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Fully Functorial Resolutions of Complex Analytic Singularities

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"I am not really doing research,
just trying to cultivate myself"
Alexander Grothendieck

Alla mia famiglia & Federico,

Contents

Abstract

This is a new demonstration of functorial resolution of singularities of complex analytic spaces following the new method introduced in [MM19]. In particular there are no hypothesis of compactness, or even relative compactness.

Preface

i Introduction

The object of this manuscript is the resolution of complex analytic singularities, *i.e.* we prove the existence of a resolution functor in the holomorphic case, ii.b, *cf.* theorem [MM19, I.a], albeit in the 2-category of complex analytic (Deligne-Mumford) champs. As we will clarify in the prologue, §ii, a fully functorial procedure for the resolution of singularities in characteristic zero is impossible, *i.e.* there cannot exist a smooth centre determined by purely local data blowing up in which must improve some discrete measure of how far the singularity is from being smooth.

The present work may perfectly well be seen as a corollary of the existence of a resolution functor for excellent Deligne-Mumford champ of characteristic zero (every complex analytic ring is excellent), *cf.* [MM19], however, the actual contribution we provide is to present a series of simplifications due to the absence of many technicalities which, on the contrary, naturally appears in the excellent case. We adopt the new method of [MM19], hence there is a significant non-empty intersection with this work and *op.cit.* , *i.e.* §I - §IV, where the invariant is constructed and its useful properties, *e.g.* III.c, pointed out. Nevertheless in the holomorphic case we can simplify a few thorny issues such as the convergence of the weighted centre, *i.e.* whether such weighted centre defining ρ is well-defined in A, rather than only in its completion, $\widehat{A}$. In particular we provide an easier proof, ii.f, of convergence wherein any intervention of champs is just a categorical tool which allows us to work with quotient singularities while doing linear algebra. In second place, to go from convergence to ii.b one needs the upper semi-continuity of the invariant which is just a consequence, V.d, of its definition and the properties peculiar to the analytic topology, which makes life even easier than in the geometric case. Lastly, in the final assembly, V.m & V.q, we work without any compactness assumption For the convenience of the reader and with opportune modifications we reproduce the preface of [MM19] in the following chapter.

ii Prologue

It is a known fact that resolution of singularities, already in characteristic zero, cannot be achieved in a way that is both étale local and independent of the resolution process itself while blowing up in smooth centres. More precisely one would like in the category of complex analytic spaces (all locally ringed spaces which are locally isomorphic to an open subset of the vanishing locus of a finite set of holomorphic functions or in general of reduced excellent algebraic spaces, [MM19]) a modification functor

$$U \longmapsto M(U)$$

and an invariant $\mathrm{inv}(U) \in \Gamma_{\geqslant 0}$ in a (preferably discrete) ordered group such that
(M.1) $M(U) \longrightarrow U$ is a blow up in a smooth centre.
(M.2) $U = M(U)$ iff U is smooth.
(M.3) M commutes with étale base change $U' \longrightarrow U$, *i.e.* $M(U') = M(U) \times_U U'$ whenever U, U' are connected and $\mathrm{inv}(U') = \mathrm{inv}(U)$.
(M.4) For any $U' \longrightarrow U$ étale, $\mathrm{inv}(U') \leq \mathrm{inv}(U)$.
(M.5) $\mathrm{inv}\big(M(U)\big) < \mathrm{inv}(U)$ whenever $\mathrm{inv}(U) > 0$.

The impossibility of this is shown by the example, *cf.* [Kol07, pg. 142],
ii.a Example. Let K be any field of characteristic 0, and consider

$$U : x^2 + y^2 + (zt)^2 = 0 \hookrightarrow \mathbb{A}^4_K, \tag{2.1}$$

wherein the singular locus is the union of the two lines,

$$L_1 : x = y = z = 0 \;\;\&\;\; L_2 : x = y = t = 0.$$

On the other hand if $M(U) \longrightarrow U$ were to exist then by (M.1), (M.2) & (M.3) it must be a blow up in a smooth centre contained in the singular locus, so the only possibilities are L_1, L_2 or their intersection, *i.e.* the origin. Now the latter operation leaves (2.1) unchanged where the proper transform of either line meets the exceptional divisor, while a choice amongst L_1, L_2 is inadmissible because the process must respect, (M.3), the symmetry $z \longleftrightarrow t$, and that's without even addressing the issue that (2.1) might only be valid after completion, so that globally the L_i could be branches of the same curve.

The traditional get out from this difficulty is to change the problem, *e.g.* the argument of the modification functor becomes not just varieties but varieties with marked divisor, so, in particular, blowing up (2.1) in the origin creates a marked divisor and amongst the new singular lines one of them is marked.

However, we change the traditional paradigm, (M.1)-(M.5), to one which adapts the modification to the problem, so that (M.1) is replaced by,

(M.1') $M(U) \longrightarrow U$ is a smoothed weighted blow up in a regular centre, [MP13, I.iv.3].

The paradigm shift works and the existence of a resolution functor for varieties and spaces over $\mathbb{C}$ is provided by,

ii.b Theorem. [*cf.* V.q] In the 2-category of analytic champ there is a modification functor $U \longrightarrow M(U)$, V.p, satisfying (M.1'), (M.2), (M.3), (M.4), (M.5), albeit inv takes values in $\mathbb{Q}^{\infty}_{\geqslant 0} = \varinjlim \mathbb{Q}^{N}_{\geqslant 0}$. Nevertheless, the invariant has self-bounding denominators, II.a.

Obviously the goal of the construction of inv is that it should go down under an appropriate weighted blow up (*cf.* III.c), so taking values in $\mathbb{Q}_{\geqslant 0}$ may be a little disconcerting. The invariant has, however, self-bounding denominators, which is a certain technical condition, II.a, which has all the effects, II.b, of defining the invariant in $\mathbb{Z}_{\geqslant 0}$ while allowing us to define the invariant and perform various construction, *e.g.* II.m, where they naturally occur, *i.e.* $\mathbb{Q}_{\geqslant 0}$. Specifically for I an ideal of a m-dimensional regular local ring, A, of characteristic zero, with maximal ideal $\mathfrak{m}$ we construct an invariant, §II, $\mathrm{inv}_A(I)$ with self bounding denominators in $\mathbb{Q}^{2m}_{\geqslant 0}$ ordered lexicographically.

Better there is a yoga for constructing inv that makes the resolution process more widely applicable to more difficult problems such as vector field singularities, which, essentially views the resolution process as a diagram chase, and manifest itself as follows,

(Y.1) Generically most thing are smooth, a.k.a. $I = \mathcal{O}$, so the invariant is $\underline{0}$ and there is nothing to do.

(Y.2) If (Y.1) didn't happen then generically most things have an isolated singularity at the closed point, and after a single blow up in the same the multiplicity will decrease,

(Y.3) If (Y.2) didn't happen then there is proper sub-space of the tangent space where the multiplicity did not decrease and its annihilator in $\mathfrak{m}/\mathfrak{m}^2$ gives us the start of a filtration of A which depends only on I.

(Y.4) Construct inductively, II.f - II.g, a sequence of filtrations, $F^{\bullet}_s(I)$, according to the dichotomy,
ii.c Case(A). Something generic happens, case (A), II.p, then $s \mapsto s+1$;
ii.d Case(B). Nothing generic happens, case (B), II.q, then at worst, $F^{\bullet}_s(I)$ converges $\mathfrak{m}$-adically.

Proceeding in this way leads to the key,
ii.e Fact. [*cf.* V.b] There is an invariant, $\mathrm{inv}_A(I) \in \mathbb{Q}^{2m}_{\geqslant 0}$, of regular m- dimensional characteristic zero local rings and their ideals with self bounding denominators such that if $\mathfrak{U}$ is the completion of its spectrum at the closed point, then there is a smoothed weighted blow up $\rho : \widetilde{\mathfrak{U}} \longrightarrow \mathfrak{U}$ such that at every closed point of $\mathfrak{U}$ the invariant strictly decreases provided $I \neq A$.

At this point the only remaining issue is whether the weighted centre defining ρ is well defined in A, rather than only in its completion, $\widehat{A}$.
ii.f Proposition. [*cf.* V.i] If the centre in ii.e is of dimension 0 or, A is the local ring of holomorphic functions of a polydisc V, then, V.j, the (canonically defined) smoothed weighted blow up of V.b is the completion in the exceptional divisor of a smoothed weighted blow up of V. Similarly if A is the ring of functions of a

complex analytic space V, $\mathfrak{U}$ the completion in the closed point, and $\rho : \widetilde{\mathfrak{U}} \longrightarrow \mathfrak{U}$ the modification of ii.e obtained after a choice of an embedding of $\mathfrak{U}$ in a smooth formal scheme, V.n, then there is a smoothed weighted blow up, V.p, of V whose completion in the exceptional divisor is ρ.

Convergence of ii.e, while true more generally for excellent rings *cf.* [MM19, VII.d], is much easier when we were to work in the holomorphic world, V.g & V.h, and whence this manuscript offers an attractive alternative even to the pure geometric case, [MM19][VI.i & VII.f].

The manuscript is organised as follow,

§I. This contains some linear algebra about weighted projective spaces (technically champs because we want them to be smooth) which describes the manifestation of item (Y.3) above in the generality necessary for the distinctions between generic and non-generic phenomena in item (Y.4).

§II. This is the inductive definition of the invariant as outlined in (Y.1)-(Y.4). The key step is the sub-induction II.m whose illustration by way of its Newton polyhedron, figure 1 of page 13, should facilitate its understanding.

§III. Calculates the invariant for ideals on weighted projective champs. It is the proof that the invariant goes down on blowing up in its weighted centre.

§IV. This begins to address the aforesaid convergence issues, and related questions such as upper semi-continuity of the invariant by calculating it in a suitably general, IV.a, relative setting.

§V. Is the final assembly of the preceeding into a modification functor. First we prove the following weak principalisation statement,

ii.g Theorem. [V.m] There is a modification functor from the 2-category whose objects, $(U, \mathcal{I})$, are ideals on complex analytic (Deligne-Mumford) champs whose value

$$M_{(U,\mathcal{I})} = (\widetilde{U}, \widetilde{\mathcal{I}}) \tag{2.2}$$

is the proper (rather than total) transform $\widetilde{\mathcal{I}}$ on a smoothed weighted blow up $\widetilde{U} \to U$, satisfying (in the obvious change of notation) (M.1'), (M.3), (M.4), (M.5), while (M.2) becomes, $M_{(U,\mathcal{I})} = 0$ iff $\mathcal{I} = \mathcal{O}_U$, and, again, inv takes values in $\mathbb{Q}^{\infty}_{\geqslant 0} = \varinjlim \mathbb{Q}^{N}_{\geqslant 0}$ with self-bounding denominators.

After which we remove the condition of the embedding of singular variety in a smooth one, implicit in ii.g, to obtain V.q. En passant, we provide a new proof of the convergence of the centre and remove any conditions of (quasi) compactness and/or Noetherianity.

Resolution of Singularities for Complex Analytic Spaces

I Weighted Projective Champs

I.a Set Up/Definition. Throughout this section, k is a ring of characteristic 0, and $A_k := \mathbb{A}_k^{N+1} \setminus \{\underline{0}\}$. For $n \leqslant N$, let $\underline{a} = (\underline{a}_0, \underline{a}_1, ..., \underline{a}_n) \in \mathbb{Z}_{>0}^{N+1}$ with each $\underline{a}_i = \big(a_i, ..., a_i\big) \in \mathbb{Z}_{>0}^{c_i}$, $c_i \geqslant 1$ and $N+1 = c_0 + ... + c_n$. We denote the coordinates of $\mathbb{A}_k^{N+1}$ by x_{ij} for $0 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant c_i$, and we will call the set of variables with the same weight a_i, *i.e.* $\{x_{i\,1}, ..., x_{i\,c_i}\}$, a *block*, or a *block of weight* a_i, and often abbreviate it by X_i, similarly, consistent with this decomposition, we will abbreviate monomials $\prod x_{ij}^{e_{ij}}$ by $X_i^{E_i}$, where $|E_i| = \sum_j e_{ij}$ (*i.e.* the degree of the monomial in the relevant block); while $X_i = 0$ means $x_{ij} = 0$, $\forall\, 1 \leqslant j \leqslant c_i$.

I.b Definition. The weighted projective champ $\mathbb{P}_k(\underline{a}) := \mathbb{P}(\underline{a}_0, \underline{a}_1, ..., \underline{a}_n)$ is defined to be the classifying champ $[A_k/\mathbb{G}_m]$ of the action

$$\mathbb{G}_m \times A_k \underset{\text{id}}{\overset{\lambda}{\rightrightarrows}} A_k,\ (\lambda^{a_0} X_0, ..., \lambda^{a_n} X_n) \overset{\lambda}{\longleftarrow\!\shortmid} (X_0, ..., X_n) \overset{\text{id}}{\longmapsto} (X_0, ..., X_n) \tag{1.1}$$

on which the tautological bundle $\mathscr{O}_{\mathbb{P}_k(\underline{a})}(1)$ corresponds to the character:

$$\mathbb{G}_m \longrightarrow \mathbb{G}_m : \lambda \longmapsto \lambda^{-1}. \tag{1.2}$$

In particular, functions on $\mathbb{A}_k^{N+1}$ are naturally graded by the action (1.1) and we denote the grading by $\mathbf{wt}$, *i.e.*

$$\mathbf{wt}(X_i) = a_i, \quad \mathbf{wt}(X_i^{E_i}) = a_i|E_i|. \tag{1.3}$$

and by S the graded algebra of $\mathbb{G}_m$-homogeneous equivariant functions on $\mathbb{A}_k^{N+1}$ whose graded pieces, $S_e = \{f \in S \mid \mathbf{wt}(f) = e\}$, $e \in \mathbb{Z}_{>0}$ are, by [McQ17, I.c.3], canonically isomorphic (so without loss of generality we may suppose equal) to the global sections of $\mathscr{O}_{\mathbb{P}_k(\underline{a})}(e)$, to wit:

$$S_e \cong \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(e)\big). \tag{1.4}$$

Similarly, derivations of S, $\mathrm{Der}(S)$, are naturally graded by the action of $\mathbb{G}_m$ and we consider the graded sub-algebra of $\mathbb{G}_m$-equivariant homogeneous derivations on S of strictly negative weight,

$$\mathrm{Der}_{<0}(S) := \coprod_{-e\in\mathbb{Z}_{<0}} \mathrm{Der}(S)_{-e} = \coprod_{-e\in\mathbb{Z}_{<0}} \big\{\partial \in \mathrm{Der}(S) \,\big|\, \partial^{\lambda} = \lambda^{-e}\partial\big\}. \tag{1.5}$$

Finally if $\underline{r} = \big(\underline{r}_0, ..., \underline{r}_n\big) \in \mathbb{Q}_{>0}^{N+1}$, $\underline{r}_i := (r_i, ..., r_i) \in \mathbb{Q}_{>0}^{c_i}$, and $\underline{a} \in \mathbb{Z}_{>0}^{N+1}$ is the unique integer tuple parallel to $\underline{r}$ without common factors we define,

$$\mathrm{P}_k(\underline{r}) := \mathbb{P}_k(\underline{a}) \tag{1.6}$$

to which we add the hypothesis specific to our situation *i.e.*

I.c Hypothesis. Suppose $a_0 < a_1 < ... < a_n$ and let V_d be a k-submodule of $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(d)\big) = S_d$, $d \geqslant 0$, such that if $\mathbb{P}_k(\underline{a}') = \mathbb{P}_k(\underline{a}_1, ..., \underline{a}_n)$ is the sub-weighted projective champ defined by the block of variables $X_0 = 0$ of weight a_0, S' the associated graded algebra, and V'_d is the image of V_d in $S'_d = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}'), \mathscr{O}_{\mathbb{P}_k(\underline{a}')}(d)\big)$ then for all quotients $k \twoheadrightarrow k'$, $-b < 0$ and negative weighted derivations $\partial \in \mathrm{Der}_{<0}(S)(-b)$,

$$\partial(f') = 0, \ \forall\, f' \in V'_d \otimes_k k' \iff \partial = 0. \tag{1.7}$$

In the presence of such a supposition we have,

I.d Lemma. Let be everything as in I.a-I.c, and for $-b < 0$ a strictly negative integer define

$$\mathrm{L}_{-b}(V_d) := \big\{\, \partial \in \mathrm{Der}_{<0}(S)(-b) \,\big|\, \partial(V_d) = 0 \,\big\} \tag{1.8}$$

the sub-module of negative weighted derivations of weight $-b$ which vanish on V_d. Then If $b \neq a_0$, $\mathrm{L}_{-b}(V_d) = 0$, otherwise there is a natural injective map,

$$\mathrm{L}_{-a_0}(V_d) \longrightarrow \big(S_{a_0}^{\oplus c_0}\big)^{\vee} := \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0)^{\oplus c_0}\big)^{\vee}, \tag{1.9}$$

Better still if for every quotient $k \twoheadrightarrow k'$,

$$\mathrm{L}_{-a_0} \otimes_k k' = \big\{\, \partial \in \mathrm{Der}_{<0}(S')(-a_0) \,\big|\, \partial\big(V_d \otimes_k k'\big) = 0 \,\big\}\,, \tag{1.10}$$

then (1.9) remains an injection on tensoring with k'.

