Quiver Grassmannians: Schubert Varieties, Linear Degenerations and Tropical Geometry

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   [IS23] Giulia Iezzi and Victoria Schleis. Tropical Quiver Grassmannians. arXiv preprint arXiv:2311.18648, 2023.

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Zusammenfassung

Köcher und ihre Darstellungen sind ein wichtiges Instrument der Darstellungstheorie: Sie wurden eingeführt, um Probleme der linearen Algebra zu behandeln, bieten aber auch reiche Verbindungen zu anderen mathematischen Bereichen. Ein wesentlicher Vorteil der Verwendung von Köchern für die Untersuchung von Problemen geometrischer Natur ist die Möglichkeit, kombinatorische und algebraische Methoden zu nutzen, um geometrische Eigenschaften der zugehörigen projektiven Varietät, der sogenannten Köcher-Grassmannian, abzuleiten. Der Schwerpunkt der in dieser Arbeit vorgestellten Forschung liegt an der Schnittstelle zwischen Algebra, Geometrie und Darstellungstheorie.

Das Ziel dieser Arbeit ist es zunächst, eine Realisierung von Schubert-Varietäten als Köcher-Grassmannians zu finden, mittels eines Köchers und einer Köcherdarstellung mit bestimmten sinnvollen Eigenschaften. Anschließend wird diese Realisierung ausgenutzt, um lineare Degenerierungen von Schubert-Varietäten zu definieren. Darüber hinaus verallgemeinern wir die Konstruktion des Fahnen-Dressian durch das Konzept des Köcher-Dressian und vergleichen es mit der Tropikalisierung der entsprechenden Köcher-Grassmannian.

Kapitel 1 deckt den notwendigen Hintergrund für die Untersuchung von Köchern und Köcherdarstellungen ab, sowohl aus kategorialer als auch aus geometrischer Sicht.

In Kapitel 2 betrachten wir einen speziellen Köcher mit Relationen und konstruieren eine starre Darstellung dieses Köchers. Wir untersuchen eine bestimmte Untervarietät der Varietät der Darstellungen, beschreiben die Zerlegungen in Unzerlegbare für die Elemente dieser Untervarietät und parametrisieren die B-Isomorphismenklassen, wobei B die Borel-Untergruppe der oberen Dreiecksmatrizen darstellt.

Kapitel 3 fasst grundlegende Fakten und Definitionen über Köcher-Grassmannians, Fahnenvarietäten und ihre Schubert-Varietäten zusammen. Wir beweisen, dass jede Köcher-Grassmannian, die mit der in Kapitel 2 definierten Köcher-Darstellung assoziiert ist, glatt und irreduzibel ist, und ihre Dimension kann leicht mit Hilfe der Euler-Ringel-Form berechnet werden.

Kapitel 4 enthält mehrere unserer wichtigsten Ergebnisse. Wir beweisen, dass jede Permutation eine geometrisch kompatible Zerlegung zulässt - wir führen diesen Begriff in Abschnitt 4.1 ein - und realisieren die Bott-Samelson-Auflösung einer festen Schubert-Varietät unter Verwendung der zuvor betrachteten Köcherdarstellung und Köcher-Grassmannian. Schließlich geben wir durch die Wahl eines anderen, geeigneten Dimensionsvektors für unseren Köcher einen expliziten Isomorphismus zwischen einer beliebigen glatten Schubert-Varietät und der entsprechenden Köcher-Grassmannian.

In Kapitel 5 untersuchen wir lineare Degenerierungen. Der erste Abschnitt fasst kurz lineare Degenerierungen von Fahnenvarietäten zusammen, während wir im zweiten Abschnitt auf den Konstruktionen und Ergebnissen aus den Kapiteln 2, 3 und 4 aufbauen und lineare Degenerierungen von Schubert-Varietäten

definieren. Wir zeigen, wie eine der in Abschnitt 2.3 betrachteten Parametrisierungen die Beziehungen zwischen den B-Bahnen (und ihren Abschlüssen) in der in Abschnitt 2.2 definierten Untervarietät beschreibt. Danach führen wir die Bedingungen auf, die notwendig und hinreichend dafür sind, dass ein Tupel von nichtnegativen ganzen Zahlen die Parametrisierung einer Darstellung in dieser Untervarietät ist. Schließlich eröffnen wir die Diskussion über den flachen Locus der Projektion von der universellen linearen Degenerierung auf den betrachteten Darstellungsraum. Wir präsentieren und motivieren eine Vermutung über diesen flachen Locus.

Kapitel 6 bildet eine Brücke zwischen der Köcherdarstellungstheorie und der tropischen Geometrie, insbesondere durch die Einführung von Köchern von bewerteten Matroiden und die Untersuchung ihrer tropischen Parameterräume. Wir definieren Köcher-Dressians, welche die Teilraumbeziehung von tropischen linearen Räumen nach tropischer Matrixmultiplikation parametrisieren, und zeigen, dass Tropikalisierungen von Köcher-Grassmannians das realisierbare Analogon parametrisieren. Des Weiteren führen wir affine Morphismen von bewerteten Matroiden ein und zeigen die Kompatibilität mit schwach monomialen Köcherdarstellungen. Schließlich zeigen wir, dass Köcher-Dressians ab der Vektorraumdimension 2 nicht realisierbare Punkte haben können.

Abstract

Quivers and their representations theory provide powerful tools, in particular for studying representations of finite-dimensional algebras; they were introduced to treat problems of linear algebra, but present rich connections to diverse mathematical subjects. A key advantage of using quivers for studying problems of geometrical nature is the possibility to exploit combinatorial and algebraic tools to deduce geometric properties of the associated projective variety, the quiver Grassmannian. The main focus of the research presented in this thesis lies at the intersection of algebra, geometry and representation theory.

The aim of this thesis is, firstly, to find a realisation of Schubert varieties as quiver Grassmannians by means of a quiver and quiver representation with certain reasonable properties. Subsequently, this realisation is exploited to define linear degenerations of Schubert varieties. Furthermore, we generalise the construction of the flag Dressian by defining the concept of quiver Dressian and compare it to the tropicalisation of the corresponding quiver Grassmannian.

Chapter 1 covers the necessary background for the study of quivers and quiver representations, both from a categorical and a from a geometric point of view.

In Chapter 2, we consider a special quiver with relations and construct a rigid representation of this quiver. We study a certain subvariety of the variety of representations, describing the decompositions into indecomposables for the elements of this subvariety and parametrising the B-isomorphism classes, where B represents the Borel subgroup of upper-triangular matrices.

Chapter 3 summarises basic facts and definitions about quiver Grassmannians, flag varieties and their Schubert varieties. We prove that any quiver Grassmannian associated to the quiver representation defined in Chapter 2 is smooth and irreducible, and its dimension can be easily computed by means of the Euler-Ringel form.

Chapter 4 contains several of our main results. We prove that every permutation admits a geometrically compatible decomposition - we introduce this notion in Section 4.1 - and realise the Bott-Samelson resolution of a fixed Schubert variety using the quiver representation and quiver Grassmannian considered previously. Lastly, by choosing a different, appropriate dimension vector for our quiver, we give an explicit isomorphism between any chosen smooth Schubert variety and the corresponding quiver Grassmannian.

In Chapter 5, we explore linear degenerations. The first section briefly recalls linear degenerations of flag varieties, while in the second section we build upon the constructions and results obtained in Chapter 2, 3 and 4 and define linear degenerations of Schubert varieties. We show how one of the parametrisations considered in Section 2.3 describes the relations between the B-orbits (and their closures) in the subvariety defined in Section 2.2 and list the conditions that are necessary and sufficient for a tuple of non-negative integers to be the parametrisation of some representation in this subvariety. Finally,

we open the discussion on the flat locus of the projection from the universal linear degeneration onto the considered representation space. We present and motivate a conjecture about this flat locus.

Chapter 6 is dedicated to a bridge between quiver representation theory and tropical geometry, in particular to the introduction of quivers of valuated matroids and the study of their tropical parameter spaces. We define quiver Dressians, which parametrise containment of tropical linear spaces after tropical matrix multiplication, and show that tropicalisations of quiver Grassmannians parametrise the realisable analogue. We further introduce affine morphisms of valuated matroids and show compatibility with weakly monomial quiver representations. Finally, we show that, starting in ambient dimension 2, quiver Dressians can have nonrealisable points.

Contents

Introduction

Flag varieties are a significant class of geometric objects in algebraic geometry. They are central in several mathematical disciplines, including commutative algebra, representation theory, and combinatorics, and their rich geometric and combinatorial structure make them a fascinating subject of study. The term flag, introduced around the middle of the 20th century, stands for a sequence of subspaces inside an ambient vector space. The definition of generalised flag variety, however, refers to the quotient of a semisimple algebraic group over a Borel subgroup. The orbits of the Borel subgroup form a stratification of this quotient, where the cells - called *Schubert cells* - are isomorphic to affine spaces.

Their Zariski closures, the *Schubert subvarieties*, were first analysed in 19th-century classical projective geometry in the context of Schubert calculus [Sch79]. Today, Schubert varieties are among the most thoroughly studied examples of complex projective varieties in the literature and occupy a central role in contemporary mathematical research as well as outside of mathematics [HLR18, LMP22, BL24] [Kos95, TH91, Tel92]. In 1934, Ehresmann ([Ehr34]) demonstrated that the Schubert subvarieties in the Grassmannian form an additive basis for its cohomology ring. In 1956, Chevalley ([Che94]) further advanced this connection, proving that the Schubert variety classes in the generalised flag variety form a $\mathbb{Z}$-basis for the Chow ring of the generalised flag variety. Mathematicians have been investigating smoothness and singularities in Schubert varieties since the 1970s. In particular, considerable attention has been given to the problem of determining the singular loci of Schubert varieties, which lie at the intersection of multiple mathematical areas of study, for instance in works by Demazure, Chevalley, Lakshmibai and Seshadri ([Dem74, Che94, LS84]).

Quivers and their representations theory provide powerful tools, in particular for studying representations of finite-dimensional algebras. *Quiver representations* are assignments of vector spaces and linear maps to the vertices and arrows, respectively, of a directed graph. According to Brion [Bri08], they were introduced to treat problems of linear algebra, for instance the classification of tuples of subspaces of a prescribed vector space, but present rich connections to diverse mathematical subjects (quantum groups, Coxeter groups, and geometric invariant theory among others).

A *quiver Grassmannian* is the projective variety which arises when we fix a quiver representation and parametrise the subrepresentations with fixed dimensions. Quiver Grassmannians first appeared in works of Crawley-Boevey and Schofield [CB89, Sch92] and were later employed in the theory of cluster algebras by Fomin, Zelevinsky, Caldero, Chapoton, Keller, Derksen and Weyman ([FZ02a, FZ02b, FZ07, CC06, CK08, CK06, DWZ10]). Among other aspects, their rationality, cell decompositions and cohomology were investigated by Cerulli Irelli, Esposito, Franzen and Reineke in [IEFR21, Fra19].

A key advantage of using quivers for studying problems of geometrical nature is the possibility to exploit combinatorial and algebraic tools to deduce geometrical properties of the considered projective variety. A few examples of this can be found in results by Cerulli Irelli, Esposito, Franzen, Feigin and Reineke, for instance in [CIFR17, IEFR21, IFR13]. However, as proven by Reineke in [Rei13] and, more generally, by Ringel in [Rin18], every projective variety arises as the quiver Grassmannian of any wild acyclic quiver. This implies that the study of quiver Grassmannians needs to be restricted in order to be meaningful, for instance by considering particular quivers or quiver representations.

The fruitful investigation of *linear degenerations of flag varieties* using quiver Grassmannians took place in this framework in the last fifteen years, appearing in several papers by Feigin, Finkelberg, Cerulli Irelli, Reineke, Fang, Fourier [Fei12, CIFR12, FF13, CIFF$^+$17, CIFF$^+$20]. The word "degeneration" refers to a construction that allows us to regard a variety as a specific element in a family of varieties, or as a chosen fibre of a certain morphism; the above-cited works characterise several geometric and combinatorial aspects of linear degenerations of flag varieties, such as their defining equations, cellular decompositions and - making use of rank tuples - flatness, irreducibility and normality.

More recently, linear degenerations have been introduced and studied in the context of tropical geometry. *Tropical algebraic geometry* transforms questions about algebraic varieties into questions about polyhedral complexes, which are combinatorial objects that encode some of the geometry of the original algebraic variety. Due to the close relationship between classical and tropical geometry, results and techniques from one domain can often be translated to the other. Algebraic varieties can be mapped to their tropical counterparts, and this process preserves certain geometric properties of the original variety. As a result, tropical geometry provides a valuable framework for proving and generalizing classical results in algebraic geometry, such as the Brill–Noether theorem and the computation of Gromov–Witten invariants, as done by Mikhalkin, Cools, Draisma, Payne, and Robeva in [Mik05, CDPR12].

Tropical linear spaces correspond to valuated matroids. They are foundational objects which are not only of intrinsic importance for the area of tropical geometry, in which they appear as the building blocks for tropical manifolds and tropical ideals (introduced and studied by Maclagan and Rincón in [MR18]) and parametrise hyperplane arrangements, but also connect tropical geometry and matroid theory. The tropical analogue of flag varieties, the *flag Dressian*, was defined by Haque [Haq12, Definition 1] and further analysed in [BEZ21] by Brandt, Eur and Zhang. In [BS23], Borzì and Schleis consider linear degenerate valuated flag matroids and their associated linear degenerate flags of tropical linear spaces.

The aim of this thesis is, firstly, to find a realisation of Schubert varieties as quiver Grassmannians by means of a quiver and quiver representation with certain reasonable properties. Subsequently, this realisation is exploited to define linear degenerations of Schubert varieties. Furthermore, we generalise the construction of the flag Dressian by defining the concept of quiver Dressian and compare it to the tropicalisation of the corresponding quiver Grassmannian.

Outline

We describe the structure of this thesis and present the main results.

Chapter 1 covers the necessary background for the study of quivers and quiver representations. We define and provide examples of path algebras $\mathbb{K}Q$ associated to a quiver Q - with relations as well as without - and discuss the category $\mathrm{rep}_{\mathbb{K}}(Q)$ of Q-representations. This includes indecomposable representations, direct sums, and the Krull-Schmidt theorem, as well as the characterisation of quivers of finite representation type (Gabriel's theorem, see Figure 1.1). We recall some concepts from homological algebra, in particular projective and injective representations, resolutions, and extensions of Q-representations. The last section is dedicated to the variety $R_{\mathbf{d}}$ of Q-representations and its orbits under the action of a base change group, which is a product of general linear groups.

Our first results appear in Chapter 2; from this point forward, we work on the field of complex numbers $\mathbb{C}$. In the first section, we consider a specific quiver with relations (Γ, I) and prove the rigidity of a certain (Γ, I)-representation, denoted by M, in Proposition 2.1.5. In the second section, we begin the study of the subvariety $R^{\iota}_{\mathbf{d}}$ in the variety of representations of (Γ, I). Each representation in $R^{\iota}_{\mathbf{d}}$ is determined by a tuple of linear maps in $\prod_{j=1}^{n-1} U_{n+1}$, where U_{n+1} is the subset of Mat_{n+1} (the set of square matrices of size $n+1$) consisting of upper-triangular matrices. One notable property of this subvariety is the uniformity of the decompositions of any of its representations. We define a class of indecomposables of (Γ, I), denoted by $U^{(h_1,\ldots,h_n)}$, and prove in Theorem 2.2.12 that all representations in $R^{\iota}_{\mathbf{d}}$ can be decomposed as direct sums of such indecomposables.

The last section of Chapter 2 answers the question of the representation type of (Γ, I) - that is, whether there are finitely many isomorphism classes of indecomposable representations of (Γ, I)- when we restrict to the representations in $R^{\iota}_{\mathbf{d}}$ and to certain isomorphisms. In order to do so, we consider the orbits in $R^{\iota}_{\mathbf{d}}$ under the action defined as

$$h \cdot M^f = (h_2 f^1_{n+1} h_1^{-1}, h_3 f^2_{n+1} h_2^{-1}, \ldots, h_n f^{n-1}_{n+1} h_{n-1}^{-1}),$$

for some $h \in \prod_{j=1}^{n-1} B_{n+1}$, where B_{n+1} is the Borel subgroup of invertible upper-triangular matrices inside the general linear group, and $M^f \in R^{\iota}_{\mathbf{d}}$. To simplify notation, we refer to this action as B-action and to such orbits as B-orbits or, equivalently, as B-isomorphism classes in $R^{\iota}_{\mathbf{d}}$. Notice, however, that the group acting on $R^{\iota}_{\mathbf{d}}$ is the product $\prod_{j=1}^{n-1} B_{n+1}$, acting via base change as given above. In [ADF85], Abeasis and Del Fra parametrise the isomorphism classes of the representations of any quiver of type $\mathbb{A}_n$. We employ an analogous parametrisation, denoted by $\mathbf{r}$, and make use of our previous results to prove the following:

Theorem A. *Two representations M^f, M^g in $R^{\iota}_{\mathbf{d}}$ are in the same B-isomorphism class if and only if $\mathbf{r}^f = \mathbf{r}^g$.*

Finally, we provide an alternative parametrisation for such B-isomorphism classes. We notice that our B-action can be regarded as an expansion of the one considered in [MS05] by Miller and Sturmfels in the context of matrix Schubert varieties. There, the orbits are described in terms of ranks of certain submatrices, namely the north-west ranks. We adapt this parametrisation to our B-action by defining the south-west arrays, denoted by $\mathbf{s}$, and showing the following:

Theorem B. *Two representations M^f, M^g in $R^{\iota}_{\mathbf{d}}$ are in the same B-isomorphism class if and only if $\mathbf{s}^f = \mathbf{s}^g$.*

Chapter 3 summarises basic facts and definitions about quiver Grassmannians, flag varieties and their Schubert varieties. Additionally, we prove the following result:

Theorem C. *Given (Γ, I) and M as in Chapter 2, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is a smooth and irreducible projective variety for any dimension vector $\mathbf{e}$. Its dimension is $\langle \mathbf{e}, \mathbf{dim}(M) - \mathbf{e} \rangle$.*

Chapter 4 contains several of our main results. We consider the Schubert varieties X_w in the flag variety Fl_{n+1}, which are indexed by permutations w in the symmetric group S_{n+1}, and define a dimension vector $\mathbf{r}^w$ for the quiver (Γ, I) according to a fixed w. The principal goal of this chapter is to give an explicit isomorphism between the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ and certain resolutions of X_w. For this purpose, we introduce the notion of geometrically compatible decompositions and show that all permutations admit such a reduced decomposition. We prove this result by means of several lemmas and Theorem 4.1.14. Then, we are able to prove he following theorem:

Theorem D. *Given a permutation $w \in S_{n+1}$ and a geometrically compatible decomposition of w, the corresponding Bott-Samelson resolution of the Schubert variety X_w is isomorphic to the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$.*

In the second section of Chapter 4, we only consider permutations w in S_{n+1} that yield smooth Schubert varieties X_w and define a different dimension vector $\mathbf{e}^w$ based on w. Here, we show that the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M)$ is isomorphic to the Schubert variety X_w and provide an explicit isomorphism.

In chapter 5, we explore linear degenerations. In the first section, we recall the construction of linear degenerations of flag varieties following [CIFF$^+$17]: given an equioriented quiver of type $\mathbb{A}_n$, the linear degeneration of Fl_{n+1}, denoted by Fl^f_{n+1}, is defined as the quiver Grassmannian consisting of all subrepresentations of the following representation:

$$\overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{f_1} \overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} \overset{\mathbb{C}^{n+1}}{\bullet}$$

for the dimension vector $(1, 2, \dots, n)$. The quiver Grassmannian corresponding to the choice of f can be regarded as the fibre of the projection $\pi : Y \to R$ over f, where R is the variety Mat^{n-1}_{n+1} and Y is the universal quiver Grassmannian, i.e. the variety of compatible pairs of sequences of maps and sequences of subspaces. In particular, we recall a result about the flat locus of π, namely that it is the union of all orbits that degenerate to the orbit parametrised by a certain rank tuple.

In the following section, we build upon the constructions and results obtained in Chapter 2, 3 and 4. We define the linear degeneration of the Schubert variety X_w, denoted by X^f_w, as the fibre of $\pi : Y \to R^\iota_{\mathbf{d}}$ over f, where $R^\iota_{\mathbf{d}}$ is the subvariety of (Γ, I)-representations considered in Section 2.2 and Y is the universal quiver Grassmannian. We show in Corollary 5.2.4 that it is possible to deduce the partial ordering on the B-orbits in $R^\iota_{\mathbf{d}}$ from the south-west parametrisation investigated in section 2.3. This is the partial ordering induced by the relations of the form $\mathcal{O}^\iota_{M^g} \subseteq \overline{\mathcal{O}}^\iota_{M^f}$, where $\overline{\mathcal{O}}^\iota_{M^f}$ denotes the Zariski closure of $\mathcal{O}^\iota_{M^f}$ in $R^\iota_{\mathbf{d}}$. To conclude the discussion on the south-west parametrisation, we list the conditions that are necessary and sufficient for a tuple of non-negative integers to be the south-west array of some representation in $R^\iota_{\mathbf{d}}$.

Lastly, we present a strategy to determine the flat locus of the projection $\pi : Y \to R^\iota_{\mathbf{d}}$ and, together with some motivations, our conjecture:

Conjecture 0.0.1. A tuple $f \in R^\iota_{\mathbf{d}}$ is in the flat locus of $\pi : Y \to R^\iota_{\mathbf{d}}$ if and only if $\dim(\mathrm{Gr}_{\mathbf{e}^w}(M^f)) = \dim(X_w)$.

The final chapter of this thesis, Chapter 6, is dedicated to a bridge between quiver representation theory and tropical geometry and is based on a joint project between the author of this thesis and Victoria Schleis, who already contributed to the study of linear degenerations of tropical flag varieties in [BS23].

We generalise the construction of the flag Dressian by defining the concept of quiver Dressian, that is, the projective tropical prevariety cut out by the tropicalised quiver Plücker relations. The idea behind this generalisation is to parametrise the tropical linear spaces satisfying the containment conditions described by the arrows of the fixed quiver. Alongside this idea, we studied tropicalisations of quiver Grassmannians (i.e. pointwise tropicalisations) and compared them to the corresponding quiver Dressians. The result we obtained answers the question of the realisability of the quiver Dressian, that is, whether we can find a point in the quiver Dressian that is not the tropicalisation of any quiver representation. We found that the tropicalised quiver Grassmannian is always contained in the quiver Dressian, and the first nonrealisable quiver subrepresentation occurs in ambient dimension 2 - as opposed to the case of tropical linear spaces, where the first example of nonrealisability occurs in ambient dimension 8, and to that of flags of tropical linear spaces, where the first example of nonrealisability occurs in ambient dimension 6.

For a finite quiver Q, a Q-representation M and a fixed dimension vector $\mathbf{d} = (d_1, \ldots, d_k)$ we define the *projective quiver Dressian* $\overline{\mathrm{QDr}}(R, \mathbf{d})$ as the projective tropical prevariety cut out by the tropicalised quiver Plücker relations (given in Definition 3.1.4). We prove the following main result, generalising [BS23, Theorem A]:

Theorem E. *Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ be valuated matroids and Q be a finite quiver. In 6.2.5, we give necessary and sufficient conditions such that $\boldsymbol{\mu} \in \overline{\mathrm{QDr}}(R, \mathbf{d})$.*

In words, the quiver Dressian parametrises tropical linear spaces satisfying the containment conditions described by the quiver.

Furthermore, for quivers with maps defined by *weakly monomial* matrices (c.f. Definition 6.1.18) we connect in Theorem 6.2.7 quiver Dressians to morphisms of valuated matroids.

Then, we obtain an analogous theorem for the tropicalisation of the quiver Grassmannian and realisable tropical linear spaces:

Theorem F. *Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ be valuated matroids and Q be a finite quiver. Let all realisations be in algebraically closed fields with nontrivial valuation. We show in 6.2.2 that the tropicalised quiver Grassmannian $\overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(R))$ parametrises containment of tropicalised linear spaces under tropical matrix multiplication*

Lastly, we analyse the realisability of points in quiver Dressians, i.e. the relation between tropicalised quiver Grassmannians and quiver Dressians.

Theorem G. *For any Q-representation R, $\overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(R)) \subseteq \overline{\mathrm{QDr}}(R, \mathbf{d})$. Further, for any finite quiver Q and any Q-representation R assigning dimension 1 to each vertex, $\overline{\mathrm{QDr}}(R, \mathbf{d}) = \overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(R))$. The same is not true in higher dimension: for any pair $n_1, n_2 \geq 2$ there exist a quiver Q and a Q-representation R containing an arrow α with $\dim(s(\alpha)) = n_1$ and $\dim(n_{t(\alpha)}) = n_2$ where the above containment is strict.*

Part of the results contained in Chapters 2 and 3, as well as the results of Chapter 4, can be found in [Iez25]. The results in Chapter 6 are available at [IS23] and are the product of joint work carried out by the author of this thesis and Victoria Schleis as part of the project A11: “Linear degenerate flag varieties and their tropical counterparts” of the SFB-TRR 195 of the German Research Foundation.

At the end of Chapters 5 and 6, we present a few possible research directions and interesting questions that arise naturally from the results presented in this thesis.

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Chapter 1

Quivers and quiver representations

We introduce quivers and quiver representations in their generality and fix for this section an algebraically closed field $\mathbb{K}$; later in this chapter, we will restrict our study to the field of complex numbers. Standard references are [Bri05, CB92, Sch14].

1.1 The path algebra

Definition 1.1.1. A **quiver** $Q = (Q_0, Q_1, s, t)$ is given by a set of vertices Q_0, a set of arrows Q_1 and two maps $s, t : Q_1 \to Q_0$ assigning to each arrow its source, resp. target. We say that a quiver Q is finite if Q_0 and Q_1 are finite.

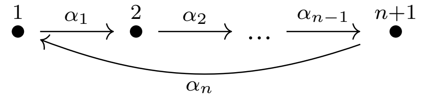
A non-trivial **path** in Q is a sequence $\alpha_1, \ldots, \alpha_n$ of arrows that satisfies $s(\alpha_{i+1}) = t(\alpha_i)$ for $1 \leq i \leq n-1$, i.e. the arrows can be concatenated:

$$\overset{1}{\bullet} \xrightarrow{\alpha_1} \overset{2}{\bullet} \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} \overset{n+1}{\bullet} .$$

For any vertex $i \in Q_0$, the trivial path at i is the path of length zero that starts and terminates at i.

Definition 1.1.2. The **path algebra** $\mathbb{K}Q$ is the $\mathbb{K}$-algebra whose basis consists of all paths in Q. The product of two paths p_1, p_2 is given by concatenation if $s(p_2) = t(p_1)$ and zero otherwise, and the identity element in $\mathbb{K}Q$ is the sum of all trivial paths in Q.

Example 1.1.3. A path of the form



is called **oriented cycle** of length n. An oriented cycle of length 1 is a **loop**.

Example 1.1.4. If $Q = \bullet \circlearrowleft$, the quiver consisting of one vertex and one loop, then the path algebra $\mathbb{K}Q$ is isomorphic to $\mathbb{K}[x]$, the polynomial algebra in one variable. If Q has one vertex and r loops, then the path algebra $\mathbb{K}Q$ is the free algebra $\mathbb{K}\langle x_1, \ldots, x_r\rangle$ whose basis is given by all the words (or non-commutative monomials) in $x_1, \ldots, x_r$.

It is straightforward to check that a path algebra $\mathbb{K}Q$ is finite-dimensional if and only if Q is finite and has no oriented cycles.

Definition 1.1.5. A **relation** on a quiver Q is a subspace of the path algebra of Q spanned by linear combinations of paths with common source and target, of length at least 2. Given a two-sided ideal I of $\mathbb{K}Q$ generated by relations, the pair (Q, I) is a **quiver with relations** and the quotient algebra $\mathbb{K}Q/I$ is the path algebra of (Q, I).

A system of relations for I is defined as a subset R of $\cup_{i,j\in Q_0} iIj$, where i denotes the trivial path on vertex i, such that R, but no proper subset of R, generates I as a two-sided ideal. For any two vertices i and j, we denote by $r(i, j, R)$ the cardinality of $R \cap iIj$, which contains those elements in R that are linear combinations of paths starting in i and ending in j. If Q contains no oriented cycle, then the numbers $r(i, j, R)$ are independent of the chosen system of relations (see for instance [Bon83]), and can therefore be denoted by $r(i, j)$.

Example 1.1.6. If Q has one vertex and r loops, a system of relations is for instance given by the ideal I of $\mathbb{K}\langle x_1, \ldots, x_r\rangle$ generated by all commutators $x_i x_j - x_j x_i$. Then, the path algebra of (Q, I) is the polynomial algebra $\mathbb{K}[x_1, \ldots, x_r]$.

Remark 1.1.7. The representations of any arbitrary finite-dimensional algebra can be obtained by considering an appropriate quiver with relations, while finite-dimensional path algebras (without relations) represent only some special cases of all finite-dimensional algebras.

1.2 The category of Q-representations

Definition 1.2.1. Given a quiver Q, we define a Q-**representation** M over $\mathbb{K}$ as the ordered pair $((M_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1})$, where M_i is a $\mathbb{K}$-vector space attached to vertex $i \in Q_0$ and $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a $\mathbb{K}$-linear map for any $\alpha \in Q_1$. A representation M is called finite-dimensional if each vector space M_i is finite-dimensional, and in this case the **dimension vector** of M is $\mathbf{dim}\, M := (\dim_{\mathbb{K}} M_i)_{i\in Q_0} \in \mathbb{Z}_{\geq 0}^{|Q_0|}$.

Example 1.2.2. To define a finite-dimensional representation $M = (M_1, M_\alpha)$ of $Q = \bullet \circlearrowleft$ means choosing the dimension m of the vector space $M_1 = \mathbb{K}^m$ and an endomorphism M_α of $\mathbb{K}^m$.

Definition 1.2.3. Given a quiver Q and two Q-representations M and M', a **morphism** $\psi : M \to M'$ is a collection of $\mathbb{K}$-linear maps $(\psi_i : M_i \to M'_i)_{i \in Q_0}$ such that

$$\begin{array}{ccc} \overset{M_{s(\alpha)}}{\bullet} & \xrightarrow{M_\alpha} & \overset{M_{t(\alpha)}}{\bullet} \\ \Big\downarrow \psi_{s(\alpha)} & /\!\!/\!\!/ & \Big\downarrow \psi_{t(\alpha)} \\ \overset{M'_{s(\alpha)}}{\bullet} & \xrightarrow{M'_\alpha} & \overset{M'_{t(\alpha)}}{\bullet} \end{array},$$

that is $M'_\alpha \circ \psi_{s(\alpha)} = \psi_{t(\alpha)} \circ M_\alpha$ for any $\alpha \in Q_1$.

Proposition 1.2.4. *([Sch14, Proposition 1.1]) The set* $\mathrm{Hom}_Q(M, M')$ *of morphisms between two Q-representations M and M' is a $\mathbb{K}$-vector space with respect to addition and scalar multiplication.*

Example 1.2.5. Let Q be the quiver $\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}$ and consider its two representations $M : \overset{\mathbb{K}}{\bullet} \xleftarrow{\mathrm{id}} \overset{\mathbb{K}}{\bullet} \xrightarrow{\mathrm{id}} \overset{\mathbb{K}}{\bullet}$ and $M' : \overset{\mathbb{K}}{\bullet} \xleftarrow{\mathrm{id}} \overset{\mathbb{K}}{\bullet} \xrightarrow{0} \overset{0}{\bullet}$. The vector space $\mathrm{Hom}_Q(M, M')$ has dimension one and is generated by the morphism $(\mathrm{id}, \mathrm{id}, 0)$, whereas $\mathrm{Hom}_Q(M', M) = 0$: the only assignment of linear maps (ψ_1, ψ_2, ψ_3) such that

$$\begin{array}{ccccc} \overset{\mathbb{K}}{\bullet} & \xleftarrow{\mathrm{id}} & \overset{\mathbb{K}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow \psi_1 & /\!\!/\!\!/ & \downarrow \psi_2 & /\!\!/\!\!/ & \downarrow \psi_3 \\ \overset{\mathbb{K}}{\bullet} & \xleftarrow{\mathrm{id}} & \overset{\mathbb{K}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{K}}{\bullet} \end{array}$$

is the zero morphism $(0, 0, 0)$.

The composition of two morphisms $\psi : M \to M'$ and $\psi' : M' \to M''$ is defined as $\psi'_i \circ \psi_i : M_i \to M''_i$ for all $i \in Q_0$; it is associative, bilinear, and has as identity element the morphism $(\mathrm{id}_{M_i})_{i \in Q_0}$. Together with such morphisms, the finite-dimensional Q-representations over $\mathbb{K}$ form a category, denoted by $\mathrm{rep}_{\mathbb{K}}(Q)$. Similarly, the category $\mathrm{rep}_{\mathbb{K}}(Q, I)$ consists of the finite-dimensional representations of Q that satisfy the given relations in I. It is known (see [Sch14, Theorem 5.4] for a proof) that $\mathrm{rep}_{\mathbb{K}}(Q)$ is equivalent to the category $\mathbb{K}Q$-mod of finite-dimensional left modules over the path algebra of Q, and that it is abelian, Krull-Schmidt and hereditary. Because of this equivalence, we sometimes apply to quiver representations terminology that originally refers to modules. Two relevant functors on the category of quiver representations are the Hom functors $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$, for any fixed $X \in \mathrm{rep}_{\mathbb{K}}(Q)$. $\mathrm{Hom}(X, -)$ is the covariant functor from $\mathrm{rep}_{\mathbb{K}}(Q)$ to the category of $\mathbb{K}$-vector spaces that sends an object Y to the vector space $\mathrm{Hom}(X, Y)$ of all morphisms from X to Y and a morphism $(f : Y \to Z)$ to the map $f_* : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $f_*(g) = f \circ g$ (the push forward of f). Similarly, $\mathrm{Hom}(-, X)$ is the contravariant functor from $\mathrm{rep}_{\mathbb{K}}(Q)$ to the category of $\mathbb{K}$-vector spaces that sends an object Y to the vector space Hom(Y, X) of all morphisms from Y to X and a morphism $(f : Y \to Z)$ to the map $f_* : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Y, X)$, $f^*(g) = g \circ f$ (the pull back of f).

Definition 1.2.6. Let Q be a finite quiver and $M = ((M_i)_{i \in Q_0}, (M_\alpha)_{\alpha \in Q_1})$, $M' = ((M'_i)_{i \in Q_0}, (M'_\alpha)_{\alpha \in Q_1})$ two representations in $\mathrm{rep}_{\mathbb{K}}(Q)$. The **direct sum** of M and M' is the Q-representation defined as

$$M \oplus M' = \Big((M_i \oplus M'_i)_{i \in Q_0}, (\begin{bmatrix} M_\alpha & 0 \\ 0 & M'_\alpha \end{bmatrix})_{\alpha \in Q_1}\Big).$$

Example 1.2.7. [Sch14, Example 1.7] We fix $Q = \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longleftarrow \overset{3}{\bullet}$ and two Q-representations:

$$M : \overset{\mathbb{K}}{\bullet} \xrightarrow{\mathrm{id}} \overset{\mathbb{K}}{\bullet} \xleftarrow{0} \overset{0}{\bullet}$$

$$M' : \overset{\mathbb{K}^2}{\bullet} \xrightarrow{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]} \overset{\mathbb{K}^2}{\bullet} \xleftarrow{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]} \overset{\mathbb{K}}{\bullet} .$$

Then, the direct sum of M and M' is the representation

$$M \oplus M' : \overset{\mathbb{K} \oplus \mathbb{K}^2}{\bullet} \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right]} \overset{\mathbb{K} \oplus \mathbb{K}^2}{\bullet} \xleftarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]} \overset{0 \oplus \mathbb{K}}{\bullet} ,$$

which is isomorphic to

$$M \oplus M' : \overset{\mathbb{K}^3}{\bullet} \xrightarrow{\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right]} \overset{\mathbb{K}^3}{\bullet} \xleftarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]} \overset{\mathbb{K}}{\bullet} .$$

Definition 1.2.8. A representation $M \in \mathrm{rep}_{\mathbb{K}}(Q)$ is called **indecomposable** if it is nonzero and cannot be written as a direct sum of nonzero representations, that is: if $M \cong M' \oplus M''$ with $M', M'' \in \mathrm{rep}_{\mathbb{K}}(Q)$, then $M' = 0$ or $M'' = 0$.

Remark 1.2.9. Indecomposable Q-representations are supported on connected subquivers of Q, since any representation of a nonconnected subquiver of Q can be written as the direct sum of two representations whose supports are the connected components of the subquiver.

