ON A RELATION BETWEEN λ-FULL WELL-ORDERED SETS AND WEAKLY COMPACT CARDINALS

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Dédié à personne

"One should use common words to say uncommon things"
Arthur Schopenhauer

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Abstract

We prove the existence, inside any linearly ordered set of regular cardinality λ equipped with the interval topology (*linearly ordered topological spaces*, so on called LOTS), of a particular kind of well-ordered subsets characterized by the property of λ-*fullness*.
For the countably ordered and λ-ordered cases we will explicitly construct these well-ordered sets, while for α-ordered cases, with $\aleph_0 < |\alpha| < \lambda$, we will present non-constructive proofs.
Let $\mathfrak{H}$ be a set of regular cardinals: we say that a LOTS X is $\mathfrak{H}$-compact if it is $[\kappa, \kappa]$-compact for every $\kappa \in \mathfrak{H}$, where $[\kappa, \kappa]$-compact means that for every open covering of X of cardinality κ, there is a sub-covering of cardinality smaller than κ.
For $A \subset X, B \subset X$ open sets not both empty, we call the ordered pair (A, B) a *gap* for X if $A \cup B = X$, $a < b \ \forall a \in A, b \in B$ and A has no maximum and B no has no minimum.
A *type* of a gap is κ if the cofinality of A is κ or the coinitiality of B is κ.
By using our results about well-ordered λ-full sets we show, in the last section, that if $\inf \mathfrak{H}$ is a weakly compact cardinal (or ω), then for every LOTS X $\mathfrak{H}$-compactness is equivalent to the nonexistence of gaps of types in $\mathfrak{H}$.

Key Words: linear orders, order topology, well-orders, $[\kappa, \lambda]$-compactness, complete accumulation points, κ-gaps, λ-fullness, regular cardinals, weakly compact cardinals.

Introduction

Overview of Linear Orders Theory

In this thesis we study linearly ordered spaces - and their subspaces, the generalized ordered one - together with their (possibly existing) sub-well-orders and the relation of this subsets with the notion of weakly compact cardinals (for this notion see Definition 3.2). There is an extensive and important literature about linearly ordered spaces and their influence in other fields of mathematics such as order theory, set theory, model theory, general topology and the theory of Banach spaces. For all these and several other applications, we refer to [6].

In particular, the theory of linear orders has been deeply studied since the work of Hausdorff and Souslin (amongst others); we refer to [42] for a quite exaustive presentation of combinatorial, model theoretic and set theoretic aspects of linear orderings.

A *generalized ordered topological* space (GOTS or GO-space for short) is a Hausdorff space X equipped with a linear order and having a base of order-convex sets. If the topology of X coincides with the open interval topology generated by the given linear order, we say that X is a *linearly ordered topological space* (LOTS for short).

Čech (see, for example, [37]) characterized the class of GO-spaces as the spaces that can be topologically embedded in some LOTS. Moreover it has been shown that for any GO-space X one can canonically construct a LOTS X^* (the smallest one) that contains X as a closed subspace.

Many properties of these spaces have been studied since the '40s. The first problem of this field of research asked for topological characterizations of the spaces whose topology can be given as the open interval topology of some linear ordering of the ground set. Such a question is known as the *orderability problem*. This problem has been solved by Eilenberg ([15]) who gave a necessary and sufficient condition for the orderability of connected and locally connected spaces. Purish [41] observed that this problem has a long history, going back to the early topological characterizations of the unit interval. The orderability problem for zero-dimensional metric spaces was solved by Herrlich in [26] and later Purish [40] gave necessary and sufficient conditions for the orderability of any metric space. The general orderability problem was solved by Van Dalen and Wattel in [49]. During the '80s Van Mill and Wattel [50] sharpened some results about orderability of compact spaces by characterizing an orderable space X in terms of existence of continuous functions (called *selections*) defined on the space 2^X of all closed non-empty subsets of X, equipped with the Vietoris topology. This led to further research about homeomorphisms between compact Hausdorff spaces and compact *ordinal spaces* (i.e. spaces of the form $[0, \alpha)$, α an ordinal number, with the order topology).

Other problems studied in this field are related to topological notions such as *perfect property, first countability, metrizability, Lindelöf property, Tychonoff property* and the conservation and hereditariness of these properties between GOTS and LOTS. In particular, many properties that are completely unrelated in other topological spaces turn out to be equivalent in GO-spaces.

