)} + \sigma \|\nabla P\|_{L^\infty(\Sigma)} \leq C\sigma^{-\frac{3}{2}-\delta}$;

(ii) *$\mathcal{H}_x$ is well defined for every $x \in \Sigma$;*

(iii)
$$\sup_\Sigma |\mathcal{H} - H| \leq C\sigma^{-1-\frac{1}{2}q-q\delta}, \qquad |h - \hbar| \leq C\sigma^{-1-\frac{1}{2}q-q\delta};$$

(iv) *If H is constant on Σ, i.e. $H \equiv h$, then $\|\mathcal{H} - \hbar\|_{L^\infty(\Sigma)} + \sigma\|\nabla\mathcal{H}\|_{L^\infty(\Sigma)} \leq \mathrm{c_{in}}\sigma^{-1-\frac{1}{2}q-q\delta}$.*

In the following, we will mainly use the H^1-estimate on $\mathcal{H} - \hbar$, which follows from the $W^{1,\infty}$ bound in the above statement, and we will continue to call $\mathrm{c_{in}}$ the constant at the right-hand side. Observe that this is the only case in which we use the lowercase in order to indicate a constant depending on the setting and not on the "roundness of the surface" (in a sense we will make more clear later, see Definition 2.3.1 below).

Proof. Point (i) and point (ii) follow from $|P_x| \leq 2|\overline{\mathrm{K}}|_{\overline{\mathrm{g}}} \leq 2\overline{c}r_\Sigma^{-\frac{3}{2}-\delta} = \mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$ and

$$(\nabla P)_i = g^{ik}g^{jl}\nabla_k\overline{\mathrm{K}}_{jl},$$

since $|\nabla_k\overline{\mathrm{K}}_{jl}| \leq \overline{c}\sigma^{-\frac{5}{2}-\delta}$, using also Lemma 2.0.2. Thus, point (iii) follows from the Lagrange mean value theorem. Also point (iv) follows in a similar way, using the constancy of the mean curvature and the equation

$$\mathcal{H}^{q-1}\nabla\mathcal{H} = H^{q-1}\nabla H - |P|^{q-1}\left(\frac{P}{|P|}\right)\nabla P. \tag{2.15}$$

□

2.1.1 Geometry of surfaces in asymptotically flat manifolds

The definition of asymptotically flat manifold implies that the decay rate between the physical and the Euclidean metric is controlled outside a compact set. In this Section, we show that this fact is inherited by the induced metrics g and g^e, by the induced connections and the other (extrinsic) geometric objects of a surface $\Sigma \hookrightarrow M$. The facts stated and proved here are well-known, see for example [Met07] or [CS21, Lemma 11]. We give a proof of these results for sake of completeness.

Lemma 2.1.4. *Let $(M, \overline{g})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and let $\iota : \Sigma \hookrightarrow M$ be an immersed surface. Then there exists $C = C(\overline{c}) > 0$ such that*

$$|g - g^e|_g \leq C|\vec{x}|^{-\frac{1}{2}-\delta}, \qquad |\Gamma^k_{ij} - (\Gamma^e)^k_{ij}| \leq C|\vec{x}|^{-\frac{3}{2}-\delta},$$

where Γ^k_{ij} and $(\Gamma^e)^k_{ij}$ are the Christoffel symbols of Σ with respect to g and g^e, respectively.

Proof. The first inequality is straightforward, using the asymptotically flatness of the 3-manifold and the restriction of the metrics to Σ. For the other inequality, consider in an arbitrary point $x \in \Sigma$ and the frames $\{e_1, e_2, \nu\}$ and $\{e_1, e_2, \nu^e\}$ where ν and ν^e are orthogonal to $T_x\Sigma$ with respect to $\overline{g}$ and $\overline{g}^e$, respectively. Using the definition of (vector) second fundamental form and since $\nabla_{e_i} e_j \in T_x\Sigma$, we have

$$\begin{aligned}\Gamma^k_{ij} - (\Gamma^e)^k_{ij} &= \langle \nabla_{e_i} e_j, e_k \rangle_g - \langle \nabla^e_{e_i} e_j, e_j \rangle_{g^e} \\ &= \langle \overline{\nabla}_{e_i} e_j - A(e_i, e_j)\nu, e_k \rangle_{\overline{g}} - \langle \overline{\nabla}^e_{e_i} e_j - A^e(e_i, e_j)\nu^e, e_k \rangle_{\overline{g}^e} \\ &= \langle \overline{\nabla}_{e_i} e_j - \overline{\nabla}^e_{e_i} e_j, e_k \rangle_{\overline{g}} = \mathrm{O}(|\vec{x}|^{-\frac{3}{2}-\delta}).\end{aligned} \tag{2.16}$$

□

As we measured the distance between the two induced metrics in an suitable coordinate system, we analyze now how the *volume form* changes from the physical to the Euclidean point of view.

Lemma 2.1.5 (Volume forms)**.** *Let $(M, \overline{g}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and consider an surface $\iota : \Sigma \hookrightarrow M$. Consider $g = \iota^*\overline{g}$ and $g^e = \iota^*\overline{g}^e$. Then, there exist $C = C(\overline{c}) > 0$ and $\sigma_0 = \sigma_0(\overline{c})$ such that, if $r_\Sigma \geq \sigma \geq \sigma_0$,*

(i) *The volume forms satisfy*

$$|d\mu_g - d\mu_{g^e}| \leq C\sigma^{-\frac{1}{2}-\delta} d\mu_g;$$

(ii) *For every $\psi \in C^\infty(\Sigma; \mathbb{R})$ it holds*

$$\left| \int_\Sigma \psi \; d\mu_g - \int_\Sigma \psi d\mu_{g^e} \right| \leq C\sigma^{-\frac{1}{2}-\delta} \|\psi\|_{L^1(\Sigma)}.$$

Proof. Let $\{e_1, e_2\}$ be a local frame on Σ with respect to a local coordinate system. In the following we will use the abuse of notation of identifying the metric coordinates g_{ij} and g^e_{ij} with their matrix representation (g_{ij}) and (g^e_{ij}) respectively. Using the mean value theorem to obtain

$$\left| \sqrt{\det(g_{ij})} - \sqrt{\det(g^e_{ij})} \right| \leq |\det(g_{ij}) - \det(g^e_{ij})|,$$

and Lemma 2.1.4, we conclude that

$$d\mu_g - d\mu_{g^e} = \mathrm{O}(|\vec{x}|^{-\frac{1}{2}-\delta}) d\mu_g.$$

Point (ii) follows integrating the volume form against a smooth function ψ. □

Lemma 2.1.6. *Let $(M, \overline{\mathrm{g}}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold, and $\Sigma \hookrightarrow$ be a surface. Then, there exist $C = C(\overline{c}) > 0$ and $\sigma_0 = \sigma_0(\overline{c}) > 1$ such that, if $r_\Sigma \geq \sigma \geq \sigma_0$, then*

$$|\nu - \nu^e|_g \leq C\sigma^{-\frac{1}{2}-\delta}, \qquad |\nabla\nu - \nabla^e\nu^e|_g \leq C\sigma^{-\frac{3}{2}-\delta}.$$

Proof. Set $X := \nu^e - \nu$, and consider an adapted coordinate frame on Σ, say $\{e_1, e_2\}$, such that $\overline{\mathrm{g}}^e(d\iota(e_i), d\iota(e_j)) = \delta_{ij}$. By construction $\overline{\mathrm{g}}^e(\nu^e, \nu^e) = 1$, and $\overline{\mathrm{g}}^e(\nu^e, d\iota(e_i)) = 0$. Then

$$1 = \overline{\mathrm{g}}^e(\nu + X, \nu + X) = \overline{\mathrm{g}}^e(\nu, \nu) + 2\overline{\mathrm{g}}^e(\nu, X) + \overline{\mathrm{g}}^e(X, X). \tag{2.17}$$

Let $\{\overline{e}_\alpha\}_\alpha$ an orthonormal coordinate system in $(M, \overline{\mathrm{g}}^e)$, and consider the coordinates $X = X_\alpha\overline{e}_\alpha$, $\nu = \nu_\alpha\overline{e}_\alpha$. The equation (2.17) then becomes

$$\sum_\alpha X_\alpha^2 + 2\sum_\alpha X_\alpha\nu_\alpha + (\overline{\mathrm{g}}^e(\nu, \nu) - 1) = 0.$$

This is a second order equation in $\mathbb{R}^3$, of the form $|\vec{X}|^2 + 2\langle\vec{X}, \vec{\nu}\rangle_{\mathbb{R}^3} + \epsilon = 0$, where $\vec{X} = (X_1, X_2, X_3)$, $\vec{\nu} = (\nu_1, \nu_2, \nu_3)$ and $\epsilon := \overline{\mathrm{g}}^e(\nu, \nu) - 1$. Writing $\vec{X} = t\vec{\omega}$, with $t > 0$ and $|\vec{\omega}| = 1$, we have

$$t^2 + 2t\langle\vec{\omega}, \vec{\nu}\rangle_{\mathbb{R}^3} + \epsilon = 0,$$

which implies $t = \frac{-2\langle\vec{\omega},\vec{\nu}\rangle \pm\sqrt{4\langle\vec{\omega},\vec{\nu}\rangle^2 - 4\epsilon}}{2}$. This implies the thesis.

For the second part, we want to study the decay of $|\nabla X|$. Consider the identity $\overline{\mathrm{g}}(\nu, d\iota(e_i)) \equiv 0$. Deriving this expression with respect to e_j, we get that

$$\overline{\mathrm{g}}_{\alpha\beta}\nabla_{e_j}\nu^\alpha\langle d\iota(e_i), \overline{e}_\beta\rangle_{\overline{\mathrm{g}}} \equiv 0.$$

Using $\nabla_{e_j}\nu = \nabla_{e_j}\nu^e - \nabla_{e_j}X$, it turns out that, in order to estimate $|\nabla X|_g$ it is sufficient to estimate $\overline{\mathrm{g}}_{\alpha\beta}\nabla_{e_j}(\nu^e)^\alpha\langle d\iota(e_i), \overline{e}_\beta\rangle_{\overline{\mathrm{g}}}$. Since

$$(\nabla_j \mathrm{v} - \nabla^e_j\mathrm{v})^i = \left(\frac{\partial \mathrm{v}^i}{\partial x_j} + \Gamma^i_{jk}\mathrm{v}^k\right) - \left(\frac{\partial \mathrm{v}^i}{\partial x_j} + (\Gamma^e)^i_{jk}\mathrm{v}^k\right) = (\Gamma^i_{jk} - (\Gamma^e)^i_{jk})\mathrm{v}^k,$$

and $\overline{\mathrm{g}} - \overline{\mathrm{g}}^e = \mathrm{O}(\sigma^{-\frac{1}{2}-\delta})$ and also $\overline{\mathrm{g}}^e_{\alpha\beta}\nabla_{e_j}(\nu^e)^\alpha\langle d\iota(e_i), \overline{e}_\beta\rangle_{\overline{\mathrm{g}}^e} \equiv 0$, because by definition $\overline{\mathrm{g}}^e(\nu^e, d\iota(e_i)) \equiv 0$, we get the thesis. □

We end this Section reviewing a result that compares extrinsic curvatures on Σ in the case of the physical metric and the Euclidean metric.

Lemma 2.1.7 (Lemma 11, [CS21])**.** *Let $(M, \overline{\mathrm{g}}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and let $\iota : \Sigma \hookrightarrow$ be a surface. Then, there exists $C = C(\overline{c})$ and $\sigma_0 = \sigma_0(\overline{c})$ such that if $r_\Sigma \geq \sigma \geq \sigma_0$ then*

$$|H - H^e| \leq C\left(\sigma^{-\frac{3}{2}-\delta} + \sigma^{-\frac{1}{2}-\delta}|A|\right), \quad |A - A^e| \leq C\left(\sigma^{-\frac{3}{2}-\delta} + \sigma^{-\frac{1}{2}-\delta}|A|\right). \tag{2.18}$$

Moreover, if $|A^e| \leq \frac{6}{\sigma}$, then $|A - A^e| \leq C\sigma^{-\frac{3}{2}-\delta}$. Finally, if $|H^e| \leq \frac{6}{\sigma}$ and $|\mathring{A}| \leq c\sigma^{-\frac{3}{2}-\delta}$, then there exists $\sigma_1 = \sigma_1(c, \overline{c})$ such that if also $\sigma \geq \sigma_1$ then $|H - H^e| \leq C\sigma^{-\frac{3}{2}-\delta}$.

Proof. By definition, we have that

$$\begin{aligned} h_{ij} - h^e_{ij} &= \langle \overline{\nabla}_{e_i}\nu, e_j\rangle_{\bar{g}} - \langle \overline{\nabla}^e_{e_i}\nu, e_j\rangle_{\bar{g}} + (\bar{g} - \bar{g}^e)\left(\overline{\nabla}^e_{e_i}\nu, e_j\right) \\ &= \langle (\overline{\nabla}_{e_i} - \overline{\nabla}^e_{e_i})\nu, e_j\rangle_{\bar{g}} + \mathrm{O}(\sigma^{-\frac{1}{2}-\delta})h_{ij} + \mathrm{O}(\sigma^{-\frac{3}{2}-\delta}). \end{aligned} \tag{2.19}$$

Since also the first addend is of order $\mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$, (2.18) follows.

We now prove the other results. From the second inequality in equation (2.18) it follows immediately

$$|A - A^e| \le C\left(\sigma^{-\frac{3}{2}-\delta} + \sigma^{-\frac{1}{2}-\delta}|A - A^e| + \sigma^{-\frac{1}{2}-\delta}|A^e|\right)$$

Moreover, using that

$$|A| \le \left|A - \frac{H}{2}g\right| + \left|\frac{H}{2}g - \frac{H^e}{2}g\right| + \left|\frac{H^e}{2}g\right| = |\mathring{A}| + \frac{\sqrt{2}}{2}|H - H^e| + \frac{\sqrt{2}}{2}|H^e|,$$

and using equation (2.18),

$$|H - H^e| \le C\left(\sigma^{-\frac{3}{2}-\delta} + \sigma^{-\frac{1}{2}-\delta}\left(|\mathring{A}| + |H - H^e| + |H^e|\right)\right).$$

Since by the hypothesis $|\mathring{A}| \le c\sigma^{-\frac{3}{2}-\delta}$, it follows that, for σ large enough,

$$|H - H^e| \le 2C\sigma^{-\frac{3}{2}-\delta} + C\sigma^{-\frac{1}{2}-\delta}|H - H^e| + 6C\sigma^{-\frac{3}{2}-\delta}.$$

Then, being σ large, $|H - H^e| \le 16C\sigma^{-\frac{3}{2}-\delta}$. □

Corollary 2.1.8. *Let $(M, \bar{g}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and let $\iota : \Sigma \hookrightarrow$ be a surface. Then, there exists $C = C(\bar{c})$ and $\sigma_0 = \sigma_0(\bar{c})$ such that if $r_\Sigma \ge \sigma \ge \sigma_0$, $|h_{ij}| \le \frac{6}{\sigma}$ and $\nabla^e H^e = 0$, then*

$$|\nabla H| \le C\sigma^{-\frac{5}{2}-\delta}. \tag{2.20}$$

Proof. Deriving equation (2.19) and using Lemma 2.0.2 and the hypothesis $\nabla^e H^e = 0$ we get the thesis. Observe moreover that in general

$$|\nabla A - \nabla A^e| \le C\left(\sigma^{-\frac{5}{2}-\delta} + \sigma^{-\frac{1}{2}-\delta}|\nabla A^e|\right),$$

and the same holds for ∇H instead of ∇A. □

Remark 2.1.9. *If Σ^e is an Euclidean sphere, then (2.20) holds.*

Lemma 2.1.10. *Let $(M, \bar{g}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and let $\iota : \Sigma \hookrightarrow$ be a surface. Then, there exist $C = C(\bar{c})$ and $\sigma_0 = \sigma_0(\bar{c})$ such that if $r_\Sigma \ge \sigma \ge \sigma_0$ and $|A| \le 6\sigma^{-1}$ then*

$$\left||\mathring{A}| - |\mathring{A}^e|\right| \le C\sigma^{-\frac{3}{2}-\delta}.$$

Proof. By definition, it follows that

$$\begin{aligned} \left||\mathring{A}| - |\mathring{A}^e|\right| &\le \left|A - A^e - \frac{H}{2}g + \frac{H^e}{2}g^e\right| \\ &\le |A - A^e| + \frac{\|H\|_{L^\infty}}{2}|g - g^e| + \frac{1}{2}|H^e - H||g^e|, \end{aligned} \tag{2.21}$$

and thus, since $\|H\|_{L^\infty} \leq C\sigma^{-1}$, $|A - A^e| \leq C\sigma^{-\frac{3}{2}-\delta}$, $|H - H^e| \leq C\sigma^{-\frac{3}{2}-\delta}$, the thesis follows. □

2.2 Umbilical surfaces

The traceless second fundamental form turns out to be a powerful geometric tool. It measures the distance between the principal curvatures κ_1, κ_2 of a surface and thus the umbilicality of its points. In 2005 De Lellis-Müller [DLM05] showed that the nearly umbilicality of a surface (in particular, in L^2-norm) in $\mathbb{R}^3$ implies that the surface is near, in a suitable norm, to a Euclidean sphere with radius which equals the area radius of the surface.

In this Section, we will always consider surfaces in $\mathbb{R}^3$. However, we keep in mind that the Euclidean surfaces we consider arise as the Euclidean image of a surface in an asymptotically flat manifold, and, at end of the Section, we will remark the consequences of this fact at the light of Lemma 2.1.10.

2.2.1 Roundness of Euclidean surfaces

De Lellis and Müller proved that, in the Euclidean space, umbilical surfaces are close to round spheres. We state their Theorem 1.1 from [DLM05], then we recall some corollaries of it.

Theorem 2.2.1 (Theorem 1.1 [DLM05], Theorem [Met07]). *There exists a universal constant* $c_{\mathrm{DM}} > 0$ *such that for each surface* $\Sigma^e \hookrightarrow \mathbb{R}^3$*, setting* $\sigma = \sigma_{\Sigma^e}$*, the following estimate holds*

$$\|A^e - \sigma^{-1}g^e\|_{L^2(\Sigma^e)} \leq c_{\mathrm{DM}}\|\overset{\circ}{A}{}^e\|_{L^2(\Sigma^e)}. \tag{2.22}$$

If in addition $\|\overset{\circ}{A}{}^e\|_{L^2(\Sigma^e)} < 8\pi$*, then* Σ^e *is topologically a sphere and there exists a conformal parametrization* $\psi : \mathbb{S}_\sigma(\vec{z}) \to \Sigma^e$*, with* $\vec{z} := |\Sigma^e|^{-1}\int_{\Sigma^e}\vec{x}\,d\mu^e$*, such that* $\psi^* g^e = u\overset{\circ}{g}_{\mathbb{S}_\sigma}$ *for some scalar function* $u : \mathbb{S}_\sigma(\vec{z}) \to \mathbb{R}$ *and*

$$\|\psi - \mathrm{id}\|_{H^2(\mathbb{S}_\sigma(\vec{z}))} \leq c_{\mathrm{DM}}\sigma^2\|\overset{\circ}{A}{}^e\|_{L^2(\Sigma^e)},$$

where id *is the identity on* $\mathbb{S}_\sigma(\vec{z})$ *and* $\overset{\circ}{g}_{\mathbb{S}_\sigma}$ *is the round metric on spheres.*

L^∞-type estimates. In his Ph.D. thesis, Nerz [Ner14] proposed an L^∞-version of the De Lellis-Müller estimate that, in a certain sense, looks also like a nearly Alexandroff theorem. In fact, in order to have a suitable $W^{2,\infty}$-vicinity to a sphere, he also requires a control on how much the surface fails to be a constant mean curvature (CMC) surface. In order to understand Nerz's statement, we introduce a definition.

Definition 2.2.2 (Graph on a Euclidean sphere). *Let* $f : \mathbb{S}_\sigma(\vec{z}_0) \to \mathbb{R}$ *be a function, for some* $\sigma \geq 1$ *and* $\vec{z}_0 \in \mathbb{R}^3$*. We define the graph of* f *over* $\mathbb{S}_\sigma(\vec{z}_0)$ *as*

$$\mathrm{graph}(f) := \{\vec{x} + f(\vec{x})\nu^e_{\vec{x}} : \ \vec{x} \in \mathbb{S}_\sigma(\vec{z}_0)\} = \{\vec{z}_0 + \sigma\vec{y}_{\vec{x}} + f(\vec{x})\vec{y}_{\vec{x}} : \ \vec{y}_{\vec{x}} \in \mathbb{S}_1\}, \tag{2.23}$$

where $\nu^e_{\vec{x}}$ *is the normal to* $\mathbb{S}_\sigma(\vec{z}_0)$ *in* $\vec{x}$*. We remark that we can also write* $\vec{x} = \vec{z}_0 + \sigma\vec{y}_{\vec{x}}$ *for some* $\vec{y}_{\vec{x}} \in \mathbb{S}_1(\vec{0})$ *and* $\nu^e_{\vec{x}} = \frac{\vec{x}-\vec{z}_0}{\sigma} = \vec{y}_{\vec{x}}$*.*

Remark 2.2.3. *We remark that, if* Σ^e *is a graph on a sphere, scalar functions defined on* Σ^e *can also be read as functions on the (approximating) sphere* $\mathbb{S}_\sigma$*. With an abuse of notation, we will indicate these two kind of functions with the same symbol, omitting the (bijective) map* $\vec{x} \mapsto \vec{x} + f(\vec{x})\nu^e_{\vec{x}}$*.*

We then re-state [Ner15, Cor. E.1] in the version we need.

Corollary 2.2.4. *Let* $\tilde{\Sigma}^e \hookrightarrow \mathbb{R}^3$ *be a closed surface in* $\mathbb{R}^3$ *and* $|\tilde{\Sigma}^e| = 4\pi\sigma^2$. *Suppose that there exists a constant* $c > 0$ *such that*

$$\epsilon := \left\| \overset{\circ}{\tilde{A}}{}^e \right\|_{L^\infty(\tilde{\Sigma}^e)} + \left\| \tilde{H}^e - \frac{2}{\sigma} \right\|_{L^\infty(\tilde{\Sigma}^e)} \leq c\sigma^{-\frac{3}{2}-\delta}.$$

Then there exist $\sigma_0(c) > 0$ *and* $\tilde{c} = \tilde{c}(c) > 0$ *such that if* $\sigma > \sigma_0$ *then there exist a point* $\vec{z}_0 \in \mathbb{R}^3$ *and a function* $\tilde{f} : \mathbb{S}_\sigma(\vec{z}_0) \to \mathbb{R}$ *such that* $\tilde{\Sigma}^e = \mathrm{graph}(\tilde{f})$ *and*

$$\|\tilde{f}\|_{L^\infty(\tilde{\Sigma}^e)} + \sigma\|\nabla\tilde{f}\|_{L^\infty(\tilde{\Sigma}^e)} + \sigma^2\|\nabla^2\tilde{f}\|_{L^\infty(\tilde{\Sigma}^e)} \leq \tilde{c}\sigma^2\epsilon.$$

2.2.2 Pseudo-spheres

At the light of the results of the previous Subsection, in particular Corollary 2.2.4, we continue to study Euclidean surfaces, with particular interest in surfaces of the following type.

Definition 2.2.5. *A surface* Σ^e *of* $\mathbb{R}^3$ *is said to be a pseudo-sphere if it is the graph of a function on the sphere* $\mathbb{S}_\sigma(\vec{z}_0)$ *for some* $f \in C^2(\mathbb{S}_\sigma(\vec{z}_0);\mathbb{R})$ *and* f *satisfies* $\|f\|_{W^{2,\infty}(\mathbb{S}_\sigma(\vec{z}_0))} \leq c\sigma^{\frac{1}{2}-\delta}$.

Since the extrinsic curvatures of Euclidean surfaces are invariant under translations, we will always consider, in this Section, the sphere $\mathbb{S}_\sigma(\vec{0})$.

Lemma 2.2.6. *Let* Σ^e *be a pseudo-sphere on the sphere* $\mathbb{S}_\sigma(\vec{0})$, $\sigma > 1$, *with* $f \in C^2(\mathbb{S}_\sigma(\vec{0});\mathbb{R})$. *We equip it with the metric induced by the immersion* $\mathrm{graph}(f) \hookrightarrow \mathbb{R}^3$, *and we call it* $g_{\mathrm{graph}(f)}$. *Moreover we indicate with* $A \equiv A^{\mathrm{graph}(f)}$ *its second fundamental form. Then, if for some* $c > 0$ *it holds*

$$\sup_{\mathbb{S}_\sigma(\vec{0})} |f| \leq c\sigma^{\frac{1}{2}-\delta}, \quad \sup_{\mathbb{S}_\sigma(\vec{0})} |\nabla f| \leq c\sigma^{-\frac{1}{2}-\delta}, \quad \sup_{\mathbb{S}_\sigma(\vec{0})} |\nabla^2 f| \leq c\sigma^{-\frac{3}{2}-\delta},$$

we find that there exists $\tilde{c}(c) > 0$ *such that*

$$|\sigma^2\overset{\circ}{g}_{\mathbb{S}} - g^{\mathrm{graph}(f)}|_{g_{\mathrm{graph}(f)}} \leq \tilde{c}\sigma^{-\frac{1}{2}-\delta}, \quad \left|\nu^{\mathrm{graph}(f)} - \nu^{\mathbb{S}_1(\vec{0})}\right| \leq \tilde{c}\sigma^{-\frac{1}{2}-\delta},$$
$$\left| |A^{\mathrm{graph}(f)}|_{g_{\mathrm{graph}(f)}} - \frac{\sqrt{2}}{\sigma} \right| \leq \tilde{c}\sigma^{-\frac{3}{2}-\delta}.$$

The proof is standard and uses the formulas for the first and second fundamental forms of surfaces in $\mathbb{R}^3$.

The smallness of the error committed comparing the geometric quantities of a pseudo-sphere and that of a sphere leads to the following straightforward generalization of Poincaré inequality on spheres.

Lemma 2.2.7 (Poincaré inequality on pseudo-spheres)**.** *Let* $(M, \overline{g}, \vec{x})$ *a* $C^2_{\frac{1}{2}+\delta}$*-asymptotically flat manifold, and let* $(\Sigma, g) \hookrightarrow M$ *be a surface such that its Euclidean image* $\Sigma^e := \vec{x}(\Sigma)$ *is a pseudo-sphere, i.e.* $\Sigma^e = \mathrm{graph}(f)$, *with* $f : \mathbb{S}_\sigma(\vec{z}_0) \to \mathbb{R}$ *for some* $\vec{z}_0 \in \mathbb{R}^3$ *and for* $\sigma = \sqrt{(4\pi)^{-1}|\Sigma|_g}$. *Suppose that there exists* $c > 0$ *such that*

$$\sup_{\mathbb{S}_\sigma(\vec{z}_0)} |f| \leq c\sigma^{\frac{1}{2}-\delta}, \quad \sup_{\mathbb{S}_\sigma(\vec{z}_0)} |\nabla f| \leq c\sigma^{-\frac{1}{2}-\delta}, \quad \sup_{\mathbb{S}_\sigma(\vec{z}_0)} |\nabla^2 f| \leq c\sigma^{-\frac{3}{2}-\delta}. \tag{2.24}$$

Fix $p \geq 2$. *Then they exist* $\sigma_0 = \sigma_0(c, \overline{c}, \delta) > 0$ *and* $\overline{c}_{S,p} > 0$ *such that, if* $\sigma > \sigma_0$ *and* $\phi \in C^1(\Sigma;\mathbb{R})$ *then*

$$\int_\Sigma |\phi|^p \, d\mu_g \leq \overline{c}_{S,p}\sigma^p \int_\Sigma |\nabla\phi|^p \, d\mu_g. \tag{2.25}$$

Inspired by the scaling factor obtained when integrating the derivatives of a function on a Euclidean sphere, we give the following definition of Sobolev norm.

Definition 2.2.8 (Sobolev norms). *Let $(M, \bar{g}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold, and consider the surface $\iota : (\Sigma, g) \hookrightarrow M$ with area radius σ_Σ. Let T be a smooth tensor defined on Σ, with L^p-norm given by $\|T\|_{L^p(\Sigma,\mu_g)}$. Then we define the Sobolev norm $\|\cdot\|_{W^{k,p}(\Sigma)}$ of T as*

$$\|T\|_{W^{0,p}(\Sigma)} := \|T\|_{L^p(\Sigma)}, \qquad \|T\|_{W^{k,p}(\Sigma)} := \|T\|_{L^p(\Sigma)} + \sigma_\Sigma \|\nabla T\|_{W^{k-1,p}(\Sigma)},$$

for every $k \in \mathbb{N}$ and $p \in [1, \infty]$.

Remark 2.2.9. *In the light of this definition, equation (2.24) takes the form $\|f\|_{W^{2,\infty}} \leq c\sigma^{\frac{1}{2}-\delta}$.*

We furthermore notice that the definition above is coherent with the Simon-Sobolev inequality when the mean curvature of the surface is comparable with that of a Euclidean sphere.

2.2.3 Some useful nearly umbilical-type results in literature

We will also need two further results on nearly umbilical surfaces. The first result we state has been proved by Nerz [Ner15, Prop. 4.1] and relies on the Stampacchia's iteration. It has a different fashion from the other (Euclidean) results presented in this Section, since it shows that, in a Riemannian 3-manifold with some asymptotic decay assumptions on the curvatures, surfaces with small traceless second fundamental form (in L^2) must satisfy higher-regularity estimates.

The second result, proved by Perez [Per11], is a supercritical version of the De Lellis-Müller theorem, and it will play an important role in the next Chapters.

Nerz's bootstrap. The (bootstrapping) regularity theory developed by Nerz (see [Ner15]) implies that a control on the smallness of the L^2-norm of $\mathring{A}$ and a decay on the the curvatures and the norm $\|H - h\|_{W^{1,p}(\Sigma)}$, with $p > 2$, imply that also $\mathring{A}$ and $\nabla\mathring{A}$ have a controlled decay in L^2-norm. The result presented in [Ner15, Prop. 4.1] is very general, and we adapt its statement to our particular case.

Lemma 2.2.10 (Nerz's bootstrap, [Ner15]). *Suppose that $(M, \bar{g})$ is a* 3-dimensional $C^2_{\frac{1}{2}+\delta}$-asymptotically flat Riemannian manifold*, and let $(\Sigma, g) \hookrightarrow (M, \bar{g})$ be a closed surface. Suppose that there exist $c_1, c_2 > 0$, $p > 2$. such that*

$$\|H - h\|_{W^{1,p}(\Sigma)} \leq \frac{c_1}{\sigma_\Sigma^{\frac{3}{2}+\delta-\frac{2}{p}}}, \qquad \left|h - \frac{2}{\sigma_\Sigma}\right| \leq \frac{c_2}{\sigma_\Sigma^{\frac{3}{2}+\delta}},$$

where σ_Σ is the area radius. Suppose finally that on Σ it holds the inequality

$$\|\psi\|_{L^2(\Sigma)} \leq \frac{\mathrm{c_s}}{\sigma_\Sigma}\|\psi\|_{W^{1,1}(\Sigma)}, \qquad \forall\psi \in W^{1,1}(\Sigma) \tag{2.26}$$

for some $\mathrm{c_s} > 0$. Then, there exist constants $\sigma_0(c_1, c_2, \mathrm{c_s}, p, \delta)$ and $c(c_2, \mathrm{c_s}, p, \delta)$ such that, if $\sigma \geq \sigma_0$,

$$\|\mathring{A}\|_{L^2(\Sigma)} \leq \frac{2}{9\mathrm{c_s}} \quad \Longrightarrow \quad \|\mathring{A}\|_{L^\infty(\Sigma)} + \sigma_\Sigma^{-1}\|\mathring{A}\|_{H^1(\Sigma)} \leq \frac{c_1 c}{\sigma_\Sigma^{\frac{3}{2}+\delta}}. \tag{2.27}$$

p-supercritical regularity (à la De Lellis-Müller). Finally, the following result is a generalization of DeLellis-Müller's result, proved by Perez in his Ph.D. thesis [Per11]. This is a bit different from the results proposed until now in this Section. It implies that, if the a surface is round in L^p-sense (with $p > n$), then the oscillation of the mean curvature are controlled. This would be a fundamental step in the definition of our *roundness class*.

Theorem 2.2.11 (Thm. 1.1, [Per11]). *Let $n \geq 2$, $p \in (n, \infty)$ and $c_0 > 0$. There exists $c_{\mathrm{Per}} = c_{\mathrm{Per}}(n, p, c_0) > 0$ such that if $\Sigma^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth, closed and connected surface with induced metric g^e and such that*

(i) $\mathrm{Vol}_{g^e}(\Sigma) = 1$*;*

(ii) $\|A\|_{L^p(\Sigma,\mu^e)} \leq c_0$*;*

then

$$\min_{\lambda\in\mathbb{R}} \|A^e - \lambda g^e\|_{L^p(\Sigma,\mu^e)} \leq c_{\mathrm{Per}} \|\overset{\circ}{A}{}^e\|_{L^p(\Sigma,\mu^e)}.$$

This estimate looks implicit, if written in the current form. In fact, it essentially gives a bound on $\|A^e - \lambda^\Sigma g^e\|_{L^p(\Sigma,\mu_e)}$, for some $\lambda^\Sigma \in \mathbb{R}$ for which the minimum is achieved. However, λ^Σ strictly depends on Σ, and so it is not a universal value. We replace this specific real number with geometric quantities, which obviously depend on the geometry of the surface, but in an explicit (extrinsic) way.

In the case $n = 2$, by the Cauchy-Schwarz inequality we have $|A^e - \lambda^\Sigma g^e|^2 \geq \frac{1}{2}\left(H^e - 2\lambda^\Sigma\right)^2$. This implies that

$$\|H^e - 2\lambda^\Sigma\|_{L^p(\Sigma,\mu_e)} \leq \sqrt{2} c_{\mathrm{Per}} \|\overset{\circ}{A}{}^e\|_{L^p(\Sigma,\mu_e)}$$

Since

$$\|H^e - h^e\|_{L^p(\Sigma,\mu^e)} \leq \|H^e - 2\lambda^\Sigma\|_{L^p(\Sigma,\mu^e)} + \|h^e - 2\lambda^\Sigma\|_{L^p(\Sigma,\mu^e)}$$

and using the definition of integral mean and the Hölder's inequality, we have

$$\leq \|H^e - 2\lambda^\Sigma\|_{L^p(\Sigma,\mu^e)} + |h^e - 2\lambda^\Sigma||\Sigma|_e^{\frac{1}{p}} \leq \|H^e - 2\lambda^\Sigma\|_{L^p(\Sigma,\mu^e)} + |\Sigma|_e^{\frac{1}{p}-1} \int_\Sigma |H^e - 2\lambda^\Sigma| \, d\mu^e$$

$$\leq \|H^e - 2\lambda^\Sigma\|_{L^p(\Sigma,\mu_e)} + |\Sigma|_e^{\frac{1}{p}-1} |\Sigma|_e^{1-\frac{1}{p}} \|H^e - 2\lambda^\Sigma\|_{L^p(\Sigma,\mu_e)} \leq 2\|H^e - 2\lambda^\Sigma\|_{L^p(\Sigma,\mu_e)}.$$

Finally we get

$$\|H^e - h^e\|_{L^p(\Sigma,\mu^e)} \leq 2\sqrt{2} c_{\mathrm{Per}} \|\overset{\circ}{A}{}^e\|_{L^p(\Sigma,\mu^e)}. \tag{2.28}$$

Therefore we re-write Theorem 2.2.11 in an asymptotically flat version, at the light of the estimates on $|H - H^e|$ and $|A - A^e|$.

Theorem 2.2.12. *Let $(M, \bar{g})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and consider a surface $\Sigma \hookrightarrow M$, with induced metric g. Suppose that there exists $\sigma > 1$ such that $|A| \leq \frac{10}{\sigma}$ and $r_\Sigma \geq \frac{\sigma}{10}$. Then there exist $\sigma_0(\bar{c}, \delta) > 0$ and $c_p(\bar{c}, \delta, p) > 0$ such that if $\sigma > \sigma_0$ then*

$$\|H - h\|^p_{L^p(\Sigma)} \leq c_p \|\overset{\circ}{A}\|^p_{L^p(\Sigma)} + c_p \sigma_\Sigma^{-p-\delta p}.$$

If $(M, \bar{g}) = (\mathbb{R}^3, \delta_{\mathrm{eucl}})$, the second addend would not appear. Its presence is due to the asymptotically flatness of the ambient manifold.

2.3 Round surfaces

We give the definition of *round surface*. As we can see comparing with [HY96], [Hua10], [Met07], [Ner15], there are some similarities between our definition of roundness and the ones proposed in literature up to this point. While the class of round surfaces in [HY96] is defined in terms of pointwise properties, our definition includes assumptions which involve integral norms of the curvature. The integral form is more suitable to study the invariance properties under the volume preserving mean curvature flow under our weaker hypotheses on the ambient space.

Definition 2.3.1. *Let $(M, \overline{g}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and let $\iota : \Sigma \hookrightarrow M$ be a surface, with induced metric $g := \iota^* \overline{g}$.*

For a given radius $\sigma > 1$ and parameters $\eta, B_1, B_2 > 0$ we say that (Σ, g) is a round surface in $(M, \overline{g})$, and we write quantitatively $\Sigma \in \mathcal{W}^\eta_\sigma(B_1, B_2)$ if the following inequalities are satisfied

$$(7/2)\pi\sigma^2 < |\Sigma|_g < 5\pi\sigma^2, \qquad |A| < \sqrt{\frac{5}{2\sigma_\Sigma^2}}, \tag{2.29}$$

$$\frac{3}{4} < \frac{r_\Sigma}{\sigma} \leq \frac{R_\Sigma}{\sigma} < \frac{5}{4}, \tag{2.30}$$

$$\|\mathring{A}\|_{L^4(\Sigma,\mu)} < B_1 \sigma^{-1-\delta}, \tag{2.31}$$

$$\eta\sigma^{-4}\|H - h\|^4_{L^4(\Sigma)} + \|\nabla H\|^4_{L^4(\Sigma)} < B_2 \sigma^{-8-4\delta}. \tag{2.32}$$

For a given radius $\sigma > 1$ and $\eta, B_1, B_2, B_{\text{cen}} > 0$, we moreover say that (Σ, g) is a well-centered round surface, and we write $\Sigma \in \mathcal{B}^\eta_\sigma(B_1, B_2, B_{\text{cen}})$ if it satisfies the above properties and in addition

$$|\vec{z}_\Sigma| < B_{\text{cen}} \sigma^{1-\delta}. \tag{2.33}$$

Finally, we write $\overline{\mathcal{W}}^\eta_\sigma(B_1, B_2)$ or $\overline{\mathcal{B}}^\eta_\sigma(B_1, B_2, B_{\text{cen}})$ when (Σ, g) satisfies the bounds above with at least one $<$ replaced by $\leq$.

