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A Clifford Algebra Approach to Gesture Recognition

Lorenzo Luzzi

RELATORE
Dott. Paolo Roselli

COORDINATORE
Prof. Andrea Braides

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Contents

Chapter 1

Introduction

In the last years, the development of the so called Artificial Intelligence (AI), and the need to optimize computer processes, got the attention of many Information Technology (IT) researchers, engineers and mathematicians on "gesture recognition" techniques. As in this work we address to an audience that is not necessarily introduced to advanced mathematical notions (such as, for instance, tensor products, ideals of rings, etc.), we try to use a mathematical formalism as accessible as possible, trying to approach the problem through tools as elementary as possible.

A few years ago, during the study of Clifford algebras, already used by other IT applications (see [4]), a research team from the Université catholique de Louvain (UCL), born from a collaboration between P. Roselli and J. Vanderdonck, has seen as possible the application of Clifford Algebras to the field of "gesture recognition", which investigates efficient algorithms to make an AI device able to evaluate similarity between two different motions (that is, gestures).
Until now, most of the recognition algorithms rescale and move the input gestures in a predetermined position before starting the effective computations. Such data preprocessing is time consuming. So, the study of suitable similarity recognizers implies that their algorithms should be invariant with respect to almost[1] all transformations that preserve similarity, like translations, rotations, or scaling. Clifford algebra can solve this problem. As a matter of fact, it allowed us to initially write an algorithm for the 2 dimensional case, the !FTL algorithm, where this preliminary data processing is not needed, thanks to the intrinsic invariance properties of our notion of "shape" given through a ratio in a Clifford algebra.
In mathematical terms, the foregoing research team implemented an opera-

[1]We point out that, in this work, we don't consider some similarities as, for example, reflections.

tor, the *shape distance* $D_S(f, g)$, where f, g are the input motions, such that $D_S(f, g) = D_S(f, T(g))$ and T is a transformation that preserves similarity.
More informations about !FTL[2] can be found in [16]; however, those results are just a starting point to approach the recognition problem and an interested reader can see possible future development in the chapter dedicated to our conclusions. Despite the important improvements, there are some topics that need a deeper mathematical investigation. My work started here.

As a matter of fact, in [16] any reference concerning the use of geometric algebra has been intentionally bypassed, avoiding excessive information about Clifford algebras not requested by a public mainly interested to IT algorithms.
However, if this has been possible for the particular development of !FTL, any extension of it to other contexts cannot be separated from the use of Clifford algebra.
Here, we can use a more general formalism than that used in [16], and we can go deeper in mathematical details to move on, beyond !FTL algorithm. This work can be divided into four main parts.

1) A particular axiomatics for the Clifford algebra, mathematically equivalent to the non-degenerate case, (one can refer to [3]); however our approach can profit readers having just some basic knowledge about complex numbers and elementary linear algebra.

2) A mathematical definition of "shape" of a 2D planar gesture followed by a proper definition of the !FTL, the proof of its convergence, an improved version (!WFTL) and, as a consequence, a well defined operator that measures the "dissimilarity" of two gestures (also called "gesture recognizer").

3) The extension of the two-dimensional results obtained in part 2) to "2.5D spaces", i.e. the recognition of gestures located no longer on a Euclidean plane but on any regular surface.

4) One of the possible generalizations of all previous definitions and results seen in 2D to higher dimensions, and the reason why we chose to go toward this specific direction.

We point out that, in this work, our main original contributions are: Theorem 3.1.9, Theorem 3.1.10, Corollary 3.1.11, Definition 3.1.12, Theorem 3.1.14,

[2]FTL stands for "Faster Than Light". Such an acronym, was chosen by one of the authors of [16] for his amazement facing the first astonishing speed results. To compensate for this exuberance, the others authors chose to use the negation mark "!" as a prefix.

Proposition 4.1.5, Lemma 4.1.6, Section 4.1.3, Section 4.2 and Chapter 5.
As we have already explained, Section 2.1 and, as a consequence, the appendix A, are our slightly original approaches to Clifford algebras, necessary for the development of the aforementioned results.
Besides, all these results open up to other possible researches. We postpone this discussion to our conclusions in Chapter 6.

Chapter 2

Clifford Algebras

Clifford algebras have been defined and generalized in several different ways (see for instance [3],[4],[8],[13]).
The varieties of approaches depends on the different applications of those algebras. This multiplicity of approaches and definitions, besides witnessing the richness and adaptability of Clifford Algebras, has also produced some problems concerning conflicting notations and deductions of results. In order to avoid such problems and to keep this work self-contained as much as possible, we explicit here our particular approach to Clifford Algebras. This is mainly suited for the results presented in this work and it is addressed to the particular audience those results were presented to.

2.1 Our Axiomatics for Clifford Algebras

Definition 2.1.1. Let S be a set. An algebra *generated* by S, is an associative algebra $\mathfrak{A}$ where every element can be expressed as a polynomial in the elements of S.

Definition 2.1.2. Let V be a n-dimensional vector space on the field $\mathbb{K}$, whose characteristic is not equal to 2.
An *algebra generated by* V is an unitary associative algebra $\mathfrak{A}$ over $\mathbb{K}$ generated by V. Let $r : V \to \mathfrak{A}$ be the linear monomorphism (an injective homomorphism) corresponding to the injection of V as a linear subspace of $\mathfrak{A}$.

Remark 2.1.1. If $s : \mathbb{K} \to \mathfrak{A}$ is a linear function such that $s(\alpha) = \alpha 1$, the unit element 1 in $\mathfrak{A}$ then s is an algebra monomorphism.

Remark 2.1.2. With abuse of notation we will ignore the r, s when we will talk about $s(\mathbb{K})$ and $r(V)$ within $\mathfrak{A}$, referring to these simply as $\mathbb{K}$ (scalars) and V (vectors).

Definition 2.1.3. Let V be a n-dimensional vector space on $\mathbb{K}$, with a non-degenerate quadratic form Q. Then the **Clifford algebra** (or *geometric algebra*) over V, $Cl_Q(V)$, is an algebra generated by V satisfying these two axioms:

(C) $\forall\, v \in V,\ v^2 = Q(v)$ (Clifford or contraction axiom)

(G) $\forall\, e_1, \cdots, e_j$ mutually orthogonal (i. e. $\forall m \neq n,\quad e_m \perp e_n$[1]) and linearly independent, $e_1 \cdots e_j \notin \mathbb{K}$ (Grassmann[2] or extension axiom)

<u>*Remark*</u> 2.1.3.

- In (G), if Q is positive definite and $e_1, \cdots, e_j$ are mutually orthogonal, then $e_1, \cdots, e_j$ are necessarily l.i. (linearly independent), too. So, in this case, we may just require in (G) the orthogonality condition. (This is still true if Q is negative definite too.)
- We can however not require the quadratic form Q at the beginning. In this case, one can replace the (C) with

$$\forall\, v \in V,\ v^2 \in \mathbb{K}\ ,$$

and then observe that the map $v \to v^2$ induces a quadratic form.

<u>*Remark*</u> 2.1.4.

- We must demonstrate the existence and uniqueness of $Cl_Q(V)$. Evidence of this will be provided later.
- As every single vector $v \in V$ verifies (G),this implies that $\mathbb{K} \cap V = \{0\}$.
- $\forall \alpha \in \mathbb{K}, v \in V \qquad v\alpha = \alpha v \quad (v\alpha = v(\alpha 1))$

<u>*Example*</u> 1. The axiom (G) is very interesting. Not only because it is a useful tool to prove some of the main features of the algebra, but also because it is needed to avoid degenerate examples like the following one:
$\mathbb{K}$ can be an algebra generated by $\mathbb{K}$ itself, but it cannot be a Clifford algebra thanks to (G), unless V is trivial, that is $V = \{0\}$ (This makes sense only if $\mathbb{K}$ is not trivial).

<u>*Remark*</u> 2.1.5. If we consider the bilinear form $\langle x, y\rangle$ associated to Q, we have that:
$2\langle x, y\rangle = Q(x+y) - (Q(x)+Q(y)) = (\text{in } Cl_Q(V)) = (x+y)^2 - (x^2+y^2) = xy+yx$, $\forall\, x, y \in V$.

[1]With respect to the bilinear $\langle\, ,\, \rangle$ form induced by Q, hence with $u \perp v$ we mean that $\langle u, v\rangle = 0$.

[2]This axiom is related to Grassmann, because of definition 52, at pag. 29 in [7].

Notation. We will use the following notations:

- $\alpha, \beta, \gamma \cdots$ for *scalars*;
- $a, b, c, \cdots, u, v, w, x, y, z, \cdots$ for *vectors*;
- $A, B, C, \cdots$ for *elements of* $Cl_Q(V)$, also named *multivectors*;
- AB (juxtaposition) for the algebra product that we call *Clifford (or geometric) product* ;
- $\langle x, y\rangle$ or $x \cdot y$ for the unique symmetric *bilinear form* such that $\langle x, x\rangle = Q(x)$.

Remark 2.1.6. It is useful to remark that $x \cdot y = 0 \;\Leftrightarrow\; xy = -yx$.

Definition 2.1.4. If $x_1, \cdots, x_k \in V$, we define the **completely antisymmetric product** of k ordered vectors $x_1, \cdots, x_k$ as

$$[x_1, \cdots, x_k] = \frac{1}{k!} \sum_{\sigma \in S_k} sign(\sigma) x_{\sigma(1)} \cdot \cdots \cdot x_{\sigma(k)},$$

where S_k is the symmetric group, i.e. the space of all the permutations of k elements and $sign(\sigma)$ is the sign of $\sigma \in S_n$ (Sometimes, to facilitate the reading, we denote the sign of σ with $|\sigma|$).

Remark 2.1.7.

- $[x, y] = \frac{xy-yx}{2}$
- $[x, x] = 0$
- $[x, y] = -[y, x]$
- $\alpha[x, y] = [\alpha x, y] = [x, \alpha y]$
- $[x + z, y] = [x, y] + [z, y]$

We can extend the foregoing results to any dimension.

Proposition 2.1.5. *For each* $x_1, \cdots x_h, y \in V$, $\forall \alpha \in \mathbb{R}$, *we have that*

(i) $[x_{\tau(1)}, \cdots, x_{\tau(h)}] = (-1)^{|\tau|}[x_1, \cdots, x_h] \qquad \forall \tau \in S_h$

(ii) $[x_1, \cdots, x_h] = 0$ *if* $x_i = x_j$ *for some* $1 \leqslant i < j \leqslant h$

(iii) The completely antisymmetric product is multilinear in its components.

(iv) $e_1, \cdots, e_h$ *mutually orthogonal, then* $[e_1, \cdots, e_h] = e_1 \cdots e_h$.

Proof. (i) The function $f: S_h \to S_h$ such that $\sigma \to \tau \circ \sigma$ is a bijection. Then it follows from: $(-1)^{|\tau\circ\sigma|} = (-1)^{|\tau|}(-1)^{|\sigma|}$.

(ii) As a transposition, that is a permutation τ that changes only $i \to j$, is an odd permutation; then (ii) follows easily from (i).

(iii) This is due to distributivity of the geometric product..

(iv) $e_i e_j = -e_j e_i$, then $e_{\sigma(1)} \cdots e_{\sigma(h)} = (-1)^{|\sigma|} e_1 \cdots e_h$.

□

Corollary 2.1.6.

$$x_1, \cdots, x_h \in V \text{ are linear independent} \iff [x_1, \cdots, x_h] \neq 0$$

Proof. ($\Leftarrow$) It follows from the previous proposition.

($\Rightarrow$) The previous proposition implies that $\exists\, e_1, \ldots, e_h$ mutually orthogonal such that $span\{x_1, \ldots, x_h\} = span\{e_1, \ldots, e_h\}$, and

$$[x_1, \ldots, x_h] = \alpha[e_1, \ldots, e_h] = \alpha\, e_1 \cdots e_h, \text{ with } \alpha = det\begin{pmatrix} \chi_{1,1} & \cdots & \chi_{1,h} \\ \vdots & & \vdots \\ \chi_{h,1} & \cdots & \chi_{h,h} \end{pmatrix},$$

where $x_i = \sum_{j=1}^{h} \chi_{i,j} e_j$.
Then, by hypothesis, $\alpha \neq 0$.
The thesis then follows, using axiom (G).

□

Remark 2.1.8. The proof points out that, if we are in the Euclidean space, for every couple of l.i. vectors u, v, $[u, v] = \sin\theta e_1 e_2$, where e_1, e_2 is an orthogonal basis for the two-dimensional linear space $span\{u, v\}$ and θ is the angle between u and v, oriented by the ordered basis $\{e_1, e_2\}$.

Remark 2.1.9. Hence, every non-degenerate completely antisymmetric product can be expressed as the Clifford product of n mutually orthogonal vectors, and vice versa.

Definition 2.1.7. We call **k-blade** the geometric product of k mutually orthogonal and linearly independent vectors. Thus, a k-blade coincides with the non-degenerate completely antisymmetric product of its factors.

Notation. If $g_1, \cdots g_n$ is an ordered basis for V and $\mathcal{A} \subseteq \{1, ..., n\}$, we will denote with $g_{\mathcal{A}} := \prod_{i\in\mathcal{A}} g_i$, where indexes in $\mathcal{A}$ are taken, in the product, in their natural order.
If $\mathcal{A} = \varnothing$, then $g_{\mathcal{A}} := 1$.

Remark 2.1.10. By the anticommutative property of the orthogonal vectors, we have that:

$$span\{g_{\mathcal{A}}\}_{\mathcal{A}\subseteq\{1,\dots,n\}} = Cl(V)$$

Furthermore, if we call $Cl^k(V) := span\{k\text{-blades}\}$ the space of k-multivectors, we obtain the following result.

Notation. In $Cl(V)$, $\dim \mathbb{K} = \dim Cl^{\dim V}(V) = 1$, but we have just shown that 1 and $e_1 \cdots e_{\dim V}$ represent different elements in the space (they are linearly independent). So, we call *pseudo-scalar* an element in $Cl^{\dim V}(V)$.
If $\{e_1, ..., e_n\}$ is an orthonormal basis of V (an orthogonal basis such that $e_i^2 = \pm 1$ in the relative Clifford algebra), then we call *pseudo-unit* the element $e_1 \cdots e_n \in Cl^{\dim V}(V) \subseteq Cl(V)$. (also called orientation of V).
We can observe that I is invertible.

Theorem 2.1.8. *If $e_1, .., e_n$ is an orthonormal basis of a non-degenerate quadratic space V, then*

i) $\{e_{\mathcal{A}}\}_{\substack{\mathcal{A}\subseteq\{1,\dots,n\}\\ \#\mathcal{A}=k}}$ *is a basis for* $Cl^k(V)$.

ii) $\{e_{\mathcal{A}}\}_{\mathcal{A}\subseteq\{1,\dots n\}}$ *is a basis for* $Cl(V)$.

iii) $Cl(V)$ *can be decomposed as follows,*
$Cl(V) = \bigoplus_{k=0}^{n} Cl^k(V)$.
Thus, $\dim Cl(V) = 2^{\dim V}$.

The proof of this theorem is left in the appendix.

2.1.1 A Classical Way to Define the non-Degenerate Clifford Algebra

In this subsection, we will refer to $Cl_Q(V)$ with $\mathcal{G}$. We do that because in this subsection we want to observe that, if we work with non-degenerate quadratic forms, our definition of Clifford algebra is equivalent to the following one, that is the usual way to define it (See [13] or [3] for example).
Moreover, in this chapter we take for granted some properties of tensor algebras. For more details see [9].

Definition 2.1.9. Let V be a n-dimensional vector space over the field $\mathbb{K}$ and Q a quadratic form on V. The Clifford algebra $Cl_Q(V)$ associated with (V, Q) is the associative algebra with unit, defined by

$$Cl_Q(V) := V^{\otimes}/\mathcal{I}(V, Q)$$

where $V^{\otimes} = \bigoplus_{i\geqslant 0} V^{\otimes i}$ is the tensor algebra of V and $\mathcal{I}(V, Q)$ the two-sided ideal generated by all the elements of the form $x \otimes x - Q(x)1$, for $x \in V$.

We don't choose to use this definition because it requires a more depth knowledge about astract algebra like tensor product and universal properties (as we can see right now). So, in order to prove the equivalence we need to show a result, respecting the classic definition.

Remark 2.1.11. There is a natural map $i : V \to Cl_Q(V)$ obtained by considering the natural embedding $j : V \hookrightarrow V^{\otimes}$, followed by the projection $\pi : V^{\otimes} \to Cl_Q(V)$. Viewing V as a subset of $Cl_Q(V)$ in that way, the algebra $Cl_Q(V)$ is generated by V (and the unit 1), subject to the relations

$$v \cdot v = Q(v, v)1.$$

Remark 2.1.12. $V^{\otimes}$ the tensor algebra with the inclusion $j : V \to V^{\otimes}$ is the essentially unique[3] pair that verifies the following universal property:
for any other associative algebra with unit $\mathcal{A}$, and any linear map $f : V \to \mathcal{A}$, there exists a unique $\mathbb{K}$-algebra homomorphism

$$\hat{f} : V^{\otimes} \to \mathcal{A}$$

satisfying

$$\hat{f} \circ j = f.$$

Also the Clifford algebra checks its own universal property.

Proposition 2.1.10. *Let $\mathcal{A}$ be an associative algebra with unit and $f : V \to \mathcal{A}$ a linear map such that for all $v \in V$*

$$f(v)^2 = Q(v)1_{\mathcal{A}}. \tag{2.1}$$

Then there exists a unique $\mathbb{K}$-algebra homomorphism

$$\tilde{f} : Cl_Q(V) \to \mathcal{A}$$

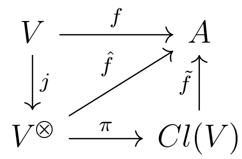
satisfying

$$\tilde{f} \circ i = f$$

Furthermore, if $\mathcal{C}$ is an associative $\mathbb{K}$-algebra with unit carrying a linear map $i' : V \to \mathcal{C}$ satisfying $i'(v)^2 = Q(v)1_{\mathcal{C}}$, with the property above, then $\mathcal{C}$ is isomorphic to $Cl_Q(V)$.

[3] Up to isomorphisms of algebras that commute with inclusions.

Proof. To prove this proposition we refer back to the universal property of the tensor algebra.



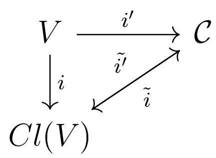
If $[a] \in Cl(V)$ (a, element of tensor algebra, is a representative of the equivalence class $[a]$, element of the Clifford algebra),

$$\tilde{f}([a]) := \hat{f}(a),$$

where $\hat{f}$ is the unique homomorphism granted by the universal property of tensor algebra. By Proposition 2.1, $\tilde{f}$ is well defined. Moreover,

$$\tilde{f}(i(v)) = \tilde{f}([j(v)]) = \hat{f}(j(v)) = v.$$

For construction we have that this homomorphism is unique (It follows from the uniqueness in the universal property of tensor algebra).
In the end, if we have $\mathcal{C}$ and i' as in the hypothesis, then we obtain $\tilde{i'} : Cl(V) \to \mathcal{C}$ and $\tilde{i} : \mathcal{C} \to Cl(V)$ that commute with i and i'. Hence they are the inverse of each other, and this implies that $Cl(V)$ and $\mathcal{C}$ are isomorphic.



□

With this result we can show the equivalence.