Many problems in this research area are still open: for example one can ask which properties can be true just in *Zermelo-Fraenkel-Choice Theory* (ZFC, the set theoretic foundation commonly assumed for mathematics). Indeed many results use the Continuum Hypothesis (CH, i.e. $2^{\aleph_0} = \aleph_1$), some of these results are even equivalent to it (see, for example, [1]) and others are related to strong (forcing) axioms as *Martin Axiom*, *Open Coloring Axiom* and *Proper Forcing Axiom*, as shown by Burke and Moore in [10]. We refer to [28] for all the (very technical) definitions of the axioms above.

Moreover there is a plethora of "consistent-but-independent" results about linearly ordered space: it is worth to mention Souslin trees and Souslin line which are examples of LOTS whose existence is known to be independent of ZFC, and have been studied during the last century.

One of the fundamental problems in the theory of LO sets was to understand whether every compact *monotonically normal space* X (meaning that for every $x \in A$ open in X, there is an open set $B(x, A) \subseteq A$ such that $x \in B(x, A)$ and if $B(x, A) \cap B(y, C) \neq \emptyset$, then either $x \in C$ or $y \in A$) is the continuous image of a compact LOTS. This is now known as *Nikiel's problem* which has been solved by M. E. Rudin in [43]. The great importance of

Nikiel's problem is also due to its relation with the Hahn-Mazurkiewicz problem that asks for characterizations of topological spaces that are continuous images of some connected compact non-separable LOTS.

At last we mention two examples about the interest of LOTS in the theory of Banach spaces:

I) it has been shown by Jayne, Namioka, Rogers and Haydon in [25] and [27] that if X is a compact LOTS, it is possible to uderstand the vector space $C(X)$ of continuous real-valued functions on X in great depth;

II) one can prove that when X is a GO space, then it has the *Dugundji Extension Property*; this is a property regarding the existence of linear functions from $C^*(A)$ to $C^*(X)$, $A \subseteq X$ closed, which "extend" in a proper way the elements of $C^*(A)$. Here we recall that $C^*(Y)$ is defined as the space of real valued bounded functions on Y.

Starting Point of our Work

The starting point of this thesis has been given by the work done by Lipparini in [36], who characterized *ultrafilter convergence* and *ultrafilter compactness* (see below for these definitions) into LOTS and GOTS.
This author has produced an extensive literature about the study of covering and converging properties of several kinds of spaces, and the properties of the product of these spaces (see [32], [33], [34] and [35]). These topics are also strongly linked with pure logical and set theoretical aspects (see [2], [5], [10], [12] and [34]). The mentioned characterizations are shown to be deeply connected with the notions of *gap* and *pseudo-gap*. For these definitions see Definition 1.2.2 and Definition 3.1.5 in the present work.
We recall that

Definition
A *filter* over a set I is a subset of $\mathcal{P}(I)$ which contains I, is closed under finite intersection of its elements and is upward closed with respect to $\subseteq$. A maximal filter is called *ultrafilter*.

Given a LOTS X, a set I and an ultrafilter D over I, we say that a sequence $(x_i)_{i\in I}$ in X *D-converges* to some $x \in X$ if, for every open neighbourhood U

of x, the set $\{i \in I : x_i \in U\}$ is an element of D.

Lipparini, in Theorem 3.1 of [36] showed that there is a relation between D-convergence and the notion of gap:

Theorem A
if X is a LOTS, D an ultrafilter over a set I and

$$A := \{x \in X : \{i \in I : x < x_i\} \in D\}, B := \{x \in X : \{i \in I : x > x_i\} \in D\},$$

then a sequence $(x_i)_{i \in I}$ D-converges in X iff (A, B) is not a gap.

He also characterized the non-existence of gaps in terms of the following notion:

Definition
A space X is *D-compact* if every I-indexed sequence in X D-converges to some $x \in X$. An ultrafilter D over a set I is *λ-decomposable*, for a *cardinal* λ (see 1.1.6 of the present work), if there is a function $f : I \to \lambda$ such that $f^{-1}(Y) \notin D$ for every $Y \subseteq \lambda$ with $|Y| < \lambda$.