Throughout the Thesis, when deriving estimates on geometric quantities on a surface Σ, we denote by $C, C_1, C_2, \dots$ constants which only depend on properties of the ambient manifold, such as $\bar{c}, \delta$ in (1.11) or the mass E_{ADM} and by $c, c_1, c_2, \dots$ constants which in addition depend on the constants B_1, B_2, B_{cen} in the previous conditions. We say that a constant is universal if it is independent on any other parameter of our problem. As usual, the letters c or C will often denote constants which may change from one line to the other, but each time depending on the same parameters.

Remark 2.3.2. *Property (2.29) implies that the Euclidean radius and the area radius are comparable. Because of the asymptotic flatness of $(M, \overline{g})$, we obtain the following bound on the Riemannian tensor*

$$|\overline{\text{Rm}}|_{\overline{g}} \leq C\sigma^{-\frac{5}{2}-\delta} \text{ on } \Sigma. \tag{2.34}$$

Remark 2.3.3. *In the following, the constant η will be fixed in an explicit way, see Lemma 4.1.11 below, depending only on the ambient manifold. For this reason, even if in the following computations some quantities will depend on η, we will omit these dependencies, considering in a certain sense η as already fixed.*

Remark 2.3.4. *The decay rates in conditions (2.31)-(2.32) are modelled on the ones of the Euclidean coordinate spheres. In fact, by Lemma 2.1.7, it is easy to check that if B_1, B_2 are large enough, depending on $\bar{c}$ in (1.11), then $\mathbb{S}_r(0)$ belongs to $\mathcal{B}^\eta_\sigma(B_1, B_2, B_{\text{cen}})$ for r large enough and r/σ enough close to 1. Conversely, we will see in Lemma 2.3.5(iv) that a round surface is close to a sphere in Euclidean coordinates.*

We briefly recall the Michael-Simon inequality in Euclidean space which, together with the curvature bound in (4.11) and Lemma 2.1.7, implies the existence of a universal Sobolev constant $c_S > 0$ such that

$$\|\psi\|_{L^2(\Sigma)} \leq \frac{\mathrm{c_S}}{\sigma}\|\psi\|_{W^{1,1}(\Sigma)}, \qquad \forall\ \psi \in W^{1,1}(\Sigma), \tag{2.35}$$

provided $\sigma \geq \sigma_0 = \sigma_0(\overline{c}, \delta) > 0$. From this, the other Sobolev inequalities can be deduced. In particular (see e.g. Lemma 12 in [CS21] and the references therein) we have, for every $p > 2$,

$$\|\psi\|_{L^\infty(\Sigma)} \leq 2^{\frac{2(p-1)}{p-2}}\mathrm{c_S}\sigma^{-\frac{2}{p}}\|\psi\|_{W^{1,p}(\Sigma)}, \qquad \forall \psi \in W^{1,p}(\Sigma), \tag{2.36}$$

and also

$$\|\psi\|_{L^\infty(\Sigma)} \leq 32\mathrm{c_S^2}\sigma^{-1}\|\psi\|_{H^2(\Sigma)}, \qquad \forall \psi \in H^2(\Sigma). \tag{2.37}$$

We now state and prove a Lemma which lists various properties of a *round surface*, from different points of view. The results summarized by this Lemma are easy consequence of known results, but we give detail for sake of completeness.

Lemma 2.3.5. *Let $(M, \overline{\mathrm{g}})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold. Let $\iota : \Sigma \to M$ be a surface, and let $g := \iota^*\overline{g}$ be the induced metric. Fix a weight $\eta > 0$ and $B_1 > 0$, $B_2 > 0$. There exists $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta, \eta) > 0$ such that whenever $\sigma > \sigma_0$, if $(\Sigma, g) \in \overline{\mathcal{W}}^\eta_\sigma(B_1, B_2)$, then the following conclusions hold:*

(i) *There exists $\mathrm{c_s} > 0$ such that it holds*

$$\|\psi\|_{L^2} \leq \frac{\mathrm{c_S}}{\sigma}\|\psi\|_{W^{1,1}(\Sigma)} \qquad \forall\ \psi \in W^{1,1}(\Sigma), \tag{2.38}$$

and, for every $p > 2$,

$$\|\psi\|_{L^\infty} \leq 2^{\frac{2(p-1)}{p-2}}\mathrm{c_s}\sigma^{-\frac{2}{p}}\|\psi\|_{W^{1,p}} \qquad \forall \psi \in W^{1,p}(\Sigma). \tag{2.39}$$

Moreover, there exists a constant $c(B_2, \eta) > 0$ such that

$$\|H - h\|_{L^\infty} \leq c(B_2, \eta)\sigma^{-\frac{3}{2}-\delta}. \tag{2.40}$$

(ii) *It holds the estimate*

$$\left|h - \frac{2}{\sigma_\Sigma}\right| \leq c(B_1, B_2, \overline{c})\sigma^{-\frac{3}{2}-\delta}, \tag{2.41}$$

and the principal curvatures κ_1, κ_2 satisfy

$$\frac{1}{2\sigma_\Sigma} < \kappa_i < \frac{\sqrt{5}}{2\sigma_\Sigma} \tag{2.42}$$

for $i \in \{1, 2\}$.

(iii) *There exists a constant $B_\infty = B_\infty(B_1, B_2, \eta, \mathrm{c_s}, \delta, \overline{c})$ such that $\|\mathring{A}\|_{L^\infty(\Sigma)} \leq B_\infty\sigma^{-\frac{3}{2}-\delta}$.*

(iv) *There exists $c = c(\delta, \overline{c}, B_1, B_2, \eta)$, $c_0 = c(B_1, \overline{c}, \delta)$, $\vec{z}_0 \in \mathbb{R}^3$, and $f : \mathbb{S}_{\sigma_\Sigma}(\vec{z}_0) \to \mathbb{R}$ such that*

$$\Sigma^e = \mathrm{graph}(f), \qquad \|f\|_{W^{2,\infty}} \leq c\sigma^{\frac{1}{2}-\delta}, \qquad |\vec{z}_0 - \vec{z}_\Sigma| \leq c_0\sigma^{\frac{1}{2}-\delta}. \tag{2.43}$$

(v) *There exists $\mathrm{c_s} > 0$, a Sobolev constant, such that the following inequality holds*

$$\int_\Sigma (\psi - \overline{\psi})^4\ d\mu_g \leq \mathrm{c_s}\sigma^4 \int_\Sigma |\nabla\psi|^4\ d\mu_g, \qquad \forall\ \psi \in W^{1,4}(\Sigma), \tag{2.44}$$

where $\overline{\psi} := \fint_{\Sigma} \psi \, d\mu_g$.

(vi) *It holds* $\|A\|_{L^4(\Sigma)} \leq 4^{\frac{1}{4}}\sqrt{5/2}\pi^{\frac{1}{4}}\sigma^{-\frac{1}{2}}$ *and thus there exists a constant* $c_{\mathrm{Per}} = c_{\mathrm{Per}}(\delta, \overline{c})$ *such that*

$$\|H - h\|_{L^4(\Sigma)} \leq c_{\mathrm{Per}}\|\mathring{A}\|_{L^4(\Sigma)} + c_{\mathrm{Per}}\sigma^{-1-\delta}.$$

Remark 2.3.6. *Point (iv) explicitly says that each point of* $\Sigma^e := \vec{x}(\Sigma)$ *can be written as*

$$\vec{x}(\iota(x)) = \vec{z}_0 + \sigma_\Sigma \nu^e(x) + f(x)\nu^e(x), \quad \forall x \in \Sigma.$$

Thanks to (2.43)*, modulo modifying* f*, we can also write, without loss of generality,*

$$\vec{x}(\iota(x)) = \vec{z}_\Sigma + \sigma_\Sigma \nu^e(x) + \hat{f}(x)\nu^e(x), \quad \forall x \in \Sigma. \tag{2.45}$$

Since by the asymptotic flatness we also have $|\vec{z}_\Sigma - \vec{z}_{\Sigma^e}| \leq C\sigma^{\frac{1}{2}-\delta}$*, we can give a completely Euclidean description of* Σ^e *as*

$$\vec{x}(\iota(x)) = \vec{z}_{\Sigma^e} + \sigma_\Sigma \nu^e(x) + \hat{f}(x)\nu^e(x), \quad \forall x \in \Sigma.$$

However, we will mainly use identity (2.45)*.*

Remark 2.3.7. *It follows from the elliptic regularity theory and the Simons' identity that* $f \in C^2(\mathbb{S}_\sigma(\vec{z}_0))$*, and so, in the* $W^{2,\infty}$*-norm, the two derivatives have to be meant as classical derivatives. This result is a consequence of* $|\overline{\partial}_l\overline{\partial}_k\overline{g}_{ij}| \leq \overline{c}|\overline{x}|^{-\frac{5}{2}-\delta}$ *(see [Ner16, pg. 8, footnote]), since in this case it holds* $f \in W^{3,p}(\mathbb{S}_\sigma(\vec{z}_0))$*, where* $p > 2$ *is such that we have a control on* $\|H - h\|_{W^{1,p}}$*. However, [Ner16, Prop. 2.4] shows that* $f \in C^2$ *also if we only assume the decay of the Ricci tensor, see [Ner16, Prop. 2.4.].*

Proof. (i) It is well known that there exists $c_s^e > 0$ such that

$$\left(\int_{\Sigma^e} \psi^2 \, d\mu^e\right)^{\frac{1}{2}} \leq c_s^e \int_{\Sigma^e} |\nabla^e\psi| + H^e|\psi| \, d\mu^e,$$

if $\Sigma^e \hookrightarrow \mathbb{R}^3$ and $\psi \in W^{1,1}(\Sigma^e)$. Thanks to the asymptotics of $(M, \overline{g}, \vec{x})$, this is also true omitting the apex e, possibly enlarging the constant c_s^e. Thanks to the estimate $|H| \leq \frac{\sqrt{5}}{\sigma}$ we have the thesis. The general case for $p > 2$ follows from [CS21, Lemma 12]. Moreover, $p = 4$ in (2.39), and $\psi := H - h$, we obtain

$$\|H - h\|_{L^\infty(\Sigma)} \leq 2^3 c_s \sigma_\Sigma^{-\frac{1}{2}} \|H - h\|_{W^{1,4}(\Sigma)} \leq 2^3 c_s \sigma_\Sigma^{-\frac{1}{2}} \left(\eta^{\frac{1}{4}} B_2 + B_2^{\frac{1}{4}}\right)\sigma^{-1-\delta},$$

which implies the conclusion.

(ii) Observing that $\frac{\sigma}{\sigma_\Sigma}$ is positive, bounded and bounded away from zero, proceeding as in [Ner15, Prop. 4.1] we find

$$\begin{aligned} 2\sqrt{\pi}\sigma_\Sigma \left|h^e - \frac{2}{\sigma_\Sigma}\right| \leq \sqrt{2}\left\|\frac{h^e}{2}g^e - \frac{H^e}{2}g^e\right\|_{L^2(\Sigma_e)} + \sqrt{2}\left\|\frac{H^e}{2}g^e - A^e\right\|_{L^2(\Sigma_e)} \\ + \sqrt{2}c_{\mathrm{DM}}\left\|\mathring{A}^e\right\|_{L^2(\Sigma_e)}. \end{aligned} \tag{2.46}$$

By Lemma 2.1.7, the hypothesis $|A| \leq \sqrt{\frac{5}{2}}\sigma^{-1}$, and $\|H-h\|_{L^2(\Sigma)} \leq c(B_2,\eta)\sigma^{-\frac{1}{2}-\delta}$, we conclude that

$$\left|h - \frac{2}{\sigma_\Sigma}\right| \leq c(B_1, B_2, \eta, \overline{c})\sigma^{-\frac{3}{2}-\delta},$$

using also that $\|\overset{\circ}{A}\|_{L^4(\Sigma)} \leq B_1\sigma^{-1-\delta}$. Finally, since $|H-h| \leq c(B_2,\eta)\sigma^{-\frac{3}{2}-\delta}$, it follows that

$$\left|H - \frac{2}{\sigma_\Sigma}\right| \leq c(B_1, B_2, \eta, \overline{c})\sigma^{-\frac{3}{2}-\delta}. \tag{2.47}$$

(iii) We apply Nerz's bootstrap (see Lemma 2.2.10) to Σ, with the area radius σ_Σ, $p=4$, $c_1 = c(B_2,\eta)$, $c_2 = c(B_1,B_2,\eta,\overline{c})$ as in equation (2.47). It follows that, if $\|\overset{\circ}{A}\|_{L^2}$ is sufficiently small with respect to $\frac{2}{9}\mathrm{c}_{\mathrm{s}}^{-1}$, i.e. σ is sufficiently large, then

$$\|\overset{\circ}{A}\|_{L^\infty} \leq c(B_2)c(\delta, B_1, B_2, \eta, \overline{c}, c_S)\sigma_\Sigma^{-\frac{3}{2}-\delta} \leq B_\infty\sigma^{-\frac{3}{2}-\delta},$$

choosing $B_\infty = B_\infty(\delta, B_1, B_2, \eta, \overline{c}, c_S)$. Since $H = \kappa_1 + \kappa_2$, $|\overset{\circ}{A}| = 2^{-\frac{1}{2}}|\kappa_1 - \kappa_2| \leq B_\infty\sigma^{-\frac{3}{2}-\delta}$, we find that for σ very large (depending on B_1, B_2, η and $\overline{c}$),

$$\kappa_i \simeq \sigma_\Sigma^{-1}, \quad H \simeq 2\sigma_\Sigma^{-1},$$

using again (2.47). In particular we can choose σ so large that $|A| < \frac{\sqrt{5}}{\sqrt{2}\sigma}$.

(iv) By [Ner15, Cor. E.1], or Corollary 2.2.4, since B_∞ and $c(B_2,\eta)$ control the L^∞-norm of $|\overset{\circ}{A}|$ and $H-h$, respectively, by the point (i)-(iii), the Euclidean image of Σ, i.e. Σ^e, is a graph on the sphere $\mathbb{S}_{\sigma_\Sigma}(\vec{z}_0)$, for some vector $\vec{z}_0 \in \mathbb{R}^3$. By Theorem 2.2.1, see [DLM05], applied on Σ^e, there exists a (conformal) parametrization of Σ^e, say Ψ, such that $\sigma_{\Sigma^e}^{-1}\|\Psi - \mathrm{Id}\|_{H^2(\Sigma)} \leq c(\overline{c},\delta)\sigma_{\Sigma^e}\|\overset{\circ}{A}{}^e\|_{L^2(\Sigma)} \leq c(\overline{c},\delta,B_1)\sigma_{\Sigma^e}^{\frac{1}{2}-\delta}$, and using the Sobolev inequality it follows

$$|\vec{z}_\Sigma - \vec{z}_0| \leq \fint_\Sigma |\Psi - \mathrm{Id}|\, d\mu^e \leq c_0\sigma_{\Sigma^e}^{\frac{1}{2}-\delta} \leq c_0\sigma^{\frac{1}{2}-\delta}.$$

(v) Since from the Euclidean point of view Σ is a graph on the sphere $\mathbb{S}_{\sigma_\Sigma}(\vec{z}_0)$, Lemma 2.2.7 implies equation (2.44).

(vi) This follows from the inequalities of the previous points applied to Theorem 2.2.11. See also inequality (2.28).

□

Chapter 3

Spectral Theory

In [Ner15] and [CS21], a deep study of the stability operator was carried out in the field of asymptotically flat manifolds and initial data sets. The invertibility of the stability operator L^{Σ} around a CMC surface Σ is at the basis of the continuation method employed in [Ner15] and [CS21]. In particular, in [Ner15, Prop. 4.7], Nerz characterizes, in a Fredholm-alternative fashioned statement, the eigenvalues of the stability operator of a CMC-surface. This result then implies that the leaves of the CMC-foliation constructed in his paper [Ner15] are stable or unstable according to the sign of the ADM-energy of the system. In the case of initial data sets, the result is generalized by [CS21, Prop. 2].

In this Chapter, we generalize this analysis to round surfaces where we only assume that H has a small oscillation as in (2.32). We will see that the positivity property of L^{Σ} when $E_{\mathrm{ADM}} > 0$ is no longer true, but that the error terms can be estimated in a way that will be enough for our purposes.

3.1 Hawking energy and stability operator

3.1.1 The Hawking energy

Since our aim is to mostly investigate the extrinsic geometry of surfaces, we start defining a notion of mass which gives a "weight" to surfaces immersed in 3-manifolds. It heuristically and physically measures the bending of the rays which crosses the surface enclosing the mass orthogonally. We start with a formal definition.

Definition 3.1.1. *Let $(M, \overline{g})$ be a 3-dimensional manifold, and $\iota : \Sigma \hookrightarrow M$ be a surface. Let $g := \iota^* \overline{g}$ be the induced metric. The Hawking energy of Σ is defined as*

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|_g}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \, d\mu\right). \tag{3.1}$$

Remark 3.1.2. *We use the notation $m_H(\Sigma)$ to represent the Hawking energy of Σ because of the interchangeability of the terms mass and energy. Moreover, to be more precise, the quantity in (3.1) takes the name of Geroch mass. The exact definition for the Hawking energy is given by*

$$E_{\mathcal{H}}(\Sigma) := \sqrt{\frac{|\Sigma|_g}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} \mathcal{H}^2 \, d\mu\right).$$

However, if $(M, \overline{g}, \overline{K})$ is an initial data set, and Σ satisfies hypothesis (2.14), Lemma 2.1.3, since $q \geq 2$, implies that there exists a constant $\tilde{c} = \tilde{c}(\overline{c}) > 0$ such that

$$|m_H(\Sigma) - E_{\mathcal{H}}(\Sigma)| \leq \tilde{c}\sigma^{-2\delta}.$$

Because of this estimate, the two notion are interchangeable for our purposes.

Lemma 3.1.3. *Let $(M, \overline{g}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold which satisfies the* mass condition. *Let $\iota : \Sigma \to M$ be a* surface *such that $(\Sigma, g) \in \overline{\mathcal{W}}_\sigma(B_1, B_2)$ for some B_1, $B_2 > 0$. Then there exist $\tilde{c} = \tilde{c}(B_1, B_2, \overline{c})$ and $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta)$ such that, for $\sigma > \sigma_0$,*

(i)

$$\left| m_H(\Sigma) + \frac{\sigma_\Sigma}{8\pi} \int_\Sigma \overline{\mathrm{G}}(\nu, \nu)\, d\mu \right| \leq \tilde{c}\sigma^{-2\delta}; \tag{3.2}$$

(ii)

$$\left| E_{\mathrm{ADM}} + \frac{\sigma_\Sigma}{8\pi} \int_\Sigma \overline{\mathrm{G}}(\nu, \nu)\, d\mu \right| \leq \tilde{c}\sigma^{-\delta}. \tag{3.3}$$

Proof. (i). Using the Gauss equation we find

$$\int_\Sigma \left(\frac{\overline{\mathrm{S}}}{2} - \overline{\mathrm{Ric}}(\nu, \nu) \right) d\mu = \int_\Sigma \left(\frac{\mathrm{S}_g}{2} - \kappa_1\kappa_2 \right) d\mu. \tag{3.4}$$

Since Σ is homeomorphic to a sphere (via De Lellis-Müller Theorem), the definitions of σ_Σ and $m_H(\Sigma)$, together with the Gauss-Bonnet theorem imply

$$\left| m_{\mathcal{H}}(\Sigma) - \frac{\sigma_\Sigma}{8\pi} \int_\Sigma \left(\frac{\overline{\mathrm{S}}}{2} - \overline{\mathrm{Ric}}(\nu, \nu) \right) d\mu \right| = \frac{\sigma_\Sigma}{16\pi} \left| \int_\Sigma (\kappa_1 - \kappa_2)^2\, d\mu \right|.$$

The conclusion follows from the estimate on $\|\mathring{A}\|_{L^\infty(\Sigma)}$ in Lemma 2.3.5.
(ii). By the roundness hypothesis, we have that $r_\Sigma \geq \frac{3}{4}\sigma \geq \frac{3}{2\sqrt{5}}\sigma_\Sigma$. Moreover, by Lemma 2.3.5, we have that $d\vec{x}(\nu^e) = \sigma_\Sigma^{-1}(\vec{x} - \vec{z}_0 - \vec{p})$, for some $\vec{p}$ such that $|\vec{p}| = \mathrm{O}(\sigma^{\frac{1}{2}-\delta})$. Thus

$$\begin{aligned} \overline{\mathrm{G}}(\sigma_\Sigma\nu, \nu) &= \sigma_\Sigma\overline{\mathrm{G}}(\nu^e, \nu) + \sigma_\Sigma\overline{\mathrm{G}}(\nu - \nu^e, \nu) = \sigma_\Sigma\overline{\mathrm{G}}(\nu^e, \nu) + \mathrm{O}(\sigma^{-2-2\delta}) \\ &= \overline{\mathrm{G}}(d\vec{x}^{-1}(\vec{x}) - d\vec{x}^{-1}(\vec{z}_0) - d\vec{x}^{-1}(\vec{p}), \nu) + \mathrm{O}(\sigma^{-2-2\delta}). \end{aligned}$$

This implies that

$$\left| \int_\Sigma \overline{\mathrm{G}}(\sigma_\Sigma\nu, \nu)\, d\mu - \int_\Sigma \overline{\mathrm{G}}(d\vec{x}^{-1}(\vec{x}), \nu)\, d\mu \right| \leq \left| \int_\Sigma \overline{\mathrm{G}}(d\vec{x}^{-1}(\vec{z}_0), \nu)\, d\mu \right| + \mathrm{O}(\sigma^{-2\delta}),$$

using the estimate on $|\vec{p}|$. If it also holds that

$$\left| \int_\Sigma \overline{\mathrm{G}}(d\vec{x}^{-1}(\vec{z}_0), \nu)\, d\mu \right| = \mathrm{O}(\sigma^{-2\delta}), \tag{3.5}$$

then, in order to have the thesis, it is sufficient to prove

$$\left| E_{\mathrm{ADM}} + \frac{1}{8\pi} \int_\Sigma \overline{\mathrm{G}}(d\vec{x}^{-1}(\vec{x}(x)), \nu)\, d\mu \right| = \mathrm{O}(\sigma^{-\delta}) \tag{3.6}$$

Proof of (3.5). Decomposing $d\vec{x}^{-1}(\vec{z}_0) = \vec{z}_0^\alpha \overline{\partial}_\alpha$, since $|\vec{z}_0^\alpha| \leq 2R_\Sigma + \sigma_\Sigma \leq \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right)\sigma$, it turns out that it is sufficient to prove

$$\left| \int_\Sigma \overline{\mathrm{G}}_{\vec{x}}(\overline{\partial}_\alpha, \nu)\, d\mu \right| = \mathrm{O}(\sigma^{-1-\delta}).$$

Consider the Euclidean sphere $\vec{x}^{-1}\left(\mathbb{S}_R(\vec{0})\right)$ such that R is so large that Σ is contained in

$\vec{x}^{-1}\left(\mathbb{B}_R(\vec{0})\right)$. Define U_R to be the volume enclosed between these two boundaries. Then, using the divergence theorem,

$$\left|\int_{\vec{x}^{-1}(\mathbb{S}_R(\vec{0}))} \overline{\mathrm{G}}_{\vec{x}}(\nu,\overline{\partial}_\alpha)\ d\mu_R - \int_\Sigma \overline{\mathrm{G}}_{\vec{x}}(\nu,\overline{\partial}_\alpha)\ d\mu\right| = \left|\int_{\mathrm{U}_R} \overline{\mathrm{G}}_{\overline{x}}\cdot\left(\overline{\nabla}_{\overline{x}}\overline{\partial}_\alpha\right)\ d\overline{\mathrm{x}}\right|,$$

and since $\mathrm{U}_R \subseteq \vec{x}^{-1}\left(\mathbb{B}_R(\vec{0})\right) \setminus \vec{x}^{-1}\left(\mathbb{B}_{\frac{3}{2\sqrt{5}}\sigma_\Sigma}(\vec{0})\right)$ and using polar coordinates we get

$$\leq \tilde{c}(\overline{c})\int_{\frac{3}{2\sqrt{5}}\sigma_\Sigma}^R \int_{\vec{x}^{-1}(\mathbb{S}_r(\vec{0}))} r^{-4-2\delta}\ d\mu_r\ dr,$$

where we also used that $|\overline{\nabla}_{\overline{\partial}_\alpha}\overline{\partial}_\beta| = \left|\sum_\gamma \overline{\Gamma}^\gamma_{\alpha\beta}\overline{\partial}_\gamma\right| = \mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$. We conclude computing the 1-dimensional integral and letting $R\to\infty$.

Proof of (3.6). Choosing R and U_R as above, since $\overline{\nabla}_{\vec{x}}\vec{x} = \mathrm{Id}$ as a bilinear form and thus $\overline{\mathrm{G}}\cdot\nabla_{\vec{x}}\vec{x} = \overline{\mathrm{S}}$, using the divergence theorem we get

$$\left|\int_{\vec{x}^{-1}(\mathbb{S}_R(\vec{0}))} \overline{\mathrm{G}}(d\vec{x}^{-1}(\vec{x}(x)),\nu)\ d\mu - \int_\Sigma \overline{\mathrm{G}}(d\vec{x}^{-1}(\vec{x}(x)),\nu)\ d\mu\right| = \left|\int_{\mathrm{U}_R}\overline{\mathrm{S}}\ d\overline{\mathrm{x}}\right| = \mathrm{O}(\sigma^{-\delta}),$$

where the order of the integral of the scalar curvature has been computed as in the previous point. Letting $R\to\infty$ we have the thesis. □

3.1.2 The stability operator

We now introduce the stability operator, which occurs as the second variation of the area functional.

Definition 3.1.4. *Let $(M,\overline{\mathrm{g}})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat 3-manifold. Given a surface $\iota:\Sigma\hookrightarrow M$ and a smooth function $\mathrm{f}\in H^2(\Sigma)$, we define the stability operator associated to Σ, $L^\Sigma: H^2(\Sigma)\to L^2(\Sigma)$, as*

$$L^\Sigma\mathrm{f} := -\Delta\mathrm{f} - (|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\mathrm{f}.$$

We simply write L instead of L^Σ whenever the role of the surface Σ is not ambiguous.

Consider a surface Σ and a normal variation $\hat{F}:\Sigma\times I\to\overline{M}$, with $0\in I$, satisfying

$$\begin{cases}\partial_t\hat{F}(x,t) = \eta(x,t)\nu(x,t)\\ \hat{F}(\Sigma,0) = \Sigma\end{cases} \tag{3.7}$$

A routine computation shows that the mean curvature locally represent the first variation of the area functional, that is

$$\frac{\mathrm{d}}{\mathrm{dt}}|\hat{\Sigma}_t| = \int_\Sigma H\eta\ d\mu, \tag{3.8}$$

where $\hat{\Sigma}_t := \hat{F}_t(\Sigma)$. It turns out that if the variation is *volume preserving*, i.e. $\int_\Sigma \eta\ d\mu = 0$, then a CMC-surface Σ, i.e. $t=0$, is a critical point of the area functional $t\mapsto|\hat{\Sigma}_t|$.

Moreover, if Σ is a CMC-surface, the second variation of the area functional is given by

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}|\hat{\Sigma}_t| = \int_\Sigma \left(|\nabla\eta|^2 - \left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)\eta^2\right)\ d\mu = \int_\Sigma \left(L^\Sigma\eta\right)\eta\ d\mu. \tag{3.9}$$

If the variation was not volume preserving, in (3.9) there would be a term involving the second derivative of the volume enclosed by $\hat{\Sigma}_t$.

Since the "differential part" of the stability operator is totally given by the Laplace-Beltrami operator, in order to understand the properties of L we are interested in we have to briefly review the spectral theory for the operator $-\Delta$. The following Lemma is taken from [CS21, Lemma 2]. Observe that, together with adapting the notations of the Lemma with our definition of roundness class, we remove the hypothesis of having a CMC-surface. In fact, reading the proof of [CS21, Lemma 2] with attention, one can observe that the CMC-hypothesis is just needed in order to compare the area radius with the *curvature radius* used in [Ner15] and [CS21].

Remark 3.1.5. *At the light of Lemma 2.3.5, i.e. of the De Lellis-Müller theorem [DLM05, Thm. 1.1], scalar functions on a round surface Σ can be also meant as functions on the approximating sphere $\mathbb{S}_{\sigma_\Sigma}$. With an abuse of notation, we identify such kind of functions.*

We first recall some properties of the Laplace-Beltrami operator on a round sphere $\mathbb{S}_\sigma(\vec{0}) \subset \mathbb{R}^3$ with the Euclidean metric. On a general closed surface, the eigenvalues of Δ are all positive, except the first one which is zero, with eigenspace given by the constant functions. For the Euclidean sphere, the first nonzero eigenvalue has multiplicity three and is given by

$$\lambda^e_\alpha = \frac{2}{\sigma^2}, \ \alpha = 1, 2, 3. \tag{3.10}$$

An orthonormal basis for the eigenspace is given by the normalized coordinate functions

$$f^e_\alpha(\vec{x}) = \sqrt{\frac{3}{4\pi\sigma_\Sigma^4}}\vec{x}_\alpha, \ \alpha = 1, 2, 3, \tag{3.11}$$

restricted on $\mathbb{S}_\sigma(\vec{0})$. The remaining eigenvalues satisfy the bound

$$\lambda^e_i \geq \lambda^e_4 = \frac{6}{\sigma^2}, \ \forall i \geq 4. \tag{3.12}$$

Moreover, we have

$$\overset{\circ}{\mathrm{Hess}}\,(f^e_\alpha) = 0, \qquad \langle \nabla^e f^e_\alpha, \nabla^e f^e_\beta \rangle - \frac{3\delta_{\alpha\beta}}{4\pi\sigma_\Sigma^4} + \frac{f^e_\alpha f^e_\beta}{\sigma_\Sigma^2} = 0. \tag{3.13}$$

We recall the statement of Lemma 2 of [CS21], which measures how much the first eigenvalues and the corresponding eigenfunctions of the Laplace-Beltrami operator on a round surface in the physical metric differ from the ones of the approximating sphere in the Euclidean metric.

Lemma 3.1.6. *Let $(M, \bar{g})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and let $\iota : \Sigma \hookrightarrow M$ be a* surface. *Suppose that Σ is in $\overline{\mathcal{W}}_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$. Then there exist a constant $c = c(B_1, B_2, \bar{c}) > 0$ and a radius $\sigma_0 = \sigma_0(B_1, B_2, \bar{c}, \delta) > 1$, such that, if $\sigma > \sigma_0$, there is a complete orthonormal system in $L^2(\Sigma)$ consisting of the eigenfunctions $\{f_\alpha\}_{\alpha=0}^\infty$ such that*

$$-\Delta f_\alpha = \lambda_\alpha f_\alpha, \qquad \textit{with } 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$$

Set $\mathbb{S}_{\sigma_\Sigma}$ to be the round sphere approximating Σ in the sense of Lemma 2.3.5. *Then there exists an orthonormal triple $\{f^e_1, f^e_2, f^e_3\}$ of eigenfunctions of $-\Delta^{\mathbb{S}_{\sigma_\Sigma}}$ such that, for $\alpha = 1, 2, 3$,*

$$\left| \lambda_\alpha - \frac{2}{\sigma_\Sigma^2} \right| \leq c\sigma^{-\frac{5}{2}-\delta}, \quad \|f_\alpha - f^e_\alpha\|_{W^{2,2}(\Sigma)} \leq c\sigma^{-\frac{1}{2}-\delta}. \tag{3.14}$$

Moreover

$$\left\|\overset{\circ}{\mathrm{Hess}}(f_\alpha)\right\|_{L^2(\Sigma)} \le c\sigma^{-\frac{5}{2}-\delta}, \qquad \int_\Sigma \left| \langle \nabla f_\alpha, \nabla f_\beta\rangle - \frac{3\delta_{\alpha\beta}}{\sigma_\Sigma^2 |\Sigma|_g} + \frac{f_\alpha f_\beta}{\sigma_\Sigma^2}\right| \, d\mu_g \le c\sigma^{-\frac{5}{2}-\delta}. \quad (3.15)$$

On the other hand, for $\alpha > 3$ we have

$$\lambda_\alpha > \frac{5}{\sigma_\Sigma^2}. \quad (3.16)$$

Remark 3.1.7. *The following proof is mainly based on [CS21, Lemma 2], but we rewrite it for reader's convenience. As a byproduct of the following proof, it is important to keep in mind that such a orthonormal system also satisfies the inequality $\|f_\alpha\|_{H^2(\Sigma)} \le C$.*

Observe moreover that [CS21, Lemma 2] is stated with the additional hypothesis of constant mean curvature. However, reading the proof of this Lemma, one can see that this hypothesis is only used in order to replace the area radius (compatible with our "roundness" radius introduced in Definition 2.3.1) with the curvature radius of [CS21, Lemma 2].

Proof. We have already seen that, thanks to the result of DeLellis-Müller [DLM05], the Euclidean image of Σ is a graph of a function on the Euclidean sphere $\mathbb{S}_{\sigma_\Sigma} \equiv \mathbb{S}_{\sigma_\Sigma}(\vec{z}_\Sigma)$. We consider a family $\{f_\alpha\}_{\alpha=0}^\infty$ of eigenfunctions of the Laplace-Beltrami operator $-\Delta$ on Σ associated to the eigenvalues $\{\lambda_\alpha\}_{\alpha=0}^\infty$. We choose three orthonormal eigenfunctions f_1, f_2, f_3 corresponding to the three eigenvalues λ_1, λ_2, λ_3, respectively. By the Rayleigh quotient, it is easy to see that $\left|\lambda_\alpha - \frac{2}{\sigma_\Sigma^2}\right| = \mathrm{O}(\sigma^{-\frac{5}{2}-\delta})$. In the Euclidean case, each rotation of the triple in (3.11) is a good choice of eigenfunctions for the Laplace-Beltrami operator. For our purposes, we want to choose one of these triples of Euclidean eigenfunctions in a way such that the second inequality in (3.14) holds. We proceed as follows.

We remark that

$$\int_\Sigma f_\alpha \, d\mu = \int_\Sigma \frac{1}{\lambda_\alpha}\left(-\Delta f_\alpha\right) \, d\mu = 0. \quad (3.17)$$

We choose $f_\alpha^e = f_\alpha - v_\alpha$, where v_α is a solution of the following equation

$$-\Delta^{\mathbb{S}_{\sigma_\Sigma}} v_\alpha - \lambda_\alpha^e v_\alpha = -\Delta^{\mathbb{S}_{\sigma_\Sigma}} f_\alpha - \lambda_\alpha^e f_\alpha. \quad (3.18)$$

A solution to this equation exists since the right hand side is orthogonal in $L^2(\mathbb{S}_{\sigma_\Sigma})$ to the kernel of the operator $-\Delta^{\mathbb{S}_{\sigma_\Sigma}} - \lambda_\alpha^e$ of the left hand side[1], by the Fredholm alternative. Moreover, since the kernel of the operator on the left hand side (LHS) of (3.18) is not trivial, the solutions to (3.18) form an affine space, with associated vector space $\mathrm{Ker}(-\Delta^{\mathbb{S}_{\sigma_\Sigma}} - \lambda_\alpha^e)$. The canonical choice of *a* solution to the equation (3.18) is given by the one which is orthogonal to $\mathrm{Ker}(-\Delta^{\mathbb{S}_{\sigma_\Sigma}} - \lambda_\alpha^e)$.

Lemma 13 in [CS21], combined with (3.17), implies

$$\|f_\alpha\|_{H^2} = \left\| f_\alpha - \fint_\Sigma f_\alpha \, d\mu \right\|_{H^2} \le C\sigma^2 \|\Delta f_\alpha\|_{L^2} \le C \quad (3.20)$$

since $\|\Delta f_\alpha\|_2 = \lambda_\alpha = \mathrm{O}(\sigma^{-2})$. This is enough to deduce that

$$\| -\Delta^{\mathbb{S}_{\sigma_\Sigma}} v_\alpha - \lambda_\alpha^e v_\alpha\|_{L^2} \le C\sigma^{-\frac{5}{2}-\delta}. \quad (3.21)$$

[1] This is true since if w satisfies $-\Delta^{\mathbb{S}_{\sigma_\Sigma}} w = \lambda_\alpha^e w$, then by the self-adjointness of $\Delta^{\mathbb{S}_{\sigma_\Sigma}}$,

$$\langle -\Delta^{\mathbb{S}_{\sigma_\Sigma}} f_\alpha - \lambda_\alpha^e f_\alpha, w\rangle_{L^2(\mathbb{S}_{\sigma_\Sigma})} = \lambda_\alpha^e \langle f_\alpha, w\rangle_{L^2(\mathbb{S}_{\sigma_\Sigma})} - \lambda_\alpha^e \langle f_\alpha, w\rangle_{L^2(\mathbb{S}_{\sigma_\Sigma})} = 0. \quad (3.19)$$

The elliptic theory reviewed in [Bes07, Thm. 27, Appendix H] implies that $\|v_\alpha\|_{H^2} = \|f_\alpha - f^e_\alpha\|_{H^2} \le C\sigma^{-\frac{1}{2}-\delta}$.

By $\langle f_\alpha, f_\beta\rangle_{L^2(\Sigma)} = \delta_{\alpha\beta}$ and the bound above, it follows that $|\langle f^e_\alpha, f^e_\beta\rangle_{L^2(\mathbb{S}_{\sigma_\Sigma})}| = \mathrm{O}(\sigma^{-\frac{1}{2}-\delta})$ for $\alpha \neq \beta$, while $\|f^e_\alpha\|_2 \simeq 1$. Using the Gram-Schmidt algorithm, we can modify the triple $\{f^e_1, f^e_2, f^e_3\}$ so that $\langle f^e_\alpha, f^e_\beta\rangle_{L^2(\mathbb{S}_{\sigma_\Sigma})} = 0$, while the estimate on $\|f_\alpha - f^e_\alpha\|_{H^2}$ continues to be true, as one can see writing down the difference between the new and the old basis in the Gram-Schmidt algorithm and derivating these expressions, together with the estimates on $\langle f^e_\alpha, f^e_\beta\rangle_{L^2(\mathbb{S}_{\sigma_\Sigma})}$. Thus we can assume that $\{f^e_1, f^e_2, f^e_3\}$ is an orthonormal system in $L^2(\mathbb{S}_{\sigma_\Sigma})$ of eigenfunctions of the Laplace operator on the round sphere $\mathbb{S}_{\sigma_\Sigma}$. This implies that there exists a triple of orthonormal vectors of $\mathbb{R}^3$, say $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, such that $f^e_\alpha = \sqrt{\frac{3}{4\pi\sigma^4_\Sigma}}\,(\vec{x} - \vec{z}) \cdot \vec{v}_\alpha$. In particular, modulo a rotation and in view of (3.13), the estimate on the traceless Hessian of f_α in the statement of the Lemma and (3.15) are satisfied. □

This description of the spectrum of the Laplace-Beltrami operator allows to define a decomposition of $L^2(\Sigma)$ in terms of the eigenfunctions of $-\Delta$.