Proof. Without loss of generality $\dim \mathbb{P}_k(\underline{a}) > 0$, so if we consider derivations on S of the form $\partial_{ij} := \frac{\partial}{\partial x_{ij}}$ for $0 \leqslant i \leqslant n$, $1 \leqslant j \leqslant c_i$, then, plainly, $\partial_{ij} \in \mathrm{Der}_{<0}(S)(-a_i)$, as the action of λ on ∂_{ij} is by way of the character λ^{-a_i}, and, in particular, they afford a $\mathbb{G}_m$ equivariant isomorphism of k-modules,

$$\textstyle\coprod_{i=0}^{n} S_{\underline{a}_i - b}\big(:= S_{a_i-b}^{\oplus c_i}\big) \xrightarrow{\ \sim\ } \mathrm{Der}_{<0}(S)(-b) : \ \underline{X}_i^{\underline{E}} \longmapsto \sum_{j=1}^{c_i} X_i^{E_j} \frac{\partial}{\partial x_{ij}}; \tag{1.11}$$

where each $\underline{X}_i^{\underline{E}} := \big(X_i^{E_1}, ..., X_i^{E_{c_i}}\big) \in S_{a_i-b}^{\oplus c_i}$. Therefore if we suppose $b \neq 0$ and $S_{\underline{a}_0 - b} \neq 0$, then $a_0 > b > 0$ which is equivalent to $a_0 > a_0 - b > 0$. However, for any $e > 0$,

$$S_e = \coprod_{|E_0|a_0 + ... + |E_n|a_n = e} k \cdot X_0^{E_0} \cdots X_n^{E_n} \tag{1.12}$$

thus $S_e \neq 0$ implies $e = |E_0|a_0 + ... + |E_n|a_n \geqslant a_0$ hence $S_{\underline{a}_0 - b} = 0$ for $a_0 > b > 0$. Finally, by (1.4) $S_e = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(e)\big)$, so by (1.11) both items in I.d will follow from the more general,

I.e Claim. Let $b > 0$ (so $b = a_0$ is allowed) and pr the projection,

$$S_{\underline{a}_0 - b} = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0 - b)\big) \xleftarrow{\ \mathrm{pr}\ } \mathrm{Der}_{<0}(S)(-b)$$

afforded by (1.11), then the submodule $\mathrm{L}^0_{-b} := \{\partial \in \mathrm{L}_{-b} \mid \mathrm{pr}(\partial) = 0\} \subset \mathrm{L}_{-b}$ consists only of the null derivation.

Proof. In order to emphasise their role say, by way of notation, that $\{y_1, ..., y_{c_0}\}$ is the (since any other is obtained via the action of $\mathrm{GL}_k(c_0)$) block $Y := X_0$ of weight a_0, and $\{x_{i\bullet}\}$, $i > 0$, are blocks X_i of weight $a_i > a_0$. Then $\partial \in \mathrm{Der}_{<0}(S)(-b)$ can be written as

$$\partial = \sum\nolimits_I Y^I \partial_I, \qquad \text{with } \mathbf{wt}(\partial_I) = -b - |I|a_0 < 0. \tag{1.13}$$

where by hypothesis $\partial \longmapsto 0$ in $S_{\underline{a}_0 - b} = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0 - b)\big)$, thus by (1.11) we have,

$$\partial \in \coprod\nolimits_{i \geqslant 1} S_{\underline{a}_i - b} \quad \text{where} \quad S_{\underline{a}_i - b} = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_i - b)\big)$$

where $S_{\underline{a}_i - b} = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_i - b)\big)$, and each ∂_I may be naturally identified to an element of $\mathrm{Der}_{<0}(S')(-b - |I|a_0)$, *cf.* I.c, via the $\mathbb{G}_m$-equivariance, being S' the graded algebra as in I.c. Now, suppose $0 \neq \partial \in \mathrm{L}^0_{-b}$ and let

$$i_0 = \min\{\, |I| \mid \partial_I \neq 0 \text{ for some } |I| \text{ as in (1.13)} \,\}.$$

Similarly, we can (again, wholly canonically because of the $\mathbb{G}_m$-equivariance) write each $f \in V_d$ as

$$f = f' + \sum\nolimits_{|J|>0} f_J Y^J, \quad \mathbf{wt}(f') = \mathbf{wt}(f_J Y^J) = a_0|J| + \mathbf{wt}(f_J) = d, \tag{1.14}$$

where f' and f_J are non-zero $\mathbb{G}_m$-homogeneous polynomials in the variables $X_1, ..., X_n$ (f' may be identified with its image in $V'_d \subseteq S'_d = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}'), \mathscr{O}_{\mathbb{P}_k(\underline{a}')}(d)\big)$) so, by hypothesis, $\partial(f) = 0$ and on the other hand

$$\partial(f) = \sum\nolimits_{|I|=i_0} \Big(Y^I \partial_I(f') \quad + \sum\nolimits_J Y^{I+J} \partial_I(f_J)\Big), \tag{1.15}$$

where $Y^{I+J}\partial_I(f_J)$ consists of monomials where Y is of degree $> i_0$, therefore $\partial(f) = 0$ only if $\sum_{|I|=i_0} Y^I \partial_I(f') = 0$. However, on identifying (as ever via the $\mathbb{G}_m$- equivariance) V'_d with a subspace of V_d, $\partial_I(f')$ depends only on the blocks $X_{\geqslant 1}$, so $\partial_I(f') = 0$ for all $|I| = i_0$, which, by I.c, implies the absurdity $\partial_I = 0$. $\square$

This certainly implies I.d when $b \neq a_0$, while for $b = a_0$ we have

$$\mathrm{Der}_{<0}(S)(-a_0) \xrightarrow{\ \mathrm{pr}\ } \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}(\underline{a}_0 - a_0)\big) \xrightarrow{\ \sim\ } \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}(a_0)\big)^\vee$$

so in this case the claim is exactly (1.9). Better since by construction the hypothesis I.c is stable under base change to an arbitrary quotient of k, our initial conclusions are too, so (1.9) is an injection on tensoring as soon as the definition of L_{-a_0} enjoys the stability under base change in (1.10). $\square$

To profit from the lemma, let us introduce,

I.f Notation/Definition. Let $W := W_0 \amalg \ldots \amalg W_n$ be a k-module with a $\mathbb{G}_m$-action such that $\mathbb{G}_m$ acts on W_i by the character λ^{b_i}, $b_i \in \mathbb{Z}$, for $0 \leqslant i \leqslant n$, then for $q \in \mathbb{Z}$, $\underline{\mathrm{Sym}}^q(W)$ is the subspace of the symmetric algebra $\mathrm{Sym}(W)$ where $\mathbb{G}_m$ acts by the character λ^q. Similarly, given blocks X_i, $n \geqslant i \geqslant 0$, as in I.a, with a slight abuse of notation, we define

$$\underline{\mathrm{Sym}}^q\big(X_0 \amalg \ldots \amalg X_n\big) := \coprod_{a_0|E_0|+\ldots+a_n|E_n|=e} k \cdot X_0^{E_0} \cdots X_n^{E_n}$$

which is, by [McQ17][I.c.3], (canonically) isomorphic to $S_e = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(e)\big)$ $= \underline{\mathrm{Sym}}^e(X_0 \amalg \ldots \amalg X_n)$. Finally, as in (1.6), if the weights $r_0, \ldots, r_n \in \mathbb{Q}_{>0}$ were any rationals and $(a_0, \ldots, a_n) = D(r_0, \ldots, r_n)$ the unique parallel tuple of positive integers without common factors, we define for $q \in \mathbb{Q}_{\geqslant 0}$

$$\underline{\mathrm{Sym}}^q\big(X_0 \amalg \ldots \amalg X_n\big) := \coprod_{a_0|E_0|+\ldots a_n|E_n|=Dq} k \cdot X_0^{E_0} \cdots X_n^{E_n}. \tag{1.16}$$

In any case to apply the lemma, observe that, the sub k-module of negative weighted derivations of S which vanich on V_d is just

$$\mathrm{L} := \coprod\nolimits_{b>0} \mathrm{L}_{-b} = \mathrm{L}_{-a_0} \tag{1.17}$$

moreover, it is plainly a Lie algebra wherein by (1.9) the bracket is even trivial; thus

I.g Corollary. Again let everything be as in I.a-I.c and suppose further that (1.9) is an isomorphism onto a trivial (*i.e.* admitting a basis) free k-module. As such there is a block Z associated to the annihilator of L, *i.e.*

$$\bigcap\nolimits_{\partial \in \mathrm{L}} \ker(\partial) \subset S_{a_0} = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0)\big), \text{ and,} \tag{1.18}$$

(i) there are *blocks*, *i.e.* weighted projective coordinates $X_1, \ldots, X_n$, of weight $a_1, \ldots, a_n$, generating a space of functions, X, such that

$$V_d \subset \underline{\mathrm{Sym}}^d\big(X \amalg Z\big) := \underline{\mathrm{Sym}}^d\big(X_0 \amalg \ldots \amalg X_n \amalg Z\big).$$

(ii) If $\widetilde{X}_i$, $1 \leqslant i \leqslant n$ is a system of coordinates with $\mathbf{wt}(\widetilde{X}_i) = a_i$, which generates a space of functions $\widetilde{X}$, and $\widetilde{Z} \subseteq S_{a_0} = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0)\big)$ such that (i) holds i.e. $V_d \subset \underline{\mathrm{Sym}}^d\big(\widetilde{X} \amalg \widetilde{Z}\big)$, then the k-module generated by $\widetilde{Z}$ contains Z.

(iii) If $\widetilde{X}_i$, $1 \leqslant i \leqslant n$, Z is any other system of coordinate such that I.g.(i) holds, then $\widetilde{X}_i = \widetilde{X}_i(X, Z)$, $1 \leqslant i \leqslant n$, *i.e.* unused coordinates are not involved.

Proof. Item I.g.(i) is trivial if $\mathrm{L}_{-a_0} = 0$, so suppose the image of (1.9) is non-zero, and profit from the fact that the the image is a trivial k-module to choose $0 \neq \partial \in \mathrm{L}_{-a_0}$ along with coordinates Z, y_1 where the former is a basis of

$$\ker \partial \subset S_{a_0} = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{a}_0)\big)$$

(thus empty if c_0 and the dimension of L_{-a_0} are 1), and $\partial y_1 = 1$. Again, let X_i, for $1 \leqslant i \leqslant n$, be the blocks of weight strictly greater than a_0; so Z, $\{y_1\}$, $X_i = \{x_{i\bullet}\}$, $1 \leqslant i \leqslant n$ is a basis for everything and in these of coordinates ∂ takes the form

$$\partial = \frac{\partial}{\partial y_1} + \sum_{i=1}^{n} \Bigg(\sum_{j=1}^{c_i} \lambda_{ij} \frac{\partial}{\partial x_{ij}} \Bigg), \qquad \mathbf{wt}(x_{ij}) = a_i > a_0, \tag{1.19}$$

where $\mathbf{wt}(\lambda_{ij}) - \mathbf{wt}(x_{ij}) = -a_0$, so $\mathbf{wt}(\lambda_{ij}) = \mathbf{wt}(x_{ij}) - a_0 < \mathbf{wt}(x_{ij})$, thus

$$\lambda_{ij} = \lambda_{ij}(Z, y_1, X_{<i}), \quad \text{where } \mathbf{wt}(X_{<i}) < a_i, \tag{1.20}$$

i.e. λ_{ij} only depends on variables of weight strictly less than a_i. To simplify the notation we'll write ∂_{y_1}, resp. $\partial_{x_{ij}}$, for $\frac{\partial}{\partial y_1}$, resp. $\frac{\partial}{\partial x_{ij}}$, and employ the summation convention so that (1.19) becomes:

$$\partial = \partial_{y_1} + \lambda_{ij}\, \partial_{x_{ij}}\ . \tag{1.21}$$

By increasing induction on $\mathbf{wt}(X_i)$ we will eliminate everything from (1.21), except ∂_{y_1}, by way of a global change of weighted projective coordinates. The starting point is $a_{i-1} = a_0$ which is a minor abuse of notation, but it is certainly true, so by induction we have

$$\partial = \partial_{y_1} + \lambda_{hj}\, \partial_{x_{hj}}, \quad \mathbf{wt}(x_{hj}) \geqslant a_i. \tag{1.22}$$

Thus in weight a_i we aim for a global change of coordinates of the form

$$x_{ij} \mapsto x_{ij} + G_{ij}(Z, y_1, X_{<i}), \quad \mathbf{wt}(X_{<i}) < a_i = \mathbf{wt}(G_{ij}) \tag{1.23}$$

and otherwise do nothing for weights strictly greater than a_i. Consequently we need to solve $\partial\big(x_{ij} + G_{ij}\big) = 0$, *i.e.*

$$\partial\big(x_{ij} + G_{ij}\big) = \lambda_{ij} + \partial_{y_1} G_{ij} = 0, \tag{1.24}$$

which is trivially solvable on any ring of characteristic 0 by (1.20) with,

$$\mathbf{wt}(G_{ij}) = \mathbf{wt}(\lambda_{ij}) - \mathbf{wt}(\partial_{y_1}) = a_i.$$

As such for our given ∂ we have a system of coordinates $\{\, Z, y_1, X_i \,\}$ such that $\partial = \partial_{y_1}$ and, of course, any other $D \in \mathrm{L}_{-a_0}$ can be expressed in this basis as

$$D = \nu\, \partial_Z + \mu\, \partial_{y_1} + \lambda_i\, \partial_{X_i}, \tag{1.25}$$

with $\mu, \nu, \in k$ but not λ_i if $\lambda_i \neq 0$, where $\lambda_i \partial_{X_i} := \sum_{j=1}^{c_i} \lambda_{ij} \partial_{x_{ij}}$. By (1.9) if D is linearly independent of ∂_{y_1}, replacing D by $D - \mu\,\partial_{y_1}$, $\mu = 0$ and some $\nu\,\partial_Z \neq 0$. Further, from $[\partial, D] = 0$, D is canonically a derivation of the algebra $k[Z, X_1, ..., X_n]$, which in turn inherits a $\mathbb{G}_m-$action. Consequently we may repeat the first step for D and $\ker D$ to get coordinates y_1, y_2, X_i, $n \geqslant i > 0$, and Z, which, now, is a block of coordinates of $\ker \partial \cap \ker D$, in which

$$\partial = \partial_{y_1}, \quad D = \partial_{y_2}. \tag{1.26}$$

and whence, by induction we arrive at a $\mathbb{G}_m$-equivariant system of coordinates Z, $Y = \{y_1, ..., y_\ell\}$, X_i, $n \geqslant i > 0$ with the properties

$$\begin{array}{ll} (1) & \partial_Y = \{\partial_{y_1}, ..., \partial_{y_\ell}\} \text{ is a basis of } \mathrm{L}_{-a_0},\ Y = \{y_1, ..., y_\ell\} \text{ its dual basis;} \\ (2) & Z \text{ is a basis of } \mathrm{ann}(\mathrm{L}) \text{ in } S_{a_0} = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0)\big),\ \textit{cf.}\ (1.18); \\ (3) & X_i,\ 1 \leqslant i \leqslant n,\ \text{are the other coordinates;} \\ (4) & V_d \text{ is a submodule of weight } da_0 \text{ of the } \mathbb{G}_m\text{-algebra } k[Z, X_{\geqslant 1}]; \end{array} \tag{1.27}$$

which complete the proof of part I.g.(i).

In regard to part I.g.(ii), under the hypothesis of *op.cit.* , L contains a subspace of fields, M, whose annihilator under the natural map of (1.9) is generated by $\widetilde{Z}$, while the annihilator of L is generated by Z, so from $M \subseteq \mathrm{L}$ we get Z is contained in the k-module generated by $\widetilde{Z}$.