Theorem 1.2.10 (Krull-Schmidt theorem)**.** *Let Q be a quiver and M a representation in* $\mathrm{rep}_{\mathbb{K}}(Q)$. *Then*

$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_s$$

where the summands M_i are indecomposable representations in $\mathrm{rep}_{\mathbb{K}}(Q)$ *and are unique up to order.*

A central problem of quiver representation theory is to "describe the isomorphism classes of finite-dimensional representations of a prescribed quiver, having a prescribed dimension vector" ([Bri05, Section 1.1]). The Krull-Schmidt theorem ensures that, in order to describe such isomorphism classes, it is enough to classify the isomorphism classes of indecomposable representations and the morphisms between them.

Example 1.2.11. [Bri05, Example 1.1.5] If Q is a loop, a morphism between two Q-representations $M = (M_1, M_\alpha)$ and $M' = (M'_1, M'_\alpha)$ is a linear map $\psi : M_1 \to M'_1$ such that $\psi \circ M_1 = M'_1 \circ \psi$. In particular, an endomorphism of an n-dimensional representation $M = (M_1, M_\alpha)$ is an endomorphism of M_1 that commutes with M_α. To be able to describe isomorphism classes of such representations, we fix a basis $\{b_1, \ldots, b_n\}$ of M_1 and identify M_α with an $n \times n$ matrix A. Finding a Q-representation that is isomorphic to M means then to change the basis of M_1, replacing A with a conjugate BAB^{-1}, where B is an $n \times n$ invertible matrix. It follows that the isomorphism classes of n-dimensional representations of Q correspond bijectively to the conjugacy classes of $n \times n$ matrices. In particular, there are infinitely many isomorphism classes of representations of the loop having a prescribed dimension.

More generally, if Q is the quiver with one vertex and r loops, the isomorphism classes of representations of Q having a prescribed dimension n correspond bijectively to the r-tuples of $n \times n$ matrices up to simultaneous conjugation.

We say that a quiver is of **finite representation type** if the number of isomorphism classes of its indecomposable representations is finite.

Theorem 1.2.12 (Gabriel's Theorem)**.** *A connected quiver is of finite representation type if and only if its underlying graph is one of the Dynkin diagrams of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$.*

To each Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ corresponds an extended Dynkin diagram, as shown in Figure 1.1. An acyclic quiver whose underlying graph is an extended Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ is called **tame** or **affine**. If a quiver is tame, then the isomorphism classes of its indecomposable representations can be parametrised by a finite number of 1-parameter families. In other words, the classification of the indecomposable representations is only possible if the quiver is Dynkin or tame; if the quiver is neither, then we call it **wild**.

Now we briefly discuss three notable types of quiver representations - simple, projective and injective representations - and describe them in Example 1.2.14 for the special case of the indecomposable representations of the equioriented Dynkn quiver of type $\mathbb{A}$. As shown in [Sch14, Section 2.1], these representations in the category $\text{rep}_\mathbb{K}(Q)$ are respectively simple, projective or injective objects in the categorical sense.

Definition 1.2.13. A representation $P \in \text{rep}_\mathbb{K}(Q)$ is called **projective** if the functor $\text{Hom}(P, -)$ maps surjective morphisms to surjective morphisms. Dually, $I \in \text{rep}_\mathbb{K}(Q)$ is called **injective** if the functor $\text{Hom}(-, I)$ maps injective morphisms to injective morphisms.

If Q is a quiver without oriented cycles, then to each vertex $i \in Q_0$ corresponds exactly one indecomposable projective representation, denoted by $P(i)$. Such projective representations are easy to describe: the basis of the vector

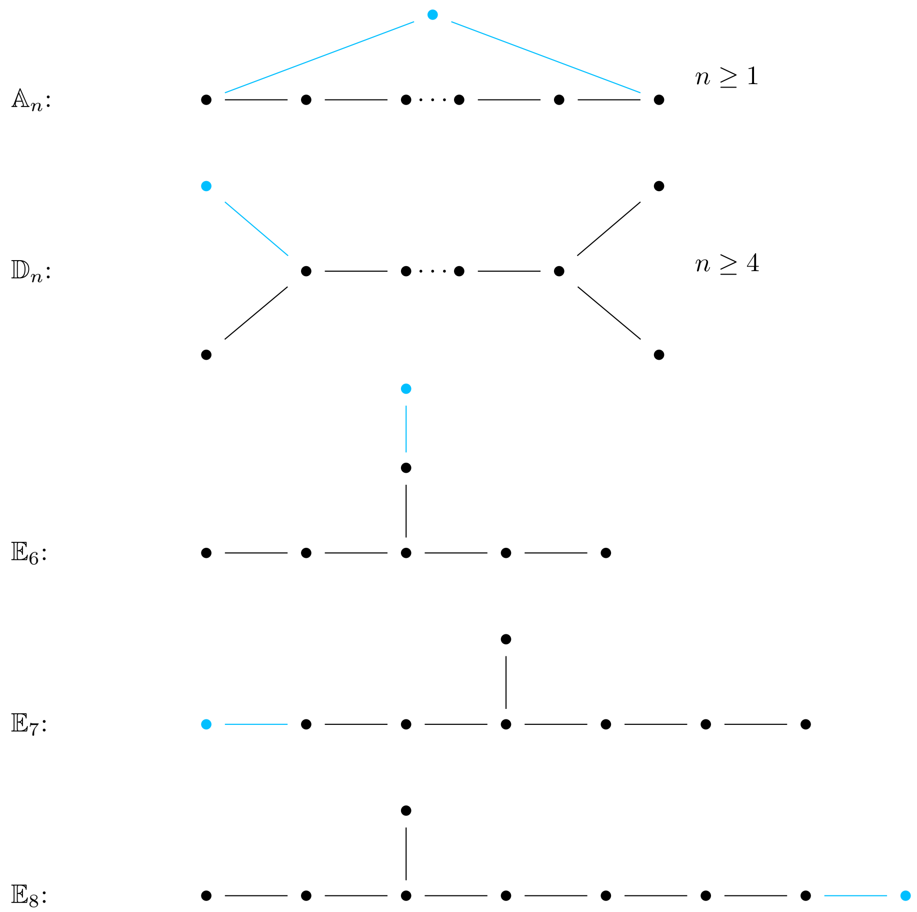


Figure 1.1: Dynkin diagrams and their extensions

space $P(i)_k$ at vertex k is given by the set of all possible paths from vertex i to vertex k, and the actions of the maps between the vector spaces are induced by the concatenation of paths. Dually, for every vertex $i \in Q_0$ there is exactly one indecomposable injective representation $I(i)$, whose basis for each vector space $I(i)_k$ is given by the set of all possible paths from vertex k to vertex i and whose maps act by concatenation of paths.

The **simple** representation $S(i)$ at vertex $i \in Q_0$ is defined by

$$S(i)_k = \begin{cases} \mathbb{K} & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}, \quad S(i)_\alpha = 0 \quad \forall \alpha \in Q_1.$$

For a proof of the fact that the representations $P(i), I(i)$ and $S(i)$ are actually indecomposable we refer to [Sch14, Proposition 2.8]. Lastly, we mention thin quiver representations: a Q-representation M is called **thin** if $\dim(M_i) \leq 1$ for all $i \in Q_0$.

Example 1.2.14. We consider the equioriented Dynkin quiver Q of type $\mathbb{A}$ with n vertices:

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n-1}{\bullet} \longrightarrow \overset{n}{\bullet}\ .$$

As proved in [Rin16, Theorem 1], any representation of a quiver of type $\mathbb{A}$ (with any orientation) is the direct sum of thin indecomposable representations. We denote the thin indecomposable representations of Q by $U_{i,j}$, where the indices $i, j : 1 \leq i \leq j \leq n$ are first and last vertex of the subquiver of Q where the indecomposable is supported:

$$U_{i,j}: \quad \underset{1}{\overset{0}{\bullet}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{i}{\overset{\mathbb{C}}{\bullet}} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \underset{j}{\overset{\mathbb{C}}{\bullet}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{n}{\overset{0}{\bullet}}\ .$$

The projective, injective and simple representations of Q then correspond, respectively, to $U_{i,n}$, $U_{1,i}$ and $U_{i,i}$:

$$P(i): \quad \underset{1}{\overset{0}{\bullet}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{i}{\overset{\mathbb{C}}{\bullet}} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \underset{n}{\overset{\mathbb{C}}{\bullet}}$$

$$I(i): \quad \underset{1}{\overset{\mathbb{C}}{\bullet}} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \underset{i}{\overset{\mathbb{C}}{\bullet}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{n}{\overset{0}{\bullet}}$$

$$S(i): \quad \underset{1}{\overset{0}{\bullet}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{i}{\overset{\mathbb{C}}{\bullet}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{n}{\overset{0}{\bullet}}$$

We will discuss in Section 2.3 a parametrisation introduced in [ADF85] for the representations of quivers of type $\mathbb{A}$, which associates to a fixed representation the indecomposables (and their multiplicities) that appear in its decomposition.

Definition 1.2.15. A vertex u in Q_0 is a **source** of the quiver Q if there are no arrows α in Q_1 such that $t(\alpha) = u$. Dually, a vertex v in Q_0 is a **sink** of the quiver Q if there are no arrows α in Q_1 such that $s(\alpha) = v$.

Remark 1.2.16. The projective representation at vertex i is the simple representation at vertex i if and only if i is a sink of Q. The injective representation at vertex i is the simple representation at vertex i if and only if i is a source of Q.

The following result holds in any additive category.

Proposition 1.2.17. *Let P, P', I and I' be representations of Q. Then:*

1. *$P \oplus P'$ is projective $\iff$ P and P' are projective;*

2. *$I \oplus I'$ is projective $\iff$ I and I' are projective.*

To conclude this section, we recall some definitions and tools from homological algebra and their application to quiver representations.

Definition 1.2.18. Given a quiver Q and a family of Q-representations M_i, a sequence of morphisms

$$\cdots \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} \ldots$$

is called **exact** at M_i if $\operatorname{Im}(f_i) = \ker(f_{i_{i+1}})$, and exact if it is exact at every M_i.

Definition 1.2.19. A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

for some Q-representations L, M and N.

It follows from the definition of exact sequence that the sequence in Definition 1.2.19 is exact if and only if $\operatorname{Im}(f) = \ker(g)$, f is injective and g is surjective. Note that every short exact sequence of finite-dimensional representations of the form given in Definition 1.2.19 satisfies

$$\mathbf{dim}\, M = \mathbf{dim}\, L + \mathbf{dim}\, N.$$

Example 1.2.20. [Sch14, Example 1.10] Consider the quiver $\mathbb{A}_2$ and its representations

$$S(2) : \overset{0}{\bullet} \xrightarrow{\;0\;} \overset{\mathbb{K}}{\bullet}$$

$$U_{1,2} : \overset{\mathbb{K}}{\bullet} \xrightarrow{\;1\;} \overset{\mathbb{K}}{\bullet}$$

$$S(1) : \overset{\mathbb{K}}{\bullet} \xrightarrow{\;0\;} \overset{0}{\bullet}$$

together with four morphisms: $f : S(2) \to U_{1,2}, f = (0,1)$, $g : U_{1,2} \to S(1), g = (1,0)$, $f' : S(2) \to S(1) \oplus S(2), f' = (0,1)$ and $g' : S(1) \oplus S(2) \to S(1), g' = (1,0)$. It is then straightforward to see that both sequences

$$0 \longrightarrow S(2) \xrightarrow{f} U_{1,2} \xrightarrow{g} S(1) \longrightarrow 0$$

and

$$0 \longrightarrow S(2) \xrightarrow{f'} S(1) \oplus S(2) \xrightarrow{g'} S(1) \longrightarrow 0$$

are short exact sequences.

Certain exact sequences of representations, the projective (respectively injective) resolutions, represent a way of describing arbitrary representations of a fixed quiver by means of projective (respectively injective) representations.

Definition 1.2.21. Given a Q-representation M, a **projective resolution** of M is an exact sequence

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is a projective Q-representation. An **injective resolution** of M is an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \dots,$$

where each I_i is an injective Q-representation.

Differently from arbitrary categories, for a representation M in $\mathrm{rep}_{\mathbb{K}}(Q)$ it is always possible to find projective and injective resolutions that are also short exact sequences (see for instance [Sch14, Theorem 2.15]). These are the so-called **standard projective resolution** of M:

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and the **standard injective resolution** of M:

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0.$$

Definition 1.2.22. The **projective dimension** of M is the smallest integer d such that there exists a projective resolution of the form

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

The **injective dimension** of M is the smallest integer d such that there exists an injective resolution of the form

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_{d-1} \to I_d \to 0.$$

As stated above, for a representation M in $\mathrm{rep}_{\mathbb{K}}(Q)$ there exists standard resolutions, meaning that projective and injective dimensions of such a representation are always at most one. However, this changes if we consider a quiver with relations (Γ, I): in this case, the projective and injective dimensions of the (Q, I)-representations can even be infinite.

Now, if we consider $M, N \in \mathrm{rep}_{\mathbb{K}}(Q)$ and a standard projective resolution

$$0 \longrightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0,$$

we can apply the functor $\mathrm{Hom}(-, N)$ to the resolution and obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}(M, N) \xrightarrow{g^*} \mathrm{Hom}(P_0, N) \xrightarrow{f^*} \mathrm{Hom}(P_1, N) \longrightarrow \mathrm{Ext}^1(M, N) \longrightarrow 0,$$

where $\mathrm{Ext}^1(M, N) = \mathrm{coker}(f^*)$ is the **first group of extensions** of M and N. Note that this definition of $\mathrm{Ext}^1(M, N)$ does not depend on the choice of the projective resolution (as shown, for example, in [RR09, Proposition 6.4]). The first group of extensions $\mathrm{Ext}^1(M, N)$ can be defined equivalently as the group whose elements are the extensions of M by N modulo an equivalence relation:

Definition 1.2.23. An **extension** ε of M by N is a short exact sequence of the form $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ for some representation E. Two extensions ε and ε' are called equivalent if there is a commutative diagram:

$$\begin{array}{ccccccccc} \varepsilon : 0 & \longrightarrow & N & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\cong} & & \downarrow{=} & & \\ \varepsilon' : 0 & \longrightarrow & N & \xrightarrow{f'} & E' & \xrightarrow{g'} & M & \longrightarrow & 0 \end{array} .$$

Example 1.2.24. [Sch14, Example 2.12] Consider the Kronecker quiver

$$Q: \quad 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \ ,$$

its simple representations $S(2)$, $S(1)$ and the following Q-representations:

$$E: \quad \mathbb{K} \overset{1}{\underset{0}{\rightrightarrows}} \mathbb{K} \ , \ E': \quad \mathbb{K} \overset{0}{\underset{1}{\rightrightarrows}} \mathbb{K} \ .$$

Then, the short exact sequences

$$\varepsilon : 0 \longrightarrow S(2) \xrightarrow{f} E \xrightarrow{g} S(1) \longrightarrow 0$$

$$\varepsilon' : 0 \longrightarrow S(2) \xrightarrow{f'} E' \xrightarrow{g'} S(1) \longrightarrow 0$$

are not equivalent, because E and E' are not isomorphic.

Definition 1.2.25. An extension of M by N is called **split** if it is split as a short exact sequence, that is, if it is equivalent to the short exact sequence:

$$0 \longrightarrow N \longrightarrow N \oplus M \longrightarrow M \longrightarrow 0 \ .$$

Then, the first group of extensions $\operatorname{Ext}^1(M, N)$ is isomorphic to the group of equivalence classes of extensions of M by N, where the zero element is given by the class of the split extension. A more detailed explanation of this group structure and a proof of the isomorphism can be found, for instance, in [Sch14, Section 2.4]. From now on, for a fixed quiver Q and $M, N \in \operatorname{rep}_{\mathbb{K}}(Q)$ we use the standard notation

$$[M, N] := \dim_{\mathbb{K}} \operatorname{Hom}(M, N), \ [M, M]^1 := \dim_{\mathbb{K}} \operatorname{Ext}^1(M, M).$$

Proposition 1.2.26. *[Bri05, Corollary 1.4.3] For any finite-dimensional representations M, N of Q with dimension vectors* $\mathbf{dim}\, M = \boldsymbol{d}^M = (d_i^M)_{i \in Q_0}$, $\mathbf{dim}\, N = \boldsymbol{d}^N = (d_i^N)_{i \in Q_0}$ *we have*

$$\dim \operatorname{Hom}(M, N) - \dim \operatorname{Ext}^1(M, N) = \sum_{i \in Q_0} d_i^M d_i^N - \sum_{\alpha \in Q_1} d_{s(\alpha)}^M d_{t(\alpha)}^N$$

Proposition 1.2.26 implies, for instance, that $\dim \mathrm{Ext}^1(S(i), S(j))$ is equal to the number of arrows from vertex i to vertex j and hence that $\dim \mathrm{Ext}^1(S(i), S(i))$ - the space of self-extensions of the simple representation $S(i)$ - is the number of oriented cycles at i. Another consequence of this equality is that the left-hand side only depends on the dimension vectors of M and N, and is a bi-additive function in these vectors. This motivates the following definition:

Definition 1.2.27. The **Euler-Ringel** form of Q is the bilinear form $\langle -, - \rangle : \mathbb{Z}^{|Q_0|} \times \mathbb{Z}^{|Q_0|} \to \mathbb{Z}$ given by

$$\langle \mathbf{d}^M, \mathbf{d}^N \rangle = \langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle := \sum_{i \in Q_0} d_i^M d_i^N - \sum_{\alpha \in Q_1} d_{s(\alpha)}^M d_{t(\alpha)}^N \qquad (1.2.28)$$

and sometimes denoted by $\langle M, N \rangle$.

Note that, for a quiver with relations (Q, I) and no oriented cycles, the Euler-Ringel form takes the relations into account and is thus given by

$$\langle \mathbf{d}^M, \mathbf{d}^N \rangle = \sum_{i \in Q_0} d_i^M d_i^N - \sum_{\alpha \in Q_1} d_{s(\alpha)}^M d_{t(\alpha)}^N + \sum_{i,j \in Q_0} r(i,j) d_i^M d_j^N \qquad (1.2.29)$$

(see for instance [Bon83] or [DW02] for details).

Definition 1.2.30. We call a representation M **rigid** if it has no self-extensions, that is $[M, M]^1 = 0$.

We will discuss later on, in Chapter 3, a few geometrical properties of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ (for some fixed quiver Q and any dimension vector $\mathbf{e}$) that follow from M being a rigid representation of Q and additional homological properties of M.

1.3 The variety of Q-representations

In Section 1.2, we defined a quiver representation $M \in \operatorname{rep}_{\mathbb{K}}(Q)$ as the assignment of a finite-dimensional $\mathbb{K}$-vector space M_i to each vertex in Q_0 and of a linear map M_α to each arrow in Q_1. By choosing bases, we can identify the vector spaces M_i with $\mathbb{K}^{d_i}$, where $d_i = \dim M_i$, and consequently we can represent each linear map M_α, for $\alpha : i \to j$, with a $d_j \times d_i$ matrix. This means that, given a quiver Q and a dimension vector $\mathbf{d}$, determining a Q-representation of dimension vector $\mathbf{d}$ is equivalent to giving the linear maps M_α.

Definition 1.3.1. The **representation space** of the quiver Q for the dimension vector $\mathbf{d}$ is

$$R_{\mathbf{d}} \cong \bigoplus_{\alpha:i\to j} \operatorname{Hom}(\mathbb{K}^{d_i}, \mathbb{K}^{d_j}) \cong \bigoplus_{\alpha:i\to j} \operatorname{Mat}_{d_j\times d_i}(\mathbb{K}).$$

The representation space $R_{\mathbf{d}}$ is isomorphic by definition to the affine space $\mathbb{A}^r$, with $r := \sum_{\alpha:i\to j} d_i \times d_j$. We consider then the group

$$G_{\mathbf{d}} = \prod_{i\in Q_0} \operatorname{GL}_{d_i}(\mathbb{K}),$$

where $\operatorname{GL}_{d_i}(\mathbb{K})$ denotes the group of invertible $d_i \times d_i$ matrices, and define the action of $G_{\mathbf{d}}$ on $R_{\mathbf{d}}$ as

$$g \cdot M = g_{t(\alpha)} M_\alpha g_{s(\alpha)}^{-1} \tag{1.3.2}$$

for some $g := (g_i)_{i\in Q_0} \in G_{\mathbf{d}}$ and $M \in R_{\mathbf{d}}$. We write $\mathcal{O}_M$ for the orbit of a representation M under this action, that is $\mathcal{O}_M = \{g \cdot M | g \in G_{\mathbf{d}}\}$.

Example 1.3.3. [Sch14, Example 8.1] Consider the quiver $Q = \overset{1}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet}$ and a dimension vector $\mathbf{d} = (d_1, d_2)$. In this case, the representation space $R_{\mathbf{d}}$ is isomorphic to $\operatorname{Mat}_{d_2\times d_1}(\mathbb{K})$, the elements $g = (g_1, g_2)$ of $G_{\mathbf{d}}$ are pairs of invertible matrices of size d_1 and d_2 respectively, and the orbits of the action defined in 1.3.2 are $\mathcal{O}_M = \{g_2 M_\alpha g_1^{-1} | (g_1, g_2) \in G_{\mathbf{d}}\}$. In other words, $\mathcal{O}_M$ is the set of all matrices whose rank is equal to the rank of M_α.

Example 1.3.4. Let Q be the loop quiver $Q = \overset{1}{\bullet} \circlearrowleft \alpha$. Then, for a dimension vector $\mathbf{d} = (d)$, we have $R_{(d)} = \operatorname{Mat}_{d\times d}(\mathbb{K})$ and $G_{(d)} = \operatorname{GL}_d(\mathbb{K})$. The action of $\operatorname{GL}_d(\mathbb{K})$ on $\operatorname{Mat}_{d\times d}(\mathbb{K})$ is simply given by conjugation:

$$g \cdot M_\alpha = g M_\alpha g^{-1} \text{ for } g \in \operatorname{GL}_d(\mathbb{K}), M_\alpha \in \operatorname{Mat}_{d\times d}(\mathbb{K}).$$

Lemma 1.3.5. *[Sch14, Lemma 8.1] The orbit $\mathcal{O}_M$ is precisely the isomorphism class of the representation M, that is*

$$\mathcal{O}_M = \{M' \in \operatorname{rep}_{\mathbb{K}}(Q) | M' \cong M\}.$$

Similarly, the stabiliser $\mathrm{Stab}(M) = \{g \in G_{\mathbf{d}} \mid g \cdot M = M\}$ of a Q-representation M is isomorphic to the automorphism group $\mathrm{Aut}(M)$ of M.

Lemma 1.3.6. *[Sch14, Lemma 8.2] Let* $\mathbf{d} \in \mathbb{Z}^n$*. Then*

1. *for any representation* M *of dimension vector* $\mathbf{d}$*, the dimensions of the varieties* $\mathcal{O}_M, G_{\mathbf{d}}$ *and* $\mathrm{Aut}(M)$ *satisfy*

$$\dim \mathcal{O}_M = \dim G_{\mathbf{d}} - \dim \mathrm{Aut}(M);$$

2. *there is one orbit of codimension zero in* $R_{\mathbf{d}}$*.*

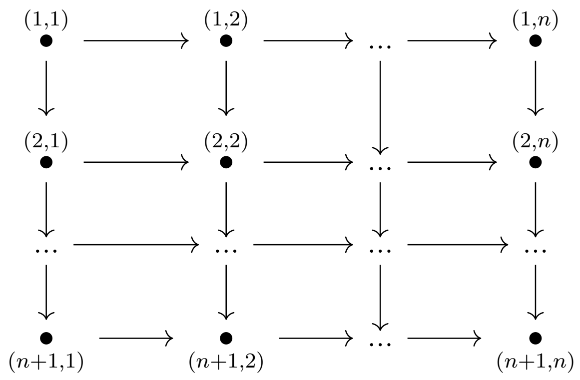
Chapter 2

A special quiver with relations

This chapter is dedicated to defining a quiver with relations, constructing a rigid representation for this quiver, and lastly studying a relevant class of its indecomposable representations. In Chapter 4 we will exploit this construction, together with two different, appropriate choices of dimension vectors, to recover the Bott-Samelson resolution for Schubert varieties and to realise smooth Schubert varieties as the corresponding quiver Grassmannian, respectively. We fix the field over which we work to be the field of complex numbers $\mathbb{C}$ and do not specify it further when doing so simplifies the notation. Part of the following construction and of the results presented in this chapter are included in the paper [Iez25], by the author of this thesis.

2.1 The representation M

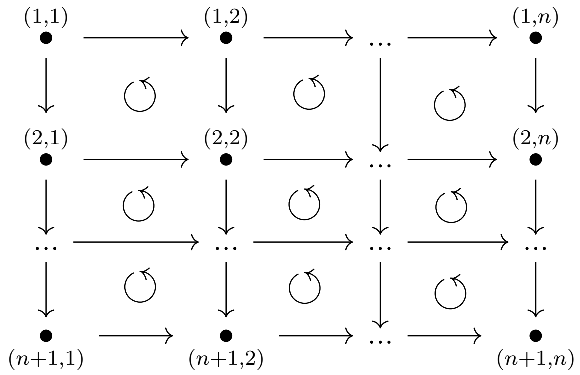
Given $n \in \mathbb{N}_{\geq 2}$, we consider the following quiver $\Gamma = (\Gamma_0, \Gamma_1)$:



where each vertex in Γ_0 is labelled by a pair (i, j), for $i = 1, \ldots, n+1$ and $j = 1, \ldots, n$. We denote by $\alpha_{(i,j)}^{(k,l)}$ the arrow going from vertex (i, j) to vertex (k, l). Then we consider the following relations on Γ:

$$\alpha_{(i,j+1)}^{(i+1,j+1)} \alpha_{(i,j)}^{(i,j+1)} = \alpha_{(i+1,j)}^{(i+1,j+1)} \alpha_{(i,j)}^{(i+1,j)} \tag{2.1.1}$$

for $i = 1, \dots, n$, $j = 1, \dots, n-1$, and denote by I the ideal of $\mathbb{C}\Gamma$ they generate. We write (Γ, I) for the quiver with relations:



(2.1.2)

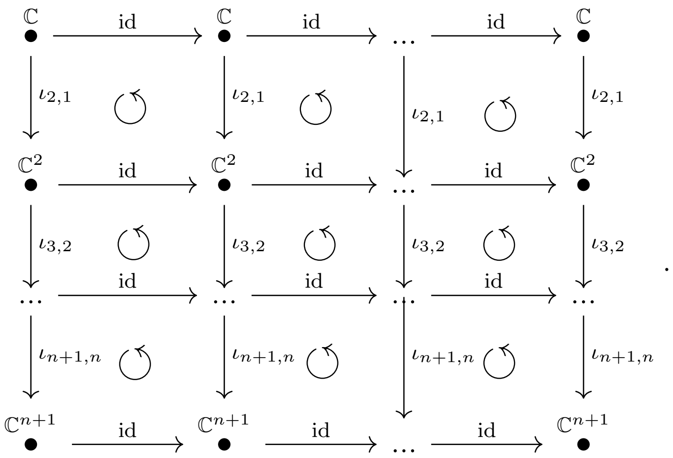
Now, to (Γ, I) we associate the representation $M = ((M_{i,j})_{(i,j)\in\Gamma_0}, (M_\alpha)_{\alpha\in\Gamma_1})$ defined as

$$M_{i,j} = \mathbb{C}^i, \quad M_\alpha = \begin{cases} \iota_{i+1,i} & \text{if } s(\alpha) = (i,j), t(\alpha) = (i+1,j) \\ \text{id} & \text{if } s(\alpha) = (i,j), t(\alpha) = (i,j+1) \end{cases}$$

where $\iota_{i+1,i}$ denotes the inclusion of $\mathbb{C}^i$ into $\mathbb{C}^{i+1}$, represented with respect to the chosen basis $\mathcal{B} = \{ b_1, b_2, \dots, b_{n+1} \}$ by the matrix

$$\iota_{i+1,i} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.1.3}$$

The relations imposed on Γ are trivially satisfied by the representation M:



.

In other words, M is an object in the category $\text{rep}_{\mathbb{C}}(\Gamma, I)$ of Γ-representations with relations. We now want to show that M is a rigid representation of (Γ, I), that is, we want to show that $\mathrm{Ext}^1(M, M) = 0$. To do so, we first consider the following subquiver Γ' of Γ

$$\overset{(1,1)}{\bullet} \xrightarrow{\iota_{2,1}} \overset{(2,1)}{\bullet} \xrightarrow{\iota_{3,2}} \overset{(3,1)}{\bullet} \xrightarrow{\iota_{4,3}} \cdots \xrightarrow{\iota_{n+1,n}} \overset{(n+1,1)}{\bullet} \quad ,$$

i.e. the equioriented Dynkin quiver $\mathbb{A}_{n+1}$, and call M' the restriction of the representation M to Γ':

$$\overset{\mathbb{C}}{\bullet} \xrightarrow{\iota_{2,1}} \overset{\mathbb{C}^2}{\bullet} \xrightarrow{\iota_{3,2}} \overset{\mathbb{C}^3}{\bullet} \xrightarrow{\iota_{4,3}} \cdots \xrightarrow{\iota_{n+1,n}} \overset{\mathbb{C}^{n+1}}{\bullet} \quad .$$

Lemma 2.1.4. *M' is a rigid representation of Γ'.*

Proof. We denote by $U_{i,j}$ the indecomposable representation of $\mathbb{A}_{n+1}$ supported on the vertices $i, \ldots, j$, for $1 \leq i \leq j \leq n+1$, that is:

$$\underset{1}{\overset{0}{\bullet}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{i}{\overset{\mathbb{C}}{\bullet}} \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \underset{j}{\overset{\mathbb{C}}{\bullet}} \xrightarrow{0} \cdots \xrightarrow{0} \underset{n+1}{\overset{0}{\bullet}} \quad .$$

First, we observe that M' can be written as the direct sum $M' = \bigoplus_{i=1}^{n+1} U_{i,n+1}$, and since $\dim_{\mathbb{C}} \mathrm{Ext}^1_{\Gamma'}(U_{k,l}, U_{i,j}) = 1$ if and only if $k+1 \leq i \leq l+1 \leq j$ and zero otherwise (see for instance [CIFF$^+$17][Section 3.1]), we conclude that $\mathrm{Ext}^1_{\Gamma'}(M', M') = 0$. □

Proposition 2.1.5. *M is a rigid representation of (Γ, I).*

Proof. Consider Γ' and M' as in Lemma 2.1.4 and the functor $\Phi : \mathrm{rep}_{\mathbb{C}}(\Gamma') \to \mathrm{rep}_{\mathbb{C}}(\Gamma, I)$ defined on $R \in \mathrm{rep}_{\mathbb{C}}(\Gamma')$ as follows. For all $i = 1, \ldots, n+1$ and $j = 1, \ldots, n$, we set $\Phi(R)_{i,j} = R_i$. For each arrow $i \to i{+}1$ in Γ' and $j = 1, \ldots, n$, the map $\Phi(R)_{i,j} \to \Phi(R)_{i+1,j}$ is defined as the map $R_i \to R_{i+1}$. Finally, for each $i = 1, \ldots, n+1$ and $j = 1, \ldots, n-1$, the map $\Phi(R)_{i,j} \to \Phi(R)_{i,j+1}$ is id_{R_i}. From the definition of Φ, it follows that $\Phi(M') = M$. As shown in [Mak19, Lemma 2.3, Lemma 2.5], Φ is an exact, fully faithful functor that takes projective objects to projective objects. This implies ([Mak19, Corollary 2.6]) that $\mathrm{Ext}^i_{(\Gamma,I)}(\Phi(V), \Phi(W)) \cong \mathrm{Ext}^i_{\Gamma'}(V, W)$ for every $V, W \in \mathrm{rep}_{\mathbb{C}}(\Gamma')$ and $i \geq 0$. In particular, we have

$$\mathrm{Ext}^1_{(\Gamma,I)}(M, M) \cong \mathrm{Ext}^1_{\Gamma'}(M', M') = 0.$$

□

It should be emphasised that M is rigid only as an element of the category of Γ-representations that satisfy the commutativity relations given above. In other words, self-extensions of M could (potentially) be constructed using Γ-representations that do not satisfy the relations imposed by I.

2.2 The variety $R^\iota_{\mathbf{d}}$ of (Γ, I)-representations

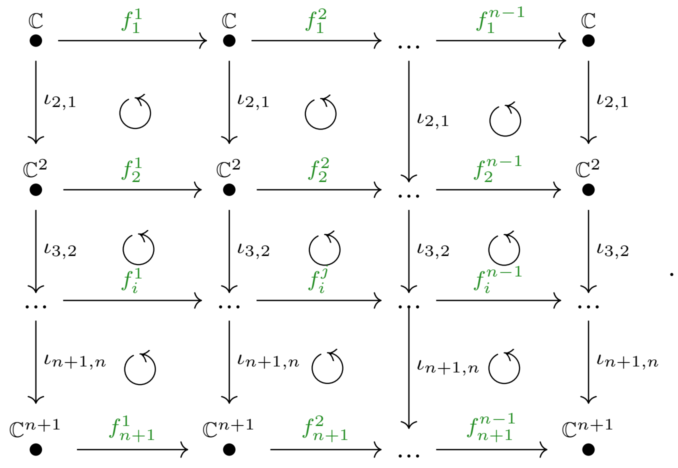
As a direct consequence of Gabriel's theorem (see Theorem 1.2.12), the representation type of Γ is infinite. Deciding whether the representation type of (Γ, I) is finite or not, on the other hand, would require its own detailed study. In this thesis, we restrict our research to certain representations in $\mathrm{rep}_{\mathbb{C}}(\Gamma, I)$ and show that they can be decomposed as direct sums of indecomposables belonging to a finite class. The motivation for such a restriction will be explained in Section 5.2, where we introduce our definition of linear degenerations of Schubert varieties using quiver Grassmannians.

Definition 2.2.1. Let (Γ, I) be the quiver defined in (2.1.2) and $R_{\mathbf{d}}$ the associated variety of representations (see Definition 1.3.1). We denote by $R^\iota_{\mathbf{d}}$ the subvariety of $R_{\mathbf{d}}$ defined as

$$R^\iota_{\mathbf{d}} = \{M \in R_{\mathbf{d}} | M_\alpha = \iota_{i+1,i} \ \forall \alpha : (i,j) \to (i+1,j), i = 1, \dots, n, j = 1, \dots, n\}.$$

In words, we obtain $R^\iota_{\mathbf{d}}$ by setting all the linear maps associated to the vertical arrows of (Γ, I) to the standard inclusion of $\mathbb{C}^i$ into $\mathbb{C}^{i+1}$ (see the definition in (2.1.3)).

Remark 2.2.2. Because of the commutativity relations on (Γ, I), the linear maps of a representation M in $R^\iota_{\mathbf{d}}$ associated to the horizontal arrows of (Γ, I) are not independent of one another. Let us denote by f_i^j the matrix representation of the linear map M_α, for α such that $s(\alpha) = (i,j)$ and $t(\alpha) = (i, j+1)$, with respect to the chosen basis $\mathcal{B} = \{ b_1, b_2, \dots, b_{n+1} \}$. The representation M is then



We write $f_1^1 = [\, a \,]$ for some $a \in \mathbb{C}$, and applying $\iota_{2,1}$ after f_1^1 we get $\iota_{2,1} f_1^1 = \left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]$. The relation $f_2^1 \iota_{2,1} = \iota_{2,1} f_1^1$ implies $f_2^1 = \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]$, for some $b, c \in \mathbb{C}$. Similarly, we see that f_3^1 must be $f_3^1 = \left[\begin{smallmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & l \end{smallmatrix}\right]$ for some $d, e, l \in \mathbb{C}$, and so forth for all $i = 1, \dots, n+1$ and $j = 1, \dots, n-1$. This means that a (Γ, I)-representation

M in the subvariety $R^{\iota}_{\mathbf{d}}$ is determined by the choice of $n-1$ upper-triangular matrices of size $n+1$. We denote this tuple by

$$f = (f^1_{n+1}, \dots, f^{n-1}_{n+1}) \in \prod_{j=1}^{n-1} U_{n+1}, \tag{2.2.3}$$

where U_{n+1} is the subset of Mat_{n+1} consisting of upper-triangular matrices (different from the subgroup B_{n+1} of invertible upper-triangular matrices). From now on, we will focus on studying the $(\Gamma, I)-$representations in the space $R^{\iota}_{\mathbf{d}}$, in particular by describing their decompositions into indecomposable representations of (Γ, I) and how they can be parametrised. Since the representations in $R^{\iota}_{\mathbf{d}}$ are determined by the choice of f, we will denote them by M^f and, depending on the context, identify M^f with the tuple $(f^1_{n+1}, \dots, f^{n-1}_{n+1})$ and therefore write $M^f = (f^1_{n+1}, \dots, f^{n-1}_{n+1})$.