Now let $\mathfrak{K}_D$ be the set of infinite *regular cardinals* κ (see Definition 1.1.8 of this work) such that an ultrafilter D is κ-decomposable.
Lipparini in Theorem 4.1 of [36] proved the following (for the definition of $[\kappa, \kappa]$-compactness see Definiton 1.2.3 in this work, while for the definiton of type of a gap see again Definition 1.2.2.):

Theorem B
For every GO-space X the following are equivalent
1) X is D-compact
2) X is $[\kappa, \kappa]$-compact for every $\kappa \in \mathfrak{K}_D$
3) X has no gaps nor pseudo-gap of type in $\mathfrak{K}_D$
4) For every $\kappa \in \mathfrak{K}_D$ and every strictly increasing (resp. decreasing) κ-indexed sequence in X, the sequence has a supremum (resp. infimum) to which it converges.

Moreover for every infinite cardinal λ and for an ultrafilter D over λ such that there is a family $(Z_\alpha)_{\alpha \in \lambda}$ of members of D with the property that the intersection of any infinite sub-family is empty, it holds the following:

Theorem C (Theorem 5.1 of [36])
For every infinite cardinal λ and every GO-space X, X is D-compact for some regular ultrafilter over λ if and only if X has no gap nor pseudo-gaps of type $\leq \lambda$.

Now, one of the many characterizations of *weakly compact cardinals* establishes that

$$\text{a regular cardinal } \kappa \text{ is weakly compact}$$

$$\Leftrightarrow$$

$$\text{for every LOTS } X,\ [\kappa, \kappa]\text{-compactness of } X \text{ is equivalent}$$

$$\text{to the nonexistence of gaps of type } \kappa \text{ in } X$$

This led Lipparini to the formulation of the statement that we call $\mathbf{P}(\mathfrak{H})$:

For every LOTS X, X is $\mathfrak{H}$-compact if and only if it has no gaps of type in $\mathfrak{H}$

where, with $\mathfrak{H}$, we identify a class of regular cardinals. We say that a space X is $\mathfrak{H}$-compact if it is $[\kappa, \kappa]$-compact for every $\kappa \in \mathfrak{H}$.
In Section 7 of [36], Lipparini proved

Theorem D
if $\mathbf{P}(\mathfrak{H})$ holds then $\inf \mathfrak{H}$ is a weakly compact cardinal (or ω).

The purpose of our work is to prove that, vice versa, if $\inf \mathfrak{H}$ is a weakly compact cardinal then $\mathbf{P}(\mathfrak{H})$ holds.

Our Results

In this work we are interested in the relation between linearly ordered set with particular well-ordered subsets (namely λ-*full* ones; for this notion see Definition 2.1 in the present thesis) and linearly ordered topological space with some appropriate covering properties (the mentioned $\mathfrak{H}$-compactness). We emphasize that while the first notion is purely set theoretical, the latter belongs to a general topological setting.
In particular, we prove the existence of λ-full well-ordered sets of various

order type (and we prove they have also a stronger version of Lipparini's λ-fullness):

Theorem 1 (sketched by P. Liparini as Lemma 7.1 in [36])
Let λ be an infinite regular cardinal. Let X be a LO set such that $|X| = \lambda$. Then X has a $\lambda^{\leftrightarrows}$-ordered subset or an $\omega^{\leftrightarrows}$-ordered subset λ-full.

For the meaning of $\alpha^{\leftrightarrows}$-ordered, for an ordinal α, see Definition 1.1.5.

By using the same methods used in proving Theorem 1 (method suggested by P. Lipparini), we extend this result to well-ordered sets of higher cardinalities:

Theorem 2
Let X be a LO set such that $|X| = \lambda > 2^{\aleph_\alpha}$ for λ regular, and let us suppose that

$$\forall Y \subseteq X \ \Big(|Y| = \lambda \Rightarrow \exists y \in Y \Big(|(-\infty, y)_Y| = |(y, \infty)_Y| = \lambda\Big)\Big),$$

then X contains an $\omega_{\alpha+1}^{\leftrightarrows}$-ordered λ-full subset.

Using the same type of technique applied in the proof of last theorem we show as follows:

Theorem 3
If X is a LO set of regular cardinality $\lambda \geq \mu$, with μ weakly compact, then X admits a $\mu^{\leftrightarrows}$-ordered λ-full subset or a $\lambda^{\leftrightarrows}$-ordered one.

We use Theorem 3 and some technique from [36] to prove

Theorem 4
If $\inf \mathfrak{H}$ is weakly compact, then $\mathbf{P}(\mathfrak{H})$ holds.