Definition 3.1.8. *Let Σ be a surface and consider the Hilbert space $L^2(\Sigma)$ equipped with the standard scalar product. Consider the orthonormal system constructed in Lemma 3.1.6. Then for every $\mathrm{w} \in L^2(\Sigma)$ we define*

$$\mathrm{w}^0 := \langle w, f_0\rangle_2 f_0 = \fint_\Sigma \mathrm{w}\; d\mu_g, \qquad \mathrm{w}^t := \sum_{\alpha=1}^{3} \langle \mathrm{w}, f_\alpha\rangle_2 f_\alpha.$$

We call w^0 the mean part of w, and w^t the translational part of w. Finally, we set

$$\mathrm{w}^d := \mathrm{w} - \mathrm{w}^t \tag{3.22}$$

the so called difference part, which obviously also contains the information about w^0.

Before starting to study the properties of the stability operator, we give a more general version of [Ner15, Lemma 4.5], which holds for round surfaces which are not necessarily CMC. Observe that, with respect to the results of [Ner15], we get some additional terms of order $\mathrm{O}(\sigma^{-\frac{5}{2}-\delta})$, which can not be absorbed by the right hand side. These terms are the crucial differences with the CMC-case of the spectral theory.

Proposition 3.1.9. *Let $(M, \overline{\mathrm{g}})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and let $\iota : \Sigma \hookrightarrow M$ be a* surface. *Suppose that Σ is in $\overline{\mathcal{W}}_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$. Then there exist a constant $c = c(B_1, B_2, \overline{c}) > 0$ and a radius $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta) > 1$, such that, if $\sigma > \sigma_0$, for every $\alpha \neq \beta$, $\alpha, \beta \in \{1, 2, 3\}$, it holds*

$$\left| \int_\Sigma \left(\overline{\mathrm{Ric}}(\nu, \nu) - \frac{H^2 - h^2}{4} \right) f_\alpha f_\beta \; d\mu \right| \le c\sigma^{-3-\delta}.$$

Moreover, for every $i \in \{1, 2, 3\}$, we have

$$\left| \lambda_\alpha - \frac{h^2}{2} - \frac{6m_H(\Sigma)}{\sigma^3_\Sigma} - \int_\Sigma \left(\overline{\mathrm{Ric}}(\nu, \nu) - \frac{H^2 - h^2}{4} \right) f^2_\alpha \; d\mu \right| \le c\sigma^{-3-\delta}.$$

Proof. Using the Bochner's formula as in [CS21, Lemma 3], we get the estimate

$$\left| \lambda^2_\alpha \delta_{\alpha\beta} - \int_\Sigma R^\Sigma \langle \nabla f_\alpha, \nabla f_\beta\rangle \; d\mu \right| \le C\sigma^{-5-\delta}. \tag{3.23}$$

By the Gauss formula and the estimate $|\mathring{A}| = \mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$, (3.23) implies

$$\left|\lambda_\alpha^2 \delta_{\alpha\beta} - \int_\Sigma \left(\overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu,\nu) + \frac{H^2}{2}\right) \langle \nabla f_\alpha, \nabla f_\beta \rangle \, d\mu \right| \leq C\sigma^{-5-\delta}. \tag{3.24}$$

Using that $H^2 = h^2 + (H^2 - h^2)$ and the variational formulation for the equation $-\Delta f_\alpha = \lambda_\alpha f_\alpha$, we get

$$\left| \left(\lambda_\alpha^2 - \lambda_\alpha \frac{h^2}{2}\right) \delta_{\alpha\beta} - \int_\Sigma \left(\overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu,\nu) + \frac{H^2 - h^2}{2}\right) \langle \nabla f_\alpha, \nabla f_\beta \rangle \, d\mu \right| \leq C\sigma^{-5-\delta}. \tag{3.25}$$

Since $H^2 - h^2 = \mathrm{O}(\sigma^{-\frac{5}{2}-\delta})$ by Lemma 2.3.5, using (3.15) we get

$$\begin{aligned} \int_\Sigma \frac{H^2 - h^2}{2} \langle \nabla f_\alpha, \nabla f_\beta \rangle \, d\mu &= \int_\Sigma \frac{H^2 - h^2}{2} \left(\frac{3\delta_{\alpha\beta}}{\sigma_\Sigma^2 |\Sigma|_g} - \frac{f_\alpha f_\beta}{\sigma_\Sigma^2} \right) d\mu + \mathrm{O}(\sigma^{-5-2\delta}) \\ &= -\int_\Sigma \frac{H^2 - h^2}{2} \left(\frac{f_\alpha f_\beta}{\sigma_\Sigma^2} \right) d\mu + \mathrm{O}(\sigma^{-5-2\delta}). \end{aligned} \tag{3.26}$$

Moreover, in [Ner15] and [CS21], it has been shown that the remaining terms in (3.25) satisfy

$$\begin{aligned} &\left(\lambda_\alpha^2 - \lambda_\alpha \frac{h^2}{2}\right) \delta_{\alpha\beta} - \int_\Sigma \left(\overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu,\nu)\right) \langle \nabla f_\alpha, \nabla f_\beta \rangle \, d\mu \\ &= \frac{2}{\sigma_\Sigma^2} \left(\lambda_\alpha - \frac{h^2}{2}\right) \delta_{\alpha\beta} - \frac{12 m_H(\Sigma)}{\sigma_\Sigma^5} \delta_{\alpha\beta} - \frac{2}{\sigma_\Sigma^2} \int_\Sigma \overline{\mathrm{Ric}}(\nu,\nu) f_\alpha f_\beta \, d\mu + \mathrm{O}(\sigma^{-5-\delta}). \end{aligned} \tag{3.27}$$

Combining (3.25), (3.26), (3.27) and dividing by $\frac{2}{\sigma_\Sigma^2}$ we get the thesis. □

The previous Lemma leads to the following.

Proposition 3.1.10. *Let $(M, \overline{g})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and let $\iota : \Sigma \hookrightarrow M$ be a* surface. *Suppose that Σ is in $\overline{\mathcal{W}}_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$. Then there exist a constant $c = c(B_1, B_2, \overline{c}) > 0$ and a radius $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta) > 1$, such that, if $\sigma > \sigma_0$, for every $f \in \operatorname{span}\{f_1, f_2, f_3\}$ and $\varphi \in \operatorname{span}\{f_\alpha : \ \alpha \geq 4\}$ the following inequalities hold*

$$\left| \langle Lf, f \rangle_2 - \frac{6 m_H(\Sigma)}{\sigma_\Sigma^3} \|f\|_2^2 + \frac{3h}{2} \int_\Sigma (H - h) f^2 \, d\mu \right| \leq c\sigma^{-3-2\delta} \|f\|_2^2, \tag{3.28}$$

$$|\langle Lf, \varphi \rangle_2| \leq c\sigma^{-\frac{5}{2}-\delta} \|f\|_2 \|\varphi\|_2, \qquad \langle L\varphi, \varphi \rangle_2 > \frac{2}{\sigma_\Sigma^2} \|\varphi\|_2^2. \tag{3.29}$$

Proof. Let $f \in \operatorname{span}\{f_1, f_2, f_3\}$, $f = \sum_{\alpha=1}^3 \langle f, f_\alpha \rangle_2 f_\alpha$. By definition and using both inequalities of Proposition 3.1.9 we get

$$\langle Lf, f \rangle_2 = \frac{6 m_H(\Sigma)}{\sigma_\Sigma^3} \|f\|_2^2 - \int_\Sigma \frac{3(H^2 - h^2)}{4} f^2 \, d\mu + \mathrm{O}(\sigma^{-3-2\delta}) \|f\|_2^2,$$

where we also used that $\|\mathring{A}\|^2_{L^\infty(\Sigma)} = \mathrm{O}(\sigma^{-3-2\delta})$ thanks to Lemma 2.3.5. Using that $H^2 - h^2 = 2h(H - h) + \mathrm{O}(\sigma^{-3-2\delta})$, we find

$$\langle Lf, f \rangle_2 = \frac{6 m_H(\Sigma)}{\sigma_\Sigma^3} \|f\|_2^2 - \frac{3h}{2} \int_\Sigma (H - h) f^2 \, d\mu + \mathrm{O}(\sigma^{-3-2\delta}) \|f\|_2^2.$$

Suppose now that $\varphi \in \mathrm{span}\{f_\alpha :\ \alpha \geq 4\}$. Then

$$\langle L\varphi, \varphi\rangle_2 \geq \left(\lambda_4 - \sup_\Sigma \left||A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right|\right) \int_\Sigma \varphi^2 \, d\mu,$$

using the characterization of λ_4. Thus, since $\lambda_4 \geq \frac{5}{\sigma_\Sigma^2}$ by Lemma 3.1.6 and since, by the roundness hypothesis $\left||A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right| \leq \frac{11}{4\sigma_\Sigma^2}$, we get the second part of equation (3.29). On the other hand

$$\langle Lf, \varphi\rangle_2 = \int_\Sigma \left(-|A|^2 - \overline{\mathrm{Ric}}(\nu,\nu)\right) f\varphi \, d\mu = \int_\Sigma \left(-|A|^2 - \overline{\mathrm{Ric}}(\nu,\nu) + \frac{h^2}{2}\right) f\varphi \, d\mu$$

concludes the proof, together with $|A|^2 - \frac{h^2}{2} = |\mathring{A}|^2 + \frac{H^2-h^2}{2}$ and, by Lemma 2.3.5, $\|H^2 - h^2\|_{L^\infty(\Sigma)} = \mathrm{O}(\sigma^{-\frac{5}{2}-\delta})$. □

We conclude with an auxiliary estimate that will be needed in the following.

Corollary 3.1.11. *Let $(M, \overline{g})$ a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold such that $|E_{\mathrm{ADM}}| \neq 0$. Let $\iota : \Sigma \hookrightarrow M$ be a* surface *in $\overline{\mathcal{W}}_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$. Then there exist a constant $c = c(B_1, B_2, \overline{c}) > 0$ and a radius $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta, |E_{\mathrm{ADM}}|) > 1$, such that, if $\sigma > \sigma_0$ then, for every $v, w \in \mathrm{span}\{f_1, f_2, f_3\}$, it holds*

$$\left|\int_\Sigma (Lv) w \, d\mu\right| \leq c\sigma^{-\frac{5}{2}-\delta}\|v\|_2\|w\|_2.$$

Proof. It follows from the identity,

$$\int_\Sigma (Lf_\alpha) f_\beta \, d\mu = \left(\lambda_\alpha - \frac{h^2}{2}\right)\delta_{\alpha\beta} + \int_\Sigma \frac{h^2 - H^2}{2} f_\alpha f_\beta \, d\mu + \mathrm{O}(\sigma^{-\frac{5}{2}-\delta}),$$

for every $\alpha, \beta \in \{1, 2, 3\}$. This, combined with Proposition 3.1.9 and $|m_H(\Sigma)| \leq 2E_{\mathrm{ADM}}$ for σ large (in view of Lemma 3.1.3), leads to the thesis. □

In the next Chapter, we will investigate the role of the stability operator in the evolution of round surfaces (maintaining the volume constant and decreasing the area). In order to do this, we estimate here some stability operator-related functions.

Lemma 3.1.12. *Let $(M, \overline{g})$ a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold such that $|E_{\mathrm{ADM}}| \neq 0$. Let $\iota : \Sigma \hookrightarrow M$ be a* surface *in $\overline{\mathcal{W}}_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$ and consider the setting of* Lemma 3.1.6*, with $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ as in its proof. Then there exist a constant $c = c(B_1, B_2, \overline{c}) > 0$ and a radius $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta, |E_{\mathrm{ADM}}|) > 1$, such that, if $\sigma > \sigma_0$, then for $\alpha \in \{1, 2, 3\}$*

$$\left|\left\langle L(H-h), \sqrt{\frac{4\pi}{3}} f_\alpha - \frac{\nu \cdot \vec{v}_\alpha}{\sigma_\Sigma}\right\rangle_{L^2(\Sigma)}\right| \leq c\sigma^{-3-2\delta}.$$

Proof. First of all, observe that the roundness of Σ implies that $\|H - h\|_{H^1(\Sigma)} \leq c\sigma^{-\frac{1}{2}-\delta}$ for some $c = c(B_1, B_2) > 0$. Moreover, Lemma 3.1.6 implies that $\|f_\alpha - f^e_\alpha\|_{W^{2,2}(\Sigma)} \leq c\sigma^{-\frac{1}{2}-\delta}$, where $f^e_\alpha = \sqrt{\frac{3}{4\pi\sigma_\Sigma^4}}\,(\vec{x} \cdot \vec{v}_\alpha)$, restricted to Σ and with the abuse of notation of $\vec{x} \simeq \mathrm{Id}$. This, multiplied by $\frac{\sqrt{4\pi}\sigma_\Sigma}{\sqrt{3}}$, also equals the projection $\nu^{\mathbb{S}_{\sigma_\Sigma}} \cdot \vec{v}_\alpha$, where $\nu^{\mathbb{S}_{\sigma_\Sigma}}$ is the normal to the round sphere. In view of Lemma 2.3.5, it also holds $\|\nu^e - \nu^{\mathbb{S}_{\sigma_\Sigma}}\|_{W^{1,\infty}} = \mathrm{O}(\sigma^{-\frac{1}{2}-\delta})$, where ν^e

is the Euclidean normal of Σ. Thus it follows that

$$\left\| \sqrt{\frac{4\pi}{3}} f^e_\alpha - \frac{\nu^e \cdot \vec{v}_\alpha}{\sigma_\Sigma} \right\|_{W^{1,\infty}} = \mathrm{O}(\sigma^{-\frac{3}{2}-\delta}). \tag{3.30}$$

Using Lemma 2.1.6 and the estimate for $\|f^e_\alpha - f_\alpha\|_{H^2(\Sigma)}$ in Lemma 3.1.6, we conclude that

$$\left\| \sqrt{\frac{4\pi}{3}} f_\alpha - \frac{\nu \cdot \vec{v}_\alpha}{\sigma_\Sigma} \right\|_{H^1(\Sigma)} \le c\sigma^{-\frac{1}{2}-\delta}.$$

Thus, for $\alpha \in \{1,2,3\}$, we obtain

$$\left| \left\langle L(H-h), \frac{\nu \cdot \vec{v}_\alpha}{\sigma_\Sigma} - \sqrt{\frac{4\pi}{3}} f_\alpha \right\rangle_{L^2(\Sigma)} \right| =$$

$$\begin{aligned}
&= \left| \int_\Sigma \left(-\Delta(H-h) - (|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))(H-h)\right) \left(\frac{\nu \cdot \vec{v}_\alpha}{\sigma_\Sigma} - \sqrt{\frac{4\pi}{3}} f_\alpha \right) d\mu \right| \\
&\le \left| \int_\Sigma \nabla(H-h) \cdot \nabla \left(\frac{\nu \cdot \vec{v}_\alpha}{\sigma_\Sigma} - \sqrt{\frac{4\pi}{3}} f_\alpha \right) d\mu \right| \\
&\quad + \int_\Sigma \left(|A|^2 + |\overline{\mathrm{Ric}}(\nu,\nu)|\right) |H-h| \left| \frac{\nu \cdot \vec{v}_\alpha}{\sigma_\Sigma} - \sqrt{\frac{4\pi}{3}} f_\alpha \right| d\mu
\end{aligned}$$

and we conclude using Hölder's inequality. □

Corollary 3.1.13. *Let $(M, \overline{g})$ a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold such that $|E_{\mathrm{ADM}}| \neq 0$. Let $\iota : \Sigma \hookrightarrow M$ be a surface in $\mathcal{W}_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$. Then there exist a constant $c = c(B_1, B_2, \overline{c}) > 0$ and a radius $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta, |E_{\mathrm{ADM}}|) > 1$, such that, if $\sigma > \sigma_0$, then*

$$\left| \left\langle L(H-h), \frac{\nu_\alpha}{\sigma} \right\rangle_{L^2(\Sigma)} \right| \le c\sigma^{-3-2\delta}. \tag{3.31}$$

Proof. It is convenient to do the computation with f_α. Since the stability operator is self-adjoint in $L^2(\Sigma)$, we have

$$\langle L(H-h), f_\alpha \rangle_{L^2(\Sigma)} = \langle (H-h)^t, L f_\alpha \rangle_{L^2(\Sigma)} + \langle (H-h)^d, L f_\alpha \rangle_{L^2(\Sigma)}.$$

Using Corollary 3.1.11 and Proposition 3.1.10, together with $\|(H-h)^t\|_2^2 + \|(H-h)^d\|_2^2 = \|H-h\|_2^2 \le C\sigma^{-1-2\delta}$, we get $\left|\langle L(H-h), f_\alpha \rangle_{L^2(\Sigma)}\right| = \mathrm{O}(\sigma^{-3-2\delta})$ and thus we conclude by Lemma 3.1.12, the orthonormality of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and the boundedness of $\frac{\sigma_\Sigma}{\sigma}$. □

3.1.3 The translational part of the mean curvature

We analyze now an important property of the translational part of a function, which we have introduced in Definition 3.1.8.

Remark 3.1.14. *Note that, by Remark 3.1.7, $\|f_\alpha\|_{H^2(\Sigma)}$ is bounded, uniformly in σ. They, by Sobolev's embedding $H^2 \hookrightarrow L^\infty$, we have that $\|(H-h)^t\|_{L^\infty(\Sigma)} \le C\sigma^{-\frac{3}{2}-\delta}$, where C depends on the roundness class to which Σ belongs. Since by definition $(H-h)^d := (H-h) - (H-h)^t$, we also have $\|(H-h)^d\|_{L^\infty(\Sigma)} \le C\sigma^{-\frac{3}{2}-\delta}$, using also Lemma 2.3.5.*

We compute the error that we commit in a specific integral when replacing Σ with its approximating Euclidean sphere.

Lemma 3.1.15. *Let $(M, \overline{g})$ a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold. Let $\iota : \Sigma \hookrightarrow M$ be a* surface *in $\overline{\mathcal{W}}_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$. Then there exist a constant $c = c(B_1, B_2, \overline{c}) > 0$ and a radius $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta) > 1$, such that, if $\sigma > \sigma_0$,*

$$\left| h \int_\Sigma (H-h)((H-h)^t)^2 \, d\mu - h \int_{\mathbb{S}_{\sigma_\Sigma}} (H-h)((H-h)^t)^2 \, d\mu^{\mathbb{S}_\Sigma} \right| \leq c\sigma^{-3-2\delta} \|(H-h)^t\|_2^2. \tag{3.32}$$

Proof. By Lemma 2.3.5, scalar functions on Σ can be also meant as function on the Euclidean sphere $\mathbb{S}_{\sigma_\Sigma}$, and moreover $|d\mu - d\mu^{\mathbb{S}_{\sigma_\Sigma}}| = \mathrm{O}(\sigma^{-\frac{1}{2}-\delta}) d\mu$. Thus the left hand side of (3.32) is bounded by

$$\leq c\sigma^{-\frac{3}{2}-\delta} \|H-h\|_{L^\infty(\Sigma)} \int_\Sigma ((H-h)^t)^2 \, d\mu \leq c\sigma^{-3-2\delta} \int_\Sigma ((H-h)^t)^2 \, d\mu.$$

□

Roughly speaking, equation (3.32) says that we can replace, modulo an error, the integral over Σ with the same integral over the sphere $\mathbb{S}_{\sigma_\Sigma}$. Thus, we now consider the case in which the integral is computed on a Euclidean round sphere. In this case, for any $u \in \mathrm{span}\{f_1^e, f_2^e, f_3^e\}$, we find, because of symmetry reasons,

$$\int_{\mathbb{S}_\sigma} u^{2k+1} \, d\mu_{\mathbb{S}_\sigma} = 0 \tag{3.33}$$

for every $k \in \mathbb{N}$. This allows to obtain a strong bound on the corresponding integral when we consider a round surface in an asymptotically flat space. We focus here on the case of a third power, which is the one that we need in the sequel.

Lemma 3.1.16. *Let $(M, \overline{g})$ a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold. Let $\iota : \Sigma \hookrightarrow M$ be a* surface *in $\overline{\mathcal{W}}_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$. Then there exist a constant $c = c(B_1, B_2, \overline{c}) > 0$ and a radius $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta) > 1$, such that, if $\sigma > \sigma_0$,*

$$\left| h \int_\Sigma ((H-h)^t)^3 \, d\mu \right| \leq c\sigma^{-3-2\delta} \|(H-h)^t\|^2_{L^2(\Sigma)}.$$

Remark 3.1.17. *Observe that this Lemma is more accurate then the one we would obtain simply estimating $|(H-h)^t|$ with $\|H-h\|_{L^\infty(\Sigma)}$.*

Proof. Define the following auxiliary function on the sphere $\mathbb{S}_{\sigma_\Sigma}$,

$$H_T := \sum_{\alpha=1}^3 \langle H-h, f_\alpha \rangle_{L^2(\Sigma)} f_\alpha^e.$$

Since H_T is an odd function on $\mathbb{S}_{\sigma_\Sigma} \equiv \mathbb{S}_{\sigma_\Sigma}(\vec{z}_0)$, we have that

$$\int_{\mathbb{S}_{\sigma_\Sigma}} H_T^3 \, d\mu^{\mathbb{S}_{\sigma_\Sigma}} = 0. \tag{3.34}$$

Moreover, using equation (3.14) (combined with the immersion $H^2 \hookrightarrow L^\infty$) and the Cauchy-Schwarz inequality for sums, we have

$$\|(H-h)^t - H_T\|_{L^\infty(\Sigma)} \le C\sigma^{-\frac{3}{2}-\delta}\left(\sum_{\alpha=1}^{3}\langle H-h, f_\alpha\rangle_2^2\right)^{\frac{1}{2}} = C\sigma^{-\frac{3}{2}-\delta}\|(H-h)^t\|_2 = \mathrm{O}(\sigma^{-2-2\delta}). \tag{3.35}$$

Considering $(H-h)^t$ as a function on $\mathbb{S}_{\sigma_\Sigma}$, equation (3.34) implies that

$$\left|\int_{\mathbb{S}_{\sigma_\Sigma}} ((H-h)^t)^3 \, d\mu^{\mathbb{S}_{\sigma_\Sigma}}\right|$$

$$\le \|(H-h)^t - H_T\|_{L^\infty(\Sigma)}\left(\int_{\mathbb{S}_{\sigma_\Sigma}} ((H-h)^t)^2 \, d\mu^{\mathbb{S}_{\sigma_\Sigma}} + \int_{\mathbb{S}_{\sigma_\Sigma}} H_T^2 \, d\mu^{\mathbb{S}_{\sigma_\Sigma}} + 2\int_{\mathbb{S}_{\sigma_\Sigma}} ((H-h)^t)H_T \, d\mu^{\mathbb{S}_{\sigma_\Sigma}}\right).$$

Since (3.35) implies that $\big|\|(H-h)^t\|_2 - \|H_T\|_2\big| \le C\sigma^{-\frac{1}{2}-\delta}\|(H-h)^t\|_2$, replacing $d\mu^{\mathbb{S}_{\sigma_\Sigma}}$ with $d\mu$ through Lemma 3.1.15, we get

$$h\left|\int_{\mathbb{S}_{\sigma_\Sigma}} ((H-h)^t)^3 \, d\mu^{\mathbb{S}_{\sigma_\Sigma}}\right| \le C\sigma^{-3-2\delta}\int_\Sigma ((H-h)^t)^2 \, d\mu,$$

and thus we conclude again with Lemma 3.1.15. □

Lemma 3.1.18. *Let $(M, \overline{g})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold and consider a surface $\iota : \Sigma \hookrightarrow M$ in $\overline{\mathcal{W}}_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$ such that*

$$\int_\Sigma \left((H-h)^d\right)^2 \, d\mu \le \int_\Sigma \left((H-h)^t\right)^2 \, d\mu. \tag{3.36}$$

Set moreover

$$\Pi := \sum_{\alpha=1}^{3}\left\langle H-h, \frac{\nu_\alpha}{\sigma}\right\rangle_{L^2(\Sigma)}^2.$$

Then there exist a constant $c = c(B_1, B_2, \overline{c}) > 0$ and a radius $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta) > 1$ such that, if $\sigma > \sigma_0$,

$$\left|\Pi - \frac{4\pi}{3}\int_\Sigma \left((H-h)^t\right)^2 \, d\mu\right| \le c\sigma^{-\frac{1}{2}-\delta}\|H-h\|^2_{L^2(\Sigma)}.$$

Proof. Thanks to hypothesis (3.36) we have that $\|H-h\|^2_{L^2(\Sigma)} \le 2\int_\Sigma \left((H-h)^t\right)^2 \, d\mu$. Moreover

$$\begin{aligned}
&\left|\Pi - \frac{4\pi}{3}\int_\Sigma \left((H-h)^t\right)^2 \, d\mu\right| \\
&\le \sum_{\alpha=1}^{3}\left|\left\langle H-h, \frac{\nu_\alpha}{\sigma}\right\rangle_{L^2(\Sigma)}^2 - \left\langle H-h, \sqrt{\frac{4\pi}{3}}f_\alpha\right\rangle_{L^2(\Sigma)}^2\right| \\
&= \sum_{\alpha=1}^{3}\left|\left\langle H-h, \frac{\nu_\alpha}{\sigma} + \sqrt{\frac{4\pi}{3}}f_\alpha\right\rangle_{L^2(\Sigma)}\right|\left|\left\langle H-h, \frac{\nu_\alpha}{\sigma} - \sqrt{\frac{4\pi}{3}}f_\alpha\right\rangle_{L^2(\Sigma)}\right| \\
&\le c\sum_{i=1}^{3}\|H-h\|_{L^2(\Sigma)}\left\|\sqrt{\frac{4\pi}{3}}f_\alpha - \frac{\nu_\alpha}{\sigma}\right\|_{L^\infty(\Sigma)}\int_\Sigma |H-h| \, d\mu \le c\sigma^{-\frac{1}{2}-\delta}\|H-h\|^2_{L^2(\Sigma)}.
\end{aligned}$$

where in the latter estimate we used equation (3.30). This ends the proof. □

3.1.4 Regge-Teitelboim conditions

In Section 2.0.2 we have defined the strong Regge-Teitelboim conditions, which, in view of Lemma 2.0.7, allows us to prove the existence of the ADM-center of mass. A weaker version of these conditions has been introduced in [Ner15]. Under these weaker hypotheses, Nerz proved that, if the ADM-center of mass exists, it coincides with another notion of center of mass, called CMC-*center of mass*. Nerz proved that the geometric structure at the basis of this definition always exists: a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold admits a CMC-foliation. The CMC-center of mass is the limit of the barycenters of the leaves of this foliation, if it exists.

We conclude the section by observing that the translational part of the mean curvature of a coordinate sphere satisfies an improved estimate if our ambient manifold satisfies the weak Regge-Teitelboim conditions.

Definition 3.1.19 ($C^1_{1+\delta}$-Regge-Teitelboim conditions)**.** *Let $(M, \overline{\mathrm{g}}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold. We say that this manifold satisfies the $C^2_{1+\delta}$-Regge-Teitelboim conditions if there exists $\overline{c} > 0$ such that*

$$\left|\overline{\mathrm{g}}_{ij}(\overline{x}) - \overline{\mathrm{g}}_{ij}(-\overline{x})\right| + |\overline{x}| \left|\overline{\Gamma}^k_{ij}(\overline{x}) + \overline{\Gamma}^k_{ij}(-\overline{x})\right| \leq \frac{\overline{c}}{|\overline{x}|^{1+\delta}}, \tag{3.37}$$

for every $\overline{x} \in M \setminus \mathrm{C}$.

First of all, we remark that this conditions imply that

$$|\nu^\Sigma_x + \nu^\Sigma_{-x}| = \mathrm{O}(\sigma^{-1-\delta}), \qquad |H^\Sigma_x - H^\Sigma_{-x}| = O(\sigma^{-2-\delta}), \tag{3.38}$$

for $\Sigma = \vec{x}\left(\mathbb{S}_\sigma(\vec{0})\right)$. The first decay in (3.38) is a consequence of the fact that the metric is asymptotically even. The second one follows from the definition of *shape operator* and the decay of the Christoffel symbols in (3.37).

The next Lemma shows that (3.38) implies that the translational part of $H - h$ is sufficiently small. It is essentially inspired by the results in [Hua10].

Lemma 3.1.20. *Let $(M, \overline{\mathrm{g}}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat 3-manifold that satisfies the $C^2_{1+\delta}$-Regge-Teitelboim conditions. Consider the immersion $\mathbb{S}_\sigma(\vec{0}) \hookrightarrow M$, i.e. $\Sigma := \vec{x}\left(\mathbb{S}_\sigma(\vec{0})\right)$, for $\sigma > 1$ fixed but large. Then there exists a constant $C = C(\overline{c}) > 0$ such that*

$$\sum_{\alpha=1}^{3} \left\langle H - h, \frac{\nu_\alpha}{\sigma} \right\rangle^2_{L^2(\Sigma)} \leq C\sigma^{-2-2\delta}.$$

Proof. We define ψ, the reflection with respect to $\vec{0}$, i.e. $\psi : \overline{x} \mapsto \vec{x}^{-1}(-\vec{x}(\overline{x}))$, defined on $M \setminus \vec{x}\left(\mathbb{B}_{\mathrm{diam(C)}/2}(\vec{0})\right)$. With an abuse of notation, we now identify $\mathbb{S}_\sigma(\vec{0})$ with its image through $\vec{x}$, and we decompose $\mathbb{S}_\sigma(\vec{0}) = \mathbb{S}^+_\sigma(\vec{0}) \cup \mathbb{S}^-_\sigma(\vec{0})$, such that $\psi\left(\mathbb{S}^-_\sigma(\vec{0})\right) = \mathbb{S}^+_\sigma(\vec{0})$, $\sigma >$

$\frac{\mathrm{diam(C)}}{2}$. Then, also using that $d\mu_x - d\mu_{-x} = \mathrm{O}(\sigma^{-1-\delta})$ thanks to the hypothesis (3.37),

$$\begin{aligned}\int_{\mathbb{S}_\sigma(\vec{0})} \nu \, d\mu &= \int_{\mathbb{S}^+_\sigma(\vec{0})} \nu_x \, d\mu_x + \int_{\mathbb{S}^-_\sigma(\vec{0})} \nu_x \, d\mu_x = \\ &= \int_{\mathbb{S}^+_\sigma(\vec{0})} \nu_x \, d\mu_x - \int_{\mathbb{S}^-_\sigma(\vec{0})} \nu_{-x} \, d\mu_x + \int_{\mathbb{S}^-_\sigma(\vec{0})} (\nu_x + \nu_{-x}) \, d\mu_x = \\ &= \int_{\mathbb{S}^+_\sigma(\vec{0})} \nu_x \, d\mu_x - \int_{\mathbb{S}^-_\sigma(\vec{0})} \nu_{-x} \, d\mu_{-x} - \int_{\mathbb{S}^-_\sigma(\vec{0})} \nu_{-x} \, (d\mu_x - d\mu_{-x}) + \mathrm{O}(\sigma^{1-\delta})\end{aligned}$$

and changing the variable $x \mapsto -x$ we get

$$= \int_{\mathbb{S}^+_\sigma(\vec{0})} \nu_x \, d\mu_x - \int_{\mathbb{S}^+_\sigma(\vec{0})} \nu_x \, d\mu_x + \mathrm{O}(\sigma^{1-\delta}) = \mathrm{O}(\sigma^{1-\delta}).$$

This implies that

$$\int_{\mathbb{S}_\sigma(\vec{0})} (H - h)\nu \, d\mu = \int_{\mathbb{S}_\sigma(\vec{0})} H\nu \, d\mu + \mathrm{O}(\sigma^{-\delta}), \tag{3.39}$$

and using moreover (3.38), we find

$$\int_{\mathbb{S}_\sigma(\vec{0})} H\nu \, d\mu = \mathrm{O}(\sigma^{-\delta}), \tag{3.40}$$

that implies the thesis. □

3.2 Spectral theory in initial data sets

In this Section we consider closed surfaces Σ belonging to a roundness class $\overline{\mathcal{W}}^\eta_\sigma(B_1, B_2)$ for fixed parameters η, B_1, B_2 and a general large σ. Moreover, we will suppose that $E_{\mathrm{ADM}} > 0$ and that Σ is almost CMC.

Definition 3.2.1. *Let $(M, \overline{\mathrm{g}})$ a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold with energy $E_{\mathrm{ADM}} > 0$. Fix $\sigma > 1$ and $\mathrm{c_{in}} > 0$. We say that $\Sigma \hookrightarrow M$ is $(\sigma, \mathrm{c_{in}})$-almost-CMC if $(\Sigma, g) \in \overline{\mathcal{W}}^\eta_\sigma(B_1, B_2)$ for some $B_1, B_2 > 0$, and*

$$\|H - h\|_{L^2(\Sigma)} \le \mathrm{c_{in}}\sigma^{-1-\delta}. \tag{3.41}$$

We sometimes simply say that Σ is $\mathrm{c_{in}}$-*almost CMC* if Σ is $(\mathrm{c_{in}}, \sigma_\Sigma)$-almost CMC. Thus, we will tacitly mean that the constants c and σ_0 which appear in the statements below only depend on η, B_1, B_2, on the constants $\mathrm{c_{in}}, \overline{c}, \delta$ and possibly on the energy E_{ADM}. We remark that assumption (3.41) is stronger than the L^2-estimate satisfied by $H - h$ on Euclidean spheres.

Throughout these Lemmas, the setting will be the same of Lemma 3.1.6. Remember moreover that $\hbar := |\Sigma|^{-1} \int_\Sigma \mathcal{H} \, d\mu$.

Lemma 3.2.2. *There exist $c > 0$ and $\sigma_0 > 1$ such that, if $\Sigma \in \overline{\mathcal{W}}^\eta_\sigma(B_1, B_2)$ and it also is $(\mathrm{c_{in}}, \sigma)$-almost CMC with $\sigma \ge \sigma_0$, the complete orthonormal system in $L^2(\Sigma)$ given by* Lemma 3.1.6 *is such that*

$$\left| \lambda_\alpha - \frac{\hbar^2}{2} - \frac{6m_{\mathcal{H}}(\Sigma)}{\sigma^3_\Sigma} - \int_\Sigma \overline{\mathrm{Ric}}(\nu, \nu) f^2_\alpha \, d\mu_g \right| \le c\sigma^{-3-\delta}, \qquad \alpha \in \{1, 2, 3\}, \tag{3.42}$$

and the corresponding eigenfunctions f_1, f_2, f_3 satisfy

$$\left| \int_\Sigma \overline{\mathrm{Ric}}(\nu, \nu) f_\alpha f_\beta \right| \le c\sigma^{-3-\delta}, \qquad \alpha \ne \beta, \quad \alpha, \beta \in \{1, 2, 3\}. \tag{3.43}$$

Sketch of the proof. The proof is analogous to the one of [CS21, Lemma 3]. We remark that, using that $|\overset{\circ}{A}| = \mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$, we can write the Gauss equation as

$$\begin{aligned} \mathrm{S}^{\Sigma} &= \overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu,\nu) - |\overset{\circ}{A}|^2 + \frac{H^2}{2} \\ &= \overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu,\nu) + \frac{\hbar^2}{2} + \frac{(\mathcal{H}-\hbar)^2}{2} + \hbar(\mathcal{H}-\hbar) + \mathrm{O}(\sigma^{-3-\delta}). \end{aligned} \tag{3.44}$$

since, using Lemma 2.1.3 and the roundness, we find $H - \mathcal{H} = \mathrm{O}(\sigma^{-2-2\delta})$ and $\mathcal{H} = \mathrm{O}(\sigma^{-1})$. We thus set $\mathcal{R} := \frac{(\mathcal{H}-\hbar)^2}{2} + \hbar(\mathcal{H}-\hbar) + \mathrm{O}(\sigma^{-3-\delta})$ and, analogously to the proof of Lemma 3.1.9, we get

$$\begin{aligned} &\left|\lambda_\alpha^2\delta_{\alpha\beta} - \int_\Sigma \mathrm{S}^{\Sigma}\langle\nabla f_\alpha, \nabla f_\beta\rangle\, d\mu_g\right| \\ =&\left|\lambda_\alpha^2\delta_{\alpha\beta} - \int_\Sigma \left((\overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu,\nu)) + \left(\frac{\hbar^2}{2} + \mathcal{R}\right)\right)\langle\nabla f_\alpha, \nabla f_\beta\rangle\, d\mu_g\right|. \end{aligned} \tag{3.45}$$

Since the spacelike case corresponds to $\mathcal{R} \equiv 0$, in the spacetime case we just have to estimate

$$\left|\int_\Sigma \mathcal{R}\langle\nabla f_\alpha, \nabla f_\beta\rangle\, d\mu\right| = \left|\int_\Sigma \left(\frac{(\mathcal{H}-\hbar)^2}{2} + \hbar(\mathcal{H}-\hbar) + \mathrm{O}(\sigma^{-3-\delta})\right)\langle\nabla f_\alpha, \nabla f_\beta\rangle\, d\mu\right|. \tag{3.46}$$

Remember, comparing this proof with the one of Lemma 3.1.9, that the aim is to show that this remainder is of order $\mathrm{O}(\sigma^{-5-\delta})$. Since $\|f_\alpha\|_{H^2(\Sigma)} = \mathrm{O}(1)$, and thus $\|\nabla f_\alpha\|_{L^2(\Sigma)} \leq \frac{C}{\sigma}$, notice that $\left|\int_\Sigma \langle\nabla f_\alpha, \nabla f_\beta\rangle\, d\mu\right| \leq C\sigma^{-2}$, which bounds the latter term in (3.46). The other two terms can be bounded, using Young's inequality, equation (3.41) and $\mathcal{H} - H = \mathrm{O}(\sigma^{-2-2\delta})$, by

$$C\sigma^{-1}\|\mathcal{H}-\hbar\|_{L^2}\|\nabla f_\alpha\|_{L^4}\|\nabla f_\beta\|_{L^4} \leq C\sigma^{-5-2\delta} \tag{3.47}$$

using that $\|f_\alpha\|_{W^{1,4}} \leq C\sigma^{-\frac{1}{2}}\|f_\alpha\|_{H^2}$, and so $\|\nabla f_i\|_4 \leq C\sigma^{-\frac{3}{2}}$. Thus we obtain

$$\left|\left(\lambda_\alpha^2 - \frac{\hbar^2}{2}\lambda_\alpha\right)\delta_{\alpha\beta} - \int_\Sigma (\overline{\mathrm{S}} - 2\overline{\mathrm{Ric}}(\nu,\nu))\left(\frac{3\delta_{\alpha\beta}}{\sigma^2|\Sigma|_g} - \frac{f_\alpha f_\beta}{\sigma^2}\right)d\mu_g\right| \leq C\sigma^{-5-\delta}, \tag{3.48}$$

which is analogous to (3.27). □

Remark 3.2.3. *Observe that equation (3.42) gives the following bound. Since* $\delta \in (0, \frac{1}{2}]$, $|m_{\mathcal{H}}(\Sigma)| \leq 2|E_{\mathrm{ADM}}|$, $\overline{\mathrm{Ric}}_{\vec{x}} = \mathrm{O}(|\vec{x}|^{-\frac{5}{2}-\delta})$ *and* $\|f_\alpha\|_2 = 1$, *then*

$$\left|\lambda_\alpha - \frac{\hbar^2}{2}\right| = \mathrm{O}(\sigma^{-\frac{5}{2}-\delta}),$$

with a constant possibly depending on $|E_{\mathrm{ADM}}|$.