Finally as to part (iii), by definition the $\widetilde{X}_i$'s and the X_i's, $1 \leqslant i \leqslant n$, modulo S_{a_0} $= \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0)\big)$, are systems of weighted projective coordinates of the sub-weighted projective champ $\mathbb{P}_k(\underline{a}') = \mathbb{P}(a_1, ..., a_n)$, *cf.* I.c, so without loss of generality (*i.e.* after replacing say the X_i's by a weighted automorphism of themselves) with Y, Z as in (1.27)

$$\widetilde{X}_i = X_i \mod (Y, Z). \tag{1.28}$$

and we have:

$$\widetilde{X}_i = X_i + \widetilde{X}_i(Z, X_{\leqslant i}) + \sum\nolimits_{|E| = \alpha_i} Y^E \lambda_E(Z, X_{\leqslant i}) + \big(\text{higher order in } Y'\text{s}\big), \tag{1.29}$$

where $\mathbf{wt}(Y^E \lambda_E) = |E| a_0 + \mathbf{wt}(\lambda_E) = a_i$, and by definition α_i is of minimal weight amongst the monomials in Y. As such, we may, in light of our goal, I.g.(iii), without loss of generality replace $X_i + \widetilde{X}_i(Z, X_{\leqslant i})$ by X_i (which is an automorphism because it is so modulo Z) so that for $\beta = \min_i\{\alpha_i\}$ (1.29) is

$$\widetilde{X}_i = X_i + \sum\nolimits_{|E_i| = \beta} Y^{E_i} \lambda_{E_i}(Z, \underline{X}) + \big(\text{order} \geqslant \beta + 1 \text{ in } Y'\text{s}\big) =: X_i + \eta_i, \tag{1.30}$$

and by hypothesis every $f = f(Z, \underline{X}) \in V_d$ can be written as $\varphi_f(Z, \underline{\widetilde{X}})$, where $\underline{X} := (X_1, ..., X_n)$ and $\underline{\widetilde{X}} = \underline{X} + \underline{\eta}$, (1.30). However from

$$\varphi_f(Z, \underline{X} + \underline{\eta}) = f(Z, \underline{X}) \tag{1.31}$$

we must have $\varphi_f = f$ and whence

$$\begin{aligned} f(Z,\underline{X}) &= f(Z,\underline{X}+\underline{\eta}) \\ &= f(Z,\underline{X}) + \eta_i(\partial_{X_i} f)(Z,\underline{X}) + \big(\text{order} \geqslant \beta+1 \text{ in } Y'\,\big) \\ &= f(Z,\underline{X}) + \sum\nolimits_{|E_i|=\beta} Y^{E_i}\lambda_{E_i}(\partial_{X_i} f) + \big(\text{order} \geqslant \beta+1 \text{ in } Y'\,\big) \end{aligned} \tag{1.32}$$

Thus for every E_i with $|E_i| = \beta$ the term $\sum_i \lambda_{E_i}(\partial_{X_i} f)$ must be equal to 0 and, as we have said, $|E_i|a_0 + \mathbf{wt}(\lambda_{E_i}) = \mathbf{wt}(X_i)$, so the operator $\lambda_{E_i}\partial_{X_i}$ has weight

$$\mathbf{wt}(\lambda_{E_i}\partial_{X_i}) = \mathbf{wt}(\lambda_{E_i}) + \mathbf{wt}(\partial_{X_i}) = a_i - |E_i|a_0 - a_i = -|E_i|a_0 < 0 \tag{1.33}$$

and it vanishes on all of V_d, so it belongs to L_{-a_0}. Therefore by lemma I.d its image under (1.9) in $\mathrm{H}(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a_0))^\vee$ is non-zero, which is nonsense since $\lambda_{E_i}\partial_{X_i}$ has value 0 on both the Y's and the Z's. $\square$

II The Invariant on Local Rings

We are going to define an invariant of rings and their ideals which is most naturally expressed in an appropriate number of copies of $\mathbb{Q}_{\geqslant 0}$ with the lexicographic ordering. On the other hand this is not a discrete group, so to avoid fastidious statements about denominators we introduce,

II.a Definition. Let $N \in \mathbb{Z}_{\geqslant 0}$; $\mathbb{Q}^{N+1}$ ordered lexicographically; and pr_i, resp. $\mathrm{pr}_{\leqslant i}$, the projection onto the i^{th} factor, resp. first i factors, $1 \leqslant i \leqslant N$, then a function $f : E \longrightarrow \mathbb{Q}^{N+1}_{\geqslant 0}$ is said to have self bounding denominators if,

(i) $f^*\mathrm{pr}_1 : E \longrightarrow \mathbb{Q}_{\geqslant 0}$ takes values in $\mathbb{Z}_{\geqslant 0}$.

(ii) If $N \geqslant 1$, then for all $1 \leqslant i \leqslant N$ there are increasing (in the lexicographic order) functions $D_i : \mathbb{Q}^i_{\geqslant 0} \longrightarrow \mathbb{Z}_{\geqslant 0}$ such that,

$$\big(f^*\mathrm{pr}^*_{\leqslant i} D_i\big) f^*\mathrm{pr}_{i+1} \in \mathbb{Z}_{\geqslant 0}. \tag{2.1}$$

The utility of the definition results from,

II.b Fact. Let everything be as in II.a with $f : E \longrightarrow \mathbb{Q}^{N+1}_{\geqslant 0}$ a function enjoying self bounding denominators, and define a function $F : E \longrightarrow Z^{N+1}_{\geqslant 0}$ whose first projection is that of f while its $(i+1)^{\mathrm{th}}$ projection is (2.1) for $1 \leqslant i \leqslant N$, then in the lexicographic order,

$$f(x) \leqslant f(y) \quad \Longleftrightarrow \quad F(x) \leqslant F(y).$$

Proof. Manifestly II.b is true if $N = 0$, so suppose $N \geqslant 1$ and $f(x) < f(y)$, then without loss of generality, $\mathrm{pr}_{\leqslant N} f(x) = \mathrm{pr}_{\leqslant N} f(y)$ but $\mathrm{pr}_{N+1} f(x) < \mathrm{pr}_{N+1} f(y)$. Consequently, $\big(f^*\mathrm{pr}^*_{\leqslant N} D_N\big)$ is the same at x and y, so: $\mathrm{pr}_{N+1} f(x) \leqslant \mathrm{pr}_{N+1} f(y)$ iff $\mathrm{pr}_{N+1} F(x) \leqslant \mathrm{pr}_{N+1} F(y)$. □

II.c Set Up/Notation. A is a regular local ring of dimension m, with residue field k of characteristic 0, and $\mathfrak{m}$ its maximal ideal. We will employ,

II.d Definition. A regular weighted filtration (or simply a weighted filtration or even just filtration if there is no danger of confusion) on a ring A, is the filtration, $F^\bullet$, associated to a system of coordinates (*i.e.* modulo $\mathfrak{m}^2$ affords a basis of $\mathfrak{m}/\mathfrak{m}^2$) $\{ x_1, ..., x_m \}$ and non-negative numbers, $r_1, ..., r_m$, by the ideals,

$$F^p A = \{ x_1^{e_1} \cdot ... \cdot x_m^{e_m} \mid r_1 e_1 + ... + r_m e_m \geqslant p \}, \quad p \in \mathbb{Q}_{>0}. \tag{2.2}$$

In addition, since in the string of rationals $(r_1, ..., r_m) \in \mathbb{Q}^m_{\geqslant 0}$, repetitions are allowed, we define

II.e Definition. A block of coordinates, X, is a set which may be extended to a system of coordinates and, which is maximal amongst such sets with the same weight. In particular any weighted filtration can always be expressed in terms of a system of blocks $X_0, ..., X_s$, $s < m$, where each X_i has the same weight and $X_0 \amalg ... \amalg X_s$ is a system of coordinates of A.

For I an ideal of A we will define inductively a weighted filtration $F^{\bullet}(I)$ which only depends on the pairs (A, I) together with

$$\mathrm{inv}(I) = \mathrm{inv}_A(I) \in \mathbb{Q}^{2m}_{\geqslant 0} \tag{2.3}$$

where $\mathbb{Q}^{2m}_{\geqslant 0}$ is endowed with the lexicographic ordering. At each step $s \geqslant 0$ of the induction we will, actually, define two successive entries of $\mathrm{inv}(I)$, (g_s, ℓ_s), beginning with

II.f Start of the Induction. Let A be as in II.c, and $I \lhd A$ an ideal, then the multiplicity of I, is

$$\mathrm{mult}(I) := \begin{cases} \max\{\alpha \in \mathbb{Q}_{\geqslant 0} \,\big|\, I \subseteq \mathfrak{m}^{\alpha}\}, & I \neq 0 \\ \infty \ , & I = 0 \end{cases}$$

As such if $\mathrm{mult}(I) = d \in \mathbb{Z}_{>0}$,

$$V_d := I \mod \big(\mathfrak{m}^{d+1}\big) \hookrightarrow \mathrm{Sym}^d\big(\mathfrak{m}/\mathfrak{m}^2\big),$$

and we apply lemma I.d to

$$V_d \hookrightarrow \mathrm{H}^0\big(\mathbb{P}(\mathfrak{m}/\mathfrak{m}^2),\ \mathscr{O}_{\mathbb{P}(\mathfrak{m}/\mathfrak{m}^2)}(d)\big) \tag{2.4}$$

with $\ell_0(I) := \dim \mathrm{L}_{-1}(V_d)$, in notation of (1.8). Then by corollary I.g.(i) there is a unique minimal subsapce $Z = Z(I) \subseteq \mathfrak{m}/\mathfrak{m}^2$ of dimension $c_0 := m - \ell_0$ such that $V_d \subseteq \mathrm{Sym}^d(\,Z\,)$. We therefore start the induction by way of:

(S.0) The first two entries of $\mathrm{inv}(I)$ are equal to $\big(\mathrm{mult}(I), \ell_0(I)\big)$.

(S.1) If either of these entries of the invariant are zero, then so are all the subsequent ones, and the process terminates.

(S.2) The weighted filtration $F_0^{\bullet}(I)$ is the weighted filtration in which each x_i has weight 1, *i.e.* the powers of the maximal ideal $\mathfrak{m}^{\bullet}$.

(S.3) Under the hypothesis of (S.1), the definition of $F^{\bullet}(I)$ also terminates, $F^{\bullet}(I) = F_0^{\bullet}(I)$.

(S.4) The first block, X_0, of cardinality c_0 is a choice of basis of Z.

II.g Inductive Hypothesis. For $s \geqslant 1$, there is a (weighted) filtration $F_{s-1}^{\bullet}(I)$ depending only on I (and for this reason we will write just $F_{s-1}^{\bullet}$ if there is no danger of confusion) defined by blocks of coordinates $X_{s-1}^0, ..., X_{s-1}^{s-1}$, respectively Y of cardinality $c_0, c_1, ..., c_{s-1}$, respectively ℓ_{s-1}, where, for $0 \leqslant i < s-1$,

$$\ell_i := m - (c_0 + ... + c_i) \ \text{ or equivalently } \ell_{i+1} := \ell_i - c_{i+1}, \tag{2.5}$$

and rationals weights $g_{s-1}^0 > g_{s-1}^1 > ... > g_{s-1}^{s-1} \in \mathbb{Q}^s_{>0}$, $g_{s-1}^i \geqslant 1$ such that:

(F.0) If Y is any block completing $X_{s-1}^0, ..., X_{s-1}^{s-1}$ to a system of coordinates then $1 = \mathbf{wt}(Y) \leqslant g_{s-1}^{s-1}$.

(F.1) $I \subseteq F_{s-1}^{dg_{s-1}^0}$.

(F.2) For $V_{s-1}^{da_{s-1}^0} := I \mod F_{s-1}^{>da_{s-1}^0}$, $V_{s-1}^{da_{s-1}^0} \subseteq \underline{\mathrm{Sym}}^{da_{s-1}^0}\big(X_{s-1}^0 \amalg \ldots \amalg X_{s-1}^{s-1}\big)$, *cf.* I.f.

(F.3) There are no derivations of strictly negative weight on the $\mathrm{P}_k(\underline{g})$, *cf.* (1.6), associated to the graded algebra

$$\mathrm{gr}_{s-1}A = \coprod\nolimits_{q\geqslant 0} F_{s-1}^q \Big/ F_{s-1}^{q+1} \tag{2.6}$$

leaving $V_{s-1}^{da_{s-1}^0}$ invariant.

(F.4) There are strictly positive integers d_i^t, $0 \leqslant i \leqslant t \leqslant s-1$, $d_0^0 = d$ as in II.f, such that the weights g_t^i are derived from $g_t \in \mathbb{Q}_{>0}$ according to the following rules: if given g_t, we define $g_t^i = g_{i+1} \ldots g_t$, $g_t^t = 1$, then

$$\begin{aligned} g_0^0 = g_0 &= 1 \\ g_1^0 d_0^0 - (g_1^0 d_0^1) &= d_1^1 \\ g_2^0 d_0^0 - (g_2^0 d_0^2 + g_2^1 d_1^2) &= d_2^2 \\ \vdots \qquad\qquad & \quad \vdots \\ g_{s-1}^0 d_0^0 - \big(g_{s-1}^0 d_0^{s-1} + g_{s-1}^1 d_1^{s-1} + \ldots + g_{s-1}^{s-2} d_{s-2}^{s-1}\big) &= d_{s-1}^{s-1}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \text{and, } g_t^0 d_0^{t+1} + g_t^1 d_1^{t+1} + \ldots + g_t^{t-1} d_{t-1}^{t+1} + d_t^{t+1} + d_{t+1}^{t+1} \; &> g_t^0 d_0^0, \\ \text{for every } \; 0 \leqslant t \leqslant s-2\,. & \end{aligned} \tag{2.8}$$

Notice that by (2.7) & (2.8), $g_t > 1$ for every $1 \leqslant t \leqslant s-1$.

(F.5) The function $\underline{g} = (d, g_1, \ldots, g_{s-1})$ of rings and their ideals has self bounding denominators, II.a.

II.h Induction Defining $F_s^\bullet$ from $F_{s-1}^\bullet$. The induction is divided as follows:

II.i Step. If $c_0 + \ldots + c_{s-1} = m$, or equivalently, by 2.5, if $\ell_{s-1} = 0$, then stop and define $F^p(I) := F_{s-1}^p(I)$, together with the invariant:

$$\mathrm{inv}(I) := \begin{cases} (\, d,\ \ell_0, \underline{0}\,)\ , & s = 1; \\ (\, d,\ \ell_0,\ g_1,\ \ell_1, \ldots,\ g_{s-1}, \ell_{s-1} = 0,\ \underline{0}\,)\ , & s \geqslant 2, \end{cases} \tag{2.9}$$

wherein ℓ_{s-1} and the last $2(m-s)$ entries are equal to 0.

II.j Step. Otherwise $m - (c_0 + \ldots + c_{s-1}) = \ell_{s-1} > 0$, and define for $H \in \mathbb{Q}_{>1}$ a set $\Lambda_H := \{(\alpha_0, \ldots, \alpha_{s-1}, \beta)\} \subseteq \mathbb{Z}_{\geqslant 0}^{s+1}$ by the rules:

(R.1) $H \cdot \big(g_{s-1}^0\, \alpha_0 + \ldots + g_{s-1}^{s-1}\, \alpha_{s-1}\big) + \beta \geqslant H \cdot \big(g_{s-1}^0\, d\big)$;

(R.2) $g_{s-1}^0\, \alpha_0 + \ldots + g_{s-1}^{s-1}\, \alpha_{s-1} < g_{s-1}^0\, d$.

Now observe that by (R.2) the possibilities for (α_i) are finite, so if (R.1) is an actual equality for some H then the denominator of H is bounded. It therefore makes sense to introduce

II.k Fact/Definition. The discrete set of sub-inductive parameters $\Theta_{s-1}(I,A)$, contained in $\mathbb{Q}_{>1}$, is the subset of $H \in \mathbb{Q}_{>1}$ where equality occurs in (R.1) for some tuple of integers satisfying (R.2), and its predecessor $h = h(H)$ is the minimum of $\Theta_{s-1} \cap \mathbb{Q}_{<H}$ or 1 if H is already the minimum of Θ_{s-1}.

Better still, observe,
II.l Fact. Let $\underline{g} = \underline{g}(I,A)$ be as in II.g.(F.5), and $h_s = h_s(I,A)$ any function taking values in the set $\Theta_{s-1}(I,A)$ of sub-inductive parameters in II.k, then $\underline{g} \times h_s$ is a function of rings and their ideals with self bounding denominators.

Proof. By the definition of h_s there are non-negative integers α_i and a positive integer β such that II.j.(R.1) is an equality. In addition there are $D_i : \mathbb{Q}^i_{\geqslant 0} \longrightarrow Z_{\geqslant 0}$, $0 \leqslant i \leqslant s-1$ self bounding the denominators of $\underline{g}$ in the sense of II.a. Consequently we must have,

$$\big(D_0 \cdots D_{s-1}\big)(\underline{g})\beta = h_s N$$

where $N \in \mathbb{Z}_{>0}$ is an integer no greater than

$$dg^0_{s-1}\big(D_0 \cdots D_{s-1}\big)(\underline{g}) \tag{2.10}$$

so D_s the factorial of (2.10) will do. □

Having cleared any scruples about denominators, consider the following,
II.m Sub-Induction $(H \in \Theta_{s-1})$**.** For $h = h(H)$ the predecessor of H, and $h^i_{s-1} = h \cdot g^i_{s-1}$, $0 \leqslant i \leqslant s-1$, there is a weighted filtration $F^{\bullet}_{s-1}(h)$ depending only on I, in which all of II.g.(F.0)-(F.3) hold but with h^i_{s-1} instead of g^i_{s-1}.

Plainly the sub-induction II.m begins with $F^{\bullet}_{s-1}(1) = F^{\bullet}_{s-1}$, while by corollary I.g.(iii) each block X^i_{s-1}, $0 \leqslant i \leqslant s-1$, is (up to a weighted projective transformation in the X^t_{s-1}, $0 \leqslant t < i \leqslant s-1$) well defined modulo $F^{h^i_{s-1}}_{s-1}(h)$. As such if $\widetilde{X}^i_{s-1}$ and $\widehat{X}^i_{s-1}$ are any two liftings of the i-th block to A, then

$$\widetilde{X}^i_{s-1} = \widehat{X}^i_{s-1} \qquad \text{mod } F^{>h^i_{s-1}}_{s-1}(h) \tag{2.11}$$

and we assert that for H as in II.m,
II.n Lemma. If $\widetilde{X}^i_{s-1}$, $0 \leqslant i \leqslant s-1$, is a lifting of the blocks from $\mathrm{gr}^{(h)}_{s-1}A$ (*cf.* 2.6), and $\widetilde{X}^s_{s-1}$ some choice of completing this to a system of coordinates, then the new filtration, $F^{\bullet}_{s-1}(H)$ say, defined by the weights

$$\begin{aligned} \mathbf{wt}_H(\widetilde{X}^i_{s-1}) &= H \cdot g^i_{s-1}, \text{ for } 0 \leqslant i \leqslant s-1, \\ \mathbf{wt}_H(\widetilde{X}^s_{s-1}) &= 1\,, \end{aligned} \tag{2.12}$$

does not depend on the aforesaid choices.

