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Adiabatic limits and renormalization in quantum spacetime

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Contents

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1 Introduction

Quantum Field Theory is an extremely successful framework for the physical models building. Its particular realization, known as the Standard Model, agrees perfectly with almost all experiments which are not affected by gravity available for the moment [1]. In particular, the electromagnetic sector of it, Quantum Electrodynamics, is often called the most precisely tested physical theory. Another notable example of the QFT triumph is the experimental discovery of the Higgs boson in 2012 [2] which was theoretically predicted at least 40 years before that [3]. Almost all modification of the Standard Model being considered (except ones including the gravity which will discuss later) also lie in the QFT framework.

One of the crucial concepts in QFT is locality. Basically, it means that all interactions happen locally and the "action at a distance is always mediated by local interaction with "force-carrying" particles (or fields). In relativistic physics, combined with the prohibition of faster than light particles, the locality forbids the spacelike-separated regions to interact in any way. It follows that observables measurable in such regions should commute. This restriction plays a crucial role in the axiomatisation of QFT [4, 5, 6]. This property is also sometimes called causality because it allows formulating theory in such a way that the cause always precedes the effect. In the physical community by locality one often understand restriction to the "local interactions", i.e. the requirement that the interaction term in Lagrangian density at a point in spacetime is a function of the fields and finitely many of their derivatives at that point (depending on the particular formulation of QFT the Lagrangian density may be replaced by the Hamiltonian density or the right hand side of the equations of motion). Then, on the level of perturbation theory, the naïve idea of all processes being series of local ones is realised in the Feynman diagrams (see e.g. [8]). Locality of interactions leads to casuality (in the above sense) at least perturbatively [7]. The converse statement is difficult to formulate precisely, because one can always make the Lagrangian non-local without affecting the theory. But informally speaking, it is very unlikely that a fixed non-local Lagrangian defines a local QFT.

On the negative side, local QFTs (with a very few exceptions) are divergent. This happens because the quantum fields are not functions, but only distributions on the spacetime. In perturbation theory (see Section 3) the ill-defined pointwise products of quantum fields among themselves and with other distributions appear. In practice it leads to the scattering sections and the observable particle masses formally presented as divergent integrals. These divergences are called ultra-violet (UV) because in the Feynman diagrams formalism they appear as divergences in the region of infinite momenta of virtual particles. To make the observable quantities finite, that divergences should be cancelled one way or another. This can be done through the so-called renormalisation. More detailed description of this procedure together with the further references are porstponed until Section 3, but we note that there are different methods of renormalization leading to the same final result. This happens essentially because the locality significantly narrows the choice of ways to cancel the divergences.

In general, the renormalization in all orders of perturbation theory requires infinitely many parameters (interaction constants and masses) to describe the resulting theory. To make any predictions using any theory one would need to find its parameters from experiments, which in the case of infinitely many parameters would require an infinite amount of data. This is why as fundamental models one usually considers only the renormalizable QFTs, a special class of theories described by finitely many parameters after the renormalization. Locality and renormalizability combined leave only finitely many allowed interactions (for a fixed set of quantum fields). In practice, physicists write down the most general local renormalizable Lagrangian with a fixed set of quantum fields, obeying the prescribed symmetries. The unknown parameters then can be found by matching the computation results to the measurements. This is how the Standard Model was built and experimentally proven.

As was mentioned in the very beginning, QFT describes successfully all the interactions except

the gravity. QFT can be as well formulated on a curved manifold [9], which in physics corresponds to quantum field theory in a background classical gravitational field. The Hawking radiation [10] of the black holes is one of the most notable non-trivial effects of gravity on quantum fields predicted. The backreaction of the quantum fields on the gravity was studied in the semi-classical approximation with applications to cosmology and to Hawking radiation corrections [11]. The self-consistent description of the world including quantum fields and gravity (which inevitably leads to quantization of the gravity itself) is still missing, though necessity of such a theory was pointed out already in in 1916 by Einstein [12] (later he changed his mind and became sceptic to the quantum physics).

In one of the first works[1] on Quantum Electrodynamics [13] Pauli and Heisenberg suggested that the gravity "carried out without any new difficulties" using the same method as they did. For linearized gravity this idea was first realized by Rosenfeld [14] in 1930 and six years later by Bronstein [15][2]. The latter author was the first to note that in the non-linear regime of quantum gravity the localization of particles and events is limited. If the localization is too sharp, the energy and momentum uncertainty will be very high due to the Heisenberg relation. But in general relativity high energy or momentum localized in small volume leads to the black hole creation. Since particles and events under the event horizon are not accessible for the external observer, such sharp localisations have no operational limit. The characteristic length of the maximal localization is of the order of the Planck length l_P suggested by Planck as a natural unit [18]. After this analysis, Bronstein expressed serious doubts that the gravity can be quantized without "profound change of the classical notions". In particular, he supposed that the spacetime manifold should be replaced by "some deeper and non-evident concepts" taking into account that no measurement localized at a point can be fulfilled. Bronstein's ideas got almost no response in the physicists' community and were forgotten for a long time. But with development of quantum gravity, the limitation of localization (the so-called "minimal length" of quantum gravity) appeared many times in different ways playing an important role [20].

If there are no spacetime points, then locality of interactions makes no sense. We note that locality understood as casuality is also not applicable to quantum gravity because the casual relation between spacetime regions is quantized there. So, one could expect that the UV-problems typical for local theories are naturally cured by the quantum gravity at high energies. This was noted by Deser (inspired by ideas of Klein) in [19][3]. It was argued that in quantum gravity the vacuum average of tensor square of a quantum ϕ

$$< 0|\phi(x)\phi(y)|0 >$$

is finite on the diagonal $x \to y$. Such a limit does not exist without gravity due to the distributional nature of the quantum fields. One could expect that other divergences would also disappear when gravity is taken into account.

Quite the opposite was found then the quantum gravity was studied beyond the linear order. It was shown that it not only UV-divergent but non-renormalizable [21]. It means that gravity can not be quantised along the same ideas as electrodynamics was and some cardinally new ideas are required[4]. At the same time, there are still no reliable experimental data in the regime where the quantum gravity effects would matter. For this reason, plenty of different competitive ways to construct quantum gravity appeared and are developing now [22]. In this situation as a complement

[1] We follow the historical review of first decades of quantum gravity [17] to which we refer for further details

[2] See also [16] for review of Matvei Bronstein's work in the historical perspective and the tragic story of his life.

[3] The idea of minimal length (or equivalently maximal energy) curing the UV divergences was discussed right after that divergences were first found but without any reference to gravity. The details can be found in [20].

[4] We note that in non-renormalisability is a problem only if we want to construct the theory to all orders of perturbation theory. In each order of perturbation theory only finitely many parameters matter even in non-renormalizable theories. The Effective Field Theory formalism [23] deals with QFTs valid only up to some finite order of perturbation theory ans hence treats non-renormalisable theories without any problem. For example, the leading order quantum corrections to the Newton potential were found in the effective field theory framework without any problem [24]. But, to take into account higher order of perturbation theory, one would need not only to perform mathematical computations, but also to get new experimental data.

to this work, it makes sense to concentrate on properties which are shared by all (or almost all) such approaches. From the above it seems that the minimal length should be one of those properties.

In the string theory the locality of interactions is broken by replacing the point-like particles with the one-dimensional objects (the strings). Due to the quantum fluctuations, strings can not be contained in too small volume. Thus it is impossible to resolve such small volumes. In [25] this was quantitatively accounted in the form of the so-called generalized uncertainty relation

$$\Delta X \Delta P \geq 1 + l_s E. \tag{1.1}$$

Here ΔX and ΔP are the coordinate and momentum uncertainties respectively, E is the energy and l_s is the dimensional parameter of the string theory with the physical sense of the characteristic string length[5]. This length is proportional to the Planck scale. In [26] it was argued that (1.1) may be not valid beyond the perturbation theory and the space-time uncertainty relation

$$\Delta X \Delta T \geq l_s^2 \tag{1.2}$$

was proposed instead. Here ΔX and l_s are as before and ΔT is the time uncertainty.

The loop quantum gravity, one of the most elaborated approaches to the gravity quantization (together with the string theory), does not introduce any non-locality by hand. Instead, it is based on choosing for quantization a suitable basis of observables, known as the loop observables, generalizing the Wilson loop of gauge theories. These observables are labelled by loops in the space and the generic state of the quantum space is labelled by a graph [52]. The "minimal length" arises here as discreteness of spectra of the area and volume operators [28] acting on that states.

The asymptotic safety program treats the quantum gravity as a usual quantum field theory with an infinite number of parameters (to make renormalization possible). A generic theory then can be pictured as a curve (due to non-uniqueness of the renormalization, see Remark 3.3) in the parametric space. Then one selects only trajectories which behave nice enough at high energy scales (see Remark 3.5). If such trajectories exist and span a finite-dimensional hypersurface, this requirement allows to reduce the number of parameters to a finite one. Even in this, rather conservative approach the presence of a minimal length was shown [29].

The model-independent estimations of the minimal length also continued to appear. The analogue of string theory generalized uncertainty principle (1.1)

$$\Delta X \Delta P \geq 1 + l_P E. \tag{1.3}$$

was proven in linearized quantum gravity [30, 31]. In [32] it is argued that any trans-Planckian (i.e. with characteristic energy above the Planck scale) scattering is dominated by soft graviton radiation. Due to decoherence, such scatterings can not be used to probe the spacetime structure. In [33] it was shown that such processes are dominated by creation of black holes[6] which also brakes sharp localization of the interactions. In [35] the processes at the scale of Planck length (or smaller) were addressed by functional integration. It was argued that the integral over all geometries is dominated by the so-called "space-time foam" spacetimes, i.e. geometries containing approximately one gravitational instanton (topologically non-trivial fluctuation) per Planck volume. So, the topology of the spacetime is predicted to become much more complicated at the Planck scale.

In [38] the spacetime uncertainty relations (STUR)

$$\Delta x^0 \Delta x^1 + \Delta x^0 \Delta x^2 + \Delta x^3 \Delta x^1 > l_P^2, \tag{1.4}$$

$$\Delta x^1 \Delta x^2 + \Delta x^2 \Delta x^3 + \Delta x^3 \Delta x^1 > l_P^2.$$

[5]Here and everywhere we use a system of units in which $\hbar = 1$, $c = 1$, but we keep the Planck length l_P.

[6]Accordance between these two results is addressed in [34].

with Δx^α staying for the αth coordinates uncertainty were derived. From now on we call them the DFR spacetime uncertainty relation (STUR) by the names of their authors Doplicher, Fredenhagen and Roberts. This idea reproduces the original Bronstein suggestion and was used by some other researchers (e.g. [36]), but before Doplicher, Fredenhagen and Roberts the area of localization was always assumed to be spherical. Most of the other approaches listed above also implicitly or explicitly assumed spherical symmetry. So, the relations (1.4) provide the first estimation of the shape of a non-spherically localized event. These relations were later supported by the work [40] not relying on the linearized gravity, but assuming the hoop conjecture [39] (in an appropriately adjusted formulation). For the spherically symmetric case similar relations were proven by solving the exact non-linear Einstein equations in [41]. Generalization of the DFR STUR to cosmological backgrounds was discussed in [42].

To sum the above short review up, the "minimal length" appears in quantum gravity from very different arguments and in diverse approaches to gravity quantization, so it seems to be an inherent property of the quantum gravity. Thus, it makes no sense to treat the spacetime as a smooth manifold. This is similar to the situation in the early years of quantum mechanics when Heisenberg decided that since coordinate and momentum of a particle can never be measured simultaneously in any experiment, then the exact values of that quantities should not be used to describe a state of that particle.

The simplest way to implement the minimal length would be to replace the smooth manifold by its discreet approximation (e.g. [43]). The immediate consequence would be the non-Lorentz invariant dispersion relation modification. Surprisingly, current astronomical observations are enough to rule out lattices with one characteristic scale even if that scale is close to the Planck length [44]. The Lorentz invariance violation at high energies also brings theoretical concerns [45], because it would require a very precise tuning of the interaction constants to preserve a Lorentz-invariant theory at low energies when quantum corrections are taken into account.

Another realisation of the minimal length follows from the fact that (1.1-1.4) look very similar to the Heisenberg uncertainty relations. Then it is natural to assume that they follow from non-trivial commutation relations between the coordinates. In particular, one can assume that the spacetime coordinate do not commute with each other.

The non-commutative coordinates can naturally appear in non-commutative geometry [46]. The key idea of this branch of mathematics is to extend the Gelfand-Naimark theorem, basically stating that the topology of a Hausdorff space is encoded in the C*-algebra of complex-valued continuous functions on it, to non-commutative C*-algebras. The non-commuting coordinates then, in general, are unbounded operators affiliated with that C*-algebra.

Despite being a thriving area of mathematics, the non-commutative geometry naturally arises in different areas of physics. For example, it naturally appears in description of quantum charged particles in a strong magnetic field and can be applied to the quantum Hall effect [47]. More relevant for us is that the non-commutative geometries emerge in the string field theory [48], in the low-energy limit of the string theory on a background with non-zero B-field [49], in the stack of D-branes description [51], in the compactifications of M-theory [50] and other (super)string- and M-theory related studies [55]. In loop quantum gravity the non-commutative geometric description of the graphs states of quantum spacetime (see above) providing a natural definition of the distance operator was suggested [53]. Another approach to loop quantum gravity based on non-commutative geometry is reviewed in [54]. Non-commutative geometry also arises in 2+1-dimensional quantum gravity[7] [56].

A remarkable realization of the Standard Model (in the Euclidean signature) in terms of the non-commutative Riemmanian geometry was suggested in [57] and of one of the Grand Unification Theory candidates in [60]. Intriguingly, the restriction on the gauge group and the particle content are much more strong in the non-commutative approach [59] than in the usual field theoretical one,

[7]Gravity in 2+1 dimensions is a very special case, because the gravitational waves are absent. So, these results can not be directly applied to the 3+1 dimensional case which is implicitly assumed everywhere in this thesis.

so a physical theory based on the former would be more predictive. The gravity can be naturally embedded into that description [58]. More recent progress in this framework can be found in [61]. In particular, realization of the minimal length (or rather minimal volume) in this framework is discussed there. Transition to the Lorentz signature is still an open problem, although proposals for non-commutative pseudo-Riemannian geometry with Lorentz signature were made in [62, 63].

We see that the non-commutative geometry is a natural realization of the non-locality of quantum physical theories. Although in the examples above the scale of that non-locality is not necessarily directly related to the Planck or string length, they still motivate the study of models of non-commutative spacetimes realising the minimal length estimations we mentioned. One can hope that such models would catch at least some properties of the quantum spacetime in the "right" quantum gravity theory, whatever it will be. One of the earliest attempts is due to Snyder [64] later developed in [65]. The commutators of the coordinates are assumed to be proportional to the Lorentz transformation generator $M^{\mu\nu}$,

$$[q^\mu, q^\nu] = il^2 M^{\mu\nu}$$

with some characteristic length (as usual proportional to the Planck scale). Since both the left hand side and the right hand side transform like a tensor under Lorentz transform, the Lorentz symmetry is preserved (as well as the translation symmetry with siutably deformed action on the coordinates). It was shown that such commutation relations produce the uncertainty relations (1.3) if the scale l is chosen appropriately.

Another, simpler, way is to set the commutator of coordinates to some constant,

$$[q^\mu, q^\nu] = i\theta^{\mu\nu} \tag{1.5}$$

with $\theta^{\mu\nu}$ being an antisymmetric matrix of real numbers. Such commutation relation lead to Heisenberg-like uncertainty relations similar to (1.2). The Lorentz symmetry is broken by the presence of a fixed tensor θ. Instead, a quantum symmetry group can be introduced [66]. This idea is quite popular around the string theory community. It was conjectured that the symmetry group of deformation can cure the UV/IR mixing problem of QFT on non-commutative spacetime (which is discussed below) and may deform particle Bose- and Fermi- statics. The latter fact means that the Pauli exclusion principle could be violated at high energies, which is considered as a possible qualitative experimental consequence of the spacetime non-commutativity [67][8]. Building an interacting theory in this setup is however problematic if possible at all as it was pointed out in [68, 69].

The original, not twisted Lorentz symmetry of the non-commutative spacetime defined by (1.5) can be preserved if the constant matrix θ is replaced by a matrix of central (with respect to the algebra generated by the coordinates q^μ) operators transforming under the Lorentz group as a tensor. This was suggested by Doplicher, Fredenhagen and Robets in the already mentioned paper [38]. The joint spectrum of $\theta^{\mu\nu}$ must be Lorentz-invariant. The minimal choice is a single Lorentz orbit. In [38] such an orbit was chosen in such a way that the DFR STUR (1.4) follow. This spacetime (see Subsection 2.1 for a more rigorous definition) will be called the Doplicher-Fredenhagen-Roberts Quantum Spacetime (DFR QST). The generalization with non-central commutators was suggested in [70], while in [42] generalization to cosmological backgrounds is discussed.

The DFR QST, being a simple Lorent-invariant model of the quantum spacetime (unlike the Snyder's model which is rather a deformation of the quantum phase space) is at least a natural toy model to develop main notions and tools for physics in such an environment. For this thesis a study of adiabatic properties of a particular model (see below) of QFT in the DFR QST was the main original goal, although the results are applicable to much more general class of non-local QFTs.

Having the quantum spacetime chosen one should define QFT on it. Unfortunately, in the absence of the locality restriction, this step is very ambiguous. There are several approaches to

[8] Current experiments rule out the twisted statistics which could be induced by non-commutativity up to energy scales of order of $250 \cdot 10^{16} - 10^{18} GeV$ depending on the invariants of θ [67], which means that the non-locality scale is below $20 l_P$.

perturbative QFT construction as shortly reviewed in Section 3. Being applied to local theories they all lead to equivalent results. It is no longer the case when the non-locality is present. QFT on a non-commutative spacetime is even less definite notion because for that one should also explain how exactly non-commutativity induce non-locality. The simplest choice is to replace point-wise products of quantum fields in interaction terms (of the Hamiltonian, the Lagrangian or the equation of motion, depending on the approach to quantization chosen) by the non-commutative product. This is, for example, the approach usually adopted by the string theory community [55]. However, as it is explained later, this choice is not always the best one.

Furthermore, there is no common agreement on the desired properties of QFT on quantum spacetime. For example, from the review above one could expect that (at least in the quantum gravity context) the non-commutativity-induced non-locality should eliminate the UV divergences. This is however not the case in the already mentioned string-theory-inspired non-commutative QFT. Instead, some of the graphs (the so-called planar graphs, i.e. graphs which can be embedded into a plane) are not affected by the non-commutativity at all. If the divergent subgraph is non-planar, the UV divergence is regularized but a new IR divergence (see for a discussion IR divergences below) appears. This phenomenon, known under the name of the UV/IR mixing, in the string theory community is considered as a manifestation of the deep interplay between UV and IR regimes of the string theory [55], while from the mathematical point of view it is an obstruction on the way to rigorous formulation of such a theory which should be avoided in some way.

We will work with the so-called Hamiltonian approach to the non-local quantum field theories quantization (introduced in [71] and reviewed in 3.1.2) and take the "quantum diagonal map" (introduced in [72] and reviewed in 2.3) on the DFR quantum spacetime as a sample of a non-local interaction kernel (the results will be applicable for more general kernels). This setup was thoroughly considered in [73]. In particular, the UV-finiteness was shown and the Feynman rules were derived. However, a problem with the adiabatic limit was observed.

The adiabatic limit is one more point which generalizes non-trivially to the case of quantum spacetime. According to the Haag theorem [4] the interaction representation, which is crucial for most of practical calculations in QFT can not exist in any interacting theory in a uniform spacetime. The usual trick to bypass this obstacle is to break spacetime uniformity by introducing an adiabatic cut-off, i.e. assuming that the interaction vanishes in the distant past and future and is constant somewhere in the middle[9]. The physical amplitudes can be achieved by considering a limit of the interaction parameter going to a constant. However, there are different possibilities to choose a stage when this limit should be taken.

One way is to find the adiabatic limit of the time-ordered correlators (called the weak adiabatic limit) and then reconstruct the scattering amplitudes using a variant of the LSZ reduction. The main virtue of this approach is that the weak adiabatic limit is known to be less sensitive to the normalization. The price we pay for that is the necessity to use the LSZ reduction based on not trivial to prove assumptions. Finiteness of the weak adiabatic limit in local QFT was shown in [7].

Another way, the so-called "strong adiabatic limit", is to keep the adiabatic cut-off until the very end of the computation and evaluate the adiabatic limit of the scattering operator. While the adiabatic cut-off is imposed the scattering operator can be defined rigorously, hence no LSZ reduction is needed. But the existence of the strong adiabatic limit is more tricky. In [74] ordinary QFT with only one massive specie of particles was shown to exist if and only if the in(out)coming fields are renormalized, so that the self-energy corrections vanish on-shell together with their derivatives (with respect to frequencies).

It is worth noting that the existence of an adiabatic limit is not just a technicality, because its absence can be manifestation of the infrared divergences present in the theory. For example, in the presence of the already mentioned UV/IR mixing the amplitudes are finite until the adiabatic limit

[9] In general one requires that the adiabatic switching parameter must be compactly supported. But for the theories we consider in the main body of the thesis the adiabatic cut-off only in time is sufficient (Pro position 3.13) as was first noticed in [73, 72].

is considered.

There is also a third option, the algebraic adiabatic limit, i.e. the adiabatic limit of the net of algebras of local observables reviewed in [5] which allows to define the adiabatic limit for theories on general (commutative) spacetime. This concept is, however, not directly applicable to the non-commutative case due to the absence of locality. Generalization of those ideas to quantum spacetime could be an interesting and perspective problem but is beyond the scope of this thesis.

From the above, it is clear that the renormalization is crucial for the strong adiabatic limit discussion. Moreover, to renormalize a theory so that the dispersion relation correction vanishes we should be able to compute that correction. This is why the renormalization and the computation of the propagator corrections are considered in this thesis alongside with the adiabatic limits. The corrected propagator is also important to decode the scattering amplitudes from the weak adiabatic limit with the LSZ reduction.

The dispersion relation correction was already computed at lowest orders in [75] through time-independent perturbation theory. One should note that a completely different description of the adiabatic cut-off is needed to use the time-independent perturbation theory, which could be convenient in the stationary states energies calculation. This method was designed to calculate corrections to the free stationary states and hence requires a time-independent Hamiltonian. Thus the Haag theorem should be bypassed by introducing a spatial cut-off instead of a temporal one. So, it is an interesting side result that our energy corrections do coincide with the ones found in [75].

The main topic of this thesis is the study of peculiarities of adiabatic limits and the renormalization in the case of non-local QFT. This thesis is organised as follows. In Second Section we introduce the DFR QST and discuss how free quantum fields and interaction terms can be defined. In Third Section we describe main methods of perturbative QFT and discuss their applicability to the non-local case. In Fourth Section we present several formulation of Feynman rules which are convenient for our purposes, and in Fifth section we discuss how they can be related to more conventional Lagrangian formulation. In Sixth Section we compute the quantum corrections to the propagator. In Seventh we prove existence of the weak adiaibatic limit and suggest necessary conditions for the strong one. The ambiguity of the renormalisation is discussed in Eighth Section. After that we summarise main results with a small disucssion and outline of further possible steps.

2 Doplicher-Fredenhagen-Roberts Quantum Spacetime

2.1 Construction and basic facts

Let us begin with reviewing the basic setup of the DFR QST following [37, 38].

As was stated in the introduction, the DFR QST is a realizaion of the space-time uncertainty relations (1.4) through quantum (i.e. non-commuative) geometry of the spacetime preserving the undeformed Poincare symmetries.

The algebra of functions on that spacetime is generated by the four self-adjoint operators q^μ subject to the commutation relations

$$[q^\mu, q^\nu] = il_P^2 Q^{\mu\nu}, \tag{2.1}$$

where l_P is a constant (the Planck scale) and $Q^{\mu\nu}$ are central operators. We assume that the full proper Poincare group $\mathfrak{P}$ (see Appendix A for conventions) acts on that generators in the usual way, i.e.

$$(M, a) \curvearrowright q^\mu = M^\mu{}_\nu q^\nu + a^\mu. \tag{2.2}$$

Then from (2.1) the centrals $Q^{\mu\nu}$ automatically transofrm as a tensor. The joint spectrum of center then should be then Lorentz-invariant and a minimal choice is a single Lorentz group orbit, guarantying the relations (1.4). Such an orbit can be always fixed by specifying the invariants of tensor $Q^{\mu\nu}$. As it was shown in [38] the most natural choice is the hypersurface Σ defined by the "quantum conditions"

$$Q^{\mu\nu} Q_{\mu\nu} = 0, \tag{2.3}$$

$$\left(\frac{1}{8}\epsilon_{\alpha\beta\mu\nu} Q^{\alpha\beta} Q_{\mu\nu}\right)^2 = 1. \tag{2.4}$$

A useful reparametrization of this hypersurface is given by two three-dimensional vectors $\vec{e}$, $\vec{m}$ defined as

$$Q^{0i} = \vec{e}^i, \quad Q^{ij} = \epsilon_{ijk}\vec{m}^k. \tag{2.5}$$

Then (2.3-2.4) take the form

$$(\vec{e}\cdot\vec{m})^2 = 1, \qquad \vec{e}^2 = \vec{m}^2. \tag{2.6}$$

The hypersurface Σ clearly consists of two diffeomorphic connected pieces $\Sigma_\pm$ with $\vec{e}\vec{m} = \pm 1$. Each of those pieces is diffeomorphic to the tangent bundle of the 2-sphere TS^2 (see Appendix B of [38]).

No bounded operator can satisfy (2.1) with central non-vanishing $Q^{\mu\nu}$, hence the coordinates q^μ can not be embedded into any C*-algebra. Moreover, from the experience of quantum mechanics and quantum field theory we know that the relations (2.1) also admit pathological realizations. In particular, the uncertainty relations can be violated (see e.g. [76](subsection 12.2 and example 14.5). To avoid that we restrict our attention to the regular realizations, i.e. such that the formally exponentiated form of (2.1) holds:

$$e^{ik_\mu q^\mu} e^{ip_\mu q^\mu} = e^{ip_\mu q^\mu} e^{ik_\mu q^\mu} e^{-k_\mu p_\nu Q^{\mu\nu}}.$$

In [37, 38] it was shown that such realizations are in one-to-one correspondence with non-degenerate representations of C*-algebra $\mathcal{E} = C_0(\Sigma, \mathcal{K})$ of continuous vanishing at infinity functions on Σ valued in the algebra of compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H})$ on a fixed separable Hilbert space. The unbounded operators q^μ then are affiliated with $\mathcal{E}$ and the symmetry action (2.2) induces an action of $\mathfrak{P}$ on $\mathcal{E}$ by automorphisms. We refer to [38] (Section 4 and appendix A) for the proof and the exact formulation of this fact.

2.2 Quantum fields on the DFR QST and the role of $Q^{\mu\nu}$

According to the introduction, the quantum spacetime effects should play an important role in the interacting quantum field theory, but it is quite a difficult an ambiguous subject. We start by discussing the quantum field on the quantum spacetime definition, considering mostly the free fields and some general ideas. Discussion of particular approaches to perturbation theory is postponed until the next section. Except for the theories with modified statistics[10] the free quantum fields are usually assumed to be not changed by the spacetime quantisation.

In ordinary quantum field theories the quantum field is an operator-valued distribution D. For simplicity let us consider a scalar quantum field ϕ. In particular, the mass m free real scalar quantum field is defined as (E.9)

$$\phi_0(x) = \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2\omega^m_{\vec{p}}}} \left(a_{\vec{p}} e^{-ipx} + a^+_{\vec{p}} e^{ipx}\right),$$

where $a^+_{\vec{p}}$ and $a_{\vec{p}}$ are creation and annihilation operator-valued distributions[11] and the 4-vector p is of the form $(\omega^m_{\vec{p}}, \vec{p})$, $\omega^m_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$[12].

The simplest generatlisation to the quantum spacetime case is to formally substitute x with the quantum coordinate q,

$$\phi_0(q) = \int d^3\vec{p} \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} \otimes e^{-ipq} + a^+_{\vec{p}} \otimes e^{ipq}\right).$$

The distributional nature of ϕ_0 can be restored if we view it as an operator-valued functional defined on a particular subset of $S(\mathcal{E})$, the statespace of $\mathcal{E}$,

$$\phi_0[\omega] = \int d^3\vec{p} \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}\omega(e^{-ipq}) + a^+_{\vec{p}}\omega(e^{ipq})\right). \tag{2.7}$$

Remark 2.1. Since any bounded functional on $\mathcal{E}$ is a linear combination of states [87], we can also say that a quantum field is a functional on a particular subset of the dual space $\mathcal{E}^*$. We should note, however, that in general $\mathcal{E}^*$ may be to large as a test-functions space. For example, in case of commutative $\mathcal{E}$ the dual space $\mathcal{E}^*$ would contain point-localised pure states which are clearly not admissible for evaluation of the quantum field.

Remark 2.2. In most of the approaches to QFT on quantum spacetimes, the non-commutativity of the underlying geometry affects only the interaction terms which become non-local. In all further steps (quantisation, perturbation theory, etc.) the theory is treated as a non-local QFT in the usual commutative spacetime. This method should lead to at least an effective theory description of QFT on QST valid when the QST effects are not too strong and catching its main peculiar properties. We will also proceed in such a way.

In DFR QST, unlike most of the other approaches listed in the introduction, as a byproduct of the Lorentz-invariance, there is an additional manifold Σ[13]. Although the free quantum field (2.7) does no vary along it, the interactions most likely will. For example, in string theory-inspired non-commutative QFT one uses the non-commutative product $\phi(q)^n$ instead of the pointwise product $\phi(x)^n$. This interaction was adopted to DFR QST (see [73, 68] and Example 3.17) unavoidably

[10] See the introduction.

[11] See Appendix E for details and normalisation agreements.

[12] In this thesis we will be also interested in generic dispersion relations, because interactions, breaking the Lorentz symmetry will be considered. But in this subsection we consider bare (i.e. non-renormalised) free field, which is assumed to be Lorentz-invariant.

[13] This is not a unique situation in generalizations of QFT, see Remark 2.6 and references wherein.

leading to dependence of the interaction term on Q. There are different ways to treat the auxiliary space Σ in the interacting theories. One natural idea is to average over it. Unfortunately it does not admit a Lorentz-invariant finite measure, so the Lorentz invariance is inevitably broken, but rotational one can be preserved. The averaging can be done inside the Lagrangian (or Hamiltonian) or for the resulting amplitudes in the very end. We refer to [73] for a comparison of these approaches. A different approach, in which the interaction term is not dependent on Q at all is discussed in the next subsection.

2.2.1 More general quantum fields in DFR QST

In the rest of this subsection we sketch how things can change if one treats the quantum spacetime $\mathcal{E}$ seriously. This discussion will not be directly related with most of the thesis, but provides interesting directions of further development of the results we achieve. This should not be considered as a complete classification, because the optimal restriction definition of the quantum field (see Remarks 2.1 and 2.12).

The field (2.7) is not the most general free scalar field in the DFR QST. This form follows from our experience in the ordinary quantum field theory which tells us that the quantum fields live in a four-dimensional world, so the extra coordinates describing position on Σ should be ignored.

As discussed in Appendix E, the most general form of a covariant bosonic free quantum field in the Minkowski spacetime is given by (E.10)

$$\phi_0^i(\vec{x},t)=\sum_\sigma\int\frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2\omega_{\vec{p}}^m}}\left(a_{\vec{p},\sigma}e^{-i(\omega_{\vec{p}}^m t-\vec{p}\vec{x})}+a^+_{\vec{p},\sigma}e^{i(\omega_{\vec{p}}^m t-\vec{p}\vec{x})}\right)\Psi_{i,\sigma}(p),$$

where the polarisation vectors are constructed by (C.5)

$$\Psi_{i,\sigma}(p)=\sum_j L_{ji}(R(p)^{-1})\Psi^{(0)}_{j,\sigma}$$

from the rest frame polarisations, satisfying (C.6)

$$\sum_j L_{ji}(O)\Psi^{(0)}_{j,\sigma}=\sum_{\sigma'}D^s_{\sigma,\sigma'}(O)\Psi^{(0)}_{i,\sigma'},\quad\forall O\in SO(3).$$

for some representation L of $SO(1,3)^\uparrow$. D^s is the spin-s representation of $SO(3)$. Here we assumed that $s\in\mathbb{N}$ and L is real, so we replaced $SL(2)$ with $SO(1,3)^\uparrow$ and $SU(2)$ with $SO(3)$ in (C.4-C.6). Finally, $R(p)$ is a boost from the rest reference frame to one where the particle has the momentum p. This form generalises the well-known electro-magnetic and Dirac fields of QED ensuring the covariance and locality. The details can be found in Appendices C and D.

The key observation is that in DFR QST a field, and thus the polarisations can depend on the central charges $Q^{\mu\nu}$,

$$\phi_0(q)=\sum_\sigma\int\frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2\omega_{\vec{p}}^m}}\left(a_{\vec{p},\sigma}e^{-ipq}+a^+_{\vec{p},\sigma}e^{ipq}\right)\Psi_\sigma(p,Q).$$

The Lorentz group of the DFR QST transforms Q as a tensor, so for covariance of $\phi_0^i(\vec{x},t)$ we require an analogue of (C.5)[14]

$$\Psi_\sigma(p,Q)=\Psi^{(0)}_\sigma\left(R(p)^{-1}\curvearrowright Q\right);\tag{2.8}$$

and of (C.6):

$$\Psi^{(0)}_\sigma(O^{-1}\curvearrowright Q)=\sum_{\sigma'}D^s_{\sigma,\sigma'}(O)\Psi^{(0)}_{\sigma'},\quad\forall O\in SO(3).\tag{2.9}$$

[14] the symbol $\curvearrowright$ denotes the action of symmetries as explained in Appendix A.

Example 2.3. Let $s = 0$. Then (2.9) requires rotational invariance of the polarisations. In the parametrisation (2.5), taking into account the quantum conditions (2.6), we have two rotational invariants, $\vec{e}^2$ and $\vec{e}\cdot\vec{m} = \pm 1$. For simplicity let us disregard the latter as non-invariant with respect to the inversions. Then the rest-frame polarisation is of the form

$$\Psi^{(0)}(Q) = \zeta(\vec{e}^2).$$

In the rest frame $p = (m, \vec{0})$ we have

$$Q^{\mu\eta}Q_{\eta\nu}p_\mu p^\nu = -m^2\vec{e}^2$$

and hence by (2.8) we have

$$\Psi(Q) = \zeta\left(-\frac{1}{m^2}Q^{\mu\eta}Q_{\eta\nu}p_\mu p^\nu\right).$$

Example 2.4. Let $s = 1$, and let D^1 be chosen as the defining representation of $SO(3)$, so that the polarization index σ which in this runs through $\{x, y, z\}$. Then (2.9) states that $\Psi^{(0)}$ transforms as a vector. We have two choices

$$\vec{\Psi^{(0)}}(\vec{e}, \vec{m}) = \vec{e}\zeta(\vec{e}^2, \vec{e}\cdot\vec{m})$$

and

$$\vec{\Psi^{(0)}}(\vec{e}, \vec{m}) = \vec{m}\zeta(\vec{e}^2, \vec{e}\cdot\vec{m}),$$

where $\vec{\Psi^{(0)}}$ symbolically denotes all three polarisations

$$(\Psi^{(0)}_x, \Psi^{(0)}_y, \Psi^{(0)}_z).$$

Example 2.5. For a generic integer spin s the representation D^s can be understood as the symmetric of the sth tensor power of D^1 [80]. Then as rest-frame polarization one should take any completely symmetric tensor formed from $\vec{e}$ and $\vec{m}$ multiplied by an arbitrary function of the invariants $\vec{e}^2$ and $\vec{e}\cdot\vec{m}$.

Remark 2.6. From the examples above we see, that, unlike the commutative case, in the DFR QST the presence of additional manifold Σ allows to define a scalar quantum field of any integer spin. In principle a single scalar quantum field can create particles of several, or even infitely many spins.

This is similar to the string theory with compactified dimensions [88] where the modes propagating in the compact dimensions (unobservable directly) are reinterpreted as different fields in the 4-dimensional spacetime, and in some sense dual to the Non-Commutative Standard Model [57], where the fields on commutative spacetime are modes on a compact (and even finite) non-commutative auxiliary space. Here instead the spacetime is non-commutative and the auxiliary space is a smooth manifold. The main difference of our situation from both the aforementioned cases is that our auxiliary space Σ fails to be compact. Because of that there is no a priori reason to assume that the spectrum of modes on it is discrete.

Finally, we note that non-compactified string theories also have an infinite tower of modes of different angular momenta. For other similarities between string theories and quantum field theories on the DFR QST we refer to [70].

We restricted ourselves to one-component quantum fields on the DFR QST because they are enough for any integer spin. Of course, multi-component fields are also possible The relevant transformation law for the rest frame polarisations is a combination of (C.6) with (2.9):

$$\sum_j L_{ji}(O^{-1})\Psi^{(0)}_{j,\sigma}(O^{-1} \curvearrowright Q) = \sum_{i,\sigma'} D^s_{\sigma,\sigma'}(O)\Psi^{(0)}_{\sigma'}(Q), \quad \forall O \in SO(3). \tag{2.10}$$

Example 2.7. Let L be the defining representation of $SO(1,3)^\uparrow$, and $s=1$. Then the most general form is[15]

$$\Psi^{(0)}_{0\sigma}(Q) = \vec{e}^{\sigma}\zeta_{0e}(\vec{e}^2,\vec{e}\cdot\vec{m}) + \vec{m}^{\sigma}\zeta_{0m}(\vec{e}^2,\vec{e}\cdot\vec{m}) + [\vec{e}\times\vec{m}]^{\sigma}\zeta_{0\times}(\vec{e}^2,\vec{e}\cdot\vec{m})$$

$$\Psi^{(0)}_{j\sigma}(Q) = \delta_{j,\sigma}\zeta_{\delta}(\vec{e}^2,\vec{e}\cdot\vec{m}) + \vec{e}^{j}\vec{e}^{\sigma}\zeta_{ee}(\vec{e}^2,\vec{e}\cdot\vec{m}) + \vec{m}^{j}\vec{m}^{\sigma}\zeta_{mm}(\vec{e}^2,\vec{e}\cdot\vec{m}) + \vec{e}^{j}\vec{m}^{\sigma}\zeta_{em}(\vec{e}^2,\vec{e}\cdot\vec{m}) + \vec{m}^{j}\vec{e}^{\sigma}\zeta_{me}(\vec{e}^2,\vec{e}\cdot\vec{m}) +$$

$$[\vec{e}\times\vec{m}]^{j}[\vec{e}\times\vec{m}]^{\sigma}\zeta_{\times\times}(\vec{e}^2,\vec{e}\cdot\vec{m}) + [\vec{e}\times\vec{m}]^{j}\vec{e}^{\sigma}\zeta_{\times e}(\vec{e}^2,\vec{e}\cdot\vec{m}) +$$

$$[\vec{e}\times\vec{m}]^{j}\times\vec{m}^{\sigma}\zeta_{\times m}(\vec{e}^2,\vec{e}\cdot\vec{m}) + \vec{e}^{j}[\vec{e}\times\vec{m}]^{\sigma}\zeta_{e\times}(\vec{e}^2,\vec{e}\cdot\vec{m}) + \vec{m}^{j}[\vec{e}\times\vec{m}]^{\sigma}\zeta_{m\times}(\vec{e}^2,\vec{e}\cdot\vec{m}).$$

If only ζ_δ is non-zero, we are left with the "conventional" spin-1 4-vector field up to an additional ζ factor, while other cases are specific DFR QST realisations of the spin-1 field.

Example 2.8. Continuing the previous example, assume that

$$\zeta_{0e}(\vec{e}^2,\vec{e}\cdot\vec{m}) = \pm(1+\vec{e}^2)\zeta_{ee}(\vec{e}^2,\vec{e}\cdot\vec{m}),$$

$$\zeta_{em}(\vec{e}^2,\vec{e}\cdot\vec{m}) = \vec{e}\cdot\vec{m}\zeta_{ee}(\vec{e}^2,\vec{e}\cdot\vec{m}),$$

$$\zeta_{\times e}(\vec{e}^2,\vec{e}\cdot\vec{m}) = \mp\zeta_{ee}(\vec{e}^2,\vec{e}\cdot\vec{m})$$

and all other ζs vanish. Then, taking into account (2.5) and (2.6) we have

$$Q_0\ {}^{\nu}\Psi^{(0)}_{\sigma}(Q) = \sum_j \vec{e}^{j}\Psi^{(0)}_{j\sigma}(Q) = (1+\vec{e}^2)\zeta_{ee}(\vec{e}^2,\vec{e}\cdot\vec{m}) = \pm\Psi^{(0)}_{0\sigma}(Q),$$

$$Q_j\ {}^{\nu}\Psi^{(0)}_{\nu\sigma}(Q) = -\sum_j \vec{e}^{j}\Psi^{j(0)}_{\sigma}(Q) = \vec{e}^{j}\Psi^{(0)}_{0\sigma}(Q) + [\vec{m}\times\vec{\Psi}^{(0)}_{\sigma}(Q)]^{j} =$$

$$\vec{e}^{j}(\zeta_{0e}(\vec{e}^2,\vec{e}\cdot\vec{m}) + \vec{e}^2\zeta_{\times e}(\vec{e}^2,\vec{e}\cdot\vec{m})) - [\vec{e}\times\vec{m}]^{j}\zeta_{ee}(\vec{e}^2,\vec{e}\cdot\vec{m})) - \vec{m}^{j},\vec{e}\cdot\vec{m}\zeta_{\times e}(\vec{e}^2,\vec{e}\cdot\vec{m})) = \pm\Psi^{(0)}_{j\sigma}(Q).$$

Here we used $\vec{\Psi}^{(0)}_{\sigma}(Q) = (\Psi^{(0)}_{1\sigma}(Q),\Psi^{(0)}_{2\sigma}(Q),\Psi^{(0)}_{3\sigma}(Q))$. Summing up, we have

$$Q_\mu\ {}^{\nu}\Psi^{(0)}_{\nu\sigma}(Q) = \pm\Psi^{(0)}_{\mu\sigma}(Q).$$

By covariance this equality holds after the action of $L(R(p)^{-1})^T$, so we have

$$Q_\mu\ {}^{\nu}\Psi_{\nu\sigma}(p,Q) = \pm\Psi_{\mu\sigma}(p,Q),$$

i.e. $\Psi_{\nu\sigma}(p,Q)$ (and hence the quantum field constructed from it) belongs to the eigenspace of (mixed components of) Q, which is one-dimensional[16]. So, in this case the quantum field describing a spin-1 particle is again essentially one-component.