Then we extend these results to GO-spaces and this leads us to the following summarizing theorem:

Theorem 5
Let $\mathfrak{H}$ be a class of infinite regular cardinals such that $\omega < \kappa = \inf \mathfrak{H}$ and let X be a GO-space. Then the following are equivalent:

i) κ is a weakly compact cardinal
ii) X is $\mathfrak{H}$-compact $\Leftrightarrow$ X has no gaps nor pseudo-gaps of type in $\mathfrak{H}$

Then the well-known characterization of weakly compact cardinals mentioned above in terms of gaps and $[\kappa, \kappa]$-compactness can be seen as the particular case of Theorem 5 when $\mathfrak{H} = \{\kappa\}$.

Organization of the work

In the first Chapter we state the essential set theoretical and topological definitions, we give an important Theorem due to Alexandroff and Urysohn [3] about the characterization of $[\lambda, \lambda]$-compactness for λ regular cardinal, and we state the property $\mathbf{P}(\mathfrak{H})$.

In the second Chapter we present our results about the existence of the mentioned well-ordered sets (Theorems 1 and 2 of this Introduction) and we fully prove a Theorem sketched by Lipparini in [36] about a connection between λ-fullness and $[\lambda, \lambda]$-compactness.

In the third and last Chapter we use the existence of well-ordered λ-full sets to prove Theorems 3, 4 and 5.

Notations

As usual we will use the first letters of the greek alphabet ($\alpha, \beta, \gamma, \delta$...) for ordinals, and the middle ones ($\kappa, \lambda, \mu, \nu$...) for cardinals. We will write $|\cdot|$ for the cardinality of a set and we will assume the Axiom of Choice (AC). We will write (G)CH for "(Generalized) Continuum Hypothesis".
We will use the symbol "$\neg$" to negate the statement which follows it.
If not specified otherwise, we will use the common symbol $<$ to represent the total order given on a set. If not differently specified, in all definitions we will assume to work with non-empty structures.

Chapter 1

Preliminaries and Motivational Setting

1.1 Basic Definitions for non Set Theorists

Here we give the foundamental set theoretic notions that we will use in what will follow. I want to stress the fact, already mentioned in the introduction, that we will always assume AC.
All the definitions, theorems and proofs of this section can be found in [14], [31] or [28].

Definition 1.1.1
A binary relation R on a set X is a *partial order* if it is reflexive, anti-symmetric and transitive.
If R is irreflexive than it is a *strict* partial order.
If R is a *connex* partial order, meaning that

$$(xRy \vee yRx) \ \forall x, y \in X$$

it is a *total* (or *linear*) *order*.
In case R is a (strict) partial order the pair (X, R) is said to be a *(strictly) partially ordered set*, while if R is a (strict) total order the pair is a *(strictly) totally* (or *linearly*) *ordered set*.

Definition 1.1.2
Let us restrict to those R which are strict partial orders on a set X. An element x of X is *minimal with respect to* R (and we say R*-minimal*) iff

$$\neg\exists y(y \in X \wedge yRx)$$

R is *well-founded* on X iff every subset of X has an R-minimal element.
If R is a well-founded strict total order on a set X we say that R *well-orders* X.

Definition 1.1.3
A set x is *transitive* iff

$$\forall y \in x(y \subseteq x)$$

A set x is an *ordinal* iff it is transitive and well-ordered by $\in$.
The *successor* $\alpha + 1$ of an ordinal α is the $\in$-smallest ordinal $\in$-bigger than α. If an ordinal is not a successor nor 0, then we call it *limit ordinal.*
On the class ON of all ordinals the relations $<, \in$ and $\subset$ coincide.

Remark 1.1.4 (see Theorem I.8.2 of [31])
Every well-ordered set is order-isomorphic to one and only one ordinal, which is then called the *order type* of the set.

Definition 1.1.5
Let α be an ordinal; we say that a set A is $\alpha^{\rightarrow}$*-ordered* (or just α-ordered) if it is well-ordered and the type of its order is equal to α.
Now, let R be a well-order on a set X; we say that $R^{\leftarrow}$ is a *reverse well-order* if

$$\forall x, y \in X(xR^{\leftarrow}y \Leftrightarrow yRx)$$

and we say $X^{\leftarrow} = (X, R^{\leftarrow})$ is the reverse of X.
A is $\alpha^{\leftarrow}$*-ordered* if it is well-ordered and the type of its order is equal to the reverse of α.
We say that a set A is $\alpha^{\leftrightarrows}$-ordered if it is α-ordered or $\alpha^{\leftarrow}$-ordered.