The proof of the following Lemma is similar to [CS21, Prop. 2].

Lemma 3.2.4. *There exist* $c > 0$ *and* $\sigma_0 > 1$ *such that, if* $\Sigma \in \overline{\mathcal{W}}_\sigma^\eta(B_1, B_2)$ *with* $\sigma \geq \sigma_0$, *for every* $\mathrm{w}, \mathrm{v} \in H^2(\Sigma)$ *it holds*

$$\left|\int_\Sigma (L\mathrm{w}^t)\mathrm{v}^t\, d\mu - \frac{6m_{\mathcal{H}}(\Sigma)}{\sigma_\Sigma^3}\int_\Sigma \mathrm{w}^t\mathrm{v}^t\, d\mu\right| \leq \frac{c\|\mathrm{w}\|_{L^2(\Sigma)}\|\mathrm{v}\|_{L^2(\Sigma)}}{\sigma^{3+\delta}}.$$

This leads to the following

Lemma 3.2.5. *There exist $c > 0$ and $\sigma_0 > 1$ such that, if Σ belongs to $\overline{\mathcal{W}}^\eta_\sigma(B_1, B_2)$ and it is $(\mathrm{c_{in}}, \sigma)$-almost CMC with $\sigma \geq \sigma_0$, for every $\mathrm{w} \in H^2(\Sigma)$ such that $\mathrm{w}^0 = 0$ we find*

(i) *The translational part, in view of* Lemma 3.2.4*, satisfies*

$$\int_\Sigma (L\mathrm{w}^t)\mathrm{w}^t \, d\mu \geq \frac{6m_{\mathcal{H}}(\Sigma)}{\sigma_\Sigma^3}\|\mathrm{w}^t\|_2^2 - c\sigma^{-3-\delta}\|\mathrm{w}\|_2^2;$$

(ii) *The remaining part satisfies*

$$\int_\Sigma (L\mathrm{w}^d)(\mathrm{w}^d) \, d\mu \geq \frac{7}{4\sigma_\Sigma^2}\int_\Sigma (\mathrm{w}^d)^2 \, d\mu.$$

Proof. Point (ii) follows from

$$\int_\Sigma (L\mathrm{w}^d)(\mathrm{w}^d) \, d\mu = \int_\Sigma \mathrm{w}^d(-\Delta \mathrm{w}^d) \, d\mu - \int_\Sigma \left(\frac{\hbar^2}{2} + \frac{(\mathcal{H}-\hbar)^2}{2} + \hbar(\mathcal{H}-\hbar) + \mathrm{O}(\sigma^{-\frac{5}{2}-\delta})\right)(\mathrm{w}^d)^2 \, d\mu.$$

Combining this with $\hbar = \frac{2}{\sigma_\Sigma} + \mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$, together also with Lemma 2.3.5, i.e. $\|\mathcal{H}-\hbar\|_{L^\infty(\Sigma)} \leq C\sigma^{-\frac{3}{2}-\delta}$, and equation (3.16), we get

$$\int_\Sigma (L\mathrm{w}^d)(\mathrm{w}^d) \, d\mu \geq \left(\frac{5}{\sigma_\Sigma^2} - \frac{2}{\sigma_\Sigma^2} + \mathrm{O}(\sigma^{-\frac{5}{2}-\delta})\right)\int_\Sigma (\mathrm{w}^d)^2 \, d\mu \geq \frac{7}{4\sigma_\Sigma^2}\int_\Sigma (\mathrm{w}^d)^2 \, d\mu,$$

where we also used the equivalence of the radii σ and σ_Σ for surfaces in the class. □

Lemma 3.2.6. *There exist $c > 0$ and $\sigma_0 > 1$ such that, if $\Sigma \in \overline{\mathcal{W}}^\eta_\sigma(B_1, B_2)$ with $\sigma \geq \sigma_0$, for every $\mathrm{w} \in H^2(\Sigma)$ it holds*

$$\|L\mathrm{w}^t\|_2^2 \leq c\sigma^{-5-2\delta}\|\mathrm{w}\|_2^2.$$

Proof. We estimate

$$\|L\mathrm{w}^t\|_2 \leq \| - \Delta \mathrm{w}^t - \frac{\hbar^2}{2}\mathrm{w}^t\|_2 + \left\|\frac{(\mathcal{H}-\hbar)^2}{2}\mathrm{w}^t + \hbar(\mathcal{H}-\hbar)\mathrm{w}^t\right\|_2 + \mathrm{O}(\sigma^{-\frac{5}{2}-\delta})\|\mathrm{w}\|_2.$$

Using the definition of w^t, and, by Remark 3.2.3, $|\lambda_i - \frac{\hbar^2}{2}|^2 = \mathrm{O}(\sigma^{-5-2\delta})$, we have

$$\left\|-\Delta \mathrm{w}^t - \frac{\hbar^2}{2}\mathrm{w}^t\right\|_2^2 \leq C\sigma^{-5-2\delta}\|\mathrm{w}\|_2^2.$$

Moreover, we conclude with the estimate

$$\left\|\frac{(\mathcal{H}-\hbar)^2}{2}\mathrm{w}^t + \hbar(\mathcal{H}-\hbar)\mathrm{w}^t\right\|_2 \leq 10\sigma^{-1}\|(\mathcal{H}-\hbar)\mathrm{w}^t\|_2 \leq C\sigma^{-\frac{5}{2}-\delta}\|\mathrm{w}\|_2,$$

using again Lemma 2.3.5. □

The previous Lemmas lead to the following conclusion.

Proposition 3.2.7. *There exist $c > 0$ and $\sigma_0 > 1$ such that, if Σ belongs to $\overline{\mathcal{W}}^\eta_\sigma(B_1, B_2)$ and it is $(\mathrm{c_{in}}, \sigma)$-almost CMC with $\sigma \geq \sigma_0$,*

$$\inf\left\{\int_\Sigma (L\mathrm{w})\mathrm{w} \, d\mu : \ \|\mathrm{w}\|_{L^2(\Sigma)} = 1, \quad \int_\Sigma \mathrm{w} \, d\mu = 0\right\} \geq \frac{2E_{\mathrm{ADM}}}{\sigma_\Sigma^3}.$$

Proof. Decomposing the operator L as follows

$$\int_\Sigma (L\mathrm{w})\mathrm{w}\ d\mu = \int_\Sigma (L\mathrm{w}^t)\mathrm{w}^t\ d\mu + 2\int_\Sigma (L\mathrm{w}^t)\mathrm{w}^d\ d\mu + \int_\Sigma (L\mathrm{w}^d)\mathrm{w}^d\ d\mu,$$

and using Lemma 3.2.5, together with the parametric Young's inequality with $\varepsilon^{-1} = (4\sigma_\Sigma^2)^{-1}$ for the intermediate term, we get

$$\begin{aligned}\int_\Sigma (L\mathrm{w})\mathrm{w}\ d\mu \geq \frac{6m_{\mathcal{H}}(\Sigma)}{\sigma_\Sigma^3}\|\mathrm{w}^t\|_2^2 - c\sigma^{-3-\delta}\|\mathrm{w}\|_2^2 + \frac{7}{4\sigma_\Sigma^2}\int_\Sigma (\mathrm{w}^d)^2\ d\mu \\ - 4\sigma_\Sigma^2\|L\mathrm{w}^t\|_2^2 - \frac{\|\mathrm{w}^d\|_2^2}{4\sigma_\Sigma^2}.\end{aligned}$$

Using (3.2) and choosing σ large we have $m_{\mathcal{H}}(\Sigma) \geq \frac{E_{\mathrm{ADM}}}{2}$, and also Lemma 3.2.6, we have

$$\int_\Sigma (L\mathrm{w})\mathrm{w}\ d\mu \geq \frac{3E_{\mathrm{ADM}}}{\sigma_\Sigma^3}\|\mathrm{w}^t\|_2^2 - c\sigma^{-3-\delta}\|\mathrm{w}\|_2^2 + \frac{3}{2\sigma_\Sigma^2}\|\mathrm{w}^d\|_2^2.$$

We conclude using $\|\mathrm{w}\|_{L^2(\Sigma)}^2 = \|\mathrm{w}^t\|_{L^2(\Sigma)}^2 + \|\mathrm{w}^d\|_{L^2(\Sigma)}^2$ choosing σ so large that $\dfrac{3}{2\sigma_\Sigma^2} \geq \dfrac{3E_{\mathrm{ADM}}}{\sigma_\Sigma^3}$. □

Chapter 4

Volume preserving mean curvature flow

4.1 Definition of the flow and evolution equations

4.1.1 Definition of the flow

Definition 4.1.1. *Let* $(M, \bar{g})$ *be a 3-dimensional manifold, and let* $\iota : \Sigma \hookrightarrow M$ *be a closed surface. A time dependent family of immersions* $F_t : \Sigma \hookrightarrow M$*, with* $t \in [0, T)$ *for some* $0 < T \leq +\infty$*, which satisfies*

$$\begin{cases} \frac{\partial}{\partial t} F_t(\cdot) = -(H(\cdot, t) - h(t))\nu(\cdot, t) \\ F_0 = \iota \end{cases} \tag{4.1}$$

is called a solution to the volume preserving mean curvature flow, with initial value ι*.*

It is well-known that this flow is parabolic and it has a smooth solution at least locally in time.

Remark 4.1.2. *In general, short-time existence and uniqueness of solutions to a general system*

$$\begin{cases} \frac{\partial}{\partial t} F_t(\cdot) = -\mathrm{f}(\kappa_1, ..., \kappa_n, t)\nu(\cdot, t) \\ F_0 = \iota \end{cases} \tag{4.2}$$

where $\kappa_1, ...\kappa_n$ *are the principal curvatures of* Σ*, are guaranteed provided that the speed satisfies*

$$\frac{\partial \mathrm{f}}{\partial \kappa_i} > 0, \qquad i \in \{1, 2\}.$$

This is proved, for example, in [Ger06, Chapter 2]. It is well-known that the uniform boundedness of $|A|$ *is sufficient to assure that also the derivatives of each order of the curvatures of the flow remain bounded, see Section 4.3. This argument allows to deduce that if* $\sup_{\Sigma \times [0,T)} |A| < \infty$ *for some* $T > 0$*, then the flow can be extended past* T*.*

In the following, we always assume that the ambient manifold $(M, \bar{g})$ is $C^2_{\frac{1}{2}+\delta}$-asymptotically flat. We write $\Sigma_t := F_t(\Sigma)$ to denote the immersed surface at time t, and we call for simplicity Σ_t the "solution of the flow" (4.1) without mentioning explicitly the immersions F_t. We call $g(t)$ the induced metric on Σ at time t and by $d\mu_t$ the corresponding measure.

4.1.2 Evolution equations

We now recall the evolution equations satisfied by the main geometric quantities on Σ_t. We choose at each fixed time a frame $\vec{e}_\alpha$ on the ambient manifold M such that $\vec{e}_1, \vec{e}_2$ are tangent vectors on Σ and $\vec{e}_3 = \nu$. Then the main geometric quantities on Σ_t satisfy the following equations along the flow, see e.g. [HP99].

Lemma 4.1.3 (Evolution of $g(t)$, $d\mu_t$ and curvatures)**.** *Let $(M, \overline{g})$ be a 3-dimensional manifold, and let $\iota : \Sigma \hookrightarrow M$ be a closed surface. Suppose that $\{F_t\}_{t\in[0,T)}$ is a solution to the equation (5.2), with initial datum Σ. Then we have*

(i) $\frac{\partial g_{ij}}{\partial t} = -2(H-h)h_{ij}$;

(ii) $\frac{\partial g^{ij}}{\partial t} = 2(H-h)h^{ij}$;

(iii) $\frac{\partial}{\partial t}(d\mu_t) = -(H-h)H d\mu_t$;

(iv) $\frac{\partial}{\partial t}\nu = \nabla H$;

(v) $\frac{\partial}{\partial t}h_{ij} = \nabla_i\nabla_j H + (H-h)\left(-h_{ik}h_j^k + \overline{\mathrm{Rm}}_{i3j3}\right)$;

(vi) $\frac{\partial H}{\partial t} = \Delta H + (H-h)(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))$.

As an immediate consequence of the above equations we also have

$$\frac{\mathrm{d}}{\mathrm{dt}}|\Sigma_t| = -\|H-h\|^2_{L^2(\Sigma_t)}, \tag{4.3}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\|H-h\|^2_{L^2(\Sigma_t)} = -2\langle L(H-h), H-h\rangle - \int_\Sigma H(H-h)^3 d\mu_t. \tag{4.4}$$

We can rewrite the term $\nabla_i\nabla_j H$ in the right-hand side of (iv) by means of the Simons identity, as in Metzger [Met07], see Lemma 2.0.1. Thus, we obtain

Lemma 4.1.4. *Along a solution of the volume preserving mean curvature flow we have*

$$\begin{aligned}
\frac{\partial}{\partial t}|\mathring{A}|^2 &= \Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + \frac{2h}{H}\{|A|^4 - H\mathrm{tr}(A^3)\} + 2|A|^2\left(\frac{H-h}{H}\right)|\mathring{A}|^2 \\
&\quad +2(H-h)\mathring{h}_{ij}\overline{\mathrm{Rm}}_{kilj}\nu^k\nu^l - 2\left(h_i^l\overline{\mathrm{Rm}}_{kjkl} + h^{lk}\overline{\mathrm{Rm}}_{lijk}\right)h_{ij} \qquad (4.5)\\
&\quad -2\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\varepsilon}\nu^\varepsilon\right) + \nabla_l\left(\overline{\mathrm{Rm}}_{\varepsilon ijl}\nu^\varepsilon\right)\right)\mathring{h}_{ij}. \\
\frac{\partial}{\partial t}|\nabla H|^2 &= \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 2(H-h)h^{ij}\nabla_i H\nabla_j H \\
&\quad +2(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))|\nabla H|^2 - 2\mathrm{Ric}^\Sigma(\nabla H, \nabla H) \qquad (4.6)\\
&\quad +2(H-h)\langle\nabla|A|^2, \nabla H\rangle + 2(H-h)\langle\nabla\left(\overline{\mathrm{Ric}}(\nu,\nu)\right), \nabla H\rangle,
\end{aligned}$$

where Ric^Σ is the Ricci tensor on Σ and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_g$.

For sake of completeness, we sketch a proof of these equations.

Proof. (i) Using Lemma 5.0.2 and Simons' identity, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - Hh_i^l h_{lj} + |A|^2 h_{ij} + (H-h)\left(-h_{ik}h_j^k + \overline{\mathrm{Rm}}_{i3j3}\right) \\
- h_i^l\overline{\mathrm{Rm}}_{kjkl} - h^{lk}\overline{\mathrm{Rm}}_{lijk} - \nabla_j\left(\overline{\mathrm{Ric}}_{i\omega}\nu^\omega\right) - \nabla^l\left(\overline{\mathrm{Rm}}_{\omega ijl}\nu^\omega\right).
\end{aligned} \tag{4.7}$$

By the definition of $|A|^2$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 = &4(H-h)h_{im}h_{mj}h_{ij} + 2\Delta h_{ij}h_{ij} - 2Hh_{il}h_{lj}h_{ij} + 2|A|^4 + \\ &- 2(H-h)h_{ik}h_{kj}h_{ij} + 2(H-h)\overline{\mathrm{Rm}}_{i3j3}h_{ij} + \\ &- 2\left(h_i^l\overline{\mathrm{Rm}}_{kjkl} + h^{lk}\overline{\mathrm{Rm}}_{lijk}\right)h_{ij} - 2\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\omega}\nu^\omega\right) + \nabla^l\left(\overline{\mathrm{Rm}}_{\omega ijl}\nu^\omega\right)\right)h_{ij}.\end{aligned}$$

Observe that, by definition, $h_{im}h_{mj}h_{ij} = \mathrm{tr}(A^3)$. Moreover,

$$\Delta|A|^2 = \Delta\langle A, A\rangle = 2\Delta h_{ij}h_{ij} + 2|\nabla A|^2,$$

and thus

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 = &\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2h\mathrm{tr}(A^3) \\ &+ 2(H-h)\overline{\mathrm{Rm}}_{i3j3}h_{ij} - 2\left(h_i^l\overline{\mathrm{Rm}}_{kjkl} + h^{lk}\overline{\mathrm{Rm}}_{lijk}\right)h_{ij} \\ &- 2\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\omega}\nu^\omega\right) + \nabla^l\left(\overline{\mathrm{Rm}}_{\omega ijl}\nu^\omega\right)\right)h_{ij}.\end{aligned}$$

Moreover, the mean curvature has a similar, but simpler, evolution. In fact

$$\frac{\partial}{\partial t}\left(\frac{H^2}{2}\right) = \Delta\left(\frac{H^2}{2}\right) - |\nabla H|^2 + H(H-h)(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)). \tag{4.8}$$

It follows that

$$\begin{aligned}\frac{\partial}{\partial t}|\mathring{A}|^2 &= \frac{\partial}{\partial t}\left(|A|^2 - \frac{H^2}{2}\right) = \\ &= \Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + 2|A|^4 - 2h\mathrm{tr}(A^3) - H|A|^2(H-h) \\ &\quad - H(H-h)\overline{\mathrm{Ric}}(\nu,\nu) + 2(H-h)\overline{\mathrm{Rm}}_{i3j3}h_{ij} - 2\left(h_i^l\overline{\mathrm{Rm}}_{kjkl} + h^{lk}\overline{\mathrm{Rm}}_{lijk}\right)h_{ij} \\ &\quad - 2\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\omega}\nu^\omega\right) + \nabla^l\left(\overline{\mathrm{Rm}}_{\omega ijl}\nu^\omega\right)\right)h_{ij}.\end{aligned} \tag{4.9}$$

Observe that equation (4.9) can be rewritten as

$$\begin{aligned}&\Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + 2|A|^4 - 2h\mathrm{tr}(A^3) - H|A|^2(H-h) \\ &= \Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + \frac{2h}{H}|A|^4 - 2h\mathrm{tr}(A^3) + 2|A|^2\left(1 - \frac{h}{H}\right)|\mathring{A}|^2,\end{aligned}$$

while it also holds

$$-H(H-h)\overline{\mathrm{Ric}}(\nu,\nu) + 2(H-h)\overline{\mathrm{Rm}}_{i3j3}h_{ij} = 2\mathring{h}_{ij}(H-h)\overline{\mathrm{Rm}}_{3i3j}.$$

We finally observe that

$$-2\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\omega}\nu^\omega\right) + \nabla^l\left(\overline{\mathrm{Rm}}_{\omega ijl}\nu^\omega\right)\right)h_{ij} = -2\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\omega}\nu^\omega\right) + \nabla_l\left(\overline{\mathrm{Rm}}_{\omega ijl}\nu^\omega\right)\right)\mathring{h}_{ij},$$

using the symmetries of the Riemannian tensor.

(ii) Deriving the evolution $\partial_t H$, taking into account also the derivative of the metric, and using the Bochner formula, we get

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla H|^2 &= \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 2(H-h)h^{ij}\nabla_i H\nabla_j H \\ &+ 2(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))|\nabla H|^2 - 2\mathrm{Ric}^\Sigma(\nabla H,\nabla H) \\ &+ 2(H-h)\langle\nabla|A|^2,\nabla H\rangle + 2(H-h)\langle\nabla\left(\overline{\mathrm{Ric}}(\nu,\nu)\right),\nabla H\rangle.\end{aligned} \tag{4.10}$$

□

4.1.3 Evolution of integral quantities

In this subsection we study the evolution of the integral quantities which appear in the definition of round surfaces, with the aim of showing that this roundness is preserved.

Hypotheses of the Chapter. For the rest of the Chapter, we will suppose that $(M, \overline{\mathrm{g}}, \vec{x})$ is a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold, $\iota : \Sigma \hookrightarrow M$ is an embedded *surface* and (Σ, F_t) is a solution to the *volume preserving mean curvature flow* system (4.1) with initial datum $F_0 = \iota$ on the time interval $[0, T]$, for some $T > 0$. We fix here, once for all, $\sigma := \sigma_\Sigma$, the area radius at the initial time $t = 0$. We suppose moreover that there exist $B_\infty > 0$ and $c_\infty > 0$ such that the flow satisfies the following hypotheses:

(i) For every $t \in [0, T]$, it holds

$$|A(t)| \le \sqrt{\frac{5}{2}}\sigma^{-1}, \qquad \kappa_i(t) \ge \frac{1}{2\sigma}, \tag{4.11}$$

where κ_i, $i \in \{1, 2\}$ are the principal curvatures of Σ_t;

(ii) For every $t \in [0, T]$ it holds

$$\|H - h\|_{L^\infty(\Sigma_t)} \le c_\infty \sigma^{-\frac{3}{2}-\delta}, \qquad \left\|\mathring{A}(t)\right\|_{L^\infty(\Sigma)} \le B_\infty \sigma^{-\frac{3}{2}-\delta}; \tag{4.12}$$

(iii) For every $t \in [0, T]$, it holds

$$\frac{\sigma_{\Sigma_t}}{r_\Sigma(t)} \le 3, \qquad (7/2)\pi\sigma^2 \le |\Sigma_t| \equiv 4\pi\sigma^2_{\Sigma_t} \le 5\pi\sigma^2. \tag{4.13}$$

We remark that, even if we are now assuming these inequalities for the Section, our approach will be that of showing that none of the above inequalities can become false first.

Remark 4.1.5. *In the following Lemmas and Propositions we will need sometimes weaker hypotheses. We will specify these cases along the statements.*

Proposition 4.1.6. *Let (Σ, F_t), $t \in [0, T]$, such that (4.11), (4.13) and*

$$\|H - h\|_{L^\infty(\Sigma_t)} \le \frac{1}{20\sigma}, \qquad \forall t \in [0, T] \tag{4.14}$$

hold. Then there exist a constant $C = C(\bar{c}, \delta) > 0$ and a radius $\sigma_0 = \sigma_0(\delta, \bar{c}) > 0$ such that if $\sigma > \sigma_0$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |\mathring{A}|^4\, d\mu_t \le -2\int_\Sigma |\mathring{A}|^2|\nabla\mathring{A}|^2\, d\mu_t - \frac{1}{2\sigma^2}\int_\Sigma |\mathring{A}|^4\, d\mu_t + C\sigma^{-6-4\delta}. \tag{4.15}$$

As a consequence, if $\int_\Sigma |\mathring{A}|^4\, d\mu_0 < B_1\sigma^{-4-4\delta}$ *and* $B_1 > 2C$*, then* $\int_\Sigma |\mathring{A}|^4\, d\mu_t < B_1\sigma^{-4-4\delta}$ *for every* $t \in [0,T]$.

Proof. Integrating (4.5) and using integration by parts,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\int_\Sigma |\mathring{A}|^4\, d\mu_t &= 2\int_\Sigma |\mathring{A}|^2\left(\frac{\partial}{\partial t}|\mathring{A}|^2\right)\, d\mu_t + \int_\Sigma |\mathring{A}|^4 H(h-H)\, d\mu_t \\
&= -2\int_\Sigma |\nabla|\mathring{A}|^2|^2\, d\mu_t - 4\int_\Sigma |\mathring{A}|^2|\nabla\mathring{A}|^2\, d\mu_t \\
&\quad + 4\int_\Sigma |\mathring{A}|^2\frac{h}{H}\left(|A|^4 - H\mathrm{tr}(A^3)\right)\, d\mu_t + 4\int_\Sigma |A|^2\left(1-\frac{h}{H}\right)|\mathring{A}|^4\, d\mu_t \\
&\quad + 4\int_\Sigma (H-h)|\mathring{A}|^2\mathring{h}_{ij}\overline{\mathrm{Rm}}_{kilj}\nu^k\nu^l\, d\mu_t - 8\int_\Sigma \overline{\mathrm{Rm}}_{1212}|\mathring{A}|^4\, d\mu_t \\
&\quad - 4\int_\Sigma \left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\omega}\nu^\omega\right) + \nabla_l\left(\overline{\mathrm{Rm}}_{\omega ijl}\nu^\omega\right)\right)\mathring{h}_{ij}|\mathring{A}|^2\, d\mu_t + \int_\Sigma |\mathring{A}|^4 H(h-H)\, d\mu_t
\end{aligned} \tag{4.16}$$

where we used that the symmetries of the Riemannian tensor imply

$$-2\left(h_i^l\overline{\mathrm{Rm}}_{kjkl} + h^{lk}\overline{\mathrm{Rm}}_{lijk}\right)h_{ij} = -2\left(h_{ji}h_{li}\overline{\mathrm{Rm}}_{lkjk} - h_{kl}h_{ji}\overline{\mathrm{Rm}}_{kjli}\right) = -4|\mathring{A}|^2\overline{\mathrm{Rm}}_{1212}. \tag{4.17}$$

In order to estimate (4.16), we note that (4.11) and (4.12) imply

$$\frac{1}{\sigma} \le H \le \frac{\sqrt{5}}{\sigma}, \qquad \left|1 - \frac{h}{H}\right| \le \frac{1}{20}, \qquad H|h-H| \le \frac{1}{4\sigma^2}. \tag{4.18}$$

Using (4.11) and the well-known identity

$$|A|^4 - H\mathrm{tr}(A^3) = -2\kappa_1\kappa_2|\mathring{A}|^2 \tag{4.19}$$

we find, using the estimate (4.14),

$$\begin{aligned}
4\int_\Sigma |\mathring{A}|^2\frac{h}{H}\left(|A|^4 - H\mathrm{tr}(A^3)\right)\, d\mu_t + 4\int_\Sigma |A|^2\left(1-\frac{h}{H}\right)|\mathring{A}|^4\, d\mu_t + \int_\Sigma |\mathring{A}|^4 H(h-H)\, d\mu_t \\
\le \left(\frac{1}{4} - \frac{19}{10} + \frac{1}{2}\right)\frac{1}{\sigma^2}\int_\Sigma |\mathring{A}|^4\, d\mu_t \le \frac{1}{\sigma^2}\int_\Sigma |\mathring{A}|^4\, d\mu_t.
\end{aligned} \tag{4.20}$$

We now consider the terms

$$\begin{aligned}
&4\int_\Sigma (H-h)|\mathring{A}|^2\mathring{h}_{ij}\overline{\mathrm{Rm}}_{kilj}\nu^k\nu^l\, d\mu_t - 8\int_\Sigma \overline{\mathrm{Rm}}_{1212}|\mathring{A}|^4\, d\mu_t \\
&\le C\int_\Sigma \sigma^{-\frac{7}{2}-\delta}|\mathring{A}|^3\, d\mu_t + C\int_\Sigma \sigma^{-\frac{5}{2}-\delta}|\mathring{A}|^4\, d\mu_t \\
&\le C\int_\Sigma \left(\frac{3\varepsilon\sigma^{-2}}{4}|\mathring{A}|^4 + C_\varepsilon 4^{-1}\left(\sigma^{-2-\delta}\right)^4\right)\, d\mu_t + C\int_\Sigma \sigma^{-\frac{5}{2}-\delta}|\mathring{A}|^4\, d\mu_t
\end{aligned} \tag{4.21}$$

where we used the parametric Young's inequality.

Estimating the remaining term we have

$$\begin{aligned}
&-4\int_\Sigma \left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\omega}\nu^\omega\right)+\nabla_l\left(\overline{\mathrm{Rm}}_{\omega ijl}\nu^\omega\right)\right)\mathring{h}_{ij}|\mathring{A}|^2\,d\mu_t\\
&=4\int_\Sigma \overline{\mathrm{Ric}}_{i\omega}\nu^\omega\nabla_j\left(\mathring{h}_{ij}|\mathring{A}|^2\right)+\overline{\mathrm{Rm}}_{\omega ijl}\nu^\omega\nabla_l\left(\mathring{h}_{ij}|\mathring{A}|^2\right)\,d\mu_t\\
&\leq C\int_\Sigma|\overline{\mathrm{Rm}}||\nabla\mathring{A}||\mathring{A}|^2\,d\mu_t\leq C\sigma^{-\frac{5}{2}-\delta}\int_\Sigma|\nabla\mathring{A}||\mathring{A}|^2\,d\mu_t,
\end{aligned}$$

also using the inequality $|\nabla|\mathring{A}||\leq|\nabla\mathring{A}|$. The latter term can be estimated as

$$C\sigma^{-\frac{5}{2}-\delta}\int_\Sigma|\nabla\mathring{A}||\mathring{A}|^2\,d\mu_t\leq\frac{\varepsilon}{2}\int_\Sigma|\nabla\mathring{A}|^2|\mathring{A}|^2\,d\mu_t+C(\varepsilon)C^2\sigma^{-5-2\delta}\int_\Sigma|\mathring{A}|^2\,d\mu_t.$$

Proceeding as above, we get a term $\varepsilon\sigma^{-2}\|\mathring{A}\|_4^4$ and a reminder $C_\varepsilon\sigma^{-6-4\delta}$.

We conclude by choosing ε suitably small and σ large, depending on $\overline{c}$ and δ. In particular, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathring{A}\|^4_{L^4(\Sigma,\mu_t)}\leq-2\int_\Sigma|\mathring{A}|^2|\nabla\mathring{A}|^2\,d\mu_t-\frac{1}{2\sigma^2}\|\mathring{A}\|^4_{L^4(\Sigma,\mu_t)}+C\sigma^{-6-4\delta}. \tag{4.22}$$

Finally, suppose that $\int_\Sigma|\mathring{A}|^4\,d\mu_t<B_1\sigma^{-4-4\delta}$ is not true for every $t\in[0,T]$. Then, there exists $t_0>0$ a first time such that $\int_\Sigma|\mathring{A}|^4\,d\mu_{t_0}=B_1\sigma^{-4-4\delta}$ and so $D=\sigma^{4+4\delta}\int_\Sigma|\mathring{A}|^4\,d\mu_{t_0}=B_1>\int_\Sigma|\mathring{A}|^4\,d\mu_0\sigma^{4+4\delta}$. Since for every $t\in[0,t_0)$ we had $\int_\Sigma|\mathring{A}|^4\,d\mu_t<B_1\sigma^{-4-4\delta}$, then

$$0\leq\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0}\int_\Sigma|\mathring{A}|^4\,d\mu_t\leq-2\int_\Sigma|\mathring{A}|^2|\nabla\mathring{A}|^2\,d\mu_{t_0}-\frac{1}{2\sigma^2}\int_\Sigma|\mathring{A}|^4\,d\mu_{t_0}+C\sigma^{-6-4\delta}.$$

It follows that $D\leq 2C<B_1$, which is a contradiction. □

Lemma 4.1.7 (Rate of change of $h(t)$)**.** *Suppose that (Σ,F_t), $t\in[0,T]$, satisfies (4.11), (4.12) and (4.13). Then there exists a constant $c=c(c_\infty,\overline{c})>0$ and $\sigma_0=\sigma_0(c_\infty,B_\infty,\overline{c})>1$ such that, if $\sigma>\sigma_0$,*

$$|\dot{h}(t)|\leq c\sigma^{-4-2\delta}. \tag{4.23}$$

Proof. By definition of h we get

$$\begin{aligned}
|\Sigma_t|\dot{h}(t)&=\int_\Sigma\frac{\partial H}{\partial t}\,d\mu_t+\int_\Sigma H^2(h-H)\,d\mu_t+h\int_\Sigma(H-h)^2\,d\mu_t\\
&=\int_\Sigma(H-h)\left(|\mathring{A}|^2+\overline{\mathrm{Ric}}(\nu,\nu)\right)\,d\mu_t-\int_\Sigma(H-h)\left(\frac{H^2}{2}-Hh+h^2\right)\,d\mu_t\\
&=\int_\Sigma(H-h)\left(|\mathring{A}|^2+\overline{\mathrm{Ric}}(\nu,\nu)\right)\,d\mu_t-\frac{1}{2}\int_\Sigma(H-h)^3\,d\mu_t,
\end{aligned}$$

using that $\int_\Sigma(H-h)\,d\mu_t=0$. In absolute value, we estimate

$$|\Sigma_t||\dot{h}(t)|\leq|\Sigma_t|\left(c_\infty\sigma^{-\frac{3}{2}-\delta}\right)\left(B_\infty^2\sigma^{-3-2\delta}+\overline{c}\sigma^{-\frac{5}{2}-\delta}\right)+\frac{|\Sigma_t|}{2}\left(c_\infty^3\sigma^{-\frac{9}{2}-3\delta}\right).$$

The thesis follows dividing by $|\Sigma_t|$ and choosing σ large depending on c_∞, B_∞ and $\overline{c}$. □

The previous preliminary Lemma leads to the following

Lemma 4.1.8 (Evolution of the oscillation)**.** *Suppose that* (Σ, F_t), $t \in [0,T]$, *satisfies* (4.11), (4.12) *and* (4.13). *Then there exist a constant* $C = C(\overline{c}) > 0$, *a constant* $c = c(c_\infty, \overline{c}) > 0$ *and a radius* $\sigma_0 = \sigma_0(B_\infty, c_\infty, \overline{c}, \delta) > 1$ *such that, if* $\sigma > \sigma_0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (H-h)^4 \, d\mu_t \leq -12\int_\Sigma (H-h)^2|\nabla H|^2 \, d\mu_t + C\sigma^{-2}\int_\Sigma (H-h)^4 \, d\mu_t + c\sigma^{-\frac{13}{2}-5\delta}$$

Proof. We consider the evolution

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (H-h)^4 \, d\mu_t = 4\int_\Sigma \left(\frac{\partial H}{\partial t} - \dot{h}\right)(H-h)^3 d\mu_t - \int_\Sigma H(H-h)^5 \, d\mu_t.$$

By Lemma 5.0.2 and integration by parts, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (H-h)^4 \, d\mu_t = &-12\int_\Sigma (H-h)^2|\nabla H|^2 \, d\mu_t + 4\int_\Sigma (H-h)^4(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)) \, d\mu_t \\ &- 4\dot{h}\int_\Sigma (H-h)^3 \, d\mu_t - \int_\Sigma H(H-h)^5 \, d\mu_t.\end{aligned}$$

By the hypothesis, since $r_\Sigma^{-1}(t) \leq 3\sigma_\Sigma^{-1}$ and $|\overline{\mathrm{Ric}}_{\overline{x}}| \leq \overline{c}|\overline{x}|^{-\frac{5}{2}-\delta}$, we find $\big||A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\big| \leq C\sigma^{-2}$. Combining this with the following consequence of Lemma 4.1.7

$$\left|\dot{h}\int_\Sigma (H-h)^3 \, d\mu_t\right| \leq c(c_\infty, \overline{c})\sigma^{-\frac{13}{2}-5\delta}, \tag{4.24}$$

we get the Thesis, also observing that, for σ large depending on c_∞, $|H(H-h)| \leq \sqrt{5}\sigma^{-2}$ in view of hypothesis (ii). □

We now estimate the evolution of $|\nabla H|$. In the proof below, observe that we do not use Hypothesis (ii).

Lemma 4.1.9. *Suppose that* (Σ, F_t), $t \in [0,T]$, *satisfies* (4.11), (4.12) *and* (4.13). *Then there exist a constant* $C = C(\overline{c}) > 0$ *and radius* $\sigma_0 = \sigma_0(\overline{c}, \delta)$ *such that if* $\sigma > \sigma_0$ *then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |\nabla H|^4 \, d\mu_t \leq -3\int_\Sigma |\nabla^2 H||\nabla H|^2 \, d\mu_t + C\sigma^{-6}\int_\Sigma (H-h)^4 \, d\mu_t + C\sigma^{-2}\int_\Sigma |\nabla H|^4 \, d\mu_t.$$

Proof. Integrating by parts (4.6) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |\nabla H|^4 \, d\mu_t = &\int_\Sigma \frac{\partial}{\partial t}\left((|\nabla H|^2)^2\right) \, d\mu_t + \int_\Sigma |\nabla H|^4 H(h-H) \, d\mu_t \\ = &-4\int_\Sigma |\nabla|\nabla H|^2|^2 \, d\mu_t - 4\int_\Sigma |\nabla^2 H|^2|\nabla H|^2 \, d\mu_t \\ &+ 4\int_\Sigma (H-h)h^{ij}\nabla_i H\nabla_j H|\nabla H|^2 \, d\mu_t \\ &- 4\int_\Sigma (H-h)\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)\nabla\cdot\left(|\nabla H|^2\nabla H\right) \, d\mu_t \\ &- 4\int_\Sigma \mathrm{Ric}^{\Sigma_t}(\nabla H, \nabla H)|\nabla H|^2 \, d\mu_t + \int_\Sigma |\nabla H|^4 H(h-H) \, d\mu_t\end{aligned} \tag{4.25}$$

By (4.11), H, $|H-h|$ and $|A|$ are all bounded by $C\sigma^{-1}$. On the other hand, the asymptotic flatness implies that $|\mathrm{Ric}^{\Sigma_t}| \le C\sigma^{-2}$ and $|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)| \le C\sigma^{-2}$. Then, for every $\varepsilon > 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma |\nabla H|^4\, d\mu_t \le& -4\int_\Sigma |\nabla^2 H||\nabla H|^2\, d\mu_t + C\sigma^{-2}\int_\Sigma |\nabla H|^4\, d\mu_t\\ &+ C\sigma^{-2}\int_\Sigma |H-h||\nabla H|^2|\nabla^2 H|\, d\mu_t\\ \le& \left(\frac{\varepsilon C}{2} - 4\right)\int_\Sigma |\nabla^2 H|^2|\nabla H|^2\, d\mu_t + C\sigma^{-2}\int_\Sigma |\nabla H|^4\, d\mu_t\\ &+ \frac{C}{2\varepsilon}\sigma^{-4}\int_\Sigma |H-h|^2|\nabla H|^2\, d\mu_t.\end{aligned} \tag{4.26}$$

We conclude choosing $\varepsilon = \frac{2}{C}$ and using Young's inequality in the following way

$$\sigma^{-4}\int_\Sigma |H-h|^2|\nabla H|^2\, d\mu_t \le \sigma^{-6}\int_\Sigma (H-h)^4\, d\mu_t + \sigma^{-2}\int_\Sigma |\nabla H|^4\, d\mu_t. \tag{4.27}$$

□

At this point, we prove the the following flow independent-inequality.