In general, given a representation L one can find real[17], covariant projection, i.e.

$$P:\Sigma\to\mathbb{C}^{\dim L},\quad P(Q)^2=1,\quad \overline{P(Q)}=P(Q) \tag{2.11}$$
$$L^*(M)P(M^{-1}\curvearrowright Q)L^*(M)^{-1} = P(Q)\,\forall M\in SO(1,3)^\uparrow.$$

Furthermore, there are always such projections with one-dimensional image.

[15] We note that Ψ transforms under the action of the Lorentz group with the representation dual to the defining, i.e. as a covector. So, usage of the lower indices is in accordance with Appendix A.

[16] This fact is easy to check for a particular choice $\vec{e}=\vec{m}=(0,0,1)$. It is also easy to see that the dimnsions of the eigenspaces of Q is Lorentz-invariant

[17] Note that orthogonality is not required and actually not defined due to absence of a preferred metric in representations of L.

Example 2.9. For L being the defining representation,

$$P_{\pm}(Q) = \frac{\pm\hat{Q} + \hat{Q}^2 \pm \hat{Q}^3 + \hat{Q}^4}{4},$$

where $\hat{Q} \in M_4(\mathbb{C})$ is a matrix formed by the mixed components of Q, i.e.[18]

$$\hat{Q}_\mu{}^\nu = Q_\mu{}^\nu.$$

This matrix is clearly covariant. By direct calculation it is a projection for $\vec{e} = \vec{m} = (1,0,0)$, where $\vec{e}_z$ is the zth basis vector and hence at any other poin of Σ by covariance.

Then we can define the reduced rest frame polarisations (2.10) .

$$\Psi^{(0,red)}_{i,\sigma}(Q) = \sum_j P(Q)_{i,j}\Psi^{(0)}_{j,\sigma}(Q),$$

which will automatically satisfy (2.10), so they can serve as rest frame polarisations[19]. The full quantum field in this setting is

$$\phi_{0,i}(q) = \sum_\sigma \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2\omega^m_{\vec{p}}}} \left(a_{\vec{p},\sigma}e^{-ipq} + a^+_{\vec{p},\sigma}e^{ipq}\right) \Psi^{(red)}_{i,\sigma}(p,Q),$$

$$\sum_j L_{ij}(R(p))\Psi^{(0)}_{j,\sigma}\left(R(p)^{-1} \curvearrowright Q\right).$$

By construction it satisfies

$$\sum_j P_{ij}(Q)\phi_{0,j}(q,Q) = \phi_{0,j}(q,Q),$$

so it is essentially one-component. But in general it can not be replaced by a scalar field because the vector bundle

$$\{(Q,v) | Q \in \Sigma, v \in \mathbb{C}^{\dim L}, P(Q)v = v\}$$

can be non-trivial. In particular, applying the projection of Example 2.9 to the polarisations of Example 2.7 we arrive to Example 2.8.

Remark 2.10. The observation above is in accordance with general idea that if the spacetime topology is non-trivial (in our case due to the auxiliary space Σ), the fields should be sections of the bundles over that spacetime[20]. Natural generalization for the non-commutative case is to seek for the quantum fields as elements of the modules over the corresponding C^*-algebra [89]. In our case such a module is

$$P\mathcal{E}^{\oplus \dim E}.$$

We note however that the bundles and modules we consider do not posses any natural hermitian product because they all come from representations of $SO(1,3)^\uparrow$.

Remark 2.11. Since generic irreducible representation of $SO(1,3)^\uparrow$ is nth symmetric tenor power of the fundamental one [77], all these arguments directly generalized to the case of arbitrary finite-dimensional L.

[18] Following the common tradition we use the same symbol for the original antisymmetric tensor $Q^{\mu\nu}$ and all three possible results of lowering one or two indices of it. In contract to that $\hat{Q}$ denotes a fixed matrix and $\hat{Q}^n$ denotes the nth power of that matrix in $M_4(\mathbb{C})$.

[19] If they do not vanish; otherwise take $1 - P$.

[20] More precisely, the classical fields are smooth sections of a vector bundle. Quantum fields are not functions at all, but we may expect that their averages at some special class of states are also smooth sections of that bundle.

Finally, we can consider spinor fields in the same way, starting from a generic representation L of $SL(2)$. Then we should just replace $SO(1,3)^\uparrow$ by $SL(2)$ and $SO(3)$ by $SU(2)$ everywhere. In particular, the observation about the projection and bundles holds. Of course, if we want the field to be approximately local in any sense, we have to use fermionic fermionic Fock space instead of the bosonic one.

Remark 2.12. In all the examples above we never stated the class of the function ζ. It depends on the particular definition of the quantum field on the non-commutative spacetime, namely on the choice of subset of "test" functionals in $\mathcal{E}^*$ and its topology (see Remark 2.1). We can consider ones of the form

$$\omega_f(A) = \int_\Sigma \tau \left(\int f(k,Q) e^{-ikQ} d^4k A(Q) \right) d\mu_\Sigma(Q)$$

where τ is the trace and $f \in \mathcal{S}(\mathbb{R}^{10})$ (as a function of 4 components of k and 10 components of an anti-symmetric tensor Q treated as independent) and require that the quantum field is an operator-valued distribution in the ordinary sense as a functional of f. Then any $\zeta \in \mathcal{S}'(\mathbb{R})$ would do. At the same such quantum fields may be to singular, because the UV-divergences may reappear in pointwise products.

2.3 Optimally localised states and the quantum diagonal map

Since the DFR quantum spacetime is designed so that it realizes the STUR (1.4), it is worth to construct the optimally localised states of the algebra $\mathcal{E}$ which saturate these limitations. Unfortunately, there is no Lorentz invariant way to estimate the localisation. So we fix a reference frame and look for a minimum of

$$\sum_{\mu=0}^{4} \left(\Delta_\omega q^\mu\right)^2, \tag{2.12}$$

where the uncertainties are defined by

$$\Delta_\omega f^2 = \omega\left(\left(f - \omega(f)\right)^2 \right).$$

For the (2.12) to be well-defined we restrict ourselves to the states in the domain of q^μ and $\left(q^\mu\right)^2$ as explained in [38].

A generic state minimising (2.12) has the form

$$\omega_{a,\mu}(F) = \int d^4\alpha d\mu(\sigma) f(\sigma,\alpha) e^{i\alpha_\mu a^\mu - \frac{1}{2}\sum_\mu \alpha_\mu^2},$$

where

$$\mathcal{E} \ni F(\sigma) = \int f(\sigma,\alpha) e^{i\alpha^\mu q_\mu} \tag{2.13}$$

and μ is a measure on Σ supported on $\Sigma' \subset \Sigma$ defined by $\vec{e} = \pm\vec{m}$. Since the operators of the form (2.13) span $\mathcal{E}$ densely, $\omega_{\alpha,\mu}$ has a unique continuation to the whole $\mathcal{E}$. The localisation map $\eta_a : \mathcal{E} \to C_0\Sigma'$ defined by extending of

$$\eta_a(F)(\sigma) = \int d^4\alpha f(\sigma,\alpha) e^{i\alpha_\mu a^\mu - \frac{1}{2}\sum_\mu \alpha_\mu^2}$$

encodes all the states localised at a in the sense that

$$\omega_{a,\mu} = \omega_\mu \circ \eta_a,$$

where

$$\omega_\mu(f) = \int f(\sigma) d\mu(\sigma)$$

is the state of $C_b(\Sigma')$ associated with the measure σ. The states $\omega_{a\mu}$ play the role of the points of the DFR QST.

In the quantum field theory context it is more important to discuss localisation of events relative to each other. This problem was addressed in [72]. To deal with coordinates of n events a new algebra

$$\mathcal{E}_n = \mathcal{E}^{\otimes_Z n} = \mathcal{E} \otimes_Z \mathcal{E} \otimes_Z \cdots \otimes_Z \mathcal{E}$$

was introduced. Here the right hand side contains n multipliers, and $\otimes_Z$ denotes the tensor product over the centre $Z(\mathcal{E})$. Geometrically it means that we are looking at the events happened at exactly the same point of Σ.

Introduce

$$q_j^\mu = 1 \otimes_{\mathcal{Z}} \cdots 1 \otimes_{\mathcal{Z}} q^\mu \otimes_{\mathcal{Z}} 1 \otimes_{\mathcal{Z}} \cdots 1,$$

with q^μ staying on the jth position. We have

$$[q_j^\mu, q_k^\mu] = i l_P^2 \delta_{ij} Q^{\mu\nu}.$$

To deal with localisation of the events relative to each other, it is natural to separate the mean coordinate

$$\overline{q}^\mu = \frac{1}{n} \sum_{j=1}^n q_j^\mu$$

and the relative coordinates

$$q_{jk}^\mu = q_j^\mu - q_k^\mu.$$

Thanks to the identification of $Q^{\mu\nu}$ for all the events, the commutation relations for the new generators are very simple,

$$[q_{jk}^\mu, \overline{q}^\nu] = 0,$$

$$[\overline{q}^\mu, \overline{q}^\nu] = \frac{i}{n} l_P^2 Q^{\mu\nu},$$

$$[q_{jk}^\mu, q_{lr}^\nu] = l_P^2 Q^{\mu\nu} (\delta_{jl} + \delta_{kr} - \delta_{jr} - \delta_{kl}),$$

i.e. the mean coordinate satisfies the same commutation relations, as the coordinates of individual events up to a rescaling, and they factorise from the algebra generated by the relative coordinates. More precisely, as it is explained in [72], there is a *-homeomorphism $\beta^{(n)}$ from $\mathcal{E}_n$ to $M(\mathcal{E}_{n+1})$, the multiplier algebra of $\mathcal{E}_{n+1}$, induced by the maps[21]

$$q_j^\mu \mapsto \tilde{q}_{n+1}^\mu + \frac{1}{n} \sum_{k=1}^n q_{kj}^\mu,$$

where $\tilde{q}_{n+1}^\mu = \frac{1}{\sqrt{n}} q_{n+1}^\mu$. In particular, for $g \in C_0(\Sigma)$, $f \in C_0(\mathbb{R}^{4n})$ the operator

$$g(Q) f(q_1, \ldots, q_n) = g(Q) \int d^4k_1 \cdots d^4k_n \tilde{f}(k_1, \ldots, k_n) e^{ik_1 q_1} \cdots e^{ik_n q_n} \in \mathcal{E}_n, \tag{2.14}$$

with $\tilde{f}$ being a Fourier transform of f, is mapped to

$$\beta^{(n)}(g(Q) f(q_1, \ldots, q_n)) = g(Q) f(q_1 + \bar{q}, \ldots, q_n + \bar{q})$$

with the right hand side defined analogously to (2.14). We see that in the context of the *-homeomorphism $\beta^{(n)}$ the $\tilde{q}_j$ of $\mathcal{E}_{n+1}$ with $j \leq n$ can be understood as displacements of the events

[21] For the sake of notation shortness we use the same symbols for operators affiliated with $\mathcal{E}_n$ and with $\mathcal{E}_{n+1}$. Also our notation differs from [72] by a permutation of factors in $\mathcal{E}_{n+1}$

with respect to the mean position $\bar{q}$. It is then natural to apply the localisation map centred at the origin η_0 to the first n factors as a realisation of the limit of n events brought the same point.

We define the quantum diagonal map

$$E_n : \mathcal{E}_n \to \mathcal{E}'$$

,

$$E_n = \eta_0 \otimes_Z \cdots \otimes_Z \eta_0 \otimes_Z \gamma,$$

where $\mathcal{E}'$ is $C_0(\Sigma', \mathcal{K})$ extended to a C*-algebra and $\gamma : \mathcal{E} \to \mathcal{E}'$ is the restriction to Σ' needed for E_n to be well-defined on the tensor product with respect to the centre. The explicit action of E_n is

$$E_n(f(q_1, \ldots, q_n)) = \int dk_1 \cdots dk_n \tilde{f}(k_1, \ldots, k_n) r_n(k_1, \ldots, k_n) e^{i \sum_{j=1}^n k_j \tilde{q}} = \tag{2.15}$$

$$f_E(\tilde{q}),$$

$$f_E(x) = c_n \int da_1 \cdots da_n \tilde{r}_n(a_1, \ldots, a_n) f(x + a_1) \cdots f(x + a_n),$$

with

$$r_n(k_1, \ldots, k_n) = \exp\left(-\frac{1}{2} \left(\sum_{j=1}^n |k_j|_E^2 - \frac{1}{n} \sum_{j,l=1}^n (k_j \cdot k_l)_E \right) \right), \tag{2.16}$$

$$\tilde{r}_n(a_1, \ldots, a_n) = \delta\left(\frac{1}{n} \sum_{j=1}^n a_j \right) \exp\left(-\frac{1}{2} \sum_{j=1}^n |a_j|^2 \right) \tag{2.17}$$

and

$$c_n = n^2 (2\pi)^{-8(n-1)}.$$

Here $|\cdot|_E = ||\cdot||$ and $(\cdot, \cdot)_E$ denote the Euclidean norm and Euclidean scalar product respectively.

E_n is a non-commutative version of the diagonal map

$$E_n^{(0)} : C_b(\mathbb{R}^{4n}) \to C_b(\mathbb{R}^4),$$

$$E_n^{(0)}(f)(x) = f(x_1, \ldots, x_n)$$

taking into account the quantum spacetime properties and hence is called the quantum diagonal map. Unlike the usual diagonal map, the action of E_n can be safely extended to the distributions because of the smooth kernel $\tilde{r}_n$. In particular, it allows to define the "pointwise" product of quantum fields on the DFR QST yielding a well-defined operator-valued distribution which is a candidate for the interaction term in the DFR quantum spacetime. For example, one can consider self-interaction of a real scalar field

$$\phi_Q^n = E^{(n)}(\phi^{\otimes_Z n}).$$

Alternatively, one can consider the quantum Wick product, which combines the diagonal map with the usual Wick product (see Appendix F). For example, the nth Wick power of the free[22] real scalar quantum field is

$$: \phi_0(q)^n :_Q =: E^{(n)}(\phi^{\otimes_Z n}) : (q) = \tag{2.18}$$

$$\sum_{n_+=0}^n \binom{n}{k} \times$$

$$\int \prod_{j=1}^n \frac{dk_j}{\sqrt{2\omega_{\vec{p}}}} r_n(k_1, \ldots, k_n) \prod_{j=1}^{n_+} a_{k_j}^+ \prod_{j=n_++1}^n a_{k_j} e^{i\left(\sum_{j=1}^{n_+} k_j - \sum_{j=n_++1}^n k_j\right) q}.$$

[22]It may look a bit strange that we define the Wick power for a free field only, while, for example, in the Hamiltonian the full interacting quantum field appears. This will be clear after 3

Remark 2.13. Unlike ordinary QFT, $\phi^n_Q(q)$ is a well-defined quantum field in the sense of Subsection 2.2 (and also a well-defined quantum field in the sense of Appendix D if q is replaced by a commutative variable x by double Fourier transform as in Example 3.16), so the Wick product is not as necessary as in ordinary QFT (see Section 3). But it is important for the adiabatic limit existence (see Remark 3.14),

The problem of the adiabatic limit existence for a theory with such an interaction originally inspired the work reported in this thesis.

3 Perturbation theory for QFT and its non-local generalization

Except a few examples of exactly solvable models, the interacting quantum field theories are constructed perturbatively, i.e. by perturbing a free quantum field theory with an interaction

There are also more general non-perturbative methods mostly based on the functional integration which as we will see in the section is not directly applicable to the non-local in time case. There are also a lot of works devoted to the non-perturbative non-commutative QFT in the Euclidean signature, e.g. [96], but unlike ordinary QFT, their significance for the physical Lorentz signature is unclear. In this section we will shortly consider some of the approaches to perturbative quantum field theory and their non-local generalisations (when available). We will group them into three groups, according to where the perturbation is added. For simplicity we assume that all the observables are generated by one real scalar quantum field.

Before proceeding let us fix some notions. We assume that the parameters of the theory depend on a parameter g which we will call the interaction constant. All our consideration can be generalised to the case of several interaction constants $g_1, g_2, \ldots$, but we will ignore this possibility in most of the cases. In perturbation theory we deal with *formal power series* of the form

$$X(g) = \sum_{n=0}^{\infty} X_n g^n,$$

there depending on particular case X_n can be numbers, functions, operators, or, generally elements of any linear space. We do not discuss their convergence[23], but use formal addition and multiplication on them defined in the obvious way. We often drop the argument g and write X instead of $X(g)$.

Although X is no function of g, it will be convenient to say that X vanishes at $g = 0$ (or vanishes at the origin) if $X_0 = 0$. We will also use notation for the truncated formal series

$$[X]_N = \sum_{n=0}^{N} X_n g^n,$$

which can be understood as either a g-dependent finite sum or as a formal power series with only finite number of non-vanishing terms. We say that X has some properties in any finite order (e.g. is a polynomial of some other parameter), meaning $[X]_N$ has these properties for any $N \in \mathbb{N}$.

The virtue to consider sequences X_n as power series is that we can define appropriate operations on them. As we agreed that X_n are at least members of a vector space, we define a sum of the formal series as

$$\sum_{n=0}^{\infty} X_n g^n + \sum_{n=0}^{\infty} Y_n g^n = \sum_{n=0}^{\infty} (X_n + Y_n) g^n.$$

If in addition to that we have power series $X(g)$ and $Y(g)$, such that the product[24] $X_n Y_m$ is well defined for any $n, m \in \mathbb{N}$, then it is natural to define

$$XY(g) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} X_k Y n - k \right) g^n.$$

[23] In fact it is known that in ordinary QFT perturbation theory leads to divergent series [90] and there is no reason to expect that the non-locality will cure it. There are however prescriptions how to get physically relevant results from finitely many terms of such series [91].

[24] Depending on the particular nature of the members of the power series, the product can be understood as a product of numbers, tensor product of distributions, multiplication of a distribution by a smooth function, action of an unbounded operator on a vector from its domain etc. In other words, it is any operation we agreed to denote as a multiplication.

Clearly, sums and product behave well with truncations, i.e. $[X+Y]_N = [X]_N[Y]_N$, $[XY]_N = [[X]_N[Y]_N]_N$.

Our main goal will be to find the so-called scattering matrix S. We expect that it is a unitary operator (because roughly speaking it is an operator, describing unitary evolution of a system from the far past to the far future), but since it will be a formal series with respect to the interaction constant we require

$$S(g)^+ S(g) = \mathbf{1}$$

in terms of formal power series only. Moreover, finite sums $[S]_N$ are unbounded operators, so we actually assume that the equation above holds on their common domain only. We say that such an operator S is *formally unitary*.

3.1 Hamiltonian perturbation theory

3.1.1 Ordinary QFT

Let us start from a Poincare-covariant local quantum field theory. In particular, it is covariant with respect to the time translation. Then we have[25]

$$\partial_t \phi(x) = -i[H, \phi(x)], \tag{3.1}$$

where $\phi(x)$ is the quantum field and H is the time translation generator. We also introduce the conjugated momentum $\pi(x)$ which can be understood as another quantum field in the sense defined in Appendix D, covariant with respect to the rotations and translation, and

$$[\phi(t,\vec{x}), \pi(t,\vec{x}')] = \delta(\vec{x}-\vec{x}'). \tag{3.2}$$

The momentum $\pi(x)$ satisfies similar evolution equation

$$\partial_t \pi(x) = -i[H, \pi(x)], \tag{3.3}$$

We assume that

$$H = H_0(t) + H_{int}(t), \tag{3.4}$$

where

$$H_0(t) = \frac{1}{2}\int d\vec{k} : \left(\tilde{\pi}(\vec{k},t)\tilde{\pi}(-\vec{k},t) + \omega_{\vec{k}}^2 \tilde{\phi}(\vec{k},t)\tilde{\phi}(-\vec{k},t)\right) :. \tag{3.5}$$

Here $\tilde{\phi}$ and $\tilde{\pi}$ are the partial (namely, in spatial directions only) Fourier transforms of ϕ and π (see D.2), and

$$H_{int}(t)$$

is a fixed functional of ϕ and π. To build a local theory, one usually assumes that H_{int} is a local functional of ϕ and π evaluated at a fixed time hypersurface, namely

$$H_{int}(t) = \int d^3\vec{x} : h_{int}(\vec{x},t) :, \tag{3.6}$$

where $h_{int}(\vec{x},t)$ is a formal series of the interaction constant g vanishing at the origin[26], such that at any finite order it is a polynomial function of the field ϕ, momentum π and finitely many of their spatial[27] derivatives all evaluated at a point $(\vec{x},t)$ and $: \cdots :$ is the Wick product (see Appendix F).

[25] Here we implicitly assume that the unbounded operator H and the quantum field (see Appendix D) $\phi(x)$ are defined on the same domain, which they map to itself.

[26] The simplest case is h_int being proportional to the first power of g. However, in the effective QFT [23] the interaction part of the Hamiltonian (or the Lagrangian) is itself found iteratively as formal power series in terms of the interaction constant. We will see that theories which we consider in this thesis in details are more naturally treated by the EFT framework.

[27] The locality would not be broken if the Hamiltonian also contains derivatives with respect to the time. The standard perturbation theory, however, would, as we will see later in this subsection.

Remark 3.1. The Wick product gives rise to a well-defined quantum field. At the same time, formally the Wick product can be rewritten as a combination of pointwise products with divergent integrals as coefficients and as so it can be considered as a first step of renormalization in the sense of Remark 3.4.

Remark 3.2. One can object that even with the Wick product regularization (3.6) does not make sense because it is not a symbolic integral of the form (B.3). We can however treat H_{int} as a distribution on $\mathbb{R}$, then the only problem remains is the absence of the fast decay of the "test function" of spatial coordinates (which is identity), but it will be also cured by the adiabatic cut-off below.

To preserve translational symmetries, one assumes that the coefficients of h_{int} do not depend on t and $\vec{x}$. Then although both $H_0(t)$ and $H_{int}(t)$ are functions of time t, they sum is not since[28]

$$\partial_t H = -i[H, H] = 0.$$

The free quantum field (E.9) and its conjugated momentum

$$\pi_0 = \partial_t \phi_0$$

satisfy (3.1) and (3.3) with $H_{int} = 0$. To avoid possible confusions, we introduce a separate denomination for the free Hamiltonian as a functional of the free field and its momentum:

$$H^{(0)}(t) = \frac{1}{2}\int d\vec{k} : \left(\tilde{\pi}_0(\vec{k},t)\tilde{\pi}_0(-\vec{k},t) + \omega_{\vec{k}}^2 \tilde{\phi}_0(\vec{k},t)\tilde{\phi}_0(-\vec{k},t)\right) :,$$

$$\partial_t \phi_0(x) = -i[H^{(0)}, \phi_0(x)], \qquad \partial_t \pi_0(x) = -i[H^{(0)}, \pi_0(x)].$$

The key ingredient of perturbation theory is the so-called interaction representation, which is just a unitary equivalence between an interacting theory and a free one given by an operator U, such that

$$\tilde{\phi}(\vec{k},t) = U(t)^{-1}\tilde{\phi}_0(\vec{k},t)U(t), \tag{3.7}$$

$$\tilde{\pi}(\vec{k},t) = U(t)^{-1}\tilde{\pi}_0(\vec{k},t)U(t). \tag{3.8}$$

Then (3.1) and (3.3) are satisfied if

$$\partial_t U(t) = -iH_I(t)U(t), \tag{3.9}$$

where

$$H_I(t) = U(t)H(t)U(t)^{-1} - H^{(0)} = U(t)H_{int}(t)U(t)^{-1}$$

is the interaction representation of H_{int}. We assumed that H_{int} is a polynomial functional of ϕ and π evaluated at the fixed time t, thus by reversing (3.7-3.8) we get that H_I is just H_{int} with ϕ and π replaced with ϕ_0 and π_0 respectively. It is convenient to introduce also $h_I(x) = U(t)h_{int}(x)U(t)^{-1}$. So, solving (3.1) and (3.3) is equivalent to solving the linear equation (3.9).

However, Haag theorem [92] establishes that a Poincare-invariant theory unitary equivalent to the free one is also free. So, such U can not exist. To bypass this contradiction one can break the Poincare covariance down by replacing (3.6) with

$$H_{int}(t) = \int d^3\vec{x}\lambda(x) : h_{int}(\vec{x},t) :$$

[28]This observation is quite formal in the light of Remark 3.2. But it will not be important for us, so we leave it as is.

with $\lambda \in \mathcal{S}(\mathbb{R}^4)$ and for simplicity we assume that its support in the variable t is compact, $\lambda(\vec{x}, t) = 0$ for $|t| > T$ for some large enough T. We will call λ the adiabatic switching or adiabatic cut-off[29]. Then we may fix the initial conditions for (3.9) as

$$U(t) = \mathbf{1}, \quad t < -T.$$

Then we have

$$U(t) = \mathbf{1} + \int_{-\infty}^{t} H_I(t')U(t')dt'.$$

The formal power series in terms of g solving this equation can be achieved iteratively. We have

$$U(t) = U(t, -\infty),$$

$$U(t', t) = \sum_n (-i)^n \int_{t'>t_n>\cdots>t_2>t_1>t_0>t} dt_1 \cdots dt_n H_I(t_n) \cdots H_I(t_1) = \tag{3.10}$$

$$\mathbf{T}\{e^{-i\int_{t<\tau<t'} H_I(\tau)d\tau}\},$$

where $\mathbf{T}$ is the time-ordering, ordering the time-dependent operators inside it according to the time stamps. The above can be understood as a rigorous definition of $\mathbf{T}$. We note that the operator $U(t, t')$ is not well-defined, because the summand contains a product of two distributions,

$$H_I(t_n) \cdots H_I(t_1)$$

and

$$\theta(t' - t_n)\theta(t_n - t_{n-1}) \cdots \theta(t_2 - t_1)\theta(t - t_1)$$

which does not exist. In practice one uses the Wick theorem (Theorem F.4 and Remark F.2) to formally rewrite $U(t, t')$ as a sum of the well-defined Wick products with numerical coefficients (more precisely, numerically-valued distributions) represented as divergent integrals. These divergences can be cancelled out by the so-called renormalisation. Let us replace (3.4) with

$$H = H_0(t) + H_{int}(t) + H_{counter}(t), \tag{3.11}$$

where $H_{counter}(t)$ is a sum of the so-called counterterms, which are also local functionals of ϕ and π in the same sense as $H_0(t)$ and $H_{int(t)}$, but with coefficients tending to infinity is such a way that $U(t, t')$ remains finite[30]. In this thesis we will deal with UV-finite theories which do not have this problem. We refer to [93] for rigorous realisation of this procedure. However, two remarks important for the present discussion can be made even at this formal level.

Remark 3.3. Since separation of an integral into a finite and a divergent parts is not unique, $H_{counter}$ is not uniquely defined by the requirement of $U(t, t')$ finiteness. To fix the counterterms one imposes the so-called renormalisation conditions. These additional restrictions are arbitrary and the result depends on them. However, any change of the renormalisation conditions can be compensated by an appropriate change of the coefficients in H_0 and H_{int}. Formally it can be seen from the fact that the separation of H into the finite and divergent parts is arbitrary and there is always a possibility to move some finite terms from one part to another.

We should note however, that above the adiabatic limit $\lambda \to 1$ is implicitly assumed. As discussed below there are several definitions of such a limit. One of them, the strong adiabatic limit exists only if H_0 describes free particles close enough to the actual particle spectrum of the theory (see Subsection 7.2 for details). This can be always achieved by (partial) fixing of the renormalisation

[29] Often λ is combined with the interaction constant g. But for the sake of clarity we keep them separated.

[30] Of course, to make it work one needs to regularise the divergent integrals so that the pure infinities are replaced by functions of some regularisation parameter, blowing in the limit corresponding to unregularized integrals.

conditions. In UV-finite theories, which we will consider, there is no need in infinite counterterms, but we still need to do finite renormalisation to make the strong adiabatic limit existing. This renormalisation is nothing but a change in separation of the full Hamiltonian into a free and an interacting parts, because when all counterterms are finite, H_{int} and $H_{counter}$ are treated on absolutely the same footing. We will keep calling this procedure renormalisation only to preserve the common terminology of ordinary QFT.

Remark 3.4. There are actually two ways to view at the renormalisation. One, explained above, is to start from a Hamiltonian with finite coefficents

$$H_{fintite} = H_0 + H_{int}$$

and then replace it by

$$H = H_{finite} + H_{counter} \tag{3.12}$$

to cure the divergences.

Instead one can start from the full Hamiltonian in the form

$$H = H_0^{(bare)} + H_{int}^{(bare)},$$

where $H_{int}^{(bare)}$ is the most generic local renormalizable covariant (and may be preserving other symmetries of theory) interaction with unknown coefficient. Then to get finite results one should allow these coefficients to be infinite (which are called "bare" coefficients opposed to the renormalized coefficients of H_{finite}). Then (3.12) can be understood as a convenient separation of the original Hamiltonian H into finite and infinite parts.

In ordinary QFT these approaches are equivalent, and one can safely switch from one to another and back (as we implicitly did in the previous Remark), because anyway the acceptable terms in all parts of the Hamiltonian are restricted by the same symmetry, locality and renormalisability requirements. We will see in Section 8 that in more general setup this equivalence may disappear.

Remark 3.5. There is also one more case when it is instructive to renormalize the theory (in addition to the aforementioned infinite renormalization to make the scattering operator finite and the finite one to make the strong adiabatic limit finite). In practical calculations one chooses renormalization conditions so, that the quantum corrections are small. This choice depends on the characteristic energy scale of the process under consideration. This leads to the notion of "running constants" depending on the energy scale and of the "renormalization group equations" allowing to find the dependency of the interaction constants on the energy scale. Both these concepts play an important role in physics, allowing to compute non-trivial logarithmic corrections to the naïve perturbation theory. We refer to standard textbooks on QFT [86, 80] for details.

In non-local case the renormalization group analysis also may help to compute the amplitudes more precisely. Especially, it is necessary if we want match a non-local QFT with, say, Standard Model at low energies and when see how the non-locality modifies physics at higher energies. Clearly, it should be renormalization, that does not change the physical result.

Identification of the interacting quantum field with a free one allows to understand the statespace as a Fock space. Since it describes the system in the past, it can be understood as a Fock space of incoming particles, which (far enough in the past) are so distant one from another that the interaction can be ignored and the free field approximation is reasonable. For this reason we may also call that free field the incoming field and denote it as $\phi_{in}(x)$

$$\phi_{in}(x) = \phi_0(x),$$

$$\phi(x) = \phi_{in}(x), \quad t < -T.$$

In the distant enough future, $t > T$, the interaction vanishes again so the quantum field is free, but in general it is a different representation of the free quantum field theory, generated by the outcoming field $\phi_{out}(x)$,

$$\phi(x) = \phi_{out}(x), \quad t > T.$$

These two representations are unitary equivalent,

$$\phi_{out}(x) = U(\infty)^{-1}\phi_{in}(x)U(\infty),$$

so the outcoming field also generates a Fock representation. The corresponding Fock basis can be understood as a basis of outcoming fields. The unitary operator $U(\infty)$ (note that $U(t)$ is constant for $t > T$, so $U(\infty)$ is just a convenient notation for that constant) maps incoming states to the outcoming ones, so we interpret it as the scattering matrix. Hence, we have

$$S = \mathbf{T}\left\{e^{-i\int_{-\infty}^{+\infty} H_I(t)dt}\right\} = \mathbf{T}\left\{e^{-i\int \lambda(x):h_I(x):d^4x}\right\}. \tag{3.13}$$

This S-matrix is a functional of a smooth function $\lambda(x)$. To get scattering amplitudes of a Poincare-invariant theory one needs to consider a limit of λ going to a constant. This limit is known under the name of an adiabatic limit. The adiabatic limit of S itself is called the strong adiabatic limit which in general does not exist (See Subsection 7.2) and references therein for details). Alternatively, one can first calculate the Green n-point function, i.e. the time-ordered vacuum expectation value

$$G_n(x_1, \ldots, x_n) = < \Omega|\mathbf{T}\{\phi(x_1)\cdots\phi(x_n)\}|\Omega > .$$

Here $|\Omega >$ is the true vacuum (i.e. the unique translationally-invariant state) of the translationally-invariant theory. In the following we will also need the following identity (originally due to Gell-Man and Low [95]):

$$G_n(x_1, \ldots, x_n) = \lim_{\lambda\to 1} \frac{< 0|\mathbf{T}\left\{\phi_0(x_1)\cdots\phi_0(x_n)e^{-i\int \lambda(x):h_I(x):d^4x}\right\}|0 >}{< 0|\mathbf{T}\left\{e^{-i\int \lambda(x):h_I(x):d^4x}\right\}|0 >}. \tag{3.14}$$

Assuming that $t_1 > t_2 > \ldots > t_n$, from (3.7) we have

$$\mathbf{T}\{\phi(x_1)\cdots\phi(x_n)\} = U(t_1)^{-1}\mathbf{T}\left\{\phi_0(x_1)\cdots\phi_0(x_n)e^{-i\int_{-\infty}^{t_1} H_I(t)}\right\},$$

so the only non-trivial ingredient in (3.14) is that the true vacuum $|\Omega >$ can be approximated by the incoming vacuum $|0 >$ as well as by $U(+\infty, -\infty)^{-1}|0 >$ up to a normalisation constant which was argued by Gell-Man and Low. Whenever the limit of the right hand side of (3.14), the so-called weak adiabatic limit, exists, it serves a definition of the left hand side. The weak adiabatic limit always exists in local theories [74]. The scattering amplitudes can be reconstructed from the Green functions via the LSZ reduction (see Subsection 3.4 and references wherein).

3.1.2 Non-local case: fixing H_I

An approach to perturbative construction of non-local quantum field theories was suggested in [37] and further developed in [71, 72, 73]. The following exposition repeats results from that works important in this thesis. But, because we will work in a more general setup, we present them with proofs. We also change the flow of the proofs to emphasise that the theories appearing in this way are actually local in time, because this fact plays a major role in our analysis.

Assume that we are given the Hamiltonian in the interaction picture H_I as a non-local functional of the free quantum field $\phi_0(x)$. Then (3.13-3.14) still hold with $U(t, t')$ again defined by (3.10). If the non-local functional is smooth enough, the UV-divergences do not appear and the operator

$U(t,t')$ is well-defined with no need of the infinite renormalisation. But, since ϕ_0 is a free field obeying a second order differential equation, any such functional can be presented as a functional of the field and its conjugated momentum at a Cauchy hypersurface. Let us now formulate these ideas more precisely.

Proposition 3.6. *Let $Y^{\alpha_1,\dots\alpha_n} \in \mathcal{S}(\mathbb{R}^{3n})$ be a family of Schwartz functions enumerated by a number $n \in \mathbb{N}$ and n signs $\alpha_i = \pm$ with only finite number of non-vanishing members. Then*

$$\sum_n \sum_{\alpha_i=\pm} \int \prod_{j=1}^n \frac{1}{n!} d^3\vec{p}_j Y^{\alpha_1\dots\alpha_n}(\vec{p}_1,\dots,\vec{p}_n)\tilde{\phi}_{0\alpha_1}(\vec{p}_1,t)\cdots\tilde{\phi}_{0\alpha_n}(\vec{p}_n,t)$$

is a well-defined unbounded operator defined on $D_{\mathcal{S}}$. It is symmetric whenever

$$Y^{\alpha_1\dots\alpha_n}(\vec{p}_1,\dots,\vec{p}_n) = \overline{Y^{-\alpha_n,\dots,-\alpha_1}(-\vec{p}_n,\dots,-\vec{p}_1)}. \tag{3.15}$$

Proof. First statement follows from Remark E.4, stating that the free quantum field evaluated at fixed time is a well-defined distribution of the remaining three arguments. Then (3.15) is nothing but evaluation of a sum of tensor product of operator-valued distributions (see Appendix B) on test functions.

The second statement follows from straightforward computations. □

We will encounter a lot of expressions like (3.15) containing sums over the signs $\alpha_i = \pm$. To simplify notation, we extend the "Einstein summation rule" also to this kind of indices. Due to the reason which will be clear from Remark 4.6 we call this index the *time-orientation*.

Proposition 3.7. *Let $Y^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,\dots,\vec{p}_n)$ be a family of formal power series in terms of the coupling constant g vanishing at the origin. The members of the family are enumerated by enumerated by an integer n and time-orientations α_i. Assume that at each finite order there is only a finite number of non-vanishing members. Assume also that $[Y^{\alpha_1\dots\alpha_n}]_N \in \mathcal{S}(\mathbb{R}^{3n+1})$ for any $n, N \in \mathbb{N}$ and any choice of time-orientations α_i. Let*

$$H_I(t) = \sum_n \frac{1}{n!} \int \prod_{j=1}^n d^3\vec{p}_j Y^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,\dots,\vec{p}_n)\tilde{\phi}_{0\alpha_1}(\vec{p}_1,t)\cdots\tilde{\phi}_{0\alpha_n}(\vec{p}_n,t). \tag{3.16}$$

Then

$$U(t',t'') = \mathbf{T}\{e^{-i\int_{t''}^{t'} H_I(t)}\}$$

is well-defined formal series in terms of g. It is formally unitary if

$$Y^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,\dots,\vec{p}_n) = \overline{Y^{-\alpha_n,\dots,-\alpha_1}(t;-\vec{p}_n,\dots,-\vec{p}_1)}$$

Proof. First we note that by Proposition 3.6 $[H_I(t)]_N$ is a well-defined unbounded operator defined o $D_{\mathcal{S}}$ for any $t \in \mathbb{R}$ and $N \in \mathbb{N}$, so (3.16) makes sense. Now,

$$U(t',t'') = \sum_{n=0}^{\infty} \int \prod_{j=1}^n dt_j \prod_{k=1}^{l_j} d^3\vec{p}_{j,k}\theta(t'-t_n)\theta(t_1-t'')\prod_{j=1}^{n-1}\theta(t_{j+1}-t_j)\times$$

$$\sum_{l_1,\dots,l_m}\prod_{j=1}^n \frac{1}{l_j!} Y^{\alpha_{j,1},\dots,\alpha_{j,l_j}}(t_j;\vec{p}_{j,1},\dots,\vec{p}_{j,l_j})\times$$

$$\tilde{\phi}_{0\alpha_{1,1}}(\vec{p}_{1,1},t_1)\cdots\tilde{\phi}_{0\alpha_{1,l_1}}(\vec{p}_{1,l_1},t_1)\dots\tilde{\phi}_{0\alpha_{n,l_n}}(\vec{p}_{n,l_n},t_n).$$

It is a sum of evaluations of well-defined (according to Remark E.4) products of distributions on test-functions and hence is well-defined.

The unitarity can be proven analogously to [73]. □

From the above, taking into account that

$$\tilde{\phi}_{0\pm}(\vec{p},t)=\frac{1}{2}\left(\tilde{\phi}_0(\vec{p},t)\pm\frac{i}{\omega_{\vec{p}}}\partial_t\tilde{\phi}_0(\vec{p},t)\right)=\frac{1}{2}\left(\tilde{\phi}_0(\vec{p},t)\pm\frac{i}{\omega_{\vec{p}}}\tilde{\pi}_0(\vec{p},t)\right), \tag{3.17}$$

we see that the Hamiltonian perturbation theory easily generalises to the case of H_I being a spatially-non-local functional of the field and its first derivative with respect to time (or, more naturally in the Hamiltonian formalism, the conjugated momentum). But if we now define

$$H_I(t)=\sum_n\frac{1}{n!}\int\prod_{j=1}^n d^3\vec{p}_j dt_j K^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)\tilde{\phi}_{0\alpha_1}(\vec{p}_1,t_1)\cdots\tilde{\phi}_{0\alpha_n}(\vec{p}_n,t_n)= \tag{3.18}$$

$$\sum_n\frac{1}{n!}\int\prod_{j=1}^n d^3\vec{p}_j dt_j e^{-i\alpha_j\omega_{\vec{p}_j}(t_j-t)}K^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)\tilde{\phi}_{0\alpha_1}(\vec{p}_1,t)\cdots\tilde{\phi}_{0\alpha_n}(\vec{p}_n,t).$$

We see, that formally H_I can be represented in the form (3.16) whenever

$$\int\prod_{j=1}^n dt_j e^{-i\alpha_j\omega_{\vec{p}_j}(t_j-t)}K^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)$$

is a Schwartz function of the rest of the variables in any order of perturbation theory. This calculation is however too formal in general and may include subtle operations with distributions which would require additional work to define. We will restrict our attention to the following case for which H_I is well-defined according to Remark E.4:

Definition 3.8. We say that a family of formal power series vanishing at the origin $K^{\alpha_1\dots\alpha_n}$, is an *admissible interaction kernel*, if for any $N\in\mathbb{N}$:

- $$[K^{\alpha_1\dots\alpha_n}]_N\in\mathcal{S}'(\mathbb{R}^{4n+1});$$
- Only finite number of $[K^{\alpha_1\dots\alpha_n}]_N$ does not vanish;
- $$[K^{\alpha_1\dots\alpha_n}]_N(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)=\sum_j[\hat{K}_j^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)A_j(t-t_1,\dots,t-t_n)]_N,$$

  where $\hat{K}_j^{\alpha_1\dots\alpha_n}$ is a family of formal power series, consisting of the Schwartz functions and A is a family of formal power series consisting of distributions. The auxiliary index j has at most countable range and to any finite order N only finite number of values of j contribute.
- $$K^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)=\overline{K^{-\alpha_n,\dots,-\alpha_1}(t;-\vec{p}_n,t_n;\dots;-\vec{p}_1,t_1)}. \tag{3.19}$$

Proposition 3.9. *If $K^{\alpha_1\dots\alpha_n}$ is an admissible interaction kernel, then H_I given by (3.18) is a well-defined unbounded operator on $D_{\mathcal{S}}$ that can be represented in a time-local form (3.16). The corresponding operator $U(t',t'')$ is unitary.*

Proof. Existence of H_I follows from Remark E.4. The local form is given by the second part of (3.18), we need only to prove that after integration over the timestamps t_j we end up with a Schwartz function of the rest of the arguments. Instead for further use we prove Lemma 3.10 below, from which the desired statement follows directly, taking into account that $\omega_{\vec{p}}\in\Theta_M(\mathbb{R}^3)$ as agreed in Remark E.2 (see Appendix B for definition of the multiplicator Θ_M), thus the possible polynomial on $\omega_{\vec{p}}$ factor would not spoil the decay properties.