Definition 1.1.6
We say that $X \approx Y$ iff there is a function $f : X \longrightarrow Y$ that is one-to-one and onto (and so we say X and Y have the same *cardinality*).
An ordinal α is called *cardinal* iff there are no $\xi < \alpha$ with $\xi \approx \alpha$.

If κ is a cardinal, we say that

$$\kappa^+ = |\inf\{\alpha \in ON : \kappa < |\alpha|\}\,|$$

is the *successor cardinal* of κ.
We define the $\aleph$*(aleph)'s hierarchy* as follows:

$$\aleph_0 = \omega$$

$$\aleph_{\alpha+1} = \aleph_\alpha^+ \text{ (successor step)}$$

$$\aleph_\lambda = \bigcup_{\gamma<\lambda} \aleph_\gamma \text{ (limit step)}$$

Under AC, there is a bijection between the class $Card$ of cardinals and the class of $\aleph$'s.

Definition 1.1.7
Let X be a partial ordered set; a set $Y \subseteq X$ is *cofinal* in X if for every $x \in X$ there is an $y \in Y$ such that $x \leq y$.
Y is *coinitial* if for every $x \in X$ there is an $y \in Y$ such that $x \geq y$.
The cardinality of the smallest cofinal (resp. coinitial) set in X is called *cofinality* (resp. *coinitiality*) of X.
We write $cf(\cdot)$ for cofinality of a set, and $ci(\cdot)$ for coinitiality.

Definition 1.1.8
A cardinal λ is *regular* if $\lambda = cf(\lambda)$, or, equivalently and more intuitively, if it is not the union of less than λ sets, each of size less than λ.
A cardinal which is not regular is said to be *singular*.

It is important to remark that the use of regularity will turn to be of crucial importance in the rest of the paper.

We conclude this section by recalling an important extention of mathematical induction which is commonly used in set theory to prove properties depending on ordinals $> \omega$:

Transfinite Induction Theorem (see, for example, Theorem 3 chp 1 section 2 of [28])
Let ON be the class of all ordinals and let $\mathfrak{A}$ a class of ordinals.
Assume that:
1) $0 \in \mathfrak{A}$
2) if $\alpha \in \mathfrak{A}$, then $\alpha + 1 \in \mathfrak{A}$
3) if $\alpha \neq 0$ is a limit ordinal and $\beta \in \mathfrak{A}$ for all $\beta < \alpha$, then $\alpha \in \mathfrak{A}$
Then $\mathfrak{A} = ON$.
$\dashv$

The following theorem allows us to *define* sets depending, again, on ordinals $> \omega$:

Transfinite Recursion Theorem (see, for example, Theorem 4 chp 1 section 2 of [28])
Let V be the class of all sets and let $G : V \to V$ a class-function (not simply a function because its domain is not a set); then there is a unique sequence $F : ON \to V$ such that

$$F(\alpha) = G(F|\alpha) \qquad \forall \alpha$$

$\dashv$

Note that $F|\alpha$ is a set thanks to the *Axiom of Replacement* (see, for example, [28] or [31] for details):

If F is a function and A is a set, then $F(A)$ is a set

1.2 The Topological Setting and its Set Theoretic Aspects

In this section we present the basic topological definitions we need, the set theoretical definitions connected with last ones, and a classical result due to Alexandroff and Urysohn which will be used in this work.

Definition 1.2.1
We say a set X is a *linearly ordered topological space* (LOTS so on) if it is (strictly) linearly ordered (by a relation that we call just $<$) and we consider on it the induced *interval topology*, namely the one generated by the open rays

$$(-\infty, a) = \{x \in X | x < a\}, \quad (b, \infty) = \{x \in X | b < x\}$$

for all $a, b \in X$.

Definition 1.2.2
Let X be a LOTS. The ordered pair (A, B) is a *gap* of X if:
1) A and B are open subset of X;
2) $X = A \cup B$;
3) $a < b$ for every $a \in A$ and for every $b \in B$;
4) A has no maximum and B has no minimum.

If κ is a cardinal and (A, B) is a gap, we say tat κ is *a type* of the gap if $cf(A) = \kappa$ or $ci(B) = \kappa$. Thus a gap has at most two types. If a gap has as a type κ we say it is a κ-gap.

As we mentioned in the introduction, LOTS have been exstensively investigated since the beginning of the XX century. Now we want to briefly mention some of the basic topological properties of LOTS. For all these properties (and many others) we refer to [6] and [16].