Lemma 4.1.10. *Let $\Sigma \hookrightarrow M$ be a* surface*. Then we have, for every $\varepsilon > 0$ and $\sigma > 1$,*

$$-\sigma^{-4}\int_\Sigma (H-h)^2|\nabla H|^2\, d\mu \le -\frac{\varepsilon}{2\sigma^2}\int_\Sigma |\nabla H|^4\, d\mu + \varepsilon^2\int_\Sigma |\nabla^2 H|^2|\nabla H|^2\, d\mu.$$

Proof. Since h is constant,

$$\begin{aligned}\sigma^{-2}\int_\Sigma |\nabla H|^4\, d\mu =& \sigma^{-2}\int_\Sigma \langle\nabla(H-h),\nabla H\rangle|\nabla H|^2\, d\mu\\ =& -\sigma^{-2}\int_\Sigma (H-h)(\Delta H)g^{kl}\nabla_k H\nabla_l H\, d\mu - 2\sigma^{-2}\int_\Sigma (H-h)g^{ij}\nabla_j H g^{kl}\nabla_i\nabla_k H\nabla_l H\, d\mu\\ \le& \frac{\sqrt{2}+2}{\sigma^2}\int_\Sigma |H-h||\nabla^2 H||\nabla H|^2\, d\mu \le 2\int_\Sigma \left(\frac{(H-h)^2}{\varepsilon\sigma^4} + \varepsilon|\nabla^2 H|^2\right)|\nabla H|^2\, d\mu,\end{aligned}$$

using also the (parametric) Young's inequality. □

Lemma 4.1.11. *Let (Σ, F_t), $t \in [0,T]$, be as above. For $\eta > 0$, let us set*

$$\mathrm{a}_k(t) := k\sigma^{-4}\|H-h\|^4_{L^4(\Sigma,\mu_t)} + \|\nabla H\|^4_{L^4(\Sigma,\mu_t)}. \tag{4.28}$$

Then there exist a constant $\eta_{\mathrm{w}} = \eta(\overline{c}) > 0$, a constant $c = c(B_1,\delta,\overline{c})$ and a radius $\sigma_0 = \sigma_0(B_\infty, B_1, c_\infty, \delta, \overline{c}) > 1$ such that, for $k = \eta_{\mathrm{w}}$, $B_2 > c(B_1,\delta,\overline{c})$ and $\sigma > \sigma_0$, we have the implication

$$\mathrm{a}_{\eta_{\mathrm{w}}}(0) < B_2\sigma^{-8-4\delta} \implies \mathrm{a}_{\eta_{\mathrm{w}}}(t) < B_2\sigma^{-8-4\delta} \text{ for every } t \in [0,T]. \tag{4.29}$$

Proof. Combining the previous Lemmas, we have that

$$\begin{aligned}\dot{\mathrm{a}}_k(t) :=& \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Sigma |\nabla H|^4\, d\mu_t + k\sigma^{-4}\int_\Sigma (H-h)^4\, d\mu_t\right)\\ \le& -3\int_\Sigma |\nabla^2 H||\nabla H|^2\, d\mu_t + C\sigma^{-6}\int_\Sigma (H-h)^4\, d\mu_t + C\sigma^{-2}\int_\Sigma |\nabla H|^4\, d\mu_t\\ &- 12k\sigma^{-4}\int_\Sigma (H-h)^2|\nabla H|^2\, d\mu_t + Ck\sigma^{-6}\int_\Sigma (H-h)^4\, d\mu_t + kc(c_\infty,\overline{c})\sigma^{-\frac{21}{2}-5\delta},\end{aligned}$$

where $C = C(\overline{c}) > 0$ is the constant introduced in the statement of Lemma 4.1.8 and Lemma 4.1.9, while $c(c_\infty, \overline{c}) > 0$ have been introduced in Lemma 4.1.8. By Lemma 4.1.10 multiplied by $12k$,

$$-\frac{12k}{\sigma^4}\int_\Sigma (H-h)^2|\nabla H|^2\, d\mu \le -\frac{6k\varepsilon}{\sigma^2}\int_\Sigma |\nabla H|^4 d\mu + 12k\varepsilon^2\int_\Sigma |\nabla^2 H|^2|\nabla H|^2\, d\mu,$$

and Sobolev inequality, we rewrite $\dot{a}_k(t)$ as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\int_\Sigma |\nabla H|^4\, d\mu_t + k\sigma^{-4}\int_\Sigma (H-h)^4\, d\mu_t\right)\\ &\le (C-6k\varepsilon)\,\sigma^{-2}\int_\Sigma |\nabla H|^4\, d\mu_t + \left(12k\varepsilon^2 - 3\right)\int_\Sigma |\nabla^2 H|^2|\nabla H|^2\, d\mu_t\\ &\quad + Ck\sigma^{-6}\int_\Sigma (H-h)^4\, d\mu_t + kc(c_\infty,\overline{c})\sigma^{-\frac{21}{2}-5\delta}.\end{aligned} \tag{4.30}$$

We thus solve the system

$$\begin{cases} C - 6k\varepsilon = -C\\ 12k\varepsilon^2 = 1\end{cases}$$

that is $k = \frac{4}{3}C^2$ and $\varepsilon = \frac{1}{4C}$. With this choice, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\int_\Sigma |\nabla H|^4\, d\mu_t + \frac{4}{3}C^2\sigma^{-4}\int_\Sigma (H-h)^4\, d\mu_t\right)\\ &\le -C\sigma^{-2}\int_\Sigma |\nabla H|^4 - 2\int_\Sigma |\nabla^2 H|^2|\nabla H|^2\\ &\quad + \frac{4}{3}C^3\sigma^{-6}\int_\Sigma (H-h)^4\, d\mu_t + \frac{4}{3}C^2 c(c_\infty,\overline{c})\sigma^{-\frac{21}{2}-5\delta}.\end{aligned} \tag{4.31}$$

So we choose $\eta_{\mathrm{w}} := \frac{4}{3}C^2$. Then point (iv) of Lemma 2.3.5 implies that

$$\int_\Sigma (H-h)^4\, d\mu_t \le \mathrm{c}_{\mathrm{Per}}^4\left(\|\mathring{A}\|^4_{L^4(\Sigma,\mu_t)} + \sigma^{-4-4\delta}\right) \le \mathrm{c}_{\mathrm{Per}}^4(B_1^4+1)\sigma^{-4-4\delta}. \tag{4.32}$$

Observe moreover that

$$c(c_\infty,\overline{c})\sigma^{-\frac{21}{2}-5\delta} = \left(c(c_\infty,\overline{c})\sigma^{-\frac{1}{2}-\delta}\right)\sigma^{-10-4\delta} \le \sigma^{-10-4\delta},$$

if $\sigma^{\frac{1}{2}+\delta} \ge \sigma_0(B_\infty, c_\infty, \overline{c})$, and $\sigma \ge \sigma_0$. This implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\int_\Sigma |\nabla H|^4\, d\mu_t + \eta_{\mathrm{w}}\sigma^{-4}\int_\Sigma (H-h)^4\, d\mu_t\right)\\ &\le -C\sigma^{-2}\int_\Sigma |\nabla H|^4 + \frac{4}{3}C^3\mathrm{c}_{\mathrm{Per}}^4(B_1^4+1)\sigma^{-10-4\delta} + \frac{4}{3}C^2\sigma^{-10-4\delta}.\end{aligned} \tag{4.33}$$

This implies

$$\dot{a}_\eta(t) \le -C\sigma^{-2}a_\eta(t) + c\sigma^{-10-4\delta}, \tag{4.34}$$

with $c = c(B_1, c_{\mathrm{Per}}, \overline{c})$, using again (4.32). The thesis follows. □

From now on, when considering the roundness class $\mathcal{W}^\eta_\sigma(B_1, B_2)$, we fix the parameter η equal to the value η_w given by the previous Lemma, and we will no longer need to specify the dependence on η of the constants in the estimates. Moreover, we will simply write $\mathcal{W}_\sigma(B_1, B_2)$ and $\mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$. See also Remark 2.3.3.

4.1.4 Evolution of the barycenter and convergence

In this Subsection, we obtain a first result which is an important block in the proof of Theorem 1.1.2. In particular, we show that by an appropriate choice of the parameters of the class $\mathcal{B}^{\eta}_{\sigma}(B_1, B_2, B_{\text{cen}})$ and under suitable conditions on the initial surface, the solution of the flow remains inside the class for arbitrary times. However, in order to control the possible drift of the barycenter, we will need an additional smallness requirement on the L^2-norm of the mean curvature of Σ: in terms of Definition 3.2.1, we prove that the flow exists for every $t > 0$ if the initial surface is almost CMC.

An important assumption in the previous results was the uniform comparability between r_{Σ_t} and σ in (2.29), which shows that Σ_t stays enough far from the coordinate origin to ensure the desired decay of the ambient curvature. To justify this assumption, we study now the evolution of the barycenter under the flow.

Proposition 4.1.12. *Let $(M, \overline{\mathrm{g}}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-*asymptotically flat manifold *with* ADM-energy $E_{\mathrm{ADM}} > 0$ *and let $\iota : \Sigma \hookrightarrow M$ be a* surface. *Let (Σ, F_t) be a solution to the* volume preserving mean curvature flow *system (4.1) with initial datum $F_0 = \iota$. Suppose that the flow exists on a compact interval of time $[0, T]$, with $T > 0$, and that $(\Sigma, g(t)) \in \overline{\mathcal{W}}_{\sigma}(B_1, B_2)$ for some $B_1, B_2 > 0$. Then there exists $\sigma_0 = \sigma_0(B_1, B_2, \overline{c}, \delta, E_{\mathrm{ADM}})$ such that, for every $\sigma > \sigma_0$ and every $t \in [0, T]$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|H-h\|^2_{L^2(\Sigma_t)} \leq -\frac{4E_{\mathrm{ADM}}}{\sigma^3_{\Sigma_t}}\|(H-h)^t\|^2_{L^2(\Sigma_t)} - \frac{2}{\sigma^2_{\Sigma_t}}\|(H-h)^d\|^2_{L^2(\Sigma_t)}.$$

Proof. In the following, c will be a positive constant that can change from line to line and that depends on the roundness constants. Combining the inequalities obtained in Proposition 3.1.10, since $(H-h)^d \in \operatorname{span}\{f_k : \ k \geq 4\}$, we get

$$\begin{aligned}
&\langle L(H-h), H-h\rangle_2 \geq \\
&\geq \frac{6m_H(\Sigma_t)}{\sigma^3_{\Sigma_t}}\|(H-h)^t\|^2_2 - \frac{3h}{2}\int_\Sigma (H-h)((H-h)^t)^2\, d\mu - c\sigma^{-3-2\delta}\|(H-h)^t\|^2_2 \\
&\quad + \frac{2}{\sigma^2_{\Sigma_t}}\|(H-h)^d\|^2_2 - c\sigma^{-\frac{5}{2}-\delta}\|(H-h)^t\|_2\|(H-h)^d\|_2 \geq \\
&\geq \frac{5m_H(\Sigma_t)}{\sigma^3_{\Sigma_t}}\|(H-h)^t\|^2_2 - \frac{3h}{2}\int_\Sigma \left((H-h)^t + (H-h)^d\right)((H-h)^t)^2\, d\mu + \frac{3}{2\sigma^2_{\Sigma_t}}\|(H-h)^d\|^2_2,
\end{aligned}$$

for σ large enough. Keep in mind that, moreover, for σ sufficiently large, $m_H(\Sigma) \geq \frac{E_{\mathrm{ADM}}}{2} > 0$. From a dynamical point of view we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|H-h\|^2_2 &= -2\langle L(H-h), H-h\rangle_2 - \int_\Sigma H(H-h)^3\, d\mu \\
&\leq -\frac{5E_{\mathrm{ADM}}}{\sigma^3_{\Sigma_t}}\|(H-h)^t\|^2_2 + 3h\int_\Sigma ((H-h)^t + (H-h)^d)((H-h)^t)^2\, d\mu \\
&\quad - \frac{3}{\sigma^2_{\Sigma_t}}\|(H-h)^d\|^2_2 - h\int_\Sigma (H-h)^3\, d\mu + c\sigma^{-3-2\delta}\int_\Sigma (H-h)^2\, d\mu,
\end{aligned}$$

where $c = c(c_\infty) > 0$. Thus, we have to study the integrals

$$h\int_\Sigma ((H-h)^t)^3\, d\mu, \quad h\int_\Sigma (H-h)^d((H-h)^t)^2\, d\mu, \quad h\int_\Sigma ((H-h)^t + (H-h)^d)^3\, d\mu. \tag{4.35}$$

The first integral in (4.35) is estimated by Lemma 3.1.16. The second integral can be estimated combining $\|(H-h)^t\|_{L^\infty(\Sigma)} = \mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$, $\|f_\alpha\|_{L^\infty(\Sigma)} = \mathrm{O}(\sigma^{-1})$ and

$$\left| h\int_\Sigma (H-h)^d((H-h)^t)^2\, d\mu \right| \leq c\sigma^{-\frac{5}{2}-\delta}\int_\Sigma |(H-h)^d||(H-h)^t|\, d\mu$$
$$\leq \frac{c\sigma^{-2}}{2}\int_\Sigma \left(\sigma^{-\delta}((H-h)^d)^2 + \sigma^{-1-\delta}((H-h)^t)^2\right)\, d\mu.$$

By the formula of the cube of a binomial, it only remains to estimate

$$\left| h\int_\Sigma ((H-h)^d)^3\, d\mu + 3h\int_\Sigma (H-h)^t((H-h)^d)^2\, d\mu \right|$$
$$\leq \|h(H-h)^d + 3h(H-h)^t\|_\infty \int_\Sigma ((H-h)^d)^2\, d\mu \leq c\sigma^{-\frac{5}{2}-\delta}\int_\Sigma ((H-h)^d)^2\, d\mu,$$

since also $\|(H-h)^d\|_{L^\infty(\Sigma)} \equiv \|(H-h)-(H-h)^t\|_{L^\infty(\Sigma)} = \mathrm{O}(\sigma^{-\frac{3}{2}-\delta})$. Putting the pieces together, we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\|H-h\|_2^2 \leq -\frac{4E_{\mathrm{ADM}}}{\sigma_{\Sigma_t}^3}\|(H-h)^t\|_2^2 - \frac{2}{\sigma_{\Sigma_t}^2}\|(H-h)^d\|_2^2.$$

□

Remark 4.1.13. *An immediate consequence, if σ is sufficiently large depending on E_{ADM}, i.e.* $\frac{2}{\sigma_{\Sigma_t}^2} \geq \frac{4E_{\mathrm{ADM}}}{\sigma_{\Sigma_t}^3}$, *is that*

$$\frac{\mathrm{d}}{\mathrm{dt}}\|H-h\|_2^2 \leq -\frac{4E_{\mathrm{ADM}}}{\sigma_{\Sigma_t}^3}\|H-h\|_2^2.$$

The next result, which is similar to Proposition 3.4 in [HY96], gives a bound on the possible change of area of the surface along the flow as long as it remains round.

Lemma 4.1.14. *Given B_1, B_2, there exist constants $c > 0$ and $\sigma_0 > 1$ such that, if $\sigma > \sigma_0$ and Σ_t is a solution of the flow (5.2) for $t \in [0,T]$ with $\Sigma_t \in \mathcal{W}_\sigma^\eta(B_1,B_2)$ for all $t \in [0,T]$ then*

$$0 \leq \sigma_{\Sigma_0} - \sigma_{\Sigma_t} \leq c\sigma^{\frac{1}{2}-\delta}$$

for every $t \in [0,T]$.

Proof. Suppose that $\sigma > 2\mathrm{diam}\,(\mathrm{C})$ and consider the sphere $\mathbb{S}_{\frac{\sigma}{2}}(\vec{0})$. Since the flow is volume preserving, we have that the volume enclosed between Σ_t and $\mathbb{S}_{\frac{\sigma}{2}}(\vec{0})$ remains the same for every $t \in [0,T]$. We call this region Ω_t, while Λ_t is the Euclidean volume of the region enclosed by Σ_t. Since Σ_t belongs to the roundness class for every $t \in [0,T]$, we find that

$$|\mathrm{Vol}_{\overline{\mathrm{g}}}(\Omega_t) - \mathrm{Vol}_{\overline{\mathrm{g}}^e}(\Omega_t)| \leq C\sigma^{\frac{5}{2}-\delta}, \tag{4.36}$$

$$\left|\mathrm{Vol}_{\overline{\mathrm{g}}^e}(\Lambda_t) - \frac{4\pi\sigma_{\Sigma_t}^3}{3}\right| = \left|\mathrm{Vol}_{\overline{\mathrm{g}}^e}(\Lambda_t) - \mathrm{Vol}_{\overline{\mathrm{g}}^e}(\mathbb{S}_{\sigma_{\Sigma_t}}(\vec{z}_{\Sigma_t}))\right| \leq C|\Sigma_t|\|f(\cdot,t)\|_{L^\infty} \leq c\sigma^{\frac{5}{2}-\delta}, \tag{4.37}$$

for every $t \in [0,T]$. Combining the identity $\mathrm{Vol}_{\overline{\mathrm{g}}^e}(\Omega_t) = \mathrm{Vol}_{\overline{\mathrm{g}}^e}(\Lambda_t) - \frac{\pi\sigma^3}{6}$ with (4.36) and (4.37), we get

$$\left|\mathrm{Vol}_{\overline{\mathrm{g}}}(\Omega_t) - \left(\frac{4\pi\sigma_{\Sigma_t}^3}{3} - \frac{\pi\sigma^3}{3}\right)\right| \leq C\sigma^{\frac{5}{2}-\delta} + \left|\mathrm{Vol}_{\overline{\mathrm{g}}^e}(\Lambda_t) - \frac{4\pi\sigma_{\Sigma_t}^3}{3}\right| \leq c\sigma^{\frac{5}{2}-\delta}. \tag{4.38}$$

We conclude noticing that $\mathrm{Vol}_{\overline{g}}(\Omega_t) = \mathrm{Vol}_{\overline{g}}(\Omega_0)$, and thus $\frac{4\pi\sigma^3_{\Sigma_t}}{3} - \frac{\pi\sigma^3}{3}$, up to an error of order $\sigma^{\frac{5}{2}-\delta}$, is constant in $[0,T]$. This implies the thesis. □

We are now ready to prove that, by an appropriate choice of the parameters of roundness class, a well-centered almost CMC round surface remains inside the class for arbitrary times. We remember that the definition of roundness class has been given in Definition 2.3.1.

Lemma 4.1.15. *Let* $(M, \overline{g}, \vec{x})$ *be a* $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold *with* $E_{\mathrm{ADM}} > 0$*. Let* $\iota : \Sigma \hookrightarrow M$ *be* surface *and set* $\sigma = \sigma_\Sigma$*. Fix* $Q > 1$ *and* $\mathrm{c_{in}} > 0$*. There exists* $C = C(Q, \mathrm{c_{in}}, E_{\mathrm{ADM}}) > 0$ *such that if* B_1 *is chosen as in* Lemma 4.1.6 *and* η *and* B_2 *as in* Lemma 4.1.11*, and* $B_{\mathrm{cen}} > C$*, then the following statement holds. Let* (Σ, F_t) *be a solution to the* volume preserving mean curvature flow *with initial datum* $F_0 = \iota$*. Suppose that the flow exists on a compact interval of time* $[0,T]$*, with* $T > 0$*, and that the following conditions hold*

(i) $(\Sigma, F_0) \in \mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$*;*

(ii) (Σ, F_0) *is* $(\sigma, \mathrm{c_{in}})$*-almost-CMC, in the sense of* Definition 3.2.1*.*

(iii) $(\Sigma, F_t) \in \overline{\mathcal{B}}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$ *for every* $t \in [0,T]$*.*

Then there exists $\sigma_0 = \sigma_0(\bar{c}, \delta, B_1, B_2, B_{\mathrm{cen}}, E_{\mathrm{ADM}}, Q)$ *such that, if* $\sigma > \sigma_0$*, then* $(\Sigma, g(t)) \in \mathcal{B}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$ *for every* $t \in [0,T]$*.*

Proof. In this proof, for the reader's convenience, we set $E \equiv E_{\mathrm{ADM}}$. Moreover, for sake of brevity, we will indicate with $\vec{z}(t)$ the barycenter $\vec{z}_{\Sigma_t}$. We have to show that no equality in the definition of $\overline{\mathcal{B}}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$ can occur. If we start in $\mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$, then there exists a maximal time $t_0 \in (0,T]$ such that $(\Sigma, g(t)) \in \mathcal{B}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$ for every $t \in [0, t_0)$ and $(\Sigma, g(t_0)) \in \overline{\mathcal{B}}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$. This means that, at $t = t_0$, at least one of the inequalities in Definition 2.3.1 is an equality, with B_{cen} replaced by QB_{cen}. We note that the conditions (2.29) are preserved: Lemma 2.3.5 says that, for σ sufficiently large, a surface in $\overline{\mathcal{W}}_\sigma(B_1, B_2)$ satisfies the strict bounds (2.42) on κ_i, and thus $|A| < \sqrt{5/2\sigma^2_\Sigma}$ must hold; moreover, thanks to the choice of σ, Lemma 4.1.14 implies that the area radius σ_{Σ_t} is controlled for every $t \in [0,T]$ in the sense of (2.29), if σ is large. Thus, it is enough to prove the strict inequalities in (2.31) and (2.32).

In a first step, we suppose that $|\vec{z}_{\Sigma_t}| \le QB_{\mathrm{cen}}\sigma^{1-\delta}$ for every $t \in [0, t_0]$ and we show that the other inequalities are strict. In a second moment we will show that also this condition holds strictly. Observe that, thanks to Lemma 2.3.5, if σ is suitably large

$$\left|\frac{\vec{x}(F_t(x))}{\sigma_{\Sigma_t}}\right| = \left|\frac{\vec{z}_{\Sigma_t}}{\sigma_{\Sigma_t}} + \nu^e_t(x) + \frac{f(x,t)}{\sigma_{\Sigma_t}}\nu^e_t(x)\right| \ge 1 - c\sigma^{-\delta} > \frac{1}{3}, \tag{4.39}$$

from which we get $\frac{\sigma_{\Sigma_t}}{r_\Sigma(t)} < 3$. Combining this with the results of Lemma 2.3.5, we see that we are in the hypothesis of the Lemmas of Section 4.1.3 and of Proposition 4.1.12. Thus, choosing B_1 as in Lemma 4.1.6, we have that $\|\mathring{A}\|_{L^4(\Sigma_t)}$ can never reach the bound $B_1\sigma^{-1-\delta}$. In the same way, choosing B_2 as in Lemma 4.1.11, depending only on B_1, and B_∞ and $c_\infty = c_\infty(B_2, \eta)$ as in Lemma 2.3.5, we also have that the left hand side of (2.32) remains strictly below $B_2\sigma^{-8-4\delta}$, if σ is sufficiently large, depending on B_1, B_2 and the universal constants.

We now suppose that there exists $t_0 \in (0,T]$, such that $|\vec{z}_{\Sigma_{t_0}}| = QB_{\mathrm{cen}}\sigma^{1-\delta}$. Also in this case, for every $t \in [0, t_0]$ we continue to be in the hypothesis of Proposition 4.1.12, and this implies that

$$\int_\Sigma (H-h)^2\, d\mu_t \le \left(\int_\Sigma (H-h)^2\, d\mu_0\right) e^{-\frac{5Et}{2\sigma^3}},$$

also using the bounds on the area radius, for every $t \in [0, t_0]$, where $d\mu_0 = d\mu^{(\Sigma,g)}$. By the hypothesis on the $(\mathrm{c_{in}}, \sigma)$-almost-CMC of Σ, we find

$$\|H - h\|_{L^2(\Sigma_t)} \leq \mathrm{c_{in}}\sigma^{-1-\delta} e^{-\frac{5Et}{2\sigma^3}}, \qquad \forall t \in [0, t_0]. \tag{4.40}$$

In the following, we indicate with $\vec{z}_0(t)$ the center of the Euclidean sphere of radius σ_Σ approximating Σ, defined as in Lemma 2.3.5. This Lemma also implies that $|\vec{z}_0(t) - \vec{z}(t)| \leq c_0(B_1, \overline{c}, \delta)\sigma^{\frac{1}{2}-\delta}$, uniformly in t. A straightforward computation shows that the barycenter evolves according to

$$\partial_t \vec{z}(t) \equiv \partial_t \vec{z}_{\Sigma_t} = \frac{\int_\Sigma (h - H)\left[\nu + H\left(F_t(x) - \vec{z}(t)\right)\right] \, d\mu_t}{|\Sigma_t|}. \tag{4.41}$$

See for example [CW08, Remark 3.1]. Since $\vec{z}_0(t) = \vec{x}(F_t(x)) - \sigma_{\Sigma_t}\nu_{\sigma_{\Sigma_t}} - f_t\nu_{\sigma_{\Sigma_t}}$, where f_t is defined in Lemma 2.3.5 and $\nu_{\sigma_{\Sigma_t}}$ is the Euclidean normal of the sphere $\mathbb{S}_{\sigma_{\Sigma_t}}(\vec{z}_0(t))$, we find that, for σ large (depending on the roundness constants), $\vec{z}(t)$ is bounded by $\mathrm{O}(\sigma)$, uniformly with respect to the roundness constant, since $\max_x |\vec{x}(F_t(x))|$ is the Euclidean radius of Σ_t^e, i.e. $R_\Sigma(t)$. Thus

$$|\partial_t \vec{z}(t)| \leq \frac{c}{|\Sigma_t|} \int_\Sigma |H - h| \, d\mu_t \leq \frac{c}{|\Sigma_t|^{\frac{1}{2}}} \|H - h\|_{L^2(\Sigma_t)} \leq c(\mathrm{c_{in}})\sigma^{-2-\delta} e^{-\frac{5Et}{2\sigma^3}}.$$

Integrating this expression in $[0, t_0]$, we get

$$|\vec{z}(t_0) - \vec{z}(0)| \leq \int_0^{t_0} |\partial_t \vec{z}| \, dt \leq c(\mathrm{c_{in}})\sigma^{-2-\delta} \left(\frac{2\sigma^3}{5E}\right) \left(1 - e^{-\frac{5Et_0}{2\sigma^3}}\right) \leq C\sigma^{1-\delta},$$

where $C = C(\mathrm{c_{in}}, E) > 0$. Using now hypothesis (i), we conclude that

$$|\vec{z}_{\Sigma_{t_0}}| \leq B_{\mathrm{cen}}\sigma^{1-\delta} + C\sigma^{1-\delta} < QB_{\mathrm{cen}}\sigma^{1-\delta},$$

if B_{cen} is sufficiently large depending on C and $Q > 1$, and thus we have a contradiction with the definition of t_0. This implies the thesis. □

Theorem 4.1.16 (Existence of the flow - Part I)**.** *Let* $(M, \overline{\mathrm{g}}, \vec{x})$ *be a* $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold *with* $E_{\mathrm{ADM}} > 0$. *Let* $\iota : \Sigma \hookrightarrow M$ *be a surface and set* $\sigma := \sigma_\Sigma$. *Fix* $Q > 1$ *and* $\mathrm{c_{in}} > 0$. *Set* B_1, B_2, B_{cen} *and* σ_0 *as in* Lemma 4.1.15, *and suppose that* $\sigma > \sigma_0$, (Σ, F_0) *belongs to* $\mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$ *and that it is* $(\sigma, \mathrm{c_{in}})$*-almost-CMC. Let* (Σ, F_t) *be a solution to the* volume preserving mean curvature flow *with initial datum* $F_0 = \iota$. *Then, this solution exists for every* $t \in [0, \infty)$ *and* (Σ, F_t) *belongs to* $\mathcal{B}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$ *for every* $t \in [0, \infty)$.

Proof. Since (Σ, F_0) belongs to $\mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$, for t small we have that (Σ, F_t) belongs to $\mathcal{B}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$. Define $T_{\max}$ as

$$\sup\left\{\overline{T} : \ (\Sigma, F_t) \text{ exists in } [0, \overline{T}) \text{ and it belongs to } \mathcal{B}_\sigma(B_1, B_2, QB_{\mathrm{cen}}) \text{ for every } t \in [0, \overline{T})\right\}. \tag{4.42}$$

Of course $T_{\max} > 0$. Suppose that $T_{\max} < \infty$. Then, we can consider the limit $(\Sigma^{T_{\max}}, g_{T_{\max}}) := \lim_{t \nearrow T_{\max}} (\Sigma, g(t))$, which is a smooth surface, since we are considering the limit of a sequence of surfaces whose second fundamental form is uniformly bounded, together with its derivative of each order, see Section 4.3. Thus, $[0, T_{\max}) \ni t \mapsto g(t)$ is smoothly extended to $[0, T_{\max}]$. We have that $(\Sigma, g(t))_{t \in [0, T_{\max}]}$ is a (smooth) solution to the flow which belongs, for every t, to $\overline{\mathcal{B}}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$ and which at $t = 0$ belongs to $\mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$ and is almost-CMC. Choosing σ_0 as in Lemma 4.1.15 with $T = T_{\max}$ (observing that the choice of σ_0 does not depends on T) and $\sigma > \sigma_0$, we have that $(\Sigma, g(t)) \in \mathcal{B}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$ for every $t \in [0, T_{\max}]$. Thanks again to the computations in Section 4.3, see also Remark 4.1.2, we can smoothly extend the solution past $T_{\max}$, that is, there exists $\tau > 0$ such that the solution can be extended to $[0, T_{\max} + \tau)$. Since $(\Sigma, g(T_{\max}))$ belongs to $\mathcal{B}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$, possibly choosing $\tau > 0$ smaller we have that $(\Sigma, g(t))$ belongs to $\mathcal{B}_\sigma(B_1, B_2, QB_{\mathrm{cen}})$ for every $t \in [0, T_{\max} + \tau)$. But this contradicts the definition of $T_{\max}$ unless $T_{\max} = \infty$. □

4.2 Proof of Theorem 1.1.2

4.2.1 Evolution of Euclidean spheres

We conclude by considering the explicit example of a Euclidean coordinate sphere $\mathbb{S}_r(0)$ as initial surface for our flow. To ensure that condition (ii) of Theorem 4.1.15 is satisfied (for an initial time which is possibly non-zero), we have to strengthen the assumptions on our ambient manifold by requiring the $C^1_{1+\delta}$-Regge-Teitelboim conditions in Definition 3.1.19. Even if under this condition the existence of the ADM-center of mass is not guaranteed, at the end we will prove the existence of an *abstract center of mass*, i.e. a CMC-foliation constructed via volume preserving mean curvature flow.

We consider the immersion $F_0 = \iota : \vec{x}^{-1}\left(\mathbb{S}_r(\vec{0})\right) \hookrightarrow M$, and we set, as usual, $\sigma := \sigma_{\mathbb{S}_r(\vec{0})}$, the area radius of the Euclidean sphere. First of all, we remember that Lemma 3.1.20 implies the following result.

Lemma 4.2.1. *Let $(M, \bar{g}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat 3-manifold that satisfies the $C^2_{1+\delta}$-Regge-Teitelboim conditions. Consider the immersion $\vec{x}^{-1}\left(\mathbb{S}_r(\vec{0})\right) \hookrightarrow M$ and set σ as above. Then there exist two universal constants $C_{\mathrm{tot}} > 0$ and $C_{\mathrm{trasl}} > 0$ such that the family of Euclidean spheres satisfies*

$$\|H - h\|_{L^2(\mathbb{S}_r(\vec{0}))} \le C_{\mathrm{tot}} \sigma^{-\frac{1}{2}-\delta}, \qquad \left| \sum_{\alpha=1}^{3} \left\langle H - h, \frac{\nu_\alpha}{\sigma} \right\rangle^2_{L^2(\mathbb{S}_r(\vec{0}))} \right| \le C_{\mathrm{trasl}} \sigma^{-2-2\delta}, \tag{4.43}$$

for every r sufficiently large, where, with an abuse of notation, we identified $\vec{x}^{-1}\left(\mathbb{S}_r(\vec{0})\right)$ and $\mathbb{S}_r(\vec{0})$.

We study the evolution of the quantity in (4.43).

Lemma 4.2.2. *Let $(M, \bar{g}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-*asymptotically flat manifold *with $E_{\mathrm{ADM}} > 0$. Let $\iota : \Sigma \hookrightarrow M$ be a* surface *and set $\sigma := \sigma_\Sigma$. Let (Σ, F_t) be a solution to the* volume preserving mean curvature flow *with initial datum $F_0 = \iota$. Suppose that the flow exists on a compact interval of time $[0, T]$, with $T > 0$, and that $(\Sigma, g(t)) \in \overline{\mathcal{W}}_\sigma(B_1, B_2)$ for every $t \in [0, T]$ and some $B_1, B_2 > 0$. Then there exist a constant $c = c(B_1, B_2, \bar{c}, \delta) > 0$ and a radius $\sigma = \sigma_0(B_1, B_2, \bar{c}, \delta) > 1$ such that if $\sigma > \sigma_0$ then*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \left\langle H - h, \frac{\nu}{\sigma} \right\rangle_{L^2(\Sigma_t)} \right| \le c\sigma^{-3-2\delta},$$

for every $t \in [0, T]$.

Proof. By the definition of scalar product in $L^2(\Sigma_t)$ we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle H - h, \frac{\nu_\alpha}{\sigma} \right\rangle_{L^2(\Sigma_t)} =$$

$$= \frac{1}{\sigma}\left(\int_\Sigma \left(\frac{\partial H}{\partial t} - \dot{h}\right)\nu_\alpha \, d\mu_t + \int_\Sigma (H-h)\left(\frac{\partial \nu_\alpha}{\partial t}\right) \, d\mu_t - \int_\Sigma (H-h)^2 H \nu_\alpha \, d\mu_t\right)$$
$$= \int_\Sigma (-L(H-h))\frac{\nu_\alpha}{\sigma} \, d\mu + \left\{-\frac{1}{\sigma}\int_\Sigma \dot{h}\nu_\alpha \, d\mu + \frac{1}{\sigma}\int_\Sigma (H-h)\left(\frac{\partial \nu_\alpha}{\partial t}\right) \, d\mu + \mathrm{O}(\sigma^{-3-2\delta})\right\},$$

using also Lemma 5.0.2. This also implies

$$\left|\frac{1}{\sigma}\int_\Sigma (H-h)\left(\frac{\partial \nu_\alpha}{\partial t}\right) \, d\mu\right| \le \sigma^{-1}\|H-h\|_2\|\nabla H\|_2 \le c\sigma^{-3-2\delta}, \tag{4.44}$$

where we used that $\sigma\|\nabla H\|_2 + \|H-h\|_2 \le c\sigma^{-\frac{1}{2}-\delta}$. Then inequality (4.44), Lemma 4.1.7 and equation (3.31) imply the thesis. □

We finally conclude the proof of Theorem 1.1.2.

Proposition 4.2.3 (Existence of the flow - Part II)**.** *Let* $(M, \overline{g}, \vec{x})$ *be a* $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold *with* $E_{\mathrm{ADM}} > 0$ *that satisfies the* $C^1_{1+\delta}$-Regge-Teitelboim conditions, *i.e.* (3.37) *holds. There exists* $r_0 = r_0(\overline{c}, \delta) > 1$ *such that for every* $r > r_0$ *the solution* (Σ, F_t) *to the* volume preserving mean curvature flow *with initial datum* $F_0 = \iota : \vec{x}^{-1}\left(\mathbb{S}_r(\vec{0})\right) \hookrightarrow M$ *exists for every* $t \in [0,\infty)$.

Remark 4.2.4. *The weak Regge-Teitelboim assumption in the hypothesis of the statement above could be replaced with assuming directly that the initial family of surfaces satisfy inequality* (4.43)*.*

Proof. In the following, r and $\sigma := \sigma_{\mathbb{S}_r(\vec{0})}$ will be arbitrary but fixed. In particular, we will require σ to be large, which translates into a requirement on the largeness of r, in view of the asymptotic flatness. Set $\mathrm{c_{in}} := \max\left\{\sqrt{\frac{2(C_{\mathrm{trasl}}+1)}{\pi}}, C_{\mathrm{tot}}\right\}$ and $Q = \frac{4}{3}$, and choose B_1, B_2 and B_{cen} as in Theorem 4.1.16, and σ_0 to be the maximum of the σ_0s obtained by Lemma 4.1.15 considering in the statement both the class $\mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$ and $\mathcal{B}_\sigma(B_1, B_2, 3B_{\mathrm{cen}})$. Suppose moreover that B_1, B_2 and B_{cen} are such that $\vec{x}^{-1}\left(\mathbb{S}_r(\vec{0})\right)$ belongs to $\mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$. Set

$$\Pi(t) := \sum_{\alpha=1}^{3}\left\langle H-h, \frac{\nu_\alpha}{\sigma}\right\rangle^2_{L^2(\Sigma_t)}.$$

Lemma 4.2.1 says that $|\Pi(0)| \le C_{\mathrm{trasl}}\sigma^{-2-2\delta}$. Consider the solution F_t starting from Σ and define

$$T_{\max} := \sup\left\{\overline{T} : \begin{array}{c} F_t \text{ exists in } [0,\overline{T}) \text{ and } \Pi(t) < (C_{\mathrm{trasl}}+1)\sigma^{-2-2\delta} \text{ for every } t \in [0,\overline{T}), \\ (\Sigma, g(t)) \in \mathcal{B}_\sigma(B_1, B_2, 2B_{\mathrm{cen}}) \text{ for every } t \in [0,\overline{T}) \end{array}\right\}.$$

If $T_{\max} = \infty$ then the Theorem is proved. Suppose then that $T_{\max} < \infty$. Thus, $T_{\max}$ is the first time such that

$$\Pi(T_{\max}) = (C_{\mathrm{trasl}}+1)\sigma^{-2-2\delta} \text{ or } (\Sigma, g(T_{\max})) \in \overline{\mathcal{B}}_\sigma(B_1, B_2, 2B_{\mathrm{cen}}). \tag{4.45}$$

Since we have chosen B_1 and B_2 as in Lemma 4.1.15, which does not require the almost-CMCness in its hypotheses, then for σ large the second case only occurs when $|\vec{z}_{\Sigma_{T_{\max}}}| = 2B_{\mathrm{cen}}\sigma^{1-\delta}$.

Claim. We show the following claim: if (4.45) occurs, then there exists a time $t_0 \le T_{\max}$ such that $(\Sigma, g(t_0)) \in \mathcal{B}_\sigma(B_1, B_2, 3B_{\mathrm{cen}})$ and $\|H-h\|_{L^2(\Sigma_{t_0})} \le \mathrm{c_{in}}\sigma^{-1-\delta}$.

In a second step, Theorem 4.1.16, with $Q = \frac{4}{3}$ and B_{cen} of the statement of Theorem 4.1.16 replaced by $3B_{\text{cen}}$, implies that $(\Sigma, g(t))$ exists for every $t \in [0,\infty)$ and $(\Sigma, g(t)) \in \mathcal{B}_\sigma(B_1, B_2, 4B_{\text{cen}})$, thanks to the choice of B_1, B_2 and B_{cen} and for σ is suitably large.