Proposition 2.1.11. *If $\mathcal{G}$ is the geometric algebra, and $Cl_Q(V)$ is the Clifford algebra defined above, then*

$$\mathcal{G} \cong Cl_Q(V).$$

Proof. It is directly implied by Theorem 2.1.8.
In fact, according to the last proposition, we only need to show that there is a unique algebra homomorphism $\tilde{f} : \mathcal{G} \to \mathcal{A}$ such that $\tilde{f} \circ i = f$, where i is the canonical inclusion of V in $\mathcal{G}$ and $f : V \to \mathcal{A}$ is a linear map with $f(v)^2 = q(v,v)1_{\mathcal{A}}$. But this is trivial, when we know that $\{e_{\mathcal{B}}\}_{\mathcal{B}\subseteq\{1,\dots n\}}$ is a basis for $\mathcal{G}$, because if $A = \sum_{\mathcal{B}} \alpha_{\mathcal{B}} e_{\mathcal{B}}$ is an its generic element, $\tilde{f}$ can only be

$$\tilde{f}(A) := \sum_{\mathcal{B}} \alpha_{\mathcal{B}} \tilde{f}(e_{\mathcal{B}})$$

where $\tilde{f}(e_{\mathcal{B}}) := f(e_{i_1}) \cdot ... \cdot f(e_{i_k})$. □

Notation. Hence, for the rest of the paper we will refer to a generic Clifford Algebra with $Cl(V)$ or G_V without any difference.

2.2 Basic Results on Clifford Algebras

In this section we will show useful results for the development of our results, that a reader can also find in [12] or in [8].

2.2.1 Basic Operators on Clifford algebras

Thanks to the last result we will identify the Clifford Algebra with $Cl(V)$ or G_V without any difference.
In this section we want to introduce the most common structures used in the geometric algebra and some of the results associated to them.

Remark 2.2.1. For every $x, y \in V$, we have that $(x + y)^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle y, y\rangle + 2\langle x, y\rangle$, but $(x + y)^2 = x^2 + y^2 + xy = \langle x, x\rangle + \langle y, y\rangle + xy + yx$, then we obtain

$$\frac{xy + yx}{2} = \langle x, y\rangle$$

Now, we can show one of the main and famous identities in the geometric algebra,

Proposition 2.2.1. *For every $x, y \in V$,*

$$xy = \langle x, y\rangle + [x, y] \tag{2.2}$$

Proof. $xy = \frac{xy+yx}{2} + \frac{xy-yx}{2}$. □

Now we want to get an insight into (2.2).
First, we need to analyze the $[x, y]$ part. We want to consider it as a binary (bilinear) operation, to focus the algebraic aspect over the geometric one. To do that we need to define two new operators.

Notation. $A \in Cl(V)$, then we call $\langle A\rangle_k$ the k-grade component of A.
In other words, $\langle A\rangle_k$ is the projection of $A \in Cl(V)$ onto $Cl^k(V)$.

Definition 2.2.2. For every pair of elements A, B : $A \in Cl^j(V)$; $B \in Cl^k(V)$,

- (**outer product**) $A \wedge B = \langle AB \rangle_{j+k}$
- (**inner product**) $A \cdot B = \langle AB \rangle_{k-j}$ (only if $k \geqslant j$; 0 otherwise.)

Of course we can extend the definition by linearity to all elements of $Cl(V)$.

Remark 2.2.2. If $a, b \in V$ the inner product is the well known scalar product.

Proposition 2.2.3. *The outer product is associative.*

Proof. Consider first a j-vector A, a k-vector B and a l-vector C:

$$(A \wedge B) \wedge C = \langle AB \rangle_{j+k} \wedge C = \langle \langle AB \rangle_{j+k} C \rangle_{j+k+l} = \langle ABC \rangle_{j+k+l}$$

Last equality holds because only $\langle AB \rangle_{j+k}$ can contribute to $\langle ABC \rangle_{j+k+l}$.
a similar calculation shows the same for $A \wedge (B \wedge C)$. □

The next proposition answers our previous problem, letting us to consider $[x, y]$ as a binary operation.

Proposition 2.2.4. $\forall x_1, ...x_k \in V$,

$$[x_1, \ldots, x_k] = x_1 \wedge \ldots \wedge x_k.$$

In particular, if $a, b \in V$,

$$ab = a \cdot b + a \wedge b. \qquad \textit{(fundamental identity)}$$

Proof. We will use the induction on j.
The base is trivial ($[x] = x$).
Now we want to show that $[x_1, ...x_{m-1}] \wedge x_m = [x_1, ..., x_m]$.
If $[x_1, ..., x_{m-1}] = 0$ it is true.
Otherwise, there exists an orthogonal base $\{e_1, ..., e_m\}$ such that
$[x_1, ..., x_{m-1}] = k(e_1 \cdots e_{m-1})$ with $k \in \mathbb{K} \neq 0$.
Now, $x_m = \sum_{i=1}^{m} \alpha_i e_i$, hence,

$$\langle [x_1, ..., x_{m-1}] x_m \rangle_m = \alpha_m k \, e_1 \cdots e_m = [x_1, ..., x_m].$$

□

Notation. Now we want to give a geometric meaning to our algebraic blades. So, we will consider the $span\{b_1, ..., b_k\} \subseteq V$ the space associated to the blade $B = b_1 \cdots b_k$.

Remark 2.2.3. Sometimes, with abuse of notation, we will call the blade and its space associated with the same name.

Proposition 2.2.5. *For every $a \in \mathbb{K}$, $\quad A, B, C \in Cl(V)$, we have*

(i) $(aA) \cdot B = A \cdot (aB) = a(A \cdot B)$

(ii) $A \cdot (B + C) = A \cdot B + A \cdot C; \quad (B + C) \cdot A = B \cdot A + C \cdot A$

(iii) $(aA) \wedge B = A \wedge (aB) = a(A \wedge B)$

(iv) $A \wedge (B + C) = A \wedge B + A \wedge C; \quad (B + C) \wedge A = B \wedge A + C \wedge A.$

(v)
$$A \cdot (B \cdot C) = (A \wedge B) \cdot C \tag{2.3}$$

Proof. i) and iii) Those are trivial from the definition.
ii) We suppose that A is a j-vector and B and C are k-vectors.
$A \cdot (B+C) = \langle A(B+C)\rangle_{k-j} = \langle AB+AC\rangle_{k-j} = \langle AB\rangle_{k-j} + \langle AC\rangle_{k-j} = A \cdot B + A \cdot C$.
If they are arbitrary multivectors, we have

$$\begin{aligned} A \cdot (B + C) &= \sum_j \langle A\rangle_j \sum_k \langle B + C\rangle_k = \\ &= \sum_{j,k} \langle A\rangle_j \cdot (\langle B\rangle_k + \langle C\rangle_k) = \\ &= \sum_{j,k} \langle A\rangle_j \cdot \langle B\rangle_k + \langle A\rangle_j \cdot \langle C\rangle_k = \\ &= A \cdot B + A \cdot C \ . \end{aligned}$$

We use a similar argument for the other equality and for the iv).
v) Both the terms are equal to $\langle ABC\rangle_{l-k-j}$ if A is a j-vector, B is a k-vector and C is a l-vector. Then, like in ii), we can extend it to arbitrary multivectors. □

We denote with I the pseudo-unit in $Cl(V)$ w.r.t. to an orientation given by a basis of the space.

Definition 2.2.6. If $A \in Cl(V)$ we call $A^* := {}^{A}/_{I} = AI^{-1}$ its dual[4]. (with respect to the orientation).

[4]It makes sense because I is always invertible.

Proposition 2.2.7. *If $n = \dim V$,*

(i) $(aA)^* = aA^*$

(ii) $(A + B)^* = A^* + B^*$

(iii) $A^{**} = \pm A$, *and* $A^{**} = (-1)^{\frac{n(n-1)}{2}} A$ *if Q is a positive definite quadratic form.*

(iv) if A is a k-vector then A^* is a $(n-k)$-vector

(v) if A is a k-blade then A^* is a $(n-k)$-blade

(vi) if U is the space associated to a blade B, then the space associated to B^* will be $U^{\perp}$

Proof. i) and ii) are true for the linearity of the Clifford product.
iii) If Q is a positive definite quadratic form it immediately follows from $I^2 = (-1)^{\frac{n(n-1)}{2}}$, otherwise I^2 is anyway equal to -1 or 1, in accord to the signature of the form.
iv) follows from v). v and vi) Let $A = a_1 \cdots a_j$, the product of members of an orthogonal invertible basis for U. Let $\{a_{j+1}, \cdots, a_n\}$ be an orthonormal basis for $U^{\perp}$. Then,

$$I = \pm \frac{a_1}{|a_1|} \cdots \frac{a_j}{|a_j|} \cdot a_{j+1} \cdots a_n.$$

Hence $A^* = \pm |a_1| \cdots |a_j| \cdot a_{j+1} \cdots a_n$. □

We can immediately see an interesting result using the dual operator.

Proposition 2.2.8. *If $\dim V = n$, every $A \in Cl_{n-1}(V)$ is a $(n-1)$-blade*

Proof. If $A \in Cl_{n-1}(V)$, from the proposition above, we have that A^* is a vector, in particularly it is a 1-*blade*. Then, A^{**} is a $(n-1)$-blade.
But $A = \pm A^{**}$, hence we got the thesis. □

Example 2. It is useful to remark that the last statement it is not true for any arbitrary $m \leqslant n$. In fact in $Cl(\mathbb{R}^4)$, the 2-multivector $e_1e_2 + e_3e_4$ it is not a 2-blade.

Now we want to endow our Clifford algebra with a norm or something similar. To do that we need to introduce the "reversion" first,

Definition 2.2.9. If A is a j-vector, the reversion $\tilde{A} := (-1)^{\frac{j(j-1)}{2}} A$ and we extend it for linearity to all G_V.

Remark 2.2.4. If $A := a_1 \wedge a_2 \cdots \wedge a_k$, we have that $\tilde{A} = a_k \wedge a_{k-1} \wedge \cdots \wedge a_1$.

Definition 2.2.10. $A \in G_V$,

$$||A||^2 = \langle A\tilde{A}\rangle_0$$

Remark 2.2.5. We can observe that if $A = \sum_J a_J e_J$ w.r.t. to an o.n. basis of G_V, then

$$||A||^2 := \sum_J |a_J|^2 ||e_J||^2.$$

Then we can also see that it can be negative despite what it might seem. Only if Q is a positive definite form, we have that this map is a norm in all respects.

Moreover, if B is a blade, $||B|| \neq 0 \Leftrightarrow B$ is invertible.

Proposition 2.2.11. *If* $A, B \in G_V$*,*

(i) $\widetilde{A \wedge B} = \tilde{B} \wedge \tilde{A}$

(ii) $||A^*|| = ||A||$

(iii) $(A \wedge B)^* = A \cdot B^*, \quad (A \cdot B)^* = A \wedge B^*$

(iv) $\widetilde{AB} = \tilde{B}\tilde{A}$

Proof. i) It follows from Remark 2.2.4.
ii) It is another consequence of the proof of Proposition 2.2.7,(v). iii) If A is a j-vector and B a k-vector,
$A \cdot B^* = \langle A(BI^{-1})\rangle_{(n-k)-j} = \langle A(BI^{-1})\rangle_{n-(j+k)} = \langle AB\rangle_{j+k} I^{-1} = (A \wedge B)^*$.
We can extend it to the general case through the linearity of the involved operators. We can use a similar argument for the other equality.
iv) For linearity, if we decompose both A and B with respect to an o.n. basis $\{e_I\}_I$, it suffices proving the thesis for $e_I e_J$.
First of all, we start to prove that $\widetilde{e_0 e_I} = \tilde{e_I} e_0$.
If $0 \notin I$ this is trivial; otherwise, if $|I| = k$, we have that,

$$\widetilde{e_0 e_I} = (-1)^{\frac{(k-1)(k-2)}{2}} e_0 e_I = (-1)^{\frac{(k-1)(k-2)}{2}} (-1)^{n-1} e_I e_0 = (-1)^{\frac{(k-1)(k)}{2}} e_I e_0 = \tilde{e_I} e_0.$$

Now we can finally prove the thesis by induction on $|J|$ simply observing the following chain of equalities, (calling e_f the "last" element of e_J: $e_{J/\{f\}} e_f = e_J$)

$$\tilde{e_J}\tilde{e_I} = \widetilde{e_{J/\{f\}} e_f}\tilde{e_I} = \tilde{e_f}\widetilde{e_{J/\{f\}}}\tilde{e_I} = e_f \widetilde{e_I e_{J/\{f\}}} = e_I \widetilde{e_{J/\{f\}e_f}} = \widetilde{e_I e_J}.$$

□

Remark 2.2.6. In $G_{\mathbb{R}^3}$ we can recognize the well known *cross product* defining it as $u \times v := (u \wedge v)^*$.
Infact, we can observe that choosing an o.n. basis of the space.

Theorem 2.2.12. *$A, B \in G_V$. If A or B is a blade, then*

$$||AB|| = ||A||\,||B||.$$

Proof. We observe that if B is a blade we have that $B\tilde{B} = \langle B\tilde{B}\rangle_0$. Then,

$$||AB||^2 = \langle AB\widetilde{AB}\rangle_0 = \langle AB\tilde{B}\tilde{A}\rangle_0 = ||B||^2\langle A\tilde{A}\rangle_0 = ||B||^2||A||^2.$$

□

Example 3. In general this is not true: $(e_1e_2 + e_3e_4)(e_1e_3 + e_2e_4) = 0$.

Now we can extend the fundamental identity,

Lemma 2.2.13. *Let $B = b_1 \cdots b_k$ a k-blade, $a_{||} \in B$ and $a_\perp \perp B$. Then,*

a) $a_{||} \cdot B = a_{||}B$ *and* $a_{||} \wedge B = 0$,
$a_{||} \cdot B$ *is a* $(k-1)$*-blade in* B *(unless* $a_{||} = 0$*)*

b) $a_\perp \wedge B = a_\perp B$ *and* $a_\perp \cdot B = 0$,
$a_\perp \wedge B$ *is a* $(k+1)$*-blade representing span*$\{a_\perp, B\}$ *(unless* $a_\perp = 0$*)*

Proof. Since $a_{||} \in B$, it is a linear combination of $\{b_1, \cdots, b_k\}$. Then it is easy to see that $a_{||}B$ is a combination of $(k-1)$-vectors in B. Thus,

$$a_{||} \cdot B = \langle a_{||}B\rangle_{k-1} = a_{||}B \qquad a_{||} \wedge B = \langle a_{||}B\rangle_{k+1} = 0.$$

Moreover, if $a_{||} \neq 0$, $a_{||}B$ is a combination of $(k-1)$-vectors, and so from Proposition 2.2.8 is a $(k-1)$-blade.
Since $a_\perp \perp B$, $a_\perp B$ is a $(k+1)$-blade. Hence as we did above, we obtain the thesis. □

Proposition 2.2.14 ((Extended fundamental identity))**.** *Let $a \in V$ and $B \in G_V$. Then*

$$aB = a \cdot B + a \wedge B$$

.

Proof. If we prove that only for blades, the thesis will follow from the linearity of all the operators (the thesis holds by definition if B is a scalar, then we will not consider this trivial case in the proof).
We decompose $a = a_{||} + a_{\perp}$. From the lemma,

$$aB = (a_{||} + a_{\perp})B = a_{||} \cdot B + a_{\perp} \wedge B = a \cdot B + a \wedge B.$$

□

Remark 2.2.7. Moreover a left vector product split every k-blade in a sum of $(k-1)$ and $(k+1)$-blades.

Remark 2.2.8. We cannot extend anymore, because

$$AB \neq A \cdot B + A \wedge B.$$

(example: e_1e_2 and e_2e_3 in $\mathbb{R}^3$)

We want to end listing some other properties of these operators.

Proposition 2.2.15. *Let A, B be two blades in G_V. Then, according to the abuse of notation announced in the Remark 2.2.3, we have that*

(i) $A \cdot B \subseteq B$ *(Provided $A \cdot B$ is a blade, or 0)*

(ii) $A \subseteq B \Rightarrow A \cdot B = AB$

(iii) $a \in A \Rightarrow a \perp A \cdot B$

(iv) $||A||^2 = ||A \cdot A||$

(v) $A \wedge B = span\{A, B\}$ *if* $A \cap B = \{0\}$, *otherwise it is* 0

(vi) $B \wedge A = (-1)^{\dim A \cdot \dim B} A \wedge B$

Proof. i) Let A be a j-blade and B a k-blade. Express $A = a_1a_2 \cdots a_{j-1}a_j$, a product of orthogonal vectors. We will start with $a_j \cdot B$, then $(a_{j-1}a_j) \cdot B$, and so on, showing that each is a blade (or 0). According to the extended fundamental identity, $a_j \cdot B$ is a $(j-l)$-blade in B (or 0). Next, $(a_{j-1}a_j) \cdot B = \langle a_{j-1}a_jB \rangle_{k-2}$.
Now, only the blade $\langle a_j \cdot B \rangle_{k-1}$ from a_jB can contribute to $(\langle a_{j-1}a_jB \rangle_{k-2}$.
Now, apply the extended fundamental identity twice to see that the right side is a $(k-2)$-blade in B (or 0). Hence,

$$\langle a_{j-1}a_jB \rangle_{k-2} = a_{j-1} \cdot \langle a_jB \rangle_{k-1} = a_{j-1} \cdot (a_j \cdot B).$$

Continuing in this way, $A \cdot B$ is a blade in B (or 0).
ii) It is trivial because in this case AB is a $(k-j)$-blade.
iii) $a \cdot (A \cdot B) = (a \wedge A) \cdot B = 0 \cdot B = 0$.
iv) Let $A = a_1 a_2 \cdots a_j$, a product of orthogonal vectors. Then both $||A||^2$ and $||A \cdot A||$ are equal to $|a_1|^2 |a_2|^2 \cdots |a_j|^2$.
v) Consider first the case $A \cap B \neq \{0\}$. Choose $c \in A \cap B$. Extend $\{c\}$ to an orthogonal basis $\{a_1, \cdots, a_{j-1}, c\}$ for A. If $A \neq a_1 \cdots a_{j-1} c$, then make it so by multiplying at by a nonzero scalar. Similarly. $B = cb_1 \cdots b_{k-1}$. Then

$$A \wedge B = \langle AB \rangle j + k = |c|^2 \langle a_1 \cdots a_{j-1} b_1 \cdots b_{k-1} \rangle j + k = 0.$$

Now consider the case $A \cap B = \{0\}$. Express $A = a_1 \cdots a_j$, a product of orthogonal vectors. We will build up $A \wedge B$ starting with $a_j \wedge B$, then $(a_{j-1} a_j) \wedge B$, and so on. Since $a_j \notin B$, $a_j \wedge B$ is a $(k+l)$-blade representing $span(a_j, B)$. Next, analogously to i),

$$(a_{j-1} a_j) \wedge B = a_{j-1} \wedge (a_j \wedge B).$$

Since $a_{j-1} \notin span(a_j, B)$, it follows that $(a_{j-1} a_j) \wedge B$ is a $(k+2)$-blade representing $span(a_{j-1}, a_j, B) = span(a_{j-1} a_j, B)$. Continuing in this way, $A \wedge B$ is a $(k+j)$-blade representing $span(A, B)$. vi) For the anticommutative property of the outer product, if $A = a_1 \cdots a_j$ and $B = b_1 \cdots b_k$, we have that

$$B \wedge A = b_1 \wedge \cdots \wedge b_k \wedge a_1 \wedge \cdots \wedge a_j = (-1)^j b_1 \wedge \cdots \wedge b_{k-1} \wedge a_1 \wedge \cdots \wedge a_j \wedge b_k =$$

$$= (-1)^{j+k} a_1 \wedge \cdots \wedge a_j \wedge b_1 \wedge \cdots \wedge b_k = A \wedge B$$

□

2.2.2 Projection, Rotation, Reflection

A first impressive Clifford algebra result can be observed through the following new ways to define the most common geometric transformations: projection, rotation and reflection.