Let us denote by $U^{(h_1,\dots,h_n)}$ the indecomposable representation of (Γ, I) defined as:

$$U^{(h_1,\dots,h_n)}_{i,j} := \begin{cases} 0 & \text{if } i \le n+1-h_j \\ \mathbb{C} & \text{if } i > n+1-h_j \end{cases}, \tag{2.2.4}$$

where $1 \le h_j \le n+1$, for $i = 1, \dots, n+1$ and $1 \le j \le n$, and whose linear maps are

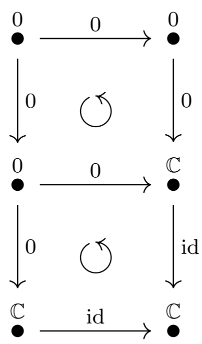
$$U^{(h_1,\dots,h_n)}_{\alpha} := \begin{cases} \mathrm{id} & \text{if } U^{(h_1,\dots,h_n)}_{s(\alpha),t(\alpha)} = \mathbb{C} \\ 0 & \text{otherwise} \end{cases}. \tag{2.2.5}$$

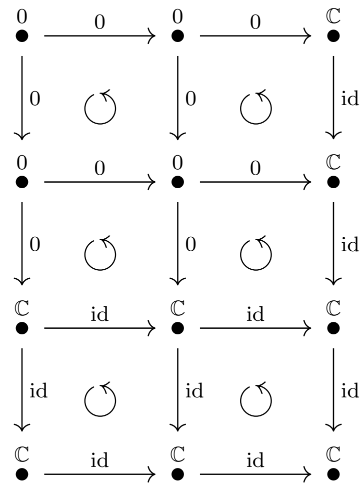
Each index h_j represents the "height" of the first nonzero vector space in the j-th column of (Γ, I): this space and all the spaces below it are isomorphic to $\mathbb{C}$ by the definition given in (2.2.4). In order to satisfy the commutativity relations of (Γ, I), we further require $h_j \le h_{j'}$ for any $j \le j'$ and $h_{j'} > 0$. It is straightforward to verify that, if $h_{j'} > 0$, the relation $h_j > h_{j'}$ implies the existence of the following diagram in (Γ, I):

$$\begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{[0]} & \overset{0}{\bullet} \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle [0]} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \end{array},$$

which does not commute. As shown in Theorem 2.2.12, the definition for the linear maps of $U^{(h_1,\dots,h_n)}$ given in (2.2.5) ensures that these representations of (Γ, I) are actually indecomposable.

Example 2.2.6. Two examples of indecomposable (Γ, I)-representations of the form $U^{(h_1,\dots,h_n)}$ are $U^{(1,2)}$ and $U^{(2,2,4)}$, respectively for $n+1=3$ and $n+1=4$:

$U^{(1,2)} =$ 

, $U^{(2,2,4)} =$ 

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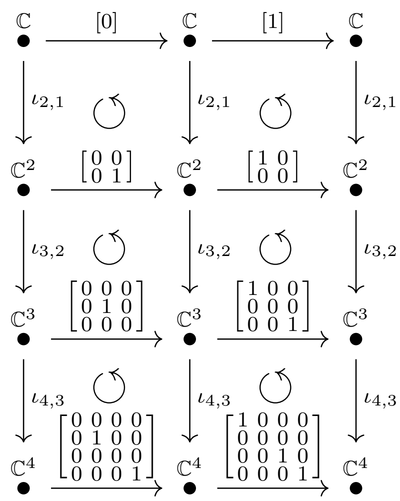
Example 2.2.7. For $n+1=4$, the representations M^f in $R^{\iota}_{\mathbf{d}}$ are determined by $f = (f_4^1, f_4^2) \in U_4 \times U_4$. To simplify our notation, when possible, we omit the row index and only specify the "column" of (Γ, I) where a linear map appears. In this case, for example, we write $f = (f_1, f_2) \in U_4 \times U_4$. A possible choice is:

$$f_1 = \begin{bmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix},$$

which corresponds to the representation

$M^f =$ 

.

The decomposition of M^f is then $M^f = U^{(4,0,0)} \oplus U^{(3,3,0)} \oplus U^{(2,0,0)} \oplus U^{(1,1,1)} \oplus U^{(0,4,4)} \oplus U^{(0,2,2)} \oplus U^{(0,0,3)}$.

Remark 2.2.8. We denote the (unique) projective, injective and simple indecomposable representations of (Γ, I) (see Definition 1.2.13) by $P(i,j), I(i,j)$ and $S(i,j)$, respectively. Among them, the ones of the form $U^{(h_1,\ldots,h_n)}$, for some $h_1,\ldots,h_n$, are $P(1,j), P(i,1), I(n+1,j)$ and $S(n+1,j)$. To simplify notation, we show them in a few examples for the specific case of $n+1=4$:

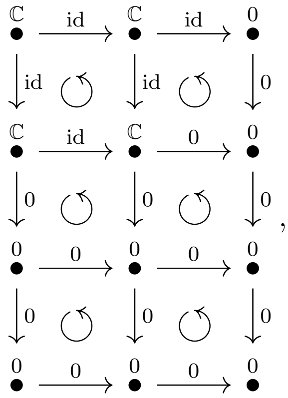
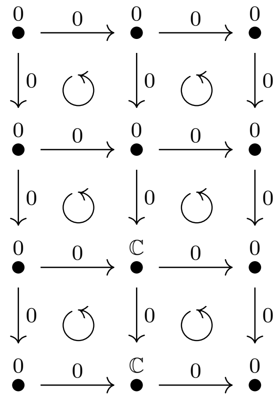
$$P(1,1) = \begin{array}{ccccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \end{array}, \quad P(1,2) = \begin{array}{ccccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \end{array},$$

$$I(4,1) = \begin{array}{ccccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \end{array}, \quad I(4,2) = \begin{array}{ccccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \end{array},$$

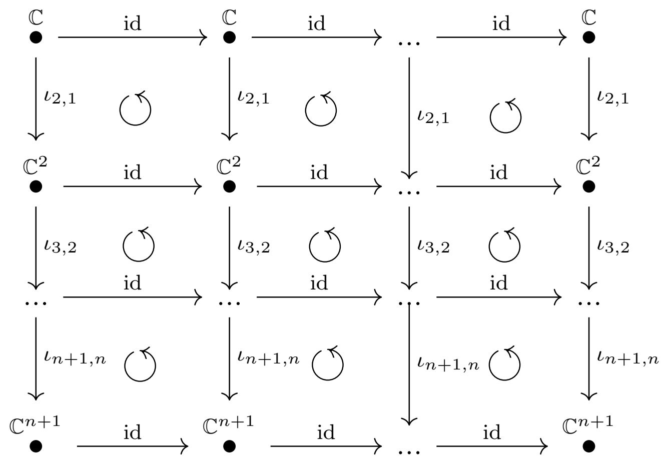
$$I(4,3) = \begin{array}{ccccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} & \circlearrowleft & \downarrow{\scriptstyle \mathrm{id}} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{\mathrm{id}} & \overset{\mathbb{C}}{\bullet} \end{array}, \quad S(4,2) = \begin{array}{ccccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \end{array},$$

that is, $P(1,1) = U^{(4,4,4)}, P(1,2) = U^{(0,4,4)}, I(4,1) = U^{(4,0,0)}, I(4,2) = U^{(4,4,0)}, I(4,3) = U^{(4,4,4)}$ and $S(4,2) = U^{(0,1,0)}$.

Some examples of injective and simple indecomposables of (Γ, I) that are not of the form $U^{(h_1,h_2,h_3)}$ are the following:

$I(2,2) =$  , $S(3,2) =$  .

Example 2.2.9. We defined in Section 2.1 the representation M:



,

which is the element of $R^{\iota}_{\mathbf{d}}$ that corresponds to the choice $f^1_{n+1} = f^2_{n+1} = \cdots = f^{n-1}_{n+1} = \mathrm{id}$. Its decomposition is $M = U^{(n+1,\dots,n+1)} \oplus U^{(n,\dots,n)} \oplus U^{(n-1,\dots,n-1)} \oplus \cdots \oplus U^{(2,\dots,2)} \oplus U^{(1,\dots,1)} = P(1,1) \oplus P(2,1) \oplus \cdots \oplus P(n,1) \oplus P(n+1,1)$. Since it is a direct sum of projective (Γ, I)-representations, M is a projective (Γ, I)-representation as well.

Remark 2.2.10. In Example 2.2.9, we found that the representation M is a projective representation of (Γ, I), which implies that the projective dimension of M is zero (see Definition 1.2.22). Furthermore, we can construct an injective resolution of M to show that the injective dimension of M is one. We define the injective representations I_0, I_1 of (Γ, I) as the following sums of indecomposable injective representations:

$$I_0 = \bigoplus_{i=1}^{n+2} I(n+1, n), \quad I_1 = \bigoplus_{i=1}^{n+1} I(i, n).$$

Then we consider the sequence

$$\varepsilon : 0 \longrightarrow M \xrightarrow{f} I_0 \xrightarrow{g} I_1 \longrightarrow 0$$

where g is the surjective map projecting I_0 onto I_1 such that $\operatorname{Im}(f) \cong M \cong \ker(g)$ and f is the corresponding embedding of M into I_0, meaning that ε is a short exact sequence.

Example 2.2.11. In dimension $n + 1 = 3$, the representations M, I_0 and I_1 of Remark 2.2.10 are:

$$M : \begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{\text{id}} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \iota_{2,1}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{2,1}} \\ \overset{\mathbb{C}^2}{\bullet} & \xrightarrow{\text{id}} & \overset{\mathbb{C}^2}{\bullet} \\ \downarrow{\scriptstyle \iota_{3,2}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{3,2}} \\ \overset{\mathbb{C}^3}{\bullet} & \xrightarrow{\text{id}} & \overset{\mathbb{C}}{\bullet} \end{array}, \; I_0 : \begin{array}{ccc} \overset{\mathbb{C}^4}{\bullet} & \xrightarrow{\text{id}} & \overset{\mathbb{C}^4}{\bullet} \\ \downarrow{\scriptstyle \text{id}} & \circlearrowleft & \downarrow{\scriptstyle \text{id}} \\ \overset{\mathbb{C}^4}{\bullet} & \xrightarrow{\text{id}} & \overset{\mathbb{C}^4}{\bullet} \\ \downarrow{\scriptstyle \text{id}} & \circlearrowleft & \downarrow{\scriptstyle \text{id}} \\ \overset{\mathbb{C}^4}{\bullet} & \xrightarrow{\text{id}} & \overset{\mathbb{C}^4}{\bullet} \end{array}, \; I_1 : \begin{array}{ccc} \overset{\mathbb{C}^3}{\bullet} & \xrightarrow{\text{id}} & \overset{\mathbb{C}^3}{\bullet} \\ \downarrow{\scriptstyle \text{p}} & \circlearrowleft & \downarrow{\scriptstyle \text{p}} \\ \overset{\mathbb{C}^2}{\bullet} & \xrightarrow{\text{id}} & \overset{\mathbb{C}^2}{\bullet} \\ \downarrow{\scriptstyle \text{p}} & \circlearrowleft & \downarrow{\scriptstyle \text{p}} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{\text{id}} & \overset{\mathbb{C}}{\bullet} \end{array}.$$

Theorem 2.2.12. *All representations in $R^{\iota}_{\mathbf{d}}$ can be decomposed as direct sums of the indecomposable (Γ, I)-representations $U^{(h_1,\dots,h_n)}$.*

Proof. First, we observe that the (Γ, I)-representations $U^{(h_1,\dots,h_n)}$ are indecomposable, because they are thin and all linear maps connecting two one-dimensional vector spaces are identity maps (the only proper subspace of a one-dimensional vector space is the trivial subspace, which is the domain or codomain of only the zero map). To show that all representations in $R^{\iota}_{\mathbf{d}}$ can be decomposed as their direct sums, we recall from (2.2.3) that each representation M^f in $R^{\iota}_{\mathbf{d}}$ is determined by the choice of $f = (f^1_{n+1}, \dots, f^{n-1}_{n+1})$ in $\prod_{j=1}^{n-1} U_{n+1}$: every other linear map f^j_i corresponding to the horizontal arrow $(i, j) \to (i, j+1)$ of (Γ, I) is represented by the appropriate submatrix of f^j_{n+1} (i.e., it is the restriction of f^j_{n+1} to $\mathbb{C}^i$), while all maps corresponding to vertical arrows of (Γ, I) are fixed as the standard inclusions $\mathbb{C}^i \hookrightarrow \mathbb{C}^{i+1}$. In other words, any indecomposable representation appearing in the decomposition of M^f must be a representation of (Γ, I) whose linear maps associated to the vertical arrows are either zero or identity maps and such that, if the linear map associated to a horizontal arrow $(i, j) \to (i, j+1)$ is zero, then all linear maps associated to arrows of the form

$(i,k) \to (i,k+1)$ for $k > l$ are zero. Then, we consider the subquivers of (Γ, I) that start at $(i,1)$ and end at (i,n), for $i = 1,\ldots,n+1$ (i.e., each row of (Γ, I) is considered as a subquiver). These are equioriented Dynkin quivers of type $\mathbb{A}_n$, and we know (see Example 1.2.14) that their indecomposables $U_{a,b}$ are thin and given by connected intervals $[a,b]$ with $1 \le a \le b \le n$. The statement then follows by applying this description of the indecomposables $U_{a,b}$ to all the considered subquivers of (Γ, I), starting from the topmost row downwards; the definition of each f_j^i as the restriction of f_{n+1}^j to $\mathbb{C}^i$ ensures the commutativity of all arising square diagrams. □

Our goal now is to find a way to parametrise the representations in $R_{\mathbf{d}}^{\iota}$ or, more precisely, to parametrise their isomorphism classes. We recall from Lemma 1.3.5 that the isomorphism classes of the representations of a given quiver coincide with the orbits under the action of $G_{\mathbf{d}} = \prod_{i\in Q_0} \mathrm{GL}_{d_i}(\mathbb{K})$, defined on the variety $R_{\mathbf{d}}$ as

$$g \cdot M = g_{t(\alpha)} M_\alpha g_{s(\alpha)}^{-1}$$

for $g \coloneqq (g_i)_{i\in Q_0} \in G_{\mathbf{d}}$. The action of $G_{\mathbf{d}}$, however, is not compatible with the restrictions we impose on the (Γ, I)-representations when we consider the subvariety $R_{\mathbf{d}}^{\iota}$ of $R_{\mathbf{d}}$: for $M^f \in R_{\mathbf{d}}^{\iota}$, the representation $g \cdot M^f \in R_{\mathbf{d}}$ is isomorphic to M^f but is not, in general, an element of $R_{\mathbf{d}}^{\iota}$. This is because the linear maps associated to the vertical arrows of $g \cdot M^f$ are not, in general, of the form $\iota_{i+1,i}$, the standard inclusion of $\mathbb{C}^i$ into $\mathbb{C}^{i+1}$ (see Definition 2.2.1 for the conditions defining $R_{\mathbf{d}}^{\iota}$). For this reason, we consider instead the action of the maximal subgroup of $G_{\mathbf{d}}$ that is compatible with the definition of the subvariety $R_{\mathbf{d}}^{\iota}$:

Definition 2.2.13. Given a representation $M^f = (f_{n+1}^1, \ldots, f_{n+1}^{n-1}) \in R_{\mathbf{d}}^{\iota}$, we denote by $\mathcal{O}_{M^f}^{\iota}$ the orbit of M^f under the action of $G_{\mathbf{d}}^{\iota} \coloneqq \prod_{i=1}^n B_{n+1}$, where B_{n+1} is the group of invertible upper-triangular matrices of size $n+1$, defined as

$$h \cdot M^f = (h_2 f_{n+1}^1 h_1^{-1}, h_3 f_{n+1}^2 h_2^{-1}, \ldots, h_n f_{n+1}^{n-1} h_{n-1}^{-1})$$

for some $h \in G_{\mathbf{d}}^{\iota}$.

Now we can reformulate our goal: we want to parametrise the orbits $\mathcal{O}_{M^f}^{\iota}$ under the action of $G_{\mathbf{d}}^{\iota}$ or, equivalently, we want to parametrise the classes of representations in $R_{\mathbf{d}}^{\iota}$ that are isomorphic via a morphism in $\prod_{i=1}^n B_{n+1}$. We denote such isomorphisms by

$$M^f \overset{B}{\cong} M^g, \tag{2.2.14}$$

and say that f and g are in the same B-isomorphism class, or B-orbit, of representations in $R_{\mathbf{d}}^{\iota}$.

Remark 2.2.15. As shown in Remark 2.2.2, the linear maps f_i^j of a representation $M^f = (f_{n+1}^1, \ldots, f_{n+1}^{n-1}) \in R_{\mathbf{d}}^{\iota}$, for $i < n+1$, are the restrictions of f_{n+1}^j to $\mathbb{C}^i$. This means that, if we denote $M^g \coloneqq h \cdot M^f$ for some $h = (h_1, \ldots, h_n) \in G_{\mathbf{d}}^{\iota}$, the linear maps g_i^j of M^g for $i < n+1$ are the restrictions of g_{n+1}^j to $\mathbb{C}^i$. In other words, the action of h on M^f on the linear maps $(f_i^1, \ldots, f_i^{n-1})$, for a fixed $i < n+1$, is given by the action of the restrictions to $\mathbb{C}^i$ of the fixed maps $(h_1, \ldots, h_n)$ on $(f_i^1, \ldots, f_i^{n-1})$. We clarify this through the following example.

Example 2.2.16. Consider the quiver (Γ, I) for $n+1 = 3$ and its representation M^f for some $f = (f_3^1) = \left[\begin{smallmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & l \end{smallmatrix}\right] \in U_3$:

$$M^f : \begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{[\,a\,]} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \iota_{2,1}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{2,1}} \\ \overset{\mathbb{C}^2}{\bullet} & \xrightarrow{\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]} & \overset{\mathbb{C}^2}{\bullet} \\ \downarrow{\scriptstyle \iota_{3,2}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{3,2}} \\ \overset{\mathbb{C}^3}{\bullet} & \xrightarrow{\left[\begin{smallmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & l \end{smallmatrix}\right]} & \overset{\mathbb{C}^3}{\bullet} \end{array} .$$

The orbit of M^f under the action of $G^{\iota}_{\mathbf{d}} = B_3 \times B_3$ is

$$\mathcal{O}^{\iota}_{M^f} = \{h_2 f_3^1 h_1^{-1} | (h_1, h_2) \in B_3 \times B_3\}$$

and, if we denote $h_1 = \left[\begin{smallmatrix} x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 \\ 0 & 0 & x_6 \end{smallmatrix}\right]$ and $h_2 = \left[\begin{smallmatrix} y_1 & y_2 & y_3 \\ 0 & y_4 & y_5 \\ 0 & 0 & y_6 \end{smallmatrix}\right]$, then the representation $M^g := h \cdot M^f$ is

$$M^g = \begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{g_1^1} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \iota_{2,1}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{2,1}} \\ \overset{\mathbb{C}^2}{\bullet} & \xrightarrow{g_2^1} & \overset{\mathbb{C}^2}{\bullet} \\ \downarrow{\scriptstyle \iota_{3,2}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{3,2}} \\ \overset{\mathbb{C}^3}{\bullet} & \xrightarrow{g_3^1} & \overset{\mathbb{C}^3}{\bullet} \end{array}$$

where

$$g_1^1 = [\,y_1\,][\,a\,][\,x_1\,]^{-1}, \quad g_2^1 = \left[\begin{smallmatrix} y_1 & y_2 \\ 0 & y_4 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 & x_2 \\ 0 & x_4 \end{smallmatrix}\right]^{-1},$$

$$g_3^1 = \left[\begin{smallmatrix} y_1 & y_2 & y_3 \\ 0 & y_4 & y_5 \\ 0 & 0 & y_6 \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & l \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1 & x_2 & x_3 \\ 0 & x_4 & x_5 \\ 0 & 0 & x_6 \end{smallmatrix}\right]^{-1}.$$

Remark 2.2.17. The row (or column) echelon form of a matrix $f \in U_{n+1}$ can be obtained via the action of $h \in B_{n+1} \times B_{n+1}$ given, as in Definition 2.2.13, by $h \cdot f = h_2 f h_1^{-1}$: the effect of this action is "sweeping upwards" and "sweeping to the right" of the pivots of f, allowing us to transform f simultaneously into its reduced row and column echelon form. We consider then this form as the standard representative for the orbit of f: it is an upper-triangular matrix whose entries are all equal to 0, except for at most one entry equal to 1 in each row and column. Such matrices (in general, not upper-triangular) are known as **partial permutation matrices**, a term employed, for instance, in [KM05] and [MS05].

Example 2.2.18. For $n+1=3$, the standard representatives of the orbits under the action given in Definition 2.2.13 are the following matrices:

$$f^1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, f^2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}, f^3=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}, f^4=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}, f^5=\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix},$$

$$f^6=\begin{bmatrix}0&0&1\\0&1&0\\0&0&0\end{bmatrix}, f^7=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}, f^8=\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix}, f^9=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, f^{10}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},$$

$$f^{11}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, f^{12}=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, f^{13}=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, f^{14}=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, f^{15}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

These represent the 15 orbits of the form

$$\mathcal{O}^{\iota}_{M^f}=\{h_2 f_3^1 h_1^{-1}|(h_1,h_2)\in B_3\times B_3\},$$

where $f_3^1\in U_3$ is the linear map defining the representation M^f of Example 2.2.16.

Remark 2.2.19. Because of our definition of (Γ, I) (see Section 2.1), considering orbits of the form given in 2.2.18 only makes sense in dimension $n+1=3$: for $n+1=2$ the quiver is not defined and, for $n+1>3$, the (Γ, I)-representations consist of sequences of more than one linear map. However, computing how many such orbits exist for a generic dimension $n+1$ is straightforward: the number of orbits $\mathcal{O}^{\iota}_{M^f}$ in dimension $n+1$, where M^f is determined by one $n+1\times n+1$ upper-triangular matrix, is the $n+2$-th Bell number, defined recursively by

$$b_{n+2}=\sum_{k=0}^{n+1}\binom{n+1}{k}b_k.$$

This number is obtained by counting (recursively) the possible configurations of pivots of an $n+1\times n+1$ upper-triangular matrix of any rank between 0 and $n+1$.

2.3 Two parametrisations for the representations in $R^{\iota}_{\mathbf{d}}$

Differently from Example 2.2.18, understanding and counting the orbits $\mathcal{O}^{\iota}_{M^f}$ in a generic dimension is a more complex question: the base changes performed by the action are not independent of one another, because the linear maps of a representation M^f share source and target vertices.

In order to prove that there are finitely many orbits $\mathcal{O}^{\iota}_{M^f}$ under the action given in Definition 2.2.13 and to parametrise such orbits, we recall a result for the Dynkin quivers of type $\mathbb{A}_m$ (with any orientation) presented in [ADF85]. Let

Q_m be a quiver of type $\mathbb{A}_m$. Consider a Q_m-representation $A = (A_1, \ldots, A_{m-1})$ and any pair of indices u, v such that $1 \le u \le v \le m$. Let φ^A_{uv} denote the linear map going from the direct sum of the vector spaces relative to all the sources to the one relative to all the sinks between u and v (included u and v) whose components are

$$\begin{aligned} V_{s_{t-1}} \oplus V_{s_{t+1}} &\to V_{s_t} \\ (z, z') &\mapsto (\bar{A}_{t-1,t}(z) - \bar{A}_{t+1,t}(z')) \end{aligned}$$

where $\bar{A}_{pt}$, for $p = t-1, t+1$, is the composition of all the maps A_i going from the sources s_{t-1} or s_{t+1} to the sink s_t.

Theorem 2.3.1. *[ADF85, Proposition 2.7] The orbits of the Q_m-representations A under the action of $G_{\mathbf{d}} = \prod_{i=1}^m \mathrm{GL}_{d_i}(\mathbb{K})$ are parametrised by the sets of non-negative integers $N^A = \{N^A_{uv}\}_{1 \le u \le v \le m}$ defined as*

$$\begin{cases} N^A_{uv} := \operatorname{rk} \varphi^A_{uv} & \textit{if } u < v \\ N^A_{uv} := \dim V_u & \textit{if } u = v \end{cases}.$$

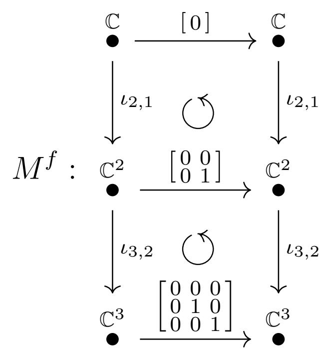
Let us now return to the question of parametrising the orbits $\mathcal{O}^\iota_{M^f}$ under the action given in Definition 2.2.13. First, we define the following sequence of non-negative integers:

Definition 2.3.2. Given a representation $M^f \in R^\iota_{\mathbf{d}}$, the **rank vector** of M^f is $\mathbf{r}^f := ((r^f_{l,s}), (r^f_{ij_1j_2k}))$, for $l, i = 1, \ldots, n+1$, $s = 1, \ldots, n$, $k = 1, \ldots, i+1$ and $1 \le j_1 \le j_2 \le n-1$, where $r^f_{l,s} := \dim(M^f_{l,s})$ and

$$\begin{cases} r^f_{ij_1j_2k} := \dim(\mathrm{Im}(f_i^{j_2} \circ \cdots \circ f_i^{j-1}) \cap \mathrm{Im}(\iota_{i,i-1} \circ \cdots \circ \iota_{k,k-1})) \text{ for } k = 1, \ldots, i \\ r^f_{ij_1j_2k} := \dim(\mathrm{Im}(f_i^{j_2} \circ \cdots \circ f_i^{j-1}) \cap \mathbb{C}^k) \text{ for } k = i+1 \end{cases}. \tag{2.3.3}$$

In words, the first part of $\mathbf{r}^f$ is the dimension vector of M^f, while the information encoded in the second part of the rank vector of M^f consists of the dimensions of all possible intersections of the images of the linear maps that determine M^f. When $k = i+1$, the number $r^f_{ij_1j_2k}$ is exactly the rank of $f_i^{j_2} \circ \cdots \circ f_i^{j-1}$.

Example 2.3.4. The rank vector associated to the following (Γ, I)-representation:

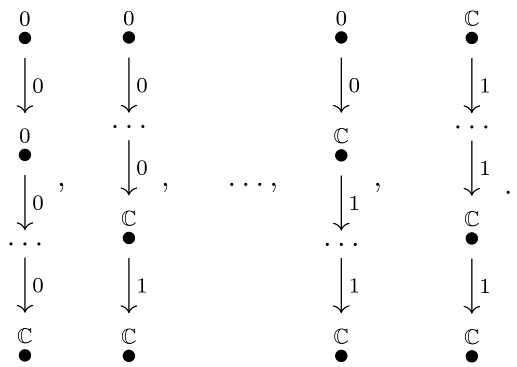


is $\mathbf{r}^f = (\dim(\mathrm{Im}(f_1^1) \cap \mathbb{C}^1), \dim(\mathrm{Im}(f_2^1) \cap \mathbb{C}^1), \dim(\mathrm{Im}(f_2^1) \cap \mathbb{C}^2), \dim(\mathrm{Im}(f_3^1) \cap \mathbb{C}^1), \dim(\mathrm{Im}(f_3^1) \cap \mathbb{C}^2), \dim(\mathrm{Im}(f_3^1) \cap \mathbb{C}^3)) = (0, 0, 1, 0, 1, 2)$.

In order to show that the rank vectors parametrise the orbits $\mathcal{O}^{\iota}_{M^f}$, we need the following technical Lemma:

Lemma 2.3.5. *The rank vectors of the indecomposable* (Γ, I)*-representations of the form* $U^{(h_1,\dots,h_n)}$*, defined in* (2.2.4)*, are independent as elements of the free* $\mathbb{Z}$*-module* $\mathbb{Z}^N$*, where* N *denotes the number of entries of any rank vector in dimension* $n+1$*.*

Proof. We proceed by induction on the number of columns in the support of the indecomposables. For the base case, we observe that the indecomposables of the form $U^{(h_1,\dots,h_n)}$ supported on only one column of (Γ, I) can be regarded as the following indecomposables of the equioriented quiver of type $\mathbb{A}_n$:



The dimension vectors of such indecomposables form a basis for the dimension vectors of all the indecomposables $U^{(h_1,\dots,h_n)}$, and therefore their set of rank vectors is an independent set. For all other indecomposables - that is, the ones supported on more than one column - in order to prove the independence we need to consider the second part of the rank vectors as well.

For the induction step, we assume the claim to be true when the number of columns in the support of the considered indecomposable V is at most $m-1$ (for some $m > 1$) and want to prove it for the indecomposables obtained by adding one column to the support of V. Let us denote such indecomposables by V'. We call $\hat{i}$ the number $n+1-h_{m-1}$, which is the first row in the $m-1$-th column where the dimension vector of V is not zero. Because of the condition $h_a \leq h_b$ for every $a \leq b$, which holds for all indecomposables of the form $U^{(h_1,\dots,h_n)}$, we know that the first row in the m-th column where the dimension vector of V' is not zero can be at most $\hat{i}$. Now, if this row is exactly $\hat{i}$, then the rank vector of V' is:

$$\begin{aligned}\mathbf{r}^{V'} = (&0, \dots, 0, r^V_{\hat{i},1,m-1,\hat{i}}, 0, \dots, 0, r^V_{\hat{i},2,m-1,\hat{i}}, 0, \dots, 0, 1, 0, \dots, 0, r^V_{\hat{i}+1,1,m-1,\hat{i}},\\ &r^V_{\hat{i}+1,1,m-1,\hat{i}+1}, \dots, 1, 0, \dots, r^V_{\hat{i}+2,1,m-1,\hat{i}}, r^V_{\hat{i}+2,1,m-1,\hat{i}+1}, r^V_{\hat{i}+2,1,m-1,\hat{i}+2}, \dots,\\ &1, 0, \dots, 1, \dots, 1)\end{aligned}$$

where the 1s in red correspond, respectively, to $r^{V'}_{\hat{\imath},m-1,m,\hat{\imath}}$, $r^{V'}_{\hat{\imath}+1,m-1,m,\hat{\imath}}$, $r^{V'}_{\hat{\imath}+2,m-1,m,\hat{\imath}}$ and so on until $r^{V'}_{n+1,m-1,m,\hat{\imath}}$. Because of the definition of the index $\hat{\imath}$, the corresponding entries of the rank vector $\mathbf{r}^V$ (and of the rank vectors corresponding to indecomposables with support smaller than the support of V) were zero, meaning that the set of all rank vectors of the indecomposables considered so far is an independent set in $\mathbb{Z}^N$ when we add $\mathbf{r}^{V'}$ to it. Similarly, we know that the entries of $\mathbf{r}^{V'}$ from $r^{V'}_{n+1,m-1,m,\hat{\imath}}$ to $r^{V'}_{n+1,m-1,m,n+1}$ (the last entries) are ones.

If, instead, the first row in the m-th column where the dimension vector of V' is not zero is $\hat{\imath}-1$, then the new 1s in the rank vector $\mathbf{r}^{V'}$ correspond to all entries labelled by $\ell, m-1, m, \ell'$ where $\ell \geq \hat{\imath}$ and $\hat{\imath}-1 \leq \ell' < \hat{\imath}$, and therefore the set of all rank vectors of the indecomposables considered so far plus $\mathbf{r}^{V'}$ is an independent set. We proceed analogously if the first row in the m-th column where the dimension vector of V' is not zero is $\hat{\imath}-2$: then the new 1s in $\mathbf{r}^{V'}$ are the entries labelled by $\ell, m-1, m, \ell'$ for $\ell \geq \hat{\imath}$ and $\hat{\imath}-2 \leq \ell' < \hat{\imath}$, and so forth until the first row in the m-th column where the dimension vector of V' is not zero is the first row of (Γ, I). Adding all such rank vectors to the preexisting set of rank vectors does not change its independency in $\mathbb{Z}^N$. This proves the statement in the induction step and concludes the proof. □

The following example illustrates the idea of the proof of Lemma 2.3.5 in dimension $n+1=3$:

Example 2.3.6. We write all the indecomposables of (Γ, I) of the form U^{h_1,h_2}

and their rank vectors for the case $n+1=3$:

$$U^1 = \begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 1} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 1} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \end{array}, \quad U^2 = \begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 1} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 1} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \end{array}, \quad U^3 = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \end{array}, \quad U^4 = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 1} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \end{array}$$

$$U^5 = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \end{array}, \quad U^6 = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \end{array}, \quad U^7 = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \end{array}, \quad U^8 = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 1} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \end{array}$$

$$U^9 = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \end{array}, \quad U^{10} = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 1} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \end{array}, \quad U^{11} = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 1} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \end{array}, \quad U^{12} = \begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{0}{\bullet} & \xrightarrow{0} & \overset{0}{\bullet} \\ \downarrow{\scriptstyle 0} & \circlearrowleft & \downarrow{\scriptstyle 0} \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{1} & \overset{\mathbb{C}}{\bullet} \end{array}$$

$$\begin{aligned} \mathbf{r}^{U^1} &= (1,1,1,1,1,1,1,1,1,1,1,1), & \mathbf{r}^{U^2} &= (1,1,1,0,0,0,0,0,0,0,0,0), \\ \mathbf{r}^{U^3} &= (0,0,0,1,1,1,0,0,0,0,0,0), & \mathbf{r}^{U^4} &= (0,1,1,0,0,0,0,0,0,0,0,0), \\ \mathbf{r}^{U^5} &= (0,0,0,0,1,1,0,0,0,0,0,0), & \mathbf{r}^{U^6} &= (0,0,1,0,0,0,0,0,0,0,0,0), \\ \mathbf{r}^{U^7} &= (0,0,0,0,0,1,0,0,0,0,0,0), & \mathbf{r}^{U^8} &= (0,1,1,1,1,1,0,1,1,1,1,1), \\ \mathbf{r}^{U^9} &= (0,0,1,1,1,1,0,0,0,1,1,1), & \mathbf{r}^{U^{10}} &= (0,1,1,0,1,1,0,0,1,0,1,1), \\ \mathbf{r}^{U^{11}} &= (0,0,1,0,1,1,0,0,0,0,1,1), & \mathbf{r}^{U^{12}} &= (0,0,1,0,0,1,0,0,0,0,0,1). \end{aligned}$$

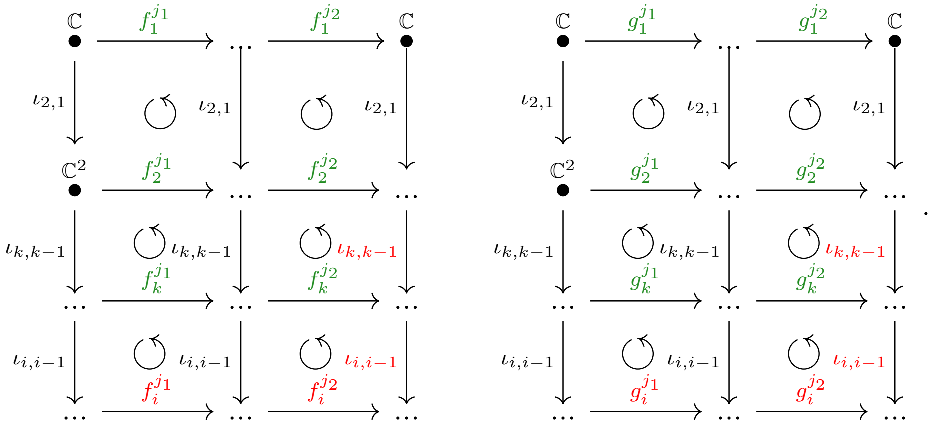
It is easy to verify that these twelve rank vectors are independent (and therefore a basis of $\mathbb{Z}^{12}$) by inserting them as rows in a matrix and using Gauss elimination.

Now we can state the first main result about the B-isomorphism classes of representations in $R^{\iota}_{\mathbf{d}}$.