Finally, (3.19) is exactly the unitarity condition from Proposition 3.7 expressed through the interaction kernel according to (3.18). □

Lemma 3.10. *Assume that $K^{\alpha_1\dots\alpha_n}$ is an admissible interaction kernel. Let*

$$\tilde{K}^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,\omega_1;\dots;\vec{p}_n,\omega_n)=\int\prod_{j=1}^{n}dt_j e^{i\omega_j t_j}K^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)$$

be its partial Fourier transform. Then $\tilde{K}^{\alpha_1\dots\alpha_n}\in C^\infty(\mathbb{R}^{4n+1})$. Moreover, there is $N\in\mathbb{N}$ such that for any $M\in\mathbb{N}$

$$|\tilde{K}^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,\omega_1;\dots;\vec{p}_n,\omega_n)|\leq C_M\left(1+\sum_{j=1}^{n}\omega_j^2\right)^N\left(1+\sum_{j=1}^{n}\vec{p}_j^{\,2}+|t|^2\right)^{-M}.$$

Analogous bounds (may be with different C_M and N) hold also for any partial derivative of any order of $\tilde{K}^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,\omega_1;\dots;\vec{p}_n,\omega_n)$.

Proof. Let us fix some n and some time-orientations α_i. Clearly, it is enough to prove the statement for

$$K^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)=\hat{K}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)A(t-t_1,\dots,t-t_n),$$

$\hat{K}\in\mathcal{S}(\mathbb{R}^{4n+1})$, $A\in\mathcal{S}'(\mathbb{R}^n)$, since in general at any finite order $K^{\alpha_1\dots\alpha_n}$ is a linear combination of such terms.

To prove that $\tilde{K}^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,\omega_1;\dots;\vec{p}_n,\omega_n)$ is a smooth function let us fix $f\in\mathcal{S}(\mathbb{R}^n)$ which does not vanish anywhere. Consider

$$(f\tilde{K}^{\alpha_1\dots\alpha_n})(t;\vec{p}_1,\omega_1;\dots;\vec{p}_n,\omega_n)=$$

$$\int\prod_{j=1}^{n}dt_j e^{i\omega_j t_j}f(\omega_1,\dots,\omega_n)\hat{K}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)A(t-t_1,\dots,t-t_n).$$

For the product

$$X(t;\vec{p}_1,t_1,\omega_1;\dots;\vec{p}_n,t_n,\omega_n)=\prod_{j=1}^{n}e^{i\omega_j t_j}f(\omega_1,\dots,\omega_n)\hat{K}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)$$

we have $X\in\mathcal{S}(\mathbb{R}^{5n+1})$ and hence $f\tilde{K}^{\alpha_1\dots\alpha_n}\in\mathcal{S}(\mathbb{R}^{4n+1})$ by Theorem B.2. Then

$$\tilde{K}^{\alpha_1\dots\alpha_n}\tilde{K}^{\alpha_1\dots\alpha_n}(t;\vec{p}_1,\omega_1;\dots;\vec{p}_n,\omega_n)=\frac{(f\tilde{K}^{\alpha_1\dots\alpha_n}\tilde{K}^{\alpha_1\dots\alpha_n})(t;\vec{p}_1,\omega_1;\dots;\vec{p}_n,\omega_n)}{f(\omega_1,\dots,\omega_n)}$$

due to $f\neq 0$.

To prove the bound, assume that

$$\int A(\tau_1,\dots,\tau_n)g(\tau_1,\dots,\tau(n))\prod_{j=1}^{n}d\tau_j\leq c\max_{|\alpha|<M}||g||_{M';\alpha},\tag{3.20}$$

where $||\cdot||_{M;\alpha}$ is the Schwartz seminorm (B.1) and $|\alpha|$ is the total order of the differential operator ∂^α with α being a multiindex (see Appendix B for details of the notation). Such M, M' and c have to exist for A to be a tempered distribution. Since $\hat{K}\in\mathcal{S}(\mathbb{R}^{4n+1})$, we may assume

$$|\hat{K}(t;\vec{p}_1,t_1;\dots;\vec{p}_n,t_n)|<c'\left(1+\sum_{j=1}^{n}\vec{p}_j^{\,2}+|t|^2\right)^{-M}\left(1+\sum_{j=1}^{}t_j^2\right)^{-M'}.$$

We can also assume the same bound for all its derivatives with respect to $t_1, \dots, t_n$ of total order not exceeding N. Then applying (3.20) to

$$g = \hat{K}(t; \vec{p}_1, t_1; \dots; \vec{p}_n, t_n) \prod_{j=1}^{n} e^{i\omega_j t_j}.$$

we get the desired bound. The proof for the derivatives of the Fourier transform goes along exactly the same lines. □

An important class of interactions is given by integration of the Hamiltonian density over the space like it was in the local case. So, similarly to the previous subsection we define

$$H_I(t) = \int_{x^0=t} \lambda(x) h_I(x) d^3\vec{x}, \tag{3.21}$$

$$h_I(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \kappa_n(x_1 - x, \dots, x_n - x) : \phi_0(x_1) \cdots \phi_0(x_n) :. \tag{3.22}$$

Analogously to Definition 3.8 we introduce:

Definition 3.11. We say that a family of formal power series vanishing at the origin κ_n, is an *admissible interaction kernel density*, if for any $N \in \mathbb{N}$:

- $$[\kappa_m]_N \in \mathcal{S}'(\mathbb{R}^{4n});$$
- Only finite number of $[\kappa_m]_N$ does not vanish;
- $$[\kappa_n]_N(x_1, \dots, x_n) = \sum_j [\hat{\kappa}_{n,j}(x_1, \dots, x_n) A_j(x_1^0, \dots, x_n^0) B_j(\vec{x}_1 + \dots + \vec{x}_n)]_N, \tag{3.23}$$

  where $\hat{\kappa}_{n,j}$ is a family of formal power series, consisting of the Schwartz functions, while A and B are families of formal power series consisting of distributions. The auxiliary index j has at most countable range and to any finite order N only finite number of values of j contribute.
- κ_n is real.
- $\kappa_n = 0$ for $n \leq 2$.

Proposition 3.12. *Let κ be an admissible interaction kernel density and $\lambda \in \mathcal{S}(\mathbb{R}^4)$. Then at any order (3.22) defines an operator-valued distribution and (3.21) defines an interaction Hamiltonian corresponding to an admissible interaction kernel with non-zero components given by*

$$K^{+_1 \cdots +_k -_{k+1} \cdots -_n}(t; \vec{p}_1, t_1; \dots; \vec{p}_n, t_n) = \tag{3.24}$$

$$\binom{n}{k} \int d^3\vec{x} \prod_{j=1}^{k} d^3\vec{x}_j e^{-i\vec{p}_j\vec{x}_j} \prod_{j=k+1}^{n} d^3\vec{x}_j e^{i\vec{p}_j\vec{x}_j} \lambda(x) \kappa(x_1 - x, \dots, x_n - x).$$

Here we adopt a convention $x = (t, \vec{x})$ and $x_j = (t_j, \vec{x}_j)$. In the left hand side K has first k time-orientations "+" and the rest $n - k$ are "-".

Proof. First we prove that $[: h_I :]_N$ is an operator-valued distribution. For that we use the decomposition (3.23). The distribution

$$\phi_0(x_1)\cdots\phi_0(x_n)$$

can be safely multiplied by a time-dependent distribution

$$A_j(t_1-t,\dots,t_n-t)$$

due to Remark E.4. The additional variable t here is treated as a parameter. Multiplication by

$$B_j(\vec{x}_1+\ldots+\vec{x}_n-n\vec{x})$$

due to Remark B.6 gives a well-defined operator-valued distribution on $\mathbb{R}^{4n+3}$, there the additional three dimensions correspond to the new variable $\vec{x}$. Multiplication by a Schwartz function $\hat{\kappa}_j$ and summation over j, of course, brings no problem, so $[: h_I :]_N$ is well-defined.

Substitution of (D.2) and (E.12) into (3.21-3.22) gives exactly (3.18) with interaction kernel (3.24). We took into account that after the normal ordering only terms with all plus on the left to all minuses survive and the combinatoric factor $\binom{n}{k}$ appears because the same ordered term can arise from several different original ones.

To see that (3.24) is admissible we use (3.23) once again. We see that the spatial distributions B_j disappear after integration over $\vec{x}$. Integration over $\vec{x}_j$ is a Fourier transform of a Schwartz function which is Schwartz again. The temporal distributions A_j do not interfere with any of these operations and are allowed by Definition 3.8. The reality of κ guarantees (3.19). □

The form (3.21) allows to consider the adiabatic limit $\lambda\to 1$ in which the translational invariance is restored. We do it in two steps. First we consider the limit of λ going to a function depending on the time only, which we symbolically denote with $\lambda(x)\to\lambda(t)$. The limit $\lambda(t)\to 1$ we consider in Section 7.

Proposition 3.13. *Let κ be an admissible interaction kernel density. Then the operator $S = U(\infty,-\infty)$ remains finite after a formal substitution $\lambda(x)\to\lambda(t)$, where $\lambda(t)$ is compactly supported.*

Proof. Following [73, 72] first formally represent S with such a formal substitution this limit in the normal-ordered form by means of the Second Wick Theorem F.4,

$$S=\mathbf{1}+\sum_{n,n'}\int\prod_{i=1}^{n}d^3\vec{p}_i\prod_{i=1}^{n'}d^3\vec{p}'_i S_{n',n}(\vec{p}'_1,\dots,\vec{p}'_{n'};\vec{p}_1,\dots,\vec{p}_n)\times \tag{3.25}$$

$$\prod_{i=1}^{n'}\frac{a^+(\vec{p}'_i)}{\sqrt{2\omega_{\vec{p}'_i}}(2\pi)^{\frac{3}{2}}}\prod_{i=1}^{n}\frac{a(\vec{p}_i)}{\sqrt{2\omega_{\vec{p}_i}}(2\pi)^{\frac{3}{2}}}.$$

The coefficients $S_{n',n}(\vec{p}'_1,\dots,\vec{p}'_{n'};\vec{p}_1,\dots,\vec{p}_n)$, symmetric in $(\vec{p}_1,\dots,\vec{p}_n)$ and in $(\vec{p}'_1,\dots,\vec{p}'_{n'})$, can be found through computing matrix elements of the both sides of (3.25),

$$S_{n',n}(\vec{p}'_1,\dots,\vec{p}'_{n'};\vec{p}_1,\dots,\vec{p}_n)=<\vec{p}'_1,\dots,\vec{p}'_{n'}|S|\vec{p}_1,\dots,\vec{p}_n>',$$

where prime means that we ignore direct contractions between the incoming and the outcoming fields which are exactly the amplitudes computed by Feynman Rules presented in Section 4 [31]. By Lemma 4.10 these scattering amplitudes have the form

$$S_{n,n'}=\sum_{\Gamma}W_\Gamma(\vec{p}'_1,\dots,\vec{p}'_{n'};\vec{p}_1,\dots,\vec{p}_n)C_\Gamma(\vec{p}'_1,\dots,\vec{p}'_{n'};\vec{p}_1,\dots,\vec{p}_n),$$

[31] More precisely, we substitute the definition (E.4) and then compute the vacuum matrix element according to the Wick Theorem (Theorem F.1 and Remark F.2), ignoring contraction between operators used to create $|\vec{p}_1,\dots,\vec{p}_n>'$ and $<\vec{p}'_1,\dots,\vec{p}'_{n'}|$. Such contractions are also ignored in the scattering amplitudes.
We also note that in Section 4 the notation introduced in Remark 3.19 is used.

where the sum goes all graphs Γ, $W_\Gamma \in \mathcal{S}(\mathbb{R}^{3(n+n')})$ and $C \in \mathcal{S}'(\mathbb{R}^{3(n+n')})$ is a distribution a accumulating all momentum conservation laws[32]. Then we immediately see that S is defined as a quadratic form on $D_{\mathcal{S}}$.

Moreover, for any $\Psi \in D_{\mathcal{S}}$,

$$\Psi = \sum_n \int \Psi(\vec{p}_1, \dots, \vec{p}_n) | \vec{p}_1, \dots, \vec{p}_n > \prod_{i=1}^n \frac{d^3\vec{p}_i}{(2\pi)^3 2\omega_{\vec{p}_i'}}$$

we can compute

$$S\Psi = \sum_{n,n',n''} c_{n,n',n''} \int \prod_{i=1}^n \frac{d^3\vec{p}_i}{(2\pi)^3 2\omega_{\vec{p}_i'}} \prod_{i=1}^{n'} \frac{d^3\vec{p}_i'}{(2\pi)^3 2\omega_{\vec{p}_i'}} \prod_{i=1}^{n''} \frac{d^3\vec{p}_i''}{(2\pi)^3 2\omega_{\vec{p}_i'}} \times$$

$$S_{n',n}(\vec{p}_1', \dots, \vec{p}_{n'}'; \vec{p}_1, \dots, \vec{p}_n) \Psi(\vec{p}_1, \dots, \vec{p}_n, \vec{p}_1'', \dots, \vec{p}_{n''}'') | \vec{p}_1', \dots, \vec{p}_{n'}', \vec{p}_1'', \dots, \vec{p}_{n''}'' > .$$

Here $c_{n,n',n''}$ is a combinatoric coefficient irrelevant for us, and we assumed that Ψ is symmetric, which is, of course, natural choice. Since all conservation laws contain both incoming and outcoming particles[33], C_Γ can be considered as a distribution in $\mathbb{R}^{3n}$ depending on $\vec{p}_i'$ and after integration over $\vec{p}_i$ it gives a Schwartz function of the rest of the variables. So, $S\Psi \in D_{\mathcal{S}}$ is well-defined for any $\Psi \in D_{\mathcal{S}}$. □

Remark 3.14. The Wick product is essential in the above, because otherwise vacuum corrections (and tadpoles in massless case) may appear, which are adiabatically divergent already at this step [73]. In the proof above it is hidden in the Feynaman rules of Section 4.

Remark 3.15. One can actually do more and prove that

$$S_l[\lambda(x)]\Psi \to S_l[\lambda'(t)]\Psi, \forall \Psi \in D_{\mathcal{S}}, \tag{3.26}$$

where $S_l[\lambda(x)]$ is the l-order contibution to the operator S computed assuming the full adiabatic cut-off λ, $S_l[\lambda'(t)]$ is formally constructed in Proposition 3.13 with only temporal adiabatic cutoff $\lambda'(t)$ (here for the sake of clearness we will use different leters to denote them). We will not present here the full proof, but let us show the key steps. For simplicity consider a sequence of the adiabatic switchings

$$\lambda_n(x,t) = \int d^3\vec{k} \tilde{\lambda}_n(\vec{k},t) e^{i\vec{k}\vec{x}},$$

there $\tilde{\lambda}(\vec{k},t)$ is a smooth function vanishing outside of a fixed (in particular, independent of n) compact set $K \in \mathbb{K}$ and that

$$a_n = \sup_t |\lambda_n(0,t) - \lambda'(t)| = \sup_t \left| \int \tilde{\lambda}_n(\vec{k},t) d^3\vec{k} - \lambda'(t) \right| \longrightarrow 0,$$

$$b_n = \sup_t \int |\tilde{\lambda}_n(\vec{k},t)| |\vec{k}| d^3\vec{k} \longrightarrow 0,$$

For example, one can take

$$\lambda_n(\vec{x},t) = \lambda(\epsilon_n \vec{x}, t),$$
$$\tilde{\lambda}_n(\vec{k},t) = \epsilon_n^{-3} \tilde{\lambda}(\epsilon_n^{-3}\vec{k}),$$

where $\tilde{\lambda}_n(\vec{k},t)$ is compactly supported and smooth, and $\epsilon_n \to 0$.

[32]For example, if Γ is a connected (in topological sense) graph, $C_\Gamma(\vec{p}_1', \dots, \vec{p}_{n'}'; \vec{p}_1, \dots, \vec{p}_n) = \delta(\vec{p}_1' + \dots + \vec{p}_{n'}' - \vec{p}_1 - \dots - \vec{p}_n)$. In general case it is a product of such delta-functions of all connected pieces.

[33]Here we use that the theory is massive, $m > 0$.

Since at any order S is a polylinear functional of λ, we may write

$$S_l[\lambda_n]\Psi(\vec{p}_1,\dots,\vec{p}_m) = \int \Phi^{(l)}_m(\vec{p}_1,\dots,\vec{p}_m;\vec{k}_1,t_1,\dots,\vec{k}_l,t_l)\prod_{i=1}^{l}\tilde{\lambda}_n(\vec{k}_i,t_i)\prod_{i=1}^{l}d^3\vec{k}_i dt_i, \tag{3.27}$$

where, by Remark 4.11 and considerations similar to the ones used in the proof of Proposition 3.13, $\Phi^{(l)}_m(\vec{p}_1,\dots,\vec{p}_m;\vec{k}_1,t_1,\dots,\vec{k}_n,t_n)$ is a smooth function which Schwartz seminorms with respect to the momenta $\vec{p}_i$ are bounded uniformly in all $\vec{k}_i$ and t_i on the support of $\tilde{\lambda}$, and the same holds for any its derivatives. In particular, we have

$$\left|\Phi^{(l)}_m(\vec{p}_1,\dots,\vec{p}_m;\vec{k}_1,t_1,\dots,\vec{k}_n,t_n)\right| \le C_{m,l}\left(1+\sum_{i=1}^{m}\vec{p}_i^{\,2}\right)^{-3m},$$

$$\left|\frac{\partial}{\partial\vec{k}_j}\Phi^{(l)}_m(\vec{p}_1,\dots,\vec{p}_m;\vec{k}_1,t_1,\dots,\vec{k}_n,t_n)\right| \le C_{m,l}\left(1+\sum_{i=1}^{m}\vec{p}_i^{\,2}\right)^{-3m},$$

for $(\vec{k}_i,t)\in K$. Moreover (again, see Remark 4.11),

$$S_l[\lambda']\Psi(\vec{p}_1,\dots,\vec{p}_m) = \int \Phi^{(l)}_m(\vec{p}_1,\dots,\vec{p}_m;0,t_1,\dots,0,t_l)\prod_{i=1}^{l}\lambda'(t_i)\prod_{i=1}^{l}dt_i.$$

Then by a straightforward application of Cauchy mean value theorem we get

$$|S_l[\lambda_n]\Psi(\vec{p}_1,\dots,\vec{p}_m) - S_l[\lambda']\Psi(\vec{p}_1,\dots,\vec{p}_m)| \le$$

$$\left|\int \Phi^{(n)}_{m,l}(\vec{p}_1,\dots,\vec{p}_m;0,t_1,\dots,0,t_l)\prod_{i=1}^{l}(\lambda'(t_i)-\lambda_n(t_i,0))\prod_{i=1}^{l}dt_i\right| +$$

$$\left|\int \left(\Phi^{(n)}_{m,l}(\vec{p}_1,\dots,\vec{p}_m;0,t_1,\dots,0,t_l) - \Phi^{(n)}_{m,l}(\vec{p}_1,\dots,\vec{p}_m;\vec{k}_1,t_1,\dots,\vec{k}_l,t_l)\right)\prod_{i=1}^{l}\tilde{\lambda}(\vec{k}_i,t_i)\prod_{i=1}^{l}dt_i d^3\vec{k}_i\right| \le$$

$$C'_{m,l}a_n^m(a_n+b_n)\left(1+\sum_{i=1}^{m}\vec{p}_i^{\,2}\right)^{-3m}.$$

which goes to zero in the norm of $\mathcal{L}^2$. Since for fixed l and fixed $\Psi\in D_{\mathcal{S}}$ only a finite number of $S_l[\lambda_n]\Psi(\vec{p}_1,\dots,\vec{p}_m)$ does not vanish, we get that (3.26) holds.

So, the spatial adiabatic limit (unlike the temporal one, see 7.2) always exists in the strong sense.

Example 3.16. Let us consider

$$h_I(x) = \frac{1}{n!} :\phi_0(x)^n :_Q,$$

where $:\cdot:_Q$ is the Quantum Wick Product defined by (2.18)[34]. It is an admissible interaction kernel density with $\kappa = \tilde{r}$ with $\tilde{r}$ defined by (2.17).

Example 3.17. Let $h_I(q) = \phi_0(q)^n$ be the "star-product" interaction, i.e. the ordinary interaction term with the pointwise product replaced by the ordinary one. Then (see Appendix C of [38])

$$h_I(x) = \frac{1}{n!}\int C(x_1-x,\dots,x_n-x)\phi_0(x_1)\cdots\phi_0(x_n)d^4x_1\cdots d^4x_n,$$

[34]The quantum Wick product is defined as a functional on $\mathcal{E}^*$ and hence can not be evaluated at a point in general. However we can formally substitute q with x in Fourier representation of ϕ_0.

where

$$C(x_1, \dots, x_n) = c_n e^{iQ_n[x_1,\dots,x_n]},$$

where Q_n is a quadratic form of the coordinates with real coefficients. Clearly, this function is not decaying at all with growth of x_i, so it does not define a nice enough interaction. In practice such interactions are known to cause serious problems. Only a special class of non-planar diagrams is UV-finite. Even some diagrams which are finite in presence of the adiabatic cut-off may become divergent in the adiabatic limit [73]. The properties of the theory can be made a bit better by means of integration over Q with some measure. For example the case of rotationally-invariant measure localized on Σ_1 was studied in [73]. The mixing effect however persists [105].

Remark 3.18. We note that the the fields defined in 2.2.1 have additional momentum-dependent factor ζ. If ζ decays fast enough, it can provide additional regularization to the star-product interaction of the example above. However, numerical calculations have shown that it is difficult to find such a ζ, that h_I has reasonable behaviour at low energies. So, we will not develop this approach here.

Remark 3.19. Let κ be an admissible interaction density. Let us find the localised form of the corresponding operator H_I in the limit $\lambda(x) \to \lambda(t)$. We have:

$$Y^{\alpha_1\dots\alpha_n}(t; \vec{p}_1, \dots, \vec{p}_n) = \lambda(t)\mathbf{1}_{:\alpha_1,\dots\alpha_n:}(2\pi)^3\delta\left(\sum_{j=1}^n \alpha_j\vec{p}_j\right)\frac{1}{\alpha_-!\alpha_+!}\sum_{\rho\in\mathfrak{S}_n} F^{\alpha_{\rho_1}\dots\alpha_{\rho_n}}(\vec{p}_{\rho_1}, \dots, \vec{p}_{\rho_n}), \tag{3.28}$$

$$F^{\alpha_1\dots\alpha_n}(\vec{p}_1, \dots, \vec{p}_n) = \int\prod_{j=1}^n d^4x_j e^{i\alpha_j p_j x_j}\kappa(x_1 - x, \dots, x_n - x), \tag{3.29}$$

where $p_j = (\omega_{\vec{p}_j}, \vec{p}_j)$, $x = (t, \vec{x})$ and $\vec{x}$ is arbitrary due to the presence of the momentum-conservation δ-function. The summation in (3.28) goes over the group $\mathfrak{S}_n$ of permutations of n elements, $\mathbf{1}_{:\alpha_1,\dots\alpha_n:}$ is equal to one if all pluses precede all minuses in $(\alpha_1, \dots, \alpha_n)$ and is zero otherwise, and finally α_- and α_+ are the number of minuses and pluses in α respectively. For simplicity we assume that F is symmetric, then we may replace

$$\frac{1}{\alpha_-!\alpha_+!}\sum_{\rho\in\mathfrak{S}_n} \to \binom{n}{\alpha_+}$$

This is the form we use.

For further reference, let us some up the key results of this exposition in the following:

Remark 3.20. The "Hamiltonian approach" [71, 73] allows to quantise the non-local quantum field theories with quite general interaction kernel in the presence of an adiabatic cut-off. Moreover, the spatial adiabatic cut-off can be weared off resulting in a theory invariant with respect to the spatial translations. However, it is worth noting, that although the interaction kernel K, or the interaction kernel κ are allowed to be non-local also in time, this is a notational issue only. Such interaction can be always replaced by an equivalent one, which is non-local only in space. This happens because H_I is a functional of the free quantum field which evolution in time is completely determined by the free wave equation. In 3.1.3 we will discuss perturbative construction of a quantum field theory which is really non-local in time by taking non-linear corrections to the free evolution into account. As a byproduct the interacting kernel (3.28-3.29) depends in a non-trivial way on the bare frequency. Recall that in ordinary QFT the choice of the bare frequency is rather arbitrary (except for the strong adiabatic limit) and its change can be always compensated by a quadratic interaction term. This is not the case anymore.

3.1.3 Hamiltonian approach: fixing H_{int}

As we have seen, the non-local Hamiltonian perturbation theory goes the same way as the local one (except that we should not care about the UV divergences anymore) whenever we are provided H_I as a functional of the free fields. It would be more natural to start from an interaction part of the Hamiltonian H_{int} given as a non-local functional of the physical quantum fields:

$$H_{int}(t) =$$

$$\sum_n \frac{1}{n!} \int \prod_{i=1}^n d^4x_i R_n(t; x_1, \dots, x_n) \prod_{i=1}^n \phi(x_i). \tag{3.30}$$

We still can define the interaction representation by (3.7). Then[35]

$$H_I(t) = U(t) H_{int}(t) U(t)^{-1} =$$

$$\sum_n \frac{1}{n!} \int \prod_{i=1}^n d^4x_i R_n(t; x_1, \dots, x_n) U(t) \left(\prod_{i=1}^n \left(U(t_i)^{-1} \phi_0(x_i) U(t_i) \right) \right) U(t)^{-1}$$

or

$$H_I(t) = \sum_n \frac{1}{n!} \int \prod_{i=1}^n d^4x_i R_n(t; x_1, \dots, x_n) U(t, t_1) \phi_0(x_1) U(t_1, t_2) \phi_0(x_2) \cdots U(t_{n-1}, t_n) \phi_0(x_n) U(t_n, t). \tag{3.31}$$

Here

$$U(t, t') = U(t) U(t')^{-1}.$$

If $t' < t$ it coincides with $U(t, t')$ defined above,

$$U(t, t') = \mathbf{T} e^{-i \int_{t'}^t H_I(t'') dt''}.$$

For $t > t'$ we have

$$U(t, t')^{-1} = U(t', t) = T e^{-i \int_{t'}^t H_I(t'') dt''},$$

$$U(t, t') = \bar{\mathbf{T}} e^{i \int_{t'}^t H_I(t'') dt''},$$

where $\bar{\mathbf{T}}$ is anti-timeordering (i.e. ordering which brings the operators with later timestamp to the right). Both cases can be symbolically united by

$$U(t, t') = \mathbf{P} e^{i(t-t') \int_0^1 H_I(t(1-s)+t's) ds} =$$

$$\sum_{n=0}^\infty \frac{(i(t-t'))^n}{n!} \int_{0<s_1<s_2<\ldots<s_n<1} H_I(t(1-s_n)+t's_n) \cdots H_I(t(1-s_1)+t's_1) ds_1 \cdots ds_n,$$

where $\mathbf{P}$ stands for the ordering along the integration path (i.e. multipliers corresponding to bigger values of s are moved to the left). By construction

$$U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3).$$

and

$$U(t, t') = U(t', t)^{-1}.$$

[35] From now and until the end of the subsection we adopt the convention $x_j = (t_j, \vec{x}_j)$.

Proposition 3.21. *Let R be an admissible interaction kernel in the sense of Definition 3.8. Then (3.31) uniquely defines $H_I(t)$ as a formal power series vanishing at the origin of the form*

$$H_I(t) = \sum_n \frac{1}{n!} \int Y^{\alpha_1 \dots \alpha_n}(t; \vec{k}_1, \dots, \vec{k}_n) \times$$

$$\tilde{\phi}_{0\alpha_1}(\alpha_1 \vec{k}_1, t) \cdots \tilde{\phi}_{0\alpha_n}(\alpha_n \vec{k}_n, t) \prod_{i=1}^n d^3 \vec{k}_i.$$

where $Y_{\alpha_1,\dots,\alpha_n}(t; \vec{k}_1, \dots, \vec{k}_m)$ is a family of power series of Schwartz functions vanishing at the origin, and in any order only a finite number of $Y_{\alpha_1 \dots \alpha_n}$ is non-zero.

Proof. Clearly, $[H_I]_0 = 0$ is of the desired form. Now assume that for some N

$$[H_I]_N(t) = \sum_n \frac{1}{n!} \int [Y_{\alpha_1,\dots,\alpha_n}(t; \vec{k}_1, \dots, \vec{k}_n)]_N \times$$

$$\tilde{\phi}_{0\alpha_1}(\alpha_1 \vec{k}_1, t) \cdots \tilde{\phi}_{0\alpha_n}(\alpha_n \vec{k}_n, t) \prod_{i=1}^n d^3 \vec{k}_i$$

for some $[Y^{\alpha_1 \dots \alpha_n}(t; \vec{k}_1, \dots, \vec{k}_m)]_N \in \mathcal{S}(\mathbb{R}^{3n+1})$. From (3.31) we have

$$[H_I(t)]_{N+1} = \sum_n \frac{1}{n!} \int \prod_{i=1}^n d^4 x_i \big[R_n(t; x_1, \dots, x_n) \times \tag{3.32}$$

$$[U(t, t_1)]_N \phi_0(x_1) [U(t_1, t_2)]_N \phi_0(x_2) \cdots [U(t_{n-1}, t_n)]_N \phi_0(x_n) U(t_n, t) \big]_{N+1}.$$

Here we used that R_n is a power series vanishing at the origin, so only lower orders of $U(t', t'')$ contribute to $[H_I(t)]_{N+1}$. We note, that

$$[U(t', t'')]_N = \Big[\sum_{n=0}^\infty \frac{(i(t'-t''))^n}{n!} \sum_{l_1,\dots,l_n} \int_{0<s_j<1} \prod_{j=1}^n \left(ds_j \prod_{i=1}^{l_j} dx_{j,i} \right) \prod_{j=1}^{n-1} \theta(s_j - s_{j+1}) \times$$

$$\prod_{j=1}^n \frac{1}{l_j!} Y^{\alpha_{j,1} \dots \alpha_{j,l_j}}(t'(1-s_j) + t''s_j; \vec{k}_{j,1}, \dots, \vec{k}_{j,l_j}) \prod_{i=1}^{l_j} \tilde{\phi}_{0\alpha_{j,i}}(\alpha_{j,i} \vec{k}_{j,i}, t'(1-s_j) + t''s_j) \Big]_N =$$

$$\Big[\sum_{n=0}^\infty \frac{(i(t'-t''))^n}{n!} \sum_{l_1,\dots,l_n} \int_{0<s_j<1} \prod_{j=1}^n ds_j \prod_{i=1}^{k_j} d\vec{x}_{j,i} \prod_{j=1}^{n-1} \theta(s_j - s_{j+1}) \times$$

$$\prod_{j=1}^n \frac{1}{l_j!} Y^{\alpha_{j,1} \dots \alpha_{j,l_j}}(t'(1-s_j) + t''s_j; \vec{k}_{j,1}, \dots, \vec{k}_{j,l_j}) \prod_{i=1}^{l_j} e^{i\alpha_{j,i}\omega_{\vec{k}_{j,i}}((t'(1-s_j)+t''s_j)-t)} \tilde{\phi}_{0\alpha_{j,i}}(\vec{k_{j,i}}, t) \Big]_N.$$

We now consider integration over the variables s_j. We see that it is an evaluation of the distribution

$$\theta(s_n)\theta(1-s_1) \prod_{j=1}^{n-1} \theta(s_j - s_{j+1})$$

on a Schwartz function of a larger number of variables. It is well-defined and yields a Schwartz function of the rest of the variables according to Theorem B.2, so

$$[U(t', t'')]_N = \sum_n \int [u^{\alpha_1 \dots \alpha_n}(t, t', t''; \vec{k}_1, \dots, \vec{k}_n)]_N \prod_{i=1}^n \tilde{\phi}_{0\alpha_i}(\vec{k}_i, t) \prod_{i=1}^n d^3 \vec{k}_i, \tag{3.33}$$

$u^{\alpha_1\dots\alpha_n} \in \mathcal{S}(\mathbb{R}^{3n+3})$ Then (3.32) gives

$$[H_I(t)]_{N+1} = \sum_n \frac{1}{n!} \int \prod_{i=1}^n d^4x_i d\vec{p}_i \big[R_n(t; x_1, \dots, x_n) \times$$

$$[U(t,t_1)]_N \tilde{\phi}_{0,\beta_1}(\vec{p}_1, t)[U(t_1,t_2)]_N \tilde{\phi}_{0\beta_2}(\vec{p}_2, t) \cdots [U(t_{n-1},t_n)]_N \phi_{0\beta_n}(\vec{k}_n, t) U(t_n, t)\big]_{N+1} \times$$

$$e^{i\sum_{i=1}^n (\beta_i \omega_{\vec{p}_i}(t_i - t) - \vec{p}_i \vec{x}_i)} =$$

$$\sum_n \frac{1}{n!(2\pi)^n} \sum_{\beta_1,\dots,\beta_n = \pm} \int \prod_{i=1}^n d\vec{p}_i d\omega_i dt_i \big[\tilde{R}_n(t; \vec{p}_1, \omega_1; \dots; \vec{p}_n, \omega_n) \times$$

$$[U(t,t_1)]_N \tilde{\phi}_{0,\beta_1}(\vec{p}_1, t)[U(t_1,t_2)]_N \tilde{\phi}_{0\beta_2}(\vec{p}_2, t) \cdots [U(t_{n-1},t_n)]_N \phi_{0\beta_n}(\vec{k}_n, t) U(t_n, t)\big]_{N+1} \times$$

$$e^{-i\sum_{i=1}^n \beta_i \omega_{\vec{p}_i} t} e^{i\sum_{i=1}^n (\beta_i \omega_{\vec{p}_i} t_i - \vec{p}_i \vec{x}_i + \omega_i t_i)}.$$

Here $\tilde{R}$ is the partial Fourier Transform defined in the same way as $\tilde{K}$ of Lemma 3.10. Substituting (3.33), evaluating integral over t_i (which is just a Fourier transform of Schwartz functions u form (3.33))[36].Applying Lemma 3.10 and taking into account that possible polynomial growth of $\tilde{R}$ as a function of frequencies is always overruled by fast decaying Fourier transforms of u, we get that the integrand (aside from the fields product) is a Schwartz function of all its arguments, thus $[H_I]_{N+1}$ has the desired form. Then the theorem is proven by induction. □

Remark 3.22. Unlike the free field, the interacting field in general is not defined at a point, so there is no analogue of Remark E.4 to justify the operator H_{int} with admissible interaction kernel R. However, the theorem above provides a way of perturbative construction of its interaction representation H_I.

Remark 3.23. In this formalism an interaction invariant with respect to the spatial translations analogous to (3.21) with $\lambda(x) \to \lambda(t)$ can also be considered. We claim that such a limit is well-defined and that H_I can be again represented as in Remark 3.19 up to some infinite vacuum energy and tadpoles renormalisation (which appear, because we cannot impose Wick product on general interaction fields), but we leave the details to be considered elsewhere.

So, we see that the non-local theories considered in the previous subsection, although being quite unnatural to begin with, can be considered as an effective theory arising from very reasonable interaction (3.30).

Remark 3.24. We note that the theory with fixed H_I can always be achieved in this formalism if we allow H_{int} depend on the canonical momentum together with the field. For that we should take

$$H_{int}(t) = \sum_n \sum_{\alpha_j = \pm} \frac{1}{n!} \int \prod_{j=1}^n d^3\vec{p}_j Y^{t;\alpha_1\dots\alpha_n}(\vec{p}_1, \dots, \vec{p}_n) \prod_{j=1}^n \frac{\tilde{\phi}(\vec{p}_j, t) + \frac{i\alpha_j}{\omega_{\vec{p}_j}} \tilde{\pi}(\vec{p}_j, t)}{2}. \tag{3.34}$$

Here we used (3.6), (3.7-3.8). It is worth noting that, although in (3.6) one can use either the canonical momentum $\tilde{\pi}_0$ or the "velocity" $\partial_t \tilde{\phi}_0$, only for the former there is a simple relation between the original and the interaction pictures. We note that Proposition 3.21 generalises easily to such interactions.

[36]There is actually one exception, namely zeroth order of $U(t_i, t_j) = \mathbf{1}$. In this case the Fourier transform is a delta-function $\delta(\omega_i + \beta\omega_{\vec{p}_i})$. In this case one should use the same argument as in Proposition 3.9.

3.2 Lagrangian perturbation theories

Again, we star from a local theory. Let it also be Poincare-invariant, then we expect that (in the adiabatic limit) the S-matrix is covariant. It is quite difficult to see that from (3.13). Indeed, in (3.13) the covariance seems to be broken in two ways. Firstly, S depends on the Hamiltonian which is not Lorentz-invariant (it is a zero-component of a four-vector, but in (3.13) it appears alone). Secondly, (3.13) contains a time-ordered product, which is defined in a fixed reference frame. At the first sight it is not a problem in a local theory, because for time- or light-like separated events the time-ordering is invariant, while for space-like separated ones it does not matter. The problem appears for coinciding points. Unfortunately, they often bring singularities, so they can not be disregarded as a zero-measure set.

In more details, assuming that the dispersion relation is covariant, $\omega_{\vec{p}} = \omega_{\vec{p}}^m$ from (F.5), we have

$$< 0|\mathbf{T}\{\phi_0(x)\phi_0(x')\}|0 >= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i0},$$

which is covariant. Hence, by the Second Wick Theorem (Theorem F.4 and Remark F.5) time-ordered product containing only the field, but not its derivatives is covariant. We note that if no derivatives appear in the interaction term we have

$$h_I(x) = -L_I(x),$$

where L_I is the interacting part of the Lagrangian, which is a Lorentz-scalar and (3.13) is Lorentz-covariant (in the adiabatic limit). In presence of the derivatives from (F.7) we see that

$$< 0|\mathbf{T}\{\partial_{x^\mu}\phi_0(x)\phi_0(x')\}|0 >= \partial_{x^\mu} < 0|\mathbf{T}\{\phi_0(x)\phi_0(x')\}|0 >$$

also is covariant. But, from (F.8), when two derivatives appear,

$$< 0|\mathbf{T}\{\partial_{x^\mu}\phi_0(x)\{\partial_{x'^\nu}\phi_0(x')\}|0 >= \partial_{x^\mu}\partial_{x'^\nu} < 0|\mathbf{T}\{\phi_0(x)\phi_0(x')\}|0 > +i\delta_{\mu,0}\delta_{\nu,0}\delta(x - x'),$$

only the first part is covariant. At the same time, connection between the Lagrangian and the Hamiltonian is also more involved in presence of the derivatives. In [8] it was shown that these two issues compensate each other in the scalar electrodynamics, so one has

$$S = \hat{\mathbf{T}}\left\{e^{-i\int L_{int}(x)\lambda(x)}\right\}, \tag{3.35}$$

$$G_n(x_1, \ldots, x_n) = \lim_{\lambda \to const} \frac{< 0|\hat{\mathbf{T}}\left\{\phi_0(x_1)\cdots\phi_0(x_n)e^{i\int g(x):L_{int}(x):d^4x}\right\}|0 >}{< 0|\hat{\mathbf{T}}\left\{e^{i\int g(x):L_{int}(x):d^4x}\right\}|0 >}. \tag{3.36}$$

there $\hat{\mathbf{T}}$ is the covariant time-ordering defined by

$$\hat{\mathbf{T}}\{\phi(x)\phi(y)\} = \mathbf{T}\{\phi(x)\phi(y)\}$$

$$\hat{\mathbf{T}}\{\partial_{x^\mu}\phi(x)\phi(y)\} = \partial_{x^\mu}\mathbf{T}\{\phi(x)\phi(y)\} = \mathbf{T}\{\partial_{x^\mu}\phi(x)\phi(y)\},$$

$$\hat{\mathbf{T}}\{\partial_{x^\mu}\phi(x)\partial_{y^\nu}\phi(y)\} = \partial_{x^\mu}\partial_{y^\nu}\mathbf{T}\{\phi(x)\phi(y)\} = \mathbf{T}\{\partial_{x^\mu}\phi(x)\partial_{y^\nu}\phi(y)\} - i\delta_{\mu,0}\delta_{\nu,0}\delta(x - y)\mathbf{1}, \tag{3.37}$$

and so on. In Section 5 we will see that this result generalises to arbitrary QFT with first-order derivatives in the interaction, but the classical connection between the Lagrangian and the Hamiltonian may be corrected by quantum effects.