Proof of the claim. At first, suppose that $T_{\max}$ is the first time such that $\Pi(T_{\max}) = (C_{\text{trasl}}+1)\sigma^{-2-2\delta}$ and that moreover it holds $(\Sigma, g(T_{\max})) \in \mathcal{B}_\sigma(B_1, B_2, 2B_{\text{cen}})$.

(i) If there exists $t_0 \in [0, T_{\max}]$ such that

$$\int_\Sigma \left((H-h)^d)\right)^2 \, d\mu_{t_0} \le \int_\Sigma \left((H-h)^t\right)^2 \, d\mu_{t_0}, \tag{4.46}$$

then $\|H-h\|^2_{L^2(\Sigma_{t_0})} \le 2\int_\Sigma \left((H-h)^t\right)^2 \, d\mu_{t_0}$, and moreover, thanks to Lemma 3.1.18, with $\Pi = \Pi(t_0)$, we get

$$\left|\Pi(t_0) - \frac{4\pi}{3}\int_\Sigma \left((H-h)^t\right)^2 \, d\mu_{t_0}\right| \le c\sigma^{-\frac{1}{2}-\delta}\|H-h\|^2_{L^2(\Sigma_{t_0})},$$

where in the latter inequality we combined the proof of Lemma 3.1.18 with the estimate on $|\sigma - \sigma_{\Sigma_t}|$ given by Lemma 4.1.14.
Since $\Pi(t_0) \le (C_{\text{transl}}+1)\sigma^{-2-2\delta}$, (4.46) implies that $\|H-h\|^2_{L^2(\Sigma_{t_0})} \le \frac{2(C_{\text{transl}}+1)}{\pi}\sigma^{-2-2\delta}$, for σ large. Moreover, $(\Sigma, g(t_0)) \in \mathcal{B}_\sigma(B_1, B_2, 2B_{\text{cen}})$ and thus we have the claim thanks to the definition of c_{in}.

(ii) Suppose now that for every $t \in [0, T_{\max}]$ it holds

$$\int_\Sigma \left((H-h)^d)\right)^2 \, d\mu_t > \int_\Sigma \left((H-h)^t\right)^2 \, d\mu_t. \tag{4.47}$$

Thus Lemma 4.1.12 implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|H-h\|_2^2 \le -\frac{2}{\sigma_\Sigma^2}\|(H-h)^d\|_2^2 \le -\frac{1}{\sigma_\Sigma^2}\|(H-h)^t\|_2^2 - \frac{1}{\sigma_\Sigma^2}\|(H-h)^d\|_2^2,$$

that is

$$\|H-h\|^2_{L^2(\Sigma_t)} \le \|H-h\|^2_{L^2(\Sigma_0)} e^{-\frac{4t}{5\pi\sigma^2}} \le C_{\text{tot}}\sigma^{-1-2\delta}e^{-\frac{4t}{5\pi\sigma^2}} \tag{4.48}$$

for every $t \in [0, T_{\max}]$. On the other hand, Lemma 4.2.2, combined with $\sqrt{\Pi(t)} \le \sqrt{C_{\text{trasl}}+1}\sigma^{-1-\delta}$ for every $t \in [0, T_{\max}]$, implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle H-h, \frac{\nu_\alpha}{\sigma}\right\rangle^2_{L^2(\Sigma_t)} \le c\sigma^{-4-3\delta},$$

for every $t \in [0, T_{\max}]$, with c depending on B_1, B_2, $\overline{c}$, δ and also on C_{trasl}. This means that, integrating and computing in $t = T_{\max}$,

$$(C_{\text{trasl}}+1)\sigma^{-2-2\delta} \overset{(4.45)}{=} \sum_\alpha \left\langle H-h, \frac{\nu_\alpha}{\sigma}\right\rangle^2_{L^2(\Sigma_{T_{\max}})} \le \sum_\alpha \left\langle H-h, \frac{\nu_\alpha}{\sigma}\right\rangle^2_{L^2(\Sigma_0)} + c\sigma^{-4-3\delta}T_{\max}.$$

Since the L^2-product at the initial time is smaller than $C_{\text{trasl}}\sigma^{-2-2\delta}$, the inequality leads to $T_{\max} \ge \frac{1}{c}\sigma^{2+\delta}$. Computing (4.48) in $T_{\max}$ we get

$$\|H-h\|^2_{L^2(\Sigma_{T_{\max}})} \le C_{\text{tot}}\sigma^{-1-2\delta}e^{-\frac{4T_{\max}}{5\pi\sigma^2}} \le C_{\text{tot}}\sigma^{-1-2\delta}e^{-\frac{4\sigma^\delta}{5\pi c}} \le C_{\text{tot}}\sigma^{-2-2\delta}, \tag{4.49}$$

for σ large. We get the claim choosing $t_0 = T_{\max}$, thanks to the definition of $\mathrm{c_{in}}$, observing also that $(\Sigma, g(t_0)) \in \mathcal{B}_\sigma(B_1, B_2, 2B_{\mathrm{cen}})$.

To conclude with the remaining case, suppose secondly that $T_{\max}$ is the first time such that $|\vec{z}_{\Sigma_{T_{\max}}}| = 2B_{\mathrm{cen}}\sigma^{1-\delta}$ and $\Pi(t) \le (C_{\mathrm{trasl}}+1)\sigma^{-2-2\delta}$ for every $t \in [0, T_{\max}]$. As above, we consider two cases.

- If $T_{\max} > \sigma^{2+\delta}$, we conclude as above, distinguishing again the two cases: the case (i) is identical to the one exposed above; in the case (ii) we skip from inequality (4.48) to (4.49) and we conclude setting $t_0 := T_{\max}$ as above.
- Otherwise, $T_{\max} \le \sigma^{2+\delta}$. Since $\Pi(t) \le (C_{\mathrm{trasl}}+1)\sigma^{-2-2\delta}$ for every $t \in [0, T_{\max}]$, it follows that $|\langle H-h, \nu\rangle_{L^2(\Sigma_t)}| \le c(C_{\mathrm{trasl}})\sigma^{-\delta}$. Thus the evolution (4.41) implies that

$$\begin{aligned}|\partial_t \vec{z}| \le |\Sigma_t|^{-1}\left|\langle h-H,\nu\rangle_{L^2(\Sigma_t)}\right| + |\Sigma_t|^{-1}\left|\int_\Sigma (H-h)^2(F_t(x)-\vec{z})\, d\mu_t\right| \\ + |\Sigma_t|^{-1}\left|h\int_\Sigma (h-H)(F_t(x)-\vec{z})\, d\mu_t\right|.\end{aligned} \tag{4.50}$$

Since, by Lemma 2.3.5 $F_t(x) - \vec{z}(t) = \sigma_{\Sigma_t}\nu_t + \mathrm{O}(\sigma^{\frac{1}{2}-\delta})$, it follows the inequality $|\partial_t\vec{z}| \le c(C_{\mathrm{trasl}}, B_1, B_2)\sigma^{-2-\delta}$ for every $t \in [0, T_{\max}]$ and for σ sufficiently large. Notice, in fact, that the second and (the second addend of the) third addend in (4.50) decay with the right order. It follows that

$$\left|\vec{z}_{\Sigma_{T_{\max}}} - \vec{z}(0)\right| \le \int_0^{T_{\max}} |\partial_t\vec{z}|\, dt \le T_{\max} c(C_{\mathrm{trasl}}, B_1, B_2)\sigma^{-2-\delta} \le c(C_{\mathrm{trasl}}, B_1, B_2).$$

Since by the asymptotic flatness of the manifold, $|\vec{z}(0)| \equiv |\vec{z}_{\mathbb{S}_\sigma(\vec{0})}| \le C(\bar{c})\sigma^{\frac{1}{2}-\delta}$, we find that $|\vec{z}_{\Sigma_{T_{\max}}}| \le C(\bar{c})\sigma^{\frac{1}{2}-\delta}$ for σ large, and thus the equality $\left|\vec{z}_{\Sigma_{T_{\max}}}\right| = 2B_{\mathrm{cen}}\sigma^{1-\delta}$ cannot hold, for σ large depending on B_{cen} and $\bar{c}$. Thus, this second scenario cannot happen.

□

4.3 Conclusions

In Section 4.2 we proved long time existence of solutions to the volume preserving mean curvature flow starting from Euclidean spheres. In this Section we review some technical details concerning the regularization of the second fundamental form, and its derivatives, along the flow. We also conclude that the flow converges, as $t \to \infty$, to a CMC-surface.

It is well-known that, if

$$C_m \equiv C_m^\sigma := \max_{0\le l\le m}\sup_{\mathcal{A}_\sigma}\left|\overline{\nabla}^l\overline{\mathrm{Rm}}\right|, \qquad \forall m \in \mathbb{N}\cup\{0\}. \tag{4.51}$$

the derivatives of the second fundamental form of Σ_t evolving by volume preserving mean curvature flow satisfy

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 \le \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \sum_{i+j+k=m}\nabla^m A * \nabla^i A * \nabla^j A * \nabla^k A \\ + (h-H)\sum_{i+j=m}\nabla^m * \nabla^i A * \nabla^j A + C_m\sum_{i\le m}\nabla^i A * \nabla A + C_{m+1}|\nabla^m A|.\end{aligned} \tag{4.52}$$

See in example [CRM07], [Hui87]. Since we are assuming that our flow lives in the roundness class $\mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$ for $\sigma = \sigma_{\Sigma_0}$ and suitably B_1, B_2 and B_{cen}, and thus

$$|A(t)| < \sqrt{\frac{5}{2}}\sigma^{-1}, \tag{4.53}$$

for every $t \in [0, \infty)$, the following (classical) Lemma says that also the derivatives of $A(t)$ are uniformly bounded.

Lemma 4.3.1. *Let $(M, \bar{g}, \vec{x})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold. Let $(\Sigma, g(t))$ be the volume preserving mean curvature flow of* Section 4.2 *in a time interval $[0, T]$. Then the second fundamental form and its derivatives remains bounded uniformly in $[0, T]$.*

Corollary 4.3.2. *Let $(M, \bar{g}, \vec{x})$ be a $C^2_{1/2+\delta}$-asymptotically flat manifold. Let $(\Sigma, g(t))$ be a surface of M evolving by volume preserving mean curvature flow of* Section 4.2*. Then there exists a constant $c > 0$, depending on σ, such that*

$$|\nabla H(t)|^2 \le c, \qquad |\nabla^2 H(t)|^2 \le c \qquad \forall t \in [0, \infty). \tag{4.54}$$

Proof. This follows immediately from Lemma 4.3.1, $|\nabla H(t)|^2 \le 2|\nabla A(t)|^2$ and $|\nabla^2 H(t)|^2 \le 2|\nabla^2 A(t)|^2$. □

We now prove that the speed of the flow goes to zero in L^2, for large times. Note that, integrating $\frac{\mathrm{d}}{\mathrm{d}t}|\Sigma_t|$ on $[0, t_0]$ we get

$$\int_0^{t_0} \int_\Sigma (H - h)^2 \, d\mu_t \, dt \le 10\pi\sigma^2. \tag{4.55}$$

This implies that $\|H - h\|_{L^2(\Sigma, \mu_t)} \to 0$ as $t \to \infty$, since $\frac{d}{dt} \int_\Sigma (H - h)^2 \, d\mu_t$ is bounded uniformly in t. Moreover, also the L^∞ norm of $H - h$ goes to zero, as the following Lemma shows.

Lemma 4.3.3. *Let $(M, \bar{g}, \vec{x})$ be a $C^2_{1/2+\delta}$-asymptotically flat manifold. Let $(\Sigma, g(t))$ be a surface of M evolving by volume preserving mean curvature flow of* Section 4.2*. Then there exists a constant $c > 0$, depending on σ, such that*

$$\|H - h\|_{L^\infty(\Sigma, \mu_t)} \le c\|H - h\|_{L^2(\Sigma, \mu_t)}, \qquad \forall t \in [0, \infty). \tag{4.56}$$

Proof. Using now the interpolation result of [Aub98, Thm. 3.69], with $p = q = r = 2 = n$, it follows that

$$\|\nabla H\|^2_{L^2(\Sigma, \mu_t)} \le \sqrt{2}\|H - h\|_{L^2(\Sigma, \mu_t)}\|\nabla^2 H\|_{L^2(\Sigma, \mu_t)}. \tag{4.57}$$

On the other hand, on Σ_t it holds the Sobolev inequality (see Corollary 2.3.5), with a constant c uniformly in t (since we are in the class of roundness) and thus

$$\|H - h\|_{L^\infty} \le c\sigma^{-\frac{1}{2}}\|H - h\|_{W^{1,4}} = c\sigma^{-\frac{1}{2}}\left(\|H - h\|_4 + \sigma\|\nabla H\|_4\right). \tag{4.58}$$

Since moreover $H - h$ has zero mean, the Poincaré inequality implies that $\|H - h\|_4 \le c\sigma\|\nabla H\|_4$, and thus

$$\|H - h\|_{L^\infty} \le c\|\nabla H\|_4 \le c\|\nabla H\|_2^{\frac{1}{2}}\|\nabla H\|_\infty^{\frac{1}{2}}, \tag{4.59}$$

when we let c to absorb the radius σ. We also used the Holder's interpolation inequality. Thus, using Corollary 4.3.2, we obtain

$$\|H - h\|_{L^\infty(\Sigma, \mu_t)} \le c\|H - h\|_{L^2(\Sigma, \mu_t)}. \tag{4.60}$$

□

Conclusions. At this point, Proposition 4.1.12 implies that also $\|H-h\|_{L^\infty(\Sigma,\mu_t)}$ converges exponentially to zero. Thus, for every $t>t_0$,

$$|F(x,t)-F(x,t_0)| = \left|\int_{t_0}^t \frac{\partial F}{\partial \tau}\, d\tau\right| \leq \int_{t_0}^t |H-h|\, d\tau \leq \bar{c}_\sigma \int_{t_0}^t e^{-\frac{E_{\mathrm{ADM}}}{\sigma^3}(\tau-t_0)}\, d\tau, \tag{4.61}$$

for some $\bar{c}_\sigma>0$. Thus, $F(\cdot,t)$ is bounded uniformly in t. Since the flow lives in $\mathcal{B}_\sigma(B_1,B_2,B_{\mathrm{cen}})$, also $h(t)$ is uniformly bounded in t. Thus, by standard arguments, see for example [Hui84] or the reference therein [Ham82, Lemma 14.2], the limit of F_t is a smooth immersion. Also $h(t)$ has a limit, since this is true along a sequence of times by the boundedness and also $\frac{d}{dt}h(t)$ is uniformly bounded in t. Finally, the fact that $H-h\to 0$ implies that the mean curvature of the limit of F_t equals the limit of $h(t)$. Since σ has been fixed at the beginning of the Section, we call Σ^σ this CMC-limit and $F^\sigma(t,\cdot)$ the immersion at time t. This concludes the proof of Theorem 1.1.2.

4.3.1 CMC-foliation

In the previous Section, we constructed, for every $\sigma>\sigma_0$ a *constant mean curvature* surface on $(M,\bar{\mathrm{g}})$, say $(\Sigma^\sigma, g^\sigma)$, where g^σ the pullback of $\bar{\mathrm{g}}$ through $\iota^\sigma(\cdot) := \lim_{t\to\infty} F^\sigma(t,\cdot)$. By the definition of roundness class, we have that there exists $C>0$, independent of σ, such that

$$|\vec{z}_{\Sigma^\sigma}| \leq C\sigma^{1-\delta}, \quad \|\mathring{A}\|_{L^\infty(\Sigma^\sigma)} \leq B_\infty \sigma^{-\frac{3}{2}-\delta}, \qquad |\Sigma^\sigma| \leq C\sigma^2, \qquad \frac{\sigma}{r_{\Sigma_\sigma}} \leq C, \tag{4.62}$$

$$|h_\sigma| \leq \left|h_\sigma - \frac{2}{\sigma}\right| + 2\sigma^{-1} \leq C\sigma^{-\frac{3}{2}-\delta} + 2\sigma^{-1} \overset{\sigma\to\infty}{\longrightarrow} 0. \tag{4.63}$$

Since, moreover, by Lemma 2.3.5 and Lemma 4.1.14, Σ^σ is a graph on the Euclidean sphere $\mathbb{S}_\sigma(\vec{z}_{\Sigma^\sigma})$, for every $\sigma\geq\sigma_0$ there exists a bijective map $\boldsymbol{F}_\sigma : \mathbb{S}_1(\vec{0}) \to \Sigma^\sigma$. Proceeding as in [Hua12, Section 5.3], using also (the spacelike version of) [CS21, Lemma 9] in order to show that the family $\{\Sigma^\sigma\}_{\sigma\geq\sigma_0}$ does not self intersect, we obtain a CMC-foliation of the asymptotic flat space $(M,\bar{\mathrm{g}})$. Moreover, thanks to the following remark, this foliation coincide with the one constructed by Nerz (because of the CMC-uniqueness in Nerz's roundness class, see [Ner15, Thm. 5.3] and [CS21, Thm. 4]).

CMC-surfaces are round in Nerz's sense. We end this Section showing that our foliation coincides with Nerz's foliation (and thus it is unique). We also recall the following definition from [Ner15].

Definition 4.3.4 (Nerz's class of roundness). *Fix $c_0\in[0,1)$, $c_1\geq 0$ and $\hat{\eta}\in(0,1]$. We say that $\Sigma\hookrightarrow M$ is asymptotically centered, $\Sigma\in\mathcal{A}^{\delta,\hat{\eta}}(c_0,c_1)$ if, setting $\mathrm{g}:=\mathrm{genus}(\Sigma)$, then*

$$|\vec{z}_\Sigma| \leq c_0\sigma_\Sigma + c_1\sigma_\Sigma^{1-\hat{\eta}}, \quad \sigma_\Sigma^{2+\hat{\eta}} \leq r_\Sigma^{\frac{5}{2}+\delta}, \qquad \int_\Sigma H^2\, d\mu - 16\pi(1-\mathrm{g}) \leq c_1 r_\Sigma^{-\hat{\eta}}. \tag{4.64}$$

Equations (4.62) and (4.63) imply that (4.64) holds with $c_0=0$, $c_1=C$ and $\hat{\eta}=\delta$. Moreover

$$\frac{\sigma^{2+\delta}}{r_\Sigma^{\frac{5}{2}+\delta}} = \left(\frac{\sigma}{r_\Sigma}\right)^{\frac{5}{2}+\delta} \sigma^{-\frac{1}{2}} \leq C\sigma^{-\frac{1}{2}}, \tag{4.65}$$

for $\sigma_0 = \sigma_0(C,\delta)$ large. Since genus$(\Sigma^\sigma) = 0$,

$$\begin{aligned}\int_{\Sigma^\sigma} H^2\, d\mu - 16\pi &= \int_{\Sigma^\sigma} H^2\, d\mu - 2\int_{\Sigma^\sigma} S_g\, d\mu = \\ &= 2\int_{\Sigma^\sigma} |\mathring{A}|^2\, d\mu + 4\int_{\Sigma^\sigma}\left(\overline{\mathrm{Ric}}(\nu,\nu) - \frac{\overline{\mathrm{S}}}{2}\right) d\mu \le C\sigma^{-\frac{1}{2}-\delta}\end{aligned} \tag{4.66}$$

possibly enlarging C. Thus $\Sigma^\sigma \in \mathcal{A}^{\delta,\delta}(0,C)$. Replacing the variable σ with $s := \frac{2}{h_{\Sigma^\sigma}}$, and using (4.63), [Ner15, Thm. 5.3] implies that our foliation coincides with the one constructed in [Ner15, Thm. 5.1].

4.3.2 The case of negative ADM-energy

We end this Chapter analyzing what happens to the flow when the mass of the system is negative. This scenario is interesting since the method employed by Nerz [Ner15] allows to prove the existence of a foliation also in this case of a negative ADM-mass. However, in Lemma 4.1.15 we observed that there is a technical obstruction in order to use the flow with the aim of constructing a foliation in the negative mass case, see equation (4.40). We remark now how this obstruction is not just technical but also substantial.

We start considering the case of negative ADM-mass when $(M,\overline{\mathrm{g}})$ is a Schwarzschild metric of mass $m<0$. That is,

$$\overline{\mathrm{g}}_S^m := \left(1+\frac{m}{2r}\right)^4 \overline{\mathrm{g}}^e,$$

where $r = r(\vec{x}) = |\vec{x}|$ for every $\vec{x}\in\mathbb{R}^3$ and $\overline{\mathrm{g}}^e$ is the Euclidean metric on $\mathbb{R}^3\setminus\{\vec{0}\}$. We consider the Euclidean sphere $\mathbb{S}_\sigma(\vec{0})$, which is a CMC. For sake of simplicity, we write $\mathbb{S}_\sigma = \mathbb{S}_\sigma(\vec{0})$. In particular, the family of immersions $\iota^\sigma : \mathbb{S}_\sigma \hookrightarrow M$, for $\sigma \ge 1$, generates a CMC-foliation of the Schwarzschild manifold. Since the Euclidean spheres have constant mean curvature, $\mathbb{S}_\sigma$ is a critical point of each volume preserving variation starting at $\mathbb{S}_\sigma$.

Following the classical approach (see for example [Hua12]), it can be proved that the *stability indicator* of $L : H^2(\mathbb{S}_\sigma) \to L^2(\mathbb{S}_\sigma)$, i.e.

$$\mu_0(L) := \inf\left\{\int_{\mathbb{S}_\sigma} (L\eta)\,\eta\, d\mu : \int_{\mathbb{S}_\sigma} \eta\, d\mu = 0,\ \|\eta\|_2 = 1\right\},$$

satisfies

$$\mu_0(L) \le -\frac{6|m|}{\sigma^3} + \mathrm{O}(\sigma^{-4}). \tag{4.67}$$

In particular, the right hand side of (4.67) is obtained choosing η as a coordinate translation in $\mathbb{R}^3$, restricted to $\mathbb{S}_\sigma$. We thus modify this translation into a volume preserving (normal) deformation setting

$$\begin{cases}\frac{\partial \hat{F}_t}{\partial \tau} = -(\eta - \hat{\eta})\nu \\ F_0(\cdot) = \iota^\sigma\end{cases} \tag{4.68}$$

with $\nu = \nu(x,\tau)$ the normal of $\hat{F}_\tau(\mathbb{S}_\sigma)$, $\eta = \eta(x)$ as above and

$$\hat{\eta} \equiv \hat{\eta}(\tau) := \frac{1}{|\hat{F}_\tau(\mathbb{S}_\sigma)|}\int_{\mathbb{S}_\sigma} \eta\, d\hat{\mu}_\tau,$$

where $d\hat{\mu}_\tau$ is the volume form on $\mathbb{S}_\sigma$ induced by the immersion $\hat{F}_\tau$. Note that $\hat{\eta}(0) = 0$, since η is a coordinate translation and $d\hat{\mu}_0$ is the standard *round* metric on the Euclidean sphere $\mathbb{S}_\sigma$. On the other hand, (4.68) is volume preserving for every τ for which it is defined, by construction. Thus, the latter formula in [Hua12, Pg. 9] implies that

$$\frac{d^2}{d\tau^2}\bigg|_{\tau=0} |\hat{F}_\tau(\mathbb{S}_\sigma)| = \int_{\mathbb{S}_\sigma} (\eta - \hat{\eta})\,|_{\tau=0} L(\eta - \hat{\eta})|_{\tau=0}\, d\mu < 0, \tag{4.69}$$

since $(\eta - \hat{\eta})\,(0) = \eta$, volume is constant (in order to let the additional term in (4.69) vanish) and (4.67). Thus, $\tau = 0$ is a point of local maximum for $\tau \mapsto |\hat{F}_\tau(\mathbb{S}_\sigma)|$, for τ in a small interval of time. In particular, $\hat{F}_\tau(\mathbb{S}_\sigma)$ perturbs the Euclidean sphere as a "volume preserving translation", at least at the first non-zero order. Thus there exists, arbitrarily near to $\mathbb{S}_\sigma$ in the C^∞-topology, a round surface $\Sigma = \hat{F}_{\overline{\tau}}(\mathbb{S}_\sigma)$, for $\overline{\tau}$ fixed but depending on how we want Σ to be near to $\mathbb{S}_\sigma$. In particular $|\Sigma| < |\mathbb{S}_\sigma|$. Thus if we consider the volume preserving mean curvature flow starting at Σ, i.e. $\Sigma_t = F_t(\Sigma)$, we find that this flow cannot converge to the Euclidean sphere $\mathbb{S}_\sigma$, since the flow is an area-non-increasing flow.

Observe moreover that $\mathbb{S}_\sigma$ is almost-CMC, in the sense of Definition 3.2.1, and thus Theorem 4.1.16 implies that this situation does not happen when the mass is positive.

Conclusions and open problems. In the case of a asymptotically Schwarzschildean manifold, Huisken-Yau proved in [HY96] that, in points of the surface where the Euclidean distance achieves its maximum, the (vector) speed of the flow, i.e. $-(H-h)\nu$, points toward the inside of the surface (respectively the outside) if the mass is positive (respectively negative). See [HY96, Prop. 2.2] and the dynamical approach of [HY96, Prop. 3.5]. This suggests that in the negative mass case an initially off-centered surface evolves drifting away.

If we show that, in the negative mass case, for *each direction* there exists a volume preserving translation of a sphere which drifts away in *that* direction along the volume preserving mean curvature flow, then we can conjecture the existence of a round surface that does not drift away, as in the positive case. In the general case of a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat manifold the situation could be more delicate, due to the presence of some additional terms when computing the stability operator on an Euclidean sphere, see Proposition 3.1.10. However, we think that this argument can be applied at least to the case of an asymptotically Schwarzschildean manifold in the sense of [HY96], providing a flow-proof of the existence of the foliation also in the negative mass case.

Chapter 5

Volume preserving spacetime mean curvature flow in initial data sets

In this Chapter we study a modification of the volume preserving mean curvature flow. Let us describe our setting in more detail. We consider an initial data set $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ which is $C^2_{\frac{1}{2}+\delta}$-asymptotically flat and constrained by the densities $(\overline{\mu}, \overline{\mathrm{J}})$. The *volume preserving spacetime mean curvature flow* (VPSTMCF) is a family of time dependent immersions $F : \Sigma \times [0, T] \to M$, with Σ a closed 2-surface, which evolves according to

$$\frac{\partial F}{\partial t}(t, \cdot) = -\left[\mathcal{H}(t, \cdot) - \hbar(t)\right] \nu(t, \cdot), \tag{5.1}$$

where $\hbar(t)$ is the integral average of $\mathcal{H}(t, \cdot)$, see Definition 2.1.2. Observe that, for $q = 2$, $\mathcal{H}$ is the spacetime mean curvature defined in [CS21]. As initial data for the flow (5.1), we consider a well-centered (in the sense of Nerz [Ner15]) CMC-surface. As in [HY96] and in the previous Chapter, the evolution is parametrized by a non-physical time parameter and takes place in a fixed spacelike slice, but now it has a speed that takes into account the spacetime texture of the initial data set. We aim to prove long-time existence of this flow, together with a convergence result. See the statement of Theorem 1.1.4 for details.

5.0.1 Definition of the flow and evolution equations

Definition 5.0.1. *Let $(M, \overline{\mathrm{g}}, \overline{\mathrm{K}})$ be an initial data set and let $\iota : \Sigma \hookrightarrow M$ be a closed surface. A time dependent family of immersions $F_t : \Sigma \hookrightarrow M$, with $t \in [0, T)$ for some $0 < T \leq \infty$, which satisfies*

$$\begin{cases} \frac{\partial}{\partial t} F_t(\cdot) = -\left(\mathcal{H}(\cdot, t) - \hbar(t)\right) \nu(\cdot, t) \\ F_0 = \iota \end{cases} \tag{5.2}$$

is called a solution to the volume preserving spacetime mean curvature flow*, with initial value ι.*

We highlight that the function $\mathcal{H}$ is an increasing function of the mean curvature H. The function $P = g^{ij}\overline{\mathrm{K}}_{ij}$ in $\mathcal{H} = \sqrt[q]{H^q - |P|^q}$ depends on the metric induced on Σ_t, which only involves first order derivatives of the immersion. Thus, without the volume preserving term, the equation is parabolic. However, this term only depends on time, and thus it does not affect the parabolicity and local existence of solutions and uniqueness are ensured.

In the following, we will assume that the ambient initial data set is $C^2_{\frac{1}{2}+\delta}$-asymptotically flat. We write $\Sigma_t := F_t(\Sigma)$ to denote the solution of the flow at time t and we call $g(t)$ the induced metric and denote by $d\mu_t$ the induced 2-dimensional measure.

We recall the evolution equations satisfied by the main geometric quantities on Σ_t. At each fixed t, we choose a frame $\{\vec{e}_\alpha(t)\}_{\alpha=1}^3$ on $(M, \overline{\mathrm{g}})$ such that $\{\vec{e}_1(t), \vec{e}_2(t)\}$ are tangent vectors on Σ_t and $\vec{e}_3(t) := \nu_t$. The following Lemma collects the equations satisfied by the main geometric quantities on Σ_t, see [HP99].

Lemma 5.0.2. *Let $\{F_t\}_{t\in[0,T)}$ be a solution to the flow (5.2). Then we have*

(i) $\frac{\partial g_{ij}}{\partial t} = -2\left(\mathcal{H}-\hbar\right)h_{ij}$;

(ii) $\frac{\partial}{\partial t}(d\mu_t) = -\left(\mathcal{H}-\hbar\right)H d\mu_t$;

(iii) $\frac{\partial}{\partial t}\nu = \nabla\mathcal{H}$;

(iv) $\frac{\partial}{\partial t}h_{ij} = \nabla_i\nabla_j\mathcal{H} + (\mathcal{H}-\hbar)\left(-h_{ik}h^k_j + \overline{\mathrm{Rm}}_{ikjl}\nu^k\nu^l\right)$;

(v) $\frac{\partial H}{\partial t} = \Delta\mathcal{H} + (\mathcal{H}-\hbar)\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)$.

Notation for the rest of the Section. In the following, it is convenient to set $\Phi = \Phi(s,\gamma) := \sqrt[q]{s^q - |\gamma|^q}$, so that $\mathcal{H} = \Phi(H,P)$. We denote by Υ the derivative of Φ computed with respect to the variable s, i.e. $\Upsilon := \partial_s\Phi\big|_{(s,\gamma)=(H,P)}$. On the other hand, we will denote by Ψ the derivative of Φ in time, due to the dependence on $P = P(t)$, i.e. $\Psi := \partial_t\left(\Phi(\rho,P(t))\right)\big|_{\rho=H}$. Thus,

$$\partial_t\left(\Phi(H,P)\right) = \Upsilon\partial_t H + \Psi. \tag{5.3}$$

This notation is particularly useful since we will mainly take trace of the term Υ. In the following, we will have

$$\Upsilon = \left(1-\left(\frac{|P|}{H}\right)^q\right)^{\frac{1}{q}-1}, \qquad \nabla\Upsilon = (q-1)\left(\frac{H}{\mathcal{H}}\right)^{q-2}\left(-\frac{|P|^q}{\mathcal{H}^2H^{q-1}}\nabla\mathcal{H} + \frac{1}{\mathcal{H}}\frac{|P|^{q-2}P}{H^{q-1}}\nabla P\right), \tag{5.4}$$

$$|\Psi| = \left|\frac{\partial\Phi(s,P)}{\partial t}\Big|_{s=H}\right| = \left|\frac{-q\Phi(H,P)}{(\Phi(H,P))^q}\frac{(\partial_t P)P}{|P|^{2-q}}\right| \le C\sigma^{q-1}|P|^{q-1}|\partial_t P| \le C\sigma^{\frac{1}{2}-\frac{1}{2}q-\delta q+\delta}|\partial_t P|, \tag{5.5}$$

where the latter inequality holds assuming (2.14), because of Lemma 2.1.3. Hypothesis (2.14) is natural in our setting since we will work solely on round surfaces. Note also that (5.4) implies

$$|\Upsilon - 1| \le C\left|\frac{P}{H}\right|^q = \mathrm{O}(\sigma^{-\frac{1}{2}q-q\delta}), \qquad \Upsilon - 1 \ge \underline{c}\sigma^q|P|^q. \tag{5.6}$$

Lemma 5.0.3. *There exists $C>0$ and $\sigma_0>0$ such that, if Σ_t satisfies $|A(t)| \le \sqrt{\frac{5}{2}}\sigma^{-1}$ and (2.14) for every $t\in[0,T]$, and $\sigma>\sigma_0$,*

$$|\partial_t P| \le C\sigma^{-\frac{5}{2}-\delta}|\mathcal{H}-\hbar| + C\sigma^{-\frac{3}{2}-\delta}|\nabla\mathcal{H}|. \tag{5.7}$$

Proof. We choose normal coordinates on a point x^* of Σ_{t^*}, for an arbitrary $t^*\in[0,T]$, say $\{x_1,x_2\}$, and normal coordinates $\{y_1,y_2,y_3\}$ on $y^* := F_{t^*}(x^*)$ in M. Thus, if $\{\frac{\partial F}{\partial x_i}\}_{i=1}^2$ is the frame induced by the immersion, we notice that

$$g_{ij} = (F^*\bar{\mathrm{g}})_{ij} = \bar{\mathrm{g}}\left(\frac{\partial F}{\partial x_i},\frac{\partial F}{\partial x_j}\right), \qquad \bar{\mathrm{K}}_{ij} = \left(F^*\bar{\mathrm{K}}\right)_{ij} = \bar{\mathrm{K}}\left(\frac{\partial F}{\partial x_i},\frac{\partial F}{\partial x_j}\right). \tag{5.8}$$

Thus in particular

$$\delta_{\alpha\beta}\frac{\partial F^\alpha}{\partial x_i}\frac{\partial F^\beta}{\partial x_j} = \bar{\mathrm{g}}\left(\frac{\partial F}{\partial x_i},\frac{\partial F}{\partial x_j}\right)\Big|_{(x^*,t^*)} = g_{ij}\big|_{(x_*,t_*)} = \delta_{ij}. \tag{5.9}$$

By direct computation, using the symmetry of $\overline{\mathrm{K}}$ we find that

$$\begin{aligned}\partial_t P = \partial_t\left(g^{ij}\overline{\mathrm{K}}_{ij}\right) &= 2(\mathcal{H}-\hbar)h^{ij}\overline{\mathrm{K}}_{ij} + \overline{\nabla}_\gamma\overline{\mathrm{K}}_{ij}\frac{\partial F^\gamma}{\partial t} + 2g^{ij}\overline{\mathrm{K}}\left(\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x_i}\right), \frac{\partial F}{\partial x_j}\right) \\ &= 2(\mathcal{H}-\hbar)h^{ij}\overline{\mathrm{K}}_{ij} + \overline{\nabla}_\gamma\overline{\mathrm{K}}_{ij}(\hbar-\mathcal{H})\nu^\gamma + 2g^{ij}\overline{\mathrm{K}}\left(\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x_i}\right), \frac{\partial F}{\partial x_j}\right).\end{aligned} \tag{5.10}$$

Since

$$\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial t}\right) = \frac{\partial}{\partial x_i}((\hbar-\mathcal{H})\nu) = -\frac{\partial \mathcal{H}}{\partial x_i}\nu + (\hbar-\mathcal{H})\frac{\partial \nu}{\partial x_i}, \tag{5.11}$$

we rewrite (5.10) as

$$\partial_t P = 2(\mathcal{H}-\hbar)h^{ij}\overline{\mathrm{K}}_{ij} + \overline{\nabla}_\gamma\overline{\mathrm{K}}_{ij}(\hbar-\mathcal{H})\nu^\gamma - 2g^{ij}\overline{\mathrm{K}}_{\alpha\beta}\nu^\alpha\frac{\partial\mathcal{H}}{\partial x_i}\frac{\partial F^\beta}{\partial x_j} + 2(\hbar-\mathcal{H})g^{ij}\overline{\mathrm{K}}_{\alpha\beta}\frac{\partial\nu^\alpha}{\partial x_i}\frac{\partial F^\beta}{\partial x_j}. \tag{5.12}$$

Note that, in normal coordinates, the Weingarten equation takes the form

$$\left.\frac{\partial \nu^\alpha}{\partial x_i}\right|_{(x^*,t^*)} = h_i^j(x^*,t^*)\left.\frac{\partial F^\alpha}{\partial x_i}\right|_{(x^*,t^*)}, \tag{5.13}$$

see [HP99, Pg. 63]. Thus, computing (5.12) in the point (x^*,t^*), and estimating, we get

$$|\partial_t P| \le C|\mathcal{H}-\hbar||A||\overline{\mathrm{K}}| + |\overline{\nabla}\ \overline{\mathrm{K}}||\mathcal{H}-\hbar| + C|\overline{\mathrm{K}}||\nabla\mathcal{H}|, \tag{5.14}$$

where we used (5.13) combined with (5.9) in order to estimate the latter term in (5.12). We conclude using that, thanks to the assumption on $|A(t)|$ and (2.14), Lemma 2.1.3 implies that $|\overline{\mathrm{K}}| \le C\sigma^{-\frac{3}{2}-\delta}$ and $|A||\overline{\mathrm{K}}| + |\overline{\nabla}\ \overline{\mathrm{K}}| \le C\sigma^{-\frac{5}{2}-\delta}$. □

The Φ-notation, together with helping us avoiding huge formulas in the following, highlights that existence and convergence of the flow could be studied in the case of more general speed functions. However, we just focus our attention on the spacetime flow. We also define $\alpha : (0,1) \to \mathbb{R}$ to be $\alpha(\rho) := \sqrt[q]{1-\rho^q}$, so that $\Phi(s,\gamma) = s\alpha\left(\frac{|\gamma|}{s}\right)$.