Proposition 2.2.16. *$a \in V$ and B an invertible blade. Then we can decompose $a = a_{||} + a_{\perp}$. Hence,*

$$a_{||} = {}^{a \cdot B}\!/_{B} \qquad \textit{(projection)}$$

$$a_{\perp} = {}^{a \wedge B}\!/_{B} \qquad \textit{(rejection)}$$

Proof. From the propositions above, $a_{||}B = a \cdot B$ and $a_{\perp}B = a \wedge B$. □

Definition 2.2.17. $v \in V$, B blade, then

- the *distance between* v and B is $d(v, B) := |v_{\perp}|$
- the *projection* of v on B is $P_B(v) := v_{||}$

<u>*Remark*</u> 2.2.9. If B is invertible,

- $d(v, B) = \frac{||v \wedge B||}{||B||}$
- $P_B(v) = {}^{v \cdot B}\!/_{B}$

Definition 2.2.18. $M \in G_V, B$ invertible, then

$$P_B(M) := {}^{(M \cdot B)}\!/_{B} \ .$$

<u>*Remark*</u> 2.2.10.

- P_B is linear. $(P_B(\alpha M + \beta N) = \alpha P_B(M) + \beta P_B(N))$
- Recalling the Remark 2.2.3, $A \subseteq B \Rightarrow P_B(A) = A$ (see (2.3))

Proposition 2.2.19.

$$P_B(M \wedge N) = P_B(M) \wedge P_B(N).$$

Proof. We have to show for blades, and for linearity we have the thesis.
First, $P_B(a \wedge b) = P_B(a_{||} \wedge b) + P_B(a_{\perp} \wedge b)$. The second term is zero, hence doing the same for b we obtain that:

$$P_B(a \wedge b) = P_B(a_{||} \wedge b_{||}) = a_{||} \wedge b_{||} = P_B(a) \wedge P_B(b)$$

. Hence, for blades $A = a_1 \wedge \cdots \wedge a_i$ and $C = c_1 \wedge \cdots \wedge c_j$ we have that

$$\begin{aligned} P_B(A \wedge C) &= P_B(a_1 \wedge \cdots \wedge a_i \wedge c_1 \wedge \cdots \wedge c_j) = \\ &= P_B(a_1) \wedge \cdots \wedge P_B(a_i) \wedge P_B(c_1) \wedge \cdots \wedge P_B(c_j) = \\ &= P_B(A) \wedge P_B(C). \end{aligned}$$

□

Now we can speak about another fundamental geometric transformation: the rotation.
We will analyze it within euclidean spaces

Notation. If A is a blade such that $A^2 = -1$ and λ is a scalar, we can write $e^{A\lambda} := \cos\lambda + A\sin\lambda$. For this reason, in this context, we often use the letter I to call such a blade A.

Definition 2.2.20. The rotation in the plane of blade I with the angle θ is denoted by $R_{I\theta}$.

Remark 2.2.11. Thanks to Remark 2.1.8 we can see that, in euclidean spaces, if u, v are two l.i. vectors, then $uv = |u||v|(\cos\theta + \sin\theta I)$ with I the 2-blade representing the plane of u, v, and θ the angle between them.
Then in our new notation, $uv = |u||v|e^{I\theta}$.
Moreover if we want to rotate u in a plane I of an angle θ, obtaining the new vector v (with the same norm of u), we have that $u^2v = u|u||v|e^{I\theta}$, and because $|u||u| = |u||v| = u^2$ we obtain that $v = ue^{I\theta}$.

Now we want to consider a general rotation of angle θ in the plane I: this time u is not necessarily lying in the plane I. We will denote it $R_{I\theta}(u)$.

Lemma 2.2.21. *If a is a vector and A is a j-blade such that $a \in A$ (it makes sense since Remark 2.2.3), then*

$$aA = (-1)^{j-1}Aa.$$

Proof. We need to decompose a in the orthogonal basis of the blade A (obviously we can because $a \in A$). Then every vector of the basis anticommutes with every element of A, except with itself (where it trivially commutates). □

Proposition 2.2.22. *With the notation above,*

$$R_{I\theta}(u) = e^{-\frac{I\theta}{2}}ue^{\frac{I\theta}{2}}.$$

Proof. We decompose u with respect to the plane of I, so $u = u_{||} + u_{\perp}$. Then, we have that $R_{I\theta}(u) = u_{||}e^{I\theta} + u_{\perp}$ (Indeed $R_{I\theta}(u_{\perp}) = u_{\perp}$). Thus, thanks to the foregoing lemma,

$$\begin{aligned} R_{I\theta}(u) &= u_{||}e^{\frac{I\theta}{2}}e^{\frac{I\theta}{2}} + u_{\perp}e^{-\frac{I\theta}{2}}e^{\frac{I\theta}{2}} = \\ &= e^{-\frac{I\theta}{2}}u_{||}e^{\frac{I\theta}{2}} + e^{-\frac{I\theta}{2}}u_{\perp}e^{\frac{I\theta}{2}} = \\ &= e^{-\frac{I\theta}{2}}ue^{\frac{I\theta}{2}}. \end{aligned}$$

□

For completeness, using the last result, we extend the operator R also to the generic elements of the Clifford algebra, even if it will not be necessary for the results of our work.

Definition 2.2.23. If $M \in G_V$,

$$R_{I\theta}(M) := e^{-\frac{I\theta}{2}} M e^{\frac{I\theta}{2}}.$$

Remark 2.2.12.

- $R_{I\theta}$ is linear.
- $R_{I\theta}(MN) = R_{I\theta}(M)R_{I\theta}(N) \qquad \forall\, M, N \in G_V.$

Lemma 2.2.24. *Let A and B be multivectors. Then $\langle AB\rangle_0 = \langle BA\rangle_0$.*

Proof. Let aE (a a scalar, E a 1-norm blade) be a term in the expansion of A with respect to a standard basis $\mathcal{B}$. Similarly, let bF be in the expansion of $\mathcal{B}$. In AB each aE is multiplied by each bF: $(aE)(bF) = abEF$. Inspection of $\mathcal{B}$ shows that $\langle EF\rangle_0 \neq 0$ only if $E = F$. Then $\langle (aE)(bF)\rangle_0 = abE^2$ (or 0), where $E^2 = \pm 1$. Similarly $\langle (bF)(aE)\rangle_0 = baE^2$. The scalar parts are equal. □

From the lemma, $\langle (AB)C\rangle_0 = \langle C(AB)\rangle_0$, this implies a cyclic reordering property: $\langle ABC\rangle_0 = \langle CAB\rangle_0$.

Proposition 2.2.25.

$$R_{I\theta}(M \wedge N) = R_{I\theta}(M) \wedge R_{I\theta}(N),$$

$$R_{I\theta}(M \cdot N) = R_{I\theta}(M) \cdot R_{I\theta}(N).$$

Moreover, the rotation preserves the grade.

Proof. We already know that the rotation operator transforms vectors in vectors. Moreover, by the previous lemma,

$$R_{I\theta}(u) \cdot R_{I\theta}(v) = \langle e^{-\frac{I\theta}{2}} uv e^{\frac{I\theta}{2}}\rangle_0 = \langle e^{-\frac{I\theta}{2}} e^{\frac{I\theta}{2}} uv\rangle_0 = \langle uv\rangle_0 = u \cdot v.$$

Then the rotation preserves the orthogonality.
For construction, we have that, for every j-blade $u_1 \cdots u_j$,

$$R_{I\theta}(u_1 \cdots u_j) = R_{I\theta}(u_1) \cdots R_{I\theta}(u_j)$$

. Since the rotation preserves the orthogonality, the right side is a j-blade again. Then we have proved that rotation preserves the grade too.
Now, suppose first that M is a j-blade and N a k-blade. Applying what we have just observed,

$$\begin{aligned} R_{I\theta}(M \wedge N) &= R_{I\theta}\langle MN\rangle_{j+k} \\ &= \langle R_{I\theta}(MN)\rangle_{j+k} \\ &= \langle R_{I\theta}(M)R_{I\theta}(N)\rangle_{j+k} \\ &= R_{I\theta}(M) \wedge R_{I\theta}(N). \end{aligned}$$

Thanks to the linearity of the involved operators, we can easily generalize this result to general multivectors. A similar argument can be used to prove the inner product part. □

Now it is time to analyze the reflection, another well known transformation.

Proposition 2.2.26. *Let a be a vector and A a j-blade. Then,*

$$a \cdot A = \frac{1}{2}(aA - (-1)^j Aa),$$

$$a \wedge A = \frac{1}{2}(aA + (-1)^j Aa).$$

Proof. $a = a_{||} + a_{\perp}$ w.r.t to A. Hence we know that $a_{\perp}A = (-1)^j Aa_{\perp}$.
And for Lemma 2.2.21, $a_{||}A = (-1)^{j-1}Aa_{||}$. Thus,

$$\begin{aligned} aA - (-1)^j Aa &= (a_{||} + a_{\perp})A - (-1)^j A(a_{||} + a_{\perp}) = \\ &= (a_{||} + a_{\perp})A - (-a_{||} + a_{\perp})A = 2a_{||}A = 2a \cdot A. \end{aligned}$$

In the same way we can prove the second expression. □

Definition 2.2.27. Let a be a vector and B an invertible k-blade. Then if $a = a_{||} + a_{\perp}$ is the well known decomposition then the *reflection* of a through B is

$$F_B(a) := a_{||} - a_{\perp}.$$

Proposition 2.2.28. *With the notation above,*

$$F_B(a) = (-1)^{k+1}BaB^{-1}.$$

Proof.
$F_B(a) = a_{||} - a_{\perp} = (a \cdot B - a \wedge B)B^{-1} =$ (previous prop.) $= (-1)^{k+1}BaB^{-1}$. □

Moreover if we are working with hyperplanes we can simplify the formula.

Proposition 2.2.29. *If a is a vector, and B is an invertible hyperplane, with $b := B^*$ its dual vector, then*

$$F_B(a) = -bab^{-1}.$$

Proof. Observing that $a_{||} \perp b$ and $a_\perp \parallel b$,

$$-bab^{-1} = -b(a_{||} + a_\perp)b^{-1} = a_{||}bb^{-1} - a_\perp bb^{-1} = a_{||} - a_\perp = F_B(a).$$

□

Definition 2.2.30. As we did for rotation and projection, we can extend the definition of reflection to multivectors, then if A is a multivector whose grade is j,

$$F_B(A) := (-1)^{j(k+1)}BAB^{-1}.$$

Proposition 2.2.31. *F_B is linear, preserves grade. Moreover,*

$$F_B(M \wedge N) = F_B(M) \wedge F_B(N),$$

$$F_B(M \cdot N) = F_B(M) \cdot F_B(N).$$

Proof. The proof is similar to that at Proposition (2.2.25). □

Remark 2.2.13. Summarizing,

- (projection) $P_B(A) = (A \cdot B)B^{-1}$
- (rotation) $R_{I\theta}(A) = e^{-\frac{I\theta}{2}}Ae^{\frac{I\theta}{2}}$
- (reflection) $F_B(A) = (-1)^{j(k+1)}BAB^{(-1)}$

Now, we can rewrite the rotation in a more useful way, using a classical theorem[5] of geometry: the Cartan-Dieudonne theorem.

Theorem 2.2.32 (Cartan-Dieudonne)**.** *Let f be an orthogonal transformation in a n-dimensional symmetric bilinear space, then f is a composition of at most n reflections.*

With the notation and the results above we can rearrange the theorem in a "new" algebraic form for our context.

[5] See details in [6]

Theorem 2.2.33. *Let f be an orthogonal transformation in a n-dimensional non-degenerate quadratic space, then there exists a $V = v_1 \cdots v_r$ with $r \leqslant n$ such that,*

$$f(v) = (-1)^r V v V^{-1}.$$

Notation. If a, b are two vectors, we denote with $\check{ab}$ the angle between them.

Remark 2.2.14. Let us to show an example: the rotation in the plane represented by the 2-blade i ($i^2 = -1$), $R_{I\theta}(v) = e^{-\frac{I\theta}{2}} v e^{\frac{I\theta}{2}}$.
If we find two unit vectors $a, b \in I$ such that $\check{ab} = \theta/2$, we have that $ab = e^{\frac{I\theta}{2}}$ then we obtain the statement above, hence $R_{I\theta}(v) = b^{-1}a^{-1}vab$.

Chapter 3

Shape Distance in 2D

In this chapter, our purpose is to find a reasonable way to describe a "similarity distance" between two smooth curves. In particular, we will find a pseudo-distance between smooth curves that can measure how much similar are two curves.
We consider two smooth curves as similar if we can obtain one composing the other one with a direct homothety (a scaled rototranslation), while we exclude inverse homothety (An interested reader may also see [11]).
We want to stress that we work with such invariances only in dimension 2, as we will choose other similarity criteria to compare gestures in higher dimensions, in Chapter 5.

3.1 Gestures and Shapes

In this section V will be a two-dimensional Euclidean space, that we shortly denote by $\mathbb{E}^2$. We start to define similarity between two ordered couples of vectors, and then a similarity distance.

Lemma 3.1.1. *a, b, c are coplanar in a Euclidean space, if and only if*

$$abc = cba.$$

Proof. As $abc = a\langle b, c\rangle + a(b \wedge c)$, using Lemma 2.2.21, we obtain

$$abc = a\langle b, c\rangle - (b \wedge c)a = \langle c, b\rangle a + (c \wedge b)a = cba.$$

□

Proposition 3.1.2. *Let (a, b, c) be an ordered triple of elements $\in \mathbb{E}^2$. Then if a, b are l.i., we have that $d = ba^{-1}c$ is the "vector fourth proportional" to the triple*

(a, b, c), that is the unique vector such that:

- $\frac{|a|}{|b|} = \frac{|c|}{|d|}$. *(scalar)*
- *the angle* $\check{ab} = \check{cd}$. *(directional)*

Proof. Since we Let $c' = R_{a \wedge b \, \check{ab}}(c)$ that we denote, with abuse of notation, $R_{\check{ab}}(c)$. For Remark (2.2.14), we have that

$$R_{\check{ab}}(c) = \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}\hat{a}c\hat{a}\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|},$$

where $\hat{x} = \frac{x}{|x|}$. Now we got the thesis iif $d = \frac{|b|}{|a|}c'$. Then,

$$\begin{aligned} \frac{|b|}{|a|}c' &= \frac{|b|}{|a|}\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}\hat{a}c\hat{a}\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|} = \quad \text{(by Lemma (3.1.1))} \\ &= \frac{|b|}{|a|}\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}\hat{a}\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}\hat{a}c = \quad \text{(because } a^{-1} = \frac{a}{|a|^2}\text{)} \\ &= \frac{|b|}{|a|}\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}a\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}a^{-1}c = |b|\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}\hat{a}\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}a^{-1}c = \\ &= |b|\; F_{\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}}(\hat{a})\; a^{-1}c = |b|\hat{b}a^{-1}c = ba^{-1}c = d. \end{aligned}$$

We recall that $F_B(x)$ is the reflection of x through the blade B. □

As $F_{\frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}}(\hat{a}) = \hat{b}$, we can reformulate this proposition in a "geometrical" way, to better focus our point of view.

<u>Notation</u>. If three consecutive vectors a, b, c trace a triangle in the plane (so that $a + b + c = 0$), we denote that triangle by $\overset{\triangle}{abc}$, as a geometric figure.
This definition does not depend on the translations (only vectors are involved in this construction); consequently, it describe the triangle as an object unrelated to the origin.

Corollary 3.1.3. *Let T_1 and T_2 be two triangles in the plane such that $T_1 := x_1\overset{\triangle}{x_2}x_3$ and $T_2 := y_1\overset{\triangle}{y_2}y_3$. Then,*

$$T_1 \sim T_2 \textit{(direct similarity)} \iff \exists\, a, b, c, d \in \{1; 2; 3\} \textit{ such that } x_a x_b^{-1} = y_c y_d^{-1}.$$

Definition 3.1.4. Let $\mathcal{S} := \{(a, b) \mid a, b \in \mathbb{R}^2,\ b \neq 0\}$.
Then, $(a, b) \approx (c, d) \iff ab^{-1} = cd^{-1}$.

<u>Remark</u> 3.1.1. We can see that if we consider b consecutive to a and d to c, $(a, b) \approx (c, d) \implies ab(\overset{\triangle}{a} + b) \sim cd(\overset{\triangle}{c} + d)$, but not the reverse!

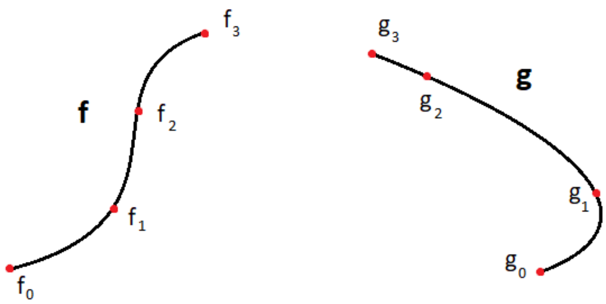
For example, $e_1e_2(\overset{\triangle}{-e_1} - e_2) \sim -e_1(e_1 \overset{\triangle}{+} e_2)(-e_2)$, but $e_1e_2^{-1} \nsim -e_1(e_1+e_2)^{-1}$.

Now we can introduce the objects we will use in this section,

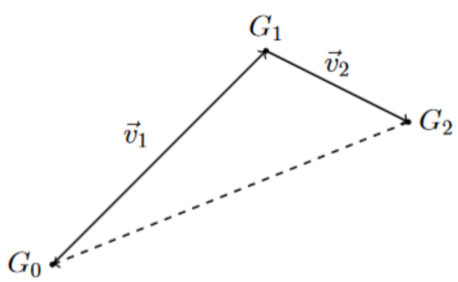
Definition 3.1.5.

- A curve $f : I = [0,1] \to \mathbb{R}^2$ is a **plane gesture** if it is regular (that is $C^2[0,1]$ such that $f' \neq 0$),
- A $(n+1)$-sample of a gesture is $\{f_0, \ \dots \ , f_n\} \subset f(I)$.

This definition is useful both to mathematicians and to IT engineers. As a matter of fact, we can interpret the samples as the input data of a basic gesture in the plane, but we can still look at them as the points associated to a generic partition of the curve.



Definition 3.1.6. A **basic gesture** in $\mathbb{R}^2$ is an ordered couple of not null vectors (v_1, v_2) commonly considered as a particular 2-sample of a plane gesture tracing a triangle.



This definition is the result of what we had shown in the last corollary, namely that the basic gesture can be interpreted as a “triangle with ordered sides”.

Definition 3.1.7. The **shape** of a basic gesture (v_1, v_2) is the Clifford number ${}^{v_1}/_{v_2}$.

From Corollary 3.1.3, the shape of a basic gestures is invariant through direct similarity transformations. Then it induces us to consider the follow “measurer” function,

Definition 3.1.8. The **Local Shape Distance** between two basic gestures (u_1, u_2) and (v_1, v_2) is the non-negative real number

$$LSD\big((u_1, u_2), (v_1, v_2)\big) = \left\| {}^{u_1}/_{u_2} - {}^{v_1}/_{v_2} \right\|$$

Thus, the Local Shape Distance is simply a proper distance between the elements representing the shapes of two basic gestures, (according to [10]), hence a measure of how much the two “ordered triangles” are far from being directly similar.

Finally, we define our operator to evaluate a shape pseudo-distance for plane gestures too.

Theorem 3.1.9.
$I = [0, 1]$ $f, g : I \to \mathbb{R}^2 \in C^3(I)$: $f'(t), g'(t) \neq 0$ $\forall t \in I$.
Let P_n be a partition of $I := \left\{ t_{k,n} := \frac{k}{n} \middle| \, k = 0, \cdots, n \right\}$ and

$$f_{k,n} := f(t_{k,n})$$

$$g_{k,n} := g(t_{k,n})$$

Then:

$$\lim_{n\to+\infty} \sum_{k=1}^{n-1} \left\| {}^{(f_{k,n} - f_{k-1,n})}/_{(f_{k+1,n} - f_{k,n})} - {}^{(g_{k,n} - g_{k-1,n})}/_{(g_{k+1,n} - g_{k,n})} \right\| =$$

$$= \int_I \left\| {}^{f''(t)}/_{f'(t)} - {}^{g''(t)}/_{g'(t)} \right\| dt \tag{3.1}$$

To prove this theorem we need to show some other results.