Proof. To this end, by (2.11), it is sufficient to prove
II.o Claim. $f \in F^{>hg^i_{s-1}}_{s-1}(h) \Longrightarrow \mathbf{wt}_H(f) \geqslant H \cdot g^i_{s-1}$, *i.e.* $f \in F^{Hg^i_{s-1}}_{s-1}(H)$.

Proof. By hypothesis f is contained in the ideal, $F^{>hg^i_{s-1}}_{s-1}(h)$, generated by monomials with total degrees α_i, resp. β, for the blocks X^i_{s-1}, $0 \leqslant i \leqslant s-1$, resp. X^s_{s-1}, such that:

$$h \cdot \big(g^0_{s-1}\,\alpha_0 + \dots + g^{s-1}_{s-1}\,\alpha_{s-1}\big) + \beta \;>\; h \cdot g^i_{s-1}; \tag{2.13}$$

while from the definition of the integers d^i_i, II.g-(F.4),

$$g^0_i\,d^i_0 + g^1_i\,d^i_1 + \dots + g^{i-1}_i\,d^i_{i-1} + (d^i_i - 1) = g^0_i\,d^0_0 - 1 \tag{2.14}$$

so multiplying this by g^i_{s-1} we get

$$g^0_{s-1}\,d^i_0 + g^1_{s-1}\,d^i_1 + \dots + g^{i-1}_{s-1}\,d^i_{i-1} + g^i_{s-1}\,(d^i_i - 1) = g^0_{s-1}\,d^0_0 - g^i_{s-1} \tag{2.15}$$

then multiplying (2.15) by h and adding it to (2.13) gives:

$$\begin{aligned} h \cdot \Big(g^0_{s-1}\,(\alpha_0 + d^i_0) + \dots + g^{i-1}_{s-1}\,(\alpha_{i-1} + d^i_{i-1}) + g^i_{s-1}\,(\alpha_i + d^i_i - 1) +\\ g^{i+1}_{s-1}\,\alpha_{i+1} + \dots + g^{s-1}_{s-1}\,\alpha_{s-1}\Big) + \beta > h \cdot g^0_{s-1}\,d^0_0 \end{aligned} \tag{2.16}$$

so from the definition of $h = h(H)$, II.k,

$$\begin{aligned} H \cdot \Big(g^0_{s-1}\,(\alpha_0 + d^i_0) + \dots + g^{i-1}_{s-1}\,(\alpha_{i-1} + d^i_{i-1}) + g^i_{s-1}\,(\alpha_i + d^i_i - 1) +\\ g^{i+1}_{s-1}\,\alpha_{i+1} + \dots + g^{s-1}_{s-1}\,\alpha_{s-1}\Big) + \beta \geqslant H \cdot g^0_{s-1}\,d^0_0. \end{aligned} \tag{2.17}$$

Now multiply (2.15) by H and subtract from 2.17 to get

$$H \cdot g^0_{s-1}\,\alpha_0 + \dots + H \cdot g^{s-1}_{s-1}\,\alpha_{s-1} + \beta \;\geqslant\; H \cdot g^i_{s-1}\,, \tag{2.18}$$

wherein the left hand side is the monomial's weight in the new H-filtration. □

Which in turn complete the poof of II.n . □

Now in the new filtration $F^\bullet_{(s-1)}(H)$, *i.e.* the filtration obtained from $F^\bullet_{(s-1)}(h)$ of (2.12) (and unambiguously by II.n), define

$$V^d_{s-1}(H) := I \mod F^{>Hg^0_{s-1}\,d}_{s-1}(H), \tag{2.19}$$

then one of the following must occur,

II.p Case(A). $\mathrm{L}\big(V^d_{s-1}(H)\big)$ (*cf.* I.d) does not have maximal dimension, *i.e.*

$$\dim \mathrm{L}\big(V^d_{s-1}(H)\big) = \ell_s < m_s := m - (c_0 + \dots + c_{s-1}).$$

Then by corollary I.g applied to $\mathrm{P}_k\big(Ha^0_{s-1}, \dots, Ha^{s-1}_{s-1}, \underline{1}\big)$, (1.6), there is a filtration satisfying (F.1)-(F.4) of II.g but with blocks X^i_s, $0 \leqslant i < s$, respectively X^s_s, liftings of the blocks X_i, respectively Z, *i.e.* the annihilator of $\mathrm{L}\big(V^d_{s-1}(H)\big)$ in corollary I.g, and $c_s = m_s - \ell_s$ while $g_{s+1} = H$ in II.g.(F.4), *i.e.* $g^i_s = H \cdot g^i_{s-1}$ with $g^s_s = 1$.

II.q Case(B). $\mathrm{L}\big(V^d_{s-1}(H)\big)$ has maximal dimension, so its annihilator in corollary I.g, Z, is the empty set. Nevertheless, *op.cit.* still applies to give new liftings, X^i_s, $0 \leqslant i \leqslant s-1$, of the blocks X_i (of *op.cit.* applied to $\mathrm{P}_k(Ha^0_{s-1}, \dots, Ha^{s-1}_{s-1}, \underline{1})$), such that the sub-inductive hypothesis II.m is valid for the successor of H in Θ_{s-1}.

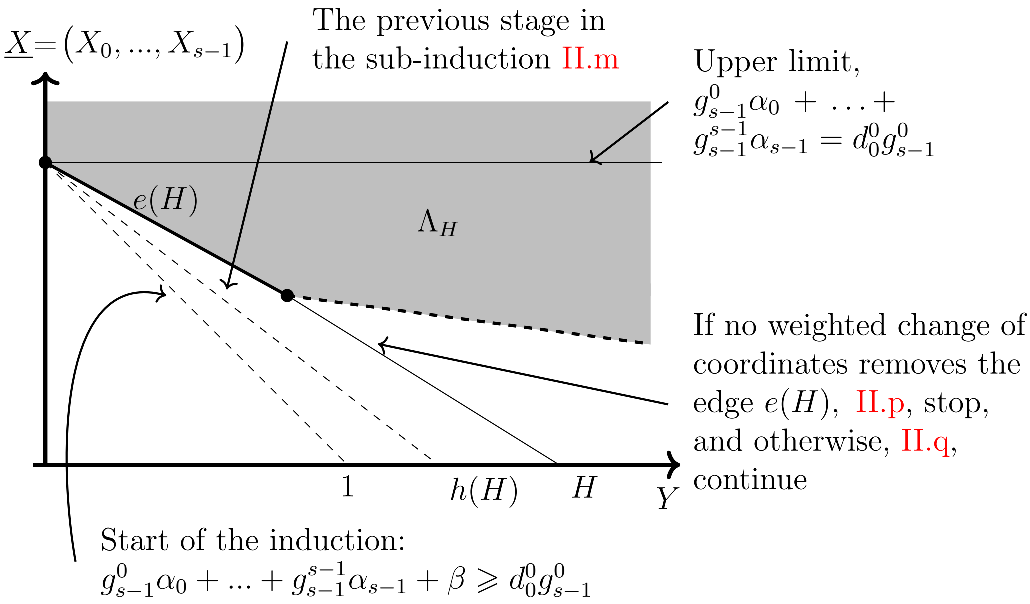

Figure 1: Newton Polyhedron for Sub-Induction II.m.

II.r Partial Finish. In case (A) ,II.p, the sub-induction II.m has terminated, and we have found our new filtration $F^\bullet_s$, to wit $F^\bullet_{s-1}(H)$, so that the induction now continues in s.

Otherwise in case (B), II.q, we either eventually fall into case (A), II.p, and, again, terminate the sub-induction, II.m, or we repeat case (B), II.q, *ad infinitum*. Suppose, therefore,

II.s Hypothesis. Case II.q occurs *ad infinitum*.

Such repetition is indexed by the possible h in Θ_{s-1} of II.k and we continue to denote by H its successor. Our aim is to calculate the coordinates $X^i_{s-1}(H)$ of I.g.(i) (whose liftings will be, again, the blocks $X^i_{s-1}(H)$) and, because we are in case (B), II.q, the relationship with the old coordinates $X^i_{s-1}(h)$ is given by:

$$X^i_{s-1}(H) - X^i_{s-1}(h) \;\in \underline{\mathrm{Sym}}^{hg^i_{s-1}}\big(X(h)\,\mathrm{II}\,\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}),\, \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{1})\big)\big), \tag{2.20}$$

where $X(h)$ is the space of function generated by $X^i_{s-1}(h)$, for every $0 \leqslant i \leqslant s-1$, and $\mathbb{P}_k(\underline{a}) = \mathrm{P}_k\big(h\underline{g}^0_{s-1}, \ldots, h\underline{g}^{s-1}_{s-1}, \underline{1}\big)$, *cf.* (1.6) & (1.16). Now without loss of generality we have equality modulo $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{1})\big)$, *i.e.* the projection of $X^i_{s-1}(H) - X^i_{s-1}(h)$ onto $\mathrm{Sym}^\bullet\big(X(h)\big)$ is always zero, thus, $X^i_{s-1}(H) - X^i_{s-1}(h)$ is a combination of monomials

$$X(h)^E \cdot Y^Q\,, \tag{2.21}$$

where $X(h)^E = \prod_i X_i(h)^{E_i}$, respectively $Y^Q = Y_1^{q_1} \ldots Y_{c_s}^{q_{c_s}}$, coming from $X(h)$ alone, respectively $\mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(\underline{1})\big)$ alone, and by construction

$$h\, g^i_{s-1} = h\,\mathbf{wt}(E) + Q, \tag{2.22}$$

where $\mathbf{wt}(E) = g^0_{s-1}|E_0| + \ldots + g^{s-1}_{s-1}|E_{s-1}|$, $Q = q_1 + \ldots + q_{\ell_{s-1}}$. Therefore, $Q =$

$h\,(g^i_{s-1} - \mathbf{wt}(E))$ while $e := \min_E\{g^i_{s-1} - \mathbf{wt}(E) > 0\}$ is attained since the weights of the $F^\bullet_{s-1}$ filtration are a discrete set. Thus $Q \geqslant h\,e$ and the right hand side of (2.22) tends to infinity. Consequently the $X^i_{s-1}(h)$ are a Cauchy sequence in the $\mathfrak{m}$-adic topology, so if II.s were to occur,

II.t Fact/Proposition. The filtrations $F^\bullet_{s-1}(h)$, $h \in \Theta_{s-1}$ converges $\mathfrak{m}$-adically as $h \to \infty$ to a filtration $F^\bullet(I)$ determined uniquely by I consisting of blocks X^i_{s-1} of weights $\mathbf{wt}(X^i_{s-1}) = g^i_{s-1}$ and cardinality c_i, where $m_s = m - (c_0 + \ldots + c_{s-1}) > 0$.

II.u Conclusion. Should the sub-induction, II.m, eventually not terminate. *i.e.* , II.s, then we arrive to a filtration $F^\bullet(I)$ of the completion $\widehat{A}$ of A in $\mathfrak{m}$ (depending only on I) with blocks X^i_{s-1} of cardinality $c_0, \ldots, c_{s-1}$ together with weights $g^0 > \ldots > g^{s-1}$, satisfying (F.1)-(F.4) of II.g and we define:

$$\mathrm{inv}(I) = \big(d, \ell_0, g_1, \ell_1, \ldots, g_{s-1}, \ell_{s-1}, \underline{0}\big) \in \mathbb{Q}^{2m}_{\geqslant 0} \tag{2.23}$$

wherein the last block $\underline{0}$ has length $2(m-s)$. Otherwise, case (A), II.p, applies for all s and the invariant is eventually defined by (2.9).

Finally it is appropriate to explicitly observe the behaviour under regular maps beginning with,

II.v Fact. The formation of the invariant is étale local, in fact better for $\widehat{A}$ the completion of our regular local ring A of II.c, and $\widehat{I} := I \otimes_A \widehat{A}$ we have,

(i) $\mathrm{inv}_A(I\,) = \mathrm{inv}_{\widehat{A}}(\widehat{I}\,)$;

(ii) If $F^\bullet(I)$, resp. $F^\bullet(\widehat{I})$, is the filtration whether of A or $\widehat{A}$ resulting whether from the termination of the induction, II.h, or the sub-induction, II.m, running *ad infinitum*, II.s, then

$$F^\bullet(\widehat{I}) = \begin{cases} F^\bullet(I)\,, & \text{should II.s occur,} \\ F^\bullet(I) \otimes_A \widehat{A}\,, & \text{otherwise.} \end{cases} \tag{2.24}$$

Proof. In the situation of the inductive hypothesis II.g,

$$\mathfrak{m}^N \subset F^N_{s-1} \quad \text{and} \quad F^p_{s-1} \subset \mathfrak{m}^{p/g^0_{s-1}}\,,$$

so if II.s never occurs everything is determined modulo a sufficiently large power of the maximal ideal, and both items (i) & (ii) are trivial. Otherwise if II.s occurs then the conclusion II.u and the reasons for it (2.21)-(2.22) are $\mathfrak{m}$-adic by definition, so this is trivial too. □

In the same vein we may prepare for replacing étale by regular via,

II.w Lemma. Suppose $B = A[\![z_1, \ldots, z_\epsilon]\!]$ is a formal power series ring over A; J the pull-back of I to A with $\widehat{A}, \widehat{B}, \widehat{I}, \widehat{J}$ their completions in the maximal ideal of A, then:

(i) The odd entries of $\mathrm{inv}_B(J)$ and $\mathrm{inv}_A(I)$ agree.

(ii) Even entries where the invariant is zero agree, and otherwise the difference $\mathrm{inv}_B(J) - \mathrm{inv}_A(I)$ at an even entry is ϵ.

(iii) The filtrations (2.24) are related by, $F^\bullet(\widehat{J}) = F^\bullet(\widehat{A}) \otimes_{\widehat{A}} \widehat{B}$.

Proof. By induction in the parameter s, we assert that the relation between the graded rings $\mathrm{gr}_{s-1}A$, $\mathrm{gr}_{s-1}B$ of (2.6) is,

$$\mathrm{gr}_{s-1}B = \mathrm{gr}_{s-1}A \otimes_k k[z_1, ..., z_\epsilon] \tag{2.25}$$

while in the sub-induction II.m, the maximal contact spaces $\mathrm{L}_B\big(V^d_{s-1}(H)\big)$, resp. $\mathrm{L}_A\big(V^d_{s-1}(H)\big)$ are related by,

$$\begin{aligned} \mathrm{L}_B\big(V^d_{s-1}(H)\big) &= \mathrm{L}_A\big(V^d_{s-1}(H)\big) \amalg k \otimes_A \mathrm{Der}_A(B) \\ &= \mathrm{L}_A\big(V^d_{s-1}(H)\big) \coprod_{1\leqslant j \leqslant \epsilon} k \frac{\partial}{\partial z_j} \end{aligned} \tag{2.26}$$

Indeed for $s = 1$, (2.25) is obvious, while for any $s \geqslant 1$, (2.25) $\Rightarrow$ (2.26) since the $\frac{\partial}{\partial z_j}$ always vanish on generators of I so the right hand side of (2.26) is always contained in the left, while modulo the $\frac{\partial}{\partial z_j}$ they are plainly equal. Consequently in case A of the sub-induction, II.p, (2.26) implies (2.25) for s, while in case B, II.q, the convergence is actually modulo the pull-back of the maximal ideal of A, equivalently the filtration is pulled back from $\widehat{A}$. □

III The Invariant on Weighted Projective Champ

III.a Set Up. Let $\mathbb{P}_k(\underline{a}) = \mathbb{P}(\underline{a}^0, ..., \underline{a}^s)$ be a $(m-1)$-dimensional weighted projective champ, with blocks of coordinates $X_0, ..., X_s$ of weights $a^0 > ... > a^s$ and cardinality $c_0, ..., c_s$ over a field k of characteristic zero. Suppose further that $d \in \mathbb{Z}_{>0}$ and $V \subset \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(da^0)\big) =: S_{da^0}$ is a space of sections such that:

III.b Hypothesis. If for every $s \geqslant i > 0$, $\mathrm{P}_i \hookrightarrow \mathbb{P}_k(\underline{a})$ is the weighted projective sub-champ defined by $X_i = ... = X_s = 0$, and S^i the associated $\mathbb{G}_m$-equivariant graded algebra of homogeneous functions, as in (I.c), with for convenience of notation $\mathrm{P}_{s+1} = \mathbb{P}_k(\underline{a})$ & $S^{s+1} := S$, then

$$\mathrm{L}_i(V) := \coprod\nolimits_{-b<0} \big\{\partial \in \mathrm{Der}_{<0}(S^i)(-b) \,\big|\, \partial(V_i) = 0\big\} = 0 \tag{3.1}$$

where V_i is the image of V in $S^i_{da^0} = \mathrm{H}^0(\mathrm{P}_i, \mathscr{O}_{\mathrm{P}_i}(da^0))$.