Theorem 2.3.7. *Two representations M^f, M^g in $R^{\iota}_{\mathbf{d}}$ are in the same orbit under the action of $G^{\iota}_{\mathbf{d}}$ given in Definition 2.2.13 if and only if $\mathbf{r}^f = \mathbf{r}^g$.*

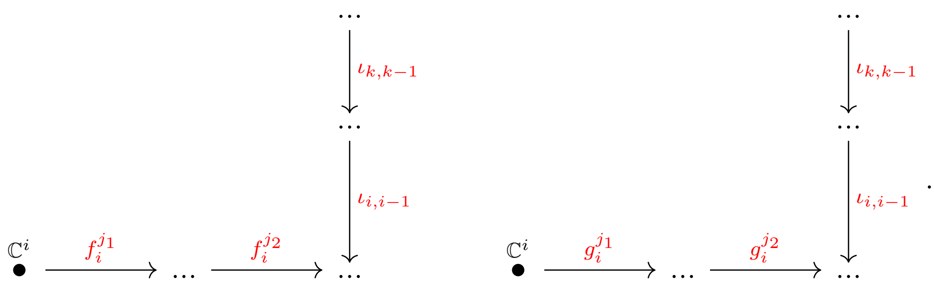
Proof. ($\Longrightarrow$) We suppose that M^f and M^g are B-isomorphic and at the same time there exist i, j_1, j_2, k such that $r^f_{ij_1j_2k} \neq r^g_{ij_1j_2k}$. If $k \neq i+1$, the parameters

$r^f_{ij_1j_2k}$ and $r^g_{ij_1j_2k}$ correspond to the dimension of the intersections of the images of the compositions of the linear maps coloured in red:



If $k = i + 1$, then $r^f_{ij_1j_2k} \neq r^g_{ij_1j_2k}$ means $\mathrm{rk}(f_i^{j_2} \circ \cdots \circ f_i^{j-1}) \neq \mathrm{rk}(g_i^{j_2} \circ \cdots \circ g_i^{j_1})$.

The hypothesis $M^f \overset{B}{\cong} M^g$ means that there exists an isomorphism φ of representations in $\mathrm{rep}_{\mathbb{C}}(Q, I)$ (represented by an an invertible, upper-triangular matrix) such that $\varphi(M^f_{i,j}) = M^g_{i,j}$ for all i, j. In particular, φ is an isomorphism when restricted to the subquivers of type $\mathbb{A}_m$:



We apply Proposition 2.7 of [ADF85] to these subquivers. The isomorphism φ between M^f and M^g then implies

$$\mathrm{rk}((f_i^{j_2} \circ \cdots \circ f_i^{j_1}) - (\iota_{i,i-1} \circ \cdots \circ \iota_{k,k-1}) = \mathrm{rk}((g_i^{j_2} \circ \cdots \circ g_i^{j_1}) - (\iota_{i,i-1} \circ \cdots \circ \iota_{k,k-1}))$$

and

$$\mathrm{rk}(f_i^{j_2} \circ \cdots \circ f_i^{j_1}) = \mathrm{rk}(g_i^{j_2} \circ \cdots \circ g_i^{j_1}),$$

contradicting the assumption $r^f_{ij_1j_2k} \neq r^g_{ij_1j_2k}$.

($\Longleftarrow$) We need to show that, if $\mathbf{r}^f = \mathbf{r}^g$, then $M^f \cong M^g$. This is equivalent to proving that to a fixed rank vector $\mathbf{r}^f$ corresponds exactly one decomposition into indecomposables: $M^f = \bigoplus U^{(h_1,\ldots,h_n)}$. This correspondence is directly implied by Lemma 2.3.5. □

Example 2.3.8. Theorem 2.3.7 provides an alternative proof to the fact that the matrices representing all orbits in Example 2.2.18 define, in fact, representations that are not B-isomorphic. We write their rank vectors without the first six entries, since the dimension vector of all (Γ, I)-representations in $R^{\iota}_{\mathbf{d}}$ is fixed (in dimension $n+1=3$) as $(1,1,2,2,3,3)$:

$$\begin{aligned} &\mathbf{r}^{f^1} = (1,1,2,1,2,3), && \mathbf{r}^{f^2} = (1,1,2,1,2,2), && \mathbf{r}^{f^3} = (1,1,1,1,1,2),\\ &\mathbf{r}^{f^4} = (0,0,1,0,1,2), && \mathbf{r}^{f^5} = (1,1,1,1,2,2), && \mathbf{r}^{f^6} = (0,0,1,1,2,2),\\ &\mathbf{r}^{f^7} = (0,1,1,1,2,2), && \mathbf{r}^{f^8} = (0,1,1,1,1,2), && \mathbf{r}^{f^9} = (1,1,1,1,1,1),\\ &\mathbf{r}^{f^{10}} = (0,0,1,0,1,1), && \mathbf{r}^{f^{11}} = (0,0,0,0,0,1), && \mathbf{r}^{f^{12}} = (0,0,0,0,1,1),\\ &\mathbf{r}^{f^{13}} = (0,0,0,1,1,1), && \mathbf{r}^{f^{14}} = (0,1,1,1,1,1), && \mathbf{r}^{f^{15}} = (0,0,0,0,0,0). \end{aligned}$$

Definition 2.3.9. We call the parametrisation of B-isomorphism classes of (Γ, I)-representations in $R^{\iota}_{\mathbf{d}}$ given in Definition 2.3.2 **r-parametrisation**.

As mentioned in Example 2.3.8, once some assumptions are made on the considered representations (for instance, their dimensions vector), part of the data provided by the r-parametrisation is redundant. We introduce now a second, more compact parametrisation for the (Γ, I)-representations in $R^{\iota}_{\mathbf{d}}$. First, let us recall the definition of Matrix Schubert varieties and a few facts from [MS05].

Definition 2.3.10. [MS05, Definition 15.1] Let $w \in \mathrm{Mat}_{k\times l}$ be a partial permutation, meaning that w is a $k \times l$ matrix having all entries equal to 0 except for at most one entry equal to 1 in each row and column. The **matrix Schubert variety** $\overline{X}_w$ inside $\mathrm{Mat}_{k\times l}$ is the subvariety

$$\overline{X}_w = \{Z \in \mathrm{Mat}_{k\times l} \,|\, \mathrm{rk}(Z_{p\times q}) \leq \mathrm{rk}(w_{p\times q}) \ \forall p, q\}$$

where $Z_{p\times q}$ is the upper-left $p \times q$ rectangular submatrix of Z.

We denote by B^-_k the Borel subgroup of invertible lower-triangular matrices in GL_k. It follows from Definition 2.3.10 that matrix Schubert varieties are preserved by the action of $B^-_k \times B_l$ on $\mathrm{Mat}_{k\times l}$ defined as $(b^-, b) \cdot Z = b^- Z b^{-1}$ (the effect of this action is "sweeping downwards" and "sweeping to the right"). The following proposition implies that $B^-_k \times B_l$ has finitely many orbits in $\mathrm{Mat}_{k\times l}$.

Proposition 2.3.11. *[MS05, Proposition 15.27] In each orbit of $B^-_k \times B_l$ on $\mathrm{Mat}_{k\times l}$ lies a unique partial permutation w, and the orbit is contained in $\overline{X}_w$.*

The orbits of $B^-_k \times B_l$ on $\mathrm{Mat}_{k\times l}$ can then be parametrised by rank arrays: given a partial permutation w, $r(w)$ is the $k \times l$ array whose entry at (p, q) is $\mathrm{rk}(w_{p\times q})$. Since this array describes the upper-left submatrices of w, we call this parametrisation **north-west parametrisation**. It follows from Proposition 2.3.11 that two matrices Z, Z' lie in the same orbit if and only if $r(Z) = r(Z')$.

Furthermore, [MS05, Lemma 15.19] and [MS05, Theorem 15.31] imply that the orbit of w' under the action of $B_k^- \times B_l$ lies in the closure of the orbit of w (with respect to the Zariski topology on $\mathrm{Mat}_{k\times l}$) if and only if $r(w') \leq r(w)$, where the "less than or equal to" relation is intended componentwise on the rank arrays.

Let us now recall the group action we considered in Definition 2.2.13. On a matrix M in Mat_{n+1}, this was defined as $(h_1, h_2) \cdot M = h_2 M h_1^{-1}$, for $(h_1, h_2) \in B_{n+1} \times B_{n+1}$. As observed in Remark 2.2.17, the effect of this action is "sweeping upwards" and "sweeping to the right", meaning that the ranks of the lower-left rectangular submatrices of M are preserved. In general, however, the group $G_{\mathbf{d}}^{\iota} = \prod_{i=1}^{n} B_{n+1}$ of Definition 2.2.13 acts on a tuple of $n-1$ matrices (in ambient dimension $n+1$). Our next goal is to show that the north-west parametrisation for matrix Schubert varieties can be adapted to our context, that is, we want to describe the orbits $\mathcal{O}^{\iota}_{M^f}$ in terms of ranks of certain submatrices obtained from $M^f = (f_{n+1}^1, \dots, f_{n+1}^{n-1}) \in R_{\mathbf{d}}^{\iota}$.

Definition 2.3.12. Given $M^f = (f_{n+1}^1, \dots, f_{n+1}^{n-1}) \in R_{\mathbf{d}}^{\iota}$, we define the **south-west array** of M^f as

$$\mathbf{s}^f = (\mathbf{s}^{f_{n+1}^{j_2} \circ \cdots \circ f_{n+1}^{j_1}}) := (\mathrm{rk}(f_{n+1}^{j_2} \circ \cdots \circ f_{n+1}^{j_1})_{p\times q})$$

for $1 \leq p \leq q \leq n+1$ and $1 \leq j_1 \leq j_2 \leq n-1$, where $(f_{n+1}^{j_2} \circ \cdots \circ f_{n+1}^{j_1})_{p\times q}$ is the lower-left $p \times q$ submatrix of $f_{n+1}^{j_2} \circ \cdots \circ f_{n+1}^{j_1}$.

In words, the south-west array of M^f contains the lower-left ranks of all possible compositions of the linear maps that define M^f in $R_{\mathbf{d}}^{\iota}$. Since all these matrices are upper-triangular, we only consider the lower-left ranks computed at entries (p, q) for $p \leq q$. In order to better visualise this information, we write the components $(\mathbf{s}^{f_{n+1}^{j_2} \circ \cdots \circ f_{n+1}^{j_1}})$ of $\mathbf{s}^f$ as upper-triangular matrices as well, so that each entry contains the value of the corresponding south-west rank.

Example 2.3.13. We consider again the (Γ, I)-representation of Example 2.3.4 and write its south-west array:

$$M^f : \begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{[0]} & \overset{\mathbb{C}}{\bullet} \\ \downarrow \iota_{2,1} & \circlearrowleft & \downarrow \iota_{2,1} \\ \overset{\mathbb{C}^2}{\bullet} & \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]} & \overset{\mathbb{C}^2}{\bullet} \\ \downarrow \iota_{3,2} & \circlearrowleft & \downarrow \iota_{3,2} \\ \overset{\mathbb{C}^3}{\bullet} & \xrightarrow{\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]} & \overset{\mathbb{C}^3}{\bullet} \end{array} .$$

In ambient dimension $n+1 = 3$, the south-west array consists of only one matrix:

$$\mathbf{s}^f = \left(\left[\begin{smallmatrix} 0 & 1 & 2 \\ * & 1 & 2 \\ * & * & 1 \end{smallmatrix}\right] \right).$$

Theorem 2.3.14. *Two representations M^f, M^g in $R^\iota_{\mathbf{d}}$ are in the same orbit under the action of $G^\iota_{\mathbf{d}}$ given in Definition 2.2.13 if and only if $\mathbf{s}^f = \mathbf{s}^g$.*

Proof. ($\Longrightarrow$) If $\mathcal{O}^\iota_{M^g} = \mathcal{O}^\iota_{M^f}$, there exists an element $h \in G^\iota_{\mathbf{d}}$ such that $h \bullet M^f = M^g$, i.e. $h_2 f^1_{n+1} h_1^{-1} = g^1_{n+1}$, $h_3 f^2_{n+1} h_2^{-1} = g^2_{n+1}$ and so forth. This means that

$$\mathcal{O}^\iota_{M^{g^j_{n+1}}} = \mathcal{O}^\iota_{M^{f^j_{n+1}}}$$

for all j, and analogously for all the possible compositions $f^{j_2}_{n+1} \circ \cdots \circ f^{j_1}_{n+1}$ and $g^{j_2}_{n+1} \circ \cdots \circ g^{j_1}_{n+1}$. By [MS05, Proposition 15.27], this implies $\mathbf{s}^f = \mathbf{s}^g$.

($\Longleftarrow$) The claim that M^f and M^g lie in the same $G^\iota_{\mathbf{d}}$-orbit when $\mathbf{s}^f = \mathbf{s}^g$ is a direct consequence of the fact that, if $\mathbf{s}^f = \mathbf{s}^g$, then $\mathbf{r}^f = \mathbf{r}^g$ (where $\mathbf{r}^f$ is the rank vector of Definition 2.3.2). This is true because, given a south-west array, the process of recovering the corresponding rank vector yields a unique result. We illustrate the (technical) operations in Figure 2.1. To simplify notation, we fix one upper-triangular matrix g and show how the rank vector $\mathbf{r}^g$ is recovered from the south-west array $\mathbf{s}^g$. The entries of $\mathbf{r}^g$ are given by $r^g_{kl} = \dim(\mathrm{Im}(g_k) \cap \mathrm{Im}(\iota_{k,k-1} \circ \cdots \circ \iota_{l,l-1}))$, where g_k denotes the restriction of g to $\mathbb{C}^k$: Then, the statement follows by applying this construction to $f^{j_2}_{n+1} \circ \cdots \circ f^{j_1}_{n+1}$ for all $1 \leq j_1 \leq j_2 \leq n-1$. $\square$

Example 2.3.15. Consider the following matrix in U_4:

$$g = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and its south-west array:

$$\mathbf{s}^g = \begin{bmatrix} 1 & 2 & 2 & 3 \\ * & 1 & 1 & 2 \\ * & * & 0 & 1 \\ * & * & * & 1 \end{bmatrix},$$

from which we want to compute the rank vector $\mathbf{r}^g$ following the method explained in Figure 2.1. First, we set $r^g_{11} := s^g_{11} = 1$, $r^g_{22} := 2$, $r^g_{33} := 2$ and $r^g_{44} := 3$. For r^g_{21}, we go over the outer **while** loop one time and find $r^g_{21} := 1$. In words, r^g_{21} can only be 0 or 1, and we decide it is 1 by looking at s^g_{11} and s^g_{12}. Similarly, we find $r^g_{31} := 1$ and $r^g_{32} := 2$ by considering s^g_{13}, s^g_{23} and s^g_{33}; notice that $r^g_{kl} \leq \min\{k, l\}$, which means that it is often not necessary to go over all the steps given in 2.1. For instance, we can set $r^g_{41} := 1$ because we already found $r^g_{11} = 1$ and it cannot be greater than 1. The same reasoning can be applied to $r^g_{42} := 2$ and $r^g_{43} := 3$. In conclusion, we find precisely

$$\mathbf{r}^g = (r^g_{11}, r^g_{21}, r^g_{22}, r^g_{31}, r^g_{32}, r^g_{33}, r^g_{41}, r^g_{42}, r^g_{43}, r^g_{44}) = (1, 1, 2, 1, 2, 2, 1, 2, 3, 3),$$

the rank vector of g.

```
input       : s^g
output      : r^g
begin
    r^g_11 := s^g_11
    r^g_kk := s^g_1k for all k                // they describe the same rank

    for 1 < l < k ≤ n + 1 do
                                              // go over k, l and determine all r^g_kl
        if s^g_1k = 0 then                    // all entries of submatrix are zero
            r^g_kl := 0
        end
        else
            dimtemp := 0                      // temporary counter
            j := 1                            // j goes over columns
            old s^g_1j := 0                   // to see where rank increases
            while j ≤ k and dimtemp ≤ l do
                while s^g_1j = old s^g_1j do
                    j++                       // to find j where s^g_1j increases
                end
                                              // now s^g_1j has increased by one
                if s^g_{l+1 j} = 0 then
                    dimtemp++                 // the increase contributes to r^g_kl
                end
                old s^g_1j := s^g_1j          // re-start check
            end
            r^g_kl := dimtemp
        end
    end
    return r^g
end
```

Figure 2.1: recovering $\mathbf{r}^g$ from $\mathbf{s}^g$

Chapter 3

Background on quiver Grassmannians, flag varieties and Schubert varieties

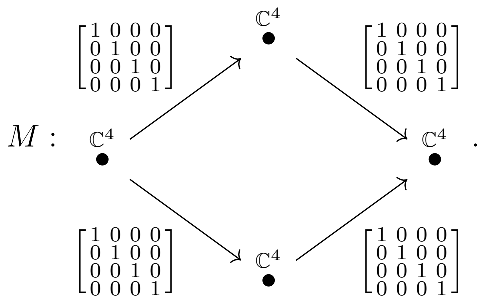
3.1 Quiver Grassmannians

In this section, we provide some background on quiver Grassmannians and a few examples and facts that will be useful later on.

Definition 3.1.1. Given a quiver Q and a Q-representation M, a **subrepresentation** of M, denoted by $N = ((N_i)_{i\in Q_0}, (M_\alpha \restriction_{N_{s(\alpha)}})_{\alpha\in Q_1})$, is a representation of Q such that $N_i \subseteq M_i$ for all $i \in Q_0$ and $M_\alpha(N_{s(\alpha)}) \subseteq N_{t(\alpha)}$ for all $\alpha \in Q_1$.

In other words, the chosen subspaces have to be compatible with the linear maps of the fixed representation M.

Example 3.1.2. We fix a basis $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$ of $\mathbb{C}^4$ and consider the following quiver with its representation $M = ((\mathbb{C}^4, \mathbb{C}^4, \mathbb{C}^4, \mathbb{C}^4), (\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}))$:



An example of a subrepresentation N of M is $N = ((\langle b_1\rangle, \langle b_1, b_2\rangle, \langle b_1, b_4\rangle, \langle b_1, b_2, b_4\rangle), ([\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}], \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}))$, which has dimension vector $\dim(N) = (1, 2, 2, 3)$. The matrices representing the linear maps appearing in N are the restrictions of the identity maps to the chosen subspaces. A sequence of subspaces that cannot belong to any subrepresentation of M is, for instance, $(\langle b_1\rangle, \langle b_1, b_2\rangle, \langle b_1, b_4\rangle, \langle b_1, b_2, b_3\rangle)$, because $\mathrm{id}(\langle b_1, b_4\rangle) \nsubseteq \langle b_1, b_2, b_3\rangle$.

Definition 3.1.3. Consider a quiver Q, a Q-representation M and a dimension vector $\mathbf{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$ such that $e_i \leq \dim M_i$ $\forall i \in Q_0$. The **quiver Grassmannian** $\mathrm{Gr}_{\mathbf{e}}(M)$ is defined as the collection of all subrepresentations N of M with $\dim N_i = e_i$ for all $i \in Q_0$.

Analogously to Grassmannians and flag varieties, non-empty quiver Grassmannians can be realised as closed subvarieties of products of Grassmannians, via the close embedding

$$\iota : \mathrm{Gr}_{\mathbf{e}}(M) \to \prod_{i\in Q_0} \mathrm{Gr}(e_i, M_i)$$

which sends a subrepresentation N of M to the collection of e_i-dimensional subspaces N_i of M_i. The relations defining (pointwise) the subvariety associated to a given quiver representation and a dimension vector are given in [LW19] and called **quiver Plücker relations**:

Definition 3.1.4. [LW19] We consider a quiver Q, a Q-representation M, a dimension vector $\mathbf{e}$ for the quiver Q and fix bases for all vertices. Let M_α be the matrix representing the linear map associated to an arrow $\alpha \in Q_1$, let $r = e_{s(\alpha)}$ and $s = e_{t(\alpha)}$. Then, the quiver Plücker relations arising from M_α are the polynomials in the variables $\{p_I : I \in \binom{\dim(M_{s(\alpha)})}{r}\} \cup \{p_J : J \in \binom{\dim(M_{t(\alpha)})}{s}\}$ with coefficients in $\mathbb{K}$:

$$\mathscr{P}_\alpha = \{ \sum_{j\in[n]\setminus I, i\in J} \mathrm{sign}(j; I, J) \cdot (M_\alpha)_{i,j} \cdot p_{I\cup j} \cdot p_{J\setminus i} \mid I \in \binom{\dim(M_{s(\alpha)})}{r-1},$$
$$J \in \binom{\dim(M_{t(\alpha)})}{s+1} \}$$

where $\mathrm{sign}(j; I, J) = (-1)^{\#\{j'\in J: j<j'\}+\#\{i\in I: i>j\}}$.

Note that, to define $\mathrm{Gr}_{\mathbf{e}}(M)$ in a product of projective spaces, we need to consider the standard Grassmann-Plücker relations (explained, for instance, in [LB09, Section 9.5]) for each vector space as well as the quiver Plücker relations. Later, in Chapter 6, we will discuss a tropicalisation of the quiver Plücker relations.

We present now a few examples of quiver Grassmannians with very different geometric properties and structures. It is known, in fact, that quiver Grassmannians can be anything: the fact that every projective variety arises as a quiver Grassmannian was proven in [Rei13], and later generalised in [Rin18] by showing that every projective variety arises as the quiver Grassmannian of every wild quiver.

Example 3.1.5. For the quiver consisting of one vertex and no arrows, defining a (finite-dimensional) representation M means to choose a dimension m for the vector space assigned to the vertex. Then, for a positive integer $k \leq m$, the quiver Grassmannian $\mathrm{Gr}_{(k)}(M)$ is isomorphic to the Grassmannian $\mathrm{Gr}(k, m)$, i.e. the projective variety that parametrises vector subspaces of dimension k in a vector space of dimension m.

Example 3.1.6. [Ire20, Example 4] Consider the quiver $Q = \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet}$ of type $\mathbb{A}_2$ and the following Q-representation:

$$M = \overset{\mathbb{C}^2}{\bullet} \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]} \overset{\mathbb{C}^2}{\bullet} .$$

If we choose $\mathbf{e} = (1, 1)$ as dimension vector, then the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is the union of two $\mathbb{P}^1 s$ crossing in one point and is therefore a connected, equidimensional curve of dimension one with two irreducible components and one singular point. We can obtain a non-equidimensional quiver Grassmannian with similar geometric structure by changing the dimension of the considered vector spaces. For the Q-representation:

$$M = \overset{\mathbb{C}^2}{\bullet} \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\\0&0\end{smallmatrix}\right]} \overset{\mathbb{C}^3}{\bullet}$$

and again dimension vector $\mathbf{e} = (1, 1)$, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is the union of a $\mathbb{P}^2$ and a $\mathbb{P}^1$ crossing in one point, and so is a connected projective variety of dimension two with two irreducible components (of dimensions one and two) and one singular point.

Example 3.1.7. [Ire20, Example 5] Non-connected quiver Grassmannians can easily be obtained by exploiting quivers with parallel edges and the compatibility conditions given in Definition 3.1.1 defining subrepresentations. Consider the Kronecker quiver $Q = 1 \rightrightarrows 2$, a 2×2 matrix A with two distinct eigenvalues and the Q-representation:

$$M = \mathbb{C}^2 \underset{A}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{C}^2 .$$

Then, for $\mathbf{e} = (1, 1)$, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ consists of two distinct points (the two eigenspaces) and hence is a zero-dimensional projective variety with two connected components.

Example 3.1.8. The complete flag variety Fl_{n+1} is defined (as a set) as

$$Fl_{n+1} = \{V_0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1} |\dim(V_i) = 0, i = 1, \ldots, n\},$$

i.e. it parametrises chains of vector subspaces in ambient dimension $n + 1$. In [CIFR12], the authors realise the (linear degenerate) flag variety as the quiver

Grassmannian associated to certain representations of the equioriented quiver of type $\mathbb{A}_n$. In particular, the complete flag variety Fl_{n+1} can be realised as follows. Consider the the equioriented Dynkin quiver Q of type $\mathbb{A}$ with n vertices. We fix the Q-representation M with $M_i = \mathbb{C}^{n+1}$ for $i = 1, \dots, n$ and $M_\alpha = \mathrm{id}$ for all arrows α:

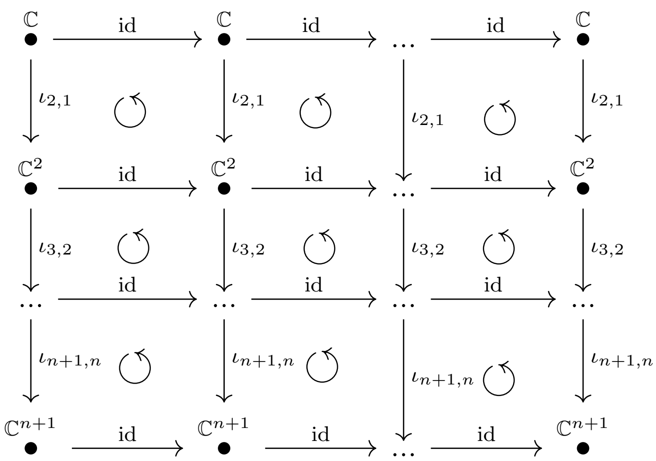
$$\overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{\mathrm{id}} \overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \overset{\mathbb{C}^{n+1}}{\bullet},$$

and the dimension vector $\mathbf{e} = (1, 2, \dots, n)$. Then, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ consists precisely of the subrepresentations N of M with $\dim(N_i) = i$, i.e. full flags of vector subspaces.

Since every projective variety arises as a quiver Grassmannian, it is clear that studying quiver Grassmannians in a general case would not be fruitful. Hence, their study needs to be restricted to particular cases, for instance by considering quivers or quiver representations with special properties. One possible, very rewarding restriction is to consider Dynkin quivers, in particular equioriented quivers of type $\mathbb{A}$. We will discuss in Chapter 5 a family of quiver Grassmannians for such quivers which realises the linear degenerations of the flag variety Fl_{n+1}.

The rigidity of a representation (see Definition 1.2.30), together with some further assumptions, yields certain geometrical properties of the associated quiver Grassmannians, for any dimension vector $\mathbf{e}$. For example, as shown in [CIFR12, Proposition 2.2], if a Q-representation M with dimension vector $\mathbf{d}$ is rigid and $\mathrm{Ext}^1(N, L)$ vanishes for two generic representations N and L of dimensions $\mathbf{e}$ and $\mathbf{d}-\mathbf{e}$, then $\mathrm{Gr}_{\mathbf{e}}(M)$ is non-empty, smooth, and has dimension $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle$ (the Euler-Ringel form given in Definition 1.2.27).

We briefly recall the representation M of the quiver with relations (Γ, I) defined in Section 2.1:



.

We showed in Proposition 2.1.5 that M is a rigid representation of (Γ, I). Now we exploit this result, together with a few homological properties of M, to characterise the quiver Grassmannians $\mathrm{Gr}_{\mathbf{e}}(M)$ associated to M for any dimension vector $\mathbf{e}$.

Proposition 3.1.9. *Given* (Γ, I) *and* M *as above, the quiver Grassmannian* $\mathrm{Gr}_{\mathbf{e}}(M)$ *is a smooth and irreducible projective variety for any dimension vector* $\mathbf{e}$. *Its dimension is* $\langle \mathbf{e}, \mathbf{dim}(M) - \mathbf{e} \rangle$.

Proof. As shown in Proposition 2.1.5, M is a rigid representation of (Γ, I), therefore the irreducibility of $\mathrm{Gr}_{\mathbf{e}}(M)$ follows directly from [IEFR21, Proposition 3.8]. In order to prove the remaining claims, we show that all hypotheses of [IFR13, Proposition 7.1] hold. The representation M has projective dimension zero, since it is a projective representation of (Γ, I) (see Example 2.2.9), and Remark 2.2.10 shows that the injective dimension of M is one. It is straightforward to verify that the quotient algebra $\mathbb{C}\Gamma/I$ has global dimension two, since it can be realised as the tensor product of two well-known path algebras. Namely, we consider the path algebra of the cartesian product of an equioriented $\mathbb{A}_n$ quiver and an equioriented $\mathbb{A}_{n+1}$ quiver and take the quotient over the commutativity relations on all resulting squares. It is known that the global dimension of the path algebra of any type $\mathbb{A}_n$ quiver (for $n \geq 2$) is one (see, for instance, [Sch14, Section 2.2]). Then, we apply [Aus55, Theorem 16] and obtain that the global dimension of $\mathbb{C}\Gamma/I$ is the sum of the global dimensions of the path algebras of the two quivers of type $\mathbb{A}_n$. □

Extensive discussions and characterisations about special classes of quiver Grassmannians can be found, for instance, in [CI11, CIE12, IFR13, CIFR12, CIFF$^+$17, CIFR17, CIFF$^+$20, Ire20, IEFR21, LP23, FLP23, FLMP24].

3.2 Flag varieties and their Schubert varieties

Flag varieties are fundamental objects of study in algebraic geometry, due to their rich structure and their connections to various fields, such as representation theory, combinatorics, and commutative algebra. We have already seen in Example 3.1.8 how the complete flag variety Fl_{n+1} is defined in $\mathbb{C}^{n+1}$: it is the algebraic variety consisting of flags, i.e. sequences of subspaces $V_0 \subset V_1 \subset \cdots \subset V_n$ where $\dim(V_i) = i$ for all i. The group of invertible matrices GL_{n+1} acts naturally on Fl_{n+1}, and the quotient $\mathrm{GL}_{n+1}/B_{n+1}$, where B_{n+1} is the Borel subgroup of GL_{n+1} consisting of upper-triangular matrices, is isomorphic to the complete flag variety. The orbits under the action of B_{n+1} on $\mathrm{GL}_{n+1}/B_{n+1}$ are the Schubert cells, which are isomorphic to affine spaces and form a stratification of $\mathrm{GL}_{n+1}/B_{n+1}$. Their closures in $\mathrm{GL}_{n+1}/B_{n+1}$ are the Schubert varieties, which are, in general, singular varieties; at the end of this section, we briefly recall their desingularisation via Bott-Samelson varieties. Our main references for this section are [Ful97, LB09, Bri05, BL00].

Definition 3.2.1. A sequence of subspaces $V_\bullet = V_0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}$ such that $\dim(V_i) = i$ for all i is a **complete flag** in $\mathbb{C}^{n+1}$. We denote by Fl_{n+1}

the set of full flags in $\mathbb{C}^{n+1}$ and, given a basis $\mathcal{B} = \{ b_1, b_2, \ldots, b_{n+1} \}$, we denote by $F_\bullet$ the **standard flag** $\langle b_1 \rangle \subseteq \langle b_1, b_2 \rangle \subseteq \cdots \subseteq \langle b_1, b_2, \ldots, b_{n+1} \rangle$

Remark 3.2.2. Since we fixed basis $\mathcal{B}$ of $\mathbb{C}^{n+1}$, we can represent a flag $V_\bullet$ in Fl_{n+1} by the invertible $n+1 \times n+1$ matrix whose first column spans V_1, first and second columns together span V_2, and so on. The group GL_{n+1} then acts transitively on Fl_{n+1} by matrix multiplication.

The set of full flags can be equivalently realised as the quotient $\mathrm{GL}_{n+1}/B_{n+1}$, and has therefore the structure of a projective variety. This can be seen by embedding the set of full flags in the appropriate product of Grassmannians:

$$Fl_{n+1} \hookrightarrow \mathrm{Gr}(1, n+1) \times \mathrm{Gr}(2, n+1) \times \cdots \times \mathrm{Gr}(n, n+1)$$

and imposing the incidence relations $V_i \subset V_{i+1}$ for $i = 1, \ldots, n$, which allow us to identify Fl_{n+1} as a closed subset of $\mathrm{Gr}(1, n+1) \times \mathrm{Gr}(2, n+1) \times \cdots \times \mathrm{Gr}(n, n+1)$ (more precisely, Fl_{n+1} is obtained as the zero locus of the Plücker relations [LB09, Section 9.5]).

The Borel subgroup $B_{n+1} \subset \mathrm{GL}_{n+1}$ acts (not transitively) on Fl_{n+1}, and this action provides a stratification of Fl_{n+1}:

Proposition 3.2.3. *[Bri05, Proposition 1.2.1]*

1. *Fl_{n+1} is the disjoint union of its B_{n+1}-orbits, and each orbit X_w° corresponds to a permutation w in S_{n+1} (the symmetric group on $n+1$ elements).*

2. *Further, denoting by X_w the closure of X_w° in Fl_{n+1} with respect to the Zariski topology, we have*

$$X_w = \bigsqcup_{u \leq w} X_u^\circ,$$

where "$\leq$" denotes the Chevalley-Bruhat order on S_{n+1}.

The relation $u \leq w$ holds under the Chevalley-Bruhat (partial) order on S_{n+1} if every reduced expression of w contains a subexpression which is a reduced expression for u.

We give now an equivalent definition for the Schubert cells - and for their closures - in Fl_{n+1}. For the purposes of this thesis, we will focus on their combinatorial aspects and on some known characterisations of singularity for Schubert varieties.

Definition 3.2.4. For $w \in S_{n+1}$, the **Schubert cell** X_w° is

$$X_w^\circ = \{ V_\bullet \in Fl_{n+1} : \dim(F_p \cap V_q) = \#\{ k \leq q : w(k) \leq p \} \text{ for } 1 \leq p, q \leq n+1 \}.$$

Definition 3.2.5. The **Schubert variety** X_w is defined as the closure in Fl_{n+1} of the cell X_w°, that is

$$X_w = \{ V_\bullet \in Fl_{n+1} : \dim(F_p \cap V_q) \geq \#\{ k \leq q : w(k) \leq p \} \text{ for } 1 \leq p, q \leq n+1 \}.$$

We observe that the conditions on the intersections between the F_p and the V_q imply, for each pair p, q, one of the following : $F_p \subset V_q$, $F_p \supset V_q$, $F_i = V_j$ or $F_p \cap V_q = U$ with $0 \leq \dim(U) < \min\{p, q\}$. A minimal set of conditions that imply all the conditions defining a Schubert variety X_w has been described in terms of essential sets of the permutation w (see [Ful92, Section 3] or [GR02, Section 4]).

Each Schubert variety X_w is an irreducible subvariety of Fl_{n+1}, and its dimension is given by the number of inversions in w, called length:

$$\ell(w) = \#\{i < j : w(i) > w(j)\}.$$

The length of a permutation w is also the minimal number of simple transpositions needed to form a decomposition of w, called **reduced decomposition**: $w = s_{\ell(w)} \cdots s_1$, where s_i denotes the swap of i and $i+1$. We recall that, in general, a permutation admits more than one reduced decomposition. As mentioned above, the Schubert variety X_w consists of the cell X_w°, which is open and dense in X_w, and of the cells corresponding to permutations that are smaller than w with respect to the Chevalley-Bruhat order on S_{n+1}.

For the purposes of this thesis, we consider the intersections $F_p \cap V_q$ instead of the standard $V_p \cap F_q$ in the definition of Schubert varieties, and to simplify the notation we write $r^w_{p,q}$ for the numbers $\#\{k \leq q : w(k) \leq p\}$. We represent a permutation w in S_n by listing its (naturally) ordered images, that is, its one-line notation $w = [w(1)w(2)\dots w(n)]$.

Example 3.2.6. For id $= [1\,2 \dots n+1]$ and $w_0 = [n+1\, n \dots 1]$ in S_{n+1}, it is easy to compute from Definition 3.2.5 the Schubert varieties of minimal and maximal dimension, respectively $X_e = \{F_\bullet\}$ and $X_{w_0} = Fl_{n+1}$.

Smooth Schubert varieties were characterised combinatorially in [LS90]: a Schubert variety X_w is smooth if and only if w avoids the patterns [4231] and [3412]. We recall that a permutation $w = [w(1)w(2)\dots w(n)]$ avoids a pattern π if no subsequence of w has the same relative order as the entries of π. In [GR02, Theorem 1.1], the authors prove that this pattern-avoiding condition is equivalent to X_w being defined by non-crossing inclusions:

Definition 3.2.7. ([GR02, Section 1]) A Schubert variety X_w is **defined by inclusions** if the defining conditions on each V_q (see Definition 3.2.5) are a conjunction of conditions of the form $V_q \subseteq F_p$ and $V_q \supseteq F_s$, for some p and s. A pair of conditions $V_q \subset F_p$ and $F_{p'} \subset V_{q'}$ is **crossing** if $q < q'$ and $p > p'$.

If X_w is defined by inclusions and its conditions do not contain any crossing pair, then X_w is **defined by non-crossing inclusions**.

Example 3.2.8. All permutations in S_3 are defined by non-crossing inclusions.

In S_5, the permutation $w = [31542]$ avoids both patterns [4231] and [3412], which means that X_w is defined by non-crossing inclusions. We can compute these inclusions using Definition 3.2.5: a flag $V_\bullet$ is in X_w if and only if

$$V_1 \subseteq F_3,\ F_1 \subseteq V_2 \subseteq F_3,\ F_1 \subseteq V_3,\ F_1 \subseteq V_4.$$

The same conditions can be described without redundancy as $F_1 \subseteq V_2 \subseteq F_3$, which is a pair of non-crossing inclusions.

A permutation in S_5 that yields crossing inclusions is, for instance, $\tau = [45312]$, which contains the pattern $[3412]$. A flag $V_\bullet$ is in X_τ if and only if $V_1 \subseteq F_4$, $F_1 \subseteq V_4$.