The form (3.35) is preferable in physics, because there the Lagrangian, rather than the Hamiltonian is the usual starting point. Besides already mentioned Lorentz invariance, which is manifest in (3.35) unlike (3.13), the field strength renormalisation is much simpler in this picture (technical difficulties arising in the Hamiltonian approach are discussed in Subsection 8.2). For this reason, it is preferable to express the perturbation theory in such a way that it leads to (3.35) directly. There are at least two such approaches.

A mathematically rigorous approach was developed by Stuckelberg, Bogolubov, Epstein and Glaser. We will follow a pedagogical exposition of [97] and will present only the general scheme without going into any details. Motivated by the Hamiltonian perturbation theory we consider a theory with adiabatic cut-off of all the interactions. Then we expect to have two representations of the free QFT, the incoming and outcoming ones, unitary equivalent via the unitary operator $S(\lambda)$ being a functional of the adiabatic switching $\lambda(x)$. As always, we expect that the operator S is a formal power series in terms of g. But we also assume that in any finite order it is a polynomial function of λ as it was in the Hamiltonian perturbation theory,

$$S(\lambda) = \sum_n \int S_n(x_1, \dots, x_n)\lambda(x_1) \cdots \lambda(x_n) d^4x_1 \cdots d^4x_n$$

with S_n being the operator-valued distributions which may be always assumed to be symmetric and S_0 the identity operator. We also require that $S(g)$ is unitary, covariant (in the sense of (D.1) naturally generalized to distribution on $\mathbb{R}^{4n}$) and casual in the sense that

$$S(\lambda + \mu_1 + \mu_2) = S(\lambda + \mu_1)S(\lambda)^{-1}S(\lambda + \mu_2) \tag{3.38}$$

whenever the support of μ_2 is in the casual past of the support of μ_1. Then it is easy to see that if we restrict S_n to some closed region $O \subset \mathbb{R}^{4n}$ such that

$$x_i \neq x_j, \quad \forall i, j = 1, \dots, n, \forall (x_1, \dots, x_n) \in O,$$

then

$$S_n(x_1, \dots, x_n) = \mathbf{T}(S_1(x_1) \dots S_n(x_n)),$$

where $\mathbf{T}$ is the time-ordered product as in the precious subsection. In general one has

$$S_n(x_1, \dots, x_n) = \hat{\mathbf{T}}_r(S_1(x_1) \dots S_n(x_n)),$$

where $\hat{\mathbf{T}}_r$ differs from the naïve time-ordering by distribution localised at hypersurfaces $x_i = x_j$. As we have already seen in (3.37), such local corrections are enough to make it covariant. In [7] it was shown that there is a recurrent construction of such time-ordered product that is also free of the UV-divergences. Based on the correspondence principle one may argue that $S_1(x) = iL_I(x)$, so we arrive to (3.35) with $\hat{\mathbf{T}}$ replaced by its renormalised version $\hat{\mathbf{T}}_r$. (3.36) can also be proven in this formalism. The freedom mentioned in Remark 3.3 reappears as non-uniqueness of $\hat{\mathbf{T}}_r$ satisfying the requirements. Change of this definition again turns out to be equivalent to a suitably chosen finite shift of the parameters.

Another approach is the so-called functional integration [97, 80]. It is based on a formal equation

$$G_n(x_1, \dots, x_n) = \frac{\int \phi(x_1) \cdots \phi(x_n) e^{iS[\phi]} d[\phi]}{\int e^{iS[\phi]} d[\phi]}, \tag{3.39}$$

where the formal integration in the right hand side goes over all functions $\phi : \mathbb{R}^4 \to \mathbb{R}$. This integral is not defined in any rigorous sense (although it can be approximated by finite-dimensional integrals, see [77]), but if the action is quadratic, the integral is Gaussian (or, more precisely, Fresnel) and can

be formally computed by the Iserlsiss theorem [98]. Because of formal similarity of the Wick and Iserlsiss's theorems, (3.39) understood in this sense holds for a free theory,

$$< 0|T\{\phi_0(x_1)\cdots\phi_n(x_n)\}|0 >= \frac{\int \phi(x_1)\cdots\phi(x_n)e^{iS_0[\phi]}d[\phi]}{\int e^{iS_0[\phi]}d[\phi]}, \tag{3.40}$$

where the free action

$$S_0 = -\frac{1}{2}\int d^4x \left(\partial_\mu\phi(x)\partial^\mu\phi(x) + m^2\phi(x)^2\right)$$

Then, taking $S = S_0 + \int \lambda(x)L_{int}(x)d^4x$ one can compute the integral in the right hand side of (3.39) perturbatively, again formally using the Iserlsiss's theorem. Because of (3.40) it coincides exactly with (3.36). In this approach the UV divergences appear in local theories and should be treated similarly to what we have in the Hamiltonian approach.

In non-local theories the casuality requirement (3.38) fails to hold, so the Epstein-Glaser method can not be used at all. The functional integration can be formally used leading to (3.36), but the corresponding S-matrix (3.35) may be not unitary [55]. Because of these problems in the string theory community, there the functional integration approach is popular, it is commonly believed that the non-local in time theories (and in particular non-commutative theories with non-central time) should be excluded from consideration [55]. This restriction seems to be neither natural nor covariant.

Although from the connection of locality with casuality one could expect that it is the space-like non-locality which would bring new problems (since it inevitably allows interaction between space-like separated regions) we see that it is the time-like non-locality which breaks perturbation theory. In fact, the main ambiguity is the definition of the time-ordered product which plays a crucial role in both Hamiltonian and Lagrangian approaches.

The main problem of the functional integration method is that for the integral each occurrence of the free field in L_{int} is treated separately. As a result, the time-ordering induced by it treats each free field as a separate operator with its own time stamp. For example, in the first order of (3.36) the time-ordered Lagrangian would appear which is not Hermitian, so the unitarity is broken already at that level.

In the Hamiltonian approach instead we have the time-ordering of the operators $H_I(t_i)$ instead with respect to the time-stamps t_i (the "Interaction Point Time-Ordering Prescription", see [106]).

Other approaches to generalization of the Lagrangian perturbation theory to the non-local case modify the time-ordered product one way or another so, that the fields are again ordered in blocks. For example, in [107] the unpleasant reorderings were forbidden just by hands. The resulting time-ordered product is just $\mathbf{T}$, so, although formulated in terms of the action, it is actually the Hamiltonian approach of 3.1.2. It is not clear, how the "Interaction Point Time-Ordering Prescription" can be made compatible with the covariance.

A more natural approach for the interaction terms which are local functionals of fields, convoluted with smooth kernels (in particular, the interaction based on the Quantum Wick Product of 2.3 can be presented in this way) was suggested in [108].

We also note that in 3.1.2 any admissible interaction can be represented in the form (3.16), which is local in time. Even more, it depends on the field and its conjugated momentum (but not on higher order derivatives with respect to time) only, as the Hamiltonian should. As such it corresponds to some Lagrangian, also non-local in space. In Section 5 we will see that (3.35-3.36) coincides with the results of the Hamiltonian approach, if quantum corrections to the Legendre transform are taken into account. It could be interesting problem to generalise this correspondence to the case of properly non-local in time theories.

3.3 Yang-Feldman quantizaion

Finally, one can start from the equations of motion in the form

$$(\Box + m_0^2)\phi(x) = F[\phi](x), \tag{3.41}$$

where $F[\phi]$ is a functional of the field proportional to the interaction constant. The free quantum field solves that equation for $F = 0$, so if the interaction is appropriately switched off in the past, we can impose the initial conditions

$$\phi(x) = \phi_0(x), x_0 < -T$$

for large enough T as before. Then (3.41) is equivalent to an integral equation (the Yang-Feldman equation)

$$\phi(x) = \phi_0(x) + E_R F[\phi](x), \tag{3.42}$$

where E_R is the retarded Green function of the Poisson equation. (3.42) can be solved iteratively to get a formal power series in the interaction constant.

The covariant time-ordered product construction of Epstein and Glaser in fact relies on iterative solving of (3.42) derived from the extremal action principle, so whenever F is a local functional, the Lagrangian approach of previous subsection is by construction equivalent to the Yang-Feldman equation approach. In the non-local case, when the Lagrangian approach is not applicable, the Yang-Feldman equation can be considered and gives rise to a unitary theory extensively considered in [68] (for the case of star-product interaction).

We note that in [109] a perturbative construction of the Hamiltonian from equations of motion in accordance with the Yang-Feldman approach was suggested.

3.4 LSZ reduction

Let us repeat the main steps of the LSZ analizis of [99] (see also [4]) in order to verify that it does not actually require neither locality (eee however Remark 3.27) nor Lorentz-invariance.

The crucial idea is the assumption that in states, relevant for the scattering theory, in both distant past and future the particles are far away from each other and hence are essentially free. Therefore, their statespaces can be considered as Fock spaces of some free quantum fields, which we call incoming and outcoming fields. This is formalised in the following way:

Assumption 3.25. *Assume that besides a real scalar quantum field $\phi(x)$ acting on the dense domain $D \subset \mathcal{H}_{phys}$ there are two general free quantum fields (E.11) with renormalisation and a dispersion function $\omega^Q_{\vec{p}}$, ϕ_{in}, ϕ_{out}, called respectively the incoming and the outcoming quantum fields, acting on the same domain $D\mathcal{H}_{phys}$,*

$$\phi_{(ex)} = \int d^3\vec{p} \frac{\sqrt{Z(\vec{p})}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega^Q_{\vec{p}}}} \left(a_{(ex),\vec{p}} e^{-ipq} + a^+_{(ex),\vec{p}} e^{ipq})\right),$$

with "ex" standing for either "in" or "out", such, that

1. *The span of images of the vector-valued distributions*

$$|\vec{p}_1, \cdots, \vec{p}_n >= a^+_{(ex)}(\vec{p}_1) \cdots a^+_{(ex)}(\vec{p}_n)\Omega$$

   is dense in $\mathcal{H}_{phys}$;

2. $$\int (\Psi_1, a^+(\vec{p}, t+T)\Psi_2) f(t) d^3\vec{p}dt \underset{T\to+\infty}{\longrightarrow} \int (\Psi_1, a^+_{(out)}(\vec{p})\Psi_2) f(\vec{p}, t) d^3\vec{p}dt,$$

$$\int (\Psi_1, a^+(\vec{p}, t+T)\Psi_2) f(t) d^3\vec{p} dt \underset{T\to-\infty}{\longrightarrow} \int (\Psi_1, a^+_{(in)}(\vec{p})\Psi_2) f(\vec{p}, t) d^3\vec{p} dt$$

for any $\Psi_1, \Psi_2 \in D$ *and* $f \in \mathcal{S}(\mathbb{R}^4$ *and*

$$a^+(\vec{p}, t) = -e^{-i\omega^Q_{\vec{p}} t} \sqrt{\frac{(2\pi)^{\frac{3}{2}}}{2\omega^Q_{\vec{p}} Z(\vec{p})}} \left(i\partial_t \tilde{\phi}(\vec{p}, t) - \omega^Q_{\vec{p}} \tilde{\phi}(\vec{p}, t) \right).$$

We note that Assumption 3.25 holds for free fields with $\phi_{in} = \phi_{out} = \phi$ and for interacting fields in presence of the adiabatic cut-off with $\phi_i n$ and $\phi_o ut$ as in the previous subsections.

Lemma 3.26 (LSZ reduction formula)**.** *Consider the distribution*

$$S_{n,n',l}(\vec{p}_1, \dots, \vec{p}_n; \vec{p}'_1, \dots, \vec{p}'_n; \vec{k}_1, \dots, \vec{k}_l; t_1, \dots, t_l) =$$

$$_{out} < \vec{p}'_1, \dots, \vec{p}'_{n'} | T\{\tilde{\phi}(t_1, \vec{k_1}) \cdots \tilde{\phi}(t_l, \vec{k_l})\} | \vec{p}_1, \dots, \vec{p}_n >_{in}$$

restricted to a region $\mathcal{O}$ *in which* $\vec{p}_i \neq \vec{p}'_j$ *for any* i, j *and considered only on functions with compact support in the variables* t_i*. Then* [37]

$$S_{n,n',l}(\vec{p}_1, \dots, \vec{p}_n; \vec{p}'_1, \dots, \vec{p}'_{n'}; \vec{k}_1, \dots, \vec{k}_l; t_1, \dots, t_l) = \tag{3.43}$$

$$\lim_{T\to\infty} \int dt H_T(t) \frac{e^{-i\omega^Q_{\vec{p}_n} t} \left(\partial_t^2 - \left(\omega^Q_{\vec{p}_n} \right)^2 \right) (2\pi)^3}{\sqrt{Z(\vec{p}_n)}} \times$$

$$S_{n-1,n',l+1}(\vec{p}_1, \dots, \vec{p}_{n-1}; \vec{p}'_1, \dots, \vec{p}'_{n'}; \vec{k}_1, \dots, \vec{k}_l, \vec{p}_n; t_1, \dots, t_l, t)$$

and

$$S_{n,n',l}(\vec{p}_1, \dots, \vec{p}_n; \vec{p}'_1, \dots, \vec{p}'_{n'}; \vec{k}_1, \dots, \vec{k}_l; t_1, \dots, t_l) = \tag{3.44}$$

$$\lim_{T\to\infty} \int dt H_T(t) \frac{e^{i\omega^Q_{\vec{p}'_{n'}} t} \left(\partial_t^2 - \left(\omega^Q_{\vec{p}_{n'}} \right)^2 \right) (2\pi)^3}{\sqrt{Z(\vec{p}_n)}} \times$$

$$S_{n,n'-1,l+1}(\vec{p}_1, \dots, \vec{p}_n; \vec{p}'_1, \dots, \vec{p}'_{n'-1}; \vec{k}_1, \dots, \vec{k}_l, -\vec{p}'_{n'}; t_1, \dots, t_l, t),$$

where

$$H_{T,\omega_{\vec{p}}}(t) = \int_{-\infty}^{t} (h(t'+T) - h(t'-T)) dt',$$

$h \in C_c^\infty(\mathbb{R})$, $\int_{-\infty}^{\infty} h(t) dt = 1$.

Proof. First note that $H_T \in C_c^\infty(\mathbb{R})$, so it is an admittable test-function. Now note, that

$$a^+(\vec{p}, t+T) - a^+(\vec{p}, t-T) = -\int dt a^+(t)(h(t+T) - h(t-T)) = \int dt a^+(t) \partial_t H_T(t) =$$

$$-\int dt H_T(t) \partial_t a^+(t) =$$

$$-\int dt e^{-i\omega^Q_{\vec{p}} t} H_T(t) \left(\partial_t - i\omega^Q_{\vec{p}} \right) (i\partial_t - \omega_{\vec{p}}) \sqrt{\frac{(2\pi)^{\frac{3}{2}}}{2\omega_{\vec{p}}}} \phi(t) =$$

[37] As always one should integrate over all free arguments of the distributions with a test function and all limits are in the weak sense. For shortness we consider integration over the only variable we deal with. We also implicitly assume that the time-ordered product of the quantum fields is a well-defined tempered distribution.

$$-i\sqrt{\frac{(2\pi)^{\frac{3}{2}}}{2\omega_{\vec{p}}}}\int dt e^{-i\omega^Q_{\vec{p}}t}H_T(t)\left(\partial_t^2-\left(\omega^Q_{\vec{p}}\right)^2\right)\phi(t).$$

Then we have

$$_{out}<\vec{p}'_1,\dots,\vec{p}'_{n'}|T\{\tilde{\phi}(t_1,\vec{k_1})\cdots\tilde{\phi}(t_l,\vec{k_l})\}|\vec{p}_1,\dots,\vec{p}_n>_{in}=$$

$$(2\pi)^{\frac{3}{2}}\sqrt{2\omega^Q_{\vec{p}_n}}\big(_{out}<\vec{p}'_1,\dots,\vec{p}'_{n'}|T\{\tilde{\phi}(t_1,\vec{k_1})\cdots\tilde{\phi}(t_l,\vec{k_l})\}a^+_{in}(\vec{p}_n)|\vec{p}_1,\dots,\vec{p}_{n-1}>_{in}-$$

$$_{out}<\vec{p}'_1,\dots,\vec{p}'_{n'}|T\{a^+_{out}(\vec{p}_n)\tilde{\phi}(t_1,\vec{k_1})\cdots\tilde{\phi}(t_l,\vec{k_l})\}a^+_{in}(\vec{p}_n)|\vec{p}_1,\dots,\vec{p}_{n-1}>_{in}\big)=$$

$$\lim_{T\to\infty}\int\frac{e^{-i\omega^Q_{\vec{p}}t}\left(\partial_t^2-\left(\omega^Q_{\vec{p}_n}\right)^2\right)(2\pi)^3}{\sqrt{Z(\vec{p_n})}}\times$$

$$_{out}<\vec{p}'_1,\dots,\vec{p}'_{n'}|T\{\tilde{\phi}(\vec{p}_n,t)\tilde{\phi}(t_1,\vec{k_1})\cdots\tilde{\phi}(t_l,\vec{k_l})\}|\vec{p}_1,\dots,\vec{p}_{n-1}>_{in}dt.$$

This gives (3.43). Proof of (3.44) is completely the same. □

We note that pointwise $H_T(t)$ converges to the identity at large T, so the expression under the limit in (3.43-3.44) is nothing but a regularisation of the Fourier transform at of a distribution at a point. So, informally (3.43) can be interpreted as

$$\int\frac{e^{i\omega t}(2\pi)^3}{\sqrt{Z(\vec{p})}}S_{n,n'-1,l+1}(\vec{p}_1,\dots,\vec{p}_n;\vec{p}'_1,\dots,\vec{p}'_{n'-1};\vec{k}_1,\dots,\vec{k}_l,\vec{p}_{n'};t_1,\dots,t_l,t)\approx$$

$$\frac{1}{\omega^2-\left(\omega_{\vec{p}^Q_n}\right)^2}S_{n,n',l}(\vec{p}_1,\dots,\vec{p}_n;\vec{p}'_1,\dots,\vec{p}'_{n'};\vec{k}_1,\dots,\vec{k}_l;t_1,\dots,t_l),\omega\to\omega_{\vec{p}^Q_n}$$

and similarly for (3.44).

Sequentially applying (3.43-3.44) we get[38]

$$\mathcal{G}(\vec{p}'_1,\omega'_1;\cdots;\vec{p}'_{n'},\omega_{n'};-\vec{p}_1,-\omega_1;\cdots;-\vec{p}_n,-\omega_n)\sim \tag{3.45}$$

$$\prod_{i=1}^{n'}\frac{i\sqrt{Z_F(\vec{p}'_i)}}{(2\pi)^4 2\omega^Q_{\vec{p}'_i}(\omega'_i-\omega^Q_{\vec{p}'_i}+i0)}\prod_{i=1}^{n}\frac{i\sqrt{Z_F(\vec{p}_i)}}{(2\pi)^4 2\omega^Q_{\vec{p}_i}(\omega_i-\omega^Q_{\vec{p}_i}-i0)}\cdot$$

$$_{out}<\vec{p}'_1,\cdots,\vec{p}'_{n'}|\vec{p}_1,\cdots,\vec{p}_n>_{in},$$

where $\mathcal{G}$ is the Fourier transform of the time-ordered vacuum correlators defined by

$$G(x_1,\dots,x_n)=\int d\vec{p}_1\cdots d\vec{p}_n d\omega_1\cdots d\omega_n\mathcal{G}(\vec{p}_1,\omega'_1;\cdots;\vec{p}_n)\prod_{i=1}^{n}e^{-i(\omega_i t_i-\vec{p}_i\vec{x}_i)}$$

with $x_i=(t_i,\vec{x}_i)$ and the sign $\sim$ means that we leave only the most singular part in the limit $\omega_i\to\omega^Q_{\vec{p}_i}$. We see that the physical spectrum of the theory is encoded in the poles of the correlators . The residues of that poles contain information about the scattering amplitudes. Finally, although the proof above is not applicable for coinciding incoming and outcoming momenta, one still can prove following the same lines that

$$\mathcal{G}(\vec{p},\omega;\vec{p}',\omega')\sim$$

$$-i\delta(\vec{p}-\vec{p}')\delta(\omega-\omega')\frac{Z_F(\vec{p})}{\omega^2-\left(\omega^Q_{\vec{p}}\right)^2},$$

which allows to get the renormalisation coefficient also from the correlators.

[38] We give the informal statement only for the sake of shortness. The rigorous one would contain a subsequent limit of $n+n'$ regularisations wearing off. We refer to [77] for a more elegant rigorous version.

Remark 3.27. We note that, although in our proof we never assumed the fields to be local, the very idea of the LSZ analysis requires the particles to be free when they are far away from each other. Otherwise there is no reason to expect that Assumption 3.25 holds.

4 Feynman rules for non-local Hamiltonian Perturbation theory

In this section we develop the Feynman rules for calculation of the S-matrix and of the correlators for the non-local theories quantised in the Hamiltonian approach of 3.1.2 with H_I provided in the form (3.16) and assuming the invariance under spatial translations (3.28), so

$$H_I(t) = \lambda(t)\sum_n (2\pi)^3 \frac{1}{n!}\int d\vec{k}_1 \cdots d\vec{k}_n \delta(\vec{k}_1 + \cdots + \vec{k}_n) \tag{4.1}$$

$$: \tilde{\phi}_{0\alpha_1}(\vec{k}_1, t)\cdots\tilde{\phi}_{0\alpha_{n-1}}(\vec{k}_{n-1}, t)\tilde{\phi}_{\alpha_n}(\vec{k}_n, t) : F^{\alpha_1\cdots\alpha_n}(\vec{k}_1, \ldots, \vec{k}_n).$$

Here F are Schwartz functions proportional to the coupling constant g, or, more generally, form a family of power series consisting of the Schwartz functions and vanishing at the origin. Due to symmetry of the Wick product, we can always assume that F is symmetric, and we do from now on. We also introduce Fourier transform of the adiabatic switching

$$\tilde{\lambda}(\Delta) = \frac{1}{2\pi}\int dt \lambda(t) e^{-i\Delta t}.$$

The Feynman rules for the scattering amplitudes in adiabatic limit were already presented in [73]. But, for our purposes we will need several different formulations and we also prefer to keep the temporal adiabatic switching explicit.

We will work in the partial Fourier Transform (D.2) for both free and interacting fields. For convenience we introduce decomposition of the interacting field analogous to (E.12)

$$\tilde{\phi}(\vec{p}, t) = \tilde{\phi}_+(\vec{p}, t) + \tilde{\phi}_-(\vec{p}, t).$$

$$\tilde{\phi}_\alpha(\vec{p}, t) = U(t, t_0)^{-1}\tilde{\phi}_{0\alpha}(\vec{p}, t)U(t, t_0).$$

We note, that, unlike the free field case, $\tilde{\phi}_\pm$ in general does not have sense of creation and annihilation parts. This decomposition is introduced for convenience only and depends on the renormalisation.

We introduce the decomposed partial Fourier transform of the correlators

$$G_n(x_1, \ldots, x_n) = \sum_{\alpha_1,\ldots,\alpha_n}\int d\vec{p}_1 \cdots d\vec{p}_{n-1} d\vec{p_n} \tilde{G}_{\alpha_1,\cdots,\alpha_n}(\vec{p}_1, x_1^0; \ldots; \vec{p}_n, x_n^0) e^{i\sum_{j=1}^n \vec{x}_j\vec{p}_j},$$

$$\tilde{G}_{\alpha_1,\ldots,\alpha_n}(\vec{p}_1, x_1^0; \ldots; \vec{p}_n, x_n^0) =< \Omega|\mathbf{T}\{\tilde{\phi}_{\alpha_1}(\vec{p}_1, x_1^0)\cdots\tilde{\phi}_{\alpha_n}(\vec{p}_n, x_n^0)\}|\Omega >, \tag{4.2}$$

as well as the complete Fourier transform

$$\tilde{G}_{\alpha_1,\ldots,\alpha_n}(\vec{p}_1, t_1; \ldots; \vec{p}_n, t_n) = \tag{4.3}$$

$$\int d\omega_1 \cdots d\omega_n \mathcal{G}_{\alpha_1,\cdots,\alpha_n}(\vec{p}_1, \omega_1; \cdots; \vec{p}_n, \omega_n) e^{-i\sum_{k=1}^n \omega_k t_k}.$$

Instead of (3.14) we then have

$$\tilde{G}_{\alpha_1,\ldots,\alpha_n}(\vec{p}_1, t_1; \ldots; \vec{p}_n, t_n)\left\langle 0\left|\mathbf{T}\left\{\exp\left(-i\int dt H_I(t)\right)\right\}\right|0\right\rangle = \tag{4.4}$$

$$\left\langle 0\left|\mathbf{T}\left\{\tilde{\phi}_{0\alpha_1}(\vec{p}_1, t_1)\cdots\tilde{\phi}_{0\alpha_n}(\vec{p}_n, t_n)\exp\left(-i\int dt H_I(t)\right)\right\}\right|0\right\rangle.$$

In addition to the "Einstein summation rule" for the time-orientations we introduce the following definition:

$$A_\alpha = A^{-\alpha}, \qquad \alpha = \pm.$$

Note, that

$$A^\alpha B_\alpha = A^+ B_+ + A^- B_- = A^+ B^- + A^- B^+ = A_\alpha B^\alpha,$$

so one can raise and lower indices in any Einstein sum as usual. The free propagator in this convention will have the diagonal form:

$$\mathcal{G}^{(0)\alpha}{}_\beta(\vec{p},\omega) = \delta_{\alpha,\beta}\frac{i\alpha}{2\omega_{\vec{p}}(2\pi)^4(\omega-\alpha(\omega_{\vec{p}}-i0))}. \tag{4.5}$$

Now we are ready to apply the Second Wick Theorem (Theorem F.4 and Remark F.5) to (3.13) and (4.4). The main difference from the ordinary QFT is the separation of quantum field in two parts, so the Feynman rules look like the Feynman rules for two species (or one specie with additional internal quantum number) with the propagator (F.2-F.3) mixing these species (changing the internal number). So, in addition to usual labeling of lines by corresponding momenta, we need to label each end of each line by corresponding time-orientation α.

Proposition 4.1 ("Time-momentum" space Feynman rules for correlators)**.** *The correlator* $\tilde{G}_{\alpha_1,\dots,\alpha_E}(\vec{p}_1,t_1,\dots,\vec{p}_E,t_E$ *can be found using the following Feynman rules:*

1. *Draw all Feynman graphs with E external lines corresponding to E triples* $(\alpha_i,\vec{p}_i,t_i)$ *without self-contractions and vacuum energy corrections;*

2. *Assign to each internal line a 3-momentum flow* $\vec{p}\in\mathbb{R}^3$ *and to each end of each line a time orientation* $\alpha=\pm$ *. The external ones will be always assumed to bring the assigned momentum from the free end to the non-free one for convenience;*

3. *To each vertex assign a time stamp* τ*.*

4. *For a line, transporting a 3-momentum* $\vec{p}$ *from an end with a time-orientation* β *and a time stamp* τ *to an end with a time-orientation* γ *and a time stamp* τ' *multiply by a factor* $\delta_{\beta,-\gamma}\theta(\beta(\tau-\tau'))\frac{1}{2\omega_{\vec{p}}(2\pi)^3}$*.*

5. *For a vertex of order m multiply by a factor*

$$-i(2\pi)^4 F^{\beta_1,\dots,\beta_m}(\pm\vec{k}_1,\cdots,\pm\vec{k}_m)\lambda(\tau)\delta^{(3)}(\pm\vec{k}_1+\ldots\pm\vec{k}_m)e^{i(\beta_1\omega_{\vec{k}_1}+\cdots+\beta_m\omega_{\vec{k}_m})\tau} \tag{4.6}$$

   where $\vec{k}_i$ *are the momenta flowing along the lines incident to the vertex, sign* $\pm$ *are determined by the direction of this flow, namely sign "+" corresponds to incoming flow and sign "-" to the outgoing,* β_i *are the time-orientations assigned to the incidences of the corresponding lines with the vertex, and* τ *is the time-stamp of the vertex;*

6. *Integrate over all 3-momenta flowing along internal lines and the timestamps of all the vertices and sum over all free time orientations;*

7. *Multiply by a usual symmetry factor.*

Note that we moved the oscillating part of the propagator $\tilde{G}^0_{\alpha\beta}(\vec{p},t)$ to the vertex factor.

Proposition 4.2 ("Energy-momentum" space Feynman rules for correlators)**.** *The correlator* $\mathcal{G}_{\alpha_1,\dots,\alpha_n}(\vec{p}_1,\omega_1;\dots;\vec{p}_E,$ *can be found using the following Feynman rules:*

1. *Draw all Feynman graphs with E external lines corresponding to E triples* $(\alpha_i,\vec{p}_i,\omega_i)$ *without self-contractions and vacuum energy corrections;*

2. *Assign to each internal line a 4-momentum flow* $(\omega, \vec{p}) \in \mathbb{R}^4$ *and to each end of each line a time orientation* $\alpha = \pm$ *. The external ones will be always assumed to bring the assigned momentum from the free end to the non-free one for convenience;*

3. *For a line, transporting the 4-momentum* $(\omega, \vec{p})$ *from an end with time-orientation* β *to an end with time-orientation* γ *multiply by a factor* $\mathcal{G}^0_{\beta\gamma}(\vec{p}, \omega)$*;*

4. *For a vertex of order* m *multiply by a factor*

$$-i(2\pi)^4 F^{\beta_1,\dots,\beta_m}(\pm\vec{k}_1, \dots, \pm\vec{k}_m)\tilde{\lambda}(\pm\omega'_1 + \cdots + \pm\omega'_n)\delta^{(3)}(\pm\vec{k}_1 + \cdots \pm \vec{k}_m),$$

where $\vec{k}_i$ *and* ω'_i *are the momenta flowing along the lines incident to the vertex, sign* $\pm$ *are determined by the direction of this flow, namely sign "+" corresponds to incoming flow and sign "-" to the outgoing, and* β_i *are the time-orientations assigned to the incidences of the corresponding lines with the vertex.*

5. *Integrate over all 4-momenta flowing along internal lines and sum over all free time orientations;*

6. *Multiply by a usual symmetry factor*

The first rather unconventional mixed (part momentum part position space) formulation will be useful to prove the weak adiabatic limit existence in Subsection 7.1.1), while the second one will be used in the rest of the thesis.

The Feynman rules for scattering amplitudes follow from the Second Wick Theorem together with the definition of the Fock vector-valued distibutions (E.4). We also present them in two forms.

Proposition 4.3 ("Time-momentum" space Feynman rules for scattering amplitudes)**.** *The scattering amplitude* $_{out} < \vec{p}'_1, \cdots, \vec{p}'_{n'} | \vec{p}_1, \dots, \vec{p}_n >_{in}$ *can be found using the following Feynman rules:*

1. *Draw all Feynman graphs with* $n + n'$ *external lines, namely* n *incoming lines, corresponding to* n *momenta* $\vec{p}_i$ *and* n' *outgoing lines corresponding to* n' *momenta* $\vec{p}_i$ *without tadpoles and vacuum energy corrections.*

2. *Assign to each internal line a 3-momentum flow* $\vec{p} \in \mathbb{R}^3$ *and to each end of each line a time orientation* $\alpha = \pm$*. An incoming (respectively outgoing) line with momentum* $\vec{p}$ *is assumed to bring this momentum towards the diagram (respectively from the diagram) and its non-free end is assumed to have time-orientation "-" (respectively "+");*

3. *Assign to each vertex a time-stamp* τ*;*

4. *For an internal line, transporting a 3-momentum* $\vec{p}$ *from an end with a time-orientation* β *and a time stamp* τ *to an end with a time-orientation* γ *and a time stamp* τ' *multiply by a factor* $\delta_{\beta,-\gamma}\theta(\beta(\tau - \tau'))\frac{1}{2\omega_{\vec{p}}(2\pi)^3}$*.*

5. *For an external line with momentum* $\vec{p}$ *multiply by a factor* $\frac{1}{\sqrt{2\omega_{\vec{p}}(2\pi)^3}}$*;*

6. *For a vertex of order* m *multiply by a factor*

$$-i(2\pi)^4 F^{\beta_1,\dots,\beta_m}(\pm\vec{k}_1, \cdots, \pm\vec{k}_m)\lambda(\tau)\delta^{(3)}(\pm\vec{k}_1 + \ldots \pm \vec{k}_m)e^{i(\beta_1\omega_{\vec{k}_1} + \cdots + \beta_m\omega_{\vec{k}_m})\tau}$$

where $\vec{k}_i$ *are the momenta flowing along the lines incident to the vertex, sign* $\pm$ *are determined by the direction of this flow, namely sign "+" corresponds to incoming flow and sign "-" to the outgoing,* β_i *are the time-orientations assigned to the incidences of the corresponding lines with the vertex, and* τ *is the time-stamp of the vertex;*

7. *Integrate over all 3-momenta flowing along internal lines and the timestamps of all the vertices and sum over all free time orientations;*

Proposition 4.4 ("Energy-momentum" space Feynman rules for scattering amplitudes)**.** *The scattering amplitude* $_{out} < \vec{p}'_1, \cdots, \vec{p}'_{n'} | \vec{p}_1, \ldots, \vec{p}_n >_{in}$ *can be found using the following Feynman rules:*

1. *Draw all Feynman graphs with* $n + n'$ *external lines, namely* n *incoming lines, corresponding to* n *momenta* $\vec{p}_i$ *and* n' *outgoing lines corresponding to* n' *momenta* $\vec{p}_i$ *without tadpoles and vacuum energy corrections.*

2. *Assign to each internal line a 4-momentum flow* $(\omega, \vec{p}) \in \mathbb{R}^4$ *and to each end of each line a time orientation* $\alpha = \pm$*. An incoming (respectively outgoing) line with momentum* $\vec{p}$ *is assumed to bring this momentum towards the diagram (respectively from the diagram) and its non-free end is assumed to have time-orientation "-" (respectively "+");*

3. *For an internal line, transporting the 4-momentum* $(\omega, \vec{p})$ *from an end with time-orientation* β *to an end with time-orientation* γ *multiply by a factor* $\mathcal{G}^0_{\beta\gamma}(\vec{p}, \omega)$*;*

4. *For an external line with momentum* $\vec{p}$ *multiply by a factor* $\frac{1}{\sqrt{2\omega_{\vec{p}}(2\pi)^3}}$*;*

5. *For a vertex of order* m *multiply by a factor*

$$-i(2\pi)^4 F^{\beta_1,\ldots,\beta_m}(\pm\vec{k}_1, \ldots, \pm\vec{k}_m)\tilde{\lambda}(\pm\omega'_1 + \cdots + \pm\omega'_n)\delta^{(3)}(\pm\vec{k}_1 + \cdots \pm \vec{k}_m),$$

   where $\vec{k}_i$ *and* ω'_i *are the momenta flowing along the lines incident to the vertex, sign* $\pm$ *are determined by the direction of this flow, namely sign "+" corresponds to incoming flow and sign "-" to the outgoing, and* β_i *are the time-orientations assigned to the incidences of the corresponding lines with the vertex;*

6. *Integrate over all 4-momenta flowing along internal lines and sum over all free time orientations;*

7. *Multiply by a usual symmetry factor.*

The following examples and remarks will help to understand the terminology used, and illustrate the relations of the Feynman rules above with the Feynman rules of [73] and ordinary Feynman rules.

Remark 4.5. In the "time-momentum" formulations of the Feynman rules a product of Heaviside functions of the same argument (or of the opposite arguments) may appear, which is not well-defined in $\mathcal{S}(\mathbb{R})$. These products should be understood in the naïve sense

$$\theta(t)^2 = \theta(t), \qquad \theta(t)\theta(-t) = 0. \tag{4.7}$$

It is worth emphasising that the above is not an artificial regularisation, but just a definition, since the timeordered product $\mathbf{T}$ in the right hand side (4.4) does not produce any singular products as we have seen in Subsection 3.1.2. The definition (4.7) brings no problem, because no derivatives with respect to the time, and no products with more singular distributions (like δ-functions) ever appear, so behaviour at one point $t = 0$ is irrelevant.

Remark 4.6. Since the propagator (with two lower indices) is anti-diagonal, only diagrams with all lines having time-orientation "+" at one end and "-" at another contribute. So, instead of summing over all graphs and all time-orientations of each incidence we can just sum over all directed graphs, orienting each line from it "+" to its "-" end[39]. In fact, this direction is the direction of time, which is clear from the "time-momentum" formulation of Proposition 4.1. We however prefer to keep the summation over time-orientations, although it may seem artificially overcomplicated at the moment.

[39]This direction should not be confused with the direction of the momentum flow along the internal lines which is completely arbitrary in our formulation

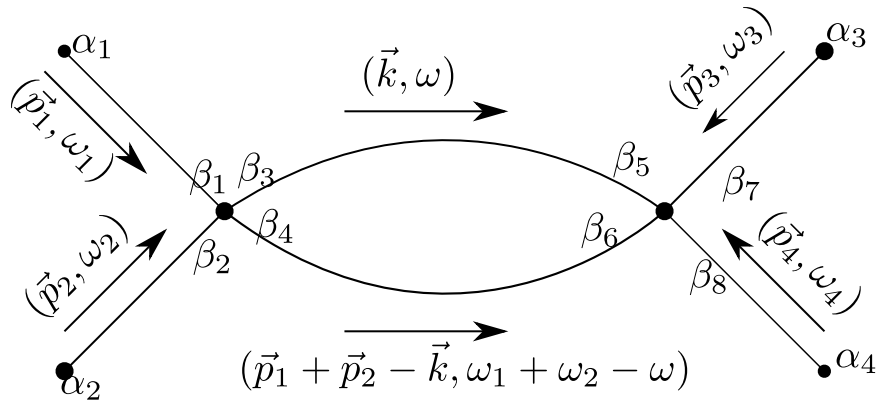


Figure 4.1: Fish-type diagram with energy-momentum flows and time-orientations assigned.

Remark 4.7. The Feynman rules of Proposition 4.4 do not coincide with the momentum space Feynman rules in [73]. To get the latter one has to evaluate the integral over the time stamps of the interaction vertices explicitly, assuming all vertices to be explicitly time-ordered (and sum over all possible orderings in the end).