Remark 3.1.2. If we consider the even subalgebra $Cl^0(\mathbb{E}^2) \oplus Cl^2(\mathbb{E}^2) \trianglelefteq Cl(\mathbb{E}^2)$ we can observe that,

$$\begin{array}{ccc} \mathbb{C} & \cong & Cl^0(\mathbb{E}^2) \oplus Cl^2(\mathbb{E}^2) \\ x+iy & \xrightarrow{j} & x + e_2 e_1 y \end{array} \tag{3.2}$$

Now $\mathbb{C} = \mathbb{E}^2$ as vectorial space (with its quadratic form), then if $u, v \in \mathbb{E}^2 \quad (u = xe_1 + ye_2)$, we can obviously consider both as elements of $\mathbb{C} \quad (\mathbf{u} = x + iy)$.
And we can finally observe that

$$u/_v = {}^{j(\mathbf{u})}/_{j(\mathbf{v})} = j\left(\frac{\mathbf{u}}{\mathbf{v}}\right)$$

The last equality follows from the foregoing definition of isomorphism j. We point out that in the even subalgebra, the product is commutative, and that we can swap indifferently between the two spaces $\mathbb{C}$ and $Cl^0(\mathbb{E}^2) \oplus Cl^2(\mathbb{E}^2)$.

Then, thanks to this isomorphism we can consider the shape of a gesture or any element of the even subalgebra as fraction of complex numbers too. More in details,

- (complex) $shape(u, v) = \frac{\mathbf{u}}{\mathbf{v}} \in \mathbb{C}$
- (complex) $LSD((u_1, u_2), (v_1, v_2)) = \left|\frac{\mathbf{u_1}}{\mathbf{u_2}} - \frac{\mathbf{v_1}}{\mathbf{v_2}}\right|_{\mathbb{C}} \geqslant 0$

Remark 3.1.3. It is better to specify that LSD is a pseudometric in the space of basic gestures $\{(u, v) \mid 0 \neq u, v \in \mathbb{R}^2\}$, while we are talking about it as "distance". This is perfectly fine, because it is a distance in the space of shapes ($\mathbb{C}$ or the even subalgebra, depending on the point of view).

Hence, Theorem 3.1.9 is equivalent to the following one,

Theorem 3.1.10. $I = [0, 1] \quad f, g : I \to \mathbb{R}^2 \in C^2(I) : \; f'(t), g'(t) \neq 0 \quad \forall t \in I.$ *Let* P_n *be a partition of* $I := \left\{ t_{k,n} := \frac{k}{n} \middle| \, k = 0, \cdots, n \right\}$ *and*

$$f_{k,n} := f(t_{k,n})$$

$$g_{k,n} := g(t_{k,n})$$

Then:

$$\lim_{n \to +\infty} \sum_{i=1}^{n-1} \left| \frac{(f_{k,n} - f_{k-1,n})}{(f_{k+1,n} - f_{k,n})} - \frac{(g_{k,n} - g_{k-1,n})}{(g_{k+1,n} - g_{k,n})} \right|_{\mathbb{C}} =$$

$$= \int_I \left| \frac{f''(t)}{(f'(t))} - \frac{g''(t)}{(g'(t))} \right|_{\mathbb{C}} dt \tag{3.3}$$

Proof of both Theorem 3.1.9 and Theorem 3.1.10.

By hypothesis, the Riemann integral $\int_0^1 \left| \frac{f''(t)}{f'(t)} - \frac{g''(t)}{g'(t)} \right|_{\mathbb{C}} dt$ exists; this implies that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{n} \left| \frac{f''(\xi_k)}{f'(\xi_k)} - \frac{g''(\xi_k)}{g'(\xi_k)} \right|_{\mathbb{C}} \frac{1}{n} - \int_0^1 \left| \frac{f''(t)}{f'(t)} - \frac{g''(t)}{g'(t)} \right|_{\mathbb{C}} dt \right| < \epsilon \ ,$$

provided $n > N_\epsilon$, and $\xi_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$, with $k = 1, \ldots, n$.
Notice that, to evaluate each shape $\frac{\Delta g_k}{\Delta g_{k+1}} := \frac{g_{k,n} - g_{k-1,n}}{g_{k+1,n} - g_{k,n}}$, the extremities of two adjacent intervals are needed. In particular, we can write

$$\sum_{k=1}^{2m-1} \frac{\Delta g_k}{\Delta g_{k+1}} = \sum_{h=1}^{m} \frac{\Delta g_{2h-1}}{\Delta g_{2h}} + \sum_{h=1}^{m-1} \frac{\Delta g_{2h}}{\Delta g_{2h+1}} \ , \tag{3.4}$$

when n is even. A similar expression holds when n is odd. Thus, to estimate the difference between shapes and terms of a Riemann sum, we have to consider the latter on couples of adjacent intervals. In order to simplify notations, we will consider only the case $n = 2m$ (n even). However, our arguments can be applied similarly to the case: n odd. If $n > 2N_\epsilon$, then the integral can be estimated both by

- $$\left| \sum_{h=1}^{m} \left| \frac{f''(\xi_h^e)}{f'(\xi_h^e)} - \frac{g''(\xi_h^e)}{g'(\xi_h^e)} \right|_{\mathbb{C}} \frac{1}{n} - \frac{1}{2} \int_0^1 \left| \frac{f''(t)}{f'(t)} - \frac{g''(t)}{g'(t)} \right|_{\mathbb{C}} dt \right| < \frac{\epsilon}{2} \ ,$$

  where $\xi_h^e \in \left[\frac{2(h-1)}{n}, \frac{2h}{n}\right]$, with $h = 1, \ldots, m$, and

- $$\left| \sum_{h=1}^{m-1} \left| \frac{f''(\xi_h^o)}{f'(\xi_h^o)} - \frac{g''(\xi_h^o)}{g'(\xi_h^o)} \right|_{\mathbb{C}} \frac{1}{n} - \frac{1}{2} \int_0^1 \left| \frac{f''(t)}{f'(t)} - \frac{g''(t)}{g'(t)} \right|_{\mathbb{C}} dt \right| < \frac{\epsilon}{2} \ ,$$

  where $\xi_h^o \in \left[\frac{2h-1}{n}, \frac{2h+1}{n}\right]$, with $h = 1, \ldots, m$.

Then, to obtain the thesis, it suffices to see how to estimate the following quantity,

$$\left| \frac{\Delta f_{2h}}{\Delta f_{2h+1}} - \frac{\Delta g_{2h}}{\Delta g_{2h+1}} - \left(\frac{g''(\xi_h^o)}{g'(\xi_h^o)} - \frac{f''(\xi_h^o)}{f'(\xi_h^o)} \right) \frac{1}{n} \right|_{\mathbb{C}} =$$
$$= \left| \left(\frac{\Delta f_{2h}}{\Delta f_{2h+1}} - 1 + \frac{f''(\xi_h^o)}{f'(\xi_h^o)} \frac{1}{n} \right) + \left(1 - \frac{\Delta g_{2h}}{\Delta g_{2h+1}} - \frac{g''(\xi_h^o)}{g'(\xi_h^o)} \frac{1}{n} \right) \right|_{\mathbb{C}} \ ,$$

for each $h = 1, \dots, m$. In particular, we can observe that, assuming $\delta = \frac{1}{n}$, then

$$1 - \frac{\Delta g_{2h}}{\Delta g_{2h+1}} = 1 - \frac{g(t_{2h}) - g(t_{2h} - \delta)}{g(t_{2h} + \delta) - g(t_{2h})} = \frac{g(t_{2h} + \delta) - 2g(t_{2h}) + g(t_{2h} - \delta)}{g(t_{2h} + \delta) - g(t_{2h})} =$$

$$= \frac{\frac{g(t_{2h}+\delta) - 2g(t_{2h}) + g(t_{2h}-\delta)}{\delta^2}}{\frac{g(t_{2h}+\delta) - g(t_{2h})}{\delta}} \delta \ .$$

By hypothesis, the function g is twice differentiable and $g' \neq 0$, thus we have that, for every $t \in [0, 1]$

$$\lim_{\delta \to 0} \frac{\frac{g(t+\delta) - 2g(t) + g(t-\delta)}{\delta^2}}{\frac{g(t+\delta) - g(t)}{\delta}} = \frac{g''(t)}{g'(t)} \ ,$$

as the limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero. So, we have that, for every $\epsilon > 0$ there exists δ_ϵ, such that if $\delta < \delta_\epsilon$, then

$$\left| 1 - \frac{\Delta g_{2h}}{\Delta g_{2h+1}} - \frac{g''(\xi_h^o)}{g'(\xi_h^o)} \delta \right|_{\mathbb{C}} = \left| \frac{\frac{g(t_{2h}+\delta) - 2g(t_{2h}) + g(t_{2h}-\delta)}{\delta^2}}{\frac{g(t_{2h}+\delta) - g(t_{2h})}{\delta}} - \frac{g''(\xi_h^o)}{g'(\xi_h^o)} \right|_{\mathbb{C}} \delta < \epsilon\delta \ ,$$

and provided $\delta < \min\{\delta_\epsilon, \frac{1}{2N_\epsilon}\}$, observing that a similar argument can be also applied for the function f, this prove the thesis.

□

The foregoing proof can also be used to prove other results, such as the following one[1].

Corollary 3.1.11. *With the same notations of the previous theorem, given a plane gesture g, then, if P_n be a partition of I, $g_{k,n} := g(t_{k,n})$, and $\Delta g_k := g_{k,n} - g_{k-1,n}$,*

$$\lim_{n \to \infty} \sum_{k=1}^{n-1} \frac{\Delta g_k}{\Delta g_{k+1}} = 2 - \int_0^1 \frac{g''(t)}{g'(t)} \ dt \ \in \mathbb{C}.$$

That is why it is reasonable to give the following definitions.

Definition 3.1.12.

- The shape of a plane gesture g, is the following function

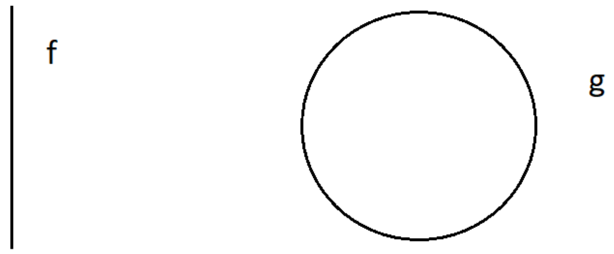
$$S(g(t)) = 1 - \frac{1}{2} \, {g''(t)} \big/ {g'(t)}$$

[1] A more general proof of it will be given with Theorem 3.1.14.

- The distance between the shape of two gestures f, g is the following operator

$$\begin{aligned} D_S(f,g) &= \int_I \left|\left|S(f(t)) - S(g(t))\right|\right| \, dt \; = \\ &= \frac{1}{2}\int_I \left|\left| \, {f''(t)}\big/{(f'(t))} \; - \; {g''(t)}\big/{(g'(t))} \, \right|\right| \, dt. \end{aligned}$$

<u>*Example*</u> 4. $f(t) = (x_0, t)$; $g(t) = (x_1 + r\cos(2\pi t - \phi), y_1 + r\sin(2\pi t - \phi))$ with $t \in [0,1]$ and $r > 0, \phi, x_1, y_1 \in \mathbb{R}$.



- $S(f(t)) \equiv 1$
- $S(g(t)) \equiv 1 - \pi\, e_1 e_2$
- $D_S(f,g) = \; \pi.$

<u>*Remark*</u> 3.1.4.

- We choose to use the Clifford product (instead of the product between complex numbers) to keep continuity with the first definitions.
- We decided to scale in half so that the shape of a rectilinear gesture would be 1, regardless of whether it is considered as “basic” or not. Indeed, Definition 3.1.12 is scaled in a half also to compensate a kind of double counting of intervals used in the sum in (3.1)[2].
- Despite the previous arguments, we preferred to avoid to scale in half the (3.1). This, in continuity with the subsequent original definition of !FTL (See Section 3.2).

[2]That double counting is clearly visible in the proof of the theorem, where we needed to divide the intervals in even and odd

With the next example we want to show that the shape of two gestures can differ considerably, although their images may appear close to each other, or even be the same. We can also notice that, for every planar gesture f, sometimes its shape can be considered as a gesture too. In fact, applying the bijection $\alpha + \beta\,(e_1 \wedge e_2) \leftrightarrow (\alpha, \beta)$, we obtain a parametrized curve which, if regular, represents a new gesture.
Thus in the next figures we will show both the imagine of the gesture and its shape together (we have chosen gestures whose shapes are regular).

Example 5. We will consider three gestures:

- $b : t \to (t, t^2)$, a quadratic function.
- $r : t \to (t, t^3)$, a cubic function, "visually close" to the previous quadratic[3].
- $g : t \to \left(\sin(\frac{\pi t}{2}), \sin(\frac{\pi t}{2})^2\right)$, a gesture which has the same image of b ($g([0,1] = b([0,1])$) but, as a gesture, different.

To avoid the overlap of gesture graphics, instead of the canonical shape, we choose to consider $1 - S(f) = \frac{1}{2}\, {}^{f''}\!/_{f'}$ for any gesture f involved in this example.

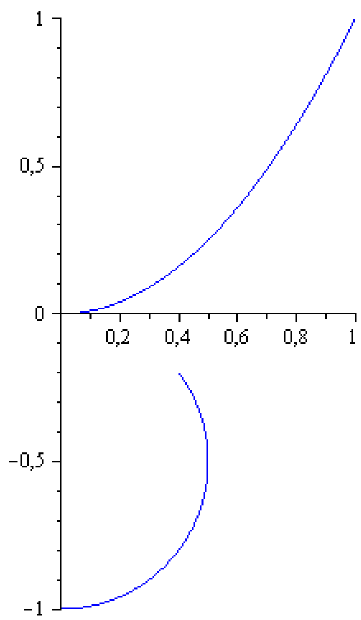


Figure 3.1: (Top) $b(t) = (t, t^2)$
(Bottom) $1 - S(b(t))$

[3]Obviously, this is a subjective parameter: a reader might consider these as very different functions, also in terms of appearance.

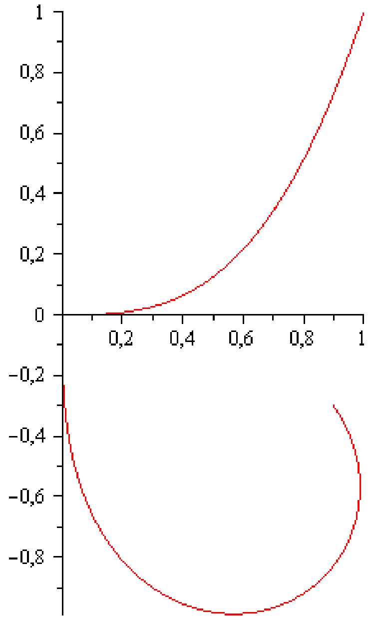


Figure 3.2: (Top) $r(t) = (t, t^3)$
(Bottom) $1 - S(r(t))$

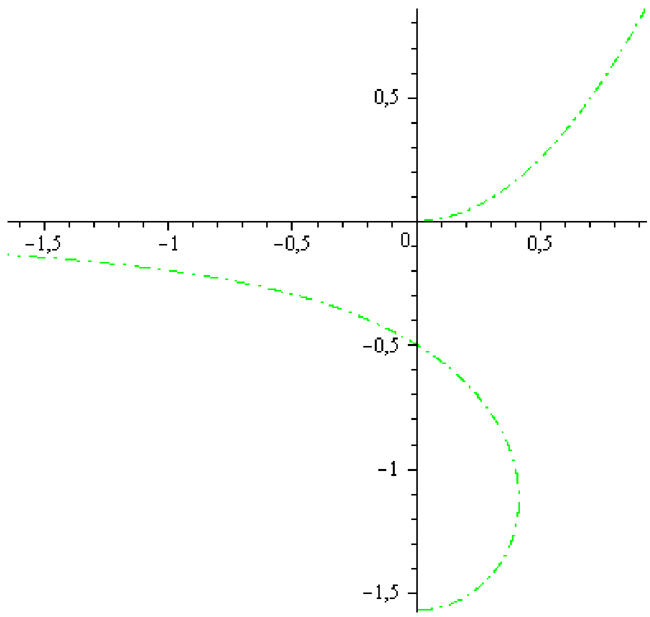


Figure 3.3: (Top) $g(t) = \left(\sin(\frac{\pi t}{2}), \sin(\frac{\pi t}{2})^2\right)$
(Bottom) $1 - S(g(t))$

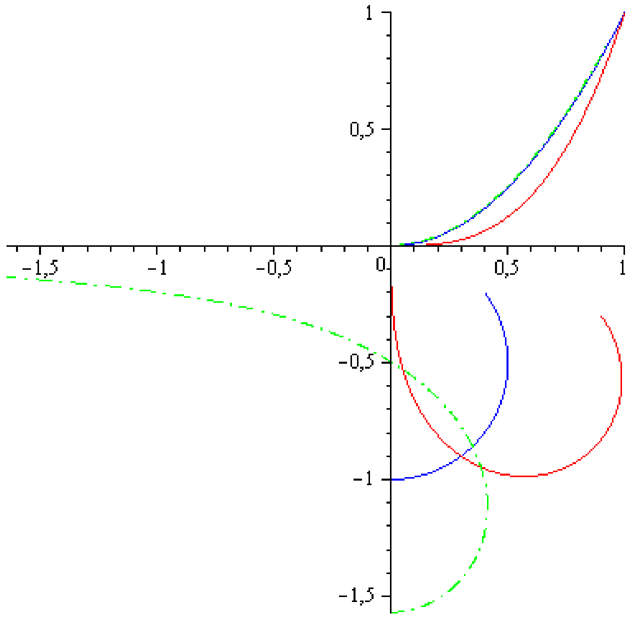


Figure 3.4: b, r, g all together with their shapes

As you can see, the graphs of the three shapes differ greatly, although with the images of three curves are close to each other, and two of them even coincide!

We have noticed that, for some planar gestures, their shapes can be considered as a gestures too.
Now we consider the generic gesture $f(t) = (x(t), y(t)) = x(t)e_1 + y(t)e_2$ in $\mathbb{R}^2$. Hence we have that its shape is

$$S(f(t)) = 1 - \frac{1}{2}\, {f''(t)}\big/{f'(t)} = 1 - \frac{1}{2}\left(\frac{x''x' + y''y'}{(x')^2 + (y')^2} + \frac{x''y' - y''x'}{(x')^2 + (y')^2}\, e_1 \wedge e_2\right)$$

Therefore, if $s(t) = 1 - \frac{1}{2}(\sigma_1(t), \sigma_2(t))$ is a given shape, one can ask to solve the following ODE system $S(f(t)) = s(t)$, that is:

$$\begin{cases} \frac{x''x'+y''y'}{(x')^2+(y')^2} = \sigma_1(t) \\ \frac{x''y'-y''x'}{(x')^2+(y')^2} = \sigma_2(t) \end{cases}$$

In particular, one can ask to look for any gesture that coincides with its shape, that is $S(f(t)) = f(t)$, or

$$\begin{cases} \frac{x''x'+y''y'}{(x')^2+(y')^2} = x \\ \frac{x''y'-y''x'}{(x')^2+(y')^2} = y \end{cases}$$

As we are considering only regular gestures (those for which $|f'| \neq 0$), we can also write

$$\begin{cases} x''x' + y''y' = x((x')^2 + (y')^2) \\ x''y' - y''x' = y((x')^2 + (y')^2) \end{cases}$$

Putting $u = x'$ and $v = y'$, we have that,

$$\begin{cases} u = x' \\ v = y' \\ u'u + v'v = x(u^2 + v^2) \\ u'v - v'u = y(u^2 + v^2) \end{cases}$$

Manipulating the two last relations, we can obtain the following equivalent ODE,

$$\begin{cases} v' = xv - yu \\ u' = xu + yv \end{cases} \longrightarrow \begin{cases} x'' = xx' + yy' \\ y'' = xy' - yx' \end{cases}$$

Of course, setting $x_1 = x, x_2 = y, x_3 = x', x_4 = y'$, the system can be reduced to a system of first order:

$$X = (x_1, x_2, x_3, x_4); \quad X' = F(X)$$

where,

$$F : \mathbb{R}^4 \to \mathbb{R}^4 \ : \quad F(x_1, x_2, x_3, x_4) = (x_3, x_4, x_1x_3 + x_2x_4, x_1x_4 - x_2x_3).$$

As F is $C^\infty(\mathbb{R}^4)$, the initial value problem has always a unique local solution. More interesting is the existence of bounded periodic solutions.