Finally, the permutation $\pi = [53421]$ in S_5 contains the pattern $[4231]$ and defines a non-trivial condition on X_π that is not an inclusion: a flag $V_\bullet$ is in X_π if and only if $\dim(F_3 \cap V_2) \geq 1$.

There exist several constructions to desingularise Schubert varieties, but for the purposes of this thesis we only introduce their Bott-Samelson resolution. These are equivariant desingularisations of Schubert varieties in G/B (or, more generally, in G/P for a parabolic subgroup $P \subset G$), and they can be thought of as "towers" of $\mathbb{P}^1$-bundles; for a more detailed background see, for instance, [AF23, Chapter 18].

Definition 3.2.9. ([HMP20, Definition 3.1]) Given a permutation $w \in S_{n+1}$ of length N and a reduced decomposition $w = s_{i_N} \cdots s_{i_1}$, the **Bott-Samelson variety** $\mathrm{BS}(s_{i_N} \cdots s_{i_1})$ is a subvariety of $(Fl_{n+1})^N$ defined as follows:

$$\begin{aligned}\mathrm{BS}(s_{i_N} \cdots s_{i_1}) = \{(V^0_\bullet, V^1_\bullet, \ldots, V^N_\bullet) \in (Fl_{n+1})^N : V_i^{k-1} = V_i^k, \forall k = 1, \ldots, N, \\ \forall i = 1, \ldots, n, i \neq i_k\}\end{aligned}$$

where $V^0_\bullet = F_\bullet = \langle b_1 \rangle \subseteq \langle b_1, b_2 \rangle \subseteq \cdots \subseteq \langle b_1, b_2, \ldots, b_{n+1} \rangle$, the standard flag in Fl_{n+1}.

Example 3.2.10. We fix the permutation $w = [43251] \in S_5$ and its reduced decomposition $w = s_1 s_2 s_3 s_1 s_2 s_1 s_4$. The elements $V^\bullet_\bullet$ of $\mathrm{BS}(s_1 s_2 s_3 s_1 s_2 s_1 s_4)$ are given by tuples of seven complete flags, each living in Fl_5, of the following form:

$$\begin{array}{ll}\langle b_1 \rangle \subseteq \langle b_1, b_2 \rangle \subseteq \langle b_1, b_2, b_3 \rangle \subseteq V_4^1, & V_1^4 \subseteq V_2^3 \subseteq V_3^5 \subseteq V_4^1, \\ V_1^2 \subseteq \langle b_1, b_2 \rangle \subseteq \langle b_1, b_2, b_3 \rangle \subseteq V_4^1, & V_1^4 \subseteq V_2^6 \subseteq V_3^5 \subseteq V_4^1, \\ V_1^2 \subseteq V_2^3 \subseteq \langle b_1, b_2, b_3 \rangle \subseteq V_4^1, & V_1^7 \subseteq V_2^3 \subseteq V_3^5 \subseteq V_4^1, \\ V_1^4 \subseteq V_2^3 \subseteq \langle b_1, b_2, b_3 \rangle \subseteq V_4^1. & \end{array}$$

Remark 3.2.11. In Chapter 4, we will consider some opportune reduced decompositions of (any permutation) w and show how the Bott-Samelson resolutions of X_w corresponding to such decompositions can be realised as a certain quiver Grassmannian. The Bott-Samelson varieties corresponding to different reduced decompositions of the same permutation w are birational (see [AF23][Chapter 18, Lemma 2.1]), and therefore they are all birational to the constructed quiver Grassmannian.

Chapter 4

Schubert varieties and quiver Grassmannians

Schubert varieties have already been linked to degenerate flag varieties and to quiver Grassmannians. Two recent examples of such connections are in [IL14], where the authors show that any type A or C degenerate flag variety is isomorphic to a Schubert variety in an appropriate partial flag manifold, and later in [CIFR17], which proves that some Schubert varieties arise as irreducible components of certain quiver Grassmannians.

In this chapter, we get back to the quiver with relations (Γ, I) and its representation M defined in Section 2.1. We will consider the quiver Grassmannian that corresponds to opportune choices of a dimension vector for the quiver (Γ, I); this projective variety, as shown in Proposition 3.1.9, is always smooth and irreducible. We will prove two separate results for Schubert varieties: for a certain choice of dimension vector, our quiver Grassmannian recovers the Bott-Samelson resolution for Schubert varieties (see Definition 3.2.9) and, for a different choice of dimension vector, it is isomorphic to any chosen smooth Schubert variety. The following results are part of the paper "Quiver Grassmannians for the Bott-Samelson resolution of type A Schubert varieties", written by the author of this thesis and made available at [Iez25]. We present them in the context of this thesis, providing useful examples and more extensive background.

4.1 Recovering the Bott-Samelson resolution for Schubert varieties

We consider the quiver (Γ, I) and its representation M constructed in Section 2.1, and fix a permutation w in S_{n+1}. The conditions that define the elements $V_\bullet$ in X_w are of the form $\dim(F_p \cap V_q) \geq \#\{\, k \leq q : w(k) \leq p \,\}$, for $1 \leq p, q \leq n+1$ (see Definition 3.2.5). Notice that for $q = n+1$ and any p these conditions are trivial, since $n+1$ is the dimension of the ambient space $\mathbb{C}^{n+1}$, and therefore it is enough to consider $q = 1, \dots, n$.

Now we define the dimension vector $\mathbf{r}^w = (r^w_{i,j})$ for the quiver (Γ, I) as

$$r^w_{i,j} := \#\{\, k \leq j : w(k) \leq i \,\}, \quad i = 1, \dots, n+1, \quad j = 1, \dots, n. \tag{4.1.1}$$

Before introducing the Bott-Samelson resolution for Schubert varieties, let us make a few remarks about this definition for the dimension vector $\mathbf{r}^w$, in particular about how its entries change as we move from w to permutations that are bigger than w with respect to the Bruhat order in S_{n+1}. The following lemma describes which (unique) row and which columns of the dimension vector are affected, and how they change, when we left-multiply by a simple transposition which increases by one the length of the permutation we are considering.

Lemma 4.1.2. *Consider $r^{\hat{w}}_{p,q} = \#\{\, j \leq q : \hat{w}(j) \leq p \,\}$ for $1 \leq p, q \leq n+1$ and a fixed $\hat{w} \in S_{n+1}$ (see Definition 3.2.5). Then, for a simple transposition s_i such that $\ell(s_i\hat{w}) = \ell(\hat{w}) + 1$, the numbers $r^{s_i\hat{w}}_{p,q} = \#\{\, j \leq q : s_i\hat{w}(j) \leq p \,\}$ are given by*

$$\begin{cases} r^{s_i\hat{w}}_{p,q} = r^{\hat{w}}_{p,q} - 1 & \text{if } p = i \text{ and } q_i \leq q < q_{i+1} \\ r^{s_i\hat{w}}_{p,q} = r^{\hat{w}}_{p,q} & \text{otherwise} \end{cases}$$

where $q_i = \hat{w}^{-1}(i)$ and $q_{i+1} = \hat{w}^{-1}(i+1)$.

Proof. It is straightforward to verify that, since s_i only swaps i and $i+1$, the count is not affected when $p \neq i$ or when $p = i$ and $q < q_i \ \vee \ q \geq q_{i+1}$.

If $p = i$ and $q_i \leq q < q_{i+1}$, there is exactly one j that satisfies $j \leq q \wedge \hat{w}(j) \leq p$ but not $j \leq q \ \wedge \ s_i\hat{w}(j) \leq p$, that is $j = q_i$, and so in this case the count decreases by one. □

Example 4.1.3. We fix $\hat{w} = [34251] \in S_5$ and compute the corresponding dimension vector $\mathbf{r}^{\hat{w}}$ according to the definition given in (4.1.1):

$$\mathbf{r}^{\hat{w}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Then we apply s_3, obtaining $w := s_3\hat{w} = [43251]$, and we know from Lemma 4.1.2 that $\mathbf{r}^w$ differs from $\mathbf{r}^{\hat{w}}$ only at entry $r^w_{3,1}$:

$$\mathbf{r}^{w} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \color{red}{0} & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Remark 4.1.4. An important consequence of Lemma 4.1.2 is that some information about the reduced decompositions of w can be read directly off the corresponding dimension vector. In Example 4.1.3, for instance, we can compare $\mathbf{r}^w$ to the dimension vector corresponding to the identity in S_5:

$$\mathbf{r}^{\mathrm{id}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

and observe that $r^w_{2,2} = r^{\mathrm{id}}_{2,2} - 2$. By Lemma 4.1.2, this can only happen if the simple transposition s_2 appears at least two times in any reduced decomposition of w. Similarly, we deduce that s_1, s_3 and s_4 appear at least one time in any reduced decomposition of w. The converse is also true: if a simple transposition s_i appears k times in all reduced decompositions of a given permutation w (that is, there are k instances of s_i that are not part of any braid $s_i s_{i+1} s_i$ or $s_{i+1} s_i s_{i+1}$), then there exists an entry in the i-th row of the dimension vector $\mathbf{r}^w$ that has decreased by k from its value in $\mathbf{r}^{\mathrm{id}}$. We do not include a proof of this statement as it is not relevant to the purpose of this section, but an idea of the strategy can be found in the proof of Theorem 4.1.14, since knowing that these k instances of s_i are not part of any braid allows us to describe which simple transpositions can appear between them.

Now, we want to show that the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$ is isomorphic to certain Bott-Samelson resolutions of X_w (see Definition 3.2.9 and Example 3.2.10). In order to do so, we consider the following type of reduced decompositions for permutations in S_{n+1}:

Definition 4.1.5. Let $w \in S_{n+1}$ and denote by $R = (R_{p,q})$, for $p = 1, \dots, n+1$ and $q = 1, \dots, n$, an element of $\mathrm{Gr}_{\mathbf{r}^w}(M)$. We call a reduced decomposition $w = s_{i_N} \cdots s_2 s_1$ **geometrically compatible** if $[i_N, \dots, i_2, i_1] = [r^w_{p,q}]$ for all the p, q such that

$$\begin{cases} r^w_{p,q} < p \\ r^w_{p,q} > r^w_{p-1,q} \\ r^w_{p,q} > r^w_{p,q-1} \end{cases}, \tag{4.1.6}$$

where the notations $[i_N, \dots, i_2, i_1]$ and $[r^w_{p,q}]$ indicate nonordered multisets.

Remark 4.1.7. The $r^w_{p,q}$ that satisfy the conditions listed in (4.1.6) are the dimensions of exactly those subspaces $R_{p,q}$ of $\mathbb{C}^p$ that are not trivial and do not coincide with a subspace to their left or above them. Fixing an element R in the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$ means precisely to make a choice for all such subspaces $R_{p,q}$. Hence the name "geometrically compatible" decomposition: it is a reduced decomposition of w from which we can read the dimensions of all the subspaces that are relevant to determine R.

Because of the geometrical significance of the conditions given in (4.1.6), we will interchangeably refer to the $r^w_{p,q}$ and to the $R_{p,q}$ that satisfy these conditions.

Example 4.1.8. We consider $w = [43251]$ and the corresponding dimension vector $\mathbf{r}^w$ as in 4.1.3. Given any subrepresentation R in $\mathrm{Gr}_{\mathbf{r}^w}(M)$, the subspaces whose dimensions satisfy all conditions in (4.1.6) are $R_{2,3}, R_{3,2}, R_{4,1}, R_{3,3}, R_{4,2}, R_{4,3}$ and $R_{5,4}$. Their dimensions are, in order, 1,1,1,2,2,3, and 4, so a geometrically compatible decomposition of w contains three s_1, two s_2, one s_3 and one s_4. The decomposition $w = s_1s_2s_3s_1s_2s_1s_4$ considered in 3.2.10 is geometrically compatible, while, for instance, $w = s_3s_1s_2s_1s_3s_2s_4$ is not.

To show that all permutations admit a geometrically compatible decomposition, we first characterise the subspaces appearing in (4.1.6), that is, what follows from the fact that a certain $R_{p,q}$ is not a trivial subspace of $\mathbb{C}^p$ in terms of the reduced decompositions of w. Recall that the length of a permutation w can be equivalently defined as the number of inversions appearing in w or as the number of simple transpositions that form any reduced decomposition of w.

Lemma 4.1.9. *Given $w \in S_{n+1}$ and a subrepresentation $R = (R_{p,q})$ in $\mathrm{Gr}_{\mathbf{r}^w}(M)$, with $p = 1, \ldots, n+1$ and $q = 1, \ldots, n$, the number of subspaces that satisfy all conditions in* (4.1.6) *is exactly the length of w.*

Proof. We denote by N the length of w and write $w = s_{i_N} \cdots s_2 s_1$. By Lemma 4.1.2, the left-multiplication of each s_{i_k} results in a new subspace (namely $R_{i_k+1,q_{i_k}}$, for q_{i_k} as in the notation of Lemma 4.1.2) satisfying the conditions in (4.1.6). On the other hand, applying one simple transposition cannot cause two new subspaces to satisfy the conditions in (4.1.6), because all the affected entries of the dimension vector decrease by the same amount. $\square$

Lemma 4.1.10. *Let $w \in S_{n+1}$ and R any subrepresentation in $\mathrm{Gr}_{\mathbf{r}^w}(M)$. If the $r^w_{p,q}$ that satisfy the conditions in* (4.1.6) *are all distinct, then w admits a geometrically compatible decomposition.*

Proof. In order for a subspace of dimension d to satisfy the conditions in (4.1.6), the simple transposition s_d must appear at least once in any reduced decomposition of w. If this weren't the case, by Lemma 4.1.2 the d-th row of $\mathbf{r}^w$ would be equal to the d-th row of $\mathbf{r}^{\mathrm{id}}$, which would imply that all subspaces appearing in R of dimension d have to coincide with $\mathbb{C}^d$ - and therefore do not satisfy the conditions in (4.1.6). The result then follows immediately from Lemma 4.1.9. $\square$

Remark 4.1.11. A straightforward consequence of Lemma 4.1.2 is that if the reduced decompositions of $w \in S_{n+1}$ consist of all distinct simple transpositions, then they are geometrically compatible. As shown in the lemma, each of these transpositions s_{i_k} affects the corresponding row of the dimension vector, resulting in the subspace R_{i_k+1,q_i} (which has dimension i_k) satisfying the conditions in (4.1.6).

We recall the following notation from Lemma 4.1.2: if we left-multiply a permutation w by a simple transposition s_j, we denote by q_j the pre-image $w^{-1}(j)$ of j via w.

Lemma 4.1.12. *Let $w \in S_{n+1}$ with reduced decomposition $w = s_{i_N} \dots s_{i_1}$, s_j a simple transposition such that $\ell(s_j w) = \ell(w) + 1$, and R_{j+1,q_j} the subspace that satisfies the conditions in (4.1.6) if R is any subrepresentation in $\mathrm{Gr}_{\mathbf{r}^{s_j w}}(M)$ (but does not satisfy them if R is in $\mathrm{Gr}_{\mathbf{r}^w}(M)$). Then, the dimension of R_{j+1,q_j} is $\hat{j}$ for some $\hat{j} \leq j$. In particular, $\hat{j} < j$ can only happen if all reduced decompositions of $s_j w$ are of the form $s_j w = s_j s_{i_N} \dots s_{i_k} \dots s_{i_1}$, where $i_k = j$ for some k such that $i_t \neq j+1$ for all $k < t < N$.*

Proof. The first statement is almost straightforward. A subspace $R_{p,q}$ can satisfy the conditions in (4.1.6) only if $p \geq \dim(R_{p,q}) + 1$, and we know from Lemma 4.1.2 that the only effect of s_j on the corresponding dimension vector is to decrease certain entries in row j by one. By Definition 4.1.1 of the dimension vector, all entries are bounded by their corresponding numbers of row and column (which implies that a dimension j' can only appear from row j' downwards), and so the dimension of R_{j+1,q_j} cannot be greater than j.

For the second statement, we know that an index k such that $i_k = j$ exists: as stated in Remark 4.1.11, if the simple transpositions appearing in the reduced decomposition of $s_j w$ were all distinct, then the dimension of R_{j+1,q_j} would be j. Then, we suppose that s_{j+1} occurs between these two instances of s_j and look at which entries of the dimension vector decrease when w is left-multiplied by s_j. According to Lemma 4.1.2, the entries in columns q_j and $q_j + 1$ (and possibly more) would decrease by one, meaning that the subspace R_{j+1,q_j}, for $R \in \mathrm{Gr}_{\mathbf{r}^{s_j w}}(M)$, cannot satisfy the conditions in (4.1.6), which contradicts the assumption. □

Remark 4.1.13. Lemma 4.1.12 provides a characterisation of when braid moves are possible in a decomposition of $w \in S_{n+1}$ in terms of the dimensions of the subspaces $R_{p,q}$ that satisfy the conditions in (4.1.6), for $R \in \mathrm{Gr}_{\mathbf{r}^w}(M)$. The second statement in Lemma 4.1.12 implies that if a transposition s_i appears k times in all reduced decompositions of w (i.e. these k instances of s_i are not part of any braid move) then there are (at least) k subspaces $R_{p,q}$ of dimension i that satisfy the conditions in (4.1.6). On the other hand, if we apply s_j after w and obtain a reduced decomposition of $s_j w$ that is not geometrically compatible, we know that it is possible to perform a braid move on $s_j s_{j-1} s_j$. This follows from the fact that we can move s_j to the right via commutation until we find an instance of s_{j-1}, and similarly move the second instance of s_j to the left until s_{j-1} (s_{j+1} cannot occur in between by Lemma 4.1.12).

For instance, we saw in Example 4.1.8 a reduced decomposition for $w = [43251]$ that is not geometrically compatible: $w = s_3 s_1 s_2 s_1 s_3 s_2 s_4$. We obtain $w = s_1 s_3 s_2 s_3 s_1 s_2 s_4$ by commutation on the two occurrences of s_3, then perform a braid move as described above and get $w = s_1 s_2 s_3 s_2 s_1 s_2 s_4$. Finally, we perform a braid move on $s_2s_1s_2$ and obtain the geometrically compatible decomposition of w shown in Example 4.1.8: $w = s_1s_2s_3s_1s_2s_1s_4$.

Theorem 4.1.14. *All permutations admit a geometrically compatible decomposition.*

Proof. Let us denote by t the total number of repetitions in a given reduced decomposition of a permutation (i.e. how many times any simple transposition is repeated) and by m the difference between the length of w and t. We prove the statement by double induction on m and t.

The base case of the induction ($m = 1$ and $t = 0$) and the induction step on m ($m \geq 1$ and $t = 0$) follow directly from Remark 4.1.11: a reduced decomposition of w without repetitions consists of distinct simple transpositions, and is therefore geometrically compatible. For the induction step on t we show that, if a permutation with $t \geq 0$ repetitions admits a geometrically compatible decomposition, then a permutation with $t+1$ repetitions admits a geometrically compatible decomposition (for any $m \geq 1$). Let $w = s_{i_N} \dots s_{i_1}$ with t repetitions be a geometrically compatible decomposition of w. We denote by d_i the number of subspaces $R_{p,q}$ of dimension i, for $R \in \mathrm{Gr}_{\mathbf{r}^w}(M)$, that satisfy the conditions in (4.1.6). Since the fixed decomposition of w is geometrically compatible, we have $\#s_i = d_i$ for all i. Let then $w' := s_j w = s_j s_{i_N} \dots s_{i_1}$ such that w' has $t+1$ repetitions, which means $j = i_k$ for some k, and such that $\ell(w') = \ell(w) + 1$. If the corresponding new free subspace appearing in R has dimension j, then this reduced decomposition of w' is already geometrically compatible. If not, then by Lemma 4.1.12 the dimension of the new free subspace must be $\hat{j} < j$, and therefore $d_{\hat{j}}$ has increased by one. As described in Remark4.1.13, we move s_j via commutation and perform a braid move: $w' = s_j s_{i_N} \dots s_{i_1} = s_{i_N} \dots s_j s_{j-1} s_j \dots s_{i_1} = s_{i_N} \dots s_{j-1} s_j s_{j-1} \dots s_{i_1}$, which decreases by one $\#s_j$ (the number of occurrences of s_j) and increases by one $\#s_{j-1}$. Now, if $\hat{j} = j-1$ we have again $\#s_i = d_i$ for each i, meaning that this reduced decomposition of w' is geometrically compatible. Otherwise, we denote by $\hat{w}$ the subword of w starting from the second instance of s_{j-1}: $\hat{w} = s_{j-1} \dots s_{i_1}$ and observe that $\hat{w}$ has t repetitions. Therefore, by the induction hypothesis, $\hat{w}$ admits a geometrically compatible decomposition. We know that the current decomposition of $\hat{w}$ is not geometrically compatible, because the number of $s_{\hat{j}}$ appearing in $\hat{w}$ is $d_{\hat{j}} - 1$. The geometrically compatible decomposition of $\hat{w}$ must then be obtained by performing a sequence of braid moves until the braid $s_{\hat{j}+1} s_{\hat{j}} s_{\hat{j}+1} = s_{\hat{j}} s_{\hat{j}+1} s_{\hat{j}}$. Each braid move decreases by one the number of s_{l+1} and increases by one the number of s_l, for $j-2 \leq l \leq \hat{j}$. Since the number of all other transpositions appearing in $\hat{w}$ (and in w') is not changed during this process, in the end we get $\#s_i = d_i$ for all i, which means that we obtained a geometrically compatible decomposition of w'. ☐

Theorem 4.1.15. *Given a permutation $w \in S_{n+1}$ and a geometrically compatible decomposition $w = s_{i_N} \cdots s_1$, the Bott-Samelson resolution $\mathrm{BS}(s_{i_N} \cdots s_{i_1})$ of the Schubert variety X_w is isomorphic to the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$.*

Proof. Given a geometrically compatible decomposition $w = s_{i_N} \cdots s_1$, we define a map φ_w according to the correspondence between the ordered set of indices of the transpositions appearing in w and the vector space $R_{p,q}$ for any $R \in \mathrm{Gr}_{\mathbf{r}^w}(M)$:

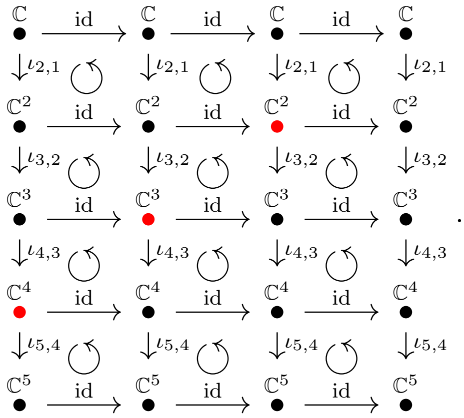
$$\begin{aligned} \varphi_w : \{i_N, \dots, i_1\} &\to \{n+1\} \times \{n\} \\ i_k &\mapsto (p(k), q(k)) \coloneqq (i_k + 1 + n_k, i_k + m_k) \end{aligned} \tag{4.1.16}$$

with $n_k \coloneqq \#\{j : j < k, i_j = i_k\}$ and $m_k \coloneqq \#\{j : j > k, q_{i_j} \leq q(k) < q_{i_j+1}\}$, where q_{i_j} and q_{i_j+1} are defined as in Lemma 4.1.2.

We prove the statement by induction on the length of $w \in S_{n+1}$. For $w = \mathrm{id}$, the corresponding Bott-Samelson resolution and quiver Grassmannian coincide since they consist of a single point. We then denote $w' = s_{i_{N-1}} \cdots s_{i_1}$ and assume $\mathrm{BS}(s_{i_{N-1}} \cdots s_{i_1}) \cong \mathrm{Gr}_{\mathbf{r}^{w'}}(M)$, where the isomorphism is given by $\varphi_{w'}$. This means that the explicit correspondence between an element $V^\bullet_\bullet \in \mathrm{BS}(s_{i_{N-1}} \cdots s_{i_1})$ and a subrepresentation $R' \in \mathrm{Gr}_{\mathbf{r}^{w'}}(M)$ is $V_a^b = R'_{p(b),q(b)}$, therefore they are defined by the same inclusion conditions. We now consider $w = s_{i_N} w'$ such that $\ell(w) = \ell(w') + 1$. The image of i_N via φ_w is $(p(N), q(N)) = (i_N + 1 + n_N, i_N + m_N)$: we need to show that the subspace $R_{p(N),q(N)}$ is isomorphic to the subspace $V_{i_N}^N$, whose defining conditions are $V_a^b \subseteq V_{i_N}^N \subseteq V_d^c$ for $b, c < N$ and $a < i_N < d$. By the induction hypothesis, the subspaces $R_{p(b),q(b)}$ and $R_{p(c),q(c)}$ realise respectively V_a^b and V_d^c for all such a, b, c, d. We observe that the dimension of $R_{p(N),q(N)}$ is i_N due to the choice of a geometrically compatible decomposition of w. The statement then follows from the fact that R is a subrepresentation of M, which implies $R_{p,q} \subseteq R_{p(N),q(N)} \subseteq R_{p',q'}$ for all $p \leq p(N), q \leq q(N)$, $p' \geq p(N), q' \geq q(N)$ and so in particular for $p = p(b), q = q(b)$, $p' = p(c), q' = q(c)$. ☐

Corollary 4.1.17. *Since the Bott-Samelson resolutions corresponding to different reduced decompositions of the same permutation are birational, they are all birational to* $\mathrm{Gr}_{\mathbf{r}^w}(M)$.

Example 4.1.18. Given a permutation w', the map φ_w defined in Equation (4.1.16) describes explicitly which subspace $R_{p,q}$, for $R \in \mathrm{Gr}_{\mathbf{r}^w}(M)$, becomes a nontrivial subspace of $\mathbb{C}^i$ when s_i is applied to w', with $\ell(w) = \ell(w') + 1$. Consider, for instance, the geometrically compatible decomposition $w = s_1 s_2 s_3 s_1 s_2 s_1 s_4$ of Example 4.1.8, where s_1 appears three times, as s_{i_2}, s_{i_4} and s_{i_7}. The images of i_2, i_4 and i_7 via the map φ_w defined in (4.1.16) are $(i_2+1+n_2, i_2+m_2) = (2,3)$, $(i_4+1+n_4, i_4+m_4) = (3,2)$ and $(i_7+1+n_7, i_7+m_7) = (4,1)$. The one-dimensional subspaces $R_{2,3}$, $R_{3,2}$ and $R_{4,1}$, which correspond respectively to the subspaces V_1^2, V_1^4 and V_1^7 considered in 3.2.10, can be visualised at the red vertices of (Γ, I):



4.2 Realisation of smooth Schubert varieties

In Section 4.1, we recovered certain Bott-Samelson resolutions for Schubert varieties by defining the dimension vector $\mathbf{r}^w$ for the quiver (Γ, I) as

$$r^w_{i,j} = \#\{\, k \leq j : w(k) \leq i \,\}$$

for all i, j. In this section we give a construction for a different dimension vector for the quiver (Γ, I), denoted by $\mathbf{e}^w$, and show how the corresponding quiver Grassmannian realises the Schubert variety X_w if it is smooth, i.e. if w is pattern-avoiding. We recall from Section 3.2 that a permutation $w \in S_{n+1}$ corresponds to a smooth Schubert variety if and only if it avoids the patterns [4231] and [3412], and that this is equivalent to X_w being defined by non-crossing inclusions (see Definition 3.2.7).

Consider again the quiver (Γ, I) and its representation M constructed in Section 2.1, and fix a permutation w in S_{n+1} that avoids the patterns [4231] and [3412]. For $i = 1, \ldots, n+1$ and $j = 1, \ldots, n$, we now define the dimension vector $\mathbf{e}^w = (e^w_{i,j})$ for the quiver (Γ, I) as:

$$\begin{cases} e^w_{i,j} := r^w_{i,j} & \text{if } r^w_{i,j} = \min\{i,j\} \\ & \text{or } r^w_{i,j} = 0 \\ e^w_{i,j} := \max\{e^w_{i-1,j}, e^w_{i,j-1}\} & \text{if } 0 < r^w_{i,j} < \min\{i,j\} \end{cases} . \tag{4.2.1}$$

Notice that the value of $r^w_{1,1}$ is either 0 or 1 (according to w) and falls therefore under the first case of Definition (4.2.1), meaning that $e^w_{1,1}$ is well-defined.

Example 4.2.2. We compute the conditions defining the flags $V_\bullet$ in X_w for $w = [65124837] \in S_8$ according to Definition 3.2.5, denoting $\dim(F_p \cap V_q)$ by $d_{p,q}$:

$$\begin{array}{cccccccc}
d_{1,1} \geq 0 & d_{1,2} \geq 0 & d_{1,3} \geq 1 & d_{1,4} \geq 1 & d_{1,5} \geq 1 & d_{1,6} \geq 1 & d_{1,7} \geq 1 & d_{1,8} \geq 1 \\
d_{2,1} \geq 0 & d_{2,2} \geq 0 & d_{2,3} \geq 1 & d_{2,4} \geq 2 & d_{2,5} \geq 2 & d_{2,6} \geq 2 & d_{2,7} \geq 2 & d_{2,8} \geq 2 \\
d_{3,1} \geq 0 & d_{3,2} \geq 0 & d_{3,3} \geq 1 & d_{3,4} \geq 2 & d_{3,5} \geq 2 & d_{3,6} \geq 2 & d_{3,7} \geq 3 & d_{3,8} \geq 3 \\
d_{4,1} \geq 0 & d_{4,2} \geq 0 & d_{4,3} \geq 1 & d_{4,4} \geq 2 & d_{4,5} \geq 3 & d_{4,6} \geq 3 & d_{4,7} \geq 4 & d_{4,8} \geq 4 \\
d_{5,1} \geq 0 & d_{5,2} \geq 1 & d_{5,3} \geq 2 & d_{5,4} \geq 3 & d_{5,5} \geq 4 & d_{5,6} \geq 4 & d_{5,7} \geq 5 & d_{5,8} \geq 5 \\
d_{6,1} \geq 1 & d_{6,2} \geq 2 & d_{6,3} \geq 3 & d_{6,4} \geq 4 & d_{6,5} \geq 5 & d_{6,6} \geq 5 & d_{6,7} \geq 6 & d_{6,8} \geq 6 \\
d_{7,1} \geq 1 & d_{7,2} \geq 2 & d_{7,3} \geq 3 & d_{7,4} \geq 4 & d_{7,5} \geq 5 & d_{7,6} \geq 5 & d_{7,7} \geq 6 & d_{7,8} \geq 7 \\
d_{8,1} \geq 1 & d_{8,2} \geq 2 & d_{8,3} \geq 3 & d_{8,4} \geq 4 & d_{8,5} \geq 5 & d_{8,6} \geq 6 & d_{8,7} \geq 7 & d_{8,8} \geq 8
\end{array}$$

Since w avoids the patterns [4231] and [3412], X_w is smooth and defined by non-crossing inclusions. These inclusions, which follow from the inequalities above, are:

$$\begin{aligned}
&V_1 \subset F_6, V_2 \subset F_6, F_1 \subset V_3 \subset F_6, F_2 \subset V_4 \subset F_6, \\
&F_2 \subset V_5 \subset F_6, F_2 \subset V_6, F_6 \subset V_7.
\end{aligned} \tag{4.2.3}$$

The corresponding dimension vector $\mathbf{e}^w$ obtained from (4.2.1) is

$$\mathbf{e}^w = \begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 2 & 2 & 2 & 3 \\
0 & 0 & 1 & 2 & 2 & 2 & 4 \\
0 & 0 & 1 & 2 & 2 & 2 & 5 \\
1 & 2 & 3 & 4 & 5 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 5 & 6 \\
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{pmatrix}.$$

By reading each entry $e^w_{i,j}$ as the dimension of the intersection $F_p \cap V_q$ and comparing $\mathbf{e}^w$ with the defining conditions in (4.2.3), we see how $\mathbf{e}^w$ encodes the same information on $V_\bullet$.

Theorem 4.2.4. *If $w \in S_{n+1}$ avoids the patterns [4231] and [3412], the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M)$ is isomorphic to the Schubert variety X_w. The isomorphism is given by*

$$\begin{aligned}
\psi : \mathrm{Gr}_{\mathbf{e}^w}(M) &\to X_w \\
N &\mapsto N_\bullet
\end{aligned} \tag{4.2.5}$$

where $N_\bullet = N_{n+1,1} \subseteq N_{n+1,2} \subseteq \cdots \subseteq N_{n+1,n}$.

Proof. By the definition of M and $\mathbf{e}^w$, we have

$$N_{n+1,j} \subseteq N_{n+1,j+1},\ \dim(N_{n+1,j}) = j$$

for all j, implying $N_\bullet \in Fl_{n+1}$. Since w avoids the patterns [4231] and [3412], all flags $V_\bullet = V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n$ in X_w are defined by conditions of the following form: for each $q \in \{1, \dots, n\}$, V_q is defined by $F_{p'_q} \subseteq V_q \subseteq F_{p_q}$ for some p_q, p'_q. These conditions are equivalent, respectively, to $\dim(F_{p'_q} \cap V_q) = \min(p'_q, q) = p'_q$ and $\dim(F_{p_q} \cap V_q) = \min(p_q, q) = q$. The definition of the dimension vector $\mathbf{e}^w$

(in the first line of (4.2.1)) imposes on $N_\bullet$ exactly these conditions, meaning that $F_{p'_q} \subseteq N_{n+1,q} \subseteq F_{p_q}$ for all q and the corresponding p'_q, p_q. The statement follows from the fact that, whenever the condition $\dim(F_i \cap V_j) \geq \#\{ k \leq j : w(k) \leq i \}$ is not defining for $V_\bullet$ (i.e. it is redundant), the corresponding subspace $N_{i,j}$ in $N_\bullet$ is set to either $N_{i-1,j}$ or $N_{i,j-1}$ (second line of (4.2.1)). □

Remark 4.2.6. The flag variety Fl_{n+1} can be defined equivalently as the quotient G/B, where $G = \mathrm{GL}_{n+1}$ and $B \subset G$ is the Borel subgroup of upper-triangular matrices (a construction explained, for instance, in [Bri05][Section 1.2]). Let T be the torus subgroup of B consisting of diagonal matrices, then the left multiplication by T on G induces a T-action on G/B. In this setting, the Schubert varieties in G/B are realised as the Zariski closures of the orbits in G/B under the action of B, and they are invariant under the T-action. From this fact and from Theorem 4.2.4, we get an action of T on the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M)$ induced by the action of T on M, which is in turn induced by the left multiplication of T on the elements of the chosen basis $\mathcal{B} = \{ b_1, b_2, \ldots, b_{n+1} \}$ of $\mathbb{C}^{n+1}$ (it is also straightforward to check that, if N is an element of $\mathrm{Gr}_{\mathbf{e}^w}(M)$, then $T \cdot N$ is still in $\mathrm{Gr}_{\mathbf{e}^w}(M)$). Furthermore, following from its definition in (4.2.5), the isomorphism $\psi : \mathrm{Gr}_{\mathbf{e}^w}(M) \to X_w$ is T-equivariant, i.e. $\varphi(t \cdot N) = t \cdot \varphi(N)$ for all $t \in T$ and $N \in \mathrm{Gr}_{\mathbf{e}^w}(M)$.

Chapter 5

Linear degenerations

This chapter is dedicated to a type of construction known as "linear degeneration". First, we discuss linear degenerations of flag varieties and recall a few results about them. Then, we introduce our definition of linear degenerations of type A Schubert varieties and summarise the research carried out so far. In general, to degenerate means to consider a family of varieties over $\mathbb{A}^1$, such that all fibres over $\mathbb{A}^1 \setminus \{0\}$ are isomorphic - the general fibres - and their limit is the special fibre over 0. The term "linear", employed for instance in the study of linear degenerations of flag varieties, refers to the linear conditions that determine the variety: we vary the defining linear maps and describe how the corresponding fibres behave.

5.1 Linear degenerations of flag varieties

In [CIFR12, Proposition 2.7], the authors realise the linear degenerate flag variety as the quiver Grassmannian associated to representations of the equioriented quiver of type $\mathbb{A}_n$. This family of varieties has then been extensively studied, for instance in [CIFF$^+$17, CIFF$^+$20]. Our main reference for this section is [CIFF$^+$17], and in particular we will focus on the notion of flatness of a certain morphism of varieties. Recall that, in Example 3.1.8, we described how to realise the flag variety Fl_{n+1} as a quiver Grassmannian. This is a special case arising from the following construction.