Example 4.8. Consider a fish-type diagram for the quartic interaction shown on the Fig. 4.1 According to the Feynman rules we have:

$$\mathcal{G}_{\Gamma;\alpha_1,\alpha_2,\alpha_3,\alpha_4}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2;\vec{p}_3,\omega_3;\vec{p}_4,\omega_4) =$$

$$\frac{1}{2}\sum_{\beta_i\in\{+,-\},i=1,\cdots,8}\int d\vec{k}d\omega\mathcal{G}^0_{\alpha_1,\beta_1}(\vec{p}_1,\omega_1)\mathcal{G}^0_{\alpha_2,\beta_2}(\vec{p}_2,\omega_2)(2\pi)^4F^{\beta_1,\beta_2,\beta_3,\beta_4}(\vec{p}_1,\vec{p}_2,-\vec{k},-\vec{p}_1-\vec{p}_2+\vec{k})\times$$

$$\mathcal{G}^0_{\beta_3,\beta_5}(\vec{k},\omega)\mathcal{G}^0_{\beta_4,\beta_6}(\vec{p}_1+\vec{p}_2-\vec{k},\omega_1+\omega_2-\omega)(2\pi)^4F^{\beta_5,\beta_6,\beta_7,\beta_8}(\vec{k},\vec{p}_1-\vec{p}_2-\vec{k},\vec{p}_3,\vec{p}_4)\times$$

$$\mathcal{G}^0_{\alpha_3,\beta_7}(\vec{p}_3,\omega_3)\mathcal{G}^0_{\alpha_4,\beta_8}(\vec{p}_4,\omega_4)\delta^{(3)}(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)\delta(\omega_1+\omega_2+\omega_3+\omega_4).$$

Now, one can see that the integrand decays as $\frac{1}{\omega^2}$ then ω goes to infinity, so the the integration contour can be closed in both upper or lower half-planes. Let us close it in the upper half-plane if $\beta_3 = -$ and in the lower otherwise, so that the pole at $\omega = -\beta_3(\omega_{\vec{k}} - i0)$ never contributes. Then the second pole at $\omega = \omega_1 + \omega_2 + \beta_4(\omega_{\vec{p}_1+\vec{p}_2-\vec{k}} - i0)$ contributes only if $\beta_4 = \beta_3$:

$$\mathcal{G}_{\Gamma;\alpha_1,\alpha_2,\alpha_3,\alpha_4}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2;\vec{p}_3,\omega_3;\vec{p}_4,\omega_4) =$$

$$\frac{1}{2}\frac{-i\alpha_1}{2\omega_{\vec{p}_1}(2\pi)^4(\omega_1+\alpha_1(\omega_{\vec{p}_1}-i0))}\frac{-i\alpha_2}{2\omega_{\vec{p}_2}(2\pi)^4(\omega_2+\alpha_2(\omega_{\vec{p}_2}-i0))}\times$$

$$\frac{-i\alpha_3}{2\omega_{\vec{p}_3}(2\pi)^4(\omega_3+\alpha_3(\omega_{\vec{p}_3}-i0))}\frac{-i\alpha_4}{2\omega_{\vec{p}_4}(2\pi)^4(\omega_4+\alpha_4(\omega_{\vec{p}_3}-i0))}\times$$

$$\int d\vec{k}\sum_{\beta_3\in\{+,-\}}(2\pi)^4F^{-\alpha_1,-\alpha_2,\beta_3,\beta_3}(\vec{p}_1,\vec{p}_2,-\vec{k},-\vec{p}_1-\vec{p}_2+\vec{k})\times$$

$$\frac{1}{2\omega_{\vec{k}}(2\pi)^4}\frac{1}{2\omega_{\vec{p}_1+\vec{p}_2-\vec{k}}(2\pi)^4}\frac{-2\pi i}{(\beta_3\omega_1+\beta_3\omega_2+\omega_{\vec{k}}+\omega_{\vec{p}_1+\vec{p}_2-\vec{k}}-i0)}\times$$

$$(2\pi)^4F^{-\beta_3,-\beta_4,-\alpha_3,-\alpha_4}(\vec{k},\vec{p}_1-\vec{p}_2-\vec{k},\vec{p}_3,\vec{p}_4)\delta^{(3)}(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)\delta(\omega_1+\omega_2+\omega_3+\omega_4)$$

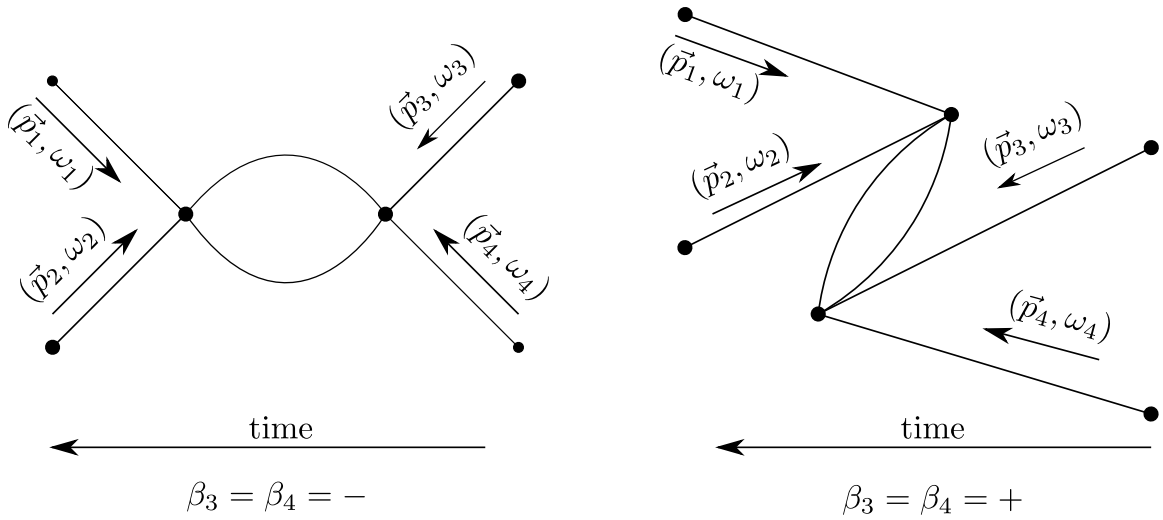


Figure 4.2: Schematic illustration of the two possible time-orderings of the fish-type diagram in a theory with quartic interaction.

Similarly, the corresponding contribution to a scattering amplitude of a process

$$\vec{p}_1, \vec{p}_2 \to \vec{p}\,'_1, \vec{p}\,'_2$$

is

$$\frac{1}{2}\frac{1}{(2\pi)^6\sqrt{2\omega_{\vec{p}_1}2\omega_{\vec{p}_2}2\omega_{\vec{p}\,'_1}2\omega_{\vec{p}\,'_2}}}$$

$$\int d\vec{k} \sum_{\beta_3\in\{+,-\}} (2\pi)^4 F^{-\alpha_1,-\alpha_2,\beta_3,\beta_3}(\vec{p}_1, \vec{p}_2, -\vec{k}, -\vec{p}_1 - \vec{p}_2 + \vec{k})\times$$

$$\frac{1}{2\omega_{\vec{k}}(2\pi)^4}\frac{1}{2\omega_{\vec{p}_1+\vec{p}_2-\vec{k}}(2\pi)^4}\frac{-2\pi i}{(\beta_3\omega_1 + \beta_3\omega_2 + \omega_{\vec{k}} + \omega_{\vec{p}_1+\vec{p}_2-\vec{k}} - i0)}\times$$

$$(2\pi)^4 F^{-\beta_3,-\beta_4,-\alpha_3,-\alpha_4}(\vec{k}, \vec{p}_1 - \vec{p}_2 - \vec{k}, -\vec{p}\,'_1, -\vec{p}\,'_2)\delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{p}\,'_1 - \vec{p}\,'_2)\delta(\omega_{\vec{p}_1} + \omega_{\vec{p}_2} - \omega_{\vec{p}\,'_1} - \omega_{\vec{p}\,'_2})$$

This result coincides with one that can be achieved by the momentum space Feynman rules of [73]. As expected, the summation over two possible time-orientations of the internal lines corresponds to the summation over the time-orderings of [73] as show on the Fig. 4.2.

Remark 4.9. If we apply this approach to an ordinary QFT without derivatives with respect to time in the interaction, the vertex factors will not depend on the time-orientations. The free propagator (F.4) will appear only summed over both its indices, which is just the Feynman propagator,

$$\mathcal{G}^{(0)}_F(\vec{p}, \omega) = \sum_{\alpha\beta} \mathcal{G}^{(0)}_{\alpha\beta}(\vec{p}, \omega) = \frac{i}{(2\pi)^4(\omega^2 - \omega^2_{\vec{p}} + i0)}.$$

So, the Feynman rules introduced here will coincide with the usual ones.

A local quantum field theory can also contain a finite number of derivatives with respect to time in the interaction. This case is considered in Section 5.

In the end of the section let us present some results we used in Proposition 3.13 and Remark 3.15.

Lemma 4.10. *Let Γ be a Feynman graph contributing to ${}_{out} < \vec{p}'_1, \cdots, \vec{p}'_{n'} | \vec{p}_1, \ldots, \vec{p}_n >_{in}$ in the sense of Propositions 4.3-4.4. Then the corresponding contribution has the form*

$$W_\Gamma(\vec{p}'_1, \ldots, \vec{p}'_{n'}; \vec{p}_1, \ldots, \vec{p}_n) C_\Gamma(\vec{p}'_1, \ldots, \vec{p}'_{n'}; \vec{p}_1, \ldots, \vec{p}_n),$$

where $W_\Gamma \in \mathcal{S}(\mathbb{R}^{3(n+n')})$, and C_Γ is a product of the over-all three-momentum conservation delta-functions of all connected components.

Proof. Let us use the formulation of Proposition 4.3. Assume that the graph Γ consists of one connected component. Then integration over all internal momenta leaves of the delta-functions in the vertex factors only the over-all 3-momentum conservation law (see also [72, 73]) which is exactly C_Γ.

The rest of the integrand is a product of continuous functions (F, λ and the oscillating exponents from the vertex factors and $\frac{1}{\omega_{\vec{p}}}$ from propagators) and theta-functions from the propagators. The latter is always a well-defined distributions, while the former is a Schwartz function of the timestamps, internal and external momenta (for that note that each internal or external momentum appears at least once in the vertex factors F and each timestamp appears in the adiabatic switching λ; finally $\frac{1}{\omega_{\vec{p}}} \in \theta_M(\mathbb{R}^3)$, see Remark E.2). So, by Theorem B.2 it is a Schwartz function of all the external momenta.

Finally, contribution of a generic graph is a product of contributions of its connected components, which finishes the proof. □

Remark 4.11. It is not difficult to reformulate the Feynman rules discussed here, and in particular the ones of Proposition 4.3 to allow the full adiabatic cut-off $\lambda(x)$. The main difference then will be that in the vertex factors instead of

$$\lambda(\tau)\delta^{(3)}(\pm\vec{k}_1 + \ldots \pm \vec{k}_m)$$

we will have

$$\frac{1}{(2\pi)^3} \int \lambda(\tau, \vec{x}) e^{i(\pm\vec{k}_1 + \ldots \pm\vec{k}_m)\vec{x}} d^3\vec{x} = \delta^{(3)}(\pm\vec{k}_1 + \ldots \pm \vec{k}_m + \kappa)\tilde{\lambda}(\tau, \kappa)$$

with the partial Fourier transform $\tilde{\lambda}$ defined as in Remark 3.15. Now if we fix a graph with V vertices and postpone integration over τ_j and κ_j with a weight $\tilde{\lambda}(\tau_j, \kappa_j)$, where j enumerates the vertices, we can compute the corresponding contribution to the amplitudes as a function of $(\tau_1, \kappa_1, \ldots, \tau_V, \kappa_V)$. Then, one can prove that the decomposition similar to observed in Lemma 4.10 is possible, but the conservation laws in C_Γ will be corrected by sums of κ_j over vertices belonging to corresponding subgraphs and W_Γ will be replaced by

$$W'_\Gamma(\vec{p}'_1, \ldots, \vec{p}'_{n'}; \vec{p}_1, \ldots, \vec{p}_n; \vec{\kappa}_1, \tau_1, \ldots, \vec{\kappa}_V, \tau_V),$$

$W'_\Gamma \in C^\infty(\mathbb{R}^{3(n+n')+4V})$. For fixed $(\tau_1, \vec{\kappa}_1, \ldots, \tau_V, \vec{\kappa}_V)$ W'_Γ is fast decaying with respect to the rest variables. If in addition the variation of κ_j is restricted to a compact set (this is necessary, since κ_j shift the dependence of the vertex factors on the external momenta), the corresponding Schwartz seminorms are bounded uniformly in all κ_j and τ_j.

The spatial adiabatic limit can be reconstracting by substituting $\vec{\kappa} = 0$ and integrating over all τ_i with a weight $\lambda(\tau_j)$.

5 Lagrangian reformulation of the Hamiltonian Feynman rules

In Subsection 3.2 we stated that if a theory is non-local in time, then it can not be quantised by means of the standard Lagrangian methods, because the Epstein-Glaser approach makes no sense, and the functional integration violates the unitarity. However, the non-local theories considered in 3.1.2 are actually non-local only in space and hence one could expect that they have an equivalent Lagrangian description. In this section we will see that it exists and the Lagrangian is the Legendre transform of the Hamiltonian (3.34) plus some quantum loop corrections. We will need several steps to reformulate the Feynman rules in the "Lagrangian-like" form. We will work with the correlators in the adiabatic limit (its existence is shown in Subsection 7.1.1) and we will use the "energy-momentum" formulation of the Feynman rules as in Proposition 4.2.

Our goal is to replace the time-ordered product $\mathbf{T}$ by its covariant version $\hat{\mathbf{T}}$. As we have seen in Subsection 3.2, $\hat{\mathbf{T}}$ commutes with the derivatives with respect to the time. In the "energy-momentum" formulation it means that each time a derivative of the quantum field appears inside the propagator, it is multiplied by the off-shell frequency ω. If a derivative appears inside the naïve time-ordering $\mathbf{T}$, an on-shell frequency $\alpha\omega_{\vec{p}}$ appears instead. This is formally summed in the following:

Lemma 5.1. *The free propagator (F.4) satisfies:*

$$\sum_{\alpha,\beta} \mathcal{G}^{(0)\alpha}{}_{\beta}(\omega,\vec{p}) = \mathcal{G}_F^{(0)}(\omega,\vec{p}) \tag{5.1}$$

$$\sum_{\alpha,\beta} \alpha\omega_{\vec{p}}\mathcal{G}^{(0)\alpha}{}_{\beta}(\omega,\vec{p}) = \omega\mathcal{G}_F^{(0)}(\omega,\vec{p}), \tag{5.2}$$

$$\sum_{\alpha,\beta} \alpha\beta\omega_{\vec{p}}^2\mathcal{G}^{(0)\alpha}{}_{\beta}(\omega,\vec{p}) = \omega^2\mathcal{G}_F^{(0)}(\omega,\vec{p}) - \frac{i}{(2\pi)^4}. \tag{5.3}$$

Combining this lemma with the following simple identity

$$\delta_{\alpha\beta} = \frac{1+\alpha\beta}{2} = \frac{1}{2}\sum_{s=0}^{1}\alpha^s\beta^s, \quad \alpha,\beta = \pm 1,$$

we get

$$\mathcal{G}^{(0)\alpha}{}_{\beta}(\omega,\vec{p}) = \frac{1}{4}\sum_{s=0}^{1}\sum_{s'=0}^{1}\sum_{\alpha',\beta'}\alpha^s\alpha'^s\beta^{s'}\beta'^{s'}\mathcal{G}^{(0)\alpha'}{}_{\beta'}(\omega,\vec{p}) =$$

$$f_S^{\alpha}(\vec{p},\omega)f_{S\beta}(-\vec{p},-\omega)\mathcal{G}_S^{(0)}(\omega,\vec{p}) + f_D^{\alpha}(\vec{p},\omega)f_{D\beta}(-\vec{p},-\omega)\mathcal{G}_D^{(0)} \quad (\omega,\vec{p}),$$

where

$$f_S^{\alpha}(\vec{p},\omega) = \frac{1+\alpha\frac{\omega}{\omega_{\vec{p}}}}{2}, \qquad f_D^{\alpha}(\vec{p},\omega) = \frac{\alpha}{2\omega_{\vec{p}}},$$

$$\mathcal{G}_S^{(0)}(\vec{p},\omega) = \mathcal{G}_F^{(0)}(\vec{p},\omega), \mathcal{G}_D^{(0)}(\vec{p},\omega) = \frac{i}{(2\pi)^4}.$$

The letters S and D stand for "solid" and "dashed", revealing that we will graphically represent the two terms in the propagator above by solid and dashed lines. For shortness we will also call the terms themselves the "solid" and the "dashed" propagator. Although this separation seems to be an unnecessary complication, due to nice factorization of both solid and dashed propagators we can get rid of sum over time orderings. The result is summarized in the Proposition below, the proof is straightforward. For the sake of simplicity we will find only the correlator summed over time-orientations of external lines, although the generalization is not too difficult.

Proposition 5.2. *The (weak adiabatic limit of the) correlator*

$$\sum_{\alpha_1\dots\alpha_n} \mathcal{G}_{\alpha_1\dots\alpha_E}(\vec{p}_1,\omega_1;\dots\vec{p}_E,\omega_E)$$

can be found using the following Feynman rules:

1. *Draw all ordinary connected Feynman graphs with E external solid lines corresponding to E pairs $(\vec{p}_i,\omega_i)$ and two possible types of internal lines (solid and dashed) without self-contractions and vacuum energy corrections;*
2. *Assign to each internal line the 4-momentum flow $(\omega,\vec{p})$;*
3. *For a solid internal or external line transporting the 4-momentum $(\omega,\vec{p})$ multiply by the propagator $\mathcal{G}_S^{(0)}(\vec{p},\omega)$*
4. *For a dashed internal line transporting the 4-momentum $(\omega,\vec{p})$ multiply by a propagator $\mathcal{G}_D^{(0)}(\vec{p},\omega)$;*
5. *For a vertex of order m add multiply by factor*

$$-i(2\pi)^4\tilde{F}^{s_1\dots s_m}(\pm\vec{k}_1,\dots,\pm\vec{k}_m)\delta(\pm\omega_1+\dots+\pm\omega_m)\delta^{(3)}(\pm\vec{k}_1+\dots\pm\vec{k}_m),$$

   where $\vec{k}_i$ and ω_i are the momenta flowing along the lines incident to the vertex, sign $\pm$ are determined by the direction of this flow, namely sign "+" corresponds to incoming flow and sign "-" to the outgoing, and $s_i\in\{S,D\}$ determines whether the ith line is solid or dashed. Finally,

$$\tilde{F}^{s_1\cdots s_n}(\vec{k}_1,\cdots,\vec{k}_n)=$$

$$\sum_{\alpha_1\dots\alpha_E}\prod_{i=1}^{n} f_{s_i}^{\alpha_i}(\vec{p}_i,\omega_i)F^{\alpha_1\cdots\alpha_n}(\vec{k}_1,\omega_1;,\cdots,\vec{k}_n,\omega_n)\delta(\omega_1+\dots+\omega_n).$$

6. *Integrate over all free (i.e. not eliminated by the momentum conservation and except the arguments of the correlator) 4-momenta;*
7. *Multiply by a usual symmetry factor.*

Remark 5.3. The only frequency-dependent factors in the Feynman rules above are the solid propagators, decreasing as ω^{-2} at high frequencies and f_S^α ($\sim\omega$), which increases at high frequencies as ω. Clearly, there are exactly two f_S factors for each solid propagator, so the integral over the loop frequencies is most likely divergent. However, if we first sum over all diagrams with the same topology, and only then integrate over the loop energies, we get back to the well-defined rules formulated in Proposition 4.2, so all the divergences must cancel. We postpone further discussion of this issue until Remark 5.7. In the meantime we implicitly assume that all the frequency integrals are regularised in UV in such a way that the mentioned cancellation is preserved.

The next step is to note that the dashed propagator is regular in the energy space (in fact, it is a constant). So, two vertices connected by such a propagator can be also considered as just one vertex. We can continue in this way until all vertices connected by dash lines are absorbed into new grouped vertices. Then the only propagator left is the usual Feynman propagator. We should be careful with the "no self-contractions" rule, because the new vertices are composed from the old ones, so some of pairs of their lines (namely, lines which were incident to different vertices before regrouping of the vertices) can be self-contracted and some (namely, ones incident to the same vertex before regrouping) can not. Our main idea here is to show that the new vertices can be interpreted as vertices of a Lagrangian QFT with Lagrangian related to the Hamiltonian (3.34) by Legendre transformation up to some quantum loop corrections. Since self-contractions are by definition loop diagrams, they are part of the mentioned corrections and thus they will not be too important for us. These ideas lead to the following:

Proposition 5.4. *The correlator $\mathcal{G}_{\alpha_1\dots\alpha_n}(\vec{p}_1,\omega_1;\dots\vec{p}_n,\omega_n)$ can be found using the ordinary Feynman rules (i.e. with only one propagator equal to the Feynman propagator)*[40] *with the interaction part of the Lagrangian L_{int} which itself is given by summing of the diagrams according to the following rules:*

1. *Draw all connected graphs with n external lines. To fix terminology we will call vertices of such graph the original vertices (because the corresponding factor can be directly read from the Hamiltonian).*

2. *Assign to ith external line the corresponding momentum $(\vec{p}_i,\omega_i)$ and the time-orientation α_i flowing from its free end to another one. Assign to each internal line a free momentum flow.*

3. *For each internal line multiply by a propagator*

$$\mathcal{G}_D^{(0)}=\frac{i}{(2\pi)^4};$$

4. *For an external line with a momentum flow $\vec{p}$ and time-orientation α multiply by a factor*

$$\frac{\tilde{\phi}(\vec{p},t)+\frac{i\alpha}{\omega_{\vec{p}}}\partial_t\tilde{\phi}(\vec{p},t)}{2};$$

5. *For a vertex of order m with l incident external lines (which we for simplicity assume to be enumerated by first l numbers) multiply by a factor*

$$-i(2\pi)^4\tilde{F}^{\alpha_1\dots\alpha_l;s_{l+1}s_m}(\pm\vec{k}_1,\dots,\pm\vec{k}_m)\delta(\pm\omega_1+\dots+\pm\omega_n)\delta^{(3)}(\pm\vec{k}_1+\dots\pm\vec{k}_m),$$

   where $\vec{k}_i$ and ω_i are the momenta flowing along the lines incident to the vertex, sign $\pm$ are determined by the direction of this flow, namely sign "+" corresponds to incoming flow and sign "-" to the outgoing, $s_i=S$ for an external line and $s_i=D$ for an internal line. Finally,

$$\tilde{F}^{\alpha_1\dots\alpha_l;s_{l+1}s_n}(\vec{k}_1,\omega_1;\dots;\vec{k}_n,\omega_n)=$$

$$\sum_{\alpha_{l+1}\dots\alpha_m}\prod_{i=1}^{m}f_D^{\alpha_i}(\vec{p}_i,\omega_i)F^{\alpha_1\dots\alpha_n}(\vec{k}_1,\dots,\vec{k}_n)\delta(\omega_1+\dots+\omega_n).$$

6. *Integrate over all free energies and momenta;*

7. *Multiply by a usual symmetry factor;*

8. *Multiply by*

$$-\frac{i}{(2\pi)^4};$$

9. *Impose Wick product on each group of the fields coming from external lines incident to the same original vertex.*

.

Proof. The statement follows from Proposition 5.2 through grouping of vertices connected by the dashed lines as explained above, we need only to comment a few issue. The external lines factor is chosen exactly so, that it produces f_S^α in the Lagrangian Feynman rules. Overall factor takes into account that in the Lagrangian Feynman rules the vertex factor contains $i(2\pi)^4$. Finally, we impose the normal ordering is imposed in such a way that it forbids self-contractions of the original vertices. □

[40] In the standard terminology these are exactly the modified Feynman rules for a spatially non-local QFT. However, in context of this thesis we call them ordinary as opposed to ones of Section 4 with the matrix propagator.

Theorem 5.5. *At the tree level the effective Lagranfian of Proposition 5.4 is related to the Hamiltonian (3.34) via the Legendre transform,*

$$L_0(t) + L_{int}(t) = \int \tilde{\pi}(\vec{p},t)\partial_t\tilde{\phi}(-\vec{p},t)d^3\vec{p} - H_0(t) \tag{5.4}$$

with $\tilde{\pi}(\vec{p},t)$ substituted by a solution of

$$\partial_t\tilde{\phi}(\vec{p},t) = \frac{\delta(H_0(t)+H_{int}(t))}{\delta\tilde{\pi}(\vec{p},t)}, \tag{5.5}$$

and

$$L_0(t) = \frac{1}{2}\int d^3\vec{p}\Big(\partial_t\tilde{\phi}(\vec{p},t)\partial_t\tilde{\phi}(-\vec{p},t) - \omega_{\vec{k}}^2\tilde{\phi}(\vec{p},t)\tilde{\phi}(-\vec{p},t)\Big)$$

By $H_0(t)$ we still mean (3.5), but, since we are dealing with the tree level (i.e. we ignore the quantum effects), field and its canonical momentum are treated as functions, and the normal ordering is ignored.

This result is a special case of the well-known fact in QFT that the Legendre transform counts the tree diagrams. Originally it was discovered in [103]. In physical literature it is usually explained through functional integration [80]. We follow the modern pure combinatoric proof of [104]. Due to difference in conventions and necessity of generalisation from functions to functionals, it is easier tor reproduce the main steps then to explain how the result of [104] can be applied.

Proof. Since we work with tree diagrams, for which $V = I + 1$, where V is number of vertices and I is the number of the internal lines, all i and $(2\pi)^4$ factors from vertices, propagators and the overall factors cancel each other leaving only the overall minus sign. Thus, in this proof we will ignore these factors.

Let us introduce

$$\eta(\vec{p},t) = \frac{\delta L_{int}(t)}{\delta\partial_t\phi(\vec{p},t)}. \tag{5.6}$$

Diagrammatically, η is equal sum over all graphs with one marked external line with momentum flow $\vec{p}$ contracted with and the external line factor replaced by

$$\frac{\partial}{\partial(\partial)_t\phi(\vec{p},t))}\frac{\tilde{\phi}(\vec{p},t) + \frac{i\alpha}{\omega_{\vec{p}}}\partial_t\tilde{\phi}(\vec{p}_j,t)}{2} = f_D^\alpha(\vec{p},t).$$

Alternatively, we can first sum over all kinds of vertices, incident to the marked line. Then each of the rest of the lines going out of such a vertex is either the external one, or the beginning of a new tree graph with a marked external line. Taking into account that the vertices are generated by H_{int} and the overall minus sign, we get

$$\eta(\vec{p},t) = -\frac{\delta H_{int}(t)}{\delta\tilde{\pi}(\vec{p},t)}\bigg|_{\tilde{\pi}(\vec{k},t)=\partial_t\tilde{\phi}(\vec{k},t)+\eta(\vec{k},t)} = 0.$$

Thus,

$$\tilde{\pi}(\vec{k},t) = \partial_t\tilde{\phi}(\vec{k},t) + \eta(\vec{k},t)$$

is the unique solution[41] of (5.5).

Finally, we put $\tilde{\pi}(\vec{k},t) = \partial_t\tilde{\phi}(\vec{k},t) + \eta(\vec{k},t)$ into the right hand side of (5.4). We have

$$\int d^3\vec{p}\Big((\partial_t\tilde{\phi}(\vec{p},t) + \eta(\vec{p},t))\partial_t\tilde{\phi}(-\vec{p},t) - \frac{1}{2}\left(\partial_t\tilde{\phi}(\vec{p},t) + \eta(\vec{p},t)\right)\left(\partial_t\tilde{\phi}(-\vec{p},t) + \eta(-\vec{p},t)\right)$$

[41] Recall, that both H_{int} and L_{int} are formal power series vanishing at the origin. Then it is easy to see that (5.5) has a unique solution in formal power series.

$$-\frac{1}{2}\omega_{\vec{p}}^2\tilde{\phi}(\vec{p},t)\tilde{\phi}(-\vec{p},t)\Big) - H_{int}(t)\big|_{\tilde{\pi}(\vec{k},t)=\partial_t\tilde{\phi}(\vec{k},t)+\eta(\vec{k},t)} =$$

$$L_0(t) - \frac{1}{2}\int d^3\vec{p}\eta(\vec{p},t)\eta(-\vec{p},t) - H_{int}(t)\big|_{\tilde{\pi}(\vec{k},t)=\partial_t\tilde{\phi}(\vec{k},t)+\eta(\vec{k},t)}.$$

We have to prove that the second and the third terms above sum to L_{int}. For that we note that the second term can be interpreted as a sum over all diagrams contributing to L_{int} with a marked internal line (indeed, we then need to sum over all diagrams to the left and to the right of that line, both sums can be expressed through η; factor $\frac{1}{2}$ due to the symmetry) with a minus sign, while the third one is sum over all diagrams with a marked vertex (the vertex factors are generated by H_{int}, each line coming from the marked vertex is either external or a root of a tree graph with marked external line, which, as we already saw, correspond to substitution $\tilde{\pi}(\vec{k},t) = \partial_t\tilde{\phi}(\vec{k},t) + \eta(\vec{k},t)$). Hence, each diagram with I internal lines and V vertices appears with the coefficients $-I$ and $+V$ in the second and the third terms respectively. Using $V = I + 1$ once again, we get the statement. □

Remark 5.6. Following [104], we note, that the Legendre transform (5.4-5.5) is well-defined and invertible in formal series sense, without any additional restrictions[42].

Remark 5.7. In the higher orders the loop corrections should be added to L_{int}, which can be interpreted as quantum corrections to the Legendre transform. This corrections are strongly divergent, since the integrand does not depend on the loop frequencies at all. At the same time, the Lagrangian Feynman rules in the presence of derivatives with respect to the time in the interaction are divergent themselves due to the δ-function in the vacuum correlator (F.8). These two kinds of divergences cancel each other according to Remark 5.3. So, a Hamiltonian theory with an admissible interaction is equivalent to a Lagrangian theory with an action given by the Legendre transform and with specific counterterms given by quantum corrections to it. A local QFT containing first order derivatives with respect to the time in the interaction can also be considered in either Hamiltonian or Lagrangian formalism. Again, at the tree level they are equivalent, while the loop corrections are divergent and local. Due to locality, they can be canceled exactly by local counterterms. Note, that in both cases there is no freedom in the counterterms choice, so the renormalisability of the theory is not affected.

[42] Recall that the usual analytic Legendre transform exists for convex functions only.

6 Corrected propagator

This section can be considered as both an example of usage of the Feynman rules described above and as a preparation to discussion of adiabatic limits and renormalization. We will compute the corrected propagator, i.e. the correlator $\mathcal{G}_{\alpha\beta}(\vec{p},\omega;\vec{p}',\omega')$ in terms of the one-particle irreducible (1PI) corrections. All calculations are done in the (weak) adiabatic limit assuming its existence (which is proven in the next section independently of the results of this one).

Remark 6.1. As it was already noted, all correlators and amplitudes computed by means of the perturbation theory usually are divergent series and recovering physically meaningful information from them requires some additional work.

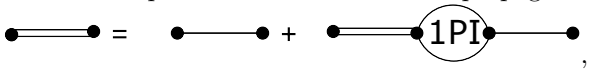
The corrected propagator is a result of formal resummation of that power series. Namely, we can regroup order of summation so, that the contributions of all diagrams can be then found as a geometric series of one-particle irreducible ones. This re-summation will be treated as a formal trick and we will not discuss actual convergence of the aforementioned geometric series.

Sum over all one-particle irreducible contributions is most likely divergent by itself and will be again treated as a formal series. To get a reasonable approximation of corrected energy one needs to truncate it at some finite order.

It is important to note that this re-summation has deep sense only if we are specially interested in details of the singularities structure of the correlators. Sometimes such re-summation is just a useful book-keeping tool, allowing to express sum over all corrections in terms of one-particle irreducible ones only. In that case the re-summed expression should be understood just a short way to write down the power series.

Alternatively, to avoid this ad hoc re-summation procedure one can require that the one-particle irreducible contribution vanishes on-shell, so that the pole is not shifted. This can be in principle achieved by remormalization procedure discussed in Section 8. This is one of the necessary conditions for the strong adiabatic limit to exist as will be discussed in Subsection 7.2.

To find the quantum corrections to the propagator let us consider the following formal equation:



,

where 1PI stand for the 1-particle-irreducible graphs.

Clearly, it is equivalent to summation of the geometric progression but is somehow simpler in our case.

We have:

$$\mathcal{G}_{\alpha\beta}(\vec{p},\omega;\vec{p}',\omega') = \mathcal{G}_{\alpha\beta}(\vec{p},\omega)(2\pi)^4\delta(\vec{p}+\vec{p}')\delta(\omega+\omega'),$$

$$\mathcal{G}_{\alpha\beta}(\vec{p},\omega) = \mathcal{G}^{(0)}_{\alpha\beta}(\vec{p},\omega) - i(2\pi)^4\mathcal{G}_{\alpha\gamma}(\vec{p},\omega)M^{\gamma\delta}(\vec{p},\omega)\mathcal{G}^{(0)}_{\delta\beta}(\vec{p},\omega), \tag{6.1}$$

there $-i(2\pi)^4M^{\gamma\delta}(\vec{p},\omega)$ is an expression corresponding to the amputated 1PI graph without the overall energy-momentum conservation delta-function. Under our assumption on the symmetry of the theory we have

$$M(\vec{p},\omega) = M^+{}_+(\vec{p},\omega) = M^-{}_-(\vec{p},-\omega) \tag{6.2}$$

and

$$N(\vec{p},\omega) = M^+{}_-(\vec{p},\omega) = M^-{}_+(\vec{p},\omega) = N(\vec{p},-\omega). \tag{6.3}$$

From now on we will say that a 1PI self-energy correction is of type M (respectively of type N) if it contributes to M (respectively to N). Then we have the following:

Proposition 6.2. *The solution of (6.1) is*

$$\mathcal{G}^{\alpha}{}_{\alpha}(\vec{p},\omega) = i\alpha\frac{1}{(2\pi)^4 2\omega_{\vec{p}}}\left(\omega+\alpha\left(\omega_{\vec{p}}+\frac{1}{2\omega_{\vec{p}}}M(\vec{p},-\alpha\omega)\right)\right)\cdot \tag{6.4}$$

$$\left[\left(\omega-\left(\omega_{\vec{p}}+\frac{1}{2\omega_{\vec{p}}}M(\vec{p},\omega)\right)+i0\right)\left(\omega+\left(\omega_{\vec{p}}+\frac{1}{2\omega_{\vec{p}}}M(\vec{p},-\omega)\right)-i0\right)+\left(\frac{1}{2\omega_{\vec{p}}}N(\vec{p},\omega)\right)^2\right]^{-1}.$$

$$\mathcal{G}^{\alpha}{}_{-\alpha}(\vec{p},\omega)=-i\frac{1}{(2\pi)^3 2\omega_{\vec{p}}}\frac{1}{2\omega_{\vec{p}}}N(\vec{p},\omega) \tag{6.5}$$

$$\left[\left(\omega-\left(\omega_{\vec{p}}+\frac{1}{2\omega_{\vec{p}}}M(\omega,\vec{p})\right)+i0\right)\left(\omega+\left(\omega_{\vec{p}}+\frac{1}{2\omega_{\vec{p}}}M(\vec{p},-\omega)\right)-i0\right)+\left(\frac{1}{2\omega_{\vec{p}}}N(\vec{p},\omega)\right)^2\right]^{-1}.$$

Remark 6.3. Recall that the time-orientation index α in terms of the Feynman rules of [73] determines whether the corresponding external line goes from the future to the past or from the past to the future. This manifests itself in the free propagator (F.4). In fact, $\mathcal{G}^{(0)+}{}_{+}=\mathcal{G}^{(0)}_{-+}$ has pole at positive frequency, corresponding to future-directed propagation, when $\mathcal{G}^{(0)-}{}_{-}=\mathcal{G}^{(0)}_{+-}$ at the negative one. The off-diagonal components vanish, because the line can not be future- (or past-) directed at both ends.

In contrast to that the corrected propagator (6.4-6.5) in general does not have any of these properties. This is not surprising, since vertices insertion can convert future-directed lines to past-directed ones and vice versa.

Remark 6.4. There are two cases when (6.4-6.5) simplifies and looks more like the ordinary propagator. First is the first order of perturbation theory, when we can assume M and N to be small. Then the off-diagonal terms are always negligible in comparison with the diagonal ones, and each diagonal component has exactly one pole at $\pm\omega^Q_{\vec{p}}$

$$\omega^Q_{\vec{p}}=\omega_{\vec{p}}+\frac{1}{2\omega_{\vec{p}}}M(\vec{p},\omega). \tag{6.6}$$

Another case is an ordinary QFT. First note, that in ordinary QFT amplitudes do not depend on the time-ordering, so $M(\pm\omega,\vec{p})=N(\omega,\vec{p})$ and the denominator is just

$$\omega^2-\omega^2_{\vec{p}}-M(\omega,\vec{p}). \tag{6.7}$$

The numerators are still quite complicated. However, in this case only the Feynman propagator

$$\mathcal{G}_F(\vec{p},\omega)=\sum_{\alpha\beta}\mathcal{G}_{\alpha\beta}(\vec{p},\omega)=\frac{i}{(2\pi)^4(\omega^2-\omega^2_{\vec{p}}+M(\vec{p},\omega))} \tag{6.8}$$

matters. It coincides with the corrected porpagator of the ordinary QFT.

Example 6.5. It is interesting to compare the dispersion relation following form poles of (6.4-6.5) and the 1-loop correction to the on-shell energy in $\lambda\phi^3$-theory in the DFR space with interaction term regularized by the quantum Wick product time found in [75] using the time-independent perturbation theory (and hence spatial adiabatic cut-off instead of temporal one we use here). In that case [73]:

$$F_{\alpha_1\cdots\alpha_n}(\vec{p}_1\cdots\vec{p}_n)=\exp\left(-\frac{l_p^2}{2}\left(\sum_{i=1}^n(\vec{p}_i-\bar{\vec{p}})^2+\sum_{i=1}^n(\alpha_i\omega_{\vec{p}_i}+\bar{\omega})^2\right)\right)$$

with

$$\bar{\vec{p}}=\frac{1}{n}\sum_{i=1}^n\vec{p}_i,\qquad \bar{\omega}=\frac{1}{n}\sum_{i=1}^n\alpha_i\omega_{\vec{p}_i}$$

and l_p standing for the Planck length. We will assume that $l_p=1$ for simplicity. In particular,

$$F^{\pm}(\vec{p},-\vec{k},-\vec{p}+\vec{k})=F^{\pm++}(\vec{p},-\vec{k},-\vec{p}+\vec{k})=$$

$$exp\left(-\frac{1}{2}\left(\vec{p}^2+\vec{k}^2+(\vec{p}-\vec{k})^2+(\pm\omega_{\vec{p}}-\omega_\pm)^2+(\omega_{\vec{k}}-\omega_\pm)^2+(\omega_{\vec{p}-\vec{k}}-\omega_\pm)^2\right)\right),$$

$$\omega_\pm=\frac{1}{3}(\pm\omega_{\vec{p}}+\omega_{\vec{k}}+\omega_{\vec{p}-\vec{k}})$$

Then, from (6.6) we have the following 1-loop correction to the on-shell energy

$$\delta\omega_{\vec{p}}=\frac{\lambda^2}{2}\frac{1}{2\omega_{\vec{p}}}\int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}2\omega_{\vec{p}-\vec{k}}}e^{-\vec{p}^2-\vec{k}^2-(\vec{p}-\vec{k})^2}\cdot$$

$$\left(\frac{e^{-(\omega_{\vec{p}}-\omega_+)^2-(\omega_{\vec{k}}-\omega_+)^2-(\omega_{\vec{p}-\vec{k}}-\omega_+)^2}}{\omega_{\vec{k}}+\omega_{\vec{p}-\vec{k}}+\omega_{\vec{p}}}+\frac{e^{-(-\omega_{\vec{p}}-\omega_-)^2-(\omega_{\vec{k}}-\omega_-)^2-(\omega_{\vec{p}-\vec{k}}-\omega_-)^2}}{\omega_{\vec{k}}+\omega_{\vec{p}-\vec{k}}-\omega_{\vec{p}}}\right),$$

which coincides with the result of [75].

To end the consideration of the propagator, let us present it in somewhat different form. At first sight (6.4-6.5) describe propagator of two different modes propagating in each of the time direction. To see that this is not the case let us diagonalize the propagator.

Proposition 6.6. *The propagator (6.4-6.5) can be written as*

$$\mathcal{G}^\alpha{}_\beta(\vec{p},\omega)=\frac{i}{(2\pi)^4 2\omega_{\vec{p}}}O^{\alpha\gamma}(\vec{p},\omega)D_\gamma{}^\delta(\vec{p},\omega)O_{\delta\beta}(\vec{p},\omega), \tag{6.9}$$

where[43]

$$O_{\alpha\beta}(\vec{p},\omega)=O^{\beta\alpha}(\vec{p},\omega)=\begin{pmatrix}\cos\left(\frac{\theta(\vec{p},\omega)}{2}\right) & -\sin\left(\frac{\theta(\vec{p},\omega)}{2}\right)\\ \sin\left(\frac{\theta(\vec{p},\omega)}{2}\right) & \cos(\frac{\theta(\vec{p},\omega)}{2})\end{pmatrix}_{\alpha\beta}, \tag{6.10}$$

$$D_\alpha{}^\beta(\vec{p},\omega)=\begin{pmatrix}\frac{1}{A(\vec{p},\omega)-B(\vec{p},\omega)+i0} & 0\\ 0 & -\frac{1}{A(\vec{p},\omega)+B(\vec{p},\omega)+i0}\end{pmatrix}_{\alpha\beta},$$

$$A(\vec{p},\omega)\sin(\theta(\vec{p},\omega))=\frac{1}{2\omega_{\vec{p}}}N(\vec{p},\omega), \tag{6.11}$$

$$A(\vec{p},\omega)\cos(\theta(\vec{p},\omega))=\omega-\frac{1}{4\omega_{\vec{p}}}\left(M(\vec{p},\omega)-M(\vec{p},-\omega)\right), \tag{6.12}$$

$$B(\vec{p},\omega)=\frac{1}{4\omega_{\vec{p}}}\left(M(\vec{p},\omega)+M(\vec{p},-\omega)\right)+\omega_{\vec{p}}. \tag{6.13}$$

$$A(\vec{p},\omega)^2=\left(\omega-\frac{1}{4\omega_{\vec{p}}}\left(M(\vec{p},\omega)-M(\vec{p},-\omega)\right)\right)^2+\left(\frac{1}{2\omega_{\vec{p}}}N(\omega,\vec{p})\right)^2 \tag{6.14}$$

$$\theta(\vec{p},-\omega)=-\theta(\vec{p},\omega),\qquad A(\vec{p},-\omega)=-A(\vec{p},\omega),B(\vec{p},\omega)=B(\vec{p},-\omega).$$

From the diagonalized form (6.9) it is clear that as in free propagator (4.5) we have one mode with positive frequency $\omega^Q_{\vec{p}}$ and another one with negative frequency $-\omega^Q_{\vec{p}}$ determined by[44]

$$A(\vec{p},\pm\omega^Q_{\vec{p}})=\pm B(\vec{p},\pm\omega^Q_{\vec{p}}), \tag{6.15}$$

[43] in the matrix notation we as usual assume that the first index enumerates rows and the second one columns. We assume that first raw/column corresponds to "+" and the second one to "-".

[44] we implicitly assume here that the interaction is small enough, so that the equation (6.15) always has exactly one solution. The fact that the same $\omega^Q_{\vec{p}}$ always solves (6.15) for both signs clearly follows from A being odd and B being even

but they are now mixed by the matrix O. Note that the case of ordinary QFT $N(\vec{p},\omega) = M(\vec{p},-\omega) = M(\vec{p},\omega)$ produces no special restriction on the angle θ, so the mixing survives in absence of non-locality.