Every LOTS X is a T_1-space, meaning that for every pair of distinct point $x, y \in X$ there is an open set $U \subseteq X$ such that $x \in U$ and $y \notin U$.

We say that a T_1-space X is *normal* (or, equivalently, a T_4-space) if for every pair of disjoint closed subsets $A, B \subseteq X$ there are two disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$. Birkhoff ([8]) proved that every LOTS is normal and moreover Bourbaki proved ([9]) that it is also *hereditarily normal* (every subspace is normal).

Haar and König showed ([22]) that a LOTS is compact iff every subset has a supremum.

We say that a cover $\{U_s\}_{s\in S}$ is a *refinement* of another cover $\{V_t\}_{t\in T}$ if for every $s \in S$ there is a $t \in T$ such that $U_s \subseteq V_t$.
A space X is said to be *paracompact* if it is a *Hausdorff* space (or, equivalently, T_2, meaning that for any two distinct points there are disjoint neighbourhoods of them) and every open cover has a locally finite open refinement. If every countable open cover has a locally finite open refinement the space is *countably paracompact*.
If the refinement $\{U_s\}_{s\in S}$ is *point-finite* (for every $x \in X$ the set $\{s \in S : x \in U_s\}$ is finite) the space is *weakly paracompact*, while if the refinement is *star-finite* (for every $\bar{s} \in S$ the set $\{s \in S : A_s \cap A_{\bar{s}} \neq \emptyset\}$ is finite), then the space is said to be *strongly paracompact*.
Ball ([4]) and Gillman and Heriksen ([19]) proved that every LOTS is countably paracompact, and Fedorčuk ([17]) and Gillman and Heriksen ([19]) proved that a LOTS is paracompact iff it is weakly paracompact iff it is strongly paracompact.

A topological space is *sequential* if a subset is closed iff together with any sequence it contains all its limits.
Every sequential LOTS is *first-countable*, i.e. at every point of the space there is a countable base.

A pair of subsets (D, E) of a LO set X is a *cut* if they are both nonempty, their union is the set X and every element of D is smaller than every element of E. A LO set such that no cut is a gap is *continuously ordered*.
A LOTS X is connected (the empty set and X itself are the only closed-and-open subsets of X) iff it is continuously ordered.

Definition 1.2.3
Let $\kappa \leq \lambda$ be cardinals. A topological space is said to be $[\kappa, \lambda]$-*compact* if every open covering of cardinality $\leq \lambda$ admits an open sub-covering of cardinality $< \kappa$.

Definition 1.2.4
An accumulation point x_0 for an infinite set A is said to be *complete* (c.a.p. so on) if for every neighbourhood U of x_0 we have that

$$|U \cap A| = |A|$$

Next theorem (see [3]) is a fundamental "bridge" between definitions 1.2.3 and 1.2.4:

Theorem 1.2.5 (Alexandroff-Urysohn)
Let λ be a regular cardinal. Then X is $[\lambda, \lambda]$-compact if and only if every subset of cardinality λ of X has a c.a.p.
$\dashv$

We will use last characterization often through this work, so sometime we will avoid to recall it.

Now, let $\mathfrak{H}$ be a class of infinite cardinals.

Definition 1.2.6
A topological space is $\mathfrak{H}$-*compact* if it is $[\kappa, \kappa]$-compact for every $\kappa \in \mathfrak{H}$.

Now we can give the statement which will be the main motivation for our investigation:

$\mathbf{P}(\mathfrak{H})$ (Lipparini [36]): For every LOTS X, X is $\mathfrak{H}$-compact $\Leftrightarrow$ X has no gaps of type in $\mathfrak{H}$.

Remark 1.2.7 (suggested by P. Lipparini)
We notice that, if λ is regular, a LOTS X $[\lambda, \lambda]$-compact has no gap of type λ.
Now, from 1.2.5, X $[\lambda, \lambda]$-compact implies that every subset $Y \subset X$ of size λ has a c.a.p.
Now, let us suppose, *ad absurdum*, that (A, B) is a gap of type λ, and suppose w.l.o.g. that A has cofinality λ (if B has coinitiality λ the argument is substantially the same). Then there is a strictly increasing (thanks to transfinite recursion) subset C of size λ which is cofinal in A.
If it had a c.a.p. $\bar{x}$, this point would be $\geq$ than every $x_i \in C$ and so we would

have three cases:
1) $\bar{x} = \max A$, which is a contradiction: (A, B) is a gap, so A has no maximum by definition;
2) $\bar{x} \in A$ but $\bar{x} \neq \max A$, so there are elements of A bigger than every element of C, which is another contradiction because C is cofinal;
3) $\bar{x} \notin A$, so $\bar{x} \in B$: in this case, from the definition of c.a.p., it follows that there are elements in C (and so in A) which are bigger than element in B (B has no minimum) and we have a contradiction with the assumption that (A, B) is a gap.
It follows that X has no gap of type λ.