Theorem 3.1.9 requires some specific samples of the gestures to work. In fact we used only $P_n := \left\{ t_{k,n} := \frac{k}{n} \middle| \, k = 0, \cdots, n \right\}$ as associated partitions.
We refer to this situation as the *isochronous* one, a model for input data in the case of the difference in scan time between two subsequent samples is always the same[4].

Before continuing we want to focus on a topic:

Remark 3.1.5. The shape of a gesture (or the distance between two shapes) is strongly dependent on its parametrization, and we want that, because it patterns the speed in making a motion.
Indeed, if one wants to study only the supports of the curves one should use other

[4] I.e. we have uniformly spaced timestamps.

instruments from Differential Geometry. In that case, one refers to curves (in the IT environment) as *strokes*, instead of gestures.

For this reason, if we have a n-sample for a gesture, we can't simply rescale it to obtain an isochronous parametrization.
However, most of sampling devices are multitasking; this implies that the Central Processing Unit is not always sampling points, so the time interval between two consecutive sampled points is not constant. Thus, it is too restrictive to require the isochronous condition, and then we need to extend Theorem 3.1.9 to more general partitions.

Fortunately, we can do it (in the next subsection), keeping a central role for Definition 3.1.12.

3.1.1 The non-Isochronous Case

Lemma 3.1.13. *Given a plane gesture $g(t)$ then, for each $t \in (0,1)$, we have that*

$$\lim_{\substack{\tau_0 \to t \\ \tau_1 \to t \\ \tau_2 \to t \\ \tau_0 \neq \tau_1 \ , \ \tau_1 \neq \tau_2 \ , \ \tau_2 \neq \tau_0}} \left(1 - \frac{\tau_2 - \tau_1}{\tau_1 - \tau_0} \frac{g(\tau_1) - g(\tau_0)}{g(\tau_2) - g(\tau_1)} \right) \frac{1}{\tau_2 - \tau_0} = \frac{1}{2} \frac{g''(t)}{g'(t)}$$

Proof.

$$\left(1 - \frac{\tau_2 - \tau_1}{\tau_1 - \tau_0} \frac{g(\tau_1) - g(\tau_0)}{g(\tau_2) - g(\tau_1)} \right) \frac{1}{\tau_2 - \tau_0} = \frac{\frac{g(\tau_2)-g(\tau_1)}{\tau_2-\tau_1} - \frac{g(\tau_1)-g(\tau_0)}{\tau_1-\tau_0}}{\tau_2 - \tau_0} \frac{1}{\frac{g(\tau_2)-g(\tau_1)}{\tau_2-\tau_1}} \quad (3.5)$$

we notice that

$$\frac{\frac{g(\tau_2)-g(\tau_1)}{\tau_2-\tau_1} - \frac{g(\tau_1)-g(\tau_0)}{\tau_1-\tau_0}}{\tau_2 - \tau_0} \quad (3.6)$$

is the second divided difference[5] of the complex valued function g at points τ_0, τ_1 and τ_2. Being the function twice continously differentiable, it is sufficient to apply the Mean Value Theorem for divided differences[6] to real and imaginary parts of g,

[5] See [5] at page 123.
[6] See Theorem 2.10 in [14], at page 60.

to obtain that

$$\lim_{\substack{\tau_0\to t\\ \tau_1\to t\\ \tau_2\to t\\ \tau_0<\tau_1<\tau_2}} \frac{\frac{g(\tau_2)-g(\tau_1)}{\tau_2-\tau_1}-\frac{g(\tau_1)-g(\tau_0)}{\tau_1-\tau_0}}{\tau_2-\tau_0} = \frac{g''(t)}{2} .$$

Notice that we can always assume condition $\tau_0 < \tau_1 < \tau_2$; as a matter of fact, the second divided difference (3.6) is symmetric with respect points τ_0, τ_1 and τ_2. As the limit of quotient (3.5) is the quotient of the limits, provided the limit of the denominator is not zero, one obtains the thesis. □

Theorem 3.1.14. *Given a plane gesture* $g(t) = (x(t), y(t)) \in \mathbb{R}^2$ *with the notation above, then*

$$\lim_{\delta\to 0^+} \sum_{k=1}^{n-1} \frac{t_{k+1}-t_k}{t_k-t_{k-1}} \left\| {}^{\Delta f_k}\!/_{\Delta f_{k+1}} - {}^{\Delta g_k}\!/_{\Delta g_{k+1}} \right\| =$$

$$= \int_I \left\| {}^{f''(t)}\!/_{(f'(t))} - {}^{g''(t)}\!/_{(g'(t))} \right\| dt \ = \ 2\, D_S(f,g) \tag{3.7}$$

where $0 = t_0 < \cdots < t_{k-1} < t_k < \cdots < t_n = 1$, *and* $\delta = \max_{1\leqslant k\leqslant n}\{t_k - t_{k-1}\}$.

Proof. As we have already done before, we can reduce all the proof to show that,

$$\lim_{\delta\to 0^+} \sum_{k=1}^{n-1} \frac{t_{k+1}-t_k}{t_k-t_{k-1}} \frac{\Delta g_k}{\Delta g_{k+1}} = 2 - \int_0^1 \frac{g''(t)}{g'(t)}\, dt \ \in \mathbb{C} ,$$

By hypothesis, the complex valued Riemann integral $\int_0^1 \frac{g''(t)}{g'(t)}\, dt$ exists; this implies that for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$\left| \int_0^1 \frac{g''(t)}{g'(t)}\, dt - \sum_{k=1}^{n} \frac{g''(\xi_k)}{g'(\xi_k)}(t_k - t_{k-1}) \right|_{\mathbb{C}} < \epsilon ,$$

provided the partition

$$0 = t_0 < \cdots < t_{k-1} < t_k < \cdots < t_n = 1$$

is such that $t_k - t_{k-1} < \delta_\epsilon$, and $\xi_k \in [t_{k-1}, t_k]$ for each $k = 1 \ldots, n$.
Notice that, to evaluate each shape $\frac{\Delta g_k}{\Delta g_{k+1}}$, the extremities of two adjacent intervals

are needed. This implies that

$$\sum_{k=1}^{2m-1} \frac{t_{k+1}-t_k}{t_k-t_{k-1}} \frac{\Delta g_k}{\Delta g_{k+1}} = \sum_{h=1}^{m} \frac{t_{2h}-t_{2h-1}}{t_{2h-1}-t_{2(h-1)}} \frac{\Delta g_{2h-1}}{\Delta g_{2h}} + \sum_{h=1}^{m-1} \frac{t_{2h+1}-t_{2h}}{t_{2h}-t_{2h-1}} \frac{\Delta g_{2h}}{\Delta g_{2h+1}} , \tag{3.8}$$

when n is even. A similar expression old when n is odd. Thus, to estimate the difference between shapes and Riemann sums, we need to consider the latter on couples of adjacent intervals; one with even-indexed extremities, the other with odd-indexed extremities. In order to simplify notations, we will consider in the following only partitions of $[0, 1]$ having an even number of points ($n = 2m$), that is

$$0 = t_0 < \cdots < t_{k-1} < t_k < \cdots < t_{2m} = 1 . \tag{3.9}$$

However, our arguments can be applied similarly to partitions of $[0, 1]$ having an odd number of points. If partition (3.9) is such that

$$\max\left\{ \max_{1\leqslant h\leqslant m} (t_{2h} - t_{2(h-1)}) \ , \ \max_{1\leqslant h\leqslant m} (t_{2h+1} - t_{2h-1}) \right\} < \delta_\epsilon \ ,$$

then we can estimate the Riemann sum both

- on "even indexed" intervals

  $$\left| \int_0^1 \frac{g''(t)}{g'(t)}\, dt - \sum_{h=1}^{m} \frac{g''(\xi_h^e)}{g'(\xi_h^e)} (t_{2h} - t_{2(h-1)}) \right| < \epsilon \ ,$$

  whatever are $\xi_h^e \in [t_{2(h-1)}, t_{2h}]$ when $h = 1, \ldots, m$, and

- on "odd indexed" intervals, where a similar estimate is possible

  $$\left| \int_0^1 \frac{g''(t)}{g'(t)}\, dt - \frac{g''(\xi_0^o)}{g'(\xi_0^o)}(t_1 - t_0) - \frac{g''(\xi_m^o)}{g'(\xi_m^o)}(t_{2m} - t_{2m-1}) - \sum_{h=1}^{m-1} \frac{g''(\xi_h^o)}{g'(\xi_h^o)}(t_{2h+1} - t_{2h-1}) \right| < \epsilon \ ,$$

  whatever are $\xi_h^o \in [t_{2h-1}, t_{2h+1}]$, with $h = 1, \ldots, m-1$, $\xi_0^o \in [t_0, t_1]$, and $\xi_m^o \in [t_{2m-1}, t_{2m}]$.

Now, let us focus on the first term of the right expression in (3.8); in order to get the thesis, we need to estimate each term

$$\frac{t_{2h}-t_{2h-1}}{t_{2h-1}-t_{2(h-1)}} \frac{\Delta g_{2h-1}}{\Delta g_{2h}} - (t_{2h} - t_{2(h-1)}) + \frac{1}{2}\frac{g''(\xi_h^e)}{g'(\xi_h^e)}(t_{2h} - t_{2(h-1)}) =$$

$$= \left(\frac{t_{2h}-t_{2h-1}}{t_{2h-1}-t_{2(h-1)}} \frac{\Delta g_{2h-1}}{\Delta g_{2h}} \frac{1}{t_{2h}-t_{2(h-1)}} - 1 + \frac{1}{2}\frac{g''(\xi_h^e)}{g'(\xi_h^e)} \right) (t_{2h} - t_{2(h-1)}) \ .$$

If one considers Lemma 3.1.13 with $\tau_0 = t_{2(h-1)}$, $\tau_1 = t_{2h-1}$, and $\tau_2 = t_{2h}$, we have then, thanks to the uniform continuity of $\frac{g''}{g'}$, the estimate

$$\left| \frac{t_{2h} - t_{2h-1}}{t_{2h-1} - t_{2(h-1)}} \frac{\Delta g_{2h-1}}{\Delta g_{2h}} \frac{1}{t_{2h} - t_{2(h-1)}} - 1 + \frac{1}{2} \frac{g''(\xi_h^e)}{g'(\xi_h^e)} \right| < \frac{\epsilon}{2} ,$$

which is independent from index h. By applying the same lemma for the odd terms involving ξ_h^o, the thesis follows. □

An interesting question may arise to the reader.
Why did we introduce the Clifford algebra if we can bypass it using complex numbers? Essentially for two reasons.

1) We used two isomorphisms to link $\mathbb{C}$ to the geometrical algebra. Firstly, through the canonical $\mathbb{R}^2 \equiv \mathbb{C}$; and, another time, when we used the j in Remark 3.1.2. That helped us in the computation simplifying the calculus, but we cannot consider vectors and shapes as the same objects.
Clifford algebra help us to differentiate them: vectors are vectors, shapes are elements of the even subalgebra, hence a scalar plus a bivector.
Moreover in this way we can do algebraic operations together. (For example, a vector times a shape is equal to another vector.)

2) The even subalgebra is isomorphic to the complex algebra. But this happens only for $\mathbb{R}^2$. This imply that we cannot extend this procedure to higher dimensions or out of this context.

3.2 Algorithms for Planar Gestures

In this subsection we will show how to "translate" all the previous results in a new algorithm for gesture recognition.
More details can be found in [16].

3.2.1 !FTL

Purpose of the algorithm:
Given the n-samples of two plane gestures f and g, we want to give a measure of their dissimilarity through an algorithm with an intrinsic invariance with respect to translation, dilation, and rotation.

Algorithm. ***!FTL***[7] *is the "solution" recently developed (2018).*

INPUT= $\{f_0, \ \dots \ , f_n, \ g_0, \ \dots \ , g_n\}$
(the samples for two gestures f, g*)*
(If $f = \big(r(t), s(t)\big)$*, we denote with* $\mathbf{f} = r(t) + i\, s(t)$*.)*

$$
\begin{aligned}
!FTL(f_0, \ \dots \ , f_n \ , \ g_0, \ \dots \ , g_n) &= \sum_{k=1}^{n-1} LSD\big((\Delta f_k, \Delta f_{k+1}), (\Delta g_k, \Delta g_{k+1})\big) \\
&= \sum_{k=1}^{n-1} \left| \frac{\Delta \mathbf{f}_k}{\Delta \mathbf{f}_{k+1}} - \frac{\Delta \mathbf{g}_k}{\Delta \mathbf{g}_{k+1}} \right|_{\mathbb{C}} .
\end{aligned}
$$

3.2.2 !WFTL

Algorithm. *For more accurate results we can extend the* $!FTL$ *to the non-isochronous case, obtaining the new* ***!WFTL*** *(Weighted !FTL).*

INPUT= $\{f_0, \ \dots \ , f_n, \ g_0, \ \dots \ , g_n, \ t_0, \ \dots \ , t_n\}$
(the samples and the timestamps)
(If $f = \big(r(t), s(t)\big)$*, we denote with* $\mathbf{f} = r(t) + i\, s(t)$*.)*

$$
\begin{aligned}
!WFTL(f_0, \ \dots \ , f_n \ , \ g_0, \ \dots \ , g_n) &= \sum_{k=1}^{n-1} \frac{t_{k+1} - t_k}{t_k - t_{k-1}} LSD\big((\Delta f_k, \Delta f_{k+1}), (\Delta g_k, \Delta g_{k+1})\big) \\
&= \sum_{k=1}^{n-1} \frac{t_{k+1} - t_k}{t_k - t_{k-1}} \left| \frac{\Delta \mathbf{f}_k}{\Delta \mathbf{f}_{k+1}} - \frac{\Delta \mathbf{g}_k}{\Delta \mathbf{g}_{k+1}} \right|_{\mathbb{C}} .
\end{aligned}
$$

Remark 3.2.1. At first impression it may seem incongruent to ignore the multiplicative factor $\frac{1}{2}$, especially after all previous considerations.
But this is only meant to be consistent with the initial development of the !FTL done in [16]. This is the reason because we are leaving unchanged the original algorithm, whereas we accorded the definition of the various shapes.

[7] As we have seen in the Introduction in Chapter 1, FTL is the acronym of "Faster than light".

Chapter 4

Shape Distance on Regular Surfaces

The next step is to expand our results on surfaces[1].
The purpose is similar to the previous section, in fact we want to understand when 2 curves (or gestures) are "similar" on any immersed regular surface.
To do that, we have to recall some useful facts about differential geometry, and after that, we need to improve that results through the Clifford Algebra tools.

4.1 Gestures on Regular Surfaces

4.1.1 Basic Results of Differential Geometry

Hereafter follows the list of some features about regular surfaces in $\mathbb{R}^3$.
All the following basic differential geometry notions can be found with more details in [1] or [15].

Remark 4.1.1. If S is a regular surface (i. e. a surface S in $\mathbb{R}^3$ that admits a smooth atlas $\{\varphi_\alpha\}$) let φ be a local parametrization: $U \to S$ such that $\varphi(x_1, x_2) = s \in S$ and centered in $p \in S$ (that is $\varphi(0) = p$).
If we define $\frac{\partial}{\partial x_j}|_p = \frac{\partial \varphi}{\partial x_j}(O)$ ($j = 1, 2;$ O is the origin), we obtain that $\{\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial x_2}|_p\}$ is the associated basis for the tangent space T_pS (The set of all tangent vectors to the surface in the point p).
This basis depends on the local parametrization.

Notation. For brevity sometimes we will use ∂_j or $\partial_{j|p}$ to denote $\frac{\partial}{\partial x_j}|_p$.

[1]Results on surfaces can have applications to gesture recognition on devices with foldable and curved input surfaces.

Remark 4.1.2. We denote the metric coefficients associated to φ with

$$E(x) = \partial_{1|\varphi(x)} \cdot \partial_{1|\varphi(x)}, \quad F(x) = \partial_{1|\varphi(x)} \cdot \partial_{2|\varphi(x)}, \quad G(x) = \partial_{2|\varphi(x)} \cdot \partial_{2|\varphi(x)} \qquad \forall\, x \in U.$$

or simplifying, with abuse of notation,

$$E = \partial_1 \cdot \partial_1, \quad F = \partial_1 \cdot \partial_2, \quad G = \partial_2 \cdot \partial_2 \qquad \forall\, x \in U.$$

Moreover we use the canonical notation for the Christoffel symbols Γ^r_{ij}, using their classical definition on surfaces:

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma^1_{ij}\partial_1 + \Gamma^2_{ij}\partial_2 + c_{ij}N,$$

where $\{\partial_1, \partial_2, N\}$ is a basis of $\mathbb{R}^3$ such that $N := \frac{\partial_1 \times \partial_2}{||\partial_1 \times \partial_2||}$
("×" is the cross product).

Remark 4.1.3. We want to express these coefficients in a different way, then doing some calculations we obtain that

$$\begin{cases} E\Gamma^1_{11} + F\Gamma^2_{11} = \left\langle \frac{\partial^2\varphi}{\partial x_1^2}, \partial_1 \right\rangle = \frac{1}{2}\frac{\partial}{\partial x_1}\langle \partial_1, \partial_1 \rangle = \frac{1}{2}\frac{\partial E}{\partial x_1} \\ F\Gamma^1_{11} + G\Gamma^2_{11} = \left\langle \frac{\partial^2\varphi}{\partial x_1^2}, \partial_2 \right\rangle = \frac{\partial}{\partial x_1}\langle \partial_1, \partial_2 \rangle - \left\langle \partial_1, \frac{\partial^2\varphi}{\partial x_1 \partial x_2} \right\rangle = \frac{\partial F}{\partial x_1} - \frac{1}{2}\frac{\partial E}{\partial x_2} \end{cases};$$

Similarly,

$$\begin{cases} E\Gamma^1_{12} + F\Gamma^2_{12} = \frac{1}{2}\frac{\partial E}{\partial x_2} \\ F\Gamma^1_{12} + G\Gamma^2_{12} = \frac{1}{2}\frac{\partial G}{\partial x_1} \end{cases};$$

$$\begin{cases} E\Gamma^1_{22} + F\Gamma^2_{22} = \frac{\partial F}{\partial x_2} - \frac{1}{2}\frac{\partial G}{\partial x_1} \\ F\Gamma^1_{22} + G\Gamma^2_{22} = \frac{1}{2}\frac{\partial G}{\partial x_2} \end{cases}.$$

Definition 4.1.1. Let $F : S_1 \to S_2$ a C^∞ map between two regular surfaces and $p \in S_1$. The **differential** of F on p is $dF_p : T_pS_1 \to T_pS_2$ such that
$dF_p(v) = (F \circ \sigma)'(0)$ where σ is a curve on S_1: $\sigma(0) = p$, $\sigma'(0) = v$.

Remark 4.1.4. $\mathbb{R}^2$ is a surface too, with $T_p(\mathbb{R}^2) = \mathbb{R}^2$
(just "translate the atlas" if necessary). In these terms, if $\gamma : S \to \mathbb{R}^2$, $d\gamma_p(v) \in \mathbb{R}^2 \quad \forall v \in T_pS$, and if $v = v_1\partial_{1|p} + v_2\partial_{2|p}$ we have that $d\gamma_p(v) = (v_1, v_2)$.