For the discussion of renormalization and scattering amplitudes reconstruction in the next sections let us find the behavior of (6.9) when $\omega \to \pm\omega^Q_{\vec{p}}$:

$$\mathcal{G}^{\alpha}{}_{\beta}(\vec{p},\omega) \sim \frac{\pm iZ(\vec{p})}{(2\pi)^4 2\omega^Q_{\vec{p}}(\omega \mp (\omega^Q_{\vec{p}} - i0))} O_{\pm,\alpha}(\vec{p},\pm\omega^Q_{\vec{p}}) O_{\pm,\beta}(\vec{p},\pm\omega^Q_{\vec{p}}), \tag{6.16}$$

where

$$Z(\vec{p}) = \frac{\omega^Q_{\vec{p}}}{\omega_{\vec{p}}\left(\frac{\partial A(\vec{p},\omega)}{\partial\omega} - \frac{\partial B(\vec{p},\omega)}{\partial\omega}\right)_{\omega=\omega^Q_{\vec{p}}}}. \tag{6.17}$$

One can note tat $O_{+,\alpha}(\omega,\vec{p}) = O_{-,-\alpha}(-\omega,\vec{p})$ and hence

$$\mathcal{G}^{\alpha}{}_{\beta}(\vec{p},\omega) \sim \frac{\pm iZ(\vec{p})}{(2\pi)^4 2\omega^Q_{\vec{p}}(\omega \mp (\omega^Q_{\vec{p}} - i0))} O_{+,\pm\alpha}(\vec{p},\omega^Q_{\vec{p}}) O_{+,\pm\beta}(\vec{p},\omega^Q_{\vec{p}}). \tag{6.18}$$

The Feynman propagator will have the form

$$\mathcal{G}_F(\vec{p},\omega) = \sum_{\alpha,\beta} \mathcal{G}^{\alpha}{}_{\beta}(\vec{p},\omega) \sim \frac{\pm iZ_F(\vec{p})}{(2\pi)^4 2\omega^Q_{\vec{p}}(\omega \mp (\omega^Q_{\vec{p}} - i0))} \tag{6.19}$$

with

$$\begin{aligned} Z_F(\vec{p}) = Z(\vec{p})\left(\cos\left(\frac{\theta(\vec{p},\omega)}{2}\right) - \sin\left(\frac{\theta(\vec{p},\omega)}{2}\right)\right)^2 = \\ = Z(\vec{p})\left(1 - \sin(\theta(\vec{p},\omega))\right). \end{aligned} \tag{6.20}$$

Remark 6.7. In the case of ordinary QFT

$$Z_F(\vec{p}) = \frac{\left(\frac{\partial\left(\omega^2 - (\omega^Q_{\vec{p}})^2\right)}{\partial\omega}\right)_{\omega=\omega^Q_{\vec{p}}}}{\left(\frac{\partial\left(\omega^2 - \omega^2_{\vec{p}} - M(\vec{p},\omega)\right)}{\partial\omega}\right)_{\omega=\omega^Q_{\vec{p}}}} \tag{6.21}$$

which is exactly the usual field strength renormalization factor for the Feynman propagator.

If moreover the dispersion relation correction vanishes, i.e. $\omega^Q_{\vec{p}} = \omega_{\vec{p}}$ (and hence $A(\omega,\omega_{\vec{p}}) = B(\omega,\omega_{\vec{p}})$) we also have[45]

$$\theta(\vec{p},\omega_{\vec{p}}) = 0,$$

i.e. the corrected propagator is diagonal on-shell.

[45] As always we choose the branch that behaves appropriately in the vanishing interaction limit.

7 Adiabatic limit

7.1 Weak adiabatic limit and the LSZ reduction

7.1.1 Existence of the weak adiabatic limit

In this subsection we prove that the weak adiabatic limit exists in a massive theory with fast enough decreasing vertex factor.

We start by introducing some notation and terminology. The argument of $\tilde{\lambda}$ in a factor corresponding to the vertex by item 4 of Proposition 4.2 measures how much the energy is not conserved at that vertex due to the adiabatic switching. We will call it the energy defect. An individual contribution of a Feynman graph Γ with V vertices is a homogeneous (of power V) functional of $\tilde{\lambda}$, so we can write

$$\mathcal{G}_{\alpha_1,\dots,\alpha_n}(\vec{p}_1,\omega_1;\dots;\vec{p}_n,\omega_n) = \tag{7.1}$$

$$\sum_\Gamma \int d\Delta_1\cdots d\Delta_V \mathcal{G}_{\Gamma;\alpha_1,\dots,\alpha_n}(\vec{p}_1,\omega_1;\dots;\vec{p}_n,\omega_n;\Delta_1,\dots,\Delta_V)\tilde{\lambda}(\Delta_1)\cdots\tilde{\lambda}(\Delta_V),$$

where Δ_i is energy defect of the ith vertex and the distribution $\mathcal{G}_{\Gamma;\alpha_1,\dots,\alpha_n}(\vec{p}_1,\omega_1;\cdots;\vec{p}_n,\omega_n;\Delta_1,\dots,\Delta_V)$ can be deduced from Proposition 4.2. The (weak) adiabatic limit then is appropriately defined valued of that distribution at $\Delta_1 = \cdots = \Delta_n = 0$. In an ordinary QFT, as was shown in [74] in absence massless particles the (contribution of a fixed graph to) correlators are smooth functions of the energy defects Δ_i in an appropriately chosen neighborhood of the origin, provided that the UV divergences were taken care of by (infinite) renormalization, so the adiabatic limit is well-defined. We want to prove that the same holds in the class of non-local theories we consider without any need of infinite renormalization.

Theorem 7.1. *Assume that the dispersion function has a mass gap,*

$$\omega_{\vec{p}} \geq M > 0,\ \forall \vec{p} \in \mathbb{R}^3,$$

and the twisting function decays faster than any polynomial of its arguments at infinity in any direction.

Let Γ be a Feynman graph with E external lines and V vertices. Take a family of Schwartz functions $f_{\alpha_1\dots\alpha_E}(\vec{p}_1,\omega_1;\dots;\vec{p}_E,\omega_E)$ and define a distribution $G^f_\Gamma \in \mathcal{D}'(\mathbb{R}^V)$:

$$G^f_\Gamma[g] = \int d\vec{p}_1\cdots d\vec{p}_E d\omega_1\cdots d\omega_E d\Delta_1\cdots d\Delta_V\cdot$$

$$\mathcal{G}_{\Gamma;\alpha_1,\dots,\alpha_V}(\vec{p}_1,\omega_1;\dots;\vec{p}_E,\omega_E;\Delta_1,\dots,\Delta_V)\cdot$$
$$g(\Delta_1,\dots,\Delta_V) f^{\alpha_1\dots\alpha_E}(\vec{p}_1,\omega_1;\dots;\vec{p}_E,\omega_E).$$

Then $G^f_\Gamma \in C^\infty(\mathbb{R}^V)$.

Before proceeding to the proof let us introduce the "time-momentum" counterpart of (7.1):

$$\tilde{G}_{\alpha_1,\dots,\alpha_E}(\vec{p}_1,t_1,\dots,\vec{p}_E,t_E) =$$

$$\sum_\Gamma \int d\tau_1\cdots d\tau_V \tilde{G}_{\Gamma;\alpha_1,\dots,\alpha_E}(\vec{p}_1,t_1,\dots,\vec{p}_E,t_E;\tau_1,\dots,\tau_V)\tilde{\lambda}(\tau_1)\cdots\tilde{\lambda}(\tau_V).$$

Now for a collection of Schwartz functions $\tilde{f}^{\alpha_1\dots\alpha_E}(\vec{p}_1,t_1;\cdots;\vec{p}_E,t_E)$ we define $\tilde{G}^{\tilde{f}}_\Gamma \in \mathcal{D}'(\mathbb{R}^V)$ as

$$\tilde{G}^{\tilde{f}}_\Gamma[\tilde{g}] = \int d\vec{p}_1\cdots d\vec{p}_E dt_1\cdots dt_E d\tau_1\cdots d\tau_V\cdot$$

$$\tilde{G}_{\Gamma;\alpha_1,\dots,\alpha_n}(\vec{p}_1,t_1;\cdots;\vec{p}_E,t_E;\tau_1,\cdots,\tau_V)\cdot$$
$$\tilde{g}(\tau_1,\dots,\tau_V)\tilde{f}^{\alpha_1\dots\alpha_E}(\vec{p}_1,t_1;\cdots;\vec{p}_E,t_E).$$

If we assume that $\tilde{f}$ and f are related by the partial Fourier transform,

$$\tilde{f}^{\alpha_1\dots\alpha_E}(\vec{p}_1,t_1;\dots;\vec{p}_E,t_E) =$$
$$\frac{1}{(2\pi)^E}\int f^{\alpha_1\dots\alpha_E}(\vec{p}_1,\omega_1;\dots;\vec{p}_E,\omega_E)e^{-i(\omega_1 t_1+\dots+\omega_E t_E)}d\omega_1\cdots d\omega_E$$

then $\tilde{G}_\Gamma^{\tilde{f}}$ is nothing but the Fourier transform of $\mathcal{G}_\Gamma^f$.

Proof. Let us compute the wave front set of $\mathcal{G}_\Gamma^f$. For that we fix a compactly supported smooth function $\chi\in C_0^\infty(\mathbb{R})$ and compute the Fourier transform of $\mathcal{G}_\Gamma^f[\chi^{\otimes V}]$ which, according to the consideration above, is just

$$\xi\mapsto\tilde{\mathcal{G}}_\Gamma^{\tilde{f}}[\tilde{\chi}_\xi^{\otimes V}]$$

with

$$\tilde{\chi}_\xi^{\otimes V}(\tau_1,\dots,\tau_V)=\tilde{\chi}(\tau_1+\xi_1)\cdots\tilde{\chi}(\tau_V+\xi_V),$$
$$\tilde{\chi}(\tau)=\frac{1}{(2\pi)}\int d\Delta\chi(\Delta)e^{-i\Delta\tau}.$$

Let us look closer to the integrals over the variables τ_i and t_i in $\tilde{\mathcal{G}}_\Gamma^{\tilde{f}}[\tilde{\chi}_\xi^{\otimes V}]$. We can subdivide the integration domain[46] into a finite number of subregions $O_{\pi,\sigma}$ enumerated by the permutations $\sigma\in\mathfrak{S}_E$ and $\pi\in\mathfrak{S}_V$ of the form

$$O_{\pi,\sigma}=\big\{(\tau_1,\dots,\tau_V;t_1,\dots,t_E)\in\mathbb{R}^{V+E}\big|\tau_{\pi_1}<\tau_{\pi_2}<\cdots<\tau_{\pi_V},$$
$$t_{\sigma_1}<t_{\sigma_2}<\ldots<t_{\sigma_E}\big\}.$$

Let us fix such a subregion. It is enough to consider only the region corresponding to the identity permutations,

$$O_{Id,Id}=\big\{(\tau_1,\dots,\tau_V;t_1,\dots,t_E)\in\mathbb{R}^{V+E}\big|\tau_1<\tau_2<\ldots<\tau_V,$$
$$t_1<t_2<\ldots<t_E\big\},$$

because integration over any other $O_{\pi,\sigma}$ can be rewritten as integration over $O_{Id,Id}$ by renumeration of the vertices and the external lines. Now we subdivide this region to even smaller subregions

$$R_s=\big\{(\tau_1,\dots,\tau_V;t_1,\dots,t_E)\in O_{Id,Id}|1\le i<j\le E\implies t_i<t_j,\tag{7.2}$$
$$1\le i<j\le V\implies\tau_i<\tau_j,$$
$$i<s_j\implies\tau_i<t_j.\qquad i\ge s_j\implies\tau_i>t_j\big\},$$

where $\{s_j\}_{j=1}^E$ is a non-decreasing sequence[47]. We restrict our attention to one such subregion.

In the region R_s only one of the choices of all time-orientation contributes and in this contribution all Heaviside functions are equal to identity, so, besides the test-functions all dependence on τ_i and t_i comes from time-dependent exponents in vertex factors.

[46] Recall that the most singular objects in the Feynman rules of Proposition 4.2 are the Heaviside functions, so we can treat all integrals as ones of the discontinuous functions. In particular, we can divide the integration regions safely.

[47] Simpler form of (7.2) is

$$R_s=\big\{(\tau_1,\dots,\tau_V;t_1,\dots,t_E)\in\mathbb{R}^{V+E}\big|$$
$$\tau_1<\tau_2<\cdots<\tau_{s_1-1}<t_1<\tau_{s_1}<\tau_{s_1+1}<\cdots<\tau_{s_2-1}<t_2<\tau_{s_2}<\cdots<\tau_V\big\},$$

but the later form is ambiguous in degenerate cases like $s_1=0$.

To begin with we examine the integrals over $\tau_1, \tau_2, \ldots, \tau_{s_1-1}$ appearing in $\tilde{\mathcal{G}}^{\tilde{f}}_{\Gamma}[\tilde{\chi}^{\otimes V}_{\xi}]$ according to Proposition 4.1 with the integration region restricted to R_s:

$$\int_{\tau_1<\tau_2<\cdots<\tau_{s_1-1}<t_1} e^{-i(\Omega_1\tau_1+\Omega_2\tau_2+\cdots\Omega_{s_1-1}\tau_{s_1-1})}\tilde{\chi}(\tau_1+\xi_1)\cdots\tilde{\chi}(\tau_{s_1-1}+\xi_{s_1-1})d\tau_1\cdots d\tau_{s_1-1} =$$

$$\frac{1}{(-2\pi i)^{s_1-1}}\int e^{-it_1(\Omega_1+\Delta_1+\cdots+\Omega_{s_1-1}+\Delta_{s_1-1})}\prod_{j=1}^{s_1-1}\frac{-ie^{-i\xi_j\Delta_j}\chi(\Delta_j)}{\sum_{k\le j}(\Omega_k+\Delta_k)+i0}d\Delta_1\cdots d\Delta_{s_1-1}.$$

Here we kept only the factors depending on the variables $\tau_1, \tau_2, \ldots, \tau_{s_1-1}$ and $-\Omega_j$ is a sum of the on-shell frequencies with appropriate signs from (4.6). We note that $\Omega_1 > M$, because only time-orientation "-" is allowed by the Heaviside functions for incidences with the earliest vertex. Ω_j, $1 < j < s_1$ can contain negative terms, since there can be lines starting at earlier vertices and ending at the jth one with "+" orientation at the latter. But such a term should also appear with positive sign in Ω_k, $k < j$. Hence if we consider $\sum_{k\le j}\Omega_k$ $(k < s_1)$ all negative terms will cancel, thus[48]

$$\sum_{k\le j}\Omega_k > M, \forall j < s_1. \tag{7.3}$$

Then we can always restrict the support of χ (say, $|\Delta_j| < M/2V$) so that the denominators are bounded from below. The integral over the earliest $s_1 - 1$ interaction timestamps then takes the form

$$\int e^{-i(\xi_1\Delta_1+\cdots+\xi_{s_1-1}\Delta_{s_1-1})}e^{-it_1(\Omega_1+\Delta_1+\cdots+\Omega_{s_1-1}+\Delta_{s_1-1})}$$

$$X(\Delta_1,\ldots,\Delta_{s_1-1};\Omega_1,\ldots,\Omega_{s_1-1})d\Delta_1\cdots d\Delta_{s_1-1} =$$

$$e^{-it_1(\Omega_1+\cdots+\Omega_{s_1-1})}\tilde{X}(\xi_1+t_1,\ldots,\xi_{s_1-1}+t_1;\Omega_1,\ldots,\Omega_{s_1-1}),$$

where $X \in C^\infty(\mathbb{R}^{2(s_1-1)})$ $\tilde{X}$ is its Fourier transform with respect to its first $s_1 - 1$ variables. X is bounded together with all its derivatives with respect to Δ_j uniformly in Ω_k (in the range of Ω_k ad functions of the internal momenta bounded by) and therefore $\tilde{X}$ is rapidly decreasing in its first $s_1 - 1$ arguments uniformly on the rest.

The latest $V - s_E + 1$ interaction vertices can be treated in the same way, contributing the factor

$$e^{-it_E(\Omega_{s_E}+\cdots+\Omega_V)}\tilde{Y}(\xi_{s_E}+t_E,\ldots,\xi_V+t_E;\Omega_{s_E},\ldots,\Omega_V)$$

where $\tilde{Y}$ is fast-decreasing function of the first half of its arguments uniformly on the others. At last we consider the integral over the remaining τ_j and t_i. We have

$$I = \Big|\int_{t_1<\tau_{s_1}<\tau_{s_1+1}<\cdots<\tau_{s_E-1}<t_E} d\tau_{s_1}\cdots d\tau_{s_E-1}dt_1\cdots dt_E$$

$$e^{-it_1(\Omega_1+\cdots+\Omega_{s_1-1})}\tilde{X}(\xi_1+t_1,\ldots,\xi_{s_1-1}+t_1;\Omega_1,\ldots,\Omega_{s_1-1})$$

$$e^{it_E(\Omega_{s_E}+\cdots+\Omega_V)}\tilde{Y}(\xi_{s_E}+t_E,\ldots,\xi_V+t_E;\Omega_{s_E},\ldots,\Omega_V)$$

$$\tilde{f}^{\alpha_1\ldots\alpha_E}(\vec{p}_1,t_1;\cdots;\vec{p}_E,t_E)\prod_{j=s_1}^{s_E-1}\tilde{\chi}(\tau_j+\xi_j)\Big| \le$$

$$\int_{t_1<\tau_{s_1}<\tau_{s_1+1}<\cdots<\tau_{s_E-1}<t_E} d\tau_{s_1}\cdots d\tau_{s_E-1}dt_1\cdots dt_E$$

[48]The situation when all contributions are cancelled is not possible since we do not consider the vacuum energy corrections.

$$|\tilde{f}^{\alpha_1\dots\alpha_E}(\vec{p}_1,t_1;\cdots;\vec{p}_E,t_E)||\tilde{X}(\xi_1+t_1,\dots,\xi_{s_1-1}+t_1;\Omega_1,\dots,\Omega_{s_1-1})|$$
$$|\tilde{Y}(\xi_{s_E}+t_E,\dots,\xi_V+t_E;\Omega_{s_E+1},\dots,\Omega_V)|\left|\prod_{j=s_1}^{s_E-1}\tilde{\chi}(\tau_j+\xi_j)\right|.$$

Introduce new variables

$$\zeta_j=\xi_j+t_1, j<s_1,$$
$$\zeta_j=\xi_j+t_E, j\geq s_E.$$

Clearly the change of variables

$$(t_1,\dots,t_E;\xi_1,\dots,\xi_{s_1-1};\xi_{s_E},\dots,\xi_V)\to(t_1,\dots,t_E;\zeta_1,\dots,\zeta_{s_1-1};\zeta_{s_E},\dots,\zeta_V)$$

is invertible and $\tilde{f}\tilde{X}\tilde{Y}$ is a fast decaying function of the latter variables. Hence, it decays fast as a function of the former. In particular, for any N, we can estimate

$$|\tilde{f}^{\alpha_1\dots\alpha_E}(\vec{p}_1,t_1;\cdots;\vec{p}_E,t_E)\tilde{X}(\xi_1+t_1,\dots,\xi_{s_1-1}+t_1;\Omega_1,\dots,\Omega_{s_1-1})\times \tag{7.4}$$
$$\tilde{Y}(\xi_{s_E}+t_E,\dots,\xi_V+t_E;\Omega_{s_E+1},\dots,\Omega_V)|\leq C_N(1+t_1^2+\cdots+t_E^2)^{-E}$$
$$(1+\tau_{s_1}^2+\cdots+\tau_{s_E-1}^2+\xi_1^2+\cdots+\xi_{s_1-1}^2+\xi_{s_E+1}^2+\cdots+\xi_V^2)^{-N}.$$

Here we took into account that inside the integration region $|\tau_j|<max(|t_1|,|t_E|)$. Now we can safely expand the integration to the whole space $\mathbb{R}^{E+s_E-s_1}$. We have

$$I\leq\int J(\xi_1,\dots,\xi_{s_1}-1;\xi_{s_E},\dots,\xi_V;\tau_{s_1},\dots_{\tau_{s_{E-1}}})\prod_{j=s_1}^{s_E-1}d\tau_j|\tilde{\chi}(\tau_j+\xi_j)|,$$

where J is the result of integration over $t_1,\dots,t_E$ which is a fast-decaying function of its arguments by (7.4). Again, we introduce new variables

$$\zeta_i=\xi_i+\tau_i, s_1\leq_i<s_E.$$

note that the transform $(\xi,\tau)\to(\zeta,\tau)$ is invertible, and the integrand is a fast decaying function of ζ,τ. Hence it is also fast-decaying as a function of the original variables and integration over τ leaves a fast-decaying function of ξ_i. It means that

$$WF(\mathcal{G}_\Gamma^f)=\varnothing$$

for an appropriately chosen neighborhood of the origin W, hence that distribution is equivalent to some smooth function whenever restricted to W. □

Note the the proof above fails if the vertex factor regularizes the loop integral by making it rapidly oscillating rather than decreasing in UV region. So, it can not be applied to the theories with star product interaction.

7.1.2 Feynman rules from LSZ reduction

Now, as we know that the correlators are well-defined in the adiabatic limit, we can use LSZ reduction reviewed in Subsection 3.4 to reconstruct the scattering amplitudes. Combining it with Proposition 4.2, we can get the Feynman rules for the scattering amplitudes, replacing Proposition 4.4 in the weak adiabatic limit. For that note, that any correlator can be written in the form

$$\mathcal{G}^{\alpha_1\cdots\alpha_n}(\vec{p}_1,\omega_1;\cdots;\vec{p}_n,\omega_n)=\prod_{i=1}^n\mathcal{G}^{\alpha_i}{}_{\beta_i}(\vec{p}_i,\omega_i)\mathcal{M}^{\beta_1\cdots\beta_n}(\vec{p}_1,\omega_1;\cdots;\vec{p}_n,\omega_n) \tag{7.5}$$

where $\mathcal{M}^{\beta_1\cdots\beta_n}(\vec{p}_1,\omega_1;\cdots;\vec{p}_n,\omega_n)$ stands for an "amputated" correlator, i.e. the correlator computed according to the Feynman rules of Proposition 4.2, but without the external propagators and their 1-particle reducible correction, which are all summed in the corrected propagators $\mathcal{G}^{\alpha_i}{}_{\beta_i}(\vec{p}_i,\omega_i)$. Now, comparing (6.17-6.19) we get:

$$\sum_\alpha \mathcal{G}^\alpha{}_\beta(\vec{p},\omega) \sim \frac{\pm i\sqrt{Z_F(\vec{p})}}{(2\pi)^4 2\omega^Q_{\vec{p}}(\omega \mp (\omega^Q_{\vec{p}} - i0))}\tilde{Z}(\vec{p})O_{+,\pm\beta}(\vec{p},\omega^Q_{\vec{p}}), \quad \omega \to \pm\omega^Q_{\vec{p}},$$

where

$$\tilde{Z}(\vec{p}) = \frac{Z(\vec{p})\sum_\alpha O_{+,\alpha}(\vec{p})}{\sqrt{Z_F(\vec{p})}} = \sqrt{Z(p)}.$$

Then (3.45) takes the form

$$\sum_{\alpha_i,\alpha'_i}\prod_{i=1}^{n'} u^{(-)}_{\alpha'_i}(\vec{p}'_i)\prod_{i=1}^{n} u^{(+)}_{\alpha_i}(\vec{p}_i)\cdot \tag{7.6}$$

$$\mathcal{M}^{\alpha'_1\cdots\alpha'_{n'}\alpha_1\cdots\alpha_n}(\vec{p}'_1,\omega^Q_{\vec{p}'_1};\cdots;\vec{p}'_{n'},\omega^Q_{\vec{p}'_{n'}};-\vec{p}_1,-\omega^Q_{\vec{p}_1};\cdots;-\vec{p}_n,-\omega^Q_{\vec{p}_n}) =$$

$$_{out}<\vec{p}'_1,\cdots,\vec{p}'_{n'}|\vec{p}_1,\cdots,\vec{p}_n>_{in},$$

where $u^{(\pm)}_\alpha(\vec{p}) = \tilde{Z}(\vec{p})O_{+,\mp\alpha}(\vec{p},\omega^Q_{\vec{p}})$.

7.2 Strong adiabatic limit

Let us very schematically see how the external line corrections make the strong adiabatic limit ill-defined. We are not aiming to find out the exact renormalization conditions guaranteeing the strong adiabatic limit existence, instead we just formulate the conditions with which one can hope that it is possible to make sense of that limit.[49]

Consider a contribution to the scattering amplitude of a graph Γ containing an external line correction. For simplicity we consider a graph with one 1PI insertion to one of the external lines only. For simplicity we start with ordinary QFT. Then such a correction will lead to a singularity in the amplitude

$$\mathcal{S}(\vec{k},\Delta) = M(\omega_{\vec{k}},\vec{k})\frac{i}{(2\pi)^4 2\omega_{\vec{k}}}\frac{1}{\Delta + i0}\mathcal{S}'(\Delta+\omega_{\vec{k}},\vec{k}),$$

where $\vec{k}$ is the momentum flowing along the corrected external line, M is the factor corresponding to the external line correction and $\mathcal{S}'$ is the amputated amplitude and Δ is the adiabatic regularizator. For shortness we have omitted all irrelevant arguments and convolutions with the incoming state. We also kept the leading order in the adiabatic limit only.

To avoid this singularity we must assume that $M(\omega_{\vec{k}},\vec{k}) = 0$, i.e. that the self-energy corrections vanish on-shell. This is however not enough. To see that we need to take into account that the first argument of M in the expression above is not exactly the on-shell frequency $\omega_{\vec{k}}$ because of the adiabatic cut-off. Instead we have something like[50]

$$\mathcal{S}(\vec{k},\Delta_1,\Delta_2) = M(\omega_{\vec{k}}+\Delta_1,\vec{k})\frac{i}{(2\pi)^4 2\omega_{\vec{k}}}\frac{1}{\Delta_1+\Delta_2 + i0}\mathcal{S}'(\Delta_1+\Delta_2+\omega_{\vec{k}},\vec{k}),$$

where Δ_1 is the energy defect of the vertex incident to the external line and Δ_2 is the sum of the energy defects of all other vertices in the external line corrections.

[49] The reason for such a vague formulation is clarified in Remark 7.2.

[50] In reality the situation is even more complicated because the expression depends on each energy defect of each vertex in the 1PI external line correction independently but we ignore that to see the key problem

Clearly, $\mathcal{S}(\vec{k}, \Delta_1, \Delta_2)$ still is not a smooth function in any neighborhood of $\Delta_1 = \Delta_2 = 0$. We need however something weaker, namely existence and independence on a compactly-supported function f, $f(0,0) \neq 0$, $\int f(\Delta_1, \Delta_2) d\Delta_1 d\Delta_2 = 1$ of a limit

$$\lim_{\epsilon \to +0} \epsilon^{-2} \int f(\epsilon^{-1}\Delta_1, \epsilon^{-1}\Delta_2) \mathcal{S}(\vec{k}, \Delta_1, \Delta_2) d\Delta_1 d\Delta_2. \tag{7.7}$$

In our case the limit exists (at least for some f) if $M(\omega_{\vec{k}}, \vec{k}) = 0$, but is dependent on f. The only way out of this situation is to require that M vanishes on-shell together with its derivative. Then the leading order will be proportional to

$$\frac{\Delta_1^2}{\Delta_1 + \Delta_2 + i0},$$

which still has no limit at $\Delta_1, \Delta_2 \to 0$, but the limit (7.7) vanishes independently of f.

Remark 7.2. The cancellation discussed above is more sensitive than it may look like, because we need it to hold (or almost hold) not only on the dispersion relation shell, but also slightly off-shell. For example, we ignored dependencies on all the other Δ_i. But in general it will spoil the cancellation . Moreover, we should ask for cancellation in each order in terms of λ separately. In [74] it was shown that in a very special approach to renormalization it is possible to make sense of the strong adiabatic limit of ordinary massive QFT.

In non-local case it is absolutely not clear how to do that.

Summing up, the strong adiabatic limit can exist only if the self-energy correction vanishes together with its first derivative on-shell. If this is the case, the external line corrections do not contribute to the scattering operator at all. The fact that the strong adiabatic limit does exist in the local, Lorentz-invariant QFT with one spinless massive specie of particles was proven in [74].

In the theories considered in this thesis there are two kinds of one-particle self-energy corrections. The one of type N is followed by the propagator with "wrong" time-orientation which is regular on the dispersion shell. But two such corrections in a row will be followed again by a singular propagator. The same happens if we have some corrections of type M between two of type N. Let us resumm all propagator corrections which may appear between two singular propagator, but contain only regular ones inside:

$$M_{sing}(\vec{p}, \omega) = M(\vec{p}, \omega) + N(\vec{p}, \omega)^2 \frac{-1}{2\omega_{\vec{p}}(\omega + \omega_{\vec{p}})} \sum_{n=0}^{\infty} \left(M(\vec{p}, -\omega) \frac{-1}{2\omega_{\vec{p}}(\omega + \omega_{\vec{p}})} \right)^n =$$

$$M(\vec{p}, \omega) - \frac{N(\vec{p}, \omega)^2}{M(\vec{p}, -\omega) + 2\omega_{\vec{p}}(\omega + \omega_{\vec{p}})} =$$

$$2\omega_{\vec{p}} \left((\omega - \omega_{\vec{p}}) - \frac{A(\vec{p}, \omega)^2 - B(\vec{p}, \omega)^2}{A(\vec{p}, \omega) \cos(\theta(\vec{p}, \omega)) + B(\vec{p}, \omega)} \right)$$

M_{sing} plays the role of "resummed" 1PI corrections. The full correction can be achieved by summing over number of insertions of M_{sing} as usual. The natural renormalization conditions then are

$$M_{sing}(\vec{p}, \omega_{\vec{p}}) = 0,$$

$$\left(\frac{\partial}{\partial \omega} M_{sing}(\vec{p}, \omega) \right)_{\omega = \omega_{\vec{p}}} = 0.$$

First one leads to[51]

$$A(\vec{p}, \omega_{\vec{p}}) = B(\vec{p}, \omega_{\vec{p}}),$$

[51] We consider only this possibility (ignoring that we can also have $A(\vec{p}, \omega_{\vec{p}}) = -B(\vec{p}, \omega_{\vec{p}})$) and the possibility for the denominator to be zero or infinite) because only this condition holds automatically in a free theory. Since we are working in the frame of the perturbation theory, only such branch can be considered.

which according to discussion in 6 means that the dispersion function remains uncorrected by the interaction. The second one simplifies drastically when we take the dispersion relation non-renormalization into account and reads as

$$Z(\vec{p})\cos\left(\frac{\theta(\vec{p},\omega_{\vec{p}})}{2}\right)^2=1.$$

Comparing it with Proposition 6.6, we see that the renormalization conditions mean that the singular parts of the components $\mathcal{G}^+{}_+(\vec{p},\omega)$ and $\mathcal{G}^-{}_-(\vec{p},\omega)$ remain uncorrected. But the off-diagonal components $\mathcal{G}^+{}_-(\vec{p},\omega)$ do admit quantum corrections in general. This happens because contributions of one correction of type N alone or followed by arbitrarily many corrections of type M does not bring any new singularities and hence does not must vanish on-shell. Due to the same reason in contrast to ordinary QFT we have regular and not vanishing external line corrections. Summing up these corrections we get the corrected external line factor

$$N(\vec{p},\omega)\frac{-1}{2\omega_{\vec{p}}(\omega+\omega_{\vec{p}})}\sum_{n=0}^{\infty}\left(M(\vec{p},-\omega)\frac{-1}{2\omega_{\vec{p}}(\omega+\omega_{\vec{p}})}\right)^n=$$

$$-\frac{N(\vec{p},\omega)}{M(\vec{p},-\omega)+2\omega_{\vec{p}}(\omega+\omega_{\vec{p}})}\underset{A=B}{=}-\tan\left(\frac{\theta(\vec{p},\omega_{\vec{p}})}{2}\right)$$

followed by amputated amplitude with a "wrong" time-orientation of an external line. So, we can formulate the following:

Conjecture 7.3. *The (appropriately defined) strong adiabatic limit of a massive theory exists if*

$$A(\vec{p},\omega_{\vec{p}})=B(\vec{p},\omega_{\vec{p}})$$

and $Z(\vec{p})=\cos\left(\frac{\theta(\vec{p},\omega_{\vec{p}}}{2}\right)^{-2}$ *(may be, together with other conditions). If this is the case, then it is given by*

$$_{out}<\vec{p}'_1,\dots,\vec{p}'_{n'}|\vec{p}_1,\dots,\vec{p}_n>^{(strong)}_{in}=u^{(-)}_{\alpha'_1}(\vec{p}'_1)\cdots u^{(-)}_{\alpha'_{n'}}(\vec{p}'_{n'})u^{(+)}_{\alpha_1}(\vec{p}_1)\cdots u^{(+)}_{\alpha_n}(\vec{p}_n)$$

$$\mathcal{M}^{\alpha'_1\dots\alpha'_{n'}\alpha_1\dots\alpha_n}(\vec{p}'_1,\omega_{\vec{p}'_1};\dots;\vec{p}'_{n'},\omega_{\vec{p}'_{n'}};-\vec{p}_1,-\omega^Q_{\vec{p}_1};\dots;-\vec{p}_n,-\omega_{\vec{p}_n}),$$

where

$$u^{(+)(strong)}_-(\vec{p})=u^{(-)(strong)}_+(\vec{p})=1,$$

$$u^{(+)(strong)}_+(\vec{p})=u^{(-)(strong)}_-(\vec{p})=-\tan\left(\frac{\theta(\vec{p},\omega_{\vec{p}})}{2}\right).$$

We note that (7.6) gives exaclty the same scattering amplitudes if the renormalization conditions are satisfied. We also note that due to Remark 6.7 $u^{(+)}_+(\vec{p})=u^{(-)}_-(\vec{p})=0$ for an ordinary QFT.

8 Renormalization

In ordinary QFT the correlators do not depend on the renormalization in adiabatic limit (up to fields rescaling), and hence scattering amplitudes reconstructed from the weak adiabatic limit is independent of the renormalization. On the other hand, the strong adiabatic limit exists only if renormalization was chosen appropriately.

As we will see, in non-local QFT even correlators are affected by renormalization. Before going to that we will first define another "formal" renormalization procedure, artificially constructed so that the correlators are preserved by it. It will be defined only diagrammatically, so it is not clear if a formally renormalized QFT is a quantum field theory in any reasonable sense at all. We note also that the "formal" renormalization may be useful for the renormalization group methods.

We postpone discussion of the results achieved here until the last section.

8.1 Formal renormalization

In this subsection we will deal with "formal" quantum field theories defined by the corresponding Feynman rules.

Let us begin with a formal definition of this notion:

Definition 8.1. A formal non-local quantum field theory A is defined by a dispersion function ${}^A\omega_{\vec{p}}$ and a collection of vertex factors ${}^AF^{\alpha_1,\cdots,\alpha_n}(\vec{p}_1,\omega_1;\cdots;\vec{p}_n,\omega_n)$ which are non-zero only for a finite number of values of n. Both dispersion function and the vertex factors are assumed to be continuous

By a free propagator of A we will mean

$$ {}^A\mathcal{G}^{(0)\alpha}{}_{\beta}(\vec{p},\omega) = i\frac{\delta^\alpha_\beta(\vec{p})}{(2\pi)^4 2\,{}^A\omega_{\vec{p}}(\omega-\alpha({}^A\omega_{\vec{p}}-i0))} $$

and by a correlator in A we will mean the correlation function

$$ {}^A\mathcal{G}_{\alpha_1,\cdots,\alpha_n}(\vec{p}_1,\omega_1;\cdots,\vec{p}_n,\omega_n) $$

formally calculated according to the Feynman rules of the Prop. 4.2 with free propagator ${}^A\mathcal{G}^{(0)}$ and vertex factors ${}^AF$ instead of $\mathcal{G}^{(0)}$ and F respectively, substituting into ${}^AF$ the frequencies ω_i assigned to the corresponding lines, taking into account the sign[52].

We say that formal quantum field theories A and B are equivalent if there is an invertible momentum-dependent matrix ${}^{A,B}Z(\vec{p}_i)$ (which will call the equivalence matrix) such that

$$ {}^A\mathcal{G}_{\alpha_1,\ldots,\alpha_n}(\vec{p}_1,\omega_1;\ldots,\vec{p}_n,\omega_n) = \prod_{i=1}^n {}^{A,B}Z(\vec{p}_i)_{\alpha_i}{}^{\beta_i}(\vec{p}_1)\,{}^B\mathcal{G}_{\beta,\ldots,\beta_n}(\vec{p}_1,\omega_1;\ldots,\vec{p}_n,\omega_n) \tag{8.1} $$

We will call such theories "formal" because in fact we do not construct a non-local Hamiltonian from which its vertex factors follow.

For simplicity we do do not discuss invariance of the formal quantum field theories with respect to rotations and inversions, so the analog of (6.2-6.3) may be violated.

For a given formal QFT we can formally apply the LSZ formula to get the scattering amplitudes. If two formal theories are equivalent, the corresponding scattering amplitudes will coincide as well as the physical spectra. The converse is not true in general.

[52] This possible (in fact polynomial) dependence of the factors ${}^AF$ on the off-shell frequencies is necessary to treat the counterterms which will appear in the "formal" renormalization. Note that the non-locality lead to dependence on the on-shell frequency, so such counterterms can never be induced by any Hamiltonian correction. We should note that the off-shell frequency can traded for the on-shell one as it is shown in Section 5.

Section 6 applies to the formal theories without any change. We get an analog of (6.1) which we prefer to write in the following functional form:

$$ {}^{A}\mathcal{G}^{\alpha}{}_{\beta}(\vec{p},\omega) = \frac{i}{(2\pi)^4} R\left(\left({}^{A}M^{\gamma}{}_{\delta}(\vec{p},\omega) \right)_{\gamma,\delta=\pm}, \omega, {}^{A}\omega_{\vec{p}} \right), \tag{8.2} $$

where ${}^{A}M^{\gamma}{}_{\delta}(\vec{p},\omega)$ is a sum of one-particle irreducible self-energy corrections and we introduced a matrix-valued function

$$ R(M,\omega,\omega') = (2\omega\ (\omega - \sigma_z\omega') - M)^{-1},\ M \in M_2(\mathbb{C}), $$

$$ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

We are now ready to formulate the main theorem of the subsection. It is nothing but the standard renormalization independence of the correlators stated in a purely diagrammatic form.

Theorem 8.2. *Let A and B be two formal non-local quantum field theories such, that*

$$ {}^{B}F^{\alpha_1,\dots,\alpha_n}(\vec{p}_1,\omega_1,\dots,\vec{p}_n,\omega_n) = \prod_{i=1}^{n} {}^{A,B}Z_{\beta_i}{}^{\alpha_i}(\vec{p}_i)\, {}^{A}F^{\beta_1,\dots,\beta_n}(\vec{p}_1,\omega_1;\dots,\vec{p}_n,\omega_n), \tag{8.3} $$

$$ n \neq 2; $$

$$ {}^{B}F^{\alpha_1\alpha_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2) = \prod_{i=1}^{2} {}^{A,B}Z_{\beta_i}{}^{\alpha_i}(\vec{p}_i) \cdot {}^{A}F^{\beta_1\beta_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2) + \tag{8.4} $$

$$ \delta M^{\alpha_1\alpha_2}(\vec{p}_1,\omega_1) $$

for some invertible 2×2 matrix ${}^{A,B}Z(\vec{p})$ and $\delta M^{\alpha_1\alpha_2}$ being such that

$$ R\left(\left(M^{\alpha}{}_{\beta} \right)_{\alpha,\beta=\pm}, \omega, {}^{A}\omega_{\vec{p}} \right) \tag{8.5} $$

$$ = R\left(\left({}^{A,B}Z_{\gamma}{}^{\alpha} \cdot {}^{A,B}Z^{\delta}{}_{\beta}(-\vec{p}) M^{\gamma}{}_{\delta} + \delta M^{\alpha}{}_{\beta}(\vec{p},\omega) \right)_{\alpha,\beta=\pm}, \omega, {}^{B}\omega_{\vec{p}} \right) $$

for any matrix $M^{\alpha}{}_{\beta}$.

Then the theories are equivalent with the equivalence matrix ${}^{A,B}Z$.

Before providing the proof let us make a simple observation on the combinatorics of the Feynman graphs. This result is standard but we reproduce it here for the sake of clarity.

Lemma 8.3. *We say that a full subgraph of a Feynman graph is a self-energy correction if it is 1PI and has exactly two external lines (we treat internal lines of the full Feynman graphs which connect the subgraph with other parts of the graph as external lines of the subgraph). We say that a graph is non-corrected if it contains no self-energy corrections.*

Then the Feynman rules of Proposition 4.2 can be modified in the following way:

- *Sum over all graphs is replaced by sum over all non-corrected graphs*
- *Free propagator is replaced by the corrected propagator.*

Of course, the statement above should be understood in the sense of formal power series. Namely, we always compute the correlator up to some fixed order in the interaction constant and we need to truncate the corrected propagator at the same order.

Proof. To any Feynman graph corresponds a unique non-corrected Feynman graph which can be achieved by cutting out all self-energy corrections and sawing back the broken links into the internal line[53]. Summation over all ways to insert arbitrarily many self-energy corrections to an internal line gives exactly the corrected propagator. □

Proof of Theorem 8.2. 1. Observe that we need the equivalence on the level of corrected propagators only, i.e.

$$ {}^{A}\mathcal{G}_{\alpha_1\alpha_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2) = \prod_{i=1}^{2} {}^{A,B}Z(\vec{p}_i)_{\alpha_i}{}^{\beta_i} \cdot {}^{B}\mathcal{G}_{\beta_1\beta_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2). \tag{8.6} $$

In fact, a graph clearly can not contain a quadratic vertex anywhere except inside the corrected propagator. Then combining (8.3) with (8.6) by Lemma 8.3 all other correlators also satisfy (8.1).