Now for consistency with II.k and II.g.(F.4), define $g_i := a^{i-1}/a^i$, $1 \leqslant i \leqslant s$, and $\ell_i = m - (c_0 + ... + c_i)$ then we assert,

III.c Proposition. If I is the sheaf of ideals generated by V, under the non-degeneracy condition III.b, then for every geometric point p of $\mathbb{P}_k(\underline{a})$ the value of the invariant $\mathrm{inv}_{\mathbb{P}_k(\underline{a})}(I)(p)$ at the stalk I_p is *strictly* less than

$$\big(d, \ell_0, g_1, \ell_1, ..., g_s, \ell_s, \underline{0}\big). \tag{3.2}$$

More precisely, if $\mathrm{inv}_{\mathbb{P}_k(\underline{a})}(I)(p) = \big(\mathrm{mult}_I(p), \ell_0(p), g_0(p), ..., \ell_s, \underline{0}\big)$ with $\ell_i(p) = m - (c_0(p) + ... + c_i(p))$ and $0 \leqslant \sigma \leqslant s$ is maximal such that $X_\sigma(p) \neq 0$, (*i.e.* there is some $1 \leqslant i \leqslant c_\sigma$, for which $x_{\sigma i}(p) \neq 0$) then:

(i) If $\sigma = 0$ the multiplicity of I at p is strictly less than d, unless $d = 0$.

(ii) If $\sigma > 0$ with, for immediate notational convenience, $g_0 = d$ and all of $g_i(p) \geqslant g_i$, $c_i(p) \leqslant c_i$, for any $0 \leqslant i \leqslant \sigma - 2$ then $g_i(p) = g_i$ and $c_i(p) = c_i$ for all $0 \leqslant i \leqslant \sigma - 2$.

(iii) If (ii) holds and $g_{\sigma-1}(p) \geqslant g_{\sigma-1}$, $c_{\sigma-1}(p) \leqslant c_{\sigma-1}$, then $g_{\sigma-1}(p) = g_{\sigma-1}$, $c_{\sigma-1}(p) = c_{\sigma-1}$, $c_\sigma \geqslant 2$, and $g_\sigma(p) < g_\sigma$; so in particular if $c_\sigma = 1$ then $g_{\sigma-1}(p) < g_{\sigma-1}$, *i.e.* $g_{\sigma-1}(p)$ goes down.

Observe that we can immediately reduce to $\sigma = s$ since,

III.d Lemma. Let $\mathcal{Q}$ be a sub-champ of $\mathbb{P}_k(\underline{a})$ containing the geometric point p and such that III.c.(i) holds, for $I\big|_{\mathcal{Q}}$, while denoting by a superscript $\mathcal{Q}$ the values of the blocks associated to the invariant of $I\big|_{\mathcal{Q}}$ calculated at p, items (ii) & (iii) of *op.cit.* hold, albeit, in the modified form:

(ii-bis) If $\sigma > 0$, $g_i(p) \geqslant g_i$, $c_i^{\mathcal{Q}}(p) \leqslant c_i$, for any $0 \leqslant i \leqslant \sigma - 2$, then $g_i(p) = g_i$, $c_i^{\mathcal{Q}}(p) = c_i$, for any $0 \leqslant i \leqslant \sigma - 2$.

(iii-bis) If (ii-bis) holds and $g_{\sigma-1}(p) \geqslant g_{\sigma-1}$, $c_{\sigma-1}^{\mathcal{Q}}(p) \leqslant c_{\sigma-1}$, then $g_{\sigma-1}(p) = g_{\sigma-1}$, $c_{\sigma-1}^{\mathcal{Q}}(p) = c_{\sigma-1}$, $c_\sigma \geqslant 2$, and $g_\sigma(p) < g_\sigma$; so in particular if $c_\sigma = 1$ then $g_{\sigma-1}(p) < g_{\sigma-1}$.

Proof. For the multiplicity $d = g_0$ this is clear, while c_0 is the minimum number of coordinates required to describe the ideal modulo $\mathfrak{m}^{d+1}(p)$, so its ambient value $c_0(p)$ is always at least that, $c_0^Q(p)$, of a subspace whenever the multiplicity of the intersection coincides. Consequently if

$$c_0 \geqslant c_0(p) \text{ and } \big(c_0 \geqslant c_0^Q(p) \Longrightarrow c_0^Q(p) = c_0 \big) \text{ then } c_0(p) = c_0. \tag{3.3}$$

Similarly the presence of a non-zero gradient g_r, $1 \leqslant r \leqslant \sigma$ reflects the necessity, or otherwise, I.g, of a new block of coordinates to describe the leading monomials in generators of the ideal, so if one needs a block after intersecting with a sub-widget one certainly needed it before hand, and should this occur $c_i^Q(p) = c_i$ will imply $c_i(p) = c_i$ exactly as in (3.3). □

In particular, therefore, after III.d, and the definition of σ it is sufficient to prove III.c on the subspace $X_{\sigma+1} = \ldots = X_s = 0$, so without loss of generality $\sigma = s$.

Proof of Proposition III.c. We proceed by induction on the number of blocks, s, starting from $s = \sigma = 0$. In this case by the action of PGL_{c_0} we may, without loss of generality suppose p is the point $[1 : 0 : \ldots : 0] \in \mathbb{P}_k^{m-1}$, in some basis $\{x_1, \ldots, x_m\}$. Consequently if the multiplicity does not go down Z of I.g is contained in the subspace generated by $x_2, \ldots, x_m$ which contradicts the definition of ℓ_0 (*i.e.* 0 under the present hypothesis) in II.f unless d were already 0.

Supposing, therefore, that $\sigma = s > 0$ let us adjust the notation accordingly by denoting the final block X_s as Y which in turn is a basis of $\mathrm{H}^0(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(a^s))$, which we write as $Y = \{y\} \cup Z$ where

$$y(p) = 1, \qquad z(p) = 0, \ \forall\, z \in Z. \tag{3.4}$$

In particular, therefore, we have an étale neighbourhood U of p obtained by slicing the groupoid $R := \mathbb{G}_m \times \mathbb{A}^m \setminus \{\underline{0}\} \rightrightarrows \mathbb{A}^m \setminus \{\underline{0}\}$ along the transversal $y = 1$, and we write the coordinate functions on U afforded by the elements of the blocks X_i as $x_{ij} + p_{ij}$, $0 \leqslant i \leqslant s-1$, $1 \leqslant j \leqslant c_i$, *i.e.*

$$U \ni p = \prod_{i=0}^{s-1} \underline{p_t} \times 1 \times \underline{0}, \quad \text{where } \underline{p_t} = p_{t1} \times \ldots \times p_{tc_t}. \tag{3.5}$$

In this notation the correspondence between a global section, $f(X_0, \ldots, X_{s-1}, Y)$ in $\underline{\mathrm{Sym}}^{da^0}\big(X_0 \amalg \ldots X_{s-1} \amalg Y\big) = \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(da^0)\big) = S_{da^0}$ and the associated function is simply

$$f \longmapsto f(x_{ij} + p_{ij},\, 1,\, \underline{z}) \in \Gamma(U, \mathscr{O}_{\mathbb{P}_k(\underline{a})}), \quad \text{for } 0 \leqslant i \leqslant s-1 \text{ and } 1 \leqslant j \leqslant c_i. \tag{3.6}$$

Furthermore, and needless to say, U is an affine space with origin p via,

$$\left(\prod_{i=0}^{s-1} \prod_{j=1}^{c_{s-1}} x_{ij} \right) \times \underline{z} = \ U \longrightarrow \mathbb{A}^{m-1}. \tag{3.7}$$

so it makes perfect sense to talk about the maximal degree in the blocks of functions $\underline{x}_t := \{x_{ti} \,|\, 1 \leqslant i \leqslant c_t\}$, $0 \leqslant t \leqslant s-1$. With this in mind we assert,

III.e Claim. The initial $2s$-part of the invariant $(g_0, \ell_0, g_1, \ell_1, ..., g_{s-1}, \ell_{s-1})$ cannot increase.

Proof. By induction in s. The starting point of the multiplicity $d = g_0$ is particular. Modulo the local functions x_{ij}, $i \geqslant 1$, $\underline{z}$, at p we have an affine space $\mathbb{A}^{c_0}$ on which the multiplicity is at most the degree in the block of functions $\underline{x}_0$ which is at most the degree in global block X_0, *i.e.* d. Furthermore were this bound to be achieved on U then the restriction I to $\mathbb{A}^{c_0}$ at p is, under the isomorphism afforded by: $X_{\bullet j} \longmapsto \underline{x}_{\bullet j}$, exactly the ideal generated under,

$$\Gamma(\mathbb{A}^m \setminus \{\underline{0}\}) = \Gamma(\mathbb{A}^m) \xrightarrow{\text{mod } X_i} \Gamma(\mathbb{A}^{c_0}), \quad i \geqslant 1 \tag{3.8}$$

at the origin, so $c_0(p) \geqslant c_0$.

Now we put ourselves in the scenario of the inductive hypothesis II.f.(F.0)-(F.4), albeit with an inductive parameter $0 \leqslant t \leqslant s-1$, rather than $s-1$ of *op.cit.* , and we add to the hypothesis:

(F.4 bis) The i^{th}-block, $0 \leqslant i \leqslant t$, is defined by the block of functions $\underline{x}_i$ and has weight $a^i/a^t = g_t^i$ (in notation of II.g.(F.4) .

Quite possibly we arrive in case (A), II.p, for a value of $H < a^t/a^{t+1}$, but, plainly should this occur then the invariant strictly decreases. If, however, we were to continue in case (B), II.q, for every $H < a^t/a^{t+1}$ by way of changes of coordinates in the blocks $\underline{x}_i$, $0 \leqslant i \leqslant t$, then this in no way changes monomials of the form

$$\underline{x}_0^{D_0} \cdots \underline{x}_{t+1}^{D_{t+1}}, \quad a^0|D_0| + ... + a^{t+1}|D_{t+1}| = a^0 d \tag{3.9}$$

since the weight of the perturbation in $\underline{x}_i$ will be

$$H \cdot \left(a^i / a^t\right) < a^i / a^{t+1}. \tag{3.10}$$

Consequently were we to eliminate all $H < a^t/a^{t+1}$, modulo $\underline{x}_i$, $i > t+1$ we would find that, mod $\underline{x}_i$, $i > t+1$, the ideal at p is exactly that generated at the origin by the image of V in the origin obtained via the isomorphism

$$\Gamma(\mathbb{A}^m \setminus \{\underline{0}\}) = \Gamma(\mathbb{A}^m) \xrightarrow[\text{mod } X_i]{\sim} \Gamma(\mathbb{A}^{c_0+...+c_{t+1}}), \quad i > t+1; \tag{3.11}$$

so the claim follows from I.g, as employed in the definition of the invariant in case (A), II.p . ☐

Suppose therefore that the extremal situation of III.e is attained (*i.e.* the invariant did not decrease), then from our original blocks of coordinates, $\underline{x}_i$, $0 \leqslant i \leqslant s-1$, $\underline{z}$

we will have performed a change of coordinates to blocks of the form

$$\begin{array}{rcll} \underline{\xi}_0 &=& \underline{x}_0 + \underline{\epsilon}_0(\underline{x}_1, ..., \underline{x}_{s-1}, \underline{z}), & \\ \underline{\xi}_1 &=& \underline{x}_1 + \underline{\epsilon}_1(\underline{x}_2, ..., \underline{x}_{s-1}, \underline{z}), & \mathbf{wt}_{\underline{x}}(\underline{\epsilon}_i) < a^i, \\ \vdots & & \vdots & \text{for } \mathbf{wt}(x_i) = a^i; \\ \underline{\xi}_{s-1} &=& \underline{x}_{s-1} + \underline{\epsilon}_{s-1}(\underline{z}); & \end{array} \tag{3.12}$$

resulting in a filtration $F^{\bullet}_{\underline{\xi}}$ around p in which the blocks $\underline{\xi}_i$, $0 \leqslant i \leqslant s-1$ have weights a^i/a^s, $\underline{z}$ has weight 1, and around p the ideal generated by V belongs to $F_{\underline{\xi}}^{a^0 d/a^s}$. In particular

III.f Warning. We allow the possibility that the sub-induction II.h may still not have terminated in case II.p and whence the invariant might even go up.

To analyse this situation we replace the coordinates x_{ij} around p by the restriction to U of the $\mathbb{G}_m$-equivariant global coordinate functions X_{ij}, $0 \leqslant i \leqslant s-1$, $1 \leqslant j \leqslant c_i$ in the various block, so that (3.12) becomes,

$$\begin{array}{rcl} \underline{\xi}_0 &=& \left(\underline{X}_0 - \underline{\varepsilon}_0(X_1, ..., X_{s-1}, Z)\right)\big|_U\ , \\ \underline{\xi}_1 &=& \left(\underline{X}_1 - \underline{\varepsilon}_1(X_2, ..., X_{s-1}, Z)\right)\big|_U\ , \qquad \mathbf{wt}_X(\underline{\varepsilon}_i) < a^i, \\ \vdots & & \vdots \\ \underline{\xi}_{s-1} &=& \left(\underline{X}_{s-1} - \underline{\varepsilon}_{s-1}(Z)\right)\big|_U\ ; \end{array} \tag{3.13}$$

and we assert

III.g Claim. In the above notation and under the hypothesis (*cf.* claim III.e) that the first $2s$ terms in the invariant at p are at least $\left(d, \ell_0, g_1, \ell_1, ..., g_{s-1}, \ell_{s-1}\right)$ the coordinate change (3.13) is global, *i.e.* there are homogeneous functions $\underline{G}_i$ on $\mathbb{A}_k^{m-1}$ of weight a^i such that,

$$\underline{\varepsilon}_i(X_{i+1}, ..., X_{s+1}, Z)\big|_U = \underline{G}_i(X_{i+1}, ..., X_{s+1}, Z). \tag{3.14}$$

Proof. We have filtrations in which the blocks X_i, $0 \leqslant i \leqslant s-1$, $X_s = \{Z, Y\}$, respectively $\underline{\xi}_i$, $\underline{z}$, with weights a^i, $0 \leqslant i \leqslant s-1$, a^s, may *a priori* be different and so we will employ the notation $\mathbf{wt}_X$, resp. $\mathbf{wt}_\xi$, to avoid ambiguity. In any case for $f \in V_d$, we have from (3.13):

$$\begin{aligned} f\big|_U = f(X_0, ..., X_{s-1}, 1, Z)\big|_U &= f\left(\underline{\xi}_0 + \underline{\varepsilon}_0, ..., \underline{\xi}_{s-1} + \underline{\varepsilon}_{s-1}, \underline{1}, \underline{z}\right) = \\ f\left(\underline{\xi}_0, ..., \underline{\xi}_{s-1}, \underline{1}, \underline{z}\right) + \sum_{i=0}^{s-1}\left(\frac{\partial f}{\partial X_i}\underline{\varepsilon}_i\right)&\left(\underline{\xi}_0, ..., \underline{\xi}_{s-1}, \underline{1}, \underline{z}\right) + \text{ stuff,} \end{aligned} \tag{3.15}$$

wherein $\frac{\partial f}{\partial X_i}\underline{\varepsilon}_i = \sum_{j=1}^{c_i} \frac{\partial f}{\partial x_{ij}}\varepsilon_{ij}$ and stuff has smaller weight in the $\underline{\xi}$-filtration than the expected top weight in

$$\left(\sum_{i=0}^{s-1} \frac{\partial f}{\partial X_i}\underline{\varepsilon}_i^{\text{top}}\right)\left(\underline{\xi}_0, ..., \underline{\xi}_{s-1}, \underline{1}, \underline{z}\right) \tag{3.16}$$

to wit: $(da^0) - \min_{0\leqslant i\leqslant s-1}\left\{a^i - \mathbf{wt}_\xi\left(\underline{\varepsilon}_i^{\mathrm{top}}\right)\right\}$, where $\underline{\varepsilon}_i^{\mathrm{top}}$ are the monomials in $\underline{\xi}, \underline{z}$ in $\underline{\varepsilon}_i$ which have maximal $\underline{\xi}$-weight,

$$\underline{\varepsilon}_i^{\mathrm{top}} := \sum_D \lambda_D\, \underline{\xi}_0^{D_0}\cdots\underline{\xi}_{s-1}^{D_{s-1}}\,\underline{z}^{D_s} + \text{ stuff}, \tag{3.17}$$

where, again, stuff is monomials with lower ξ-weight. Let us therefore define homogenous functions on the ambient space, $\mathbb{A}_k^{m-1}$ by way of the formula:

$$\Delta_i := \sum\nolimits_{D_i} \lambda_{D_i}\, X_0^{D_0}\cdots X_{s-1}^{D_{s-1}} Z^{D_s}, \tag{3.18}$$

and a homogeneous vector field,

$$\mathrm{D} = \sum_{i=0}^{s-1}\Delta_i\,\frac{\partial}{\partial X_i} \qquad \text{of } \ \mathbf{wt}_X(\mathrm{D}) = -\min_{0\leqslant i\leqslant s-1}\{\,a^i - \mathbf{wt}_\xi(\underline{\varepsilon}_i^{\mathrm{top}})\,\}. \tag{3.19}$$