Let V be a vector space of dimension $n+1$, for $n \geq 1$, and fix a basis $\mathcal{B} = \{ b_1, b_2, \dots, b_{n+1} \}$. We denote by $f = (f_1, \dots, f_{n-1})$ sequences of linear maps of the form

$$V \xrightarrow{f_1} V \xrightarrow{f_2} \dots \xrightarrow{f_{n-2}} V \xrightarrow{f_{n-1}} V \ ,$$

which can be seen as closed points of the variety $R = \mathrm{Hom}(V, V)^{n-1}$. As defined in (1.3.2), the group $G = \mathrm{GL}(V)^n$ acts on R via

$$g \cdot f \coloneqq (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \dots, g_n f_{n-1} g_{n-1}^{-1}).$$

We consider then tuples of subspaces $U = (U_1, \dots, U_n)$ in V such that $\dim(U_i) = i$ for $i = 1, \dots, n$. These can be seen as closed points of the product of Grassmannians

$$Z = \mathrm{Gr}(1, n+1) \times \mathrm{Gr}(2, n+1) \times \cdots \times \mathrm{Gr}(n, n+1),$$

on which the group G acts by translation:

$$g \cdot U := (g_1 U_1, g_2 U_2, \dots, g_n U_n).$$

Definition 5.1.1. Two tuples $f \in R$ and $U \in Z$ are called **compatible** if $f_i(U_i) \subseteq U_{i+1}$, for $i = 1, \dots, n$.

Definition 5.1.2. The **universal linear degeneration** of the flag variety Fl_{n+1} is the variety defined as

$$Y = \{(f, U) : f_i(U_i) \subseteq U_{i+1} \text{ for all } i = 1, \dots, n\},$$

i.e. the variety of compatible pairs of sequences of maps and sequences of subspaces.

Now, since G acts on R and Z, it acts componentwise on Y, and some properties of Y can be derived from the two separate actions on R and Z by considering the fibres of the projections $\pi : Y \to R$ and $p : Y \to Z$. The projection $p : Y \to Z$ is G-equivariant, which means that it commutes with the action of G: $g \cdot p((f, U)) = p(g \cdot (f, U))$ for all $g \in G, f \in R$ and $U \in Z$. Moreover, the space Z is a homogeneous space under the G-action, meaning that the action is transitive, hence Y is a homogeneous fibration over Z. If we fix a tuple $U \in Z$, we can identify its fibre via the projection p with

$$\prod_{i_1}^{n-1} (\mathrm{Hom}(U_i, U_{i+1}) \oplus \mathrm{Hom}(V_i, V)),$$

where V_i is the complement of U_i in V. These facts imply that Y is a homogeneous vector bundle over Z, and therefore it is smooth and irreducible. On the other hand, if we fix a tuple of linear maps $f \in R$ and consider its fibre via the projection π, we obtain the space consisting of all tuples $U \in Z$ that are compatible with f. In other words, each fibre $\pi^{-1}(f)$ can be viewed as a linearly degenerate version of the complete flag variety (which is the fibre over $f = (\mathrm{id}, \mathrm{id}, \dots, \mathrm{id})$).

Definition 5.1.3. [CIFF$^+$17, Definition 1] For a fixed $f \in R$, we call $Fl^f_{n+1} := \pi^{-1}(f)$ the **f-linear degenerate flag variety**, and the map $\pi : Y \to R$ is the **universal linear degeneration** of Fl_{n+1}.

In [CIFF$^+$17], among other results, the authors use rank tuples to characterise the loci in R over which π is flat and where it is flat with irreducible fibres. We recall that a morphism of varieties is called flat if the induced map on every stalk is a flat map of rings, and that, if a morphism is flat, then its fibres are equidimensional. For the morphism $\pi : Y \to R$, however, the authors exploit the following characterisation of flatness:

Proposition 5.1.4. *[Mat89, Theorem 23.1] Let $f : X \to Y$ be a morphism of varieties, where X is Cohen-Macaulay and Y is regular. Then, f is flat if and only if its fibres are equidimensional.*

We conclude this section by recalling the parametrisation of the orbits of G in R and reporting the result obtained in [CIFF$^+$17] about the flat locus of the morphism $\pi : Y \to R$.

Considering a tuple $f \in R$ is equivalent to choosing a $\mathbf{d}$-dimensional representation of the $\mathbb{A}_n$ equioriented quiver, for $\mathbf{d} = (n+1, \dots, n+1)$:

$$\overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{f_1} \overset{\mathbb{C}^{n+1}}{\bullet} \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} \overset{\mathbb{C}^{n+1}}{\bullet},$$

and the f-linear degenerate flag variety is the quiver Grassmannian associated to this representation. Then, the parametrisation of the orbits of G in R can be realised as a special case of the parametrisation by rank tuples given in [ADF85, Proposition 2.7] (see Theorem 2.3.1). In this case, the rank tuples corresponding to the orbits are of the form $\mathbf{r}^f = (r^f_{i,j})_{1 \leq i \leq j \leq n-1}$, where $r_{i,j} = \mathrm{rk}(f_j \circ \cdots \circ f_i)$: for f, g in R, the orbits $\mathcal{O}_f$ and $\mathcal{O}_g$ coincide if and only if $\mathbf{r}^f = \mathbf{r}^g$. Additionally, as proven in [ADF85, Theorem 5.2], the inclusion relations $\mathcal{O}_g \subseteq \overline{\mathcal{O}}_f$ (the Zariski closure) can be described using the same parametrisations. Such inclusion relations induce a partial ordering on the set of all G-orbits, and this partial ordering can be read off the rank tuples: the relation $\mathcal{O}_g \subseteq \overline{\mathcal{O}}_f$ holds if and only if $\mathbf{r}^g \leq \mathbf{r}^f$. In this case, we say that $\mathcal{O}_f$ **degenerates to** $\mathcal{O}_g$.

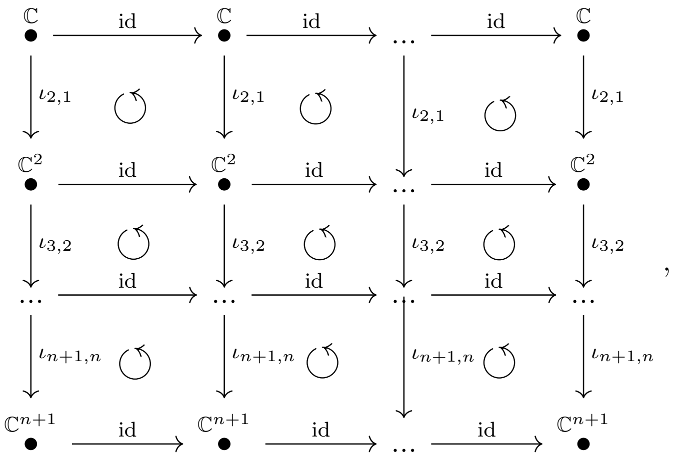
Example 5.1.5. The orbit of $f = (\mathrm{id}, \mathrm{id}, \dots, \mathrm{id})$ is parametrised by $r_{i,j} = n+1$ for all i, j, and therefore it degenerates to all other orbits. To isomorphic quiver representations correspond isomorphic quiver Grassmannians, which means that the g-linear degenerate flag varieties with $g \in \mathcal{O}_f$ are all isomorphic to the complete flag variety Fl_{n+1}. Similarly, the orbit of $f = (0, 0, \dots, 0)$ is parametrised by $r_{i,j} = 0$ for all i, j, meaning that all other orbits degenerate to this one.

Theorem 5.1.6. *[CIFF$^+$17, Theorem 3] The flat locus of π in R is the union of all orbits degenerating to the orbit of $\mathbf{r}^2$, where $r^2_{i,j} = n - j + i$ for all $i < j$.*

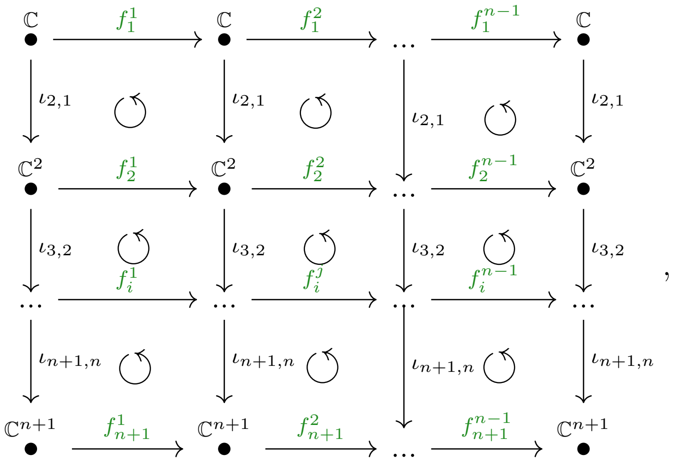
5.2 Linear degenerations of Schubert varieties

In this section, we define linear degenerations of Schubert varieties exploiting their realisation - or their desingularisation - as quiver Grassmannians described in Chapter 4.

First, we recall some notation and the quiver we discussed in Chapter 2. We defined the quiver with relations (Γ, I) and its representation M:



then considered in Section 2.2 the subvariety $R^{\iota}_{\mathbf{d}}$ inside the variety $R_{\mathbf{d}}$ of all representations of (Γ, I) of dimension vector $\mathbf{d}$. As described in Remark 2.2.2, the elements of this subvariety are tuples of linear maps $f = (f^1_{n+1}, \dots, f^{n-1}_{n+1}) \in \prod_{j=1}^{n-1} U_{n+1}$, where U is the subgroup of Mat_{n+1} of upper-triangular matrices with respect to the chosen basis $\mathcal{B} = \{ b_1, b_2, \dots, b_{n+1} \}$ of $\mathbb{C}^{n+1}$. The representation corresponding to f can then be visualised as follows:



where each map f^j_i is represented by the appropriate submatrix of f^j_{n+1} (i.e., it is the restriction of f^j_{n+1} to $\mathbb{C}^i$). Notice that in Chapter 2 we used different notations for an element f in $R^{\iota}_{\mathbf{d}}$ and for the corresponding (Γ, I)-representation M^f. In this chapter, we identify both objects with the sequence of linear maps and write "the representation f". In fact, once the dimension vector of a quiver representation is fixed, the representation is uniquely determined by the choice of linear maps.

We recall that in Chapter 4 we gave two different constructions for two dimension vectors for the quiver (Γ, I). The first one - denoted by $\mathbf{r}^w$ in Section 4.1 - allowed us to recover the Bott-Samelson resolution of the Schubert variety X_w via the quiver Grassmannian $\mathrm{Gr}_{\mathbf{r}^w}(M)$, while the second definition, denoted by $\mathbf{e}^w$ in Section 4.2 and employed only when X_w is a smooth variety, was used to find an explicit isomorphism between the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M)$ and the considered Schubert variety. The goal of this section is to introduce linear degenerations of Schubert varieties, using techniques analogous to those applied in Section 5.1 for the complete flag variety Fl_{n+1}. For this purpose, we will define a universal linear degeneration of (the considered Schubert variety) X_w, whose specific construction will depend on the fixed permutation w. For permutations w such that X_w is a singular variety (the combinatorial criterion for singularity is proven in [LS90], and we recall it in Section 3.2), we will employ the corresponding dimension vector $\mathbf{r}^w$. For permutations w that yield a smooth Schubert variety X_w, instead, the dimension vector to be considered to construct linear degenerations of X_w is $\mathbf{e}^w$. However, in order to simplify notation throughout the construction, we will simply denote by $\mathbf{e}^w$ the dimension vector that corresponds to the fixed permutation w, without subdividing the notation into cases.

Let us now fix a permutation w in S_{n+1} and denote by Z the product of Grassmannians

$$Z = \prod_{i,j} \mathrm{Gr}(e^w_{i,j}, \mathbb{C}^i),$$

for $i = 1, \dots, n+1$ and $j = 1, \dots, n$, where $\mathbf{e}^w$ is the dimension vector constructed from w, and by $U = (U_{ij})$ a closed point in Z. We consider the action of the group $G^\iota_{\mathbf{d}} := \prod_{i=1}^n B_{n+1}$ on $R^\iota_{\mathbf{d}}$ given in Definition 2.2.13:

$$h \cdot f = (h_2 f^1_{n+1} h_1^{-1}, h_3 f^2_{n+1} h_2^{-1}, \dots, h_n f^{n-1}_{n+1} h^{-1}_{n-1})$$

for some $h \in G^\iota_{\mathbf{d}}$. This action is then extended to the other linear maps f^j_i, for $i < n+1$, as explained in Remark 2.2.15: each f^j_i is acted upon by the restrictions of the maps $(h_1, \dots, h_n)$ to $\mathbb{C}^i$. The group $G^\iota_{\mathbf{d}}$ acts on Z by translation:

$$h \cdot U := (h_1 U_{n+11}, h_2 U_{n+12}, \dots, h_n U_{n+1n})$$

and this action is extended to the other subspaces U_{ij}, for $i < n+1$, by letting the restrictions of the maps $(h_1, \dots, h_n)$ to $\mathbb{C}^i$ act on U_{ij} by translation.

Definition 5.2.1. We call a pair (f, U) **compatible** if $f_\alpha(U_{s(\alpha)}) \subseteq U_{t(\alpha)}$ for all arrows α in $(\Gamma, I)_1$.

We would like to remark that, even though at first glance it might seem unnecessary to describe this trivial extension of the $G^\iota_{\mathbf{d}}$-action, considering the whole quiver (Γ, I) instead of only its last row is fundamental from a geometrical point of view: whether U is compatible with some fixed f depends on all subspaces $U_{i,j}$, which are realised in (potentially) any row of (Γ, I), according to the entries of $\mathbf{e}^w$.

Definition 5.2.2. The **universal linear degeneration** of the Schubert variety X_w is the variety defined as

$$Y = \{(f, U) : f_\alpha(U_{s(\alpha)}) \subseteq U_{t(\alpha)} \text{ for all } \alpha \in (\Gamma, I)_1\}.$$

Analogously to Section 5.1, we can then consider the two projections $\pi : Y \to R^\iota_{\mathbf{d}}$ and $p : Y \to Z$. The action of $G^\iota_{\mathbf{d}}$ on $R^\iota_{\mathbf{d}}$ and Z induces the action of $G^\iota_{\mathbf{d}}$ on Y, and the projection p is $G^\iota_{\mathbf{d}}$-equivariant, but in this case the action of $G^\iota_{\mathbf{d}}$ on Z is not transitive. This means that we cannot employ the same tools as for the universal linear degeneration of the flag variety in order to deduce smoothness or irreducibility of Y.

Now, if we fix a tuple of maps f in $R^\iota_{\mathbf{d}}$ and consider its fibre via the projection π, we obtain the space consisting of all $U \in Z$ that are compatible with f. This means that each fibre $\pi^{-1}(f)$ can be viewed as a linearly degenerate version of the fixed Schubert variety X_w, which is itself the fibre over $f = (\mathrm{id}, \mathrm{id}, \ldots, \mathrm{id})$.

Definition 5.2.3. For a fixed $f \in R^\iota_{\mathbf{d}}$, we call $X^f_w := \pi^{-1}(f)$ the **f-linear degenerate Schubert variety**, and the map $\pi : Y \to R^\iota_{\mathbf{d}}$ is the **universal linear degeneration** of X_w.

In other words, the f-linear degenerate Schubert variety is defined as the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M^f)$, where M^f is the (Γ, I)-representation associated to f in the sense of Section 2.2. By comparing the definition of linear degenerations of flag varieties and our definition of linear degenerations of Schubert varieties, the restriction we operated in Section 2.2 on the considered (Γ, I)-representations appears now more reasonable. We kept the vertical maps fixed as standard inclusions and varied the horizontal maps, meaning that we degenerate the conditions defining each flag in Fl_{n+1} but not the combinatorial conditions that determine which flags belong to X_w.

In Section 2.3, we provided two different parametrisations for the $G^\iota_{\mathbf{d}}$-orbits in $R^\iota_{\mathbf{d}}$: the rank vectors, in Definition 2.3.2, and the south-west arrays, in Definition 2.3.12. Furthermore, the south-west parametrisation allows us to describe the inclusion relations of the form $\mathcal{O}^\iota_{M^g} \subseteq \overline{\mathcal{O}}^\iota_{M^f}$: we derive the following corollary from [MS05, Lemma 15.19] and [MS05, Theorem 15.31].

Corollary 5.2.4. *The orbit $\mathcal{O}^\iota_{M^g}$ of M^g under the action of $G^\iota_{\mathbf{d}}$ lies in the closure of the orbit $\mathcal{O}^\iota_{M^f}$ of M^f (with respect to the Zariski topology on Mat_{n+1}) if and only if $\mathbf{s}^f \leq \mathbf{s}^g$, where the "less than or equal to" relation is intended componentwise on the south-west arrays. In this case, we write $\mathcal{O}^\iota_{M^g} \subseteq \overline{\mathcal{O}}^\iota_{M^f}$ and we say that $\mathcal{O}^\iota_{M^f}$ degenerates to $\mathcal{O}^\iota_{M^g}$.*

Example 5.2.5. Analogously to Example 5.1.5, the orbit of id $= (\mathrm{id}, \mathrm{id}, \dots, \mathrm{id})$ is described by the largest possible south-west ranks, and therefore it degenerates to all other orbits. Since isomorphic quiver representations yield isomorphic quiver Grassmannians, the g-linear degenerate Schubert varieties with $g \in \mathcal{O}_f$ are all isomorphic to the Schubert variety X_w. The "most degenerate" orbit is that of $0 = (0, 0, \dots, 0)$: it is parametrised by $\mathbf{s}_{i,j} = 0$ for all i, j, hence all other orbits degenerate to this one. In this case, the linear maps corresponding to the horizontal arrows of (Γ, I) are not imposing any conditions between the subspaces $U_{i,j}$ that belong to different columns of (Γ, I). This means that the 0-linear degenerate Schubert variety is given by the product of n partial flag varieties (one for each column of (Γ, I)), where the dimensions of the free subspaces are given by the corresponding entries of $\mathbf{e}^w$.

Example 5.2.6. Consider again the (Γ, I)-representation of Example 2.3.13:

$$f: \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{[0]} & \mathbb{C} \\ \downarrow \iota_{2,1} & \circlearrowleft & \downarrow \iota_{2,1} \\ \mathbb{C}^2 & \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]} & \mathbb{C}^2 \\ \downarrow \iota_{3,2} & \circlearrowleft & \downarrow \iota_{3,2} \\ \mathbb{C}^3 & \xrightarrow{\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]} & \mathbb{C}^3 \end{array},$$

whose rank vector and south-west array are, respectively:

$$\mathbf{r}^f = (0, 0, 1, 0, 1, 2), \quad \mathbf{s}^f = \left(\left[\begin{smallmatrix} 0 & 1 & 2 \\ * & 1 & 2 \\ * & * & 1 \end{smallmatrix}\right] \right).$$

The (Γ, I)-representation

$$g: \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{[0]} & \mathbb{C} \\ \downarrow \iota_{2,1} & \circlearrowleft & \downarrow \iota_{2,1} \\ \mathbb{C}^2 & \xrightarrow{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]} & \mathbb{C}^2 \\ \downarrow \iota_{3,2} & \circlearrowleft & \downarrow \iota_{3,2} \\ \mathbb{C}^3 & \xrightarrow{\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]} & \mathbb{C}^3 \end{array},$$

is evidently not in the same orbit as f, because it has rank equal to one instead of two (the rank of a matrix is the same as its south-west rank $s_{1,n+1}$). Its rank vector and south-west array are:

$$\mathbf{r}^g = (0, 1, 1, 1, 1, 1), \quad \mathbf{s}^g = \left(\left[\begin{smallmatrix} 0 & 1 & 1 \\ * & 0 & 0 \\ * & * & 0 \end{smallmatrix}\right] \right).$$

The relation between the two south-west arrays is $\mathbf{s}^g \leq \mathbf{s}^f$, which implies $\mathcal{O}^\iota_{M^g} \subseteq \overline{\mathcal{O}}^\iota_{M^f}$. Notice that comparing their rank vectors would not yield the same result: in fact, $\mathbf{r}^g$ and $\mathbf{r}^f$ are not comparable. An example of a (Γ, I)-representation that is neither above nor below f in the poset of the $G^\iota_{\mathbf{d}}$-orbits is

$$h: \quad \begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{[1]} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \iota_{2,1}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{2,1}} \\ \overset{\mathbb{C}^2}{\bullet} & \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]} & \overset{\mathbb{C}^2}{\bullet} \\ \downarrow{\scriptstyle \iota_{3,2}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{3,2}} \\ \overset{\mathbb{C}^3}{\bullet} & \xrightarrow{\left[\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right]} & \overset{\mathbb{C}^3}{\bullet} \end{array} .$$

Its south-west array is $\mathbf{s}^h = \left(\left[\begin{smallmatrix}1&1&1\\ *&0&0\\ *&*&0\end{smallmatrix}\right] \right)$, which is not comparable to $\mathbf{s}^f$. This example highlights a difference between the south-west parametrisation and the parametrisation of the G-orbits, for $G = \mathrm{GL}(V)^n$, and consequently between their posets. In the case of the G-action, it is enough to compare the ranks of (all compositions of) the matrices, while, for comparing $G^\iota_{\mathbf{d}}$-orbits, we need to check which entries of the matrices actually contribute to the rank.

Remark 5.2.7. The ordering induced by the inclusion relations between $G^\iota_{\mathbf{d}}$-orbits is a refinement of the ordering induced by the inclusion relations between G-orbits, in the sense that elements that are equal under the second ordering might not be equal under the first one. This means that the ordering induced by the $G^\iota_{\mathbf{d}}$-action is stronger than that induced by the G-action: if $a \leq b$ holds under the first ordering, then it holds under the second one.

Remark 5.2.8. In some special cases, computing the dimension of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M^f)$ (that is, of the f-linear degenerate Schubert variety) is particularly easy. If $f = 0 = (0, \ldots, 0)$, as explained in Example 5.2.5, then $\mathrm{Gr}_{\mathbf{e}^w}(M^f)$ is a product of partial flag varieties: its dimension is the sum of the dimensions of each partial flag variety. In general, however, the resulting variety is much more complicated, and so is computing its dimension. We only have a bound on the dimensions of its irreducible components: for any $f \in R^\iota_{\mathbf{d}}$ and $\mathbf{d} = \mathbf{dim}\, M^{\mathrm{id}}$, all irreducible components of $\mathrm{Gr}_{\mathbf{e}^w}(M^f)$ have dimension at least $\langle \mathbf{e}^w, \mathbf{d} - \mathbf{e}^w \rangle$, which is the dimension of $\mathrm{Gr}_{\mathbf{e}^w}(M^{\mathrm{id}})$ by Proposition 3.1.9.

Before making a few more examples of linear degenerate Schubert varieties and opening the discussion about the flat locus of the morphism $\pi : Y \to R^\iota_{\mathbf{d}}$, we would like to complete the description of the south-west parametrisation. We know that a tuple $f \in R^\iota_{\mathbf{d}}$ is parametrised by its south-west array, but which tuples of non-negative integers are the south-west array of some tuple $f \in R^\iota_{\mathbf{d}}$?

The simple answer to this question follows directly from the definition of south-west arrays: if we consider a south-west array $\mathbf{s}$ as a matrix in $U \subseteq \mathrm{Mat}_{n+1}$

(an upper-triangular matrix), then this matrix is parametrised by itself. In other words, a matrix A whose entry $a_{i,j}$ is a south-west rank has precisely those $a_{i,j}$ as its south-west ranks. This means that, in order to determine whether a tuple of non-negative integers is a south-west array, we can check if it parametrises itself. Nonetheless, we can write down the conditions that determine if a tuple of non-negative integers ω is a south-west array of some $f \in R^{\iota}_{\mathbf{d}}$, in ambient dimension $n+1$. Let us denote the entries of ω by $w^{i,j}_{a,b}$; imposing the following conditions implies that $w^{i,j}_{a,b}$ is the south-west rank computed at entry (i,j) of the matrix given by $f_b \circ \cdots \circ f_a$, for $1 \le a \le b \le n-1$ and $1 \le i \le j \le n+1$:

$$\begin{cases} w^{i,j}_{a,b} \le \min\{j-i+1, j\} \\ w^{i,j}_{a,b} \le \min\{w^{i,n}_{a,a+t}, w^{1,j}_{a+t,b}\} \\ p_{i,j} \le w^{i,j}_{a,b} \le p_{i,j}+1 \end{cases} \tag{5.2.9}$$

for all $a < t < b$, where $p_{i,j}$ is the number of pivots in the south-west $i \times j$ submatrix and $w^{i,j}_{a,b} = p_{i,j}+1$ is only allowed if there are no pivots in row i or column j. The first inequality is necessary for $w^{i,j}_{a,b}$ to be the rank of the (upper-triangular) $i \times j$ submatrix, while the second inequality represents the fact that the rank of a product $(w^{i,j}_{a,b})$ cannot be bigger than each of the ranks of the two factors; since $w^{i,j}_{a,b}$ is the result of the composition $f_b \circ \cdots \circ f_a$, this has to hold independently of how we apply associativity. The last condition ensures that $w^{i,j}_{a,b}$ counts the number of pivots in the $i \times j$ submatrix and that it does not increase by more than one in each row and column. We omit the analogous discussion for rank vectors, since it is of technical nature and falls outside the main concerns of this thesis.

Now we return to the question of the flat locus of the morphism $\pi : Y \to R^{\iota}_{\mathbf{d}}$. We would like to exploit the characterisation of the flat locus given in Proposition 5.1.4; we know that $R^{\iota}_{\mathbf{d}}$ is regular (it is a smooth subgroup of the group GL^n_{n+1}), but we need to show that Y is a Cohen-Macaulay variety. A variety X is called Cohen-Macaulay if for every point $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring of regular functions $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay. More information on Cohen-Macaulay rings can be found, for instance, in [BH98]. However, in order to prove that Y is Cohen-Macaulay, we can apply the following:

Lemma 5.2.10. *[Sta, Lemma 10.135.3] Let $\mathbb{K}$ be a field. Let S be a finite type $\mathbb{K}$-algebra. If S is locally a complete intersection, then S is a Cohen-Macaulay ring.*

In particular, we would like to make use of an analogous strategy to the one employed in the proof of Theorem 11 in [CIFF+17]: here, the authors show that the f-linear degenerate flag variety (for f such that $\mathbf{r}^f = \mathbf{r}^2$, as in Theorem 5.1.6) is locally a complete intersection. In order to do so, we realise our quiver Grassmannians as geometric quotients of the appropriate varieties; we follow [CIFR12, Section 2.3].

Let $\mathbf{e}^w$ be as above and $f \in R^\iota_{\mathbf{d}}$ such that $\dim(\mathrm{Gr}_{\mathbf{e}^w}(M^f)) = \dim(X_w)$ (as mentioned in Section 5.1, this is a necessary condition for flatness). We define the vector space

$$V := R_{\mathbf{e}^w} \times \prod_{(p,q)\in(\Gamma,I)_0} \mathrm{Hom}(\mathbb{C}^{e^w_{p,q}}, \mathbb{C}^i),$$

where $R_{\mathbf{e}^w}$ is the variety of representations of (Γ, I) with dimension vector $\mathbf{e}^w$. We denote the elements of V by $((N_\alpha), (g_{p,q}))$, where α varies over all arrows in $(\Gamma, I)_1$ and (p, q) corresponds to a vertex of (Γ, I). Then, we consider the affine variety $\mathrm{Hom}(\mathbf{e}^w, M^f)$ in V consisting of tuples $((N_\alpha), (g_{p,q}))$ that satisfy

$$\begin{cases} f_i^j \circ g_{i,j} = g_{i,j+1} \circ N_\alpha & \text{for } \alpha : s(\alpha) = (i,j), t(\alpha) = (i, j+1) \\ \iota_{i+1,i} \circ g_{i,j} = g_{i+1,j} \circ N_\alpha & \text{for } \alpha : s(\alpha) = (i,j), t(\alpha) = (i+1, j) \end{cases}. \tag{5.2.11}$$

In words, we consider all possible tuples $((N_\alpha), (g_{p,q}))$ where N is a (Γ, I)-representation of dimension vector $\mathbf{e}^w$ and $g = (g_{p,q})$, for $(p, q) \in (\Gamma, I)_0$, is a morphism of representations from N to M^f. The relations in (5.2.11) impose all commutativity relations that are necessary for g to be a morphism of representations (see Definition 1.2.3). Figure 5.1 illustrates the diagram in ambient dimension $n + 1 = 3$: the representations N and M^f satisfy the relations given by (Γ, I), while the commutativity of all other squares in the diagram, which involve the morphism g, is imposed by the relations in (5.2.11).

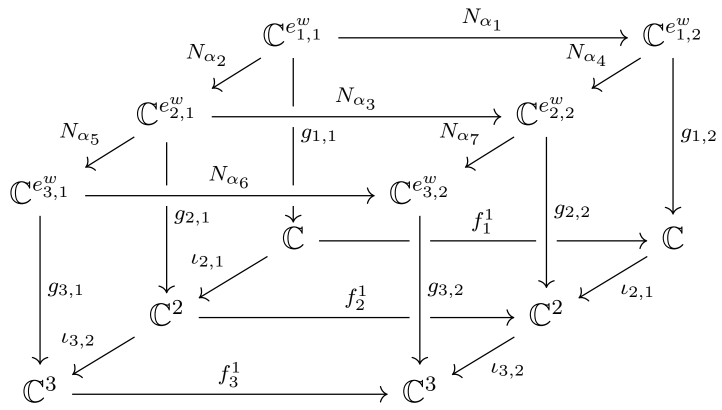


Figure 5.1: The diagram formed by M^f, N and g

Then, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M^f)$ can be realised as the following geometric quotient:

$$\mathrm{Gr}_{\mathbf{e}^w}(M^f) \cong \mathrm{Hom}^0(\mathbf{e}^w, M^f)/G_{\mathbf{e}^w},$$

where $G_{\mathbf{e}^w} = \prod_{(i,j)\in(\Gamma,I)_0} \mathrm{GL}(\mathbb{C}^{e^w_{i,j}})$, i.e. the product of the general linear groups acting naturally on the vector spaces of $N \in R_{\mathbf{e}^w}$, and $\mathrm{Hom}^0(\mathbf{e}^w, M^f)$ is the open subvariety in $\mathrm{Hom}(\mathbf{e}^w, M^f)$ defined by $g_{p,q}$ being an injective map, for all $(p, q) \in (\Gamma, I)_0$. Since $\mathrm{Hom}^0(\mathbf{e}^w, M^f)$ and $\mathrm{Hom}(\mathbf{e}^w, M^f)$ have the same codimension in V (because the first one is an open subvariety of the second one), it is enough to show that $\mathrm{Hom}^0(\mathbf{e}^w, M^f)$ is locally a complete intersection.

In order to do so, we want to make use of a dimension formula arising from the quiver Grassmannian being a geometric quotient, namely that

$$\dim(\mathrm{Gr}_{\mathbf{e}^w}(M^f)) = \dim(\mathrm{Hom}^0(\mathbf{e}^w, M^f)) - \dim(G_{\mathbf{e}^w}). \tag{5.2.12}$$

This would allow us to deduce the codimension of $\mathrm{Hom}^0(\mathbf{e}^w, M^f)$ in V and to compare it with the number of equations defining $\mathrm{Hom}(\mathbf{e}^w, M^f)$ in V: if we obtain the same number, then $\mathrm{Gr}_{\mathbf{e}^w}(M^f)$ is locally a complete intersection. In this case, by Lemma 5.2.10, we would be able to describe completely the flat locus of $\pi : Y \to R^\iota_{\mathbf{d}}$: it would consist precisely of all $f \in R^\iota_{\mathbf{d}}$ such that $\dim(\mathrm{Gr}_{\mathbf{e}^w}(M^f)) = \dim(X_w)$.

Conjecture 5.2.13. A tuple f is in the flat locus of $\pi : Y \to R^\iota_{\mathbf{d}}$ if and only if $\dim(\mathrm{Gr}_{\mathbf{e}^w}(M^f)) = \dim(X_w)$.

We present now an example in ambient dimension $n+1 = 3$ motivating our conjecture.

Let us fix the permutation $w = [231] \in S_3$ and the representation determined by the identity map f:

$$f:\quad \begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{[1]} & \overset{\mathbb{C}}{\bullet} \\ \downarrow{\scriptstyle \iota_{2,1}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{2,1}} \\ \overset{\mathbb{C}^2}{\bullet} & \xrightarrow{\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]} & \overset{\mathbb{C}^2}{\bullet} \\ \downarrow{\scriptstyle \iota_{3,2}} & \circlearrowleft & \downarrow{\scriptstyle \iota_{3,2}} \\ \overset{\mathbb{C}^3}{\bullet} & \xrightarrow{\left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right]} & \overset{\mathbb{C}^3}{\bullet} \end{array}\quad .$$

The dimension vector $\mathbf{e}^w$, constructed according to (4.2.1), is

$$\mathbf{e}^w = \left[\begin{smallmatrix}0&0\\1&1\\1&2\end{smallmatrix}\right],$$

and the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}^w}(M^f))$ is isomorphic to the Schubert variety $X_{[231]}$: it is a smooth projective variety of dimension $\ell(w) = 2$. We choose this representation specifically because we know that it is in the flat locus of $\pi : Y \to R^\iota_{\mathbf{d}}$ (its fibre is the nondegenerate Schubert variety).

Remark 5.2.14. If we choose a permutation in S_3 with length strictly smaller than 2, then $\mathrm{Gr}_{\mathbf{e}^w}(M^f)) = \mathrm{Gr}_{\mathbf{e}^w}(M))$ independently of the representation f. This can be seen by computing the corresponding dimension vectors

$$\mathbf{e}^{\mathrm{id}} = \left[\begin{smallmatrix}1&1\\1&2\\1&2\end{smallmatrix}\right],\ \mathbf{e}^{[213]} = \left[\begin{smallmatrix}0&1\\1&2\\1&2\end{smallmatrix}\right],\ \mathbf{e}^{[132]} = \left[\begin{smallmatrix}1&1\\1&1\\1&2\end{smallmatrix}\right]$$

and noticing that all free subspaces are realised in the first column of (Γ, I), meaning that the linear maps between the first and second column do not affect the quiver Grassmannian.

We consider now the geometric quotient

$$\mathrm{Gr}_{\mathbf{e}^w}(M^f) \cong \mathrm{Hom}^0(\mathbf{e}^w, M^f)/G_{\mathbf{e}^w},$$

and want to show that $\mathrm{Hom}^0(\mathbf{e}^w, M^f)$ is locally a complete intersection. Figure 5.2 represents M^f and the diagram it forms together with a representation in $R_{\mathbf{e}^w}$ and a morphism g.

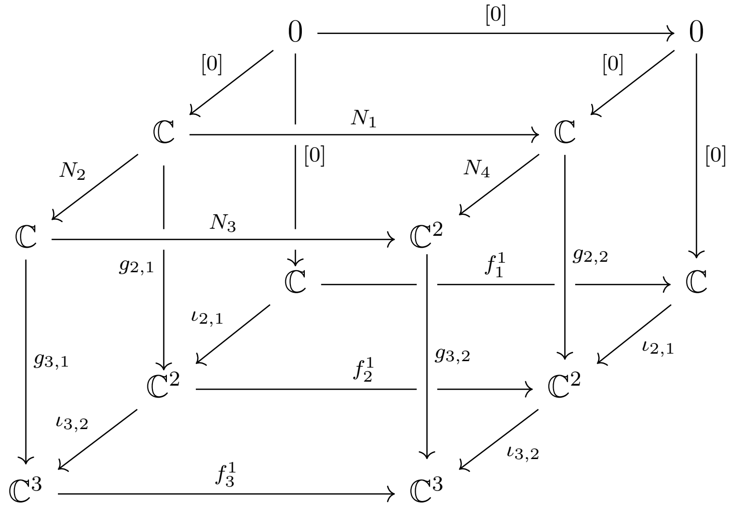


Figure 5.2: The diagram formed by M^f, N and g in the example

In this case, the group $G_{\mathbf{e}^w}$ is

$$G_{\mathbf{e}^w} = \prod_{(i,j)\in(\Gamma,I)_0} \mathrm{GL}(\mathbb{C}^{e^w_{i,j}}) = \mathrm{GL}(\mathbb{C}) \times \mathrm{GL}(\mathbb{C}) \times \mathrm{GL}(\mathbb{C}) \times \mathrm{GL}(\mathbb{C}^2)$$

and has therefore dimension $\dim(G_{\mathbf{e}^w}) = 7$. For a generic dimension vector $\mathbf{e}^w$, this dimension is given by

$$\dim(G_{\mathbf{e}^w}) = \sum_{(i,j)\in(\Gamma,I)_0} (e^w_{i,j})^2.$$

Since $\dim(\mathrm{Gr}_{\mathbf{e}^w}(M^f)) = 2$, the dimension of $\mathrm{Hom}^0(\mathbf{e}^w, M^f)$ is

$$\dim(\mathrm{Hom}^0(\mathbf{e}^w, M^f)) = \dim(\mathrm{Gr}_{\mathbf{e}^w}(M^f)) + \dim(G_{\mathbf{e}^w}) = 7 + 2 = 9. \quad (5.2.15)$$

In order to compute the codimension of $\mathrm{Hom}^0(\mathbf{e}^w, M^f)$ in V, we first need the dimension of V, which is

$$\dim(V) = \dim(R_{\mathbf{e}^w}) + \dim\Big(\prod_{(p,q)\in(\Gamma,I)_0} \mathrm{Hom}(\mathbb{C}^{e^w_{p,q}}, \mathbb{C}^i)\Big).$$

For the second summand, we have

$$\prod_{(p,q)\in(\Gamma,I)_0} \mathrm{Hom}(\mathbb{C}^{e^w_{p,q}}, \mathbb{C}^i) = \mathrm{Hom}(\mathbb{C},\mathbb{C}^2)\times\mathrm{Hom}(\mathbb{C},\mathbb{C}^2)\times\mathrm{Hom}(\mathbb{C},\mathbb{C}^3)\times\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^3)$$

and therefore

$$\dim\Big(\prod_{(p,q)\in(\Gamma,I)_0} \operatorname{Hom}(\mathbb{C}^{e^w_{p,q}}, \mathbb{C}^i)\Big) = 13. \tag{5.2.16}$$

To find the dimension of $R_{\mathbf{e}^w}$, we first observe that it is the subvariety of representations of (Γ, I) with dimension vector $\mathbf{e}^w$ that satisfy the commutativity relations. This means

$$R_{\mathbf{e}^w} \subset \operatorname{Hom}(\mathbb{C},\mathbb{C}) \times \operatorname{Hom}(\mathbb{C},\mathbb{C}) \times \operatorname{Hom}(\mathbb{C},\mathbb{C}^2) \times \operatorname{Hom}(\mathbb{C},\mathbb{C}^2),$$

that is, $R_{\mathbf{e}^w}$ is a subvariety of a six-dimensional variety of representations.