2. Note that from (8.2) and (8.5) the equivalence on the corrected propagators level (8.6) holds whenever

$$ {}^{B}M^{\alpha_1\alpha_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2) = \prod_{i=1}^{2} {}^{A,B}Z_{\beta_i}{}^{\alpha_i}(\vec{p}_i) \cdot {}^{A}M^{\beta_1\beta_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2) + \delta M^{\alpha_1\alpha_2}(\vec{p}_1,\omega_1). \tag{8.7} $$

3. By a slight modification of the proof of Lemma 8.3 we can show that the 1PI corrections ${}^{A}M^{\beta_1\beta_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2)$ and ${}^{B}M^{\beta_1\beta_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2)$ can also be found as a sum over all non-corrected graphs with propagators replaced with the corrected ones. From (8.4) we see that the theory B in comparison with A has one extra quadratic vertex (corresponding to the vertex factor $\delta M^{\alpha_1\alpha_2}(\vec{p}_1,\omega_1)$) and hence one extra non-corrected graph, consisting of that point only (this is the only connected non-corrected graph containing quadratic vertices). So, we have

$$ {}^{B}M^{\alpha_1\alpha_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2) = {}^{B}M'^{\alpha_1\alpha_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2) + \delta M^{\alpha_1\alpha_2}(\vec{p}_1,\omega_1), $$

where ${}^{B}M'$ is the contribution of all other graphs and (8.7) becomes

$$ {}^{B}M'^{\alpha_1\alpha_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2) = \prod_{i=1}^{2} {}^{A,B}Z_{\beta_i}{}^{\alpha_i}(\vec{p}_i) \cdot {}^{A}M^{\beta_1\beta_2}(\vec{p}_1,\omega_1;\vec{p}_2,\omega_2). \tag{8.8} $$

4. Since ${}^{A}M$ and ${}^{B}M'$ are sums over the same set of graphs and the vertex factors contributing to them coincide up to contraction with ${}^{A,B}Z$, (8.8), and thus (8.7) holds whenever (8.6) does.

5. We have shown that (8.6) yields (8.8), and, conversely, (8.8) yields (8.6). To finish the proof we note, that both ${}^{B}M'$ and ${}^{A}M$ computed according to Lemma 8.3 depends only on lower orders of the corrected propagator (since they always contain at least one vertex factor bringing at least ne power of the interaction constant). So, if (8.6) holds up to Nth order, then (8.8) holds for $N+1$th order and thus (8.6) also holds up to $N+1$th order. Moreover, in zeroth order both M and M' vanish, so (8.8) is trivial. Therefore, by induction (8.8), and thence (8.6) holds for all orders. But we established that this implies the statement.

□

The extra vertex δM plays the role of counterterm. Let us examine what counterterms should we add to realize the dispersion relation and field strength strength renormalization.

[53]This is close to the concept of a skeleton graph. But we separate only the propagator corrections, which makes the result unique and independent of the order in which we cut out corrections

Example 8.4 (Dispersion relation renormalization)**.** Let A be a physical theory with ${}^{A}\omega_{\vec{p}} = \omega_{\vec{p}}$ and Ω be a formal theory with ${}^{\Omega}\omega_{\vec{p}} = \Omega_{\vec{p}}$ and there are no field strength renormalization, ${}^{A,B}Z_{\alpha}{}^{\beta} = \delta_{\alpha}^{\beta}$. Then the counterterm is

$$ {}^{\Omega}\delta M^{\alpha}{}_{\beta} = 2\delta_{\beta}^{\alpha}\left(\Omega_{\vec{p}} - \omega_{\vec{p}}\right)\left(\alpha\omega - \left(\omega_{\vec{p}} + \Omega_{\vec{p}}\right)\right). $$

Example 8.5 (Field strength renormalization (general))**.** Again we let A be physical theory and $B = \mathcal{Z}$ be a formal one with the same dispersion relation, but this time ${}^{A,\mathcal{Z}}Z_{\alpha}{}^{\beta}(\vec{p}) = \mathcal{Z}_{\alpha}{}^{\beta}(\vec{p})$. Then the counterterm is

$$ {}^{\mathcal{Z}}\delta M^{\alpha}{}_{\beta} = -2\omega_{\vec{p}}((\omega_{\vec{p}} - i0)X_{\beta}^{\alpha}(\vec{p}) - \omega Y_{\beta}^{\alpha}(\vec{p})) $$

with

$$ X_{\beta}^{\alpha}(\vec{p}) = \delta_{\beta}^{\alpha} - \mathcal{Z}_{\gamma}{}^{\alpha}(\vec{p})\mathcal{Z}^{\gamma}{}_{\beta}(\vec{p}) $$

$$ Y_{\beta}^{\alpha}(\vec{p}) = \delta_{\beta}^{\alpha}\beta - \sum_{\gamma}\gamma\mathcal{Z}_{\gamma}{}^{\alpha}(\vec{p})\mathcal{Z}^{\gamma}{}_{\beta}(\vec{p}). $$

Example 8.6 (Usual field strength renormalization)**.** The usual field strength renormalization is

$$ \mathcal{Z}_{\beta}{}^{\alpha}(\vec{p}) = \delta_{\beta}^{\alpha}\sqrt{Z'_F} $$

leading to

$$ {}^{\mathcal{Z}}\delta M^{\alpha}{}_{\beta} = -2\omega_{\vec{p}}((\omega_{\vec{p}} - i0) - \beta\omega)(1 - Z'_F)\delta_{\beta}^{\alpha}. $$

Example 8.7 ("Unmixing" field strength renormalization)**.** In this example we assume that the relations (6.2-6.3) hold. Let us consider the following generalized renormalization:

$$ \mathcal{Z}_{+}{}^{+}(\vec{p}) = \mathcal{Z}_{-}{}^{-}(\vec{p}) = \sqrt{Z(\vec{p})}\cos\left(\frac{\theta(\vec{p},\omega_{\vec{p}}^{Q})}{2}\right), $$

$$ \mathcal{Z}_{-}{}^{+}(\vec{p}) = \mathcal{Z}_{+}{}^{-}(\vec{p}) = -\sqrt{Z(\vec{p})}\sin\left(\frac{\theta(\vec{p},\omega_{\vec{p}}^{Q})}{2}\right). $$

In this renormalization the corrected propagator coincides with the free one on the (corrected) dispesion relation shell.

Of course, it is possible to write down the general counterterm incorporating both field strength renormalization and dispersion relation renormalization.

8.2 The physical renormalization

In this subsection we will consider more traditional view to the renormalization, based on the ambiguity of the separation of the Hamiltonian into free and interacting parts.

In this framework the (physical) renormalization is given by considering

$$ {}^{ren}H_0 = H_0 + \delta H. $$

Moving to the interaction representation we get

$$ \delta H_I(t) = U(t)\delta H U(t)^{-1}. $$

As it is clear from Subsection 3.1, we should be very careful with non-locality in time and even with the derivative with respect to time in δH, which is the case for the field strength renormalization. In particular, the interaction representation of the derivative (with respect to time) of an interacting field is not equal to the derivative of the corresponding free field.

Instead we present δH as a local (in time) functional of the field and its canonical momentum. Then δH_I can be achieved through (3.8-3.7).

8.2.1 Dispersion relation renormalisation

We again start from the dispersion relation renormalisation. We define

$$ {}^{\Omega}H_0 = H_0 + \frac{1}{2}\int d\vec{k} : \tilde{\phi}(\vec{k},t)\tilde{\phi}(-\vec{k},t) : (\Omega^2_{\vec{k}} - \omega^2_{\vec{k}}) =$$

$$\frac{1}{2}\int d\vec{k} : \left(\partial_t\tilde{\phi}^{(0)}(\vec{k},t)\tilde{\phi}^{(0)}(-\vec{k},t) + \Omega^2_{\vec{k}}\tilde{\phi}^{(0)}(\vec{k},t)\tilde{\phi}^{(0)}(-\vec{k},t)\right) :.$$

The counterterm, compensating passing from H_0 to ${}^{\Omega}H_0$, its interaction representation and the corresponding vertex factor then are:

$$ {}^{\Omega}\delta H = -\frac{1}{2}\int d\vec{k} : \tilde{\phi}(\vec{k},t)\tilde{\phi}(-\vec{k},t) : (\Omega^2_{\vec{k}} - \omega^2_{\vec{k}}),$$

$$ {}^{\Omega}\delta H_I = -\frac{1}{2}\int d\vec{k} : \tilde{\phi}_0(\vec{k},t)\tilde{\phi}_0(-\vec{k},t) : (\Omega^2_{\vec{k}} - \omega^2_{\vec{k}}),$$

$$ {}^{\Omega}\delta M_{\alpha\beta}(\vec{k},\omega) = -(\Omega^2_{\vec{k}} - \omega^2_{\vec{k}}).$$

This is exactly the counterterm of the ordinary QFT (despite the fact that we allow now general dispersion relation, so the counterterm is in general nor Lorentz-invariant, neither spatially local). It can never (except the trivial case $\omega_{\vec{p}} = \Omega_{\vec{p}}$) coincide with ${}^{\Omega}\delta M^{\alpha}{}_{\beta}$ of Example 8.4. Moreover, the physical renormalization of the dispersion relation not only produces a counterterm but also changes all the vertex factors. Indeed, to find the vertex factors F defined by (3.29) depend not only on the interaction kernel density κ, but also on the bare dispersion relation $\omega_{\vec{k}}$. It is in principle possible, that for some very special interaction the effect of the counterterm and of the dispersion relation change could somehow compensate each other, but it would be a very special restriction on the vertex factors. Hence in general such renormalization does not preserve any of the correlators $\mathcal{G}_{\alpha_1,\dots,\alpha_n}(\vec{p_1},\omega_1;\dots;\vec{p_n},\omega_n)$. One may hope that this issue will be resolved in the theories introuduced in 3.1.3.

Remark 8.8. However, by slightly modifying the proof of Theorem 8.2, one can show that if the vertex factors do not depend on the time-orientations (i.e. we are considering a local QFT without derivatives with respect to the time in the interaction) this renormalization preserves the physical (i.e. summed over all time-orientations of external particles) correlators $\mathcal{G}(\vec{p_1},\omega_1;\cdots;\vec{p_n},\omega_n)$.

8.2.2 Field strength renormalization

The field strength renormalization is passing from $\tilde{\phi}(\vec{k},t)$ to ${}^{Z}\tilde{\phi}(\vec{k},t) = \frac{1}{\sqrt{Z(\vec{k})}}\tilde{\phi}(\vec{k},t)$. To preserve the canonical quantization relations (3.2) we need to renormalize the canonical momentum in the opposite way: ${}^{Z}\tilde{\pi}(\vec{k},t) = \sqrt{Z}(\vec{k})\tilde{\pi}(\vec{k},t)$[54].

The Hamiltonian should be rewritten via the renormalized fields. In particular, for the free quantum field we will get

$$H_0 = \frac{1}{2}\int d\vec{k} : \left(\frac{1}{Z(\vec{k})}{}^{Z}\tilde{\pi}(\vec{k},t){}^{Z}\tilde{\pi}(-\vec{k},t) + Z(\vec{k})\omega^2_{\vec{k}}{}^{Z}\tilde{\phi}(\vec{k},t){}^{Z}\tilde{\phi}(-\vec{k},t)\right) : =$$

$$\frac{1}{2}\int d\vec{k} : \left({}^{Z}\tilde{\pi}(\vec{k},t){}^{Z}\tilde{\pi}(-\vec{k},t) + \omega^2_{\vec{k}}{}^{Z}\tilde{\phi}(\vec{k},t){}^{Z}\tilde{\phi}(-\vec{k},t)\right) :$$

[54] Alternatively one could start from the Lagrangian with renormalized field substituted and define ${}^{Z}\tilde{\pi}(\vec{k},t)$ as the canonical momentum conjugated to the renormalized field. The result will be the same.

$$-\frac{1}{2}\int d\vec{k} : \left(\left(1-\frac{1}{Z(\vec{k})}\right) {}^Z\tilde{\pi}(\vec{k},t){}^Z\tilde{\pi}(-\vec{k},t) + (1-Z(\vec{k}))\omega_{\vec{k}}^{2\,Z}\tilde{\phi}(\vec{k},t){}^Z\tilde{\phi}(-\vec{k},t)\right) : =$$

$$ {}^Z H_0 - {}^Z\delta H$$

where

$$ {}^Z H_0 = \frac{1}{2}\int d\vec{k} : \left({}^Z\tilde{\pi}(\vec{k},t){}^Z\tilde{\pi}(-\vec{k},t) + \omega_{\vec{k}}^{2\,Z}\tilde{\phi}(\vec{k},t){}^Z\tilde{\phi}(-\vec{k},t)\right) :$$

is the free Hamiltonian for the "renormalized" quantum fields.The counterterm is

$$ {}^Z\delta H = \frac{1}{2}\int d\vec{k} : \left(\left(1-\frac{1}{Z(\vec{k})}\right) {}^Z\tilde{\pi}(\vec{k},t){}^Z\tilde{\pi}(-\vec{k},t) + \left(1-Z(\vec{k})\right)\omega_{\vec{k}}^{2\,Z}\tilde{\phi}(\vec{k},t){}^Z\tilde{\phi}(-\vec{k},t)\right) :,$$

$$ {}^Z\delta H_I = \frac{1}{2}\int d\vec{k} : \left(\left(1-\frac{1}{Z(\vec{k})}\right) {}^Z\dot{\tilde{\phi}}_0(\vec{k},t){}^Z\dot{\tilde{\phi}}_0(-\vec{k},t) + (1-Z(\vec{k}))\omega_{\vec{k}}^{2\,Z}\tilde{\phi}_0(\vec{k},t){}^Z\tilde{\phi}_0(-\vec{k},t)\right) :$$

and for the counterterm

$$ {}^Z\delta F_{\alpha\beta}(\vec{p},-\vec{p}) = -\omega_{\vec{p}}^2\left(\left(1-\frac{1}{Z(\vec{p})}\right)(-\alpha\beta) + (1-Z(\vec{p}))\right) = \tag{8.9}$$

$$-\omega_{\vec{p}}^2\left(\frac{1}{Z(\vec{p})}\alpha\beta + 1\right)(1-Z(\vec{p})),$$

or

$$ {}^Z\delta M(\vec{p}) = {}^Z\delta M_{+-}(\vec{p},-\vec{p}) = {}^Z\delta M_{-+}(\vec{p},-\vec{p}) = \omega_{\vec{p}}^2\frac{(Z(\vec{p})-1)^2}{Z(\vec{p})}, \tag{8.10}$$

$$ {}^Z\delta N(\vec{p}) = {}^Z\delta M_{++}(\vec{p},-\vec{p}) = {}^Z\delta M_{--}(\vec{p},-\vec{p}) = \omega_{\vec{p}}^2\frac{Z(\vec{p})^2-1}{Z(\vec{p})}. \tag{8.11}$$

Clearly, it has nothing to do with Example 8.6, so in general this renormalization also does not preserve the correlators and the scattering amplitudes reconstructed from LSZ.

In this case it is not that straightforward to compare (8.10-8.11) with the usual counterterm. To do that we would nee to use the framework of Section 5. Again, one can show that such renormalization leaves the "physical" correlators unchanged provided that the vertex factors do not depend on time-orientations.

8.3 Conluding remarks on renormalisation

Clearly, the "formal" renormalisation can not be used to regularise the strong adiabatic limit, since it replaces the non-local QFT by a set of formal Feynman rules. However, it can be used as a starting step to build an analogue of the renormalisation group, which is necessary to compute the logarithmic corrections (see Remark 3.5). Of course, it will be very different from the usual renormalisation group, not only because of its completely formal diagrammatic nature, but also because the interactions are not preserved by locality anymore, so they are not defined by a finite number of constants. This analysis, however, seems to be neccessary if we want to understand how the corrections due to the non-locality vary in large energy range, from the low energies we live in to the Planck energies, where we expect the QST effects to be important.

The "physical" renormalisation can be used to at least fulfill the necessary conditions for the strong adiabatic limit to exist we conjectured. One however has to keep in mind that it is not renormalisation in the sense of Remark 3.4, i.e. it can not be considered as a shift of the parameters and for this reason is quite artificial. This is not a surprise, since we did not actually start from the full Hamiltonian in the form

$$H = H_0 + H_{int},$$

but we took predefined interaction representation H_I. As we have seen in Subsection 3.1.3, the relation between H_I and H_{int} is highly non-linear in the non-local case, so we can not just add to H_I the counterterms of the local theory. We note also, that the "physical" renormalisation can not be used to construct a renormalisation group in any sense, since it changes the physical content of the theory.

There are other possible approaches from which we mention the following two.

First one can give up invariance of the theory under renormalisation and keeping the full quadratic part of the Lagrangian Lorentz-invariant. Instead, one starts from a free Hamiltonian with *predefined* dispersion relation which is assumed to coincide with the physical one. The counterterms then can be chosen order-by-order so, that they cancel all quantum corrections exactly. In particular, one can require the Lorentz-invariant physical dispersion relations to be preserved. The main disadvantage of this approach is that if we allow such renormalisation, there is no reason to stop at quadratic counterterms. But if we allow also higher counterterms, in absence of the locality[55], we can use this additional freedom to get absolutely any scattering operator. So, such a theory has no predictive power.

The second one is to consider the theory as time-local theory (see Remark 3.20), forgetting about the original time-nonlocal interaction kernel density. Then it can be renormalised in the usual way. It is natural to expect that this approach will be a Hamiltonian formulation of the "formal" renormalisation, because in the time-local theories renormalisation should not change the weak adiabatic limit. Physically such an approach is still not natural, because the vertex factor will depend on the bare unrenormalised dispersion relation.

There is a hope that the framework of Subsection 3.1.3 can reconcile the "physical" and "formal" approaches as it starts from H_{int} rather than from H_I.

[55]We note, that to cancel the propagator corrections we have to allow absoulutely arbitrary counterterm. In particular, they are not local, and if we start from some prescribed form of the interaction, like Example 3.16, there is no reason in general for the counterterm to be also presentable in this form.

9 Conclusions

9.1 Summary of the main results

- The results of [73] on non-local QFT in the Hamiltonian approach were generalised to more generic interactions. In particular, it was shown that it is enough to assume that the interaction kernel is a smooth function of smooth coordinates only, being sharp-localised in time. Moreover, we saw that any reasonable interaction treated by means of the Hamiltonian approach [37, 71] can be represented in this form. Under reasonable assumptions on the interaction kernel, we have proved that such theories are free from UV divergences and that the spatial adiabatic cut-off can be safely weared off leaving the perturbative scattering operator well-defined on its natural domain of definition;

- A new (in fact several versions of) version of the Feynman rules, more convenient for usual combinatoric manipulation was presented;

- Corrections to propagator were computed. The quantum dispersion relation and field strength normalisation factor were extracted from the corrected propagator. We have seen that the corrected dispersion relation are in accordance with the results of [75]. This is especially interesting because the mentioned work is based on the time-independent perturbation theory, so instead of keeping only adiabatic switching only in time it inevitably assumes only the spatial adiabatic cut-off.

- Existence of the weak adiabatic limit was shown for a quite general class of non-local Quantum Field Theories;

- Conditions of the strong adiabatic limit existence were discussed;

- A relation with the Lagrangian Feynman rules of an equivalent time-local theory was presented. In particular we have proven, that at the tree level the Lagrangian is the usual Legendre transform of time-localised form of the Hamiltonian;

- Issues of the renormalisation were discussed. More precisely, we have introduced two approaches to renormalisation, one preserving the correlators in weak adiabatic limit and another, mimicking the usual renormalisation counterterms rewritten in the Hamiltonian formalism. We have seen that the choice of approach to renormalisation is vague in absence of locality;

9.2 Outline of further directions

In this thesis we have seen several problems and ambiguities, mostly caused by us trying to treat a theory which is non-local only in space as a space-time non-local theory. A possible way out of this was presented in Subsection 3.1.3, where we perturbatively used the equations of motion so that the truly spacetime non-local theory is presented as a time-local effective one.

Another issue is Lorentz-invariance. One source of the problem is the absence of reasonable Lorentz-invariant non-local kernels. Some hint for a possible way out of this was presented in Subsection 2.2.1. Another problem is that the Hamiltonian density is not a scalar and it is difficult to build it so that it will be a component of the 4-vector in the end. In ordinary QFT it is resolved by the Legendre transform, but it is directly applicable to the theories which Lagrangian depends on at most the first-order derivative of the field. There are generalisations to higher orders of derivative and even time-non-local Lagrangians known as the Ostrogradski transform [110], so it may be interesting to see if they can be incorporated in the framework of Subsection 3.1.3.

It is worth noting, that even the truly non-local theories of 3.1.3 according to Section 5 can be (perturbatively in each order) described by an equivalent Lagrangian, not containing higher-order derivatives with respect to the time. In Effective Field Theory there is a known method of getting

rid of the higher derivatives by means of the field redefinition [111], which is often interpreted as usage of the equations of motion. It is interesting to understand if this approach is anyhow related to ideologically similar method of Subsection 3.1.3.

Finally, it is instructive to compute the non-locality corrections to the scattering amplitudes and the dispersion relations, taking into account the renormalisation. In particular, it is interesting to see if the lack of the Lorentz invariance will doom the higher loop corrections even at low energies as it is expected in local theories [45].

A Minkowski spacetime, 4-vectors and symmetry groups

We deal with the standard 4-dimensional Minkowski spacetime with the signature $(+,-,-,-)$. We use upper Greek indices $\mu, \nu, ...$[56] to denote the components of four-vectors and coordinates of the spacetime points and the lower indices to denote the same components after contraction with the metric tensor. We follow the standard Einstein summation rule

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu.$$

Then no confusion is possible (mostly in the plane waves phases) we drop the indices at all, writing px instead of $p_\mu x^\mu$ and p^2 instead of $p^\mu p_\mu$. The spatial coordinates and spatial part of the 4-vectors are denoted with the vector sign. For 3-vectors the scalar product is defined with the usual Euclidean signature. The norm $||\cdot||$ always denotes the positively-definite norm with Euclidean signature independently on dimensionality of the space.

The Lorentz group O(1,3) acts on the Minkowski space as

$$M : x^\mu \mapsto M^\mu{}_\nu x^\nu \tag{A.1}$$

for $M \in O(1,3)$, i.e.

$$M^\mu{}_\eta M_\nu{}^\eta = \delta^\mu_\nu$$

which we will abbreviate with just

$$x \mapsto Mx.$$

In all expressions like above the components with upper indices are assumed. Among the connected components of O(1,3) we will encounter only the proper ortochronous Lorentz group $SO^\uparrow(1,3)$ defined by

$$\det(M) = 1, \quad M^0{}_0 > 0.$$

The Poincare group is denoted by

$$\mathfrak{P} = \mathbb{R}^4 \rtimes SO(1,3)$$

with the semi-direct product with respect to (A.1), and the proper orthochronous Poincare group is

$$\mathfrak{P}^\uparrow_+ = \mathbb{R}^4 \rtimes SO(1,3)^\uparrow.$$

We denote the universal cover of the proper ortochronous Poincare group with

$$\overline{\mathfrak{P}^\uparrow_+} = \mathbb{R}^4 \rtimes \overline{SO(1,3)^\uparrow} = \mathbb{R}^4 \rtimes SL(2)$$

and the covering map

$$SL(2) \to SO(1,3)^\uparrow$$

with Π.

We use notation (a, M) with $a \in \mathbb{R}^4$, $M \in SO(1,3)$ to denote a generic element of $\mathfrak{P}$, as well as (a, σ) with $a \in \mathbb{R}^4$, $\sigma \in SL(2)$ for an element of $\overline{\mathfrak{P}^\uparrow_+}$.

We will also use the symbol $A \curvearrowright B$ to denote action of the transofrm A on the value B, assuming that $SL(2)$ and $SU(2)$ act on tensors and vectors via $\mathcal{P}\rangle$, e.g.

$$(\alpha \curvearrowright p)^\mu = (\Pi(\alpha) \curvearrowright p)^\mu = \Pi(\alpha)^\mu{}_\nu p^\nu.$$

[56] We reserve indices from the beginning of the Greek alphabet, $\alpha, \beta, \gamma, \delta, ...$, for the time-orientation indices introduced in the main text.

B Numerical, vector- and operator-valued distributions

For the sake of completeness and to fix the notation we list here main definitions and standard results of the distributions theory starting from the usual numerical definitions and then going to less common vector- and operator-valued distributions. All details can be found in [77, 81, 82].

Recall that a *(numerical) distribution* is a continuous map from a Frechet space of *test functions* to the complex numbers. In particular, one usually considers the space of compactly supported smooth functions $\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$ with a family of seminorms

$$||f||_\alpha = \sup_{x\in\mathbb{R}^n} |\partial^\alpha f(x)|, \tag{B.1}$$

where α is a multindex, leading to the theory of *distributions* (without any additional adjective) on X $\mathcal{D}'(X)$ and the space of *Schwartz functions*

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) \middle| ||f||_{l;\alpha} < \infty \right\}$$

with a system of the seminorms

$$||f||_{l;\alpha} = \sup_{x\in\mathbb{R}^n} \left|(1+||x||^2)^l \partial^\alpha f(x)\right|$$

leading to the space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^n)$.

Remark B.1. Without any additional difficulties arise one can always replace $\mathcal{S}(\mathbb{R}^n)$ by a direct sum $\bigoplus_r \mathcal{S}(\mathbb{R}^n)$ and get the space of *multi-component distributions*[57]

$$\left(\bigoplus_r \mathcal{S}(\mathbb{R}^n)\right)' = \bigoplus_r \mathcal{S}'(\mathbb{R}^n) \tag{B.2}$$

(and the same for compactly supported functions), so we can equally consider such a distribution F as a collection of r usual distributions F_i or as a whole distribution F.

In the following, we will use $\mathcal{T}$ to denote either $\mathcal{S}$ or $\mathcal{D}$. We will also drop $\mathbb{R}^n$ in cases then the number of dimensions is not important. Note, that we have $\mathcal{D} \subset \mathcal{S}$ and thus $\mathcal{S}' \subset \mathcal{D}'$.

We use a lot the formal notation

$$F : f \mapsto F[f] = \int f(x)F(x)d^n x, \quad F \in \mathcal{T}', \quad f \in \mathcal{T}, \tag{B.3}$$

or for multi-component distributions

$$F : f \mapsto F[f] = \sum_{i=1}^k F_i[f_i] = \sum_{i=1}^n \int f_i(x)F_i(x)d^n x, \quad F \in \bigoplus_k \mathcal{T}', \quad f \in \bigoplus_n \mathcal{T} \tag{B.4}$$

where the right hand side should be understood completely symbolically. All expressions containing distributions evaluated at a point, like $F(x)$, should be always understood as shorten versions of that expressions integrated with an arbitrary test function. In particular, we will use some operations on distributions defined by formal manipulations with (B.3)

$$(\lambda F + \mu G)[f] = \lambda F[f] + \mu G[f], \quad \forall F, G \in \mathcal{T}', \forall f \in \mathcal{T}, \forall \lambda, \mu \in \mathbb{C};$$

$$(fF)[g] = F[fg], \quad \forall F \in \mathcal{T}', \forall f \in \Theta_M, \forall g \in \mathcal{T};$$

[57] We could call them vector-valued distributions, but we preserve this name for much more special space of distributions with values in an infinite-dimensional Hilbert space.

and with (B.4)

$$(\lambda F + \mu G)[f] = \lambda F[f] + \mu G[f], \quad \forall F, G \in \bigoplus_k \mathcal{T}', \forall f \in \bigoplus_k \mathcal{T}, \forall \lambda, \mu \in M_n(\mathbb{C});$$

$$(fF)[g] = F[f^T g], \quad \forall F \in \mathcal{T}'(\mathbb{R}^n), \forall g \in \mathcal{T}(\mathbb{R}^n), f \in \Theta_M(\mathbb{R}^k, M_k(\mathbb{C})).$$

Here

$$\Theta_M(\mathcal{R}^n) = \left\{C^\infty(\mathbb{R}^n) | \forall \alpha \exists c_\alpha > 0, p_\alpha \in \mathbb{N}, \forall x \mathbb{R}^n : |\partial^\alpha f(x)| < c_\alpha (1 + ||x||^2)^{p_\alpha}\right\} \tag{B.5}$$

$$\Theta_M(\mathbb{R}^n, M_k(\mathbb{C})) =$$

$$\left\{C^\infty(\mathbb{R}^n, M_k(\mathbb{C})) | \forall \alpha \exists c_\alpha > 0, p_\alpha \in \mathbb{N}, \forall x \mathbb{R}^n : ||\partial^\alpha f(x)|| < c_\alpha (1 + ||x||^2)^{p_\alpha}\right\}$$

are the functions of polynomially limited growth which are defined so that $\Theta_M \mathcal{T} \subset \mathcal{T}$. For an affine function $\mathbb{R}^n \to \mathbb{R}^n$

$$\phi : \tau \mapsto Ax + b$$

with $b \in \mathbb{R}^n$, $A \in GL(n)$ we also define[58]

$$(F \circ \tau)[f] = F\left[|\det A|^{-1} f \circ \tau^{-1}\right], \forall F \in \mathcal{T}', \forall f \in \mathcal{T}.$$

We also use the standard notation for derivatives of the distribution based on formal integration by parts[59],

$$\partial_{x^i} F[f] = -F[\partial x_i f].$$

There is a natural embedding $\mathcal{T} \to \mathcal{T}'$ defined by (B.3) with $F(x)$ being an actual function. In this case we will identify the function $F(x)$ with the corresponding functional and write $F \in \mathcal{T} \subset \mathcal{T}'$.

We say that distributions F and G *coincide* in an open region O if for any test-function f vanishing outside of O we have $F[f] = G[f]$. The complement of the maximal set on which F *vanishes*, i.e. coincides with the zero functional, we call the *support* of F.

The tempered distributions are of great use because we can define the *Fourier transform* on them. To avoid conventional confusions, we will always write the Fourier transform as a functional integral, for example, for $F \in \mathcal{S}'(\mathbb{R})$:

$$\tilde{F}(k) = \int dx e^{ikx} F(x) \tag{B.6}$$

which should be interpreted as

$$\tilde{F}[u] = F[\tilde{u}], \quad \forall u \in \mathcal{S}(\mathbb{R}), \tag{B.7}$$

where

$$\tilde{u}(x) = \int d^k u(k) e^{ikx}. \tag{B.8}$$

Here we implicitly used the fact that the Fourier transform is a continuous map from $\mathcal{S}(\mathbb{R})$ to itself. The definition is chosen so that when restricted to $\mathcal{S}(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ (B.6) holds in the usual sense. Generalization to arbitrary number of dimensions is straightforward. We will often use the partial Fourier transform, i.e. Fourier transform only in part of the arguments.

We also have the following:

Theorem B.2 (Theorem 2.1.3 of [81][60]). *Let $F \in \mathcal{T}'(\mathbb{R}^m)$ with $\mathcal{T}$ being either $\mathcal{D}$ or $\mathcal{S}$. When F defines a continuous map*

$$\mathcal{T}(\mathbb{R}^{n+m}) \to \mathcal{T}(\mathbb{R}^m)$$

$$F[f](y) = F[f(\cdot, y)].$$

[58] This definition can be of course extended to a more general class of τ, but we will not need it.
[59] Alternatively, it can be interpreted as a usual definition, see 2.2B of [77].
[60] In [81] it is actually proven for $\mathcal{T} = \mathcal{D}(\mathbb{R}^n)$, but generalization to $\mathcal{T} = \mathcal{S}(\mathbb{R}^n)$ is straightforward

By $f(\cdot, y)$ we mean f as a function of its first n arguments with the last m fixed to $y \in \mathbb{R}^m$.

Theorem B.3 (Theorem 2.5 of [77] and V.12 of [82]). *A continuous in each argument functional on $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m)$ is jointly continuous. Moreover, it extends uniquely to a continuous functional on $\mathcal{S}(\mathbb{R}^{n+m})$.*

Theorem B.4 (Subsection 2.4B of [77]). *Let $F \in \mathcal{T}'(\mathbb{R}^n)$, $G \in \mathcal{T}'(\mathbb{R}^m)$. Then there is a unique distribution $F \otimes G \in \mathcal{T}'(\mathbb{R}^{n+m})$ such that*

$$(F \otimes G)(f \otimes g) = F(f)G(g).$$

Theorem B.5 (2.4B of [77]). *Let $F \in \mathcal{T}'(\mathbb{R}^n)$, $G \in \mathcal{T}'(\mathbb{R}^m)$. Then there is a unique distribution $F \otimes G \in \mathcal{T}'(\mathbb{R}^{n+m})$ such that*

$$(F \otimes G)(f \otimes g) = F(f)G(g).$$

Symbolically we will write $F(x)G(y)$ instead of $F \otimes G$.

Remark B.6. By combining linear non-degenerate transformation of coordinates with tensor products we can define products of distributions with linearly independent variables.

Vector-valued distributions[61] are analogously defined as continuous maps from $\mathcal{S}$ to a Hilbert space[62] $\mathcal{H}$. In this case it is enough to require that $\Psi : \mathcal{S} \to \mathcal{H}$ is a linear map and the scalar product $(\psi, \Psi) \in \mathcal{S}'$. This allows to use all the operations we have defined for the numerical distributions. It can be also shown that for a bounded operator A, $A\Psi$ defined as a composition of maps Ψ and A is also a vector-valued distribution. We keep using the symbolic integral notation

$$\Psi[f] = \int d^n x f(x)\Psi(x).$$

The scalar product of two vector-valued distribution can be defined as a numerical distribution. More precisely, if Ψ and Ψ are vector-valued distributions o $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, then there is a unique distribution $(\Psi, \Psi') \in \mathcal{T}'(\mathbb{R}^{n+m})$ such that

$$(\Psi, \Psi')[f \otimes g] = (\Psi[f], \Psi'[g]).$$

Finally, in QFT an important role is played by the operator-valued distributions. Even worse, we need distributions with values in the the unbounded operators which are unpleasant by themselves. In particular, we can not say that the distributions are continuous in the operator-norm topology. Instead we require the following:

Definition B.7. Let D be a dense subspace of the Hilbert space $\mathcal{H}$ and let A map $u\mathcal{S}(\mathbb{R}^n)$ to unbounded operators defined on D. Then A is called an *operator-valued distribution* if either of the following (equivalent)[63] requirements holds:

- $f \mapsto A[f]\psi$ is a vector-valued tempered distribution for any $\psi \in D$.
- $f \mapsto (\psi, A[f]\psi')$ is a tempered distribution for any $\psi, \psi' \in D$.

Again, all the operations (except for the tensor product) can be extended to the operator-valued ones. The tensor product in general can be not defined because product of two densely-defined unbounded operators can have non-dense domain. We will deal only with such operator-valued distributions that $A[\mathcal{S}]D \subset D$ (we say that A leaves the domain D invariant). Assume that

[61] For all proofs concerning vector- and operator-valued distributions we refer[77].
[62] We limit ourselves to Hilbert spaces because this is the case important in QFT.
[63] as proved in [77]

$A_1, \ldots, A_k$ are operator-valued distributions leaving invariant their common domain D. Then there is a unique operator-valued distribution on the same domain $A_1 \otimes \cdots \otimes A_k$ such that

$$A_1 \otimes \cdots \otimes A_k[f_1 \otimes \cdots \otimes f_k] = A_1[f_1] \cdots A_k[f_k].$$

The key step in the proof of this fact is to note that

$$||A_1(f_1) \otimes \cdots \otimes A_k(f_k)\Psi||^2$$

is a jointly continuous functional of $f_1, \ldots, f_n$ for any fixed $\Psi \in D$, by application of Theorem B.3. Then it can be uniquely extended to a vector-valued distribution, making

$$A_1 \otimes \cdots \otimes A_k[f]\Psi$$

defined for any $\Psi \in D$, and any test-function f on the product space. As always, we refer to [77] for any further details. As with the numerical distributions, we will write

$$A_1(x_1) \cdots A_k(x_k)$$

instead of

$$A_1 \otimes \cdots \otimes A_k(x_1, \ldots, x_k).$$

We will say that an operator-valued distribution F with the domain of definition D *vanishes* in an open region O if whenever evaluated on a test-function outside of O we have

$$F[f]\Psi = 0, \quad \forall \Psi \in D.$$

For a numerical distribution one can also clearly define its complex conjugate via

$$\overline{F}[f] = \overline{F[\overline{f}]}.$$

Definition of a quantum field requires a generalization of this operation to the operator-valued distributions. For that we require that the operators $A(u)^*$ are defined on D. Then one can show that

$$A^*[f] = A[\overline{f}]^*$$

defines an operator-valued distribution. Outside of this subsection we deal only with operator-valued distributions, such that $A(u)^*$ is defined on D for any u, so we will assume that this is a part of the definition.

In the end of this exposition we turn back to the numerical distributions. Sometimes it is important to have a criteria determining if a distribution is actually a smooth function. For this let us, following [84], introduce couple of notions:

Definition B.8. Let $F \in \mathcal{D}'(\mathbb{R}^n)$. Then $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is called a *regular directed point* if there is $f \in \mathcal{D}(\mathbb{R}^n)$, $f(x) = 1$ and a conical neighborhood $V \subset \mathbb{R}^n$ of u such that the Fourier transform $\widehat{Fu}$ restricted to V is decreasing faster than any polynomial.

Here by a *conical neighborhood* of u we mean a cone in $\mathbb{R}^n$ containing an open neighborhood of u and the convention for the Fourier transform is

$$\hat{F}(k) = \int e^{-ikx} F(k) d^n k.$$

The (pseudo)scalar product kx can be chosen arbitrarily. The product of compactly supported function with a Schwartz function is always compactly-supported, so $Fu \in \mathcal{S}(\mathbb{R}^n)$ and the Fourier transform is well-defined. Let us denote the set of regular points of F with $R(F)$. Then

$$WF(F) = (\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \setminus R(F)$$

is called the *wavefront set* of F. If F is a smooth function, then $\hat{Fu}$ is a Fourier transform of a smooth compactly supported function which is smooth and fast-decaying, $WF(F) = \varnothing$. Conversely, we also have:

Theorem B.9 (Theorem 11 in [84]). *A compactly-supported distribution with fast-decaying Fourier transform is smooth.*

It means that for a compact K, if we have

$$WF(F) \cap (K \times (\mathbb{R}^n \setminus \{0\})) = \varnothing,$$

then F is smooth on K (i.e. it coincides with a smooth function).

More generally, the wavefronts set of a distribution answers the question how exactly singular the distribution is. Besides other uses, it allows to determine whenever singularities of two distributions are aligned is such a way that one can make sense of their product.

Theorem B.10 (Theorem 8.2.10 of [81]). *Let $U, V \in \mathcal{D}'(\mathbb{R}^n)$ and assume that there is no $x, \xi \in \mathbb{R}^n$, such that*

$$(x, \xi) \in WF(U), \quad (x, -\xi) \in WF(V).$$

Then the product UV is well-defined and

$$WF(UV) \subset \{(x, \xi + \eta) | (x, \xi) \in WF(U), (x, \eta) \in WF(U)\}.$$

For compactly supported distributions the product above can be defined by first defining its Fourier transform as a convolution of Fourier transforms of U and V, and it is enough to define value of UV on any function in $\mathcal{D}(\mathbb{R}^n)$. This product can be also shown to be associative and behave well with the derivatives. We will need only the existence criteria of the theorem above, so we do not provide any further details.

C Action of symmetries on the statespace

Let us fix a Hilbert space $\mathcal{H}_{phys}$ of (pure) states of the quantum field theory. By Wigner's argument (see [4, 78] for references and proofs) we assume that the symmetry group acts on $\mathcal{H}_{phys}$ via a projective unitary representation[64]. As a symmetry group we take at most[65] the proper Poincare group $\mathfrak{P}_+^\uparrow$ introduced in the previous subsection. For this group any projective unitary representation is equivalent (in projective sense) to an ordinary unitary representation of its universal cover $\overline{\mathfrak{P}}_\uparrow$ [79] (see also [80] for a more general treatment) which we will denote with U.

We assume that there is a unique one-dimensional subspace (the vacuum subspace), such that U restricted to that subspace is trivial. A normed vector Ω (which we also sometimes denote in physical notation with $|\Omega>$ or $|0>$ for the free field) from that subspace we call the vacuum.

We require that the time-translations generator has a non-negative spectrum.

We assume that $\mathcal{H}_{phys}$ contains a subspace $\mathcal{H}_1$ of one-particle states which is invariant under the action of the symmetry group and the restriction of U to $\mathcal{H}_1$ (which we will denote with the same letter) is irreducible according to [79][66]. The unitary irreducible representations of $\overline{\mathfrak{P}}$ were classified (again, by Wigner [79]). We restrict our attention to ones which are acceptable as statespace of a sole massive particle. They are labelled by a real number $m > 0$ and either integer or half-integer $s \in \frac{\mathbb{N}_+}{2}$ called mass and spin respectively. This names appear by correspondence principle with quantum mechanincs, where the generators of translations and rotations play the role of the momentum and the angular momentum respectively. Then $m^2 = p^2$ is the mass and $s(s+1)$ is the squared angular momentum in the reference frame (see beolw and in [4, 77]).

By Theorem 7.3 of [77] these representations may be constructed as

$$\mathcal{H}_1^{(m,s)} = \mathcal{L}^2(\Gamma_m^+)^{\oplus(2s+1)} \tag{C.1}$$

with

$$\Gamma_m^+ = \{p \in \mathbb{R}^4 | p_\mu^\mu = m^2, p^0 > 0\}$$

being the mass hyperboloid. The scalar product on $\mathcal{H}_1^{(m,s)}$ is defined by

$$(\psi, \psi') = \int_{\Gamma_m^+} \overline{\psi}(p)\psi(p')d\mu_m(p),$$

where

$$d\mu_m(p) = \frac{d\vec{p}}{(2\pi)^3 2p^0}.$$

The action of $\overline{\mathfrak{P}}$ is given by

$$\left(U(a,\alpha)\psi\right)_\sigma(p) = e^{-ipa}\sum_{\sigma'} D^s_{\sigma,\sigma'}(V(\alpha,p))\psi_{\sigma'}(\Pi(\alpha)^{-1}p). \tag{C.2}$$

Here the index $D^s : SU(2) \to U(2s+1)$ is a $2s+1$-dimensional (spin-s) unitary irreducible representation of $SU(2)$ and the index σ enumerates direct summands in (C.1).