So that by construction and (3.13), (3.16) vanishes if and only if the top weight term in the grading of $\Gamma(\mathscr{O}_U)$ which assigns to $X_i\big|_U$ weight a^i, $0\leqslant i\leqslant s-1$, and to $Z\big|_U$ weight a^s of every $\mathrm{D}(f)\big|_U$ vanishes for every $f\in V_d$. Thus, *a fortiori*, on the weighted projective hypersurface $\mathcal{Q}$, defined by the function $Y=0$,

$$\mathrm{D}(f) = 0 \mod Y, \qquad \forall\, f\in V_d. \tag{3.20}$$

As such there are two cases: either $Z\neq\emptyset$, then since D acts trivially on $\mathrm{H}^0\big(\mathcal{Q},\mathscr{O}_{\mathcal{Q}}(a^s)\big)$ by (3.19), $\mathrm{D}=0 \mod Y$ by III.b and I.g.(ii); or $Z=\emptyset$ and $\mathrm{D}=0 \mod Y$ by the non-degeneracy hypothesis III.b and I.g.(ii). In either case $\mathrm{D}=0 \mod Y$, and whence all the $\Delta_i\equiv 0$ by virtue of their definition (3.18), which in turn is nonsense (unless claim III.g is true with $\underline{\varepsilon}_i = \underline{G}_i = 0$). Thus the top weight term in (3.16) is not zero for some $f\in V_d$. However for such a f, according to our hypothesis that the invariant does not decrease, the top $\underline{\xi}-$weight term in (3.16) must cancel with the top $\underline{\xi}-$weight of

$$f\big(\underline{\xi}_0,\dots,\underline{\xi}_{s-1},\underline{1},\underline{z}\big) \mod F_{\underline{\xi}}^{a^0 d}, \tag{3.21}$$

which in turn has weight, $a^0d - a^sn$, for some integer n. We therefore conclude,

$$a^0\,d - a^s\,n = a^0\,d - \min_{0\leqslant i\leqslant s-1}\{\,a^i - \mathbf{wt}_X(\Delta_i)\,\}, \tag{3.22}$$

i.e. for $0\leqslant i\leqslant s-1$ where the minimum in (3.22) is attained,

$$a^i = \mathbf{wt}_X(\Delta_i) + a^s\,n\,. \tag{3.23}$$

Now consider the change of variables on $\mathbb{P}(\underline{a}^0,\dots,\underline{a}^s)$ defined by,

$$X_{i,\mathrm{new}} := \underline{X}_i + Y^n\,\Delta_i(\underline{X}_{\geqslant i+1},\underline{Z}), \quad 0\leqslant i\leqslant s-1, \tag{3.24}$$

then in the *new* coordinates the invariant, $\min_{0\leqslant i\leqslant s-1}\{\,a^i - \mathbf{wt}_\xi(\underline{\varepsilon}_t^{\mathrm{top}})\,\}$, of the coordinate change (3.13) has increased and since it is an integer which is at most a^0

(*cf.* II.k), this process eventually terminates establishing the claim. $\square$

The practical upshot of III.g is when we come to compute the invariant at p we can suppose not only that all the p_{ij} are zero for $0 \leqslant i \leqslant s-1$, but that the filtration defined by $\mathbf{wt}\big(X_{ij}\big|_U\big) = a^i/a^s$, $\mathbf{wt}(Z\big|_U) = 1$ is exactly that defined by the inductive procedure II.h, albeit for the moment we remain in the situation III.f. However by claim III.g we can now just read the invariant at p from the newton polyhedron, *cf.* figure 1 pg. 13, calculated in the coordinates $X_{ij}\big|_U$, $Z\big|_U$. As such if $Z = \emptyset$ then at worst g_{s-1} goes down, whereas if $Z \neq \emptyset$ at worst g_s must go down. $\square$

IV The Relative Invariant

We proceed to construct the invariant relatively in a generality which is adequate for applications but only coincides with §II for complete local rings, to wit:

IV.a Set Up/Notation. Let $\pi : \mathfrak{U} = \operatorname{Spf} A \longrightarrow B = \operatorname{Spec} k$ be a map from an affine formal scheme to a Noetherian affine scheme, and suppose that the trace of $\mathfrak{U}$ is a regurarly embedded section σ of π of co-dimension m. Furthermore if M is the ideal of σ, suppose M/M^2 is trivial, *i.e.* $M = (x_1, ..., x_m)$ is the ideal of σ (so $A \xleftarrow{\sim} k[\![x_1, ..., x_m]\!]$) and let I be an other ideal of $\mathfrak{U}$ (so M-adically separated by definition), while for objects, over B, denote by a subscript in b the fibre (as a formal scheme, *i.e.* M-adically complete tensor product) over $b \in B$.

Plainly we begin with the multiplicity, *i.e.*

IV.b Fact. For $b \in B$, define $d_b(I) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ by,

$$d_b(I) := \sup \left\{ \alpha \in \mathbb{Z}_{\geqslant 0} \,\middle|\, M_b^\alpha \supset I_b \right\};$$

then $b \mapsto d_b(I)$ is upper semi-continuous (often abbreviated to u.s.c.).

Proof. Since I is M-adically separated, it is either zero and $d_b(I)$ is identically ∞, or there is a smallest $e \in \mathbb{Z}_{\geqslant 0}$ such that $I \subset M^e$. The former case is trivial, while in the latter case we have a non-trivial quotient of a free module, *i.e.*

$$I \longrightarrow M^e \big/ M^{e+1} \longrightarrow Q \longrightarrow 0 \tag{4.1}$$

and the condition $d_b(I) \geqslant e+1$ is equally the non-trivial closed condition,

$$\dim_{k(b)} Q_b \geqslant \operatorname{rank}\left(M^e \big/ M^{e+1}\right) \tag{4.2}$$

so we conclude by Noetherian induction. $\square$

Next we proceed to the maximal contact space by way of

IV.c Fact. Suppose the multiplicity d_b is identically $d \in \mathbb{Z}_{\geqslant 0}$ and define the sub-module V in M^d/M^{d+1} to be I modulo M^{d+1}, then the following is u.s.c. ,

$$b \longmapsto \lambda_0(b) := \begin{cases} \dim_{k(b)} \left\{ \partial \in \left(M \big/ M^2 \otimes k(b)\right)^\vee \,\middle|\, \partial(V_b) = 0 \right\}, & d > 0, \\ 0\,, & d = 0, \end{cases} \tag{4.3}$$

Proof. Plainly, without loss of generality $d > 0$, while the action of $(M/M^2)^\vee$ by derivations affords a pairing,

$$V \otimes_k \left(M^{d-1} \big/ M^d\right)^\vee \longrightarrow M \big/ M^2 \quad : \quad F \otimes \varphi \longmapsto \{\partial \mapsto \varphi(\partial F)\} \tag{4.4}$$

whose image is a k-submodule,

$$\Lambda' \hookrightarrow M \big/ M^2 \tag{4.5}$$

such that the $k(b)$-vector spaces (4.3) are the annihilators of the image of Λ'_b, so equivalently,

$$\lambda_0(b) = \dim_{k(b)} \Lambda'' \tag{4.6}$$

where Λ'' is the quotient of (4.5). $\square$

Prior to the inductive definition of the relative invariant let us make a,

IV.d Warning. In practice one wishes to take $\mathfrak{U}$ to be the completion in the diagonal of the product of B with itself whenever the latter is smooth over a field. In such a scenario if $b \in B$, then m in the sense of §II for the local ring B_b will be its dimension, $m(b)$, which will only coincide with the ambient dimension m in the sense of IV.a if b is closed.

In any case if in addition $b \hookrightarrow \lambda_0(b)$ is constant on B then generalising II.e,

IV.e Fact/Definition. In the situation of the setup IV.a, a block of (relative, should there be danger of confusion) coordinates is a subset $X \subset M$ of regular parameters whose image modulo M^2 is a subset of a k-basis. In particular whenever $b \hookrightarrow \lambda_0(b)$ is constant we have, possibly at the price of shrinking B to ensure that the implied free k-module is trivial, *cf.* hypothesis in I.g, a block X_0 consisting of the lifting of (4.5), and of course, modulo the warning IV.d,

$$\lambda_0(b) := m - c_0. \tag{4.7}$$

IV.f Inductive Hypothesis. Exactly as in II.g, with exactly the same notation up to the following minor observations consistent with IV.d,

(MO.1) I is to be understood in the sense of IV.a.

(MO.2) The definition, *cf.* (4.7), of λ_i, $0 \leqslant i \leqslant s-1$ is exactly as for the ℓ_i in (2.5) but in light of the warning IV.d we will change the notation.

(MO.3) By the definition of a relative block the graded algebra of the filtration has graded pieces free k-modules, and after clearing denominators to integers $a^0 > ... > a^{s-1} > a^s$, without common factors, defines a family, in the notation of (1.6), $\mathrm{P}_k(\underline{g}, 1) := \mathbb{P}_k(\underline{a})$ of relative weighted projective champs.

(MO.4) The starting point/initial block is X_0 of IV.e under the hypothesis that the functions $d(b)$ and $\lambda_0(b)$ of II.g.(F.1)-(F.2) are identically constant and B is sufficiently small to guarantee the triviality of Λ'' in IV.b

To which we must again adjoin,

IV.g Sub-Induction. Define the set of sub-inductive parameters Θ_{s-1} exactly as in II.k, and for $H \in \Theta_{s-1}$ we suppose the sub-inductive hypothesis II.m under which we will say that $g_s(b) \geqslant H$, $\forall\, b \in B$.

With this in mind, we have

IV.h Observation/Definition. We have a filtration $F^{\bullet}_{s-1}(H)$ defined as in (2.11) which for exactly the same reason, II.n, is independent of any choices and $V^d_{s-1}(H)$ is defined exactly as in (2.19). Finally by way of notation let Δ be the global vector fields on the associated weighted projective champ, $\mathbb{P}_k(\underline{a})$, of IV.f.(MO.3) *i.e.*

$$\Delta := \coprod_{-n<0} \mathrm{Der}_{<0}(S)(-n), \tag{4.8}$$

which in turn is a free k-module by the generalisation, [McQ17, I.c.3], of Serre's explicit calculation.

At this juncture IV.c easily generalises to,
IV.i Fact. Let everything be as in the sub-induction IV.g so in particular $m_s := m - (c_0 + \ldots + c_{s-1}) > 0$, then the following function is u.s.c.,

$$b \longmapsto \lambda_s^H(b) := \dim_{k(b)} \Big\{ \partial \in \Delta_b \,\big|\, \partial\big(V_{s-1}^d(H) \otimes k(b)\big) = 0 \Big\} \tag{4.9}$$

Proof. As in the proof of IV.c, derivation gives a pairing,

$$V \otimes_k \coprod_{-n<0} \mathrm{H}^0\big(\mathbb{P}_k(\underline{a}), \mathscr{O}_{\mathbb{P}_k(\underline{a})}(da^0 - n)\big)^\vee \to \Delta^\vee : F \otimes \varphi \mapsto \{\partial \mapsto \varphi(\partial F)\}, \tag{4.10}$$

whose image Λ' affords a short exact sequence of k-modules,

$$\Lambda' \longrightarrow \Delta^\vee \longrightarrow \Lambda'' \longrightarrow 0 \tag{4.11}$$

such that the $k(b)$-vector spaces in (4.9) are the annihilators of the image of Λ', while the fibre dimensions,

$$\lambda_s^H(b) = \dim_{k(b)} \Lambda'' \otimes k(b). \tag{4.12}$$

are plainly u.s.c.. $\square$

From which we have the corollary,
IV.j Corollary. Under the sub-inductive hypothesis IV.g, let $H' \in \Theta_{s-1}$ be the successor of H and define, $g_s(b) > H$ to mean $g_s(b) \geqslant H'$ and $g_s(b) = H$ its complement then,

(i) the conditions $g_s(b) = H$, resp. $g_s(b) > H$, are open, resp. closed.

(ii) On the open set of $b \in B$ such that $g_s(b) = H$ the function λ_s^H is u.s.c.

Equally we have the relative version of the termination of the sub-induction, *i.e.*
IV.k Case(A) (Relative, *cf.* II.p)**.** At $b \in B$, $g_s(b) = H$ (say B', by way of notation, for the open in IV.j.(ii)) then we define a function g_s to take the value H at b, and define, $\lambda_s(b)$ to be $\lambda_s^H(b)$ of (4.9). Now replace B' by the constructible subset of $b \in B'$ on which $g_s(b) = H$, and $\lambda_s(b)$ takes the constant value $m_s - c_s < m_s$; form the fibre of π, IV.a, over (the new) B'; apply I.g to get blocks $X_0, \ldots, X_s$ of cardinality $c_0, \ldots, c_s$ (thus around every $b \in B$ we replace B' by a sufficiently small Zariski neighbourhood); and continue the induction IV.f in s.
IV.l Case(B) (Relative, *cf.* II.q)**.** The complimentary closed set B'', *i.e.* $g_s > H$, is non-empty, then at $b \in B''$ apply I.g to get a Zariski neighbourhood of b, in B'', on which there are blocks $X_0, \ldots, X_{s-1}$ such that after taking the fibre of π over this open the sub-inductive hypothesis IV.g holds at the successor of H.

In so much as this procedure now involves multiple base changes to the initial set up IV.a, we can usefully observe that if case (B), IV.l, does not occur at $b \in B$ *ad infinitum* then a posteriori we can simply replace B in IV.a by a Zariski open neighbourhood of b and drop the precision of restricting to an open neighbourhood of b in case (A), IV.k. Necessarily we also want to be able to do this should case (B), IV.l, occur *ad infinitum*, and this requires a little more care, to wit:

IV.m Fact. Suppose the hypothesis of the sub-induction IV.g and let $B^\bullet \hookrightarrow B$ be the set of parameters where $g_s \geqslant H$ for all $H \in \Theta_{s-1}$ then

(i) $B^\bullet$ is closed.

(ii) Every $b \in B^\bullet$ admits a Zariski open neighbourhood $B \supset V_b \ni b$ such that on replacing B by V_b in IV.a the precision of shrinking to an open neighbourhood of b at every instance of case (B), IV.l, as H varies in Θ_{s-1}, may be omitted.

(iii) After base change of π to the constructible set $B \cap V_b \ni b$ the blocks $X_0, \ldots, X_{s-1}$ converge in the M-adic topology.

Proof. We have already proved in IV.j that for any given H, $g_s \geqslant H$ is a closed condition so not only is $B^\bullet$ closed, it is equal to $g_s \geqslant h$ for h sufficiently large. As such by base change we may suppose, without loss of generality, that $B^\bullet = B$ and case (A), IV.k, never occurs. Now the reason why we may have to restrict to an open neighbourhood of b is, in the notation of IV.i that the rank m_s k-modules,

$$\mathrm{D}(H) := \left\{\partial \in \Delta \,\middle|\, \partial\big(V^d_{s-1}(H)\big) = 0\right\} \subset \Delta \tag{4.13}$$

may not be trivial. On the other hand for any H we have a surjection,

$$M \big/ M^2 \longrightarrow\!\!\!\!\!\!\rightarrow F^1_{s-1}(H) \Big/ F^{>1}_{s-1}(H) \tag{4.14}$$

whose kernel (generated by the blocks X_i, $0 \leqslant i \leqslant s-1$) is by construction, (2.11), independent of H. Consequently the quotient (4.14) is a vector bundle independent of H, but by the better still in I.d, $\mathrm{D}(H)$ is naturally isomorphic to its dual should case (A), IV.k, never occur, so we get IV.m.(ii) by I.g. Once this is established, (iii) is exactly as in the absolute case (2.21) - (2.22). $\square$

IV.n Definition/Fact. In the set up of IV.a define the relative invariant,

$$\mathrm{INV}_{\mathfrak{U}/B}(I) : B \longrightarrow \mathbb{Q}^{2m}_{\geqslant 0} \tag{4.15}$$

starting from the rules (S.0) & (S.1) of II.f albeit with d_b, $\lambda_0(b)$ as defined in IV.b & IV.c. Subsequently if at $b \in B$ in the inductive procedure in s, every sub-induction terminates at a finite H (*i.e.* case (A), IV.k), then define

$$\mathrm{INV}_{\mathfrak{U}/B}(I)(b) := \big(d(b),\ \lambda_0(b),\ \ldots,\ \lambda_{s-1}(b),\ g_s(b),\ \underline{0}\,\big) \in \mathbb{Q}^{2m}_{\geqslant 0}; \tag{4.16}$$

where s is minimal for the property $\lambda_s(b) = 0$. Finally if case (B), IV.l, occurs *ad* *infinitum* at some $s \geqslant 1$ put,

$$\text{INV}_{\mathfrak{U}/B}(I)(b) := \big(d(b),\ \lambda_0(b),\ \dots,\ g_{s-1}(b),\ \lambda_{s-1}(b),\ \underline{0}\big) \in \mathbb{Q}^{2m}_{\geqslant 0}. \tag{4.17}$$

Consequently for $m(b)$ as in IV.d, $\epsilon = m - m_b$, and $\big(g_0 = d,\ \ell_0,\ \dots,\ \ell_t,\ g_t,\ \underline{0}\big)$ the value of the invariant, $\text{inv}_{B_b}(I_b)$ of §II, with t minimal amongst even entries ℓ_{2i} such that $\ell_{2i} = 0$, is

$$\text{INV}_{\mathfrak{U}/B}(I)(b) := \begin{cases} \big(g_0, \ell_0 + \epsilon, \dots, g_t, \ell_t + \epsilon, \underline{0}\big), & \text{if } g_t \neq 0\ , \\ \big(g_0, \ell_0 + \epsilon, \dots, \ell_{t-1} + \epsilon, \underline{0}\big), & \text{if } g_t = 0,\ t \geqslant 1\ , \\ \qquad \underline{0} & \text{if } t = 0, \text{ and } g_0 = 0. \end{cases} \tag{4.18}$$

We have already encountered a similar difference in II.w.(ii) and whence the difference merits a specific notation, to wit:

$$\text{diff}(\epsilon) := \begin{cases} \big(0,\ \epsilon,\ \dots, 0,\ \underbrace{\ \epsilon\ }_{t^{\text{th}}\text{-place}},\ \underline{0}\big), & \text{if } g_t \neq 0, t \geqslant 1, \\ \big(0,\ \epsilon,\ \dots, 0,\ \underbrace{\ \epsilon\ }_{(t-1)^{\text{th}}\text{-place}},\ \underline{0}\big), & \text{if } g_t = 0, t \geqslant 1, \\ \big(0,\ \dots,\ 0,\ \underline{0}\big), & \text{if } t = 0, g_t = 0, \end{cases} \tag{4.19}$$

Plainly the difference, (4.19), between the invariants is minimal, but it is the relative invariant that has the good properties one would expect, for example:

IV.o Fact. Let $\text{INV}_{\mathfrak{U}/B} : B \longrightarrow \mathbb{Q}^{2m}_{\geqslant 0}$ be as per IV.n, then

(i) As a function of formal neighbourhoods $\mathfrak{U}$, ideals on the same, and points on the base, $\text{INV}_{\mathfrak{U}/B}$ has self bounding denominators in the sense of II.a.