Remark 4.1.5. Now we have to recall some other definitions.

- A smooth **vector field** along a smooth curve $\sigma : I \to S$, is a C^∞ map $X : I \to \mathbb{R}^3$ such that $X(t) \in T_{\sigma(t)}S \quad \forall t \in I$.

- The **covariant derivative** for a vector field X along a smooth curve σ on the surface S is the vector field DX along σ such that $DX(t) := \pi_{\sigma(t)}\left(\frac{\partial X}{\partial t}(t)\right)$, where $\pi_{\sigma(t)} : \mathbb{R}^3 \to T_{\sigma(t)}(S)$ is the orthogonal projection on the tangent plane $T_{\sigma(t)}(S)$.

- A vector field X on a smooth curve σ is **parallel**, if $DX \equiv 0$.

- The **parallel transport** $P(\sigma)_p^s : T_p(S) \to T_s(S)$, along a smooth curve $\sigma : [0,1] \to S$ such that $\sigma(0) = p$ and $\sigma(1) = s$, is the map that moves any vector v from p to s along σ in a "parallel way",
that is $P(\sigma)_p^s(v) = X(1)$ where X is the unique parallel vector field on σ such that $X(0) = v$.

Remark 4.1.6. The required parallel field involved in the last definition is the only solution of the following Cauchy's problem,

$$\begin{cases} DX(t) = 0 & \forall t \in (0,1); \\ X(0) = v. \end{cases}$$

Hence the parallel transport is well defined.

Remark 4.1.7. Now we want express DX in local coordinates to show how the notion of covariant derivative depends only on metric coefficients.

If $\phi : U \to S$ a local parametrization whose image contains the support of a curve $\sigma : I \to S$. If X 'is a vector field along σ, we can write $\sigma(t) = \phi(\sigma_1(t), \sigma_2(t))$ and $X(t) = X_1(t)\partial_1 + X_2(t)\partial_2 \quad \forall\ t \in I$.
Then,

$$\frac{dX}{dt} = \frac{d}{dt}\left(X_1 \frac{\partial \phi}{\partial x_1} \circ \sigma\right) + \left(X_2 \frac{\partial \phi}{\partial x_2} \circ \sigma\right) =$$

$$= \sum_{k=1}^{2} \frac{dX_k}{dt} + X_k \left(\sigma_1' \frac{\partial^2 \phi}{\partial x_1 \partial x_k} \circ \sigma + \sigma_2' \frac{\partial^2 \phi}{\partial x_2 \partial x_k} \circ \sigma\right) .$$

And finally we obtain that,

$$DX = \sum_{k=1}^{2} \left[\frac{dX_k}{dt} + \sum_{i,j=1}^{2} (\Gamma_{i,j}^k \circ \sigma)\sigma_i' X_j \right] \partial_k . \tag{4.1}$$

Now we recall another main notion that it will help us for our purposes.

Definition 4.1.2. A local parametrization φ for a surface S is **isothermal** if its metric coefficients satisfy the following equalities

$$E \equiv G, \qquad F \equiv 0.$$

<u>Remark</u> 4.1.8. With an isothermal parametrization the Christoffel symbols can be expressed in a very compact way,

$$\begin{cases} \Gamma^1_{11} = \Gamma^2_{12} = -\Gamma^1_{22} = \frac{1}{2E}\frac{\partial E}{\partial x_1} \\ \Gamma^2_{11} = -\Gamma^1_{12} = \Gamma^2_{22} = \frac{1}{2E}\frac{\partial E}{\partial x_2} \end{cases}$$

Theorem 4.1.3. *Every regular surface admits an isothermal local parametrization.*

The proof of this theorem is delicate and will not be taken up here. The interested reader may consult [2].

4.1.2 Isometries between Clifford Algebras

Now it is the time to apply our informations about the isometries between vector spaces on the relative Clifford algebras.

<u>Remark</u> 4.1.9. V a vector space endowed with the scalar product $\langle \cdot, \cdot \rangle$. Let G_V be its geometric algebra.
$||A+B||^2 + ||A-B||^2 = 2||A||^2 + 2||B||^2 \qquad \forall A, B \in G_V \Rightarrow$
$\Rightarrow || \cdot ||$ induces a scalar product on G_V.
And it is the following one:

$$\langle A, B \rangle = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} (A_k \cdot B_k)$$

Lemma 4.1.4. *U, V vector spaces. $H : U \to V$ a linear map.*
Then, $\exists\, \hat{H}$ a linear map: $G_U \to G_V$ such that $\hat{H}_{|U} = H$ ($\hat{H}$ is an extension of H).
Moreover:

- *$\forall A, B \in G_U$, $\hat{H}(AB) = \hat{H}(A)\hat{H}(B)$ and $\hat{H}$ is the only extension with this property.*
- *If H is an isometry $\Rightarrow \hat{H}$ is an isometry.*

Proof. $A \in G_U$, $A = \sum_k A_k$.
$A_k = \sum_I \alpha_I e_I$, so $\hat{H}(A_k) := \sum_I \alpha_I \hat{H}(e_I)$ where $\hat{H}(e_{i_1} \cdots e_{i_k}) := H(e_{i_1}) \cdots H(e_{i_k})$.
We define $\hat{H}$: $\hat{H}(A) := \sum_k \hat{H}(A_k)$. This $\hat{H}$ proves the thesis.

Now if H is an isometry and $A_k = \sum_I \alpha_I e_I$, H transforms any orthonormal base in an orthonormal base, then:

$$\begin{aligned}\langle \hat{H}(A_k), \hat{H}(A_k)\rangle &= \langle \sum_I \alpha_I e_{H(I)}, \sum_I \alpha_I e_{H(I)}\rangle = \sum_{I,J} \alpha_I \alpha_J \langle e_{H(I)}, e_{H(J)}\rangle = \\ &= \sum_{I,J} \alpha_I \alpha_J \delta_{I,J} = \sum_I \alpha_I^2 \langle e_I, e_J\rangle = \langle A_k, A_k\rangle\end{aligned}$$

Then $\hat{H}$ is an isometry. □

<u>Remark</u> 4.1.10. It could be useful to recognize that if $H, L : U \to V$ are two linear maps with H, an isometry such that $L = \lambda H$ ($\lambda \in \mathbb{R}$), then it is not always true that $\hat{L} = \lambda \hat{H}$.
For example, if $L = 2H$, $\hat{L}(e_1) = 2\hat{H}(e_1)$ and $\hat{L}(e_1 e_2) = 4\hat{H}(e_1 e_2)$.

<u>Remark</u> 4.1.11. Then if H is nonsingular $\Rightarrow \hat{H}_{|Cl^k(U)} \subseteq Cl^k(V)$.

<u>Remark</u> 4.1.12. If $U \xrightarrow{H_1} V \xrightarrow{H_2} W$ and $H := H_2 \circ H_1$, then $\hat{H} = \hat{H}_2 \circ \hat{H}_1$

Now we will work with S surface $\subset \mathbb{R}^3$. Let $x, y \in S$.
Let $P_\sigma : T_x(S) \to T_y(S)$ be the *parallel transport* along σ, a smooth curve: $x \to y$. Then, P_σ is an isometry.

<u>Notation</u>. Let $\hat{H} : G_U \to G_V$ be a nonsingular map. Then, $\breve{H} := \hat{H}_{|Cl^0(U)+Cl^2(U)}$.

Proposition 4.1.5. *Let P_σ defined as above, and $x, y \in S$.*

$$\breve{P}_\sigma = \breve{P}_\varphi \quad \forall \sigma, \varphi : x \to y \qquad \Longleftrightarrow \qquad S \textit{ is orientable}$$

Proof. ($\Rightarrow$)
$x \in S$. Let N be a vector field defined in the following way:

$$N(s) := \breve{P}_{x \to s} \left(\frac{\partial}{\partial u_{|x}} \wedge \frac{\partial}{\partial v_{|x}} \right)^*$$

N is well defined by hypothesis. Then,

$$\frac{N(s)}{|N(s)|} \text{ is a normal unit vector field on } S \quad \Rightarrow \quad S \text{ is orientable.}$$

($\Leftarrow$)
S is orientable. First, we want to show that

$$det\, P_\sigma = 1 \quad \text{with the bases } \left\{ \frac{\partial}{\partial u_{|x}}, \frac{\partial}{\partial v_{|x}}, N_x \right\} \to \left\{ \frac{\partial}{\partial u_{|y}}, \frac{\partial}{\partial v_{|y}}, N_y \right\}.$$

Then, let $(U_k, \gamma_k)_k$ be the charts that cover $Im\,\sigma$ (in a finite number n, because $Im\,\sigma$ is a compact space.)
We prove by induction on n:

$(n = 1)$
$\frac{\delta}{\partial u}$ is a local vector field on U_1 so, $P_\sigma\left(\frac{\partial}{\partial u}_{|\sigma(0)}\right) = \frac{\partial}{\partial u}_{|\sigma(1)}$. Same for $\frac{\partial}{\partial v}$. It implies that $det\,P_\sigma = 1$.

(*inductive step*)
$\sigma = \sigma_k * \tilde{\sigma}$, where $detP_{\sigma_k} = 1$ and $P_\sigma = P_{\sigma_k} \circ P_{\tilde{\sigma}}$ with $Im\,\tilde{\sigma} \subset U_{k+1}$.
By induction, $det\,P_{\sigma_k} = det\,P_{\tilde{\sigma}} = 1$ and $det\,P_\sigma = det\,P_{\sigma_k} \cdot |M| \cdot det\,P_{\tilde{\sigma}}$ where

M is the changing base matrix between $\left\{\frac{\partial}{\partial u}_{|k}, \frac{\partial}{\partial v}_{|k}, N_k\right\} \to \left\{\frac{\partial}{\partial u}_{|k+1}, \frac{\partial}{\partial v}_{|k+1}, N_{k+1}\right\}$.

But S is orientable, so $|M| = 1 \quad \Rightarrow \quad det\,P_\sigma = 1$.

Hence, $H := P_\sigma \circ P_\varphi^{-1}$ is an isometry: $det\,H = 1 \quad \forall \sigma, \varphi$ regular paths: $x \to y$.
Therefore $\hat{H}(ab^{-1}) = H(a)/H(b) = a/b$, and then we obtain that
$\hat{P}_\sigma(ab^{-1}) = \hat{P}_\varphi(ab^{-1}) \quad \forall a, b \in T_xS$.
Finally, because $span\langle ab^{-1} \,|\, a, b \in T_xS\rangle = Cl^0(T_xS) + Cl^2(T_xS)$,
we have that $\breve{P}_\sigma = \breve{P}_\varphi$.

□

Let S be an orientable surface $\subset \mathbb{R}^3$.
We will work with φ, an isothermal parametrization of S and $\gamma := \varphi^{-1}$ (locally) the coordinate map.
We call $\alpha(x) := \sqrt{E(\gamma(x))} = \sqrt{\partial_{1|x} \cdot \partial_{1|x}} \quad (\alpha : S \to \mathbb{R}^+)$, and $H := \alpha\, d\gamma$.
Remark 4.1.13. $\forall p \in S \;\Rightarrow\; H(p) : T_pS \to T_{\gamma(p)}\mathbb{R}^2$ is an isometry.
We denote it with H_p. So, $|H_p(v)| = |v| \quad \forall v \in T_pS,\ \forall p \in S$

Lemma 4.1.6. *Let* $B_p := Cl^0(T_pS) + Cl^2(T_pS) \quad \forall p \in S$.
Let $P_{x\to y}$ *be a generic parallel transport and* H_p *defined as above.*
If we denote with $G_x := G_{T_xS}$, *then,* $\forall p \in S$, *we have:*

$$\breve{H}_y \circ \breve{P}_{|G_x} = \breve{H}_{x|G_x}$$

Proof. $b \in B_p$,

$$b = c + \lambda \frac{\partial}{\partial x_1}_{|x} \frac{\partial}{\partial x_2}_{|x} = c + \lambda \alpha^2(x) \frac{\hat{\partial}}{\partial x_1}_{|x} \frac{\hat{\partial}}{\partial x_2}_{|x}$$

Hence,

$$
\begin{aligned}
\breve{H}_y(\breve{P}(b)) &= c + \lambda \breve{H}_y \left(\alpha^2(x) \breve{P} \left(\frac{\hat{\partial}}{\partial x_1}_{|x} \frac{\hat{\partial}}{\partial x_2}_{|x} \right) \right) \\
&= c + \lambda \alpha^2(x) \breve{H}_y \left(\frac{\hat{\partial}}{\partial x_1}_{|y} \frac{\hat{\partial}}{\partial x_2}_{|y} \right) \\
&= c + \lambda \alpha^2(x) e_1 e_2 \\
&= \breve{H}_x \left(c + \lambda \alpha^2(x) \frac{\hat{\partial}}{\partial x_1}_{|x} \frac{\hat{\partial}}{\partial x_2}_{|x} \right) = \breve{H}_x(b)
\end{aligned}
$$

□

Remark 4.1.14. $\breve{H}_x(b) \in Cl^0(T_{\gamma(x)}\mathbb{R}^2) + Cl^2(T_{\gamma(x)}\mathbb{R}^2)$ while $H_y(\breve{P}(b)) \in Cl^0(T_{\gamma(y)}\mathbb{R}^2) + Cl^2(T_{\gamma(y)}\mathbb{R}^2)$, then they belong to two different spaces. But $T_z\mathbb{R}^2 \cong \mathbb{R}^2 \quad \forall z \in \mathbb{R}^2$, so with abuse of notation we have omitted the composition through the two canonical isomorphisms.

4.1.3 Shape Distance on Regular Surfaces

We keep on working with our orientable surface S immersed in $\mathbb{R}^3$ with own isothermal parametrization φ and its local inverse map γ that makes S a 2-dimensional manifold.
We consider $f : I \to S$ a regular smooth curve on S, which means $I = [a, b] \subset \mathbb{R}$ and $f'(t) \in T_{f(t)}S$ such that $f'(t) \neq 0 \quad \forall \in I$.
$f'(t)$ is a vector field on f, $f' = f'_u \frac{\partial}{\partial u}_{|f(t)} + f'_v \frac{\partial}{\partial v}_{|f(t)}$, and we locally define $\breve{f}'(t) := d\gamma(f'(t)) = (f'_u(t), f'_v(t)) \quad \forall t \in I$.

We denote with D the covariant derivative, so by (4.1) and by Remark 4.1.8 we have:

$$
\begin{aligned}
D_f(f')(t) &= \left[\frac{\delta}{\delta t} f'_u + \frac{1}{\alpha} \left(\left(\frac{\partial \alpha}{\partial u}(f(t)) \right) f'_u + \left(\frac{\partial \alpha}{\partial v}(f(t)) \right) f'_v \right) f'_u + \right. \\
&+ \left. \frac{1}{\alpha} \left(\left(-\frac{\partial \alpha}{\partial u}(f(t)) \right) f'_v + \left(\frac{\partial \alpha}{\partial v}(f(t)) \right) f'_u \right) f'_v \right] \frac{\partial}{\partial u} + \\
&+ \left[\frac{\delta}{\delta t} f'_v + \frac{1}{\alpha} \left(\left(-\frac{\partial \alpha}{\partial v}(f(t)) \right) f'_u + \left(\frac{\partial \alpha}{\partial u}(f(t)) \right) f'_v \right) f'_u + \right. \\
&+ \left. \frac{1}{\alpha} \left(\left(\frac{\partial \alpha}{\partial u}(f(t)) \right) f'_u + \left(\frac{\partial \alpha}{\partial v}(f(t)) \right) f'_v \right) f'_v \right] \frac{\partial}{\partial v}
\end{aligned}
$$

Now before continuing we observe some properties:

Remark 4.1.15.

$$(i)\quad \frac{\partial}{\partial u} f'_u + \frac{\partial}{\partial v} f'_v = \langle \nabla\alpha, \tilde{f}' \rangle \qquad \left(\nabla\alpha := \left(\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial v} \right) \right)$$

$$(ii)\quad \left(\frac{\partial}{\partial u} f'_v - \frac{\partial}{\partial v} f'_u \right) e_1 e_2 = \nabla\alpha \wedge \tilde{f}'$$

$$(iii)\quad (\nabla\alpha \wedge \tilde{f}')\tilde{f}' = \left(\left(\frac{\partial}{\partial u} f'_v - \frac{\partial}{\partial v} f'_u \right) f'_v \,,\, \left(-\frac{\partial}{\partial u} f'_v + \frac{\partial}{\partial v} f'_u \right) f'_u \right)$$

Finally we can simply the formula, obtaining that

$$d\gamma(D_f(f')) = \tilde{f}'' + \frac{1}{\alpha}\langle \nabla\alpha, \tilde{f}' \rangle \tilde{f}' - \frac{1}{\alpha}(\nabla\alpha \wedge \tilde{f}')\tilde{f}' =$$
$$= \tilde{f}'' + \frac{\tilde{f}' \nabla\alpha \tilde{f}'}{\alpha}.$$

Hence, we have proved the following lemma,

Lemma 4.1.7. *Let S be an immersed orientable surface with the isothermal coordinate map γ and $\alpha := \sqrt{E}$. If $f : I \to S$ is a regular smooth curve,*

$$d\gamma_{f(t)}(D_{f(t)}(f'(t))) = (\gamma(f(t))'' + \frac{\gamma(f(t))'\,(\nabla\alpha)\,\gamma(f(t))'}{\alpha} \qquad \forall t \in I$$

The following remark will be useful later.

Remark 4.1.16.

$$\tilde{H}_{f(t)}({}^{Df'}\!/_{f'}) = \widetilde{d\gamma f(t)}\ ({}^{Df'}\!/_{f'}) =$$
$$= d\gamma f(t)\ (Df') \cdot (\gamma(f)')^{-1} =$$
$$= {}^{\gamma(f)''}\!/_{\gamma(f)'} + \frac{1}{\alpha}\gamma(f)'\nabla\alpha$$

Finally we can define our "distance":

Definition 4.1.8. Let $f, g : I \to S$ be two regular smooth curves on S an orientable immersed surface in $\mathbb{R}^3$.

$$D_S(f,g) := \frac{1}{2}\int_I \left\| \tilde{P}_{f(t)\to g(t)}({}^{D_f f'}\!/_{f'}) - {}^{D_g g'}\!/_{g'} \right\| dt$$

Remark 4.1.17. If $S = \mathbb{R}^2 \quad \Rightarrow \quad D_S(f,g) = \int_I || {}^{f''}\!/_{f'} - {}^{g''}\!/_{g'} || \, dt.$
We can recognize the "distance" defined on the plane.

Remark 4.1.18. As we wanted, we can easily notice that if f, g are two geodesics, $D_S(f,g) = 0$.

Remark 4.1.19. As in the 2D case, this "distance" is a pseudometric too.
To prove that, we need to write it in a little different way: next theorem will help us in this regard.

Theorem 4.1.9. *With the same hypothesis of the previous lemma, if γ is the isothermal coordinate map of S with its own α, then*

$$D(f,g) = \frac{1}{2}\int_I \left\| \left({\gamma(f)''}/{\gamma(f)'} - {\gamma(g)''}/{\gamma(g)'} \right) + \left(\gamma(f)' \frac{\nabla\alpha(f(t))}{\alpha(f(t))} - \gamma(g)' \frac{\nabla\alpha(g(t))}{\alpha(g(t))} \right) \right\| dt$$

Proof. Using Lemma 4.1.6, Lemma 4.1.7 and Remark 4.1.16,

$$\begin{aligned} D_S(f,g) &= \frac{1}{2}\int_I \left\| \tilde{P}({D_f f'}/{f'}) - {D_g g'}/{g'} \right\| dt = \\ &= \frac{1}{2}\int_I \left\| \tilde{H}_g \left(\tilde{P}({D_f f'}/{f'}) - {D_g g'}/{g'} \right) \right\| dt = \\ &= \frac{1}{2}\int_I \left\| \tilde{H}_f({D_f f'}/{f'}) - \tilde{H}_g({D_g g'}/{g'}) \right\| dt = \end{aligned}$$

$$= \frac{1}{2}\int_I \left\| \left({\gamma(f)''}/{\gamma(f)'} - {\gamma(g)''}/{\gamma(g)'} \right) + \left(\gamma(f)' \frac{\nabla\alpha(f(t))}{\alpha(f(t))} - \gamma(g)' \frac{\nabla\alpha(g(t))}{\alpha(g(t))} \right) \right\| dt$$

□

Remark 4.1.20. The foregoing result makes easier to see the pseudometric properties of D_S.