In general $\mathbf{P}(\mathfrak{H})$ is not verified, in fact (suggested by P. Lipparini) let us take $X = \mathbb{R} \times \mathbb{Z}$ and consider the lexicographic order:

$$(r, n) <_{Lex} (r', n') \Leftrightarrow r <_{\mathbb{R}} r' \vee (r = r' \wedge n <_{\mathbb{R}} n')$$

and take $\mathfrak{H} = \{\mathfrak{c}\}$.

Then X is not $[\mathfrak{c}, \mathfrak{c}]$-compact because it is a space of size $\mathfrak{c}$ with discrete topology.
But it has no gap of type $\mathfrak{c}$:
for every gap (A, B) the set $(\mathbb{Q} \times \mathbb{Z}) \cap A$ is cofinal in A and the set $(\mathbb{Q} \times \mathbb{Z}) \cap B$ is coinitial in B. So every gap of X has type at most $\aleph_0$.
More in general, the example works of course also if $\mathfrak{H} = \{\lambda\}, \omega < \lambda < \mathfrak{c}$ (which is meaningfull only if CH does not hold).

But there are cases in which it holds:

1) It is known (cf Corollary 5.1 in [36]; [11] [21]) that a LOTS X is $[\kappa, \kappa]$-compact for every regular $\kappa \leq \lambda$ if and only if it has no gaps of type $\leq \lambda$. This means that $\mathbf{P}(\mathfrak{H})$ holds if $\mathfrak{H} = Reg \cap [\omega, \lambda]$.

2) Lipparini proved that (Corollary 4.1 in [36]) if $\mathfrak{H} = \mathfrak{K}_D$ (for the definition of $\mathfrak{K}_D$ see the Introduction, before Theorem) then $\mathbf{P}(\mathfrak{H})$ holds.
However, it can be shown (cf sections 5 and 7 of [33] and section 6 of [36]) that, if GCH holds or if $\mathfrak{H}$ is an interval of cardinals which does not contain the cofinality of its upper extreme, then there is no an ultrafilter D κ-decomposable.

Chapter 2

λ-Full Sets

In this section we show the existence of a particular kind of well-ordered subsets and we explain why it is interesting to have them.
We remark the fact that from this chapter to the end of the thesis when we use cardinals we assume that they are regular, and $\mathfrak{H}$ will be a class of regular cardinals.

2.1 Existence of Well-Ordered λ-Full Sets

First of all we give the crucial definition of the paper:

Definition 2.1.1 (section 7 of [36])
Let X be a linearly ordered (LO) set of cardinality λ.
Let be $Y \subseteq X$, and let $(y_1, y_2)_X$, with $y_1 < y_2 \in Y$, be the set

$$\{x \in X|\ \ y_1 < x < y_2\}$$

We say that Y is *λ-full in X on the right* if

$$\forall y \in Y \qquad \Big|\bigcup_{Y \ni y' > y} (y, y')_X\Big| = \lambda$$

Y is *λ-full in X on the left* if

$$\forall y \in Y \qquad \Big|\bigcup_{Y \ni y' < y} (y', y)_X\Big| = \lambda$$

Last definition does not depend on the existence of maximum or minimum of Y. If the context is clear, we will omit the directions "on the left"/"on the right".

We observe that, trivially, if $Y \subseteq X$, with $|X| = \lambda$, is $\lambda^{\rightarrow}$-ordered (resp. $\lambda^{\leftarrow}$-ordered), then it is λ-full on the right (resp. on the left) into itself (needing just λ is infinite), so, a fortiori, it is λ-full in X.

We use last definition in the following

Theorem 2.1.2
Let λ be an infinite regular cardinal.
Let X be a LO set such that $|X| = \lambda$.
Then X has a $\lambda^{\leftrightarrows}$-ordered subset or an $\omega^{\leftrightarrows}$-ordered λ-full subset.