Lemma 5.0.4. *Along a solution of the volume preserving spacetime mean curvature flow we have*

$$\begin{aligned}\frac{\partial}{\partial t}|\mathring{A}|^2 &= \Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + \frac{2\hbar}{H}\left\{|A|^4 - H\mathrm{tr}(A^3)\right\} + 2|A|^2\left(\frac{H-\hbar}{H}\right)|\mathring{A}|^2 \\ &\quad + 2(\mathcal{H}-\hbar)\mathring{h}_{ij}\overline{\mathrm{Rm}}_{kilj}\nu^k\nu^l - 2\left(h_i^l\overline{\mathrm{Rm}}_{kjkl} + h^{lk}\overline{\mathrm{Rm}}_{lijk}\right)h_{ij} \\ &\quad - 2\left(\nabla_j\left(\overline{\mathrm{Ric}}_{i\varepsilon}\nu^\varepsilon\right) + \nabla_l\left(\overline{\mathrm{Rm}}_{\varepsilon ijl}\nu^\varepsilon\right)\right)\mathring{h}_{ij} + 2|A|^2\left(\frac{H^2-H\mathcal{H}}{2}\right) \\ &\quad + 2(\mathcal{H}-H)\mathrm{tr}(A^3) + \langle\mathcal{T},\mathring{A}\rangle;\end{aligned} \tag{5.15}$$

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla\mathcal{H}|^2 &= \Delta|\nabla\mathcal{H}|^2 - 2|\nabla^2\mathcal{H}|^2 + 2(\mathcal{H}-\hbar)h^{ij}\nabla_i\mathcal{H}\nabla_j\mathcal{H} + 2(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))|\nabla\mathcal{H}|^2 \\ &\quad - 2\mathrm{Ric}^\Sigma(\nabla\mathcal{H},\nabla\mathcal{H}) + 2(\mathcal{H}-\hbar)\langle\nabla|A|^2,\nabla\mathcal{H}\rangle + 2(\mathcal{H}-\hbar)\langle\nabla\left(\overline{\mathrm{Ric}}(\nu,\nu)\right),\nabla\mathcal{H}\rangle \\ &\quad + 2g^{ij}\nabla_i\left((\Upsilon-1)\left(\Delta\mathcal{H} + (\mathcal{H}-\hbar)(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\right)\right)\nabla_j\mathcal{H} + 2g^{ij}\nabla_i\mathcal{H}\nabla_j\mathcal{H}\end{aligned} \tag{5.16}$$

where Ric^Σ *is the Ricci tensor of* Σ, $\beta := \alpha - 1$ *and* $\mathcal{T} := (T_{ij})$ *is the tensor defined by*

$$\begin{aligned} T_{ij} :=& (\nabla_i\nabla_j H)\,\beta\left(\frac{|P|}{H}\right) + \nabla_j H\beta'\left(\frac{|P|}{H}\right)\nabla_i\left(\frac{|P|}{H}\right) + \nabla_i H\beta'\left(\frac{|P|}{H}\right)\nabla_j\left(\frac{|P|}{H}\right) \\ &+ H\beta''\left(\frac{|P|}{H}\right)\nabla_i\left(\frac{|P|}{H}\right)\nabla_j\left(\frac{|P|}{H}\right) + H\beta'\left(\frac{|P|}{H}\right)\nabla_i\nabla_j\left(\frac{|P|}{H}\right). \end{aligned} \tag{5.17}$$

The proof is standard, and it mainly relies on the computations in [Hui87] and [HP99]. See moreover Lemma 4.1.4. Observe that the tensor T is the remainder of the Hessian of the function Φ, which, due to the introduction of the auxiliary functions α and β, is given by $\mathrm{Hess}(H)$ plus the tensor T. Finally, an easy computation shows that, since $q \geq 2$,

$$|\beta(\rho)| \leq c_q\rho^2, \qquad |\beta'(\rho)| \leq c_q\rho, \qquad |\beta''(\rho)| \leq c_q, \tag{5.18}$$

for $\rho << 1$, which is the case we are interested in, since $\rho \sim \frac{|P|}{H}$ which is small on a round surface.

Proof. Using Lemma 5.0.2, we get

$$\frac{\partial}{\partial t}h_{ij} = \nabla_i\nabla_j\left(\Phi(H,P)\right) + \left(\Phi(H,P) - \hbar\right)\left(-h_{ik}h^k_j + \overline{\mathrm{Riem}}_{i3j3}\right). \tag{5.19}$$

By $\Phi(H,P) = H\alpha\left(\frac{|P|}{H}\right)$, we have

$$\begin{aligned} \nabla_i\nabla_j\left(\Phi(H,\cdot)\right) =& \ (\nabla_i\nabla_j H)\,\alpha\left(\frac{|P|}{H}\right) + \nabla_j H\alpha'\left(\frac{|P|}{H}\right)\nabla_i\left(\frac{|P|}{H}\right) + \nabla_i H\alpha'\left(\frac{|P|}{H}\right)\nabla_j\left(\frac{|P|}{H}\right) \\ &+ H\alpha''\left(\frac{|P|}{H}\right)\nabla_i\left(\frac{|P|}{H}\right)\nabla_j\left(\frac{|P|}{H}\right) + H\alpha'\left(\frac{|P|}{H}\right)\nabla_i\nabla_j\left(\frac{|P|}{H}\right) \end{aligned} \tag{5.20}$$

We moreover define β as above, obtaining $\beta' = \alpha'$ and $\beta'' = \alpha''$. We thus get

$$\nabla_i\nabla_j\left(\Phi(H,P)\right) = \nabla_i\nabla_j H + T_{ij}. \tag{5.21}$$

Then (5.19) becomes

$$\begin{aligned} \frac{\partial}{\partial t}h_{ij} &= \nabla_i\nabla_j H + (\Phi(H,\cdot) - \hbar)\left(-h_{ik}h^k_j + \overline{\mathrm{Riem}}_{i3j3}\right) + T_{ij} \\ &= \Delta h_{ij} - Hh^l_i h_{lj} + |A|^2 h_{ij} + (\Phi(H,\cdot) - \hbar)\left(-h_{ik}h^k_j + \overline{\mathrm{Riem}}_{i3j3}\right) \\ &\quad - h^l_i\overline{\mathrm{Rm}}_{kjkl} - h^{lk}\overline{\mathrm{Rm}}_{lijk} - \nabla_j\left(\overline{\mathrm{Ric}}_{i\varepsilon}\nu^\varepsilon\right) - \nabla^l\left(\overline{\mathrm{Rm}}_{\varepsilon ijl}\nu^\varepsilon\right) + T_{ij} \end{aligned} \tag{5.22}$$

The conclusion follows remarking that

$$\begin{aligned} &\Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + 2|A|^2\left(|A|^2 - \frac{H\Phi(H,P)}{2}\right) - 2\hbar\mathrm{tr}(A^3) + H|A|^2\hbar \\ &+ 2(\Phi(H,P) - H)\mathrm{tr}(A^3) + \frac{2\hbar}{H}|A|^4 - \frac{2\hbar}{H}|A|^4 \\ =& \ \Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + \frac{2\hbar}{H}|A|^4 - 2\hbar\mathrm{tr}(A^3) + 2|A|^2\left(1 - \frac{\hbar}{H}\right)\left(|A|^2 - \frac{H^2}{2}\right) \\ &+ 2|A|^2\left(\frac{H^2}{2} - \frac{H\Phi(H,P)}{2}\right) + 2(\Phi(H,P) - H)\mathrm{tr}(A^3) \end{aligned} \tag{5.23}$$

and proceeding as in Lemma 4.1.4.

Finally, equation (5.16) follows from Lemma 5.0.2 and the Bochner formula. □

5.0.2 Evolution of integral quantities

We now study the evolution of some integral quantities along the flow. Throughout the subsection, $F_t : \Sigma \hookrightarrow M$ will be a solution to the volume preserving spacetime mean curvature flow (5.2), in a constrained initial data set $(M, \overline{g}, \overline{K})$, with $t \in [0, T]$ for some $T > 0$. We will assume that the surfaces Σ_t satisfy properties (2.29) and (2.30) of round surfaces for some given suitably large radius σ. In some results, we further assume

$$\|H - h\|_{L^\infty(\Sigma_t)} \leq B_\infty \sigma^{-\frac{3}{2}-\delta}, \qquad \left\| \overset{\circ}{A}(t) \right\|_{L^\infty(\Sigma)} \leq B_\infty \sigma^{-\frac{3}{2}-\delta}, \tag{5.24}$$

which are properties satisfied by round surfaces, see Lemma 2.3.5, and also

$$\|\mathcal{H} - \hbar\|_{H^1(\Sigma_t)} \leq c_{\text{in}} \sigma^{-\frac{q}{2}-q\delta}. \tag{5.25}$$

We do not assume apriori that Σ_t satisfy properties (2.31) and (2.32). We want to analyze the invariance of these properties along the flow. We start estimating the L^4 norm of the traceless second fundamental form of Σ_t. In this result, we replace hypothesis (5.24) by a milder assumption.

Proposition 5.0.5. *Let $\{F_t\}_{t\in[0,T]}$ be a solution to the flow satisfying (2.29) and (2.30). Suppose in addition*

$$\|H - h\|_{L^\infty(\Sigma_t)} \leq \frac{1}{20\sigma}; \tag{5.26}$$

Then there exist a constant $C = C(\overline{c}, \delta) > 0$ and a radius $\sigma_0 = \sigma_0(\delta, \overline{c}) > 0$ such that if $\sigma > \sigma_0$ then

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_\Sigma |\overset{\circ}{A}|^4 \, d\mu_t \leq -2 \int_\Sigma |\overset{\circ}{A}|^2 |\nabla \overset{\circ}{A}|^2 \, d\mu_t - \frac{1}{2\sigma^2} \int_\Sigma |\overset{\circ}{A}|^4 \, d\mu_t + C\sigma^{-6-4\delta}. \tag{5.27}$$

As a consequence, if $\int_\Sigma |\overset{\circ}{A}|^4 \, d\mu_0 < B_1 \sigma^{-4-4\delta}$ and $B_1 > 2C$, then $\int_\Sigma |\overset{\circ}{A}|^4 \, d\mu_t < B_1 \sigma^{-4-4\delta}$ for every $t \in [0, T]$.

Proof. The proof is an adaptation of the proof of Lemma 4.1.6 combined with Lemma 2.1.3, Lemma 5.0.4 and the fact that there exists C such that, for every fixed $\varepsilon > 0$ and σ_0 suitably large it holds

$$\left| \int_\Sigma \langle \mathcal{T}, \overset{\circ}{A} \rangle |\overset{\circ}{A}|^2 \, d\mu \right| \leq \frac{\varepsilon}{\sigma^2} \int_\Sigma |\overset{\circ}{A}|^4 \, d\mu + \varepsilon \int_\Sigma |\overset{\circ}{A}|^2 |\nabla \overset{\circ}{A}|^2 \, d\mu + C\sigma^{-6-4\delta}. \tag{5.28}$$

We thus prove (5.28).

Multiplying equation (5.17) by $\overset{\circ}{h}_{ij} |\overset{\circ}{A}|^2$ and integrating we get

$$\begin{aligned} \int_\Sigma T_{ij} \overset{\circ}{h}_{ij} |\overset{\circ}{A}|^2 \, d\mu = \int_\Sigma \Big\{ & \nabla_i \nabla_j H \beta \left(\frac{|P|}{H} \right) + \nabla_j H \beta' \left(\frac{|P|}{H} \right) \nabla_i \left(\frac{|P|}{H} \right) + \nabla_i H \beta' \left(\frac{|P|}{H} \right) \nabla_j \left(\frac{|P|}{H} \right) \\ & + H \beta'' \left(\frac{|P|}{H} \right) \nabla_i \left(\frac{|P|}{H} \right) \nabla_j \left(\frac{|P|}{H} \right) + H \beta' \left(\frac{|P|}{H} \right) \nabla_i \nabla_j \left(\frac{|P|}{H} \right) \Big\} \overset{\circ}{h}_{ij} |\overset{\circ}{A}|^2 \, d\mu. \end{aligned}$$

Integration by parts, Lemma 2.1.3 and $\nabla_i\left(\mathring{h}_{ij}|\mathring{A}|^2\right) = \nabla_i\mathring{h}_{ij}|\mathring{A}|^2 + 2\mathring{h}_{ij}|\mathring{A}|\nabla_i|\mathring{A}|$ imply

$$\left|\int_\Sigma (\nabla_i\nabla_j H)\beta\left(\frac{P}{H}\right)\mathring{h}_{ij}|\mathring{A}|^2\,d\mu\right| \leq \quad C\sigma^{-\frac{1}{2}-\delta}\int_\Sigma \nabla_j H\left|\frac{(\nabla_i P)H - P\nabla_i H}{H^2}\right||\mathring{A}|^3\,d\mu + C\sigma^{-1-2\delta}\int_\Sigma |\nabla H||\mathring{A}|^2|\nabla\mathring{A}|\,d\mu. \tag{5.29}$$

Using again Lemma 2.1.3, the parametric Young's inequality, $|\nabla H|^2 \leq C|\nabla\mathring{A}|^2 + C|\overline{\mathrm{Ric}}|^2$ (see [Hui86]) and (1.11) we get

$$\left|\int_\Sigma (\nabla_i\nabla_j H)\beta\left(\frac{P}{H}\right)\mathring{h}_{ij}|\mathring{A}|^2\,d\mu\right| \leq \frac{\varepsilon}{\sigma^2}\int_\Sigma |\mathring{A}|^4\,d\mu + \varepsilon\int_\Sigma |\mathring{A}|^2|\nabla\mathring{A}|^2\,d\mu + C\sigma^{-6-4\delta} \tag{5.30}$$

Moreover, we estimate

$$\left|\int_\Sigma (\nabla_i H)\beta'\left(\frac{P}{H}\right)\nabla_j\left(\frac{P}{H}\right)\mathring{h}_{ij}|\mathring{A}|^2\,d\mu\right| \leq C\sigma^{-2-2\delta}\int_\Sigma\left(|\nabla H||\mathring{A}|\right)|\mathring{A}|^2\,d\mu + C\sigma^{-2\delta}\int_\Sigma |\nabla H|^2|\mathring{A}|^3\,d\mu,$$

The second addend can be estimated as in (5.30), while the first addend, using Young's inequality, is bounded by

$$C\sigma^{-2-2\delta}\int_\Sigma\left(\frac{|\nabla H|^2}{2} + \frac{|\mathring{A}|^2}{2}\right)|\mathring{A}|^2\,d\mu. \tag{5.31}$$

Again, the first addend of (5.31) can be treated as in (5.30), for σ large.

We can also bound the term

$$\left|\int_\Sigma H\beta''\left(\frac{P}{H}\right)\nabla_i\left(\frac{P}{H}\right)\nabla_j\left(\frac{P}{H}\right)\mathring{h}_{ij}|\mathring{A}|^2\,d\mu\right| \leq C\sigma^{-2\delta}\int_\Sigma |\nabla H|^2|\mathring{A}|^2|\mathring{A}|\,d\mu + C\sigma^{-4-2\delta}\int_\Sigma |\mathring{A}|^3\,d\mu + 2\int_\Sigma \frac{|P||\nabla H||\nabla P|}{H^2}|\mathring{A}|^3\,d\mu.$$

We conclude as in (5.30), also using $|\mathring{A}| \leq C\sigma^{-\frac{1}{2}}$, for σ large and Young's inequality.

Finally, integrating by parts and using the decay of β we get

$$\left|\int_\Sigma H\beta'\left(\frac{P}{H}\right)\nabla_i\nabla_j\left(\frac{P}{H}\right)\mathring{h}_{ij}|\mathring{A}|^2\,d\mu\right| \leq C\sigma^{-\frac{1}{2}-\delta}\int_\Sigma |\nabla H|\left|\frac{(\nabla P)H - (\nabla H)P}{H^2}\right||\mathring{A}|^3\,d\mu + C\int_\Sigma H\left|\frac{(\nabla P)H - (\nabla H)P}{H^2}\right|^2|\mathring{A}|^3\,d\mu + C\sigma^{-\frac{1}{2}-\delta}\int_\Sigma H\left|\frac{(\nabla P)H - (\nabla H)P}{H^2}\right|\left|\nabla|\mathring{A}|\right||\mathring{A}|^2\,d\mu. \tag{5.32}$$

The first addend can be dealt with as (5.30), while the second as in (5.31). The third addend in (5.32) is bounded by

$$C\sigma^{-\frac{1}{2}-\delta}\int_\Sigma |\nabla P|\left|\nabla|\mathring{A}|\right| |\mathring{A}|^2\; d\mu + C\sigma^{-1-2\delta}\int_\Sigma |\nabla H|\left|\nabla|\mathring{A}|\right| |\mathring{A}|^2\; d\mu,$$

and we conclude by Young's inequality, combined with the estimate $|\nabla H|^2 \leq C|\nabla\mathring{A}|^2 + C|\overline{\mathrm{Ric}}|^2$. □

We next estimate the rate of change of the volume preserving term $\hbar(t)$ and of the L^4 norm of $\mathcal{H}-\hbar$. In particular the following Lemma employs some techniques learned in [Li09].

Lemma 5.0.6. *Let (Σ, F_t) be a solution to the volume preserving spacetime mean curvature flow for $t\in[0,T]$, satisfying properties (2.29), (2.30), (5.24) and (5.25). Then, there exist $C = C(\overline{c}) > 0$ and $\sigma_0 = \sigma_0(\overline{c}) > 1$, such that, if $\sigma > \sigma_0$,*

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Sigma (\mathcal{H}-\hbar)^2\; d\mu_t \leq -\frac{1}{2}\int_\Sigma |\nabla\mathcal{H}|^2\; d\mu_t + C\sigma^{-2}\int_\Sigma(\mathcal{H}-\hbar)^2\; d\mu_t. \tag{5.33}$$

Moreover, there exists a constant $c = c(\mathrm{c_{in}}, \overline{c}) > 0$ and a universal constant $C = C(\overline{c}) > 0$ such that

$$|\dot{\hbar}(t)| \leq c\sigma^{-\frac{7}{2}-\frac{1}{2}q-\delta-\delta q}, \tag{5.34}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Sigma (\mathcal{H}-\hbar)^4\; d\mu_t \leq -12\int_\Sigma (\mathcal{H}-\hbar)^2|\nabla\mathcal{H}|^2\; d\mu_t + C\sigma^{-2}\int_\Sigma (\mathcal{H}-\hbar)^4\; d\mu_t + cB_\infty\sigma^{-5-\frac{3}{2}q-2\delta-3\delta q},$$

provided $\sigma \geq \sigma_0$, for a suitably $\sigma_0 = \sigma_0(B_\infty, \mathrm{c_{in}}, \overline{c}, \delta)$.

Proof. We first prove (5.34). By definition of $\hbar$, we get

$$\begin{aligned}
|\Sigma_t|\dot{\hbar}(t) &= \int_\Sigma\left(\frac{\partial H}{\partial t} + (\Upsilon-1)\frac{\partial H}{\partial t} + \Psi\right)\; d\mu_t + \int_\Sigma \mathcal{H}H(\hbar-\mathcal{H})\; d\mu_t - \hbar\int_\Sigma H(\hbar-\mathcal{H})\; d\mu_t \\
&= \int_\Sigma(\mathcal{H}-\hbar)\left(|\mathring{A}|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)\; d\mu_t + \int_\Sigma\left(\frac{H^2}{2} - \mathcal{H}H\right)(\mathcal{H}-\hbar)\; d\mu_t \\
&\quad - \hbar\int_\Sigma H(\hbar-\mathcal{H})\; d\mu_t + \int_\Sigma\left((\Upsilon-1)\frac{\partial H}{\partial t} + \Psi\right)\; d\mu_t \\
&= \int_\Sigma(\mathcal{H}-\hbar)\left(|\mathring{A}|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)\; d\mu_t - \int_\Sigma \frac{H^2}{2}(\mathcal{H}-\hbar)\; d\mu_t \\
&\quad - \int_\Sigma(\mathcal{H}-H)H(\mathcal{H}-\hbar)\; d\mu_t - \hbar\int_\Sigma H(\hbar-\mathcal{H})\; d\mu_t + \int_\Sigma\left((\Upsilon-1)\frac{\partial H}{\partial t} + \Psi\right)\; d\mu_t \\
&= \int_\Sigma(\mathcal{H}-\hbar)\left(|\mathring{A}|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)\; d\mu_t - \int_\Sigma \frac{\mathcal{H}^2}{2}(\mathcal{H}-\hbar)\; d\mu_t \\
&\quad + \int_\Sigma\left(\frac{\mathcal{H}^2-H^2}{2}\right)(\mathcal{H}-\hbar)\; d\mu_t - \int_\Sigma(\mathcal{H}-H)H(\mathcal{H}-\hbar)\; d\mu_t - \hbar\int_\Sigma H(\hbar-\mathcal{H})\; d\mu_t \\
&\quad + \int_\Sigma\left((\Upsilon-1)\frac{\partial H}{\partial t} + \Psi\right)\; d\mu_t
\end{aligned}$$

To estimate the above terms, we first note that

$$\left|\int_\Sigma(\mathcal{H}-\hbar)\left(|\mathring{A}|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)\; d\mu_t\right| \leq c\sigma^{-\frac{3}{2}-\frac{q}{2}-q\delta}, \tag{5.35}$$

and that, in addition, the following identity holds

$$
\begin{aligned}
&-\int_\Sigma \frac{\mathcal{H}^2}{2}(\mathcal{H}-\hbar)\ d\mu_t - \hbar\int_\Sigma H(\hbar-\mathcal{H})\ d\mu_t \\
&= -\int_\Sigma \frac{\mathcal{H}^2}{2}(\mathcal{H}-\hbar)\ d\mu_t + \hbar\int_\Sigma (\mathcal{H}-\hbar)^2\ d\mu_t - \hbar\int_\Sigma \{(H-h)-(\mathcal{H}-\hbar)\}(\hbar-\mathcal{H})\ d\mu_t \\
&= -\frac{1}{2}\int_\Sigma (\mathcal{H}-\hbar)^3\ d\mu_t - \hbar\int_\Sigma \{(H-h)-(\mathcal{H}-\hbar)\}(\hbar-\mathcal{H})\ d\mu_t,
\end{aligned}
\tag{5.36}
$$

using also $\int_\Sigma (\mathcal{H}-\hbar)\ d\mu_t = 0$, where, thanks to Lemma 2.1.3,

$$
\left|\hbar\int_\Sigma \{(H-h)-(\mathcal{H}-\hbar)\}(\hbar-\mathcal{H})\ d\mu_t\right| \le c\sigma^{-1-q-2q\delta}. \tag{5.37}
$$

Since the remaining addend can be estimated in a similar way to (5.37), we get

$$
|\Sigma_t||\dot{\hbar}(t)| \le \left|\frac{1}{2}\int_\Sigma (\mathcal{H}-\hbar)^3\ d\mu_t\right| + c\sigma^{-\frac{3}{2}-\frac{q}{2}-q\delta} + \left|\int_\Sigma \left((\Upsilon-1)\frac{\partial H}{\partial t}+\Psi\right)\ d\mu_t\right| \tag{5.38}
$$

Observe that the term $\int_\Sigma (\mathcal{H}-\hbar)^3\ d\mu_t$ can be easily bounded using (5.24) and (5.25). Finally, we estimate

$$
\begin{aligned}
&\int_\Sigma \left((\Upsilon-1)\frac{\partial H}{\partial t}+\Psi\right)\ d\mu_t \\
&\qquad = -\int_\Sigma \nabla\Upsilon\cdot\nabla\mathcal{H}\ d\mu_t + \int_\Sigma (\Upsilon-1)\left(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu)\right)(\mathcal{H}-\hbar)\ d\mu_t + \int_\Sigma \Psi\ d\mu_t.
\end{aligned}
$$

To estimate this term, we observe that equation (5.4), together with the inequalities

$$
\left|\frac{P^p}{\mathcal{H}^2 H^{p-1}}\right| \le c_q\sigma^{1-\frac{1}{2}q-q\delta}, \qquad \left|\frac{1}{\mathcal{H}}\frac{|P|^{q-2}P}{H^{q-1}}\right| \le c_q\sigma^{\frac{3}{2}-\frac{1}{2}q-\delta(q-1)},
$$

imply

$$
\left|\int_\Sigma \nabla\Upsilon\cdot\nabla\mathcal{H}\ d\mu_t\right| \le c\sigma^{-\frac{3}{2}-\frac{q}{2}-\delta q-\delta}.
$$

Similarly, we also have

$$
\left|\int_\Sigma (\Upsilon-1)\left(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu)\right)(\mathcal{H}-\hbar)\ d\mu_t\right| \le c\sigma^{-1-q-q\delta}. \tag{5.39}
$$

We conclude combining (5.5), (5.7) and assumption (5.25), in order to get

$$
\int_\Sigma |\Psi|\ d\mu_t \le c\sigma^{-1-\frac{1}{2}q-\delta q}\|\mathcal{H}-\hbar\|_{L^2(\Sigma_t)} + c\sigma^{-\frac{1}{2}q-\delta q}\|\nabla\mathcal{H}\|_{L^2(\Sigma_t)} \le c\sigma^{-1-q-2\delta q}. \tag{5.40}
$$

Equation (5.34) follows dividing by $|\Sigma_t| \ge (7/2)\pi\sigma^2$.

We now prove (5.33). We compute the evolution of $\|\mathcal{H}-\hbar\|^2_{L^2(\Sigma,\mu_t)}$, obtaining

$$
\frac{\mathrm{d}}{\mathrm{dt}}\int_\Sigma (\mathcal{H}-\hbar)^2\ d\mu_t = 2\int_\Sigma (\mathcal{H}-\hbar)\left(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu))\right)\ d\mu_t
$$

$$+2\int_\Sigma (\mathcal{H}-\hbar)(\Upsilon-1)\left(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu))\right)\,d\mu_t$$
$$+2\int_\Sigma \Psi(\mathcal{H}-\hbar)\,d\mu_t-\int_\Sigma H(\mathcal{H}-\hbar)^3\,d\mu_t.$$

Using integration by parts, the estimate $H+|H-h|+|A|\le C\sigma^{-1}$, and the inequality

$$\begin{aligned}\left|\int_\Sigma (\mathcal{H}-\hbar)(\Upsilon-1)\Delta\mathcal{H}\,d\mu_t\right| &= \left|-\int_\Sigma (\Upsilon-1)|\nabla\mathcal{H}|^2\,d\mu_t-\int_\Sigma(\mathcal{H}-\hbar)\nabla\Upsilon\nabla\mathcal{H}\,d\mu_t\right|\\ &\le \varepsilon\int_\Sigma|\nabla\mathcal{H}|^2\,d\mu_t+C\int_\Sigma\sigma^{-\frac{1}{2}q+1-q\delta}|\mathcal{H}-\hbar||\nabla\mathcal{H}|^2\,d\mu_t \\ &\quad + C\int_\Sigma\sigma^{-\frac{1}{2}q-1-q\delta}|\mathcal{H}-\hbar||\nabla\mathcal{H}|\,d\mu_t\end{aligned} \tag{5.41}$$

and Young's inequality we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma(\mathcal{H}-\hbar)^2\,d\mu_t\le-(2-2\varepsilon)\int_\Sigma|\nabla\mathcal{H}|^2\,d\mu_t+C\sigma^{-2}\int_\Sigma(\mathcal{H}-\hbar)^2\,d\mu_t, \tag{5.42}$$

where we estimated $\int_\Sigma \Psi(\mathcal{H}-\hbar)\,d\mu_t$ combining (5.5) and (5.7), i.e.

$$\begin{aligned}\left|\int_\Sigma(\mathcal{H}-\hbar)\Psi\,d\mu_t\right| &\le C\left(\sigma^{-2-\frac{1}{2}q-\delta q}\int_\Sigma(\mathcal{H}-\hbar)^2\,d\mu+\sigma^{-1-\frac{1}{2}q-\delta q}\int_\Sigma|\mathcal{H}-\hbar||\nabla\mathcal{H}|\,d\mu\right)\\ &\le C\sigma^{-2}\int_\Sigma(\mathcal{H}-\hbar)^2\,d\mu+\varepsilon\int_\Sigma|\nabla\mathcal{H}|^2\,d\mu.\end{aligned} \tag{5.43}$$

We conclude choosing ε suitably small.

We finally compute, using Lemma 5.0.2, the evolution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma(\mathcal{H}-\hbar)^4\,d\mu_t &= 4\int_\Sigma(\mathcal{H}-\hbar)^3\left(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu))-\dot{\hbar}\right)\,d\mu_t\\ &\quad+4\int_\Sigma(\mathcal{H}-\hbar)^3(\Upsilon-1)(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu)))\,d\mu_t\\ &\quad-\int_\Sigma H(\mathcal{H}-\hbar)^5\,d\mu+\int_\Sigma\Psi(\mathcal{H}-\hbar)^3\,d\mu_t.\end{aligned}$$

We obtain the desired inequality (5.0.6) using integration by parts for the term

$$\int_\Sigma(\mathcal{H}-\hbar)^3\Delta\mathcal{H}\,d\mu_t \tag{5.44}$$

as in Lemma 4.1.8, together with the estimate

$$|\dot{\hbar}|\int_\Sigma|\mathcal{H}-\hbar|^3\le c\sigma^{-\frac{7}{2}-\frac{1}{2}q-\delta-\delta q}\left(B_\infty\sigma^{-\frac{3}{2}-\delta}\right)\|\mathcal{H}-\hbar\|_2^2\le cB_\infty\sigma^{-5-\frac{3}{2}q-2\delta-3\delta q},$$

and

$$\begin{aligned}&\left|\int_\Sigma(\mathcal{H}-\hbar)^3(\Upsilon-1)\Delta\mathcal{H}\,d\mu\right|\\ &\le c\sigma^{-\frac{1}{2}q-q\delta}\int_\Sigma(\mathcal{H}-\hbar)^2|\nabla\mathcal{H}|^2\,d\mu+\left|-\int_\Sigma(\mathcal{H}-\hbar)^3\nabla\Upsilon\nabla\mathcal{H}\,d\mu\right|\\ &\le c\sigma^{-\frac{1}{2}q-q\delta}\int_\Sigma(\mathcal{H}-\hbar)^2|\nabla\mathcal{H}|^2\,d\mu+c\int_\Sigma\sigma^{-\frac{q}{2}-1-q\delta}|\nabla\mathcal{H}||\mathcal{H}-\hbar|^3\,d\mu.\end{aligned}$$

Finally, we conclude combining (5.5) and (5.7), and thus estimating

$$\begin{aligned}\left|\int_\Sigma \Psi(\mathcal{H}-\hbar)^3\, d\mu_t\right| &\leq C\sigma^{\frac{1}{2}-\frac{1}{2}q-\delta q+\delta}\int_\Sigma \sigma^{-\frac{5}{2}-\delta}|\mathcal{H}-\hbar|^4+\sigma^{-\frac{3}{2}-\delta}|\mathcal{H}-\hbar|^3|\nabla\mathcal{H}|\, d\mu_t\\ &\leq C\sigma^{-2}\int_\Sigma(\mathcal{H}-\hbar)^4\, d\mu_t + C\sigma^{-1-\frac{1}{2}q-\delta q}\int_\Sigma(\mathcal{H}-\hbar)^2|\mathcal{H}-\hbar||\nabla\mathcal{H}|\, d\mu_t\\ &\leq C\sigma^{-2}\int_\Sigma(\mathcal{H}-\hbar)^4\, d\mu_t + C\sigma^{-q-2\delta q}\int_\Sigma(\mathcal{H}-\hbar)^2|\nabla\mathcal{H}|^2\, d\mu_t,\end{aligned} \tag{5.45}$$

where we used Young's inequality in the latter estimate. The conclusion holds for σ suitably large. □

A similar estimate, but independent of the evolution of $\hbar$, can be also given for $\nabla\mathcal{H}$. The hypothesis on $\mathring{A}$ and the H^1-norm of $\mathcal{H}-\hbar$ are not needed in order to prove the following Lemma.

Lemma 5.0.7. *Let (Σ, F_t), $t\in[0,T]$, such that (2.29) and (2.30) hold for every $t\in[0,T]$. Then there exists a constant $C=C(\overline{c})>0$ such that*

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Sigma|\nabla\mathcal{H}|^2\, d\mu_t \leq -\frac{1}{2}\int_\Sigma|\nabla^2\mathcal{H}|^2\, d\mu_t + C\sigma^{-2}\int_\Sigma|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-4}\int_\Sigma(\mathcal{H}-\hbar)^2\, d\mu_t, \tag{5.46}$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Sigma|\nabla\mathcal{H}|^4\, d\mu_t \leq -\int_\Sigma|\nabla^2\mathcal{H}|^2|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-2}\int_\Sigma|\nabla\mathcal{H}|^4\, d\mu_t + C\sigma^{-6}\int_\Sigma(\mathcal{H}-\hbar)^4\, d\mu_t. \tag{5.47}$$

Proof. We start proving inequality (5.46). From Lemma 5.0.4, after integration we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Sigma|\nabla\mathcal{H}|^2\, d\mu_t &= 2\int_\Sigma\langle\nabla\left(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu)\right),\nabla\mathcal{H}\rangle\, d\mu_t\\ &+2\int_\Sigma\langle\nabla\left((\Upsilon-1)\left(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu))\right)\right),\nabla\mathcal{H}\rangle\, d\mu_t\\ &+\int_\Sigma|\nabla\mathcal{H}|^2H(\hbar-\mathcal{H})\, d\mu_t + 2\int_\Sigma(\mathcal{H}-\hbar)|\nabla\mathcal{H}|^2h^{ij}\nabla_i\mathcal{H}\nabla_j\mathcal{H}\, d\mu_t\\ &+2\int_\Sigma\langle\nabla\Psi,\nabla\mathcal{H}\rangle d\mu_t.\end{aligned}$$

Since H, $|\mathcal{H}-\hbar|$ are bounded by $C\sigma^{-1}$ and $\left||A|^2+\overline{\mathrm{Ric}}(\nu,\nu)\right|\leq C\sigma^{-2}$ and $|\mathrm{Ric}^\Sigma|\leq C\sigma^{-2}$, using Bochner's identity and integration by part we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Sigma|\nabla\mathcal{H}|^2\, d\mu_t &\leq C\sigma^{-2}\int_\Sigma|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-4}\int_\Sigma(\mathcal{H}-\hbar)^2\, d\mu_t - \int_\Sigma|\nabla^2\mathcal{H}|^2\, d\mu_t\\ &+2\left|\int_\Sigma(\Upsilon-1)\left(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu))\right)\Delta\mathcal{H}\, d\mu_t\right|\\ &+2\left|\int_\Sigma\Psi\Delta\mathcal{H}\, d\mu_t\right|.\end{aligned} \tag{5.48}$$

Note that, combining (5.5) and (5.7), we get

$$\begin{aligned}\left|\int_\Sigma \Psi\Delta\mathcal{H}\, d\mu_t\right| &\le C\sigma^{\frac{1}{2}-\frac{1}{2}q-\delta q+\delta}\int_\Sigma\left(\sigma^{-\frac{5}{2}-\delta}|\mathcal{H}-\hbar|+\sigma^{-\frac{3}{2}-\delta}|\nabla\mathcal{H}|\right)|\nabla^2\mathcal{H}|\, d\mu_t\\ &\le C\sigma^{-4}\int_\Sigma(\mathcal{H}-\hbar)^2\, d\mu_t + C\sigma^{-2}\int_\Sigma|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-q-2\delta q}\int_\Sigma|\nabla^2\mathcal{H}|^2\, d\mu_t.\end{aligned} \tag{5.49}$$

Since $|\Upsilon-1| = \mathrm{O}(\sigma^{-\frac{1}{2}q-q\delta})$ and using Young's inequality, we conclude, for σ large,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma|\nabla\mathcal{H}|^2\, d\mu_t &\le C\sigma^{-2}\int_\Sigma|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-4}\int_\Sigma(\mathcal{H}-\hbar)^2\, d\mu_t - \int_\Sigma|\nabla^2\mathcal{H}|^2\, d\mu_t\\ &\quad + C\sigma^{-\frac{1}{2}q-q\delta}\int_\Sigma|\nabla^2\mathcal{H}|^2\, d\mu_t + C\sigma^{-2-\frac{1}{2}q-q\delta}\int_\Sigma|\mathcal{H}-\hbar||\nabla^2\mathcal{H}|\, d\mu_t\\ &\quad + C\sigma^{-q-2q\delta}\int_\Sigma|\nabla^2\mathcal{H}|^2\, d\mu_t\\ &\le C\sigma^{-2}\int_\Sigma|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-4}\int_\Sigma(\mathcal{H}-\hbar)^2\, d\mu_t - \frac{1}{2}\int_\Sigma|\nabla^2\mathcal{H}|^2\, d\mu_t.\end{aligned} \tag{5.50}$$

We now prove (5.47). From (5.16) we get, after integrating by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma|\nabla\mathcal{H}|^4\, d\mu_t = 4\int_\Sigma(\mathcal{H}-\hbar)|\nabla\mathcal{H}|^2h^{ij}\nabla_i\mathcal{H}\nabla_j\mathcal{H} \tag{5.51}$$

$$\begin{aligned}&+4\int_\Sigma\left\langle\nabla\left(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu))\right),\nabla\mathcal{H}\right\rangle d\mu_t + \int_\Sigma|\nabla\mathcal{H}|^4H(\hbar-\mathcal{H})\, d\mu_t\\ &-4\int_\Sigma(\Upsilon-1)\left(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu))\right)\Delta\mathcal{H}|\nabla\mathcal{H}|^2\, d\mu_t + 4\int_\Sigma\langle\nabla\Psi,\nabla\mathcal{H}\rangle\, d\mu_t.\end{aligned}$$

To estimates the terms above, note that, if σ_0 is so large that $|\Upsilon-1|\le\varepsilon$ (see (5.6)), then

$$\begin{aligned}&\left|4\int_\Sigma(\Upsilon-1)\left(\Delta\mathcal{H}+(\mathcal{H}-\hbar)(|A|^2+\overline{\mathrm{Ric}}(\nu,\nu))\right)\Delta\mathcal{H}|\nabla\mathcal{H}|^2\, d\mu_t\right|\\ &\qquad\le\varepsilon\int_\Sigma|\nabla^2\mathcal{H}|^2|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-2}\int_\Sigma|\mathcal{H}-\hbar||\nabla^2\mathcal{H}||\nabla\mathcal{H}|^2\, d\mu_t.\end{aligned}$$

Moreover, using again integration by parts on $\langle\nabla\Psi,\nabla\mathcal{H}\rangle = \nabla\cdot(\Psi\nabla\mathcal{H})-\Psi\Delta\mathcal{H}$, we estimate

$$\left|\int_\Sigma\langle\nabla\Psi,\nabla\mathcal{H}\rangle|\nabla\mathcal{H}|^2\, d\mu_t\right| \le C\sigma^{-1-\frac{1}{2}q-\delta q}\left(\sigma^{-1}\int_\Sigma|\mathcal{H}-\hbar||\nabla^2\mathcal{H}||\nabla\mathcal{H}|^2\, d\mu_t + \int_\Sigma|\nabla^2\mathcal{H}||\nabla\mathcal{H}|^3\, d\mu_t\right)$$

where we also used (5.5) combined with (5.7). We conclude using Bochner's formula, the inequality $H+|H-h|+|A|\le C\sigma^{-1}$ and that $|\mathrm{Ric}^\Sigma|\le C\sigma^{-2}$, obtaining

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma|\nabla\mathcal{H}|^4\, d\mu_t \le &-4\int_\Sigma|\nabla^2\mathcal{H}|^2|\nabla\mathcal{H}|^2 + C\sigma^{-2}\int_\Sigma|\nabla\mathcal{H}|^4\, d\mu + C\sigma^{-2}\int_\Sigma|\mathcal{H}-\hbar||\nabla^2\mathcal{H}||\nabla\mathcal{H}|^2\, d\mu_t\\ &+\varepsilon\int_\Sigma|\nabla^2\mathcal{H}|^2|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-1}\int_\Sigma|\nabla^2\mathcal{H}||\nabla\mathcal{H}|^3\, d\mu_t.\end{aligned} \tag{5.52}$$

The desired inequality appears when using Young's inequality

$$\begin{aligned}&C\sigma^{-2}\int_\Sigma|\mathcal{H}-\hbar||\nabla^2\mathcal{H}||\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-1}\int_\Sigma|\nabla^2\mathcal{H}||\nabla\mathcal{H}|^3\, d\mu_t\\ &\qquad\le\varepsilon\int_\Sigma|\nabla^2\mathcal{H}|^2|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-6}\int_\Sigma(\mathcal{H}-\hbar)^4\, d\mu_t + C\sigma^{-2}\int_\Sigma|\nabla\mathcal{H}|^4\, d\mu_t\end{aligned}$$

and choosing ε suitably small. □

The next simple inequality will be useful in the following Lemma. The proof is analogous to the one of Lemma 4.1.10.