4.2 Algorithms for 2.5D Gestures

In computer science, sometimes the recognition algorithms for a surface are referred as "2.5D algorithms". We adopt such terminology here.

Theorem 4.1.9 can move our focus on other scenarios; as a matter of fact, we have expressed our distance D without using the parallel transport.
Moreover we can work with this formula to not involve the Clifford operators too, founding a nice implementable algorithm to compute this "distance".

Remark 4.2.1. If we choose the same partition used in the bidimensional case (and the same notation), thanks to Theorem 4.1.9 we have that, for the isochronous case,

$$
\begin{aligned}
D_S(f,g) &= \\
&= \lim_{P_n} \left\| {}^{(\tilde f_{i+1}-\tilde f_i)}\!\big/_{(\tilde f_i-\tilde f_{i-1})} - {}^{(\tilde g_{i+1}-\tilde g_i)}\!\big/_{(\tilde g_i-\tilde g_{i-1})} + \right. \\
&\quad \left. + (t_{i+1}-t_{i-1})\left((\tilde f_{i+1}-\tilde f_{i-1})\frac{\nabla\alpha(f_i)}{\alpha(f_i)} - (\tilde g_{i+1}-\tilde g_{i-1})\frac{\nabla\alpha(g_i)}{\alpha(g_i)} \right) \right\|
\end{aligned}
$$

As we have done for the 2D case, we can still extend the algorithm to the non-isochronous case.
Now, before we can formulate our new two algorithms for gesture recognition on any regular surface, we need to do some modifications.

Remark 4.2.2. We can notice that in the last formula we expressed D_S only with elements of the even Clifford algebra of $\mathbb{R}^2$.
Then we want to use again the complex numbers to express our algorithm so that is easier to implement it.

We know that if $u, v \in \mathbb{R}^2$ $(u = xe_1 + ye_2)$, and $\mathbf{u} = x + iy$, then ${}^{u}\!/_{v} = j\left(\frac{\mathbf{u}}{\mathbf{v}}\right)$.
Hence,

$$
uv^{-1} = {}^{u}\!/_{v} = j\left(|\mathbf{v}|^2\frac{\mathbf{u}}{\mathbf{v}}\right) = j\left(\mathbf{u}\bar{\mathbf{v}}\right).
$$

Then we can finally show the algorithm,

Algorithm. ***!SFTL*** *("S" stands for surface).*

INPUT= $\{f_0, \ldots, f_n, g_0, \ldots, g_n\}$
(the samples for two gestures f, g)
$\{\gamma;\ \alpha\}$ *(The isothermal coordinate map with* $\alpha(t) := \sqrt{E_\gamma}$*)*
(If $\gamma(f) = \big(r(t), s(t)\big)$*, we denote with* $\mathbf{f} = r(t) + i\,s(t)$*.)*

$$
!SFTL(f_0, \ldots, f_n, g_0, \ldots, g_n) =
$$

$$
\sum_{k=1}^{n-1}\left| \frac{\Delta\mathbf{f}_k}{\Delta\mathbf{f}_{k+1}} - \frac{\Delta\mathbf{g}_k}{\Delta\mathbf{g}_{k+1}} + \frac{1}{n}\left((\mathbf{f}_{i+1}-\mathbf{f}_{i-1})\frac{\overline{\nabla\alpha(f_i)}}{\alpha(f_i)} - (\mathbf{g}_{i+1}-\mathbf{g}_{i-1})\frac{\overline{\nabla\alpha(g_i)}}{\alpha(g_i)} \right) \right|_{\mathbb{C}}
$$

And similarly we can extend it to the non-isochronous case,

Algorithm. ***!WSFTL***.

INPUT= $\{f_0,\ \ldots\ , f_n,\ g_0,\ \ldots\ , g_n,\ t_0,\ \ldots\ , t_n\}$
(The samples and the timestamps)
$\{\gamma,\ \alpha\}$ *(The isothermal coordinate map with* $\alpha(t) := \sqrt{E_\gamma}$*)*
(If $\gamma(f) = \big(r(t), s(t)\big)$*, we denote with* $\mathbf{f} = r(t) + i\, s(t)$*.)*

$$\begin{aligned} &!WSFTL(f_0,\ \ldots\ , f_n\ ,\ g_0,\ \ldots\ , g_n) = \\ &\sum_{k=1}^{n-1} \left| \frac{t_{k+1} - t_k}{t_k - t_{k-1}} \left(\frac{\Delta \mathbf{f}_k}{\Delta \mathbf{f}_{k+1}} - \frac{\Delta \mathbf{g}_k}{\Delta \mathbf{g}_{k+1}} \right) + \right. \\ + \quad & \left. \frac{t_{k+1} - t_{k-1}}{2} \left((\mathbf{f}_{i+1} - \mathbf{f}_{i-1}) \frac{\overline{\nabla \alpha(f_i)}}{\alpha(f_i)} - (\mathbf{g}_{i+1} - \mathbf{g}_{i-1}) \frac{\overline{\nabla \alpha(g_i)}}{\alpha(g_i)} \right) \right|_{\mathbb{C}} \end{aligned}$$

Chapter 5

Shape Distance in Higher Dimensions

In this chapter we will extend some results to dimension 3, and then to higher dimension.
Of course, we cannot anymore use the isomorphism with complex numbers (See Remark 3.1.2) to develop our results. The Clifford algebra instruments are now essential to the new developments.

5.1 Similarity Invariance in Higher Dimensions

When we step into the tridimensional case, it is reasonable to lose some similarity invariances.
More precisely, we want to distinguish space gestures lying in different planes, for several reasons. The first reason is that some gestures can be obtained, one from the other, by both direct and indirect similarities. For example if g is a space gesture lying in a plane and f is its symmetrical image through the line directed by a vector v, lying on that same plane, we can get f from g, rotating the last one $180°$ degree about the line (See the figure 5.1).
However, it is reasonable to consider g to be distinct from f, as a space gesture. As we want to use a "3D shape" to compare space gestures, we want it not to be too invariant with respect to 3D simmetries.

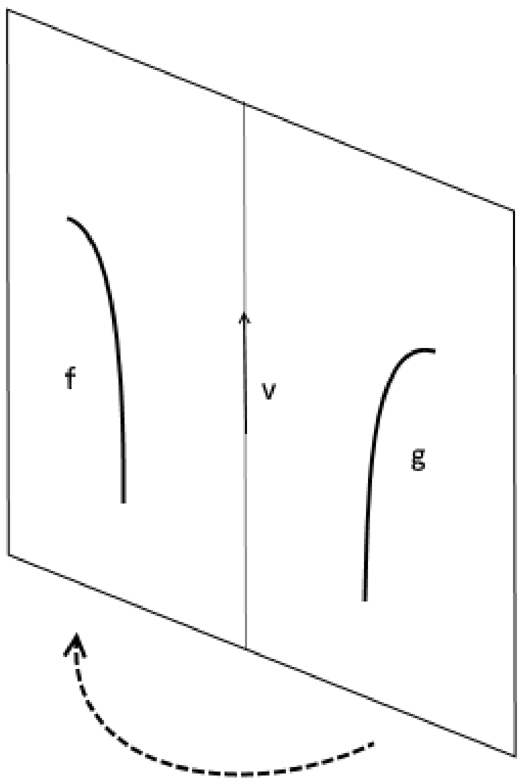


Figure 5.1: f and g

So, we have a choice between considering all the similarities, both direct and indirect, as invariant, or losing this invariance in most cases.
We adopt to the last option, because, for our purposes, we reject an invariance between symmetric opposite gestures.
Nevertheless, the generalized local shape distance will still be invariant for direct similarities in the same plane, while it will be sensible not only to plane shape differences, but also it will be able to measure of how much two gestures are far from being locally coplanar.

To do that we will start simply extending our previous definition in the general case,

Definition 5.1.1.

- A **gesture** is a regular curve $f : I = [0, 1] \to \mathbb{E}^n$,
- A $(m + 1)$-sample of a gesture is $\{f_0, \ \dots \ , f_m\} \subset f(I)$.

- A **basic gesture** in $\mathbb{E}^n$ is an ordered couple of not null vectors (v_1, v_2) considered again as a particular 2-sample of a gesture tracing a triangle.
- The **shape** of a basic gesture (v_1, v_2) is the Clifford ratio ${}^{v_1}\!/\!_{v_2}$.
- The **Local Shape Distance** between two basic gestures (u_1, u_2) and (v_1, v_2) is the non-negative real number

$$LSD\big((u_1,u_2),(v_1,v_2)\big) = \left\| \, {}^{u_1}\!/\!_{u_2} \; - \; {}^{v_1}\!/\!_{v_2} \, \right\|$$

Obviously, we can't use anymore the Lemma 3.1.1 outside of the plane and then $ab^{-1} = cd^{-1}$ is not anymore an equivalent condition of similarity between basic gestures, but with the next lemma we recover the same properties required before, making easier the extension in higher dimensions of our algorithms.

Lemma 5.1.2. *Let (a, b) and (c, d) two basic gestures in a euclidean space $\mathbb{E}^n$. Then,*

$$ab^{-1} = cd^{-1} \iff a, b, c, d \text{ are coplanar, and } (a, b) \approx (c, d)$$

Proof. $(\Leftarrow)$
If a, b, c, d are coplanar and $(a, b), (c, d)$ are direct similar we can reduce everything at the planar case, and then easily, $ab^{-1} = cd^{-1}$.
$(\Rightarrow)$
The other direction is the key result, that can be easily obtained as follows. Observe that $ab^{-1} = cd^{-1}$ implies that the bivector parts are equal too, hence $a \wedge b = c \wedge d$. Then a, b, c, d are coplanar and then for the 2−dimensional case we have that $(a, b) \approx (c, d)$ too. □

Next result is the extended version of Theorem 3.1.9.

Theorem 5.1.3.
$I = [0, 1] \quad f, g : I \to \mathbb{E}^n \in C^2(I) : \; f'(t), g'(t) \neq 0 \quad \forall t \in I.$
Let P_n be a partition of $I := \left\{ t_{k,n} := \frac{k}{n} \,\middle|\, k = 0, \cdots, n \right\}$ *and*

$$f_{k,n} := f(t_{k,n})$$

$$g_{k,n} := g(t_{k,n})$$

Then:

$$\lim_{n\to+\infty} \sum_{k=1}^{n-1} \left\| \, {}^{(f_{k,n}-f_{k-1,n})}\!/\!_{(f_{k+1,n}-f_{k,n})} \; - \; {}^{(g_{k,n}-g_{k-1,n})}\!/\!_{(g_{k+1,n}-g_{k,n})} \, \right\| =$$

$$= \int_I \left\| {}^{f''(t)}\!\big/_{f'(t)} - {}^{g''(t)}\!\big/_{g'(t)} \right\| dt \tag{5.1}$$

Now we can't use the complex isomorphism, so we will going straight with the use of Clifford numbers.
The proof is quite similar to the planar case. What we have done is just to substitute the complex product with the Clifford one, despite the commutativity of the first respect than the not-commutative of the last.
This is the reason for which we here will not comment as before and we will almost rewrite the proof.

Proof. By hypothesis, the Riemann integral $\int_0^1 \left\| {}^{f''(t)}\!\big/_{f'(t)} - {}^{g''(t)}\!\big/_{g'(t)} \right\| dt$ exists; this implies that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{n} \left\| {}^{f''(\xi_k)}\!\big/_{f'(\xi_k)} - {}^{g''(\xi_k)}\!\big/_{g'(\xi_k)} \right\| \frac{1}{n} - \int_0^1 \left\| {}^{f''(t)}\!\big/_{f'(t)} - {}^{g''(t)}\!\big/_{g'(t)} \right\| dt \right| < \epsilon \,,$$

provided $n > N_\epsilon$, and $\xi_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$, with $k = 1, \dots, n$.
Notice that, to evaluate each shape ${}^{\Delta g_k}\!\big/_{\Delta g_{k+1}} := {}^{g_{k,n}-g_{k-1,n}}\!\big/_{g_{k+1,n}-g_{k,n}}$, the extremities of two adjacent intervals are needed. In particular, we can write

$$\sum_{k=1}^{2m-1} {}^{\Delta g_k}\!\big/_{\Delta g_{k+1}} = \sum_{h=1}^{m} {}^{\Delta g_{2h-1}}\!\big/_{\Delta g_{2h}} + \sum_{h=1}^{m-1} {}^{\Delta g_{2h}}\!\big/_{\Delta g_{2h+1}} \,, \tag{5.2}$$

when n is even.
If $n > 2N_\epsilon$, then the integral can be estimated both by

$$\left| \sum_{h=1}^{m-1} \left\| {}^{f''(\xi_h^o)}\!\big/_{f'(\xi_h^o)} - {}^{g''(\xi_h^o)}\!\big/_{g'(\xi_h^o)} \right\| \frac{1}{n} + \right.$$
$$\left. - \frac{1}{2} \int_0^1 \left\| {}^{f''(t)}\!\big/_{f'(t)} - {}^{g''(t)}\!\big/_{g'(t)} \right\| dt \right| < \frac{\epsilon}{2} \,,$$

where $\xi_h^o \in \left[\frac{2h-1}{n}, \frac{2h+1}{n}\right]$, with $h = 1, \dots, m$.
A similar expression holds when n is even.
Then, to obtain the thesis, it suffices to see how to estimate the following quantity,

$$\left\| {}^{\Delta f_{2h}}\!\big/_{\Delta f_{2h+1}} - {}^{\Delta g_{2h}}\!\big/_{\Delta g_{2h+1}} - \left({}^{g''(\xi_h^o)}\!\big/_{g'(\xi_h^o)} - {}^{f''(\xi_h^o)}\!\big/_{f'(\xi_h^o)} \right) \frac{1}{n} \right\| =$$
$$= \left\| \left({}^{\Delta f_{2h}}\!\big/_{\Delta f_{2h+1}} - 1 + {}^{f''(\xi_h^o)}\!\big/_{f'(\xi_h^o)} \frac{1}{n} \right) + \right.$$
$$\left. + \left(1 - {}^{\Delta g_{2h}}\!\big/_{\Delta g_{2h+1}} - {}^{g''(\xi_h^o)}\!\big/_{g'(\xi_h^o)} \frac{1}{n} \right) \right\| \,,$$

for each $h = 1, \ldots, m$.
Now, assuming $\delta = \frac{1}{n}$, then, by hypothesis, the function g is twice differentiable and $g' \neq 0$, thus we have that, for every $t \in [0,1]$

$$\lim_{\delta \to 0} \frac{g(t+\delta)-2g(t)+g(t-\delta)}{\delta^2} \Big/ \frac{g(t+\delta)-g(t)}{\delta} = g''(t) \big/ g'(t) \quad .$$

So, we have that, for every $\epsilon > 0$ there exists δ_ϵ, such that if $\delta < \delta_\epsilon$, then

$$\begin{aligned} &\left\|1 - \Delta g_{2h} \big/ \Delta g_{2h+1} - g''(\xi_h^o) \big/ g'(\xi_h^o)\ \delta\right\| = \\ &= \left\| \frac{g(t_{2h}+\delta)-2g(t_{2h})+g(t_{2h}-\delta)}{\delta^2} \Big/ \frac{g(t_{2h}+\delta)-g(t_{2h})}{\delta} - \frac{g''(\xi_h^o)}{g'(\xi_h^o)} \right\| \delta < \epsilon\delta \ , \end{aligned}$$

and this prove the thesis, provided $\delta < \min\{\delta_\epsilon, \frac{1}{2N_\epsilon}\}$. □

□

Hence, we can also extend Definition 3.1.12.

Definition 5.1.4.

- The shape of a gesture g, is the following function

$$S(g(t)) = 1 - \frac{1}{2}\ g''(t) \big/ g'(t)$$

- The distance between the shape of two gestures f, g is the following operator

$$\begin{aligned} D_S(f,g) &= \int_I \left\|S(f(t)) - S(g(t))\right\| dt = \\ &= \frac{1}{2} \int_I \left\| f''(t) \big/ (f'(t)) - g''(t) \big/ (g'(t)) \right\| dt. \end{aligned}$$

Moreover, as announced before, a same adaptation can be done with the non-isochronous case.

Unlike the 2D case, shapes and gestures have not anymore the same dimension. In fact, the shape belongs to $Cl^0 \oplus Cl^2$, then its dimension is $1 + \binom{n}{2}$, different from n if this is greater than 2.
However, it is still possible to set up the ODE system $S(f(t)) = s(t)$ for any given

shape s, but it will differ based on dimension.

Remark 5.1.1. If a gesture f has constant speed (that is $\frac{d}{dt}|f'| = 0$), we have that $\langle f'', f' \rangle = \frac{1}{2}\frac{d}{dt}(|f'|^2) = 0$.

The last remark show that the shape of f has always the scalar component equal to 1, and this let us to go towards different scenarios.
For example, as the 2D case, we can consider the shape of 3D gesture with constant speed as a gesture too (once proven its regularity).

Example 6. Let $g(t) := (x(t), y(t), z(t))$ be the following 3D gesture:

$$x(t) := 4\cos(t)$$
$$y(t) := 2t + \sin(2t)$$
$$z(t) := \cos(2t)$$

According to Example 5, instead of the canonical shape, we continue to consider $1 - S(g) = \frac{1}{2} {}^{g''}\!/_{g'}$ in this example.

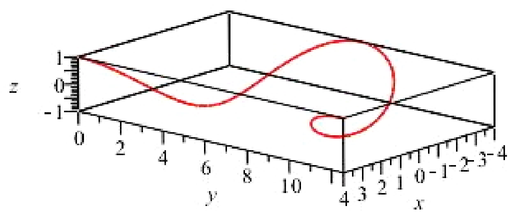


Figure 5.2: The gesture $g(t)$

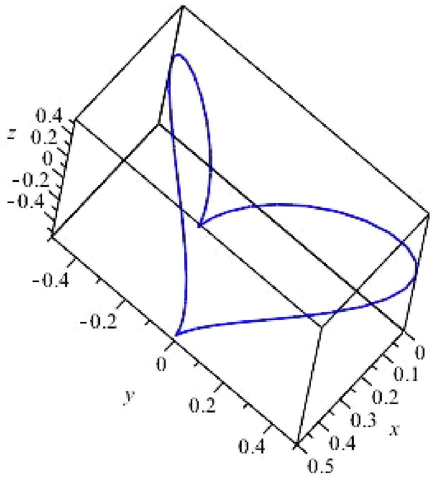


Figure 5.3: the "adapted" shape $1 - S(g(t))$

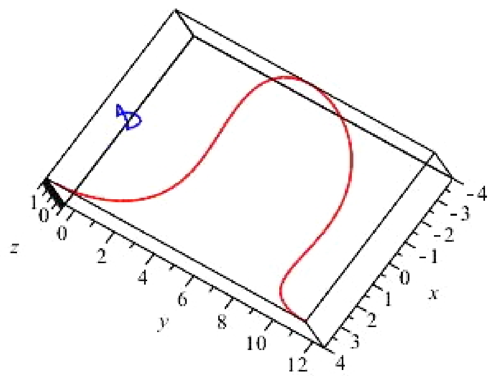


Figure 5.4: $g(t)$ and $1 - S(g(t))$ together

To draw the shape, if $1 - S(g) = \alpha_x\, e_2 \wedge e_3 + \alpha_y\, e_3 \wedge e_1 + \alpha_z\, e_1 \wedge e_2$, we considered it as the gesture $(\alpha_x, \alpha_y, \alpha_z)$.