Remark C.1. In the following we assume that we have fixed some canonical representations D^s acting on the vector spaces

$$W^s \cong \mathbb{C}^{2s+1},$$

but we do not specify particular basis and particular realisation. Moreover, we do not specify explicitly the range of the index σ, assuming only that it may take $2s+1$ values. We refer, for example, to [77, 80, 86] for possible natural choices.

[64] For simplicity we ignore the discrete symmetries for which the antiunitary representations would arise instead

[65] More precisely, we start from a Poincare symmetric field theory, but the interactions may break some of the symmetries.

[66] Here we actually assumed that there is only one kind of particles. In general one such subspace corresponds to each particle specie.

Finally,

$$V(\alpha, p) = R(p)^{-1}\alpha R(\Pi(\alpha)^{-1}p),$$

$$R(p) = \sqrt{\frac{p_\alpha \sigma^\alpha}{m}} \in SL(2), \tag{C.3}$$

with $\sigma^\alpha \in M_2(\mathbb{C})$ being the Pauli matrices. We not also that $\Pi(R(p))$, which we identify with $R(p)$ when no confusion is possible, is a Lorentz boost that transforms a particle with momentum $(m, \vec{0})$ into a particle with momentum p.

The description of $\mathcal{H}_1^{(m,s)}$ above is not always convenient, because it is not completely covariant. This happens because the spin states are transformed by the so-called little group $SU(2)$ rather than by the full symmetry group $SL(2)$. In particular, it makes the transformation law (C.2) rather complicated. To bypass that one introduces the *covariant wave-functions* [77]. This trick is standard for QFT, but to make discussion of 2.2.1 we will formulate it here in a slightly more general form, than one usually needs.

Definition C.2. Let $L : SL(2) \to GL(\dim L)$ be a representation of $SL(2)$. Let We say, that

$$\Psi : \Gamma_m^+ \to \mathbb{C}^r \otimes \mathbb{C}^{2s+1}$$

is a *spin-s polarisations basis* in L if

1.
$$\sum_j \overline{L_{ji}(\alpha)}\Psi_{j,\sigma}(p) = \sum_{\sigma'} D^s_{\sigma,\sigma'}(V(\alpha,p))\Psi_{i,\sigma'}(\Pi(\alpha)^{-1}p), \qquad \forall \alpha \in SL(2); \tag{C.4}$$

2. $\operatorname{rank}(\Psi(p)) = 2s+1, \qquad \forall p \in \Gamma_m^+$

Taking $\alpha = R(p)$ in (C.4) we get

$$\Psi_{i,\sigma}(p) = \sum_j \overline{L_{ji}(R(p)^{-1})}\Psi^{(0)}_{j,\sigma}, \tag{C.5}$$

where $\Psi^{(0)}_{i,\sigma}$ is evaluation of the polarisations basis at $p = (m, \vec{0})$ which we call the *rest frame polarisations basis*. From (C.4) we have

$$\sum_j \overline{L_{ji}(\alpha)}\Psi^{(0)}_{j,\sigma} = \sum_{\sigma'} D^s_{\sigma,\sigma'}(\alpha)\Psi^{(0)}_{i,\sigma'}, \quad \forall \alpha \in SU(2). \tag{C.6}$$

It is easy to see that any rank-$(2s+1)$ $r \times (2s+1)$ matrix, satisfying (C.6) is a rest frame polarisations basis, i.e. it defines a unique polarisation basis via (C.5). One can also note that (C.6) just says that $\Psi^{(0)}$ intertwines representations of $\overline{L}|_{SU(2)}$ with D_s. Since any representations of $SU(2)$ can be embedded into siutable representation of $SL(2)$, there are plenty of examples of polarisation basisses for any spin.

The name comes from the fact that for a fixed σ $\Psi_{i,\sigma}$ plays the role of polarisation vector or spinor (depending on the field) in QED.

This formal notion allows to realise the one-particle statespace in a more covariant form.

Lemma C.3. *Let $L : SL(2) \to GL(r)$ be a representation of $SL(2)$ and Ψ be a spin-s polarisations basis in L. Let*

$$\mathcal{X}_{\Psi,m} = \bigoplus_r \mathcal{S}(\mathbb{R}^4)$$

be a space with a non-negative sesquilinear product

$$(f, f')_{\Psi,m} = \sum_{i,j,\sigma} \int_{\Gamma_m^+} d\mu(p)\overline{f_j(p)}\Psi_{j,\sigma}(p)\overline{\Psi_{i,\sigma}(p)}f'_i(p). \tag{C.7}$$

Then the mass-m spin-s one-particle statespace can be constructed from $\mathcal{X}$ by taking the quotient with respect to the zero-norm vectors and completing the result,

$$\mathcal{H}^{(s,m)} = \overline{\frac{\mathcal{X}_{\Psi,m}}{\{f \in \mathcal{X}_{\Psi,m} | (f,f)_{\Psi,m} = 0\}}} \tag{C.8}$$

with a unitary mapping $\rho_{\Psi,m} : \mathcal{X}_{\Psi,m} \to \mathcal{H}^{(s,m)}$

$$\rho_{\Psi,m}[f]_\sigma(p) = \sum_i \overline{\Psi_{i,\sigma}(p)} f_i(p). \tag{C.9}$$

Furthermore, define the representation $T_{\Psi,m}$ of $\overline{\mathfrak{P}}^\uparrow_+$ on $\mathcal{X}_{\Psi,m}$

$$(T_{\Psi,m}(\alpha,a)f)(p) = e^{-ipa} L(\alpha) f(\Pi(\alpha)^{-1}p) \tag{C.10}$$

then $\rho_{\Psi,m}$ intertwines U with $T_{\Psi,m}$.

Proof. First note that the image of $\rho_{\Psi,m}$ is dense (recall that rank of $\psi(p)$ is always $2s+1$) and

$$(\rho_{\Psi,m}[f], \rho_{\Psi,m}[f']) = (f,f')_{\Psi,m},$$

in particular

$$\ker \rho_{\Psi,m} = \{f \in \mathcal{X}_{\Psi,m} | (f,f)_{\Psi,m} = 0\}.$$

so (C.8) holds. By substituting (C.9) into (C.10) we get

$$(U(a,\alpha)\rho_{\Psi,m}[f])_\sigma(p) = \sum_{\sigma'} e^{-ipa} D^s_{\sigma,\sigma'}(V(\alpha,p)) \rho_{\psi,m}[f]_{\sigma'}(\Pi(\alpha)^{-1}p) =$$

$$\sum_{\sigma',j} e^{-ipa} D^s_{\sigma,\sigma'}(V(\alpha,p)) \overline{\Psi_{j,\sigma'}(\Pi(\alpha)^{-1}p)} f_j(\Pi(\alpha)^{-1}p) =$$

$$\sum_{j,k} e^{-ipa} L(\alpha)_{j,k} \overline{\Psi_{k,\sigma}(\Pi(\alpha)^{-1}p)} f_j(\Pi(\alpha)^{-1}p) =$$

$$\sum_{j,k} e^{-ipa} \overline{\Psi_{k,\sigma}(\Pi(\alpha)^{-1}p)} L(\alpha)_{kj} f_j(\Pi(\alpha)^{-1}p) =$$

$$(U(a,\alpha)\rho_{\Psi,m}[T_{\Psi,m}(\alpha,a)f])_\sigma(p)$$

which is the intertwining property. □

Remark C.4. The map $\rho_{\Psi,m}$ is a vector-valued distribution in the sense of Appendix B, so we may symbolically write

$$\rho_{\Psi,m}[f] = \sum_{i=1}^r \int d^4p \rho^i_{\Psi,m}(p) f_i(p).$$

We also have

$$U(\alpha,a)\rho_{\Psi,m}(p) = e^{-ipa} L(\alpha)^T \rho_{\Psi,m}(\Pi(\alpha)p),$$

Proof. $R(p)$ grows as a square root of p and all representations of $SL(2)$ are of polynomial growth [77], so we may assume that $||L(R(p))|| < C(1+||\vec{p}||^2)^n$. Then for $f \in \mathcal{X}$, such that $|f(p)| < C'(1+||\vec{p}||^2)^{-n-2}$ we have

$$(\psi,\psi)_{\Psi,m} < C'' \int_{\Gamma_m} d\mu(p)(1+||\vec{p}||^2)^{-4} < \infty,$$

which proves that ρ is continuous in the Schwartz norm.

To prove the transformation law take $f \in \mathcal{X}_{L,Y,m}$ and consider

$$U(\alpha, a)\rho_{\Psi,m}[\psi](p) = \rho_{\Psi,m}[T_{\Psi,m}\psi](p).$$

Then the desired statement follows from (C.10) and definition of operations on the distributions from (B). □

Remark C.5. The functions $\psi \in \mathcal{K}$ coincide with the covariant wave-functions in the terminology of [77] up to different notation and restriction to the cases when Y is an isometry. There it was shown, that the construction in the Lemma above is essentially the only way to realize the spin-s mass-m irreducible representation as a completion of a quotient (C.8) of a space of covariant (i.e. transforming according to (C.10)) smooth functions.

Remark C.6. In the interacting theories we deal with in this thesis only the translational invariance is preserved (we also assume some discrete symmetries in Sections 6 and 8 for convenience, but this restriction can be easily removed). In that case the projective representation of the symmetry group is just an ordinary unitary representation of $\mathbb{R}^4$ (up to projective equivalence) and almost nothing of the above (except for the vacuum existence and the energy positivity) applies. We however will treat one-particles state-spaces as perturbed ones of the Lorentz-invariant theories.

D General quantum fields

Most generally, we define a *quantum field* as an operator-valued distribution[67] with the test-function space $\bigoplus_r \mathcal{S}(\mathbb{R}^4)$ (see Remark B.1) and the domain $D \subset \mathcal{H}_{phys}$ in the physical statespace, which leaves D invariant. If $r = 1$ we will say that ϕ is scalar. There can be also several quantum fields with a common domain, but we will consider only one.

We say that a quantum field is real if

$$\phi[f] = \phi[\overline{f}]^*\big|_D.$$

We say that the quantum field ϕ is covariant with respect to the group G acting on the spacetime, if there is a unitary representation U of G acting on $\mathcal{H}_{phys}$ such that

$$L(g)\phi(g^{-1}(x)) = U(g)^{-1}\phi(x)U(g). \tag{D.1}$$

Here $L : G \to GL(r)$ is an r-dimensional representation of G which we assume to be trivial on translations.

For the interacting theories considered in the main body of this thesis, $G = \mathbb{R}^4$ and the definition of such quantum field theory stops here. But for use in 2.2.1 we also briefly discuss the Poincare-covariant (i.e. $G = \overline{\mathfrak{P}^{\uparrow}_{+}}$, see Appendix C) local quantum fields.

We say that ϕ is a *local bosonic quantum field*, if[68]

$$\phi(x)\phi(y) - \phi(y)\phi(x)$$

vanishes in any open region O such that

$$(x-y)^2 < 0, \forall x, y \in O.$$

If instead in all such regions

$$\phi(x)\phi(y) + \phi(y)\phi(x)$$

vanishes, we will say that ϕ is a *local fermionic quantum field*.

The locality and covariance, together with the definition of the operator-valued distributions and positivity of the energy[69] together are known under the name of the *Whightman(-Garding) axioms* [85]. We refer to [77, 4] for important results concerning such fields.

In the absence of locality and Lorentz-invariance the position-dependent quantum field is not very convenient to work with. Instead we use the partial Fourier transform

$$\phi(\vec{x}, t) = \int d^3\vec{p}\tilde{\phi}(\vec{p}, t)e^{i\vec{p}\vec{x}}. \tag{D.2}$$

[67] See the previous subsection

[68] The product of operator-valued distributions below should be understood as the tensor products introduced in Appendix B. The same applies to the notion of a distribution vanishing in an open region.

[69] More precisely, one requires that the generator of time-translations has positive spectrum

E Fock space and the free quantum fields

We start from an one-particle statespace $\mathcal{H}_1$ carrying an irreducible representation of the Poincare group as explained in Appendix C. Then the standard way to construct a *bosonic Fock space* (see e.g. [83]) is to consider

$$\mathcal{H}_{bos} = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n, \tag{E.1}$$

where

$$\mathcal{H}_n = \mathcal{H}_1^{\otimes_{Sym} n} = S_n \mathcal{H}_1^{\otimes n}$$

stands for the symmetric nth power of $\mathcal{H}_1$ and the symmetrisation operator S_n defined as

$$S_n = \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} \rho_n(s)$$

with summation going over the permutation group $\mathfrak{S}_n$ and ρ_n being its representation acting on $\mathcal{H}^{\otimes n}$ by the permutation of factors. In the presence of at least translational symmetries the vector Ω is identified with the vacuum introduced in Appendix C.

Let $D_F \subset \mathcal{H}_{bos}$ be a subspace formed by vectors which have finitely many non-zero components in the decomposition (E.1).

For any vector $\psi \in \mathcal{H}_1$ we can define the *annihilation*

$$b_-[\psi] : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes (n-1)}$$

$$b_-[\psi]\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n = \sqrt{n}(\psi, \psi_1) \otimes \psi_2 \otimes \cdots \otimes \psi_n$$

and *creation*

$$b_+[\psi] : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes (n+1)}$$

$$b_+[\psi]\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n = S_{n+1}\sqrt{n+1}\psi \otimes \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n$$

operators[70]. One can verify that the symmetric part of the tensor product is invariant under their action. Hence they induce unbounded operators on $\mathcal{H}_{bos}$ defined on $D_F$which they leave invariant.

Now, let us assume that we are dealing with a fully Poincare covariant quantum field theory and $\mathcal{H}_1$ carries a mass-m spin-s irreducible representation of $\mathfrak{P}$ (see C). Then from (C.1) and from the fact that the Schwartz functions are continuously embedded into $\mathcal{L}^2$ we may consider the operator-values distributions with test-functions space $\bigoplus_{2s+1} \mathcal{S}(\mathbb{R}^3)$

$$a[f] = b_- \left[\vec{p} \mapsto \frac{1}{\sqrt{(2\pi)^3 2\omega_{\vec{p}}^m}} \overline{f(\vec{p})} \right] \tag{E.2}$$

and

$$a^+[f] = b_+ \left[\vec{p} \mapsto \frac{1}{\sqrt{(2\pi)^3 2\omega_{\vec{p}}^m}} f(\vec{p}) \right]. \tag{E.3}$$

where $\omega_{\vec{p}}^m = \sqrt{m^2 + \vec{p}^2}$. In the first line the complex conjugation was added because of anti-linearity of b, and the scaling factor was introduced to achieve the standard commutational relations[71]

$$[a_\sigma(\vec{p}), a^+_{\sigma'}(\vec{p'})] = \delta(\vec{p} - \vec{p'})\delta_{\sigma,\sigma'}.$$

[70] In these definitions we treat the vacuum space $\mathbb{C}\Omega$ as a zeroth tensor power of $\mathcal{H}_1$

[71] The commuatators should be understood in the sense of a tensor product of operator valued distributions as defined in B and the equality of course holds only on the domain of definition of that distributions.

These operators are adjoints of one another as quadratic forms on D_F. From B we can define vector-valued distributions

$$|\vec{p}_1,\sigma_1;\cdots;\vec{p}_n,\sigma_n>=(2\pi)^{\frac{3n}{2}}\left(\prod_{i=1}^n\sqrt{\omega^m_{2\vec{p}_i}}\right)a_{\sigma_1}(\vec{p}_1)^+\cdots a^+_{\sigma_n}(\vec{p}_n)|\Omega>. \tag{E.4}$$

with scalar product

$$<\vec{p}'_1,\sigma'_1;\cdots;\vec{p}'_{n'},\sigma'_{n'}|\vec{p}_1,\sigma_1;\cdots;\vec{p}_n,\sigma_n>=\delta_{n,n'}\sum_{s\in\mathfrak{S}_n}\prod_{i=1}^n(2\pi)^3 2\omega^m_{\vec{p}_i}\delta(\vec{p}_i-\vec{p}'_{s_i})\delta_{\sigma_i,\sigma'_{s_i}}$$

To define a covariant quantum field, we use the covariant wavefunctions introduced in Lemma C.3 and Remarks C.5-C.4. In particular, from the latter for a fixed r-dimensional representation L of $GL(2)$, its dual L^*,

$$L^*(\alpha)=L(\alpha^{-1})^T$$

and a spin-s polarisations basis we have a vector-valued field

$$\rho_{\Psi,m}:\bigoplus_r\mathcal{S}(\mathbb{R}^4)\to\mathcal{H}_1.$$

Let

$$\Phi(x)=\int d^4p e^{ipx}\rho_{\Psi,m}(p) \tag{E.5}$$

be its Fourier transform. From Remark C.4, we have

$$L(g)\Phi(\Pi(\alpha^{-1}(x-a))=U(a,\alpha)^{-1}\Phi(x)$$

Then the free creation field defined as

$$\phi_{0+}=b_+\circ\Phi \tag{E.6}$$

is a covariant quantum field. The free annihilation field can be defined as its adjoint restricted to D_F

$$\phi_{0-}=\overline{\phi_{0+}}. \tag{E.7}$$

This field is also covariant, but it transforms according to the complex-conjugated representation $\overline{L}$, $\overline{L}(\alpha)=\overline{L(\alpha)}$. Neither of these fields is local (for all direct computations of the commutators showing locality or non-locality of the free fields we refer to [80]). For a scalar spinless field, $r=1$, $s=0$ the representations L are both trivial and hence identical, so we can form a sum

$$\phi_0(x)=\phi_{0+}+\phi_{0-} \tag{E.8}$$

which is a real local bosonic quantum field. Equarions (E.5-E.8) with (C.9) and (E.2-E.3) together give the standard form

$$\phi_0(\vec{x},t)=\int\frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2\omega^m_{\vec{p}}}}\left(a_{\vec{p}}e^{-i(\omega^m_{\vec{p}}t-\vec{p}\vec{x})}+a^+_{\vec{p}}e^{i(\omega^m_{\vec{p}}t-\vec{p}\vec{x})}\right). \tag{E.9}$$

The same idea works for L being a representation of $SO(1,3)$ (rather than its cover), then all matrices $L(\alpha)$ are real in a suitable basis, so $L\cong\overline{L}$ and (E.8) can be combined into

$$\phi^i_0(\vec{x},t)=\sum_\sigma\int\frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2\omega^m_{\vec{p}}}}\left(a_{\vec{p},\sigma}e^{-i(\omega^m_{\vec{p}}t-\vec{p}\vec{x})}+a^+_{\vec{p},\sigma}e^{i(\omega^m_{\vec{p}}t-\vec{p}\vec{x})}\right)\Psi_{i,\sigma}(p). \tag{E.10}$$

Here we also used that for such L spin is always integer and matrices Y can be assumed to be real. Such fields are also automatically local bosonic.

For half-integer spin fields locality is possible only in the fermionic case, according to the famous spin-statistics theorem [4] (in the free case we discuss here it can be also proven directly, [80]). Free fermionic space and free fermionic fields can be constructed along the same lines, replacing the symmetrisation operator with one of alteration. We refer to [77, 83] for the details. To make a local field one still will need to sum creation and annihilation operators. To make it possible, we assume that $L \cong \overline{L}$. Since this is not the case for a generic representation of $SL(2)$, we may need to replace $L \to \overline{L}$.

Remark E.1. Usually the polarisation vectors are also restricted by a convenient normalisation condition. In our discussion it will not play any role.

Remark E.2. The interacting quantum fields we are discussing in this thesis are neither local nor Lorentz-invariant. As a result after the renormalisation the free fields can also loose that properties. But, as it was already mentioned in Remark C.6, we consider only the fields which are perturbations of the usual one. So, as a generic real scalar quantum field we take (E.9) with broken Lorentz symmetry,

$$\phi_0(\vec{x},t) = \int \frac{d^3\vec{p}\sqrt{Z(\vec{p})}}{\sqrt{(2\pi)^3 2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{-i(\omega_{\vec{p}}t - \vec{p}\vec{x})} + a^+_{\vec{p}} e^{i(\omega_{\vec{p}}t - \vec{p}\vec{x})} \right), \tag{E.11}$$

where $\omega_{\vec{p}}$ is a generic dispersion relation function and $Z(\vec{p})$ is an arbitrary renormalization factor. We require that $Z(\vec{p}), \omega_{\vec{p}}, \omega^{-1}_{\vec{p}} \in \Theta_M(\mathbb{R}^3)$ (see Appendix B).

We will also often use the notation

$$\tilde{\phi}_0(\vec{p},t) = \tilde{\phi}_{0+}(-\vec{p},t) + \tilde{\phi}_{0-}(\vec{p},t), \tag{E.12}$$

where the partial Fourier transform $\tilde{\phi}_0(\vec{p},t)$ is defined by (D.2), while

$$\tilde{\phi}_{0+}(\vec{p},t) = \frac{\sqrt{Z(\vec{p})}}{\sqrt{(2\pi)^3 2\omega_{\vec{p}}}} a^+_{\vec{p}} e^{-i\omega_{\vec{p}}t} = \tilde{\phi}_{0+}(\vec{p},0) e^{-i\omega_{\vec{p}}t} \tag{E.13}$$

and

$$\tilde{\phi}_{0-}(\vec{p},t) = \frac{\sqrt{Z(\vec{p})}}{\sqrt{(2\pi)^3 2\omega_{\vec{p}}}} a_{\vec{p}} e^{-i\omega_{\vec{p}}t} = \tilde{\phi}_{0-}(\vec{p},0) e^{+i\omega_{\vec{p}}t} \tag{E.14}$$

are its *creation* and *annihilation* parts respectively.

Remark E.3. We have constructed the quantum fields and the creation and annihilation operator-valued distributions acting on the domain D_F of states with finite number of particles. In practice it is more convenient to work with the domain $D_{\mathcal{S}}$ of states with finite number of particles and wave-functions of Schwartz class, that is the subspace spanned by image of vector-valued distributions

$$|\vec{p}_1, \sigma_1; \ldots; \vec{p}_n, \sigma_n >,$$

since they are more regular. Outside of this appendix we always assume the all operator-valued distributions are considered on $D_{\mathcal{S}}$. Clearly, if $\psi \in \mathcal{H}_1 \cap D_{\mathcal{S}}$, then $b_\pm[\psi]$ leaves $\mathcal{S}$ invariant. Since only such ψ appear in (E.2- E.3), all operator-valued distributions we defined leave $D_{\mathcal{S}}$.

Remark E.4. It is well-known[72] that unlike general quantum fields, the free fields are well-defined at fixed moments of time. This is especially clear in the "partial" Fourier transform form. In

[72] A standard treatment of this matter can be found, for example, in [83]. It is however not appropriate for our discussion.

fact formally substituting $t = 0$ into (E.13-E.14), we see that $\tilde{\phi}_{0-}(\vec{p}, 0)$ are well-defined operator distributions (as they are the products of well defined distributions $a_{\vec{p}}$ and $a^+_{\vec{p}}$ with a function in $\Theta_M(\mathbb{R}^3)$) on $\mathbb{R}^3$. Moreover, extending the distributions $\tilde{\phi}_{0+}(\vec{p}, 0)$ $\tilde{\phi}_{0-}(\vec{p}, 0)$ to $\mathbb{R}^4$ we can define

$$\tilde{\phi}'_{0+}(\vec{p}, t) = \tilde{\phi}_{0+}(\vec{p}, 0)e^{-i\omega_{\vec{p}}t} \tag{E.15}$$

and

$$\tilde{\phi}'_{0-}(\vec{p}, t) = \tilde{\phi}_{0-}(\vec{p}, 0)e^{+i\omega_{\vec{p}}t}. \tag{E.16}$$

Moreover, by evaluating $\tilde{\phi}'_{0\pm}$ on an arbitrary test-function one immediately gets that $\tilde{\phi}'_{0\pm} = \tilde{\phi}_{0\pm}$ Similarly, using the definition of a tensor product of operator-valued distributions (see Appendix B) and its uniqueness one can prove that

$$\tilde{\phi}_{\alpha_1}(\vec{p}_1, t_1) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t_n) = e^{-i\sum_{j=1}^n \alpha_j \omega_{\vec{p}_j}(t_j - t)} \tilde{\phi}_{\alpha_1}(\vec{p}_1, t) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t) \tag{E.17}$$

with $\alpha_j = \pm$. Again, here the fields $\tilde{\phi}_{\alpha_i}(\vec{p}_i, t)$ in the right hand side should be understood as distributions on $\mathbb{R}^3$. This allows to define a product of the distribution above and a generic distribution $A \in \mathcal{S}'(\mathbb{R}^n)$ depending on time stamps as

$$A(t_1, \dots, t_n)\tilde{\phi}_{\alpha_1}(\vec{p}_1, t_1) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t_n) = \tag{E.18}$$

$$e^{-i\sum_{j=1}^n \alpha_j \omega_{\vec{p}_j}(t_j - t)} \big(A(t_1, \dots, t_n)\tilde{\phi}_{\alpha_1}(\vec{p}_1, t) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t)\big),$$

where the product in the brackets is the tensor product of two distributions, while the one outside of the brackets is multiplication of a distribution by a function from $\Theta_M(\mathbb{R}^{4n})$.

In the above we have shown that the distribution in the right hand sides (E.18) is well-defined, but it may be enough to convince one that it is a natural interpretation of left hand side. The same definition can be also motivated in one more way. If we fix two vectors $\Psi, \Psi' \in D_{\mathcal{S}}$ (or D_F), then by (E.17) we get

$$(\Psi, \tilde{\phi}_{\alpha_1}(\vec{p}_1, t_1) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t_n)\Psi') = e^{-i\sum_{j=1}^n \alpha_j \omega_{\vec{p}_j}(t_j - t)} (\Psi, \tilde{\phi}_{\alpha_1}(\vec{p}_1, t) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t)\Psi'). \tag{E.19}$$

Applying Theorem B.10 twice (first to compute the wavefronts set of (E.17) and when to prove existence of the product) we get that the product

$$A(t_1, \dots, t_n)(\Psi, \tilde{\phi}_{\alpha_1}(\vec{p}_1, t_1) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t_n)\Psi')$$

is a well-defined distribution. At the same time, assuming the definition (E.18), we see that[73]

$$(\Psi, A(t_1, \dots, t_n)\tilde{\phi}_{\alpha_1}(\vec{p}_1, t_1) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t_n)\Psi') =$$

$$e^{-i\sum_{j=1}^n \alpha_j \omega_{\vec{p}_j}(t_j - t)} (\Psi, A(t_1, \dots, t_n)\tilde{\phi}_{\alpha_1}(\vec{p}_1, t_1) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t_n)\Psi') =$$

$$e^{-i\sum_{j=1}^n \alpha_j \omega_{\vec{p}_j}(t_j - t)} A(t_1, \dots, t_n)(\Psi, \tilde{\phi}_{\alpha_1}(\vec{p}_1, t) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t)\Psi').$$

So, (E.18) defines the product

$$A(t_1, \dots, t_n)\tilde{\phi}_{\alpha_1}(\vec{p}_1, t_1) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t_n)$$

such, that it agrees with the standard product of numerical distributions on the level of the quadratic forms,

$$(\Psi, A(t_1, \dots, t_n)\tilde{\phi}_{\alpha_1}(\vec{p}_1, t_1) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t_n)\Psi') = A(t_1, \dots, t_n)(\Psi, \tilde{\phi}_{\alpha_1}(\vec{p}_1, t_1) \cdots \tilde{\phi}_{\alpha_n}(\vec{p}_n, t_n)\Psi'). \tag{E.20}$$

[73]The last line is straightforward if integrated with a factorised Schwartz function of the form $f(t_1, \dots, t_n)g(\vec{p}_1, \dots, \vec{p}_n)$, $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in \mathcal{S}(\mathbb{R}^{3n})$ and then using the definition and uniqueness of the tensor product of operator-valued distributions (see Appendix B).

Finally, by evaluating this matrix element on a test-function f and using the Wick Theorem (Theorem F.1) we get

$$\int e^{-i\sum_{j=1}^n \alpha_j \omega_{\vec{p}_j}(t_j-t)} A(t_1,\dots,t_n)(\Psi, \tilde{\phi}_{\alpha_1}(\vec{p}_1,t)\cdots\tilde{\phi}_{\alpha_n}(\vec{p}_n,t)\Psi')f(t_1,\dots,t_n,\vec{p}_1,\dots,\vec{p}_n)\prod_{i=1}^n dt_i d\vec{p}_i =$$

$$\sum_{r=0}^n\sum_{s=0}^n \int (\Psi, \prod_{j=1}^r a^+_{\vec{k}_j}\prod_{j=1}^s a_{\vec{k}_j}\Psi')f'_{r,s}(\vec{k}'_1,\dots,\vec{k}'_r;\vec{k}_1,\dots,\vec{k}_s)\prod_{i=1}^r d\vec{k}'_i\prod_{i=1}^s d\vec{k}_i,$$

where $f'_{r,s}$ are results of evaluation of the product of the distribution A and all commutators appearing in the normal ordering according to the Wick theorem[74] on f. By Theorem B.2 $f'_{r,s}$ is Schwartz. But then by theorem X.44 of [83] the operator, agreeing with

$$\int e^{-i\sum_{j=1}^n \alpha_j \omega_{\vec{p}_j}(t_j-t)} A(t_1,\dots,t_n)\tilde{\phi}_{\alpha_1}(\vec{p}_1,t)\cdots\tilde{\phi}_{\alpha_n}(\vec{p}_n,t)f(t_1,\dots,t_n,\vec{p}_1,\dots,\vec{p}_n)\prod_{i=1}^n dt_i d\vec{p}_i$$

is unique, and thus (E.20) fixes the product in its left hand side uniquely.

The argument above can be generalised to arbitrary spin straightforwardly.

[74] This product is clearly well-defined, as all distributions are in different variavles

F Wick ordering, Wick theorems and propagators

Let us start from very generic setting. Suppose that there is a family of unbounded operators A_i acting on the same dense domain $D \in \mathcal{H}$ which they leave invariant, let $\Omega \in D$ be a vector of norm one, and assume that each operator A_i can be decomposed into a "creation part" $A_{i,+}$ and $A_{i,i}$,

$$A_i = A_{i,+} + A_{i,-},$$

$$A_{i,-}\Omega = 0, \qquad A_{i,+}^*\Omega = 0.$$

Here creation, annihilation parts and their adjoints are assumed to be defined on the same domain D. Finally, we assume that all creation operators and all annihilation operators commute among their classes, while commutator between creation and annihilation operators is proportional to the identity[75]

$$[A_{i,+}, A_{j,+}] = [A_{i,-}, A_{i,-}]$$

$$[A_{i,+}, A_{j,-}] = C_{ij}\mathbf{1}_{\mathcal{H}}, C_{ij} \in \mathbb{C}.$$

Then we define the *Wick product*, or the *normal oredered product* $: A_1 \cdots A_n :$ as ordinary product of the operators $A_1 \dots A_n$ in with all creation operators moved to the left by hands,

$$: A_1 \cdots A_n := \sum_{I \subset \{1,2,\dots,N\}} \prod_{i \in I} A_{i,+} \prod_{i \in \{1,2,\dots,n\} \setminus I} A_{i,-}.$$

By construction,

$$(\Omega, : A_1 \cdots A_n : \Omega) = 0.$$

The following statement, despite being very simple, plays a fundamental role in QFT

Theorem F.1 (Wick theorem [94])**.** *Let operators $A_1, \dots, A_n$ be as above. Then*

$$A_1 \cdots A_n = \sum_{\substack{\{1,\dots,n\} = J \sqcup \bigsqcup_{j=1}^k \{i_{j,1}, i_{j_2}\} \\ i_{j,1} < i_{j_2}}} \prod_{j=1}^k C_{i_{j,1} i_{j,2}} : \prod_{l \in J} A_l : . \tag{F.1}$$

Here the sum goes over all "pairngs", i.e. all possible ways to select k (with summation over k) k non-intersecting pairs among the operators $A_1, \dots A_n$ and J denotes the set of all operators that did not appear in any of the pairs.

An obvious consequence is that

$$(\Omega, A_1 \cdots A_n \Omega) = \sum_{\substack{\{1,\dots,n\} = \bigsqcup_j \{i_{j,1}, i_{j_2}\} \\ i_{j,1} < i_{j_2}}} \prod_j C_{i_{j,1} i_{j,2}} =$$

$$\sum_{\substack{\{1,\dots,n\} \bigsqcup_j \{i_{j,1}, i_{j_2}\} \\ i_{j,1} < i_{j_2}}} \prod_j (\Omega, A_{i_{j,1}} A_{i_{j,2}} \Omega),$$

i.e. vacuum expectation vale of a product of such operators is products of expectation value of pairs of that operators, summed over all possible pairings.

[75] For this subsection all relations should be understood as holding on the domain D.

Remark F.2. A natural example of operators A_i is provided by the free quantum fields (see Appendix E) evaluated on the test-functions decomposed as in (E.12). Then using the uniqueness of a tensor product of the operator-valued distributions (Appendix B) one gets a well-defined notion of normal-ordered (tensor) product of the free quantum fields themselves. So, Theorem F.1 holds for $A_i = \tilde{\phi}(\vec{p}_i, t_i)$ and

$$C_{ij} = [\tilde{\phi}_{0+}(\vec{p}_i, t_i), \tilde{\phi}_{0-}(\vec{p}_j, t_j)] = (\Omega, \tilde{\phi}_{0+}(\vec{p}_i, t_i)\tilde{\phi}_{0-}(\vec{p}_j, t_j)\Omega).$$

By reverse partial Fourier transform we can get a similar expression for $A_i = \phi(x_i)$.

Remark F.3. The Wick product of quantum fields is important, because in the decomposition of the tensor product

$$\phi(x_1) \cdots \phi(x_n)$$

according to the Wick theorem all singularities in the limit of coinciding points $x_i \to x$ are absorbed in the commutators, while the normal-ordered product has a well-defined limit [100]

$$: \phi(x)^n : .$$

Theorem F.1 can be also formulated for the time-ordered product (with straightforward generalization to anyhow ordered product, including the covariant time-ordered product $\hat{\mathbf{T}}$ of 3.2 and the integrated-path-ordered product $\mathcal{P}$ introduced in 3.1.3):

Theorem F.4 (Second Wick theorem [94])**.** *Let operators $A_1, \dots, A_n$ be as above. Then*

$$\mathbf{T}\{A_1(t_1) \cdots A_n(t_n)\}$$

$$= \sum_{\substack{\{1,\dots,n\} = J \sqcup \bigsqcup_{j=1}^{k} \{i_{j,1}, i_{j_2}\} \\ i_{j,1} < i_{j_2}}} \prod_{j=1}^{k} \left(\Omega, \mathbf{T}\left\{A_{i_{j,1}}(t_{i_{j,1}}) A_{i_{j,2}}(t_{i_{j,2}})\right\}\Omega\right) : \prod_{l \in J} A_l : .$$

Here we formulated the theorem for operators depending on t, but we use its generalization to the case of operator-valued distributions along the same lines as explained above for Theorem F.1.

Again, Theorem F.4 allows one to calculate the vacuum expectation values of time-ordered products easily.

Remark F.5. As in Remark F.2, Theorem F.4 can be applied to timeordered product of the quantum fields. In particular, we need it for $A_i = \tilde{\phi}_{0\alpha_i}(\vec{p}_i, t_i)$. Then

$$< 0|\mathbf{T}\left\{\tilde{\phi}_{0\alpha_i}(\vec{p}_i, t_i)\tilde{\phi}_{0\alpha_j}(\vec{p}_j, t_j)\right\}|0 >= \tilde{G}^0_{\alpha_i\alpha_j}(\vec{p}_i, t_i - t_j)\delta(\vec{p}_i + \vec{p}'_j),$$

where

$$\tilde{G}^0_{\alpha\beta}(\vec{p}, t) = \delta_{\alpha,-\beta}\theta(-\alpha t)\frac{1}{2\omega_{\vec{p}}(2\pi)^3}e^{-i\omega_{\vec{p}}|t|} \tag{F.2}$$

is the so-called propagator.

It is convenient to introduce also the propagator in 4-momentum space (rather than the mixed "time-momentum" space in which (F.2) is written)

$$\tilde{G}^0_{\alpha\beta}(\vec{p}, t) = \int d\omega \mathcal{G}^0_{\alpha\beta}(\vec{p}, \omega)e^{-i\omega t}, \tag{F.3}$$

with

$$\mathcal{G}^0_{\alpha\beta}(\vec{p}, \omega) = \delta_{\alpha,-\beta}\frac{-i\alpha}{2\omega_{\vec{p}}(2\pi)^4(\omega + \alpha(\omega_{\vec{p}} - i0))}. \tag{F.4}$$

In ordinary QFT a crucial role is played by the Feynman propagator

$$< 0|\mathbf{T}\left\{\phi(x)\phi(x')\right\}|0> .$$

Substituting (D.2) and (E.12) we have

$$< 0|\mathbf{T}\left\{\phi(x)\phi(x')\right\}|0>=\int d^3\vec{p}d\omega e^{-i(\omega t+\vec{p}\vec{x})}\mathcal{G}_F^0(\vec{p},\omega), \tag{F.5}$$

$$\mathcal{G}_F^{(0)}(\vec{p},\omega)=\sum_{\alpha\beta}\mathcal{G}_{\alpha\beta}^{(0)}(\vec{p},\omega)=\frac{i}{(2\pi)^4(\omega^2-\omega_{\vec{p}}^2+i0)}. \tag{F.6}$$

We also need the vacuum correlators, including the first derivatives of the fields. Clearly for $j,k\in\{1,2,3\}$,

$$< 0|\mathbf{T}\left\{\partial_{x^j}\phi(x)\phi(x')\right\}|0>=\partial_{x^j}< 0|\mathbf{T}\left\{\phi(x)\phi(x')\right\}|0>$$

and

$$< 0|\mathbf{T}\left\{\partial_{x^j}\phi(x)\phi(x')\right\}|0>=\partial_{x^j,x'^k}< 0|\mathbf{T}\left\{\phi(x)\partial_{x'^k}\phi(x')\right\}|0> .$$

From

$$\partial_t\tilde{\phi_0}(\vec{p},t)=i\omega_{\vec{p}}(\tilde{\phi}_{0+}(\vec{p},t)-\tilde{\phi}_{0-}(\vec{p},t))$$

it follows

$$< 0|\mathbf{T}\left\{\partial_{x^0}\phi(x)\phi(x')\right\}|0>=i\omega_{\vec{p}}\int d^3\vec{p}d\omega-\vec{p}^j e^{-i(\omega t+\vec{p}\vec{x})}\sum_{\alpha\beta}\alpha\mathcal{G}_{\alpha\beta}^0(\vec{p},\omega).$$

But

$$\omega_{\vec{p}}\sum_{\alpha\beta}\alpha\mathcal{G}_{\alpha\beta}^{(0)}(\vec{p},\omega)=-\frac{i\omega}{(2\pi)^4(\omega^2-\omega_{\vec{p}}^2+i0)}.$$

for all values of $\mu\in\{0,1,2,3\}$. Similarly, we have

$$< 0|\mathbf{T}\left\{\partial_{x^0}\phi(x)\partial_{x'^j}\phi(x')\right\}|0>=\partial_{x^0x'^j}< 0|\mathbf{T}\left\{\phi(x)\phi(x')\right\}|0> .$$

Finally, for the correlator of two derivatives with respect to the time we have:

$$< 0|\mathbf{T}\left\{\partial_{x^0}\phi(x)\partial_{x'^0}\phi(x')\right\}|0>=$$

$$-\omega_{\vec{p}}^2\int d^3\vec{p}d\omega\vec{p}^j e^{-i(\omega t+\vec{p}\vec{x})}\sum_{\alpha\beta}\alpha\beta\mathcal{G}_{\alpha\beta}^0(\vec{p},\omega)=\omega_{\vec{p}}^2\int d^3\vec{p}d\omega\vec{p}^j e^{-i(\omega t+\vec{p}\vec{x})}\mathcal{G}_F^0(\vec{p},\omega)=$$

$$\int d^3\vec{p}d\omega\vec{p}^j\omega^2 e^{-i(\omega t+\vec{p}\vec{x})}\mathcal{G}_F^0(\vec{p},\omega)-\int d^3\vec{p}d\omega\vec{p}^j(\omega^2-\omega_{\vec{p}}^2e^{-i(\omega t+\vec{p}\vec{x})}\mathcal{G}_F^0(\vec{p},\omega)=$$

$$\partial_{x^0}\partial_{x'^0}< 0|\mathbf{T}\left\{\phi(x)\phi(x')\right\}|0>+i\delta(x-x').$$

Summing up,

$$< 0|\mathbf{T}\left\{\partial_{x^\mu}\phi(x)\phi(x')\right\}|0>=\partial_{x^\mu}< 0|\mathbf{T}\left\{\phi(x)\phi(x')\right\}|0> \tag{F.7}$$

$$< 0|\mathbf{T}\left\{\partial_{x^\mu}\phi(x)\partial_{x'^\nu}\phi(x')\right\}|0>=\partial_{x^\mu}\partial_{x'^\nu}< 0|\mathbf{T}\left\{\phi(x)\phi(x')\right\}|0>+i\delta(x-x')\delta_{\mu,0}\delta_{\nu,0} \tag{F.8}$$

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