(ii) The function $\text{INV}_{\mathfrak{U}/B}$ is upper semi-continuous in the Zariski topology.

(iii) Let $\beta : B' \longrightarrow B$ be a map of schemes, and $\pi : \mathfrak{U}' \longrightarrow B'$ the base change of π, IV.a, *qua* formal scheme with I' the pull-back of I then,

$$\text{INV}_{\mathfrak{U}'/B'} = \beta^* \text{INV}_{\mathfrak{U}/B}$$

The proof will require some topological trivialities, to wit:

IV.p Lemma. Let X be a topological space,

$$\underline{F} := F_1 \times F_2\ : X \longrightarrow \mathbb{Z}^{n_1}_{\geqslant 0} \times \mathbb{Z}^{n_2}_{\geqslant 0}$$

a function and equip each $\mathbb{Z}^{n_i}_{\geqslant 0}$, respectively the aforesaid product, with the the lexicographic order then for $\underline{f} := f_1 \times f_2 \in \mathbb{Z}^{n_1}_{\geqslant 0} \times \mathbb{Z}^{n_2}_{\geqslant 0}$, the set $X_{\geqslant \underline{f}}$, of those $x \in X$ such that $\underline{F}(x) \geqslant \underline{f}$, is closed if the followings hold:

(i) F_1 is upper semi-continuous on $Y_0 := X$;

(ii) $Y_1' := \big\{x \in Y_1 \,\big|\, \underline{F}_2(x) \geq \underline{f}_2 \big\}$ is closed in the constructible set $Y_1 := \{x \in Y_0 \,|\, F_1(x) = f_1\}$.

Proof. By item (i) Y_1 is an open subset of $Y := \{x \in X \,|\, F_1(x) \geq f_1\}$, so Y_1 is constructible. Now, by construction

$$X_{\geqslant \underline{f}} = Y_1' \cup \{x \in X \,|\, F_1(x) > f_1\} = Y_1' \cup \big(Y \setminus Y_1\big) \subseteq Y, \tag{4.20}$$

where the latter is closed in X, so it is sufficient to prove that $Y_1' \cup \big(Y \setminus Y_1\big)$ is closed in Y. However its closure in Y is

$$\overline{Y_1'} \cup \big(Y \setminus Y_1\big) = \big(\overline{Y_1'} \cap Y_1\big) \cup \big(Y \setminus Y_1\big) = Y_1' \cup \big(Y \setminus Y_1\big), \tag{4.21}$$

where $\big(\overline{Y_1'} \cap Y_1\big) = Y_1'$ by item (ii), and we conclude. □

We will apply this in the form:

IV.q Corollary. Let X be a topological space, $F_i : X \longrightarrow \mathbb{Z}_{\geqslant 0}^{n_i}$ functions, respectively $f_i \in \mathbb{Z}_{\geqslant 0}^{n_i}$, for $n_i \in \mathbb{Z}_{>0}$, $1 \leqslant i \leqslant N$, such that if $N > r \geqslant 0$, with $Y_r := \{x \in X \,\big|\, F_i(x) = f_i,\ 1 \leqslant i \leqslant r\}$, $Y_0 := X$, and for all $0 \leqslant t \leqslant r$ the function F_{t+1} is u.s.c. on the set Y_t, then Y_r is constructible while

$$\underline{F}_{r+1} := (F_1, ..., F_{r+1}) : X \longrightarrow Z_{\geqslant 0}^{n_1+...+n_{r+1}} \quad \text{is u.s.c.}$$

Proof. By induction on $r \in \mathbb{Z}_{\geqslant 0}$, with the case $r = 0$ being trivial. As such let $r \geqslant 1$, and suppose the proposition for $r - 1$, then we may apply IV.p to

$$\underline{F}_r \times F_{r+1} : X \longrightarrow \mathbb{Z}_{\geqslant 0}^{n_1+...+n_r} \times \mathbb{Z}_{\geqslant 0}^{n_{r+1}}. \tag{4.22}$$

to conclude by induction. □

Proof of IV.o. The difference between INV and inv is given by (4.18), so in particular their difference is integer valued, thus self bounding denominators for inv, II.l, implies self bounding denominators for INV while the pre-requisites for deducing the u.s.c. by way of IV.q have already been done in IV.b, IV.c, IV.i and IV.j. Finally, as to itemn (iii), by way of notation let M' be the pull-back of M, then the condition $I \subset M^e$ plainly implies $I' \subset (M')^e$. At which point we just need to check that the conditions that the dimension of the modules (since the odd entries of INV are determined by whether this is maximal or not) (4.6) & (4.12) are stable under base change which is indeed the case since tensor products are right exact. □

V Analytic Principalisation

To begin with let us make

V.a Observation/Definition. Let $\mathcal{U}$ be an analytic champ, and $\mathscr{I}$ a sheaf of ideals on $\mathcal{U}$ then for x a closed point the invariant $\mathrm{inv}_{\mathcal{U}}(\mathscr{I})(x)$ is defined to be $\mathrm{inv}_{\mathscr{O}_{\mathcal{U},x}}(\mathscr{I}_x)$ where $\mathscr{I}_x$ is the stalk of $\mathscr{I}$ in the strictly Henselian ring $\mathscr{O}_{\mathcal{U},x}$. In particular therefore by II.v if,

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[dr, "x"'] & \mathrm{V} \arrow[d] \\ & \mathcal{U} \end{tikzcd}$$

is any factorisation through an étale neighbourhood, with y the image on V then,

$$\mathrm{inv}_{\mathcal{U}}(\mathscr{I})(x) = \mathrm{inv}_{\mathscr{O}_{\mathrm{V},y}}(\mathscr{I}_y), \tag{5.1}$$

and we will vary this construction in the obvious way for the variants $\mathrm{inv}^!$, resp. $\mathrm{inv}^\sharp$. With this in mind we have the key,

V.b Fact. Let $\mathfrak{U} = \mathrm{Spf} A$ be the formal spectrum of a complete regular ring of characteristic zero, I an ideal of A, $F^\bullet(I)$ as in II.v and $\rho : \widetilde{\mathfrak{U}} \longrightarrow \mathfrak{U}$ the smoothed weighted blow up [MP13, I.iv.3] associated to the aforesaid weighted filtration, then for $\widetilde{I}$ the proper transform of I, at every closed geometric point x of $\widetilde{\mathfrak{U}}$,

$$\mathrm{inv}_{\widetilde{\mathfrak{U}}}(\widetilde{I})(x) < \mathrm{inv}_{\mathfrak{U}}(I).$$

Proof. Upon clearing denominators the blocks of the filtration have weights a^i, and we have a $\left(A\,/\,_{F^{>0}}\right)$-module,

$$\bar{I} := I \mod F^{>da^0}$$

such that if $\mathscr{I}$ is the resulting sheaf of ideals on the associated weighted projective champ, equivalently the exceptional divisor $\mathscr{E} \hookrightarrow \widetilde{\mathfrak{U}}$, then,

$$\widetilde{I}\,\big|_{\mathscr{E}} = \mathscr{I}.$$

Consequently we can conclude by III.c provided that

$$\mathrm{inv}_{\mathscr{E}}\big(\widetilde{I}\,\big|_{\mathscr{E}}\big)(x) \geq \mathrm{inv}_{\widetilde{\mathfrak{U}}}\big(\widetilde{I}\big)(x)$$

at closed geometric points x. As far as the odd entries of the invariant are concerned, *cf.* the proof of III.c, this is clear. There is, however, need for caution at the even entries which is provided by items (ii-bis) & (iii-bis) of III.d, which are satisfied for the inclusion $\mathscr{E} \hookrightarrow \mathfrak{U}$, *i.e.* replace $\mathcal{Q}$ by $\mathscr{E}$ in *op.cit.* and the values of c_i on the ambient space by their value on $\mathfrak{U}$. □

Plainly, therefore, the convergence or otherwise of the filtration $F^\bullet$ of $\widehat{A}$ of II.v should II.s occurs is the only obstruction to constructing an analytic resolution of singularities from the invariant, and to address this problem we will relate varieties over $\mathbb{C}$ to spectra of complete local analytic rings by way of a particular instance of the relative invariant, to wit:

V.c Construction. Let V be a complex polydisc of dimension m and let $\mathscr{P}^n_V$ be the sheaf of n-jets defined as in [EGAIV.4, 16.7] *mutatis mutandis,* on replacing scheme by complex analytic space, then, for any map $\tau : T \longrightarrow \mathrm{V}$ from an other complex space, T, of dimension m', we have a formal analytic space equipped with a projection,

$$\begin{tikzcd} \mathrm{V} & \mathfrak{P}_T := \mathrm{Spf}\left(\varprojlim_n \tau^* \mathscr{P}^n_{\mathrm{V}}\right) \arrow[l, "\mathrm{pr}"'] \arrow[d, shift left, "\pi"'] \\ & T \arrow[u, bend right, "\sigma"'] \end{tikzcd} \tag{5.2}$$

whose trace is a regularly embedded section σ - in fact $\mathfrak{P}_T$ is the completion of the graph of τ.

In light of IV.o, we therefore make,

V.d Fact/Definition. Let everything be as in V.c, then for $\tau : T \longrightarrow \mathrm{V}$ a map from a complex space T, we define,

$$\mathrm{inv}^!_T(I) : T \longrightarrow \mathbb{Q}^{2m}_{\geqslant 0} \;:\; t \longmapsto \mathrm{INV}_{\mathfrak{P}_T/T}(\mathrm{pr}^* I)(t) \tag{5.3}$$

so by IV.o $\mathrm{inv}^!_T(I)$ is u.s.c. (in the Zariski topology of T).

Furthermore, if $T = \mathrm{V}$ and I is an ideal on V then, exactly as in IV.n,

$$\mathrm{inv}^!_V(I) = \mathrm{inv}_{\Gamma(V)_x}(I_x) + \mathrm{diff}(\epsilon), \quad \text{where } \epsilon = \dim \mathrm{V} - \dim \mathscr{O}_{\mathrm{V},x}. \tag{5.4}$$

Notice also,

V.e Corollary/Tautology. Let V be a complex polydisc, A its local ring of holomorphic functions around a point $x \in \mathrm{V}$, and I an ideal of A. If $I_Z \subset A$ is the ideal of

$$Z := \{p \in V \mid \mathrm{inv}^!_A(I)(p) \geqslant \mathrm{inv}^!_A(I)(x)\} \tag{5.5}$$

and $\mathfrak{p} \in \mathrm{Spec}\,\widehat{A}$ such that $\mathfrak{p} \supset I_Z$ then, for y a generic closed point of the sub-variety defined by $A \cap \mathfrak{p}$,

$$\mathrm{inv}_{A_\mathfrak{p}}(I_\mathfrak{p}) + \mathrm{diff}(\epsilon) = \mathrm{inv}_A(I)(y). \tag{5.6}$$

V.f Remark. We can replace inequality by equality in (5.5) and $\mathrm{inv}_A(I)(y)$ by $\mathrm{inv}_A(I)(x)$ in 5.6, on shrinking V as necessary in order to have Z closed in V (rather then just locally closed).

V.g Fact. Let A be the local ring of holomorphic functions of a polydisc V around a point $x \in \mathrm{V}$, $\widehat{A}$ its completion in the maximal ideal. If $\mathfrak{q}$ is the prime ideal,

$(X_0 = ... = X_s = 0)$, of $\widehat{A}$ defined by the blocks of the filtration $F^\bullet$ of $\widehat{A}$ of II.v, then

$$\mathrm{inv}^!_{\widehat{A}}(\widehat{I})(\mathfrak{q}) \geqslant \mathrm{inv}_A(I)(x), \tag{5.7}$$

In particular by V.f & V.e

$$\mathrm{inv}^!_{\widehat{A}}(\widehat{I})(\mathfrak{q}) = \mathrm{inv}_A(I)(x), \tag{5.8}$$

Proof. The in particular, (5.8), is clear, while by II.g.(F.1), $I \subseteq F^{da^0}(I)$, thus the value of the $\mathrm{inv}^!$ at $\mathfrak{q}$ is at least that of inv at x, (5.7). ☐

V.h Fact. Let everything be as in V.g If, moreover, $X_0, ..., X_s$ are the blocks defining the filtration $F^\bullet(\widehat{I})$ of $\widehat{A}$ afforded by $\widehat{I} = \widehat{A} \otimes_A I$, *cf.* II.v, then the (formal) sub-variety of $\mathrm{Spec}\, \widehat{A}$, $X_0 = ... = X_s = 0$, is a component of the locus, $\widehat{Z}$, where the invariant is maximal.

Proof. By V.e, the sub-scheme $\widehat{Z} := \left\{ \mathfrak{p} \in W \,\middle|\, \mathrm{inv}^!_W(I)(\mathfrak{p}) = \mathrm{inv}^!_V(I)(x) \right\}$ is the pre-image of Z, *cf.* (5.5), under $W \longrightarrow \mathrm{V}$ so, if $I_{\widehat{Z}}$ is the ideal of $\widehat{Z}$, and I_Z the ideal of (5.5) then,

$$I_{\widehat{Z}} = I_Z \otimes_A \widehat{A}.$$

Now, let $\widehat{Z} = \widehat{Z_1} \cup ... \cup \widehat{Z_r}$ be a decomposition of $\widehat{Z}$ into irreducible (formal) sub-varieties and $\mathfrak{p}_i$, $1 \leqslant i \leqslant r$, the associated prime ideal in $\mathrm{Spec}\, \widehat{A}$, then the value of the invariant is given by V.e, to wit $\mathrm{inv}_{\widehat{A}_{\mathfrak{p}_i}}(I_{\mathfrak{p}_i}) = \mathrm{inv}_A(I)(x) - \mathrm{diff}(\epsilon)$, *i.e.* the value of $\mathrm{inv}^!(I)$ calculated in the ideals, $\mathfrak{p}_i$, $1 \leqslant i \leqslant r$. Therefore, by (5.4), $\dim \widehat{A}_{\mathfrak{p}_i} \geqslant c_0 + ... + c_s$ which implies,

$$\dim(\widehat{Z}_i) \leqslant m - (c_0 + ... + c_s) = \dim \mathrm{V}(\mathfrak{q}),$$

so, although $\widehat{Z}$ is not of pure dimension each component $\widehat{Z}_i$ has at most dimension $m - (c_0 + ... + c_s)$ thus $\mathrm{V}(\mathfrak{q})$ is actually an irreducible component of $\widehat{Z}$. However the irreducible components of $\widehat{Z}$ are the completions of the irreducible components of Z. ☐

All of which can be combined to establish

V.i Corollary. Let everything be as in V.g, I an ideal of A then, in the situation of V.f, $\mathrm{V}(\mathfrak{q})$ is convergent. Better still the filtration $\widehat{F}^\bullet(I)$ of II.v is convergent, *i.e.* there exists a filtration $F^\bullet(I)$ of A such that $\widehat{F}^\bullet(I) = F^\bullet(I) \otimes_A \widehat{A}$.

Proof. The convergence of $\mathrm{V}(\mathfrak{q})$ is just the outcome of V.g & V.h. While, for the convergence of $\widehat{F}^\bullet(I)$, by V.g it remains to find the blocks themselves rather than just the centre, $\mathrm{V}(\mathfrak{q})$ of V.h, on which they are supported. To do this it is sufficient to do II.t $\mathfrak{q}$-adically rather than $\mathfrak{m}(x)$-adically. By the implicit function theorem in the analytic topology we can choose a projection π and a section σ,

$$\begin{tikzcd} \mathrm{V} \arrow[r, "\pi"'] & \mathrm{V}(\mathfrak{q}) \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$

such that σ is the embedding of $\mathrm{V}(\mathfrak{q}) \overset{\sigma}{\hookrightarrow} \mathrm{V}$, so $\mathfrak{q}$-adic convergence of the blocks follows from IV.m.(iii). □

The discussion also reveals that we can make numerous improvements to V.b to wit:

V.j Fact/Definition. Shrinking V as necessary in order to guarantee the convergence of $\mathrm{V}(\mathfrak{q})$, V.f, we obtain a convergent regular weighted filtration $F^\bullet(I)$, as guaranteed by V.i, and whence an associated smoothed weighted blow, *cf.* [MP13, I.iv.3],

$$\rho : \mathcal{V} \longrightarrow \mathrm{V}, \tag{5.9}$$

whose completion would be $\widetilde{\mathfrak{U}}$ of V.b. Consequently the smoothed weighted blow up is unique.