In particular, since $e^w_{1,1} = e^w_{1,2} = 0$, a representation N in $R_{\mathbf{e}^w}$ is determined by four linear maps N_1, N_2, N_3 and N_4 of the following form:

$$\begin{array}{ccc} \overset{\mathbb{C}}{\bullet} & \xrightarrow{N_1} & \overset{\mathbb{C}}{\bullet} \\ \downarrow N_2 & \circlearrowleft & \downarrow N_4 \\ \overset{\mathbb{C}}{\bullet} & \xrightarrow{N_3} & \overset{\mathbb{C}^2}{\bullet} \end{array},$$

and we can write the matrices representing these linear maps as

$$N_1 = [x_1], N_2 = [x_2], N_3 = \left[\begin{smallmatrix} x_3 \\ x_4 \end{smallmatrix}\right], N_4 = \left[\begin{smallmatrix} x_5 \\ x_6 \end{smallmatrix}\right].$$

The commutativity relation implies that $N \in R_{\mathbf{e}^w}$ if and only if the two independent relations $x_3x_2 = x_5x_1$ and $x_4x_2 = x_6x_1$ are satisfied. Thus, the dimension of $R_{\mathbf{e}^w}$ is

$$\dim(R_{\mathbf{e}^w}) = 6 - 2 = 4. \tag{5.2.17}$$

Now, we put together (5.2.16) and (5.2.17) and obtain $\dim(V) = 13+4 = 17$. From this and from (5.2.15), it follows that the codimension of $\operatorname{Hom}^0(\mathbf{e}^w, M^f)$ in V is equal to $\dim(V) - \dim(\operatorname{Hom}^0(\mathbf{e}^w, M^f)) = 17 - 9 = 8$.

The space $\operatorname{Hom}^0(\mathbf{e}^w, M^f)$ is locally a complete intersection if its codimension in V is equal to the number of equations that define $\operatorname{Hom}(\mathbf{e}^w, M^f)$ in V. A point $((N_\alpha), (g_{p,q}))$ is in $\operatorname{Hom}(\mathbf{e}^w, M^f)$ if and only if it satisfies the relations given in (5.2.11), which, in the case of this example, are

$$\begin{cases} f^1_2 g_{2,1} = g_{2,2}N_1 \\ \iota_{3,2}g_{2,1} = g_{3,1}N_2 \\ f^1_3 g_{3,1} = g_{3,2}N_3 \\ \iota_{3,2}g_{2,2} = g_{3,2}N_4 \end{cases}. \tag{5.2.18}$$

We represent the linear maps $g_{p,q}$ with the following matrices:

$$g_{2,1} = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right], g_{3,1} = \begin{bmatrix} y_3 \\ y_4 \\ y_5 \end{bmatrix}, g_{3,2} = \begin{bmatrix} y_6 & y_7 \\ y_8 & y_9 \\ y_{10} & y_{11} \end{bmatrix}, g_{2,2} = \left[\begin{smallmatrix} y_{12} \\ y_{13} \end{smallmatrix}\right]$$

and make the relations in (5.2.18) explicit, obtaining the following 11 relations:

$$\begin{cases} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} & = \begin{bmatrix} y_{12}x_1 \\ y_{13}x_1 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} & = \begin{bmatrix} y_3x_2 \\ y_4x_2 \\ y_5x_2 \end{bmatrix} \\ \begin{bmatrix} y_3 \\ y_4 \\ y_5 \end{bmatrix} & = \begin{bmatrix} y_6x_3+y_7x_4 \\ y_8x_3+y_9x_4 \\ y_{10}x_3+y_{11}x_4 \end{bmatrix} \\ \begin{bmatrix} y_{12} \\ y_{13} \\ 0 \end{bmatrix} & = \begin{bmatrix} y_6x_5+y_7x_6 \\ y_8x_5+y_9x_6 \\ y_{10}x_5+y_{11}x_6 \end{bmatrix} \end{cases}. \tag{5.2.19}$$

By imposing $x_3x_2 = x_5x_1$ and $x_4x_2 = x_6x_1$ (the relations that follow from N being a representation of (Γ, I)) and operating two substitutions in (5.2.19) we notice that 3 out of those 11 relations can be derived from the other 8. In particular, a set of independent, generating relations is given either by

$$\begin{cases} \begin{bmatrix} y_3 \\ y_4 \\ y_5 \end{bmatrix} & = \begin{bmatrix} y_6x_3+y_7x_4 \\ y_8x_3+y_9x_4 \\ y_{10}x_3+y_{11}x_4 \end{bmatrix} \\ \begin{bmatrix} y_{12} \\ y_{13} \\ 0 \end{bmatrix} & = \begin{bmatrix} y_6x_5+y_7x_6 \\ y_8x_5+y_9x_6 \\ y_{10}x_5+y_{11}x_6 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} & = \begin{bmatrix} y_{12}x_1 \\ y_{13}x_1 \end{bmatrix} \end{cases}$$

or by

$$\begin{cases} \begin{bmatrix} y_3 \\ y_4 \\ y_5 \end{bmatrix} & = \begin{bmatrix} y_6x_3+y_7x_4 \\ y_8x_3+y_9x_4 \\ y_{10}x_3+y_{11}x_4 \end{bmatrix} \\ \begin{bmatrix} y_{12} \\ y_{13} \\ 0 \end{bmatrix} & = \begin{bmatrix} y_6x_5+y_7x_6 \\ y_8x_5+y_9x_6 \\ y_{10}x_5+y_{11}x_6 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} & = \begin{bmatrix} y_3x_2 \\ y_4x_2 \end{bmatrix} \end{cases}.$$

We found - as expected - that the codimension of $\mathrm{Hom}^0(\mathbf{e}^w, M^f)$ in V is equal to the number of equations that define $\mathrm{Hom}(\mathbf{e}^w, M^f)$ in V. What motivates Conjecture 5.2.13 is the fact that the strategy and methods employed here do not depend on the specific choice of f. Therefore, we expect them to provide the same result for any f such that $\dim(\mathrm{Gr}_{\mathbf{e}^w}(M^f)) = \dim(X_w)$ and in generic ambient dimension $n+1$.

Perspectives

There are several possible directions to pursue after considering our definition of linear degenerations of Schubert varieties. A few examples concerning additional geometric properties of the varieties X_w^f are:

1. Finding estimates about the dimension of X_w^f;
2. Computing the equations that determine X_w^f;
3. Understanding whether X_w^f admits a cellular decomposition, possibly induced by the action of an appropriate torus on the representation M^f.

Chapter 6

Towards tropicalisations of quiver Grassmannians

This final chapter is dedicated to the exploration of a connection between quiver representation theory and tropical geometry. Some standard references for an introduction to tropical geometry are [SS04, Spe08].

Tropical analogues for certain projective varieties have already been extensively studied. For instance, the (projective) tropicalisation $\overline{\text{trop}}(\text{Gr}(r;n))$ of the Grassmannian (as a tropical subvariety of the tropical projective space $\mathbb{P}\left(\mathbb{T}^{\binom{n}{r}}\right)$, see Section 6.1) parametrises realisable valuated matroids of rank r on n elements, or equivalently realisable tropical linear spaces (i.e. tropicalisations of linear spaces) of dimension r in $\mathbb{P}(\mathbb{T}^n)$. On the other hand, the object parametrising all valuated matroids of rank r on n elements, or equivalently all tropical linear spaces of dimension r inside $\mathbb{P}(\mathbb{T}^n)$, is a tropical prevariety $\overline{\text{Dr}}(r,n)$ called the projective Dressian (see [SS04] for the definition in very affine space and [BEZ21] for the extension to the projective case). The relation between the tropicalisation of the Grassmannian and the Dressian is $\overline{\text{trop}}(\text{Gr}(r;n)) \subseteq \overline{\text{Dr}}(r,n)$, which is a strict inclusion for large enough n.

Another example is the flag Dressian $\text{FlDr}(\mathbf{d};n)$, the tropical analogue for flag varieties. This was defined by Haque [Haq12, Definition 1] and further analysed in [BEZ21], showing that $\overline{\text{FlDr}}(\mathbf{d};n)$ parametrises valuated flag matroids or, equivalently, flags of projective tropical linear spaces (see [BEZ21, Theorem A] and [Haq12, Theorem 1]). Then, in [BS23], the authors studied the case of flag matroids for linear degenerate valuated flag matroids and their associated linear degenerate flags of tropical linear spaces.

In this chapter, we would like to present some concepts and results that were developed by the author of this thesis together with Victoria Schleis as part of the project A11: "Linear degenerate flag varieties and their tropical counterparts" of the SFB-TRR 195 of the German Research Foundation. They are included in the paper [IS23], co-written by the author of this thesis and Victoria Schleis. We define the projective quiver Dressian $\overline{\text{QDr}}(R,\mathbf{d})$, which parametrises the tropical linear spaces satisfying the containment conditions described by a fixed quiver representation, and study its relation to the tropicalisation of the corresponding quiver Grassmannian.

First, we provide some background on tropical geometry and valuated matroids. In the second part of the first section, we summarise the results on images of linear spaces under matrix multiplication, affine morphisms of valuated matroids and weakly monomial matrices obtained during the collaboration mentioned above. These results are necessary for the second section, which focuses on combining quiver Grassmannians and tropical geometry.

6.1 Background and first results

In this section, we recall projective tropical linear spaces and their linear maps. Some standard references for basic matroid theory are [Oxl11, Wel76]. We fix the notation $[n] = \{1, 2, \ldots, n\}$ and $\binom{[n]}{r} = \{S \subset [n] : |S| = r\}$, and write $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. Every valuation is assumed to be non-Archimedean.

6.1.1 Tropical Geometry

We discuss an extension of tropical geometry corresponding to projective algebraic geometry, which is covered, for instance, in [MS15, Section 6] and [Sha13]. A deeper and more detailed background on tropical geometry can be found in [MS15] or [Jos21]. Throughout this chapter, we will follow the min-convention.

We define the tropical numbers as $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ and consider the semifield $(\mathbb{T}, \oplus, \odot)$, called **tropical semifield**, where the operations are $a \oplus b = \min\{a, b\}$ and $a \odot b = a + b$ for every $a, b \in \mathbb{T}$. For multiplication with the tropical multiplicative inverse, we write $a \oplus b = a - b$. The **tropical projective space** is $\mathbb{P}(\mathbb{T}^n) = (\mathbb{T}^n \setminus \{(\infty, \ldots, \infty)\})/\mathbb{R}\mathbf{1} = (\mathbb{T}^n \setminus \{(\infty, \ldots, \infty)\})/\sim$. Here, $\sim$ is the equivalence relation $\mathbf{u} \sim \mathbf{v}$ if $\mathbf{u} = \mathbf{v} + c\mathbf{1}$ for some $c \in \mathbb{R}$. Topologically, the tropical projective space is homeomorphic to the n-simplex equipped with the relative Euclidean topology (see [Sha13, Definition 2.18] and [Mik06]). This topology on the tropical semiring is often employed in linear tropical geometry and used, for instance, in [BEZ21]. However, it differs from similar constructions in max-plus linear algebra (see for instance [AGG09]), whose applications range from graph theory to asymptotic analysis (more details in the survey paper [GP97]). In this related area, the "usual" topology on the max-plus semiring is commonly defined via the exponential metric (see [GK07, Section 2]) and induced by Hilbert's projective distance, used for instance in [AGNS11].

Definition 6.1.1. A **tropical polynomial** is an element of the semiring $\mathbb{T}[x_1, \ldots, x_n]$ in the variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{T}$. The **tropical hypersurface** of a tropical polynomial $F = \bigoplus_{u \in \mathbb{N}^n} c_u \odot x^u \in \mathbb{T}[x_1, \ldots, x_n]$ is defined as

$$V(F) = \left\{ x \in \mathbb{P}(\mathbb{T}^n) : \min_{u \in \mathbb{N}^n} \left\{ c_u + \sum_{i=1}^{n} u_i \cdot x_i \right\} \text{is achieved at least twice} \right\}$$

where, whenever $\min_{u\in\mathbb{N}^n}\{c_u + \sum_{i=1}^n u_i \cdot x_i\} = \infty$, we adopt the convention that the minimum is achieved at least twice, even if the expression is a tropical monomial.

The **tropical variety** of an ideal of tropical polynomials $J \subseteq \mathbb{T}[x_1, \dots, x_n]$ is then defined by

$$V(J) = \bigcap_{F\in J} V(F) \subseteq \mathbb{P}(\mathbb{T}^n).$$

In the following, let $\mathbb{K}$ be a field with valuation val : $\mathbb{K} \to \mathbb{T}$. The tropicalisation of a polynomial $f = \sum_{u\in\mathbb{N}^n} a_u x^u \in \mathbb{K}[x_1, \dots, x_n]$ is the tropical polynomial

$$\operatorname{trop}(f) = \bigoplus_{u\in\mathbb{N}^n} \operatorname{val}(a_u) \odot x^u \in \mathbb{T}[x_1, \dots, x_n].$$

The (projective) tropicalisation trop(I) of an ideal $I \subseteq \mathbb{K}[x_1, \dots, x_n]$ is the ideal of tropical polynomials generated by the tropicalisations of all polynomials in I:

$$\operatorname{trop}(I) = \{\operatorname{trop}(f) : f \in I\} \subseteq \mathbb{T}[x_1, \dots, x_n].$$

Over an algebraically closed field $\mathbb{K}$ with a non-trivial valuation, the tropicalisation $\overline{\operatorname{trop}}(X)$ of a subvariety $X \subseteq \mathbb{P}^n$ is defined by

$$\overline{\operatorname{trop}}(X) = \overline{\{(\operatorname{val}(x_1), \dots, \operatorname{val}(x_n)) \in \mathbb{P}(\mathbb{T}^n) : [x_1 : \cdots : x_n] \in X\}},$$

where the closure is with respect to the Euclidean topology induced on $\mathbb{P}(\mathbb{T}^n)$. Note that the projective tropicalisation is *not necessarily* the closure of the classical tropicalisation.

There are two possible ways of constructing a tropical variety from a homogeneous ideal $I \subseteq \mathbb{K}[x_1, \dots, x_n]$: we can first tropicalise the ideal and then take its tropical variety $V(\operatorname{trop}(I))$, or we can consider the projective variety $V(I)$ and tropicalise it to obtain $\overline{\operatorname{trop}}(V(I))$. The Fundamental Theorem of Tropical Geometry assures us that, over algebraically closed fields with nontrivial valuation, these two operations yield the same result, i.e. that $\overline{\operatorname{trop}}(V(I)) = V(\operatorname{trop}(I))$ (see [MS15, Theorem 6.2.15]). Note that $\overline{\operatorname{trop}}(X)$ can contain points in which some of the coordinates are ∞. For a thorough description of projective tropical varieties, we refer to [BEZ21, Section 2] and [MS15, Section 6.2], and to [Sha13] for the construction of tropical projective spaces.

6.1.2 Valuated matroids and tropical linear spaces

Definition 6.1.2. A **valuated matroid** of rank r on the ground set $[n]$ is a function $\nu : \binom{[n]}{r} \to \mathbb{T}$ such that $\nu(B) \neq \infty$ for some $B \in \binom{[n]}{r}$ and, for all $I, J \in \binom{[n]}{r}$ and $i \in I \setminus J$, there exists $j \in J \setminus I$ satisfying

$$\nu(I) + \nu(J) \geq \nu((I \setminus i) \cup j) + \nu((J \setminus j) \cup i).$$

Given a valuated matroid $\nu : \binom{[n]}{r} \to \mathbb{T}$, the set $\{B \in \binom{[n]}{r} : \nu(B) \neq \infty\}$ forms the bases of a matroid N, which we call the underlying matroid of ν.

We say that two valuated matroids μ and ν on a common ground set $[n]$ are equivalent if there exists $a \in \mathbb{R}$ such that $\mu(B) = \nu(B) + a$ for every $B \in \binom{[n]}{r}$. In other words, every equivalence class of a valuated matroid $\nu : \binom{[n]}{r} \to \mathbb{T}$ can be seen as a point in $\mathbb{P}\left(\mathbb{T}^{\binom{n}{r}}\right)$. From now on, we only consider valuated matroids up to equivalence.

The **realisable** valuated matroids (over $\mathbb{K}$) arise from linear spaces in the following way. Consider a field $\mathbb{K}$ with valuation $\mathrm{val} : \mathbb{K} \to \mathbb{T}$ and an r-dimensional vector subspace L of K^n given as the minimal row span of a matrix A. Denote by $(p_I)_I$ the Plücker coordinates of A, where p_I is the minor of A indexed by $I \in \binom{[n]}{r}$. Then, the function $\mu(A) : \binom{[n]}{r} \to \mathbb{T}$ defined by $I \mapsto \mathrm{val}(p_I)$ is a valuated matroid. We denote its underlying matroid by $M(A)$.

Definition 6.1.3 ([BEZ21, Bry86, Definition 4.2.2, Proposition 7.4.7])**.** Let μ and ν be two valuated matroids on the ground set $[n]$ of rank $r \leq s$ respectively. We say that μ is a **valuated matroid quotient** of ν, denoted $\mu \twoheadleftarrow \nu$, if for every $I \in \binom{[n]}{r}$, $J \in \binom{[n]}{s}$ and $i \in I \setminus J$, there exists $j \in J \setminus I$ such that

$$\mu(I) + \nu(J) \geq \mu(I \cup j \setminus i) + \nu(J \cup i \setminus j).$$

In words, matroid quotients describe containment of linear spaces. If $L_1 \subseteq L_2$ are two linear subspaces of $\mathbb{K}^n$ generated as the row span of two matrices A_1 and A_2 respectively, the induced matroids form a quotient, $\mu(A_1) \twoheadleftarrow \mu(A_2)$ (see [BEZ21, Example 4.1.2]). Valuated matroid quotients arising in this way are called realisable (over $\mathbb{K}$). In general, a matroid quotient of two realisable matroids is not necessarily realisable (see [BGW03, Section 1.7.5, Example 7]).

Definition 6.1.4. Let μ be a valuated matroid of rank r on $[n]$. For each $I \in \binom{[n]}{r+1}$ define an element $C_\mu(I) \in \mathbb{T}^n$ by

$$C_\mu(I)_i = \begin{cases} \mu(I \setminus i) & i \in I, \\ \infty & i \notin I. \end{cases}$$

The set of **valuated circuits** $\mathcal{C}(\mu)$ of μ is defined as the image of

$$\left\{ C_\mu(I) : I \in \binom{[n]}{r+1} \right\} \setminus \{(\infty, \dots, \infty)\}.$$

in $\mathbb{P}(\mathbb{T}^n)$.

Definition 6.1.5 ([BEZ21, Definition 3.2.5])**.** Let μ be a valuated matroid on $[n]$. The **projective tropical linear space** of μ is the (projective) tropical variety

$$\overline{\mathrm{trop}}(\mu) = \bigcap_{C \in \mathcal{C}(\mu)} V\left(\bigoplus_{i \in [n]} C_i \odot x_i \right) \subseteq \mathbb{P}(\mathbb{T}^n).$$

This definition of a projective tropical linear space coincides with [BEZ21, Definition 3.2.5] and is a slight generalisation of the more commonly used notion of a tropical linear space. This allows us to take the tropical linear space of any valuated matroid, not only of uniform or loopless valuated matroids.

Definition 6.1.6. Let $r \leq n$ be a nonnegative integer. The **Grassmann-Plücker relations** are the following polynomials in the variables $\{p_I : I \in \binom{[n]}{r}\}$ with coefficients in $\mathbb{K}$:

$$\mathscr{P}_{r;n} = \left\{ \sum_{j \in J \setminus I} \operatorname{sign}(j; I, J) p_{I \cup j} p_{J \setminus j} : I \in \binom{[n]}{r-1}, J \in \binom{[n]}{r+1} \right\},$$

where $\operatorname{sign}(j; I, J) = (-1)^{|\{j' \in J : j < j'\}| + |\{i \in I : i > j\}|}$.

The Grassmann-Plücker relations describe the image of the Grassmannian $\operatorname{Gr}(r;n)$ in the projective space $\mathbb{P}^{\binom{n}{r}-1}$ via the Plücker embedding. The tropicalisations of the Plücker relations are denoted by $\mathscr{P}^{\text{trop}}_{r;n}$.

Remark 6.1.7. Let $\mathbb{K}$ be an algebraically closed field with nontrivial valuation. As mentioned at the beginning of this chapter, Grassmannians over $\mathbb{K}$ have two analogues in tropical geometry. The **tropical Grassmannians** $\overline{\operatorname{trop}}(\operatorname{Gr}(r;n))$ are tropicalisations of their classical analogues. These tropical varieties parametrise tropicalised objects, i.e. tropicalisations of linear subspaces of $\mathbb{K}^n$ of dimension r (see [SS04, Theorem 3.8]).

On the other hand, the **Dressians** $\overline{\operatorname{Dr}}(r;n)$ are the intersections of the tropical hypersurfaces given by the Plücker relations. They are tropical prevarieties, i.e. intersections of tropical hypersurfaces. Dressians parametrise tropical objects, i.e. tropical linear spaces in $\mathbb{P}(\mathbb{T}^n)$ as in Definition 6.1.5.

Remark 6.1.8. In tropical geometry, the Dressian $\operatorname{Dr}(r;n)$ often refers to the non-projective tropical prevariety cut out by the Grassmann-Plücker relations in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. This polyhedral complex is the parameter space of uniform valuated matroids, i.e. valuated matroids whose underlying matroid is the uniform matroid or, equivalently, Plücker vectors with no infinite coordinates. To distinguish this slightly different definition, we will refer to it as the open Dressian, as it can be used to describe the tropical analogue of the open part of the projective Grassmannian.

Interpreting valuated matroids as matroid subdivisions of matroid polytopes (see, for instance, [AFR10]) allows us to define the open Dressian $\operatorname{Dr}(r;n)$ of rank r over n as the subfan of the secondary fan Σ_2 of the hypersimplex $\Delta(r,n)$ given by matroidal subdivisions (see, for instance, [Jos21]).

6.1.3 Morphisms of valuated matroids, matrix multiplication and tropical linear spaces

We fix now a field $\mathbb{K}$ with (potentially trivial) valuation $\operatorname{val} : \mathbb{K} \to \mathbb{T}$.

Images of linear spaces under matrix multiplication

Hence, we study the behaviour of tropical linear spaces under matrix multiplication, and in order to do this we need to study tropical convexity. This concept was introduced by Develin-Sturmfels [DS04] and has been used in tropical combinatorics since. As we have discussed in Section 6.1.1, there are some adaptations we need to make due to discussing sets in the tropical projective space instead of $\mathbb{R}^n/\mathbb{R}\mathbf{1}$.

Definition 6.1.9. A set $S \subseteq \mathbb{P}(\mathbb{T}^n)$ is **tropically convex** if for all $v, w \in S$ and $\lambda, \rho \in \mathbb{R}$ the combination $\lambda \odot v \oplus \rho \odot w \in S$.

In [Ham15, Theorem 1.1], it is shown that all subsets of $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ supported on a tropical linear space are tropically convex. We now give the analogue statement in our setting. The first part of our proof (concerning the non-monomial case) follows similarly to the proof of [Ham15, Proposition 2.14], whereas the monomial case in the second part of the proof is particular to our setting.

Lemma 6.1.10. *Let* $\overline{\operatorname{trop}}(\mu) \subseteq \mathbb{P}(\mathbb{T}^n)$ *be a projective tropical linear space. Then* $\overline{\operatorname{trop}}(\mu)$ *is tropically convex.*

Proof. Let $v, w \in \overline{\operatorname{trop}}(\mu)$ and $\lambda, \rho \in R$. Let $z := \lambda \odot v \oplus \rho \odot w$.

Let $C_\mu(I) \subseteq \mathbb{T}^n$ be a valuated circuit of μ. There are two cases. First, assume that $\bigoplus_{i\in[n]} C_\mu(I)_i \odot x_i \in \mathbb{T}[x_1, \dots, x_n]$ is not a monomial. Then, both $\min_{i\in[n]}\{v_i \odot C_\mu(I)_i\}$ and $\min_{i\in[n]}\{w_i \odot C_\mu(I)_i\}$ are attained at least twice, i.e. there exist $i_1 \neq i_2, j_1 \neq j_2$ such that

$$v_C := \min_{i\in C}\{v_i \odot C_\mu(I)_i\} = v_{i_1} \odot C_\mu(I)_{i_1} = v_{i_2} \odot C_\mu(I)_{i_2}$$

$$w_C := \min_{i\in C}\{w_i \odot C_\mu(I)_i\} = w_{j_1} \odot C_\mu(I)_{j_1} = w_{j_2} \odot C_\mu(I)_{j_2}$$

We assume without loss of generality that $\rho \odot w_C \geq \lambda \odot v_C$. Thus,

$$z_{j_1} \odot C_\mu(I)_{j_1} \;=\; z_{j_2} \odot C_\mu(I)_{j_2} \;=\; \rho \odot w_C.$$

Let $k \in C$ be arbitrary. Then,

$$\begin{aligned} z_k \odot C_\mu(I)_k \;&=\; (\lambda \odot v_k \odot C_\mu(I)_k) \oplus (\rho \odot w_k \odot C_\mu(I)_k) \\ &\leq\; (\lambda \odot v_C) \oplus (\rho \odot w_C) \;=\; \rho \odot w_C \;=\; z_{j_1} \odot C_\mu(I)_{j_1}. \end{aligned}$$

In particular, the minimum $\bigoplus_{i\in[n]} z_i \odot C_\mu(I)_i$ is attained at least twice (at j_1 and at j_2).

Now, assume that $\bigoplus_{i\in[n]} C_\mu(I)_i \odot x_i \in \mathbb{T}[x_1, \dots, x_n]$ is a monomial. Without limitation of generality, assume that the monomial is in variable x_k, i.e. that $C_\mu(I)_k$ is the only non-infinite circuit entry. Then, by the definition of tropicalisation in Section 6.1.1, $C_\mu(I)_k \odot v_k = C_\mu(I)_k \odot w_k = \infty$. Since $C_\mu(I)_k \neq \infty$, this means that $v_k = w_k = \infty$, so $z_k = \lambda \odot v_k \oplus \rho \odot w_k = \infty$.

Thus, z satisfies all tropical polynomials generated by the circuits, so, by Definition 6.1.5, $z \in \overline{\operatorname{trop}}(\mu)$. □

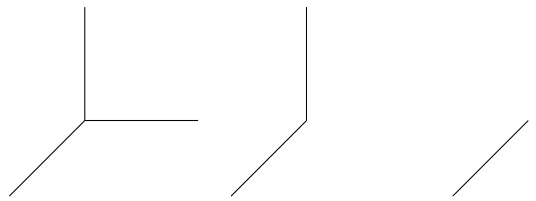
Lemma 6.1.11. *[IS23, Lemma 2.7] Let $\overline{\operatorname{trop}}(\mu)$ be a tropical linear space in $\mathbb{P}(\mathbb{T}^n)$ and $A \in \mathbb{T}^{n \times m}$. Then $A \odot \overline{\operatorname{trop}}(\mu)$ is tropically convex.*

In general, images of tropical linear spaces under pointwise matrix multiplication are not tropical linear spaces, as we can see in the following example.

Example 6.1.12. We consider the trivially valued matroid given by the map $\mu : \binom{[3]}{2} \to \mathbb{R} \cup \{\infty\}, I \mapsto 0$, and the matrices

$$A_1 = \begin{bmatrix} 0 & 0 & \infty \\ \infty & 0 & \infty \\ \infty & \infty & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & \infty \\ \infty & 0 & 0 \\ \infty & \infty & 0 \end{bmatrix}.$$

The tropical linear space $\overline{\operatorname{trop}}(\mu)$ and the polyhedral complexes $A_1 \odot \overline{\operatorname{trop}}(\mu)$ and $A_2 \odot \overline{\operatorname{trop}}(\mu)$ are depicted below. Both $A_1 \odot \overline{\operatorname{trop}}(\mu)$ and $A_2 \odot \overline{\operatorname{trop}}(\mu)$ are not tropical linear spaces, as these polyhedral complexes cannot be assigned balanced weights.



$\overline{\operatorname{trop}}(\mu)$ $\quad A_1 \odot \overline{\operatorname{trop}}(\mu)$ $\quad A_2 \odot \overline{\operatorname{trop}}(\mu)$

Affine morphisms of valuated matroids

So far, we described tropical linear maps via tropical matrix multiplication. As tropical linear spaces can be equivalently given as valuated matroids, we now define **affine morphisms of valuated matroids** as another characterisation of linear maps between tropical linear spaces. However, in order to consider projection maps, it is necessary to add an additional distinguished element to the ground set - intuitively, this imitates the origin of vector spaces.

Definition 6.1.13. Let μ be a valuated matroid over $[n]$. The **pointed valuated matroid** μ_o over $[n] \cup \{o\}$ is the valuated matroid $\mu \oplus U_{0,1}$, obtained by adding a loop o to the matroid μ. By a slight abuse of notation, we set $\overline{\operatorname{trop}}(\mu_o) := \overline{\operatorname{trop}}(\mu_o)|_{[n]} = \overline{\operatorname{trop}}(\mu)$, since the tropical linear spaces only differ by removing the ∞-entry in the o-coordinate.

Proposition 6.1.14 ([BS23, BEZ21])**.** *Let ν and μ be valuated matroids over the ground sets $[n]$ and $[m]$ respectively, and let $f : [n] \cup \{o\} \to [m] \cup \{o\}$ be a map of sets satisfying $f(o) = o$. Then, we can define the induced matroid $f^{-1}(M)$ on $[n]$ via the rank function*

$$\operatorname{rk}_{f^{-1}(M)}(S) \ := \ \operatorname{rk}_M\big(f(S)\big) \text{ for all } S \subseteq [n] \cup \{o\}$$

where M is the underlying matroid of μ_o. Further, μ induces a valuation $f^{-1}(\mu)$ with underlying matroid $f^{-1}(M)$, given as $f^{-1}(\mu)(B) = \mu|_{f([n]\cup\{o\})}(f(B))$. We say that $f : \nu \to \mu$ is a ***morphism of valuated matroids*** *if $f^{-1}(\mu) \twoheadleftarrow \nu$ is a quotient of valuated matroids, i.e. for all $I \in \mathcal{B}(f^{-1}(M))$, $J \in \mathcal{B}(N)$ and $i \in I \setminus J$, there exists $j \in J \setminus I$ such that*

$$\mu\big(f(I)\big) + \nu(J) \;\geq\; \mu\big(f(I \cup j \setminus i)\big) + \nu(J \cup i \setminus j).$$

We now extend this notion slightly, keeping track of additional scaling factors.

Definition 6.1.15. Let ν be a valuated matroid on the ground set $[n]$ and a map f defined as

$$(f_1, f_2) : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}.$$

We define the **affine induced valuated matroid** as

$$f^{-1}(\mu)(B) \;=\; \mu|_{f_1([n]\cup\{o\})}\big(f_1(B)\big) + \sum_{i\in B} f_2(i),$$

where $\mu|_{f_1([n]\cup\{o\})}$ denotes the restriction of μ_o to the set $f_1([n]\cup\{o\})$. The affine induced valuated matroid is a pointed valuated matroid as in Definition 6.1.13, hence its tropical linear space is defined as $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) := \overline{\mathrm{trop}}\big(f^{-1}(\mu)\big)|_{[n]}$.

Lemma 6.1.16. *[IS23, Lemma 2.12] Let μ be a valuated matroid on the ground set $[m]$ and $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$. Then, the affine induced valuated matroid is a valuated matroid as defined in Definition 6.1.2.*

For an example of an affine induced valuated matroid, see Example 6.1.30.

Definition 6.1.17. Let μ and ν be valuated matroids over $[m]$ and $[n]$ respectively, and let

$$f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$$

be a map. By abuse of notation, we say that $f : \nu \to \mu$ is an **affine morphism of valuated matroids** if $f^{-1}(\mu) \twoheadleftarrow \nu$ as in Definition 6.1.15 is a quotient of valuated matroids.

Additionally, if both μ and ν are realisable, we say that f is a realisable affine morphism of valuated matroids if both μ and the quotient $f^{-1}(\mu) \twoheadleftarrow \nu$ are realisable.

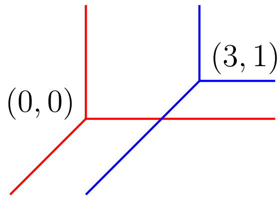
Weakly monomial matrices and associated affine maps of matroids

We point out that the two different notions of linear maps between tropical linear spaces we considered in Sections 6.1.3 and 6.1.3 are not fully compatible, as tropicalisation commutes with monomial maps, but not with linear maps. We now restrict to specific matrices for which matrix multiplication yields a tropical linear space, and show that these correspond to affine morphisms of valuated matroids.

Definition 6.1.18. Let $A \in K^{n\times m}$. We call A a **weakly monomial** matrix if A has at most one nonzero entry in each row.

Example 6.1.19. We consider the trivially valued matroid from Example 6.1.12. Its tropical linear space is the classical tropical line with vertex at $(0,0,0)$ in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ depicted in red below. Multiplying by a tropical matrix below yields a shifted tropical linear space to the right, depicted in blue. The permutation further switches the y and z rays.

$$A = \begin{bmatrix} 3 & \infty & \infty \\ \infty & \infty & 1 \\ \infty & 0 & \infty \end{bmatrix}$$



Now, to show the compatibility of morphisms of valuated matroids with the multiplication of a tropical linear space by a weakly monomial matrix, we need the following equivalent characterisation of tropical linear spaces using vectors generated by cocircuits:

Proposition 6.1.20 ([MR18, MT01])**.** *Let μ be a valuated matroid over the ground set $[n]$ of rank r. For each $I \in \binom{[n]}{r-1}$, we define $C^*_\mu(I) \in \mathbb{T}^n$ by*

$$C^*_\mu(I)_i = \begin{cases} \mu(I\cup i) & i \notin I \\ \infty & i \in I \end{cases}$$

*The **valuated cocircuits** of μ are defined as*

$$\mathcal{C}^*(\mu) = \left\{ C^*_\mu(I) : I \in \binom{[n]}{r-1} \right\} \setminus \{(\infty,\dots,\infty)\}.$$

The support of a cocircuit $C^* \in \mathcal{C}^*(\mu)$ is the set $\operatorname{supp}(C^*) = \{i \in [n] : C^*_i \neq \infty\}$. *The tropical span of cocircuits of a valuated matroid equivalently defines* $\overline{\operatorname{trop}}(\mu)$, *see [BEZ21, Theorem B].*

$$\overline{\operatorname{trop}}(\mu) = \left\{ \bigoplus_{C^*\in\mathcal{C}^*(\mu)} \lambda_{C^*} \odot C : \lambda_{C^*} \in \mathbb{R} \right\}.$$

Lemma 6.1.21. *[IS23, Lemma 2.17] Let μ be a valuated matroid on $[m]$ and $f : [n]\cup\{o\} \to [m]\cup\{o\}\times\mathbb{T}$. Let $I \in \binom{[n]}{\operatorname{rk}(f^{-1}(\mu))-1}$. Then, the coordinate entries of the valuated cocircuits of $f^{-1}(\mu)$ are given as follows:*

$$C^*_{f^{-1}(\mu)}(I)_i \;=\; \begin{cases} C^*_{\mu|_{f_1([n]\cup\{o\})}}\big(f_1(I)\big)_{f_1(i)} \odot \bigodot_{k\in I\cup i} f_2(k) & o \notin f_1(I\cup i), \\ \infty & o \in f_1(I\cup i). \end{cases}$$

Definition 6.1.22. Let $A_f \in \mathbb{T}^{n\times m}$ be a weakly monomial matrix. We define an **associated map** $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$ by

$$i \mapsto \begin{cases} (o, \infty) & i = o \text{ or } A_{ij} = 0 \text{ for all columns } j \\ (j, A_{ij}) & A_{ij} \neq 0. \end{cases}$$

If $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$ is a map of sets, we construct an **associated matrix** $A_f \in \mathbb{K}^{n\times m}$ by setting

$$A_{ij} = \begin{cases} f_2(i) & \text{if } f_1(i) = j \text{ and } i, j \neq o \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.1.23. We explicitly give the associated map for some special matrices:

(a) If A_f is a *permutation matrix*, then it is quadratic and the associated map f consists of a permutation map f_1 that fixes o, and $f_2(i) = 0$ for $i \in [n]$, and $f_2(o) = \infty$.