Lemma 5.0.8. *Let $\Sigma \hookrightarrow M$ be a* surface. *Then we have, for every $\varepsilon > 0$ and $\sigma > 1$,*

$$-\sigma^{-4}\int_\Sigma (\mathcal{H}-\hbar)^2|\nabla\mathcal{H}|^2\,d\mu \le -\frac{\varepsilon}{2\sigma^2}\int_\Sigma |\nabla\mathcal{H}|^4\,d\mu + \varepsilon^2\int_\Sigma |\nabla^2\mathcal{H}|^2|\nabla\mathcal{H}|^2\,d\mu.$$

This leads to the following Lemma.

Lemma 5.0.9. *Let (Σ, F_t), $t\in[0,T]$, such that (2.29), (2.30), (5.24), (5.25) and $\|\mathring{A}\|_{L^4(\Sigma_t)} \le B_1\sigma^{-1-\delta}$ for every $t\in[0,T]$. For $\eta>0$, let us set*

$$a_\eta(t) := \eta\sigma^{-4}\|\mathcal{H}-\hbar\|^4_{L^4(\Sigma_t)} + \|\nabla\mathcal{H}\|^4_{L^4(\Sigma_t)}. \tag{5.53}$$

Then there exist a universal constant $\eta_w>0$ and a radius $\sigma_0=\sigma_0(B_1, \mathrm{c_{in}}, \overline{c}, \delta) > 1$ such that if $\eta=\eta_w$ and $\sigma>\sigma_0$ the following statements hold.

(i) *There exists a constant $c = c(B_1, \eta_w, \overline{c})$ such that if $B_2 > c$ we have the implication*

$$a_{\eta_w}(0) < B_2\sigma^{-8-4\delta} \implies a_{\eta_w}(t) < B_2\sigma^{-8-4\delta} \text{ for every } t\in[0,T].$$

(ii) *If in addition we suppose (5.24), there exists a constant $c = c(\mathrm{c_{in}}, B_\infty)$ such that if we choose $B_{\mathrm{in}} > c(\mathrm{c_{in}}, B_\infty)$ we have the implication*

$$\mathrm{a}_\eta(0) < B_{\mathrm{in}}\sigma^{-7-\frac{3}{2}q-2\delta-3\delta q} \implies \mathrm{a}_\eta(t) < B_{\mathrm{in}}\sigma^{-7-\frac{3}{2}q-2\delta-3\delta q} \text{ for every } t\in[0,T].$$

Proof. Combining Lemma 5.0.6, Lemma 5.0.7 and Lemma 5.0.8, we get

$$\dot{a}_\eta(t) \le -C\sigma^{-2}a_\eta(t) + \tilde{C}\sigma^{-6}\int_\Sigma (\mathcal{H}-\hbar)^4\,d\mu_t + \tilde{c}B_\infty\sigma^{-9-\frac{3}{2}q-2\delta-3\delta q}, \tag{5.54}$$

for some $\tilde{C}$ universal constant and $\tilde{c} = \tilde{c}(\mathrm{c_{in}}, \overline{c})$. We will use inequality (5.54) in order to prove two different conclusions.

(i) Since $q\ge 2$, choosing σ suitably large depending on B_∞ so that $\tilde{c}B_\infty\sigma^{-9-\frac{3}{2}q-2\delta-3\delta q} \le \sigma^{-10-4\delta}$ we have

$$\dot{a}_\eta(t) \le -C\sigma^{-2}a_\eta(t) + \tilde{C}\sigma^{-6}\int_\Sigma (\mathcal{H}-\hbar)^4\,d\mu_t + \sigma^{-10-4\delta}. \tag{5.55}$$

Moreover, Lemma 2.3.5 implies that

$$\int_\Sigma (H-h)^4\,d\mu_t \le \mathrm{c}^4_{\mathrm{Per}}\left(\|\mathring{A}\|^4_{L^4(\Sigma,\mu_t)} + \sigma^{-4-4\delta}\right) \le \mathrm{c}^4_{\mathrm{Per}}(B_1^4+1)\sigma^{-4-4\delta}.$$

and thus (5.55) becomes

$$\dot{a}_\eta(t) \le -C\sigma^{-2}a_\eta(t) + c\sigma^{-10-4\delta} \tag{5.56}$$

with $c = c(B_1, \overline{c}, \mathrm{c_{Per}})$. Thus, if $B_2 > c/C$, we get the thesis.

(ii) Since we are assuming (5.25) for every $t\in[0,T]$, the Sobolev's immersion (see [CS21, Lemma 12]) implies that

$$\|\mathcal{H}-\hbar\|_{L^4(\Sigma_t)} \le \tilde{c}\sigma^{-\frac{1}{2}-\frac{q}{2}-q\delta} \quad \text{for every } t\in[0,T],$$

where $\tilde{c} = \tilde{c}(\mathrm{c_{in}}, \bar{c})$. Thus (5.54) becomes

$$\begin{aligned}\dot{a}_\eta(t) \leq& -C\sigma^{-2}a_\eta(t) + \tilde{c}\sigma^{-8-2q-4q\delta} + \tilde{c}B_\infty\sigma^{-9-\frac{3}{2}q-2\delta-3\delta q}\\ \leq& -C\sigma^{-2}a_\eta(t) + 2\tilde{c}B_\infty\sigma^{-9-\frac{3}{2}q-2\delta-3\delta q}\end{aligned} \tag{5.57}$$

for σ large, since $q \geq 2$ and $\delta \in (0, \frac{1}{2}]$. Choosing $B_{\mathrm{in}} > 2\tilde{c}B_\infty/C$ we have the thesis.

□

From now on, when considering the roundness class $\mathcal{W}^\eta_\sigma(B_1, B_2)$, we fix the parameter η equal to the value η_w given by the previous Lemma, and we will no longer need to specify the dependence on η of the constants in the estimates.

5.0.3 Evolution of $\|\mathcal{H} - \hbar\|_{L^2(\Sigma_t)}$ and convergence

An important assumption in the previous results was the comparability between r_Σ and σ in (2.30) which assures that on Σ_t the ambient curvature decays with the right order, as highlighted in Remark 2.3.2. To justify this assumption, we study now the evolution of L^2-norm of $\mathcal{H} - \hbar$, which relies on the spectral analysis of Section 3. The following Lemma is an improvement of inequality (5.33). Under an additional hypothesis, this inequality shows that the negative term in the evolution of $\|\mathcal{H} - \hbar\|^2_{L^2(\Sigma_t)}$ is dominant.

Lemma 5.0.10. *Let (Σ, F_t), $t \in [0, T]$, be such that (2.29), (2.30), (5.24), (5.25) hold for every $t \in [0, T]$. For every $\Omega > 0$ there exists $\sigma_0(\bar{c}, \Omega) > 1$ such that if*

$$\|\mathcal{H} - \hbar\|_{L^\infty(\Sigma_t)} \leq \Omega\sigma^{-\frac{5}{4}-\frac{3}{8}q-\frac{\delta}{2}-\frac{3\delta q}{4}}, \quad \forall t \in [0, T] \tag{5.58}$$

and $\sigma > \sigma_0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (\mathcal{H} - \hbar)^2 \, d\mu_t \leq -\frac{E_{\mathrm{ADM}}}{\sigma^3}\int_\Sigma (\mathcal{H} - \hbar)^2 d\mu_t,$$

for every $t \in [0, T]$.

Proof. We easily compute the evolution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (\mathcal{H} - \hbar)^2 \, d\mu_t =& -2\int_\Sigma (\mathcal{H} - \hbar)L(\mathcal{H} - \hbar) \, d\mu_t\\ &+ 2\int_\Sigma (\mathcal{H} - \hbar)(\Upsilon - 1)\left(\Delta\mathcal{H} + (\mathcal{H} - \hbar)(|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu))\right) d\mu_t\\ &+ \int_\Sigma \Psi(\mathcal{H} - \hbar) \, d\mu_t - \int_\Sigma H(\mathcal{H} - \hbar)^3 \, d\mu_t.\end{aligned} \tag{5.59}$$

Combining (5.5) and (5.7) we get

$$|\Psi| \leq C(\sigma|P|)^{q-1}\left(\sigma^{-\frac{5}{2}-\delta}|\mathcal{H} - \hbar| + \sigma^{-\frac{3}{2}-\delta}|\nabla\mathcal{H}|\right), \tag{5.60}$$

which implies, using that $\sigma|P| \leq C\sigma^{-\frac{1}{2}-\delta}$,

$$\begin{aligned}\left|\int_\Sigma (\mathcal{H} - \hbar)\Psi \, d\mu_t\right| \leq& C\left(\sigma^{-2-\frac{1}{2}q-\delta q}\int_\Sigma (\mathcal{H} - \hbar)^2 \, d\mu_t + \sigma^{-\frac{3}{2}-\delta}\int_\Sigma (\sigma|P|)^{q-1}|\mathcal{H} - \hbar||\nabla\mathcal{H}| \, d\mu_t\right)\\ \leq& \frac{\varepsilon E_{\mathrm{ADM}}}{\sigma^3}\int_\Sigma (\mathcal{H} - \hbar)^2 \, d\mu_t + \varepsilon\int_\Sigma (\sigma|P|)^{2q-2}|\nabla\mathcal{H}|^2 \, d\mu_t.\end{aligned}$$

where in the latter inequality we used parametric Young's inequality and we have chosen σ suitably large.

We now estimate, using integration by parts and formula (5.4) for $\nabla\Upsilon$, together with the fact that $|\frac{H}{\mathcal{H}}| \le C$ and $H \sim \mathcal{H} \sim \frac{2}{\sigma}$,

$$\int_\Sigma (\mathcal{H} - \hbar)(\Upsilon - 1)\Delta\mathcal{H}\, d\mu_t$$
$$\le -\int_\Sigma (\Upsilon - 1)|\nabla\mathcal{H}|^2\, d\mu_t + \int_\Sigma \left(\sigma^{q+1}|P|^q|\nabla\mathcal{H}| + \sigma^q|P|^{q-1}|\nabla P|\right)|\mathcal{H} - \hbar||\nabla\mathcal{H}|\, d\mu_t. \quad (5.61)$$

Since $|\mathcal{H} - \hbar| \le \epsilon\sigma^{-1}$ for σ suitably large, and $|\nabla P| \le \sigma^{-\frac{5}{2}-\delta}$ because of Lemma 2.1.3,

$$\left(\sigma^{q+1}|P|^q|\nabla\mathcal{H}| + \sigma^q|P|^{q-1}|\nabla P|\right)|\mathcal{H} - \hbar||\nabla\mathcal{H}|$$
$$\begin{aligned} &\le \epsilon\sigma^q|P|^q|\nabla\mathcal{H}|^2 + \sigma^{q-\frac{5}{2}-\delta}|P|^{q-1}|\mathcal{H} - \hbar||\nabla\mathcal{H}| \\ &\le \epsilon(\sigma|P|)^q|\nabla\mathcal{H}|^2 + \sigma^{-\frac{3}{2}-\delta}(\sigma|P|)^{q-1}|\mathcal{H} - \hbar||\nabla\mathcal{H}| \\ &\le \epsilon(\sigma|P|)^q|\nabla\mathcal{H}|^2 + C\sigma^{-3-2\delta}|\mathcal{H} - \hbar|^2 + \epsilon(\sigma|P|)^{2q-2}|\nabla\mathcal{H}|^2 \end{aligned} \quad (5.62)$$

where in the latter inequality we used parametric Young's inequality. Since $(\sigma|P|)^{2q-2} = (\sigma|P|)^{q-2}(\sigma|P|)^q \le C(\sigma|P|)^q$, combining (5.61) and (5.62) we get

$$\int_\Sigma (\mathcal{H} - \hbar)(\Upsilon - 1)\Delta\mathcal{H}\, d\mu_t$$
$$\le -\int_\Sigma (\Upsilon - 1)||\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-3-2\delta}\int_\Sigma (\mathcal{H} - \hbar)^2\, d\mu_t + \epsilon C\int_\Sigma (\sigma|P|)^q|\nabla\mathcal{H}|^2\, d\mu_t, \quad (5.63)$$

Note furthermore that (5.58) implies

$$\left|\int_\Sigma H(\mathcal{H} - \hbar)^3\, d\mu_t\right| \le \Omega\sigma^{-\frac{9}{4}-\frac{3}{8}q-\frac{3q\delta}{4}-\frac{\delta}{2}}\int_\Sigma (\mathcal{H} - \hbar)^2\, d\mu_t \le \frac{\varepsilon E}{\sigma^3}\int_\Sigma (\mathcal{H} - \hbar)^2\, d\mu_t,$$

if $\sigma > \sigma_0$, for some $\sigma_0 = \sigma_0(\Omega)$.

We conclude from (5.59), using Proposition 3.2.7, together with $|\Upsilon - 1||A|^2 \le C\sigma^{-2-\frac{q}{2}-q\delta} \le \varepsilon E_{\mathrm{ADM}}\sigma^{-3}$ and $\Upsilon - 1 \ge \underline{c}\sigma^q|P|^q$ because of (5.6), obtaining

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (\mathcal{H} - \hbar)^2\, d\mu_t &\le -\frac{2E_{\mathrm{ADM}}}{\sigma^3}\int_\Sigma (\mathcal{H} - \hbar)^2\, d\mu_t + \frac{\varepsilon E_{\mathrm{ADM}}}{\sigma^3}\int_\Sigma (\mathcal{H} - \hbar)^2\, d\mu_t \\ &\quad + (\epsilon C - \underline{c})\int_\Sigma (\sigma|P|)^q|\nabla\mathcal{H}|^2\, d\mu_t + C\sigma^{-3-2\delta}\int_\Sigma (\mathcal{H} - \hbar)^2\, d\mu_t \\ &\le -\frac{E_{\mathrm{ADM}}}{\sigma^3}\int_\Sigma (\mathcal{H} - \hbar)^2\, d\mu_t \end{aligned} \quad (5.64)$$

for ϵ small with respect to $\underline{c}$, $\varepsilon < 1$ and σ suitably large. □

The next result, which is similar to Proposition 3.4 in [HY96], gives a bound on the possible change of area of the surface along the flow as long as it remains round. The proof is analogous to that of Lemma 4.1.14.

Lemma 5.0.11. *Given B_1, B_2, there exist constants $c > 0$ and $\sigma_0 > 1$ such that, if $\sigma > \sigma_0$ and Σ_t is a solution of the flow (5.2) for $t \in [0, T]$ with $\Sigma_t \in \mathcal{W}^\eta_\sigma(B_1, B_2)$ for all t then*

$$|\sigma_{\Sigma_0} - \sigma_{\Sigma_t}| \le c\sigma^{\frac{1}{2}-\delta}$$

for every $t \in [0, T]$.

We are now ready to prove that, by an appropriate choice of the parameters of roundness class, a well-centered round surface remains inside the class for arbitrary times. Remember that hypotheses (1.16) are in particular satisfied by Nerz's foliation.

Remark 5.0.12. *The decay rates in conditions (1.16) are modelled on the properties of the leaves of the CMC-foliation constructed by Nerz in [Ner15]. In particular, in his Theorem 5.1, Nerz proved the existence of an exhaustive family of constant mean curvature surfaces which foliate an asymptotically flat manifold with non-zero ADM-energy. We remark that such foliation has been constructed via volume preserving mean curvature flow in Chapter 4, under the additional (weak) Regge-Teitelboim conditions and the hypothesis* $E_{\text{ADM}} > 0$. *See also Section 4.3.1.*

We denote by $\{\Sigma^s\}_{s \geq s_0}$, *for a certain* $s_0 > 1$, *the CMC-foliation (constructed as in [Ner15] or as in Chapter 4). Note that we use a different letter in order to parametrize the foliation with respect to [Ner15]. This CMC-foliation satisfies*

$$H^{\Sigma^s} = \frac{2}{s}, \qquad \|\overset{\circ}{A}\|_{H^1(\Sigma^s)} \leq C_{\text{Nerz}} s^{-\frac{3}{2}-\delta}, \qquad |\vec{z}_{\Sigma^s}| \leq C_{\text{Nerz}} s^{1-\delta}, \tag{5.65}$$

for some $C_{\text{Nerz}} > 0$. *Moreover, [Ner15, Prop. 4.4] proves that* $|s - \sigma_s| \leq C \sigma_s^{\frac{1}{2}-\delta}$, *where* $\sigma_s := \sigma_{\Sigma^s}$. *Note also that [Ner15, Prop. 4.4], combined with (5.65), implies*

$$\sigma_{\Sigma^s} - C\sigma_{\Sigma^s}^{1-\delta} \leq |\vec{x}| = |\vec{z}^s + \sigma_{\Sigma^s}\nu^s + f^s\nu^s| \leq \sigma_{\Sigma^s} + C\sigma_{\Sigma^s}^{1-\delta}, \tag{5.66}$$

that is $|r_{\Sigma^s} - \sigma_{\Sigma^s}| \leq C\sigma_{\Sigma^s}^{1-\delta}$. *Then*

$$\frac{r_{\Sigma^s}}{\sigma_s} \geq 1 - C\sigma_s^{-\delta}. \tag{5.67}$$

Thus, for s *large, this foliation satisfies (1.16) with* $\sigma = \sigma_s$. *Finally observe that, by Lemma 2.1.3, the leaves* Σ^s *also satisfy*

$$\|\mathcal{H} - \hbar\|_{W^{1,2}(\Sigma^s)} \leq \mathrm{c}_{\text{in}} \sigma^{-\frac{q}{2}-q\delta},$$

for some $\mathrm{c}_{\text{in}} = \mathrm{c}_{\text{in}}(\bar{\mathrm{c}}) > 0$. *For this reason, we will use a fixed leaf of Nerz's foliation as the initial datum of our flow.*

Remark 5.0.13. *We remark that, using the fundamental result of DeLellis-Müller [DLM05], the assumptions in (1.16) imply that* $\frac{\sigma}{r_\Sigma}$ *is also bounded away from zero. In fact, combining Lemma 2.1.10 with the latter assumption in (1.16), and using the DeLellis-Müller's Theorem, Point (iv) of Lemma 2.3.5, combined with the second assumption in (1.16), implies that* $\frac{\sigma}{r_\Sigma} \geq 1 - C_0^{-1}$, *provided that* σ *is suitably large.*

The following Theorem is the key step in the proof of Theorem 1.1.4.

Theorem 5.0.14. *Let* $(M, \bar{\mathrm{g}}, \overline{\mathrm{K}})$ *be a* $C^2_{\frac{1}{2}+\delta}$*-asymptotically flat initial data set, with* $E_{\text{ADM}} > 0$. *Choose* B_1 *as in Lemma 5.0.5 and* B_2 *and* η *as in Lemma 5.0.9. For every* $C_0 > 0$ *there exist* $\overline{B} = \overline{B}(C_0)$ *and* $\sigma_0 = \sigma_0(\bar{\mathrm{c}}, \delta, E_{\text{ADM}}, B_1, B_2, C_0)$ *such that the following holds. Let* (Σ, F_t) *be a solution to the volume preserving spacetime mean curvature flow for* $t \in [0, T]$ *such that* Σ_0 (i) *belongs to* $\mathcal{B}_\sigma(B_1, B_2, B_{\text{cen}})$ *with* $\sigma = \sigma_{\Sigma_0}$, (ii) *is a CMC-surface and* (iii) $|\vec{z}_{\Sigma_0}| \leq C_0 \sigma^{1-\delta}$. *Then, if* $B_{\text{cen}} \geq \overline{B}$ *and* $\sigma \geq \sigma_0$, Σ_t *belongs to* $\mathcal{B}_\sigma(B_1, B_2, B_{\text{cen}})$ *for every* $t \in [0, T]$.

Remark 5.0.15. *Note that the following proof also works when* Σ *is almost CMC and not exactly CMC.*

Proof. Note that, since $\Sigma_0 = \Sigma$ belongs to $\mathcal{B}_\sigma(B_1, B_2, B_{\text{cen}})$, then it satisfies

$$\frac{1}{2} \leq \frac{r_\Sigma}{\sigma} \leq 2, \qquad 1 \leq \sigma H \leq \sqrt{5}.$$

Thus Lemma 2.1.3 implies that the initial (CMC) surface satisfies

$$\|\mathcal{H}-\hbar\|_{H^1(\Sigma_0)} \le C_1\sigma^{-\frac{1}{2}q-q\delta} \tag{5.68}$$

for some $C_1 = C_1(\bar{c}) > 0$. Thus, we define the maximal time

$$T_{\max} := \sup\left\{\overline{T} \le T : \begin{array}{c} F_t \text{ exists in } [0,\overline{T}],\ \|\mathcal{H}-\hbar\|_{L^2(\Sigma_t)} < (C_1+1)\sigma^{-\frac{1}{2}q-q\delta} \text{ and} \\ \Sigma_t \text{ belongs to } \mathcal{B}_\sigma(B_1,B_2,B_{\text{cen}}) \text{ for every } t\in[0,\overline{T}] \end{array}\right\} > 0. \tag{5.69}$$

Thus, $\Sigma_{T_{\max}}$ belongs to $\overline{\mathcal{B}}_\sigma(B_1,B_2,B_{\text{cen}}) \subset \overline{\mathcal{W}}_\sigma(B_1,B_2)$. By Lemma 2.3.5, Lemma 5.0.11 and Definition 2.3.1, the conditions (2.29) and (2.30) hold for every $t\in[0,T_{\max}]$. See again Remark 5.0.12 for a direct estimate of the Euclidean radius.

Claim: There exists $\mathrm{c}_{\text{in}} = \mathrm{c}_{\text{in}}(\bar{c}) > 0$ such that (5.25) holds for every $t\in[0,T_{\max}]$.
Proof of the Claim. Combining together (5.33) and (5.46), if C is the maximum between the two constants involved, we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Sigma |\nabla\mathcal{H}|^2\,d\mu_t + 4C\sigma^{-2}\int_\Sigma(\mathcal{H}-\hbar)^2\,d\mu_t\right) \le -\frac{1}{2}\int_\Sigma|\nabla^2\mathcal{H}|^2\,d\mu_t - C\sigma^{-2}\int_\Sigma|\nabla\mathcal{H}|^2\,d\mu_t \\ +(4C^2+C)\sigma^{-4}\int_\Sigma(\mathcal{H}-\hbar)^2\,d\mu_t.\end{aligned} \tag{5.70}$$

Setting $a(t) := \|\nabla\mathcal{H}\|^2_{L^2(\Sigma_t)} + 4C\sigma^{-2}\|\mathcal{H}-\hbar\|^2_{L^2(\Sigma_t)}$, since by definition of $T_{\max}$ it holds $\|\mathcal{H}-\hbar\|_{L^2(\Sigma_t)} < (C_1+1)\sigma^{-\frac{1}{2}q-q\delta}$ for every $t\in[0,T_{\max}]$,

$$\begin{aligned}\dot{a}(t) &\le -C\sigma^{-2}a(t) + (8C^2+C)\sigma^{-4}\int_\Sigma(\mathcal{H}-\hbar)^2\,d\mu_t \\ &\le -C\sigma^{-2}a(t) + 2(8C^2+C)(C_1+1)^2\sigma^{-4-q-2q\delta}\end{aligned} \tag{5.71}$$

Since, by (5.68), $a(0)\le(1+4C)C_1\sigma^{-2-q-2\delta q}$, (5.71) implies that $a(t)\le C(\bar{c},C_1)\sigma^{-2-q-2\delta q}$ for every $t\in[0,T_{\max}]$. Since also $C_1 = C_1(\bar{c})$, this proves that there exists $\mathrm{c}_{\text{in}} = \mathrm{c}_{\text{in}}(\bar{c})$ such that the claim holds.

Now, (2.29), (2.30), (5.24) and (5.25) imply that we are in the hypotheses of Proposition 5.0.5 and of point (i) of Lemma 5.0.9. Thus, the choices of B_1 and B_2 imply that $\Sigma_{T_{\max}}$ belongs to $\mathcal{W}_\sigma(B_1,B_2)$ for σ large. Moreover, (5.24) holds for some $B_\infty = B_\infty(B_1,B_2)$, thanks again to Lemma 2.3.5.

We conclude showing that, if also B_{cen} is chosen suitably large, then $\Sigma_t\in\mathcal{B}_\sigma(B_1,B_2,B_{\text{cen}})$ for every $t\in[0,T_{\max}]$. Since Σ_0 is a CMC-surface, it is easy to verify that $a_{\eta_w}(0) < B_{\text{in}}\sigma^{-7-\frac{3}{2}q-2\delta-3\delta q}$ for a constant B_{in} suitably large. We remember that the function a_η has been defined in Lemma 5.0.9. Moreover, Lemma 5.0.9, point (ii), implies that if B_{in} is chosen suitably large, depending on c_{in} and B_∞, then $\|\mathcal{H}-\hbar\|_{L^\infty(\Sigma_t)} \le B_{\text{in}}\sigma^{-\frac{5}{4}-\frac{3}{8}q-\frac{\delta}{2}-\frac{3\delta q}{4}}$ holds for every $t\in[0,T_{\max}]$. Thus Lemma 5.0.10, with $\Omega := B_{\text{in}}$, combined with Gronwall's Lemma, implies that

$$\|\mathcal{H}-\hbar\|_{L^2(\Sigma_t)} \le \|\mathcal{H}-\hbar\|_{L^2(\Sigma_0)}e^{-\frac{E_{\text{ADM}}t}{2\sigma^3}} < (C_1+1)\sigma^{-\frac{1}{2}q-q\delta}e^{-\frac{E_{\text{ADM}}t}{2\sigma^3}}, \tag{5.72}$$

for every $t \in [0, T_{\max}]$. Setting, $\vec{z}(t) = \vec{z}_{\Sigma_t}$, we show that the behavior of the barycenter is controlled. Analogously to [CW08], we have the evolution

$$\partial_t \left(|\Sigma_t|\vec{z}(t)\right) = \int_\Sigma (\hbar - \mathcal{H})\nu \, d\mu_t + \int_\Sigma F_t(x) H(\hbar - \mathcal{H}) \, d\mu_t. \tag{5.73}$$

Combining this with the estimates $H \leq \frac{5}{\sigma}$, $|F_t(x)| \leq R_\Sigma(t) \leq 3\sigma$ and (5.72), we obtain

$$\partial_t \left(|\Sigma_t||\vec{z}(t)|\right) \leq C\sigma \|\mathcal{H} - \hbar\|_{L^2(\Sigma,\mu_t)} < C(C_1+1)\sigma^{1-\frac{q}{2}-q\delta} e^{-\frac{E_{\mathrm{ADM}} t}{2\sigma^3}}. \tag{5.74}$$

Integrating (5.74) over $[0, T_{\max}]$, we get

$$|\Sigma_{T_{\max}}||\vec{z}(T_{\max})| - |\Sigma_0||\vec{z}(0)| \leq C(C_1+1)\sigma^{1-\frac{q}{2}-q\delta} \left(\frac{2\sigma^3}{E_{\mathrm{ADM}}}\right)\left(1 - e^{-\frac{E_{\mathrm{ADM}} T_{\max}}{2\sigma^3}}\right).$$

By the hypotheses $|\vec{z}_{\Sigma_0}| \leq C_0 \sigma^{1-\delta}$, we find

$$|\vec{z}(T_{\max})| \leq \frac{2}{7\pi}\left(5\pi(C_0\sigma^{1-\delta}) + \frac{2C(C_1+1)}{E_{\mathrm{ADM}}}\sigma^{2-\frac{q}{2}-q\delta}\right) < B_{\mathrm{cen}}\sigma^{1-\delta} \tag{5.75}$$

if B_{cen} suitably large, depending on C_0, C, C_1 and E_{ADM}. Thus $\Sigma_{T_{\max}}$ belongs to the class $\mathcal{B}_\sigma(B_1, B_2, B_{\mathrm{cen}})$, and combining this with (5.72) we obtain that necessarily $T_{\max} = T$. □

Local regularity of the flow. We now review the regularity theory of the non-linear flow we are considering. Since in a local interval of existence $[0, t_0)$ the principal curvatures are uniformly bounded (by the preservation of the roundness), it follows that Σ_t can be locally written as a graph. Suppose in particular that $\Sigma_t \cap B_\epsilon(x_0) = \{(x_1, x_2, u(t, x_1, x_2)) : (x_1, x_2) \in \mathcal{A}\}$, with $\mathcal{A} \subset \mathbb{R}^2$ open. Since the metric, the unit normal vector, and the mean curvature of Σ_t are locally given by

$$\begin{aligned} g_{ij} &= \delta_{ij} + D_i u D_j u, \qquad \nu = \frac{(-D_1 u, -D_2 u, 1)}{\sqrt{1+|Du|^2}}, \\ H &= \frac{1}{\sqrt{1+|Du|^2}}\left(\delta^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\right) D^2_{ij} u, \end{aligned} \tag{5.76}$$

the equation (5.1), written in a tangential fashion, translates into an equation for u

$$\partial_t u = \sqrt{1+|Du|^2}\left(\Phi\left(\frac{1}{\sqrt{1+|Du|^2}}\left(\delta^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\right) D^2_{ij} u, P\right) - \hbar\right), \tag{5.77}$$

where $P = g^{ij}\overline{\mathrm{K}}_{ij}$ is a smooth function and $\Phi(s, \gamma) = \sqrt[q]{s^q - |\gamma|^q}$. We rewrite equation (5.77) as

$$\partial_t u = \mathcal{F}(D^2 u, Du, x, t). \tag{5.78}$$

Note that

$$\dot{\mathcal{F}}^{ij} := \frac{\partial \mathcal{F}}{\partial D^2_{ij} u} = \left(\delta^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\right)\partial_s \Phi \tag{5.79}$$

and $\partial_s \Phi = q^{-1}\left(s^q - |\gamma|^q\right)^{\frac{1}{q}-1}\left(q s^{q-1}\right) > 0$. Thus, as a matrix,

$$|w|^2 \left(\inf_{\mathcal{A}} \partial_s \Phi\right)\left(1 - \frac{\sup|Du|^2}{1+\sup|Du|^2}\right) \leq \dot{\mathcal{F}}^{ij} w_i w_j = \left(|w|^2 - \frac{(Du \cdot w)^2}{1+|Du|^2}\right)\partial_s \Phi \leq |w|^2 \left(\sup_{\mathcal{A}} \partial_s \Phi\right). \tag{5.80}$$

Finally note that if $\dot{\mathcal{F}}^{ij}M_{ij} = 0$ then $\left(\delta^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\right) M_{ij} = 0$. Thus, computing $\ddot{\mathcal{F}}^{ij,kl} := \frac{\partial^2 \mathcal{F}}{\partial D^2_{ij} u \partial D^2_{kl} u}$, it follows that this implies that $\ddot{\mathcal{F}}^{ij,kl} M_{ij} M_{kl} = 0$.

This means that we are in the hypothesis of Theorem 6 in [And04], which let us obtain a $C^{2,\alpha}$ estimate on u, for some $\alpha \in (0,1)$. By standard arguments, this means that the coefficients of the linearization of the non-linear equation are $C^{0,\alpha}$-Hölder, and thus the standard theory (see for example [LSU68]) implies uniform bounds on all higher derivatives of u. Covering Σ_t with graphs over balls of the same radius, we obtain Hölder estimates on the curvature and its derivatives.

Proof of Theorem 1.1.4. Consider a CMC surface Σ such that, setting $\sigma = \sigma_\Sigma$,

$$\|\mathring{A}\|_{L^4(\Sigma)} \le C_0 \sigma^{-1-\delta}, \qquad |\vec{z}_\Sigma| \le C_0 \sigma^{1-\delta}, \qquad \frac{\sigma}{r_\Sigma} \le 1 + C_0^{-1}, \tag{5.81}$$

for some $C_0 > 3$. Notice that, for B_1, B_2 and B_{cen} suitably large Σ belongs to $\mathcal{B}_\sigma(B_1, B_2, B_{\text{cen}})$. See also Remark 5.0.12. As in the proof of Theorem 4.1.16, suppose that the maximal time of existence of the flow, say $T_{\max}$, is finite. Then, by Theorem 5.0.14 we find that also $\Sigma_{T_{\max}}$ belongs to $\mathcal{B}_\sigma(B_1, B_2, B_{\text{cen}})$ and thus, by the regularity theory, we can extend the flow past $T_{\max}$, which contradicts the maximality. Thus $T_{\max} = \infty$.

Convergence. From Lemma 5.0.10 we see that $\|H - h\|_{L^2(\Sigma_t)}$ decays exponentially as $t \to +\infty$. Since the derivatives of any order of H are uniformly bounded, interpolation estimates imply that they also decay exponentially. Then Sobolev immersion implies that $\|H - h\|_{L^\infty(\Sigma_t)}$ decays exponentially as well. By the bootstrapping argument described in the paragraph above, the boundedness of the curvatures and [Hui84, Lemma 8.2] show that $F(\cdot, t)$ converges to a smooth immersion $F_\infty(\cdot)$. In particular, since $\mathcal{H} - \hbar \to 0$, the limit surface $\Sigma_\infty := F_\infty(\Sigma)$ satisfies $\mathcal{H} \equiv \hbar$. Finally, Theorem 5.0.14 also shows that the requirements in the definition of $\mathcal{B}_\sigma(B_1, B_2, B_{\text{cen}})$ still hold as strict inequalities on Σ_∞.

5.0.4 CSTMC foliation and centers of mass

In conclusion, we remark that the computation carried out in the above Lemmas also have some consequences on the center of mass of the foliation we constructed. In this Section, we suppose that the initial datum of our flow is a leaf of Nerz's foliation, as recalled in Remark 5.0.12. In particular, we assume that there exists the CMC-center of mass of Nerz's foliation, i.e. the limit as $s \to \infty$ of the Euclidean barycenters of the foliation $\{\Sigma^s\}_{s \ge s_0}$ constructed by Nerz (see [Ner15]). In the following, we will suppose the change of variable $s \longleftrightarrow \sigma$, with $\sigma(s) := \sigma_{\Sigma^s}$. Thus, we have

Corollary 5.0.16. *Let $(M, \overline{g}, \overline{K})$ be a $C^2_{\frac{1}{2}+\delta}$-asymptotically flat initial data set which is constrained and with positive ADM-energy $E_{\text{ADM}} > 0$. Let $\iota^\sigma : \Sigma^\sigma \hookrightarrow M \setminus C$ the inclusion of the family $\{\Sigma^\sigma\}_{\sigma \ge \sigma_0}$ of CMC-surfaces as above and suppose that there exists the CMC-center of mass of Σ^σ, i.e.*

$$\vec{\mathcal{C}}_{\text{CMC}} := \lim_{\sigma \to \infty} \fint_{\Sigma^\sigma} \vec{x}\, d\mu^\sigma, \tag{5.82}$$

where $d\mu^\sigma$ is the 2-dimensional measure induced by $\overline{g}$ on Σ^Σ. Consider the CSTMC foliation constructed above, and let $\vec{z}_{\Sigma^\sigma_{\text{st}}}$ be the barycenter of $\Sigma^\sigma_{\text{st}} := \lim_{t \to \infty} F_t(\Sigma^\sigma)$.

(i) *If $q > \frac{2}{\frac{1}{2}+\delta}$ then*

$$\lim_{\sigma \to \infty} \vec{z}_{\Sigma^\sigma_{\text{st}}} = \vec{\mathcal{C}}_{\text{CMC}}. \tag{5.83}$$

(ii) *If* $2 \leq q \leq \frac{2}{\frac{1}{2}+\delta}$, *then there exists* $C > 0$ *such that*

$$|\vec{z}_{\Sigma^\sigma_{\mathrm{st}}} - \vec{z}_{\Sigma^\sigma}| \leq C\sigma^{2-\frac{q}{2}-q\delta}. \tag{5.84}$$

Proof. Integrating (5.73) in $[0,t]$ we get

$$||\vec{z}(t)| - |\vec{z}(0)|| \leq C\int_0^t |\Sigma_t|^{-1}\|\mathcal{H} - \hbar\|_{L^1(\Sigma_t)}\, dt \leq C\sigma^{2-\frac{q}{2}-q\delta}\left(1 - e^{-\frac{E_{\mathrm{ADM}}t}{\sigma^3}}\right). \tag{5.85}$$

Since, by construction, $\vec{z}(0) = \vec{z}_{\Sigma^\sigma}$ and $\vec{z}_{\Sigma^\sigma_{\mathrm{st}}} := \lim_{t\to\infty}\vec{z}(t)$, which exists since the flow converges, letting $t \to \infty$ in (5.85) we get

$$|\vec{z}_{\Sigma^\sigma_{\mathrm{st}}} - \vec{z}_{\Sigma^\sigma}| \leq C\sigma^{2-\frac{q}{2}-q\delta}.$$

□

Remark 5.0.17. *(i) Since* $\frac{1}{2}+\delta \in \left(\frac{1}{2}, 1\right]$, *we have that*

$$2 \leq \frac{2}{\frac{1}{2}+\delta} < 4. \tag{5.86}$$

Thus, if $q \geq 4$, *the volume preserving spacetime mean curvature flow recovers the center of mass* $\vec{\mathcal{C}}_{\mathrm{CMC}}$ *for every* $\delta \in (0, \frac{1}{2}]$.

(ii) For $q = 2$, *we recover the foliation constructed in [CS21]. In this case, the right hand side of equation* (5.84) *is divergent, and the theory developed by Cederbaum and Sakovich in [CS21] let us to conclude that* $\{\vec{z}_{\Sigma^\sigma_{\mathrm{st}}}\}_{\sigma\geq\sigma_0}$ *converges if and only if the correction term converges*

$$\lim_{r\to\infty}\int_{\mathbb{S}^2_r} \frac{x^i\left(\sum_{k,l}\pi_{kl}x^kx^l\right)^2}{r^3}\, d\mu^e, \tag{5.87}$$

under the additional hypothesis that $|\overline{\mathrm{K}}| \leq \overline{c}|\vec{x}|^{-2}$.

(iii) Finally, also in the case $q \in \left(2, \frac{2}{\frac{1}{2}+\delta}\right]$ *equation* (5.84) *holds with a positive exponent, and thus, in a case in which the CMC-barycenter does not converge, this does not necessarily imply the non convergence of the CSTMC-barycenter. However, differently from point (ii), where the convergence of the limit* (5.87) *allows to deduce a relation between the two barycenters, for a general* q *we do not know if a similar correction term can be found.*

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