V.k Corollary. Let everything be as in V.e I an ideal of A, $F^\bullet(I)$ the filtration of V.i, $\rho : \mathcal{V} \longrightarrow \mathrm{V}$ the associated smoothed weighted blow up V.j, so *inter alia* its moduli $|\mathcal{V}|$ is a complex space, then:

(i) For all $p \in \mathrm{V}$,

$$\mathrm{inv}^!_{\mathrm{V}}(I)(p) \leqslant \mathrm{inv}^!_{\mathrm{V}}(I)(x).$$

(ii) If $\widetilde{I}$ is the proper transform of I, and $I \neq A$, on $\mathcal{V}$ then for all geometric point v of $\mathcal{V}$,

$$\mathrm{inv}^!_{\mathcal{V}}(\widetilde{I})(v) < \mathrm{inv}^!_{\mathrm{V}}(I)(x).$$

(iii) If $X_0, ..., X_s$ are the blocks defining the filtration $F^\bullet(I)$ then the closed analytic sub-variety $X_0 = ... = X_s = 0$ is exactly

$$\widehat{Z} = \left\{ p \in \mathrm{V} \,\middle|\, \mathrm{inv}^!_{\mathrm{V}}(I)(p) = \mathrm{inv}^!_{V}(I)(x) \right\} \tag{5.10}$$

Proof. Since A is a local ring, V.k.(i) is just the u.s.c. of $\mathrm{inv}^!$ in V.d. Similarly we already know V.k.(ii) after completing in the exceptional divisor $\mathcal{E} \hookrightarrow \mathcal{V}$ by V.b and since $\mathrm{inv}^!_{\mathcal{V}}(\widetilde{I})$ is also u.s.c. in the Zariski topology of the moduli by V.d, we have it everywhere since ρ is proper. Consequently if in V.k.(iii) $\widehat{Z}$ were not contained in $X_0 = ... = X_s = 0$, then by the u.s.c. of V.d we would have the absurdity that the invariant would not go down. Conversely the inclusion of $X_0 = ... = X_s = 0$ in $\widehat{Z}$ is V.h. □

V.l Fact/Definition. Let V be a complex polydisc of dimension m and $A := \mathscr{O}_{\mathrm{V},x}$, with $x \in \mathrm{V}$, as in V.e; I an ideal on V; $\underline{i} \in \mathbb{Q}^{2m}_{\geqslant 0}$ the maximum value of $\mathrm{inv}^!_{\mathrm{V}}(I)$ over V; $\mathcal{V} \longrightarrow \mathrm{V}$ the smoothed weighted blow up (whose existence respectively uniqueness is guaranteed by V.g, respectibvely by V.j and II.v) associated to the canonical filtration $F^\bullet(I)$; while for $\underline{q} \in \mathbb{Q}^{2m}_{\geqslant 0}$ define a modification functor

$$M_{I,\underline{q}}(\mathrm{V}) := \begin{cases} \mathcal{V}, & \text{if } \underline{i} = \underline{q}, \\ \mathrm{V}, & \text{otherwise;} \end{cases} \tag{5.11}$$

and extend $M_{I,\underline{q}}$ to a disjoint union of germs of polydiscs $\coprod_\alpha \mathrm{V}_\alpha$ by way of,

$$M_{I,\underline{q}}(\mathrm{V}) := \coprod_\alpha M_{I,\underline{q}}(\mathrm{V}_\alpha). \tag{5.12}$$

Then by II.v the modification functor $M_{I,\underline{q}}$ is étale local, *i.e.* if $\mathrm{V}' \longrightarrow \mathrm{V}$ is étale and I' the pull-back of I to V then there is a fibre square,

$$\begin{tikzcd} M_{I,\underline{q}}(\mathrm{V}) \arrow[d] & M_{I',\underline{q}}(\mathrm{V}') \arrow[l] \arrow[d] \\ \mathrm{V} \arrow[ur, phantom, "\square"] & \mathrm{V}' \arrow[l] \end{tikzcd} \tag{5.13}$$

In particular if $\mathcal{X}$ is a regular (Deligne-Mumford) analytic champ, $\mathscr{I}$ a sheaf of ideals on the same and $\underline{q}$ the maximum at geometric points of $\mathrm{inv}^!_{\mathcal{X}}(\mathscr{I})$, (5.1) *et seq.*, then for $\mathrm{V} \longrightarrow \mathcal{X}$ an étale atlas and $R = \mathrm{V} \times_{\mathcal{X}} \mathrm{V} \overset{t}{\underset{s}{\rightrightarrows}} V$ the implied groupoid,

$$\begin{tikzcd} M_{\mathscr{I}_R,\underline{q}}(R) \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] \arrow[d] & M_{\mathscr{I}_{\mathrm{V}},\underline{q}}(\mathrm{V}) \arrow[d] \\ R \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & \mathrm{V} \end{tikzcd} \tag{5.14}$$

is a map of groupoids in which $M_{\mathscr{I}_R,\underline{q}}(R)$ (which we may abusively consider unique since it's a modification) is equally the fibre of the rightmost vertical arrow over either s or t by (5.13), *i.e.* the $M_{\mathscr{I}_{\mathrm{V}},\underline{q}}$ patch to a smoothed weighted blow up,

$$M_{\mathscr{I}}(\mathcal{X}) \longrightarrow \mathcal{X} \tag{5.15}$$

depending only on $\mathscr{I}$. We therefore get our first global results, to wit:

V.m Construction. (Resolution via Exhaustion by Relative Compacts) Let X be a complex space and $\coprod_{i\geqslant 0} X^{(i)}$ an exhaustion by relative compact open sub-spaces $X^{(i)} \hookrightarrow X$, $i \geqslant 0$, $\mathscr{I}$ a (coherent analytic) sheaf of ideals on X, denote by $\mathscr{I}^{(i)} := \mathscr{I}|_{V_i}$ its restriction to the sub-space $X^{(i)}$, and, finally, define inductively a sequence of smoothed weighted blow ups in convergent weighted centres of $X^{(i)}$ by

$$\big(X_0^{(i)}, \mathscr{I}_0^{(i)}\big) := \big(X^{(i)}, \mathscr{I}^{(i)}\big) \quad \text{and} \quad \big(X_{r+1}^{(i)}, \mathscr{I}_{r+1}^{(i)}\big) := \big(M_{\mathscr{I}_r^{(i)}} X_r^{(i)}, \widetilde{\mathscr{I}_r^{(i)}}\big), \quad r > 0 \tag{5.16}$$

where $\widetilde{\mathscr{I}_r^{(i)}}$ is the proper transform of $\mathscr{I}_r^{(i)}$, then for $r = r(i) \gg 0$, $\mathscr{I}_r^{(i)}$ is trivial. Moreover if we define,

$$\mathcal{X}_i := X_{r(i)}^{(i)} \longrightarrow X^{(i)}, \tag{5.17}$$

then by (5.12), for $i > j$ we have a fibre square,

$$\begin{tikzcd} \mathscr{X}_j \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathscr{X}_i \arrow[d] \\ X^{(j)} \arrow[r, hook] & X^{(i)} \end{tikzcd} \tag{5.18}$$

so that for $\mathscr{X} := \varinjlim \mathscr{X}_i \xrightarrow{\rho} X$, the proper transform of $\mathscr{I}$ along ρ is trivial.

This of course affords a resolution of singularities of anything admitting an embedding in something smooth, but this is not a very satisfactory hypothesis so we improve it by way of

V.n Construction. Let Y be a connected germ of a complex space of dimension n, $y \in Y$ a closed point, $A := \mathscr{O}_{Y,y}$ the local ring of holomorphic functions around the point y. Then, for e the embedding dimension, *i.e.* by definition,

$$e := e_Y(y) = \dim_{\mathbb{C}} \mathfrak{m}(y) \Big/ \mathfrak{m}^2(y) \tag{5.19}$$

we can choose a presentation,

$$0 \longrightarrow I_y \longrightarrow A := \mathbb{C}\{z_1, ..., z_e\} \longrightarrow \mathscr{O}_{Y,y} \longrightarrow 0 \tag{5.20}$$

and observe that any 2 such presentations are related by a commutative diagram of exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & I_y \arrow[r] & A := \mathbb{C}\{z_1, ..., z_e\} \arrow[r] & \mathscr{O}_{Y,y} \arrow[r] & 0 \\ 0 \arrow[r] & I_0 \arrow[r] \arrow[u] & \mathbb{C}\{w_1, ..., w_e\} \arrow[r] \arrow[u, "\wr"] & \mathscr{O}_{Y,y} \arrow[r] \arrow[u, equal] & 0\,. \end{tikzcd} \tag{5.21}$$

As such $\mathrm{inv}_Y(y) := \mathrm{inv}_A(I_y)$ is well defined, and for m the maximum over all embedding dimensions we correct this to

$$\mathrm{inv}^!_Y(y) := \big(\mathrm{inv}_Y(y) + \mathrm{diff}(m - e_Y(y))\big) \times \underline{0} \in \mathbb{Q}^{2m} \tag{5.22}$$

with an implies block of zeroes whenever $e_Y(y) < m$. At the same time in the local ring $\mathscr{O}_{Y,y}$, we introduce

$$d_Y(y) := \min_{\mathfrak{q}} \dim \frac{\mathscr{O}_{Y,y}}{\mathfrak{q}}$$

where the minimum is taken over all the minimal primes in $\mathscr{O}_{Y,y}$, which in turn affords the invariant,

$$\mathrm{inv}^\sharp_Y(y) := \big(\delta_Y(y) := e_Y(y) - d_Y(y)\big) \times \mathrm{inv}^!_Y(y) \;\in \mathbb{Q}^{2m+1}_{\geqslant 0} \tag{5.23}$$

together with a unique, by V.j & (5.21), smoothed weighted blow up,

$$\mathcal{Y}_y \longrightarrow Y, \tag{5.24}$$

obtained by taking the proper transform of Y along $\mathcal{V} \longrightarrow \mathrm{V}$ after choosing, as in (5.20), an embedding $Y \hookrightarrow \mathrm{V}$ in a polydisc. Now observe that the leading term δ_Y in $\mathrm{inv}^\sharp_Y$ is just,

$$\dim_{\mathbb{C}} \Omega_Y \otimes k(y) - \min_{Y_0} \dim Y_0 \tag{5.25}$$

where the minimum is taken over components $Y_0 \ni y$ on a small neighbourhood of y. Consequently δ_Y is the difference of an upper semi-continuous function and a lower semi-continuous one so δ_Y is u.s.c. .

To conclude from here that $\mathrm{inv}^\sharp_Y$ is u.s.c. we require by IV.p to establish that $\mathrm{inv}^!_Y$ is u.s.c. where δ_Y is constant. To this end say $\delta_Y(x) = \delta_Y(z)$, then we may as well say that we're on a neighbourhood Y' of a closed set Z where Ω_Y has constant rank and around Z we have an embedding $Y' \hookrightarrow M$ into a smooth $\mathbb{C}$-variety of fixed dimension $e = e_Y(x)$, independently of $x \in Z$. Consequently,

$$\mathrm{inv}^!_Y(x) = \mathrm{inv}^!_M(I_{Y'}) + \mathrm{diff}(m - \dim M) \tag{5.26}$$

so it's upper semi continuous by V.d.

Notice that en passant we have established,

V.o Fact. Let everything be as in V.n and take $Y \ni y$ sufficiently small such that the set of points,

$$Z = \left\{ z \in Y \;\middle|\; \mathrm{inv}^\sharp_Y(z) = \mathrm{inv}^\sharp_Y(y) \right\}$$

is closed, then (possibly after shrinking Y) for all $z \in Z$, the modifications $\mathcal{Y}_z \longrightarrow Y$ of (5.24) can be identified with the modification $\mathcal{Y}_y \longrightarrow Y$

Proof. Exactly as prior to (5.26) we have (shrinking Y as necessary) an embedding $Y \hookrightarrow \mathrm{V}$ in a polydisc of dimension the embedding dimension $e = e_Y(y) = e_Y(z)$, for $z \in Z$, so that by (5.21) the modifications

$$\mathcal{Y}_z \longrightarrow Y$$

are the proper transform of Y along the smoothed weighted blow up $\mathcal{V} \longrightarrow \mathrm{V}$ associated to the ideal of Y on V. □

We can put all of this together to conclude,

V.p Summary/Definition. Let Y be the germ of a complex space, $\underline{i}(Y) \in \mathbb{Q}^{2m+1}_{\geqslant 0}$ the maximum value of $\mathrm{inv}^\sharp_Y$ then,

(E.1) By V.o every point $y \in Y$ has an open neighbourhood N_y admitting the smoothed weighted blow up $\mathcal{N}_y \longrightarrow N_y$ (5.24).

(E.2) These patch to a smoothed weighted blow up $\mathcal{Y} \longrightarrow Y$. Indeed, again by V.o, if $x \in N_y$, and $\mathrm{inv}^\sharp_Y(x) = \mathrm{inv}^\sharp_Y(y)$, then the formal fibre of $\mathcal{N}_y$ at x is that of y.

(E.3) Better still if for $\underline{i}(Y) \leqslant \underline{q}$ in $\mathbb{Q}_{\geqslant 0}^{2m+1}$ and Y connected we define

$$M_{\underline{q}}(Y) := \begin{cases} \mathcal{Y}, & \text{if } \underline{i}(Y) = \underline{q} \\ Y, & \text{if } \underline{i}(Y) < \underline{q} \end{cases} \tag{5.27}$$

and extend to direct sums of connected components as in (5.12), then (since it is enough to check the formal fibres) the smoothed weighted blow up $M_{\underline{q}}(Y)$ commutes with analytic étale maps, *i.e.* if $Y' \longrightarrow Y$ is étale or even just smooth then, by V.p.(E.2) we have a fibre square

$$\begin{tikzcd} M_{\underline{q}}(Y) \arrow[d] & M_{\underline{q}}(Y') \arrow[l] \arrow[d] \\ Y \arrow[ur, phantom, "\square"] & Y' \arrow[l] \end{tikzcd} \quad . \tag{5.28}$$

In particular therefore, *cf.* (5.14)-(5.15), if $\mathcal{Y}$ is an analytic reduced (Deligne-Mumford) champ then there is a smoothed weighted blow up,

$$M_{\underline{q}}(\mathcal{Y}) \longrightarrow \mathcal{Y} \tag{5.29}$$

supported in the singular locus whose fibre over an étale atlas is the blow up functor (5.27), and which itself commutes with étale maps, *i.e.* replace $Y \to Y'$ by an étale map of champ $\mathcal{Y}' \longrightarrow \mathcal{Y}$ in (5.28). Finally for $\mathrm{inv}^\sharp_{\mathcal{Y}}$ defined as in (5.23), let $\underline{i}(\mathcal{Y})$ be the maximum value of $\mathrm{inv}^\sharp_{\mathcal{Y}}$ and $M(\mathcal{Y}) := M_{\underline{i}(\mathcal{Y})}(\mathcal{Y})$ then by construction,

$$\underline{i}(\mathcal{Y}) = 0 \quad \Longleftrightarrow \quad \mathcal{Y} \text{ is smooth.} \quad \Longleftrightarrow \quad M(\mathcal{Y}) = \mathcal{Y}.$$

All of which is easily assembled into a resolution algorithm, to wit:

V.q Proposition. For Y a connected germ of complex space and $\coprod_{i\geqslant 0} Y^{(i)}$ an exhaustion by relative compact opens of the same, define a sequence of smoothed weighted blow ups for every $Y^{(i)}$, $i \geqslant 0$ by,

$$\mathcal{Y}_0^{(i)} = Y^{(i)}, \qquad \mathcal{Y}_{r+1}^{(i)} = M(\mathcal{Y}_r^{(i)}), \quad r \geqslant 0 \tag{5.30}$$

and let $N = N(i) \geqslant 0$ be the smallest integer such that $\mathcal{Y}_{N+1}^{(i)} \longrightarrow \mathcal{Y}_N^{(i)}$ is the identity then the chain of smoothed weighted blow ups,

$$\mathcal{Y}^{(i)} = \mathcal{Y}_0^{(i)} \longleftarrow \mathcal{Y}_1^{(i)} \longleftarrow \cdots \longleftarrow \mathcal{Y}_{N-1}^{(i)} \longleftarrow \mathcal{Y}_N^{(i)} \tag{5.31}$$

is a resolution of singularities in the 2-category of Deligne-Mumford champs enjoying the functorial resolution properties (E.1)-(E.3) of V.p, for germs of complex spaces. Moreover if we define,

$$\widetilde{\mathcal{Y}}_i := \mathcal{Y}_{N(i)}^{(i)} \longrightarrow Y^{(i)}, \tag{5.32}$$

then by (5.12), for $i > j$ we have a fibre square,

$$\begin{tikzcd} \widetilde{\mathcal{Y}}_j \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{Y}}_i \arrow[d] \\ Y^{(j)} \arrow[r, hook] & Y^{(i)} \end{tikzcd} \tag{5.33}$$

so the resolution of singularities (5.32) glue to a resolution $\mathcal{Y} := \varinjlim \mathcal{Y}_i \xrightarrow{\rho} Y$.

Proof. From the definition (5.23) and IV.o.(i) $\mathrm{inv}^{\sharp}_{\mathcal{Y}}$ has self bounding denominators, II.a, so it suffices to check,

$$\underline{i}(M_{\mathcal{Y}}) < \underline{i}(Y)$$

Plainly, however, the embedding dimension cannot increase under a smoothed weighted blow up and since (5.24) is the formal fibre around any point, this is immediate from the corresponding proposition, V.k.(ii), for $\mathrm{inv}^!$. $\square$

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