(b) If A_f is a *projection matrix* of rank $s < n$, then the associated map f is given as $f(i) = (i, 0)$ on $[s]$ and $f(i) = (o, \infty)$ otherwise.

(c) If A_f is a *diagonal matrix*, it is again quadratic and the associated map f is $f(i) = (i, A_{ii})$ for $i \in [n]$ and $f(o) = (o, \infty)$.

For the types of maps associated to matrices given above, we can again construct an associated matrix.

(a') If f_1 is a permutation map fixing o and $f_2(i) = 0$ for $i \in [n]$ and $f_2(o) = \infty$, then A_f can be chosen as the (square) tropical permutation matrix associated to the permutation f_1.

(b') If pr_S is a projection map satisfying $f(i) = (i, 0)$ for $i \in S^c$ and $f(i) = (o, \infty)$ for $i \in S \cup \{o\}$, a matrix associated to pr_S is given as the tropical projection matrix $A_{S_{ii}} = 0$ if $i \notin S$, and $A_{ij} = \infty$ otherwise.

(c') If f_1 is the identity map, an associated (square) matrix A_f is a diagonal matrix with entries $A_{f,ii} = f_2(i)$.

The following lemma develops the correspondence between affine morphisms of valuated matroids and matrix multiplication with weakly monomial matrices.

Lemma 6.1.24. *[IS23, Lemma 2.21] Let* $g : [n_1] \cup \{o\} \to [n_2] \cup \{o\} \times \mathbb{T}$ *and* $h : [n_2] \cup \{o\} \to [n_3] \cup \{o\} \times \mathbb{T}$ *be maps with* $g(o) = h(o) = (o, \infty)$*, and let* $A_g \in \mathbb{T}^{n_1\times n_2}$ *and* $A_h \in \mathbb{T}^{n_2\times n_3}$ *be their associated weakly monomial matrices. Assume that for any tropical linear space* $\overline{\mathrm{trop}}(\mu)$*,* $A_g \odot \overline{\mathrm{trop}}(\mu) = \overline{\mathrm{trop}}(g^{-1}(\mu))$ *and* $A_h \odot \overline{\mathrm{trop}}(\mu) = \overline{\mathrm{trop}}(h^{-1}(\mu))$*. Then, for* $h \circ g(i) = (h_1(g_1(i)), g_2(i) + h_2(g_1(i)))$*, we have* $(A_g \cdot A_h) \odot \overline{\mathrm{trop}}(\mu) = \overline{\mathrm{trop}}\big((h \circ g)^{-1}(\mu)\big) \subseteq \mathbb{P}(\mathbb{T}^{n_3})$*.*

Lemma 6.1.25. *[IS23, Lemma 2.22] Let μ be a valuated matroid over $[m]$. Let $f : [n] \cup \{o\} \to [m] \cup \{o\} \times \mathbb{T}$ be a map where $f_2(i) = 0$ if $f_1(i) \neq o$. Then, for any associated weakly monomial matrix A_f (as constructed in Definition 6.1.22),*

$$\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) \;=\; A_f \odot \overline{\mathrm{trop}}(\mu). \tag{6.1.26}$$

Conversely, if $A_f \in \mathbb{T}^{n\times m}$ is a weakly monomial matrix with entries in $\{0,\infty\}$, the associated map f satisfies $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = A_f \odot \overline{\mathrm{trop}}(\mu)$.

Lemma 6.1.27. *[IS23, Lemma 2.23] Let μ be a valuated matroid over $[n]$. Let $f : [n]\cup\{o\} \to [n]\cup\{o\}\times\mathbb{T}$ be a map satisfying $f_1(i) = i$. Then, for the associated weakly monomial matrix A_f defined in Definition 6.1.22, $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = A_f \odot \overline{\mathrm{trop}}(\mu)$. Conversely, if $A_f \in \mathbb{T}^{n\times n}$ is a full rank diagonal matrix, the associated map f satisfies $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = A_f \odot \overline{\mathrm{trop}}(\mu)$.*

Proposition 6.1.28. *Let μ be a valuated matroid over $[m]$. Let $f : [n]\cup\{o\} \to [m] \cup \{o\} \times \mathbb{T}$ be a map. Then, for the associated weakly monomial matrix A_f defined in Definition 6.1.22, $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = A_f \odot \overline{\mathrm{trop}}(\mu)$. Conversely, if $A_f \in K^{n\times m}$ is a weakly monomial matrix, the associated map f satisfies $\overline{\mathrm{trop}}\big(f^{-1}(\mu)\big) = A_f \odot \overline{\mathrm{trop}}(\mu)$.*

Proof. By Lemma 6.1.25, the statement holds for a weakly monomial matrix with entries in $\{0,\infty\}$, and by Lemma 6.1.27 it holds for diagonal matrices of full rank. Since every weakly monomial matrix can be written as the product of these two types, by Lemma 6.1.24, the claim follows for all matrices. □

Corollary 6.1.29. *Let μ be a valuated matroid over $[m]$ and let $A \in \mathbb{T}^{n\times m}$ be a weakly monomial matrix with no zero rows. Then, $A \odot \overline{\mathrm{trop}}(\mu) \subseteq \mathbb{P}(\mathbb{T}^n)$ is a tropical linear space.*

Example 6.1.30. In Example 6.1.19, we saw that matrix multiplication with a weakly monomial matrix induced a permutation and translation of the tropical linear space. Its matrix can be decomposed into a permutation matrix and a full rank diagonal matrix,

$$A = \begin{bmatrix} 3 & \infty & \infty \\ \infty & \infty & 1 \\ \infty & 0 & \infty \end{bmatrix} = \begin{bmatrix} 3 & \infty & \infty \\ \infty & 1 & \infty \\ \infty & \infty & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & \infty & \infty \\ \infty & \infty & 0 \\ \infty & 0 & \infty \end{bmatrix}.$$

The associated map of matroids is

$$f : [n] \to [n] \times \mathbb{R}; 1 \mapsto (1,3); 2 \mapsto (3,1); 3 \mapsto (2,0); o \mapsto (o,\infty).$$

The map f is the composition of a permutation map $g : 1 \mapsto (1,0); 2 \mapsto (3,0); 3 \mapsto (2,0)$ and an identity map with noninfinite values in h_2, $1 \mapsto (1,3), 2 \mapsto (2,1), 3 \mapsto (3,0), o \mapsto (o,\infty)$, and $f = h \circ g$ as defined above. Hence the valuations of sets in $f^{-1}(\mu)$ are $f^{-1}(\mu)(12) = 0 + 3 + 1 = 4$, $f^{-1}(\mu)(13) = 0 + 3 + 0 = 3$, $f^{-1}(\mu)(23) = 0 + 0 + 1 = 1$, and $f^{-1}(\mu)(I) = \infty$ if $o \in I$. The following table gives the cocircuits of $f^{-1}(\mu)$ and the vectors $A \odot C^*_\mu(I)$.

I	$C^*_{f^{-1}(\mu)}(I)$	$A \odot C^*_\mu(I)$
1	$(\infty, 4, 3)$	$(\infty, 1, 0)$
2	$(4, \infty, 1)$	$(3, 1, \infty)$
3	$(3, 1, \infty)$	$(3, \infty, 0)$

Note that $C^*_{f^{-1}(\mu)}(1) = A \odot C^*_\mu(1) \odot 3 = A \odot C^*_\mu(1) \odot f_2(1)$, and that analogously, $C^*_{f^{-1}(\mu)}(2) = A \odot C^*_\mu(3) \odot 1 = A \odot C^*_\mu(1) \odot f_2(2)$ and $C^*_{f^{-1}(\mu)}(2) = A \odot C^*_\mu(2) = A \odot C^*_\mu(1) \odot f_2(3)$, as in the proof of Lemma 6.1.27.

6.2 Tropical quiver (pre)varieties

6.2.1 Tropicalised quiver Grassmannians

Now, we come to the main part of this chapter: we combine quiver Grassmannians and tropical geometry. We will replace the linear spaces that are assigned to the vertices of a quiver by tropical linear spaces, and the matrix multiplication by the tropical matrix multiplication with valuated matrices we discussed in Section 6.1.3. For the background on quiver representations and quiver Grassmannians, we refer to Chapter 1 and Chapter 3, and to the material suggested previously. We will denote by $\mathscr{P}_\alpha^{\text{trop}}$ the tropicalisation of the quiver Plücker relations given in Definition 3.1.4.

Definition 6.2.1. Let Q be a quiver, M a Q-representation, and consider the quiver Grassmannian $\text{Gr}_{\mathbf{d}}(M)$ for a dimension vector $\mathbf{d}$. The **tropicalised quiver Grassmannian**

$$\overline{\text{trop}}\big(\,\text{Gr}_{\mathbf{d}}(M)\big) \;\subseteq\; \mathbb{P}\big(\mathbb{T}^{\binom{\dim(M_1)}{d_1}}\big) \times \cdots \times \mathbb{P}\big(\mathbb{T}^{\binom{\dim(M_{|Q_0|})}{d_{|Q_0|}}}\big)$$

is the projective tropicalisation of $\text{Gr}_{\mathbf{d}}(M)$.

We now show that the tropicalised quiver Grassmannian parametrises containment of tropicalised linear spaces under tropical matrix multiplication, i.e. we prove the equivalence $(a) \Leftrightarrow (b)$ of Theorem F.

Given a matrix A, we write $\text{val}(A) := \big(\text{val}(A_{ij})\big)_{ij}$ for the matrix with valuation applied to all entries, and write $\text{val}(A) \odot \overline{\text{trop}}(\mu)$ for the pointwise tropical matrix multiplication of $\text{val}(A)$ with $\overline{\text{trop}}(\mu)$.

Proposition 6.2.2. *Let $\mathbb{K}$ be an algebraically closed field with nontrivial valuation, and let M be a quiver representation of a quiver Q with quiver Grassmannian $\text{Gr}_{\mathbf{d}}(M)$, for some dimension vector $\mathbf{d}$. Then, $p \in \overline{\text{trop}}\big(\,\text{Gr}_{\mathbf{d}}(M)\big)$ if and only if there exists a tropical linear space $\overline{\text{trop}}(\mu_i)$ for each vertex $i \in Q_0$ such that $\text{val}(M) \odot \overline{\text{trop}}(\mu_{s(f)}) \subseteq \overline{\text{trop}}(\mu_{t(f)})$ for each arrow f, and there exists a quiver subrepresentation $N = \big((N_i)_{i \in Q_0}, (M_\alpha|_{N_{s(\alpha)}})_{\alpha \in A}\big)$ over $\mathbb{K}$ such that $\overline{\text{trop}}(\mu_i) = \overline{\text{trop}}(N_i)$ for all $i \in Q_0$.*

Proof. For ease of notation, we restrict to the case where Q is a graph with two vertices and one arrow f, and we write $\mathrm{Gr}_{\mathbf{d}}(M_f)$ for the corresponding quiver Grassmannian. All other cases follow similarly. If $\mu \times \nu \in \overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(M_f))$, from the Fundamental Theorem of Tropical Geometry [MS15, Theorem 6.2.15], there exist realisations U of μ and V of ν such that the Plücker coordinates of U and V are a point of $\mathrm{Gr}_{\mathbf{d}}(M_f)$. By the main theorem in [LW19], points in $\mathrm{Gr}_{\mathbf{d}}(M_f)$ satisfy $M_f \cdot U \subseteq V$, thus $\mathrm{val}(M_f) \odot \overline{\mathrm{trop}}(U) \subseteq \overline{\mathrm{trop}}(V)$.

Conversely, assume that $\mathrm{val}(M_f) \odot \overline{\mathrm{trop}}(U) \subseteq \overline{\mathrm{trop}}(V)$, and that there exist realisations U and V such that $M_f \cdot U \subseteq V$. Then, $U \times V \in \mathrm{QGr}(M_f, \mathbf{d})$, hence $p_{\overline{\mathrm{trop}}(U)} \times p_{\overline{\mathrm{trop}}(V)} \in \overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(M_f)))$ by the Fundamental Theorem. □

6.2.2 Quiver Dressians

After parametrising tropicalised linear spaces and their containment relations, we now consider parameter spaces of their tropical analogues and show Theorems E and **??**.

Definition 6.2.3. Given a quiver Q, we can define a **tropical Q-representation** as the ordered pair $\big(([n_i]_{i\in Q_0}), (M_\alpha)_{\alpha\in A}\big)$ where all n_i are natural numbers and the $M_\alpha \in \mathbb{T}^{s(\alpha)\times t(\alpha)}$ are tropical matrices.

Definition 6.2.4. Let Q be a quiver, M a Q-representation over an algebraically closed field $\mathbb{K}$ and $\mathbf{d}$ a dimension vector for Q. The **quiver Dressian** $\overline{\mathrm{QDr}}(R, \mathbf{d}) \subseteq \mathbb{P}\big(\mathbb{T}^{d_1}\big) \times \cdots \times \mathbb{P}\big(\mathbb{T}^{d_m}\big)$ is the projective tropical prevariety cut out by the tropical Plücker relations and the tropical quiver Plücker relations, $\big\{\mathscr{P}^{\mathrm{trop}}_{d_i;n_i}\big\}_{i\in Q_0} \cup \big\{\mathscr{P}^{\mathrm{trop}}_{\alpha}\big\}_{\alpha\in A}$ (see Definition 3.1.4).

Now we show Theorem E, i.e. that the quiver Dressian parametrises containment of projective tropical linear spaces under matrix multiplication.

Theorem 6.2.5. *Let μ and ν be valuated matroids over the ground sets $[m]$ and $[n]$ and of rank r and s respectively, and let Q be a quiver consisting of two vertices connected by one arrow f. Let M denote a tropical Q-representation assigning the tropical matrix $M_f \in \mathbb{T}^{n\times m}$ to f. Then,*

$$\mu \times \nu \in \overline{\mathrm{QDr}}(M, (r, s)) \Leftrightarrow M_f \odot \overline{\mathrm{trop}}(\mu) \subseteq \overline{\mathrm{trop}}(\nu).$$

Proof. The standard Grassmann-Plücker relations associated to the vertices vanish if and only if μ and ν are valuated matroids. Thus, we only focus on the quiver Plücker relations. By definition, $\mu \times \nu \in \overline{\mathrm{QDr}}(M, (r, s))$ if and only if for all $I \in \binom{[m]}{r-1}$ and $J \in \binom{[n]}{s+1}$, the minimum in

$$\bigoplus_{j\in[n]\setminus I, i\in J} \Big((M_f)_{i,j} \odot p_{I\cup j} \odot p_{J\setminus i} \Big)$$

is attained at least twice. Equivalently, for all I and J as above, the minimum in

$$\bigoplus_{\substack{j\in[n]\setminus I,\\ i\in J}} \Big((M_f)_{i,j} \odot \mu(I\cup j) \odot \nu(J\setminus i)\Big) \;=\; \bigoplus_{\substack{j\in[n]\setminus I,\\ i\in J}} \Big((M_f)_{i,j} \odot C^*_\mu(I)_j \odot C_\nu(J)_i\Big) \tag{6.2.6}$$

is attained at least twice. We write $M_f \odot C^*_\mu(I)$ for the vector with coordinate entries

$$\big(M_f \odot C^*_\mu(I)\big)_j \;:=\; \big(m_{f,1,j} \odot C^*_\mu(I)_j\big) \oplus \cdots \oplus \big(m_{f,n,j} \odot C^*_\mu(I)_j\big).$$

By distribution, the minimum in (6.2.6) is attained twice if and only if

$$M_f \odot C^*_\mu(I) \;\in\; V\big(\bigoplus_{i\in[n]} C_\nu(J)_i \odot x_i\big) \;=\; \overline{\operatorname{trop}}(\nu).$$

Finally, by Proposition 6.1.11, $M_f \odot \overline{\operatorname{trop}}(\mu)$ is tropically convex. Using Proposition 6.1.20, the above is thus equivalent to

$$\Big\{ \bigoplus_{C^*_\mu(I)\in\mathbb{C}^*(\mu)} \lambda_{C^*_\mu(I)} \odot M_f \odot C^*_\mu(I) \;:\; \lambda_{C^*_\mu(I)} \in \mathbb{R}\Big\} \;=\; M_f \odot \overline{\operatorname{trop}}(\mu) \subseteq \overline{\operatorname{trop}}(\nu).$$

□

Corollary 6.2.7. *Let M_f be a weakly monomial matrix, and let μ and ν be matroids of ranks r and s over $[n]$. Then, $M_f \odot \overline{\operatorname{trop}}(\mu) \subseteq \overline{\operatorname{trop}}(\nu)$ if and only if $f : \nu \to \mu \times \mathbb{R}$ as constructed in Proposition 6.1.28 is an affine morphism of valuated matroids. Further, $M_f \odot \overline{\operatorname{trop}}(\mu) \subseteq \overline{\operatorname{trop}}(\nu)$ is realisable, i.e. there are two $\mathbb{K}$-vector spaces $L_1 \subseteq \mathbb{K}^n$ and $L_2 \subseteq \mathbb{K}^m$ and a map $f : \mathbb{K}^n \to \mathbb{K}^m$ given by a matrix M_K such that $\mu(L_1) = \nu, \mu(L_2) = \mu$ and $M_f = \operatorname{val}(M_K)$ if and only if $f : \nu \to \mu \times \mathbb{R}$ is a realisable affine morphism of valuated matroids.*

Proof. By Proposition 6.1.28, there exists a map $f : [n]\cup\{o\} \to [n]\cup\{o\}\times\mathbb{T}$ such that $M_f \odot \overline{\operatorname{trop}}(\mu) = \overline{\operatorname{trop}}\big(f^{-1}(\mu)\big)$. By [BEZ21, Theorem A], $\overline{\operatorname{trop}}\big(f^{-1}(\mu)\big) = M_f \odot \overline{\operatorname{trop}}(\mu) \subseteq \overline{\operatorname{trop}}(\nu)$ implies that $f^{-1}(\mu)|_{[n]} \twoheadleftarrow \nu$, i.e. by Definition 6.1.17, that f^{-1} is an affine morphism of valuated matroids. The realisability statement follows from Definition 6.1.17. □

This proves Theorem **??** and the equivalence $(b) \Leftrightarrow (c)$ in Theorem F.

Remark 6.2.8. While the projective quiver Dressian and the projective tropicalised quiver Grassmannian describe projective tropical linear spaces, we can instead consider their classical tropicalisations.

As before, we obtain two different tropical analogues: The *open tropicalised quiver Grassmannian* $\operatorname{trop}(\operatorname{Gr}_{\mathbf{d}}(M))$ is the classical tropicalisation of the intersection of the quiver Grassmannian with an appropriate algebraic torus. The *open quiver Dressian* QDr is the tropical prevariety in $\mathbb{R}^n/\mathbb{R}1$ cut out by the (classical) tropicalisations of the quiver Plücker relations.

Corollary 6.2.9. *Let $M_f \in \mathbb{T}^{n \times m}$ be a tropical matrix. Let μ and ν be valuated matroids over $[m]$ and $[n]$ respectively whose underlying matroid is the uniform matroid. Then,*

$$\mu \times \nu \in \mathrm{QDr}(M, (r, s)) \Leftrightarrow M_f \odot \mathrm{trop}(\mu) \subseteq \mathrm{trop}(\nu).$$

Further, $\mu \times \nu \in \mathrm{Gr}^{\circ}_{(r,n)}(M)$ if and only if there exist tropical linear spaces $\mathrm{trop}(\mu)$ ans $\mathrm{trop}(\nu)$ such that $\mathrm{val}(M) \odot \mathrm{trop}(\mu_{)} \subseteq \mathrm{trop}(\nu)$, and there exists a quiver subrepresentation V_1, V_2 over $\mathbb{K}$ such that $\mathrm{trop}(\mu) = \mathrm{trop}(V_1)$ and $\mathrm{trop}(\nu) = \mathrm{trop}(V_2)$.

Proof. The first statement follows directly from Theorem E, the second from Theorem F. □

Remark 6.2.10. We remark that, with notation as above, $\mathrm{QDr}(M, (r, s))$ and $\mathrm{trop}(\mathrm{Gr}_{(r,n)}(M))$ are empty if the matrix M has a zero column or if the tropical matrix M_f has an infinity-column.

6.3 Realisability in quiver Dressians

In Section 6.1, we remarked the difference between intrinsically tropical and tropicalised objects, and distinguished the Dressian, parametrising *tropical* linear spaces, from the tropicalised Grassmannians, parametrising *tropicalised* linear spaces. We observe a similar distinction for quiver Dressians and tropicalised quiver Grassmannians.

The first example of a nonrealiseable tropical linear space, i.e. a tropical linear space that is not the tropicalisation of any linear space, occurs in ambient dimension 8. The first nonrealisable flag of tropical linear spaces already occurs for ambient dimension 6 (see [BEZ21, Example 5.2.4]). For arbitrary quivers, we show that the ambient dimension of the first nonrealisable quiver subrepresentation is even smaller.

Remark 6.3.1. If M is a Q-representation assigning a vector space of dimension 1 to each vertex, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ is a point for any dimension vector $\mathbf{d}$; there are no classical Plücker relations, and the only quiver Plücker relations are the monomial relations corresponding to the coordinates of the only point in the quiver Grassmannian. Hence, $\overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(R)) \subseteq \mathbb{P}(\mathbb{T}^1) \times \cdots \times \mathbb{P}(\mathbb{T}^1)$ is a point. Since $\mathbb{P}(\mathbb{T}^1) \times \cdots \times \mathbb{P}(\mathbb{T}^1)$ is also just a point, the containment is an equality, and therefore $\overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(R)) = \overline{\mathrm{QDr}}(R, \mathbf{d})$.

Theorem 6.3.2. *For any Q-representation M, $\overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(M)) \subseteq \overline{\mathrm{QDr}}(M, \mathbf{d})$. Further, for any finite quiver Q and any Q-representation M assigning dimension 1 to each vertex, $\overline{\mathrm{QDr}}(M, \mathbf{d}) = \overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(M))$. The same is not true in higher dimension: for any pair $n_1, n_2 \geq 2$ there exist a quiver Q and a Q-representation M containing an arrow α with $\dim(M_{s(\alpha)}) = n_1$ and $\dim(M_{t(\alpha)}) = n_2$ where the above containment is strict.*

The example for $\overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(M)) \neq \overline{\mathrm{QDr}}(M, \mathbf{d})$ where the two vertices adjacent to α are of dimension 2 relies on the nontrivial valuation of the base field. Similar constructions can be given for higher ambient dimension, as described in Example6.3.4. However, the examples we construct afterwards for higher ambient dimension ($n_i \geq 4$) already occur for fields with trivial valuation and for quivers with no parallel edges.

Example 6.3.3. We construct an example for $\overline{\mathrm{trop}}(\mathrm{Gr}_{\mathbf{d}}(M)) \neq \overline{\mathrm{QDr}}(M, \mathbf{d})$ where both vertices adjacent to α are of dimension 2. The quiver we consider is known as the Kronecker quiver (see Example 3.1.7). We define its representation M as shown in Figure 6.1, with quiver Grassmannian $\mathrm{Gr}_{(1,1)}(M)$; in this case, we replace $\mathbb{C}$ with $\mathbb{C}\{\{t\}\}$, the field of Puiseux series. It is an example of a reduced quiver Grassmannian of dimension 0 with two connected components (the two eigenspaces of the map corresponding to the lower arrow).

Let v_1 and v_2 denote the Plücker variables of the space corresponding to the left vertex, and let w_1 and w_2 denote the Plücker variables of the right vertex. Since $\mathrm{Gr}(1;2)$ and $\mathrm{Gr}(2;2)$ have no Grassmann-Plücker relations, the only relations are the quiver Plücker relations (see Definition 3.1.4), which are $v_1w_2 + v_2w_1$ and $v_1w_2 + (1+t)v_2w_1$. We have

$$V(\langle v_1w_2+v_2w_1, v_1w_2+(1+t)v_2w_1\rangle) = \{((1:0),(1:0)),((0:1),(0:1)) \subset \mathbb{P}^1\times\mathbb{P}^1\}$$

and

$$\overline{\mathrm{trop}}(\mathrm{Gr}_{(1,1)}(M))) = \{((0:\infty),(0:\infty)),((\infty:0),(\infty:0)) \subset \mathbb{P}(\mathbb{T}^2)\times\mathbb{P}(\mathbb{T}^2)\}.$$

Tropicalising the generators, we have that $V(\overline{\mathrm{trop}}(v_1w_2 + v_2w_1))$ is the set

$$\begin{aligned} W =&\{((v_1:v_2),(w_1:w_2)) \in \mathbb{P}(\mathbb{T}^2)\times\mathbb{P}(\mathbb{T}^2) |\min(v_1+w_2, v_2+w_1) \\ &\text{is attained at least twice}\}. \end{aligned}$$

Since $\mathrm{val}(1+t) = \mathrm{val}(1) = 0$, we further have $W = V(\overline{\mathrm{trop}}(v_1w_2 + (1+t)v_2w_1))$, thus $W = \overline{\mathrm{QDr}}(R, (1,1))$. Now, W can be rewritten as

$$W = \{((v_1:v_2),(w_1:w_2)) \in \mathbb{P}(\mathbb{T}^2)\times\mathbb{P}(\mathbb{T}^2) \mid v_1+w_2 = v_2+w_1\},$$

which is a connected 1-dimensional space (that contains the two points above), whereas $\overline{\mathrm{trop}}(\mathrm{Gr}_{(1,1)}(M))$ is not.

Example 6.3.4. To obtain an analogous example for ambient dimension 3, we can consider the same quiver as in Example 6.3.3. We assign $\mathbb{C}\{\{t\}\}^3$ to each vertex, the matrix $\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ to the upper arrow and $\begin{bmatrix} 1&0&0\\0&1+t&0\\0&0&1+t^2\end{bmatrix}$ to the lower arrow. Again, $\mathrm{Gr}(1;3)$ has no Grassmann-Plücker relations, so the only Plücker relations are

$$v_1w_2 - v_2w_1, v_1w_3 + v_3w_1, v_2w_3 - v_3w_2$$

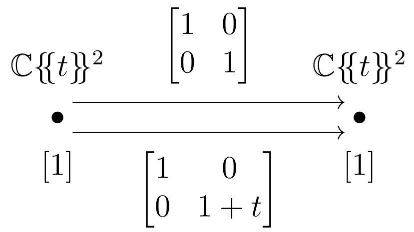


Figure 6.1: A quiver Q with Q-representation M for $n = 2$ where $\overline{\text{trop}}(\text{Gr}_{(1,1)}(M)) \neq \overline{\text{QDr}}(M, (1,1))$.

for the upper arrow, and

$$v_1w_2 - (1+t)v_2w_1, v_1w_3 + (1+t^2)v_3w_1, (1+t)v_2w_3 - (1+t^2)v_3w_2$$

for the lower arrow. The zero locus of the six equations is zero-dimensional and consists of three points: $((1:0:0),(1:0:0)), ((0:1:0),(0:1:0))$ and $((0:0:1),(0:0:1))$, so the tropicalisation of the quiver Grassmannian does too. Again, as the valuations of all nonzero matrix entries is zero, the quiver Dressian is the set

$$\{(\mathbf{v},\mathbf{w}) \in \mathbb{P}(\mathbb{T}^3)\times\mathbb{P}(\mathbb{T}^3) \mid v_1+w_2 = v_2+w_1, v_1+w_3 = v_3+w_1 \text{ and } v_2+w_3 = w_2+v_3\}.$$

This set is 1-dimensional, thus the tropicalised quiver Grassmannian and the quiver Dressian differ. This example can similarly be extended to an example for higher ambient dimension n. Here, we assign $\mathbb{C}\{\{t\}\}^n$ to both vertices, consider the dimension vector $(1,1)$ and assign the matrices to the two arrows as follows: one arrow is assigned the identity matrix, and the other arrow gets the diagonal matrix with entries $(1, 1+t, 1+t^2, \ldots, 1+t^{n-1})$. An example for a quiver representation of trivially valued fields over a quiver with no parallel edges can also be found, though it is significantly more complicated.

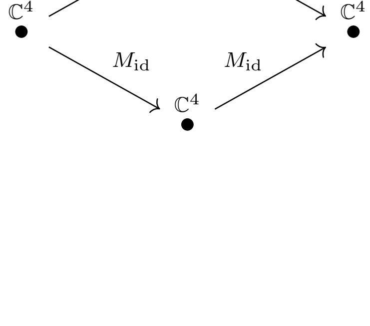
Now we give an example of a quiver representation M satisfying

$$\overline{\text{trop}}(\text{Gr}_{\mathbf{d}}(M)) \neq \overline{\text{QDr}}(M, \mathbf{d})$$

over a field with trivial valuation, using a quiver without parallel edges where all vector spaces associated to vertices have dimension at least 4. Afterwards, we will extend this to a family of such examples for higher vertex dimensions.

Example 6.3.5. We recall the quiver and the quiver representation given in Example 3.1.2:

$Q, M :$



$$, \quad M_{\text{id}} = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \quad \mathbf{d} = (1,2,2,3).$$

The quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ parametrises the arrangement of four tropical objects: two tropical lines that are contained in a common tropical plane, and a common point lying on all of them. In Figure 6.2, we give an example of such an arrangement.

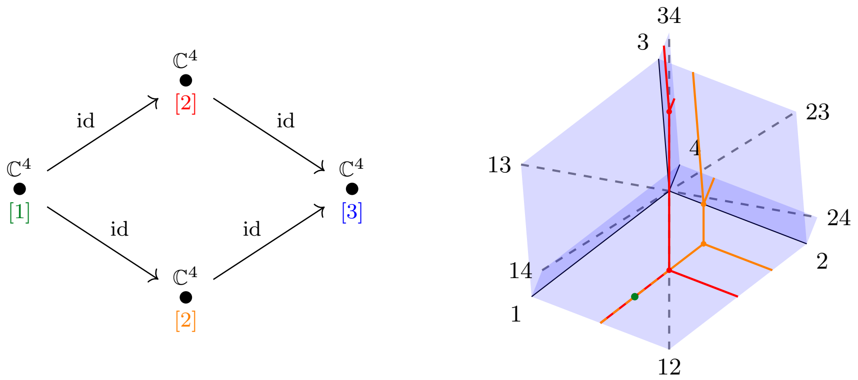


Figure 6.2: The numbers below the vertices of the quiver represent the fixed dimensions of the corresponding subspaces. Any subrepresentation consisting of such subspaces describes the containment of a point in two lines, which are both contained in a common plane. On the right, a collection of tropical linear spaces satisfying these conditions.

We obtain from Definition 3.1.4 the following equations for the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ inside the product of Grassmannians $\prod \mathrm{Gr}(d_i; 4)$:

$$\begin{array}{ccc}
p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} & p'_{12}p'_{34} - p'_{13}p'_{24} + p'_{14}p'_{23} & p_{12}p_{134} + p_{13}p_{124} + p_{14}p_{123} \\
p_{12}p_{234} + p_{23}p_{124} + p_{24}p_{123} & p_{13}p_{234} + p_{23}p_{134} + p_{34}p_{123} & p_{14}p_{234} + p_{24}p_{134} + p_{34}p_{124} \\
p'_{12}p_{134} + p'_{13}p_{124} + p'_{14}p_{123} & p'_{12}p_{234} + p'_{23}p_{124} + p'_{24}p_{123} & p'_{13}p_{234} + p'_{23}p_{134} + p'_{34}p_{123} \\
p'_{14}p_{234} + p'_{24}p_{134} + p'_{34}p_{124} & p_1p_{23} + p_2p_{13} + p_3p_{12} & p_1p_{24} + p_2p_{14} + p_4p_{12} \\
p_1p_{34} + p_3p_{14} + p_4p_{13} & p_2p_{34} + p_3p_{24} + p_4p_{23} & p_1p'_{23} + p_2p'_{13} + p_3p'_{12} \\
p_1p'_{24} + p_2p'_{14} + p_4p'_{12} & p_1p'_{34} + p_3p'_{14} + p_4p'_{13} & p_2p'_{34} + p_3p'_{24} + p_4p'_{23}
\end{array}$$

where we denote by p'_{ij} the Plücker coordinates corresponding to the two-dimensional subspace corresponding to the vertex at the bottom of Q, which is a point in $\mathrm{Gr}(d_3; 4)$.

We compute the quiver Dressian and the tropicalised quiver Grassmannian in `gfan` [Jen], and do some auxiliary computations in Oscar [OSC22].

The quiver Dressian has dimension 12 and f-Vector $(1, 58, 466, 1156, 858, 3)$. The tropicalised quiver Grassmannian has dimension 10, as does the ideal generated by the polynomials. Since the dimensions of the tropical (pre-)varieties differ, they cannot be equal, showing the second part of Theorem G for $\mathbf{n} = \mathbf{4}$.

As a polyhedral complex, the tropicalised quiver Grassmannian is the union of the tropicalisation of the 46 primary components of the quiver Grassmannian. Of these components, 37 tropicalise to linear components of dimensions 8, 7, 6 and 5 in different coordinate directions. Each of the remaining nine components

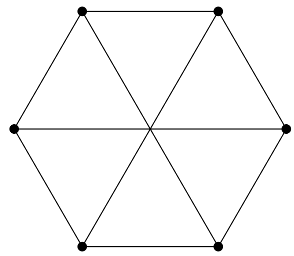


Figure 6.3: The nonlinear irreducible components of Example 6.3.5 are linear spaces of dimension 10 and 8 over the graph above.

has, after quotienting out lineality, six rays and ten facets, whose incidences are depicted in the graph in Figure 6.3.

Corollary 6.3.6. *For any $n_1, n_2, n_3, n_4 \geq 4$ there exists a quiver Q and a Q-representation M' assigning each n_i to a vertex such that $\overline{\text{trop}}(\text{Gr}_{\mathbf{d}}(M')) \subsetneq \overline{\text{QDr}}(M', \mathbf{d})$.*

Proof. We consider the quiver representation M of Example 6.3.5, and construct a quiver representation M' by substituting each base set on the vertices by $[n_i]$. For each matrix, we append an appropriate amount of zero rows or columns. This way, $\text{Gr}_{\mathbf{d}}(M')$ has the same Plücker relations as $\text{Gr}_{\mathbf{d}}(M)$. Since $\dim(\overline{\text{trop}}(\text{Gr}_{\mathbf{d}}(M)) < \dim(\overline{\text{QDr}}(M, \mathbf{d}))$, we obtain that $\dim(\overline{\text{trop}}(\text{Gr}_{\mathbf{d}}(M'))) < \dim(\overline{\text{QDr}}(M', \mathbf{d}))$. Hence,

$$\overline{\text{trop}}(\text{Gr}_{\mathbf{d}}(M')) \subsetneq \overline{\text{QDr}}(M', \mathbf{d}).$$

□

This concludes the proof of Theorem G.

Perspectives

This bridge between quiver representation theory and tropical geometry offers new strategies and points of view to deal with current open questions of tropical geometry. The following are possible questions to be answered:

1. We know that, for Dressians and for flag Dressians, there exists an additional characterisation in terms of regular matroidal subdivisions of a certain polytope. Can we find an analogous for quiver Dressians?

2. Could we give a meaningful definition of tropical Schubert varieties, by exploiting the realisation of smooth Schubert varieties described in Chapter 4 and considering the corresponding quiver Dressians?

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