Summing up, with the loss of some similarity invariance properties, we mantain the same key theorems and consequently the same algorithms:

Algorithm. ***!FTL***
INPUT= $\{f_0, \ \dots\ , f_n, \ g_0, \ \dots\ , g_n\}$
(the samples for two gestures f, g*)*

$$\begin{aligned} !FTL(f_0, \ \dots\ , f_n \ , \ g_0, \ \dots\ , g_n) &= \sum_{k=1}^{n-1} LSD\big((\Delta f_k, \Delta f_{k+1}), (\Delta g_k, \Delta g_{k+1})\big) \\ &= \sum_{k=1}^{n-1} \left\| \ {}^{\Delta f_k}\!/_{\Delta f_{k+1}} - {}^{\Delta g_k}\!/_{\Delta g_{k+1}} \right\| \ . \end{aligned}$$

Remark 5.1.2. We can explicity express the algorithm in basic operations observing that

$$LSD(a, b)(c, d)^2 = \frac{1}{|b|^2|d|^2}(|a|^2|d|^2+|b|^2|c|^2-2(\langle a, b\rangle\langle c, d\rangle-\langle a, d\rangle\langle b, c\rangle+\langle a, c\rangle\langle b, d\rangle).$$

This is obviously true for the complex notation too.

Algorithm. ***!WFTL***

INPUT= $\{f_0,\ \dots\ ,f_n,\ g_0,\ \dots\ ,g_n,\ t_0,\ \dots\ ,t_n\}$
(the samples and the timestamps)

$$\begin{aligned} !WFTL(f_0,\ \dots\ ,f_n\ ,\ g_0,\ \dots\ ,g_n) &= \sum_{k=1}^{n-1} \frac{t_{k+1}-t_k}{t_k-t_{k-1}} LSD\big((\Delta f_k, \Delta f_{k+1}), (\Delta g_k, \Delta g_{k+1})\big) \\ &= \sum_{k=1}^{n-1} \frac{t_{k+1}-t_k}{t_k-t_{k-1}} \left\| \ {}^{\Delta f_k}\!/_{\Delta f_{k+1}} - {}^{\Delta g_k}\!/_{\Delta g_{k+1}} \ \right\| \ . \end{aligned}$$

with,

$$(LSD(a,b)(c,d))^2 = \frac{1}{|b|^2|d|^2}(|a|^2|d|^2+|b|^2|c|^2-2(\langle a,b\rangle\langle c,d\rangle-\langle a,d\rangle\langle b,c\rangle+\langle a,c\rangle\langle b,d\rangle).$$

Chapter 6

Conclusions

There are various implications of the work done here, but now we want to focus only on three different aspects that can be generalized starting from our results.

ODE features for shape problems

A first reasonable way to develop this work is, for example, to deepen the bond between a gesture and its shape through the use of other ODE systems, expanding what we have already done at the end of Section 3 and partially in Section 5 too. For example, we can search for periodic solutions, for fixed points, or for the stability of these systems (which are mostly autonomous).

Transformations on regular surfaces

Another interesting way to continue this work could be to focus on Definition 4.1.8. It is a measurer of the "dissimilarity" of two paths on a surface, ignoring the various curvatures of the surface.
That can help, for example, to define some of the most well-known geometrical transformations on any regular surface, letting "move" the curve on the surface keeping its peculiarities.

Non-smooth planar gestures

The software engineer Nathan Magrofuoco is currently working for his Ph.D. thesis at the Belgian university UCLouvain on a new algorithm (named $C). This algorithm is adopting the Local Shape Distance as a metric to recognize gestures that are not necessarily smooth. As the rate of recognition of $C and its speed are comparable[1] to those of the well established algorithm $P, it would be interesting to investigate the convergence of $C. Is there a Sobolev-like framework within which $C is convergent? If this is the case, does such convergence phenomenon correspond to a notion of shape for non-smooth gestures? In that case, what would be its relation with the notion of shape given in this work and in [10]?

[1] While we are writing this work, N.Magrofuoco is working to improve such already satisfactory results.

Appendix A

Proof of Theorem 2.1.8

Here we give a complete proof of Theorem 2.1.8 in line with the axiomatics introduced in this paper.
We use the same notation used in the first chapter.
Hence, let V be a n-dimensional vector space on $\mathbb{K}$, with a non-degenerate quadratic form Q, $Cl_Q(V)$ the Clifford algebra over V, and $Cl^k(V)$ the space of k-multivectors.

Lemma A.0.1. *If $g_1 \cdots g_n$ is a blade of $Cl(V)$ (hence these are mutually orthogonal), then for every set $\mathcal{A} \subseteq \{1, ..., n\}$ we have that:*

$$g_i g_{\mathcal{A}} = \begin{cases} (-1)^{\#\mathcal{A}}\ g_{\mathcal{A}} g_i & \textit{if} \quad i \notin \mathcal{A} \\ -(-1)^{\#\mathcal{A}}\ g_{\mathcal{A}} g_i & \textit{if} \quad i \in \mathcal{A}. \end{cases}$$

That implies that

$$\frac{1}{2}(g_i g_{\mathcal{A}} + (-1)^{\#\mathcal{A}}\ g_{\mathcal{A}} g_i) = \begin{cases} g_i g_{\mathcal{A}} = \pm g_{\mathcal{A} \cup \{i\}} & \textit{if} \quad i \notin \mathcal{A} \\ 0 & \textit{if} \quad i \in \mathcal{A}. \end{cases}$$

and

$$\frac{1}{2}(g_i g_{\mathcal{A}} - (-1)^{\#\mathcal{A}}\ g_{\mathcal{A}} g_i) = \begin{cases} 0 & \textit{if} \quad i \notin \mathcal{A} \\ g_i g_{\mathcal{A}} = \pm g_i^2 g_{\mathcal{A} \cup \{i\}} & \textit{if} \quad i \in \mathcal{A}. \end{cases}$$

Proof. In order g_i "passes through" $g_{\mathcal{A}}$ from left to right, it suffices to apply the property of anti-commutativity for mutually orthogonal vectors. If $i \notin \mathcal{A}, g_i$ anti-commutes with every element g_j of monomial $g_{\mathcal{A}}$; if $i \in \mathcal{A}, g_i$ anti-commutes with every element g_j of monomial $g_{\mathcal{A}}$, except when $j = i$; in this case it trivially commutes with itself, and $g_i^2 \in \mathbb{K}$ (by Clifford's axiom). □

<u>*Remark*</u> A.0.1. If $\varnothing \neq \mathcal{A} \subset \{1, ..., n\}$, there always exists $i_{\mathcal{A}+}, i_{\mathcal{A}-} \in \{1, ..., n\}$ such

that

$$g_{i_{\mathcal{A}+}} g_{\mathcal{A}} = g_{\mathcal{A}} g_{i_{\mathcal{A}+}}$$
$$g_{i_{\mathcal{A}-}} g_{\mathcal{A}} = -g_{\mathcal{A}} g_{i_{\mathcal{A}-}}$$

Proposition A.0.2.

$$\{g_{\mathcal{A}}\}_{\substack{\mathcal{A} \subseteq \{1,...,n\} \\ \#\mathcal{A} = k}} \quad \textit{is a basis for} \ \ Cl^k(V)$$

Proof. We have only to show that it is a linearly independent set, that is, if

$$\sum_{\substack{\mathcal{A} \subseteq \{1,...,n\} \\ \#\mathcal{A} = k}} \alpha_{\mathcal{A}} g_{\mathcal{A}} = 0,$$

then every coefficient $\alpha_{\mathcal{A}}$ must be zero.
Let it be a fixed $\mathcal{B} \subseteq \{1, ..., n\}$ with k elements, and rewrite the foregoing relation as

$$\alpha_{\mathcal{B}} g_{\mathcal{B}} + \sum_{\substack{\mathcal{A} \subseteq \{1,...,n\} \\ \#\mathcal{A} = k \\ \mathcal{A} \neq \mathcal{B}}} \alpha_{\mathcal{A}} g_{\mathcal{A}} = 0 \tag{A.1}$$

we want to show that $\alpha_{\mathcal{B}} = 0$. Relation (A) implies that, for each $i \in \{1, ..., n\}$

$$\frac{\alpha_{\mathcal{B}}}{2}(g_i g_{\mathcal{B}} + (-1)^k g_{\mathcal{B}} g_i) + \sum_{\substack{\mathcal{A} \subseteq \{1,...,n\} \\ \#\mathcal{A} = k \\ \mathcal{A} \neq \mathcal{B}}} \frac{\alpha_{\mathcal{A}}}{2}(g_i g_{\mathcal{A}} + (-1)^k g_{\mathcal{A}} g_i) = 0$$

If $i \in \{1, ..., n\}$ is such that $i \in \mathcal{A}, i \notin \mathcal{B}$, then by the previous lemma, we have that

$$g_i g_{\mathcal{B}} + (-1)^k g_{\mathcal{B}} g_i = \pm g_{\lfloor \cup \{i\}} \quad \text{and} \quad g_i g_{\mathcal{A}} + (-1)^k g_{\mathcal{A}} g_i = 0$$

producing the new relation for ever $i \in \mathcal{B}^C$

$$\pm \alpha_{\mathcal{B}} g_{\mathcal{B} \cup \{i\}} + \sum_{\substack{\mathcal{A} \subseteq \{1,...,n\} \\ \#\mathcal{A} = k \\ \mathcal{A} \neq \mathcal{B} \\ i \notin \mathcal{A}}} \pm \alpha_{\mathcal{A}} g_{\mathcal{A} \cup \{i\}} = 0$$

Now we can easily recognize the (A) again and then, we obtain that:

$$\frac{\alpha_{\mathcal{B}}}{2}(g_i g_{\mathcal{B} \cup \{i\}} + (-1)^k g_{\mathcal{B} \cup \{i\}} g_i) + \sum_{\substack{\mathcal{A} \subseteq \{1,...,n\} \\ \#\mathcal{A} = k \\ \mathcal{A} \neq \mathcal{B} \\ i \notin \mathcal{A}}} \frac{\alpha_{\mathcal{A}}}{2}(g_i g_{\mathcal{A} \cup \{i\}} + (-1)^k g_{\mathcal{A} \cup \{i\}} g_i) = 0$$

Then we can iterate this foregoing elimination for every $\imath \in \mathcal{B}^C$and observing that $\mathcal{A} \cap \mathcal{B}^C = \varnothing \Leftrightarrow \mathcal{A} = \mathcal{B}$, we find that

$$\pm\alpha_{\mathcal{B}} g_{\mathcal{B}\cup\mathcal{B}^C} = 0 \quad \Longleftrightarrow \quad \alpha_{\mathcal{B}} g_1 \cdots g_n = 0$$

Hence, $\alpha_{\mathcal{B}} = 0$ by axiom (G).

□

Corollary A.0.3. *Given a finite dimensional quadratic space (V,Q), then*

$$\dim Cl^k(V) = \binom{\dim V}{k}$$

Now we only need to prove that $Cl^k(V) \cap Cl^h(V) = \varnothing$ for every $h \neq k$. This is trivial, by (G), if $h = 0$ and $k = \dim V$.

Remark A.0.2. For every (non-degenerate) quadratic space V, we can find a pseudo-orthonormal basis $\{e_i\}_i$.

Notation. In $Cl(V)$, $\dim \mathbb{K} = \dim Cl^{\dim V}(V) = 1$ but we have just shown that they represent different elements in the space. So we call *pseudo-scalar* an element in $Cl^{\dim V}(V)$.
If $\{e_1, ..., e_n\}$ is a pseudo-orthonormal basis of V, then we call *pseudo-unit* the element $e_1 \cdots e_n \in Cl^{\dim V}(V) \subseteq Cl(V)$ (called orientations of V).
If A, B are two sets, then the *symmetric difference* is
$A \triangle B := (A \cup B) \backslash (A \cap B)$.

Remark A.0.3. If $e_1, .., e_n$ is a pseudo-orthonormal basis, then for each $\mathcal{A}, \mathcal{B} \subseteq \{1, ..., n\}$, it is trivial to prove that

$$e_{\mathcal{A}} e_{\mathcal{B}} = \pm e_{\mathcal{A} \triangle \mathcal{B}}$$

Lemma A.0.4. *For every $\mathcal{B} \subseteq \{1, ..., n\}$ let $i_{\mathcal{B}} : \mathcal{P}(\{1, ..., n\}) \longrightarrow \mathcal{P}(\{1, ..., n\})$ such that $i_{\mathcal{B}}(\mathcal{A}) = \mathcal{A} \triangle \mathcal{B}$.*
Then, $i_{\mathcal{B}}$ is a bijection.

Proof. It is trivial if we observe that $(\mathcal{A} \triangle \mathcal{B}) \triangle \mathcal{B} = \mathcal{A}$. (So, $i_{\mathcal{B}}$ is an involution). □

Proposition A.0.5. *If $e_1, .., e_n$ is a pseudo-orthonormal basis of a non-degenerate quadratic space V, then*

$$\{e_{\mathcal{A}}\}_{\mathcal{A} \subseteq \{1,...n\}} \quad \textit{is a basis for } Cl(V)$$

Thus, $\quad \dim Cl(V) = 2^{\dim V}$.

Proof. We have to show that if $\sum_{\mathcal{A}\subseteq\{1,...,n\}} \alpha_{\mathcal{A}} e_{\mathcal{A}} = 0$, then every coefficient $\alpha_{\mathcal{A}}$ must be zero. we can assume (by contradiction) that there exists at least a coefficient $\alpha_{\mathcal{B}} \neq 0$. Let us rewrite the foregoing relation as:

$$\alpha_{\mathcal{B}} e_{\mathcal{B}} + \sum_{\substack{\mathcal{A} \subseteq \{1,...,n\} \\ \mathcal{A} \neq \mathcal{B}}} \alpha_{\mathcal{A}} e_{\mathcal{A}} = 0,$$

Now (last remark) every $u_{\mathcal{A}}$ is pseudo-invertible and for Lemma A.0.4) we have that, multiplying both terms for $e_{\mathcal{B}}^{-1}$, we have that (with a change of sign if needed)

$$\alpha_{\mathcal{B}} + \sum_{\substack{\mathcal{A} \subseteq \{1,...,n\} \\ \mathcal{A} \neq \varnothing}} \beta_{\mathcal{A}} e_{\mathcal{A}} = 0.$$

Now we can rewrite this expression in the following way,

$$\alpha_{\mathcal{B}} + \beta_{\{1,...,n\}} e_{\{1,...,n\}} + \beta_{\mathcal{C}} e_{\mathcal{C}} + \sum_{\substack{\mathcal{A} \subset \{1,...,n\} \\ \mathcal{A} \neq \varnothing;\mathcal{C}}} \beta_{\mathcal{A}} e_{\mathcal{A}} = 0, \tag{A.2}$$

for some proper $\mathcal{C} \subset \{1,...,n\}$. By Remark A.0.1 (and observing that $e_i^{-1} = e_i$ because we have a pseudo-orthonormal basis) we know that exist $i_{\mathcal{C}+}, i_{\mathcal{C}-}$ such that:

$$(e_{i_{\mathcal{C}+}}) e_{\mathcal{C}} (e_{i_{\mathcal{C}+}}) = (e_{i_{\mathcal{C}+}})^2 e_{\mathcal{C}} \quad \text{and} \quad (e_{i_{\mathcal{C}-}}) e_{\mathcal{C}} (e_{i_{\mathcal{C}-}}) = -(e_{i_{\mathcal{C}-}})^2 e_{\mathcal{C}}$$

Now we have to distinguish two cases:

(i) $s := (e_{i_{\mathcal{C}+}})^2 = (e_{i_{\mathcal{C}-}})^2$

In this situation, the "concord case", multiplying (A.2) from left and right by $(u_{i_{\mathcal{C}+}})$ we obtain:

$$s\ \alpha_{\mathcal{B}} - s(-1)^n \beta_{\{1,...,n\}} e_{\{1,...,n\}} + s\ \beta_{\mathcal{C}} e_{\mathcal{C}} + \sum_{\substack{\mathcal{A} \subset \{1,...,n\} \\ \mathcal{A} \neq \varnothing;\mathcal{C}}} \pm\beta_{\mathcal{A}} e_{\mathcal{A}} = 0.$$

Analogously, multiplying (A.2) from left and right by $(e_{i_{\mathcal{C}-}})$, we obtain:

$$s\ \alpha_{\mathcal{B}} - s(-1)^n \beta_{\{1,...,n\}} e_{\{1,...,n\}} - s\ \beta_{\mathcal{C}} e_{\mathcal{C}} + \sum_{\substack{\mathcal{A} \subset \{1,...,n\} \\ \mathcal{A} \neq \varnothing;\mathcal{C}}} \pm\beta_{\mathcal{A}} e_{\mathcal{A}} = 0.$$

Making the arithmetic mean (and dividing for s), we obtain the new relation

$$\alpha_{\mathcal{B}} - (-1)^n \beta_{\{1,\dots,n\}} e_{\{1,\dots,n\}} + \sum_{\substack{\mathcal{A} \subset \{1,\dots,n\} \\ \mathcal{A} \neq \emptyset; \mathcal{C}}} \gamma_{\mathcal{A}} e_{\mathcal{A}} = 0.$$

(ii) $s := (e_{i_{\mathcal{C}+}})^2 = -(e_{i_{\mathcal{C}-}})^2$

Here we can observe that, multiplying (A.2) from left and right by $(u_{i_{\mathcal{C}+}})$ we obtain:

$$s\ \alpha_{\mathcal{B}} - s(-1)^n \beta_{\{1,\dots,n\}} e_{\{1,\dots,n\}} + s\ \beta_{\mathcal{C}} e_{\mathcal{C}} + \sum_{\substack{\mathcal{A} \subset \{1,\dots,n\} \\ \mathcal{A} \neq \emptyset; \mathcal{C}}} \pm \beta_{\mathcal{A}} e_{\mathcal{A}} = 0.$$

Also here, multiplying (A.2) from left and right by $(e_{i_{\mathcal{C}-}})$, we obtain:

$$-s(\alpha_{\mathcal{B}}) + s(-1)^n \beta_{\{1,\dots,n\}} e_{\{1,\dots,n\}} + s\ \beta_{\mathcal{C}} e_{\mathcal{C}} + \sum_{\substack{\mathcal{A} \subset \{1,\dots,n\} \\ \mathcal{A} \neq \emptyset; \mathcal{C}}} \pm \beta_{\mathcal{A}} e_{\mathcal{A}} = 0.$$

This time, doing the difference (instead of the sum), in the arithmetic mean, we obtain the same relation above:

$$\alpha_{\mathcal{B}} - (-1)^n \beta_{\{1,\dots,n\}} e_{\{1,\dots,n\}} + \sum_{\substack{\mathcal{A} \subset \{1,\dots,n\} \\ \mathcal{A} \neq \emptyset; \mathcal{C}}} \gamma_{\mathcal{A}} e_{\mathcal{A}} = 0.$$

So, we have obtained the same equation in both cases.
Thus, following the foregoing procedure, we can eliminate from (A.2) almost all the terms, obtaining:

$$\alpha_{\mathcal{B}} \pm \beta_{\{1,\dots,n\}} u_{\{1,\dots,n\}} = 0$$

But that cannot holds unless $\alpha_{\mathcal{B}} = \beta_{\{1,\dots,n\}} = 0$ for the axiom (G). □

Corollary A.0.6. *$Cl(V)$ is a graduate algebra with the following decomposition,*

$$Cl(V) = \bigoplus_{k=0}^{n} Cl^k(V).$$

With this last corollary, we have finally completely proved Theorem 2.1.8.

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