Proof:
Case (1): initially let us suppose that

$$\exists Y \subseteq X \ \Big(|Y| = \lambda \ \wedge \ \forall y \in Y \Big(|(-\infty, y)_Y| < \lambda \quad \dot{\vee} \quad |(y, \infty)_Y| < \lambda\Big)\Big)$$

Of course it is not possible that both $(-\infty, y)_Y$ and $(y, \infty)_Y$ have size less than λ.
Using a recursion of lenght λ we build a sequence of elements $(y_\alpha)_{\alpha<\lambda}$ in Y and a sequence of sets $(A_\alpha)_{\alpha<\lambda} \subseteq Y$ such that, for each α, $y_\alpha \in A_\alpha$ and the complement A^c_α of A_α in Y has size $< \lambda$ (so, of course, $|A_\alpha| = \lambda$):

-step 0:
we choose as y_0 any element in $A_0 := Y$;

-successor step:
now we suppose to have built a sequence $(y_\beta)_{\beta\leq\alpha}$ and a sequence of sets $(A_\beta)_{\beta\leq\alpha}$ as above, and we define $y_{\alpha+1}$ and $A_{\alpha+1}$.
We can have two possibilities:

$$a) \quad |(y_\alpha, \infty)| = \lambda$$
$$\dot{\vee}$$

$$b) \quad |(-\infty, y_\alpha)| = \lambda$$

In the first case we choose any $y_{\alpha+1} \in A_{\alpha+1} := A_\alpha \cap (y_\alpha, \infty)$,
in the second one we choose any $y_{\alpha+1} \in A_{\alpha+1} := A_\alpha \cap (-\infty, y_\alpha)$.

Let us check that, in case (a), $|A^c_{\alpha+1}| < \lambda$: in fact

$$A^c_{\alpha+1} = (A_\alpha \cap (y_\alpha, \infty))^c = A^c_\alpha \cup (-\infty, y_\alpha)$$

but $|A^c_\alpha| < \lambda$ and $|(-\infty, y_\alpha)| < \lambda$, and so is the same for the union.
Case (b) is, *mutatis mutandis*, completely analogous.

-limit step γ:
Again, we suppose that $\forall \alpha < \gamma \ \ |A^c_\alpha| < \lambda$.
Then $|\bigcup_{\alpha<\gamma} A^c_\alpha| < \lambda$ (because λ is regular).
Therefore we choose as y_γ any element in $A_\gamma := Y \setminus \left(\bigcup_{\alpha<\gamma} A^c_\alpha\right)$.
For what we just said, obviously $|A^c_\gamma| = |\bigcup_{\alpha<\gamma} A^c_\alpha| < \lambda$.

We divide the elements of the sequence in the following way by using two subsets $\mathcal{I}$ and $\mathcal{J}$:
if $y_{\alpha+1} > y_\alpha$ then we put y_α in $\mathcal{I}$, while if $y_{\alpha+1} < y_\alpha$ we put y_α in $\mathcal{J}$.

$\mathcal{I}$ and $\mathcal{J}$ define implicitly two disjoint sub-sequences, one strictly increasing and the other strictly decreasing.
In fact (increasingness of $\mathcal{I}$): let us suppose $y_\delta, y_\beta \in \mathcal{I}$ and $\delta > \beta$; if $y_\beta \in \mathcal{I}$ this means that $y_{\beta+1} > y_\beta$ than $|(y_\beta, \infty)| = \lambda$ and than, by construction, $y_\delta \in (y_\beta, \infty)$, so $y_\delta > y_\beta$.
Decreasingness of $\mathcal{J}$ is analogous.
So we obtain two sequences such that at least one has lenght λ.
Clearly we succeed in obtaining a sequence of lenght λ because in both eventualities (a) and (b) we choose always in the set with λ elements.

Case (2): now, let us suppose that

$$(**) \quad \forall Y \subseteq X \ \left(|Y| = \lambda \Rightarrow \exists y \in Y\Big(|(-\infty, y)_Y| = |(y, \infty)_Y| = \lambda\Big)\right)$$

We build a λ-full $\omega^{\rightarrow}$-ordered set using the following recursion:

step 0: we choose an Y of size λ (and we call it Y_0) and we choose as y_0 one of those y (surely existing thank to the hypothesis) such that
$|(-\infty, y)_Y| = |(y, \infty)_Y| = \lambda$;
successor step $n+1$: we consider the set $Y_{n+1} = (y_n, \infty)$ and we choose inside this set y_{n+1} as (one of those) y such that

$$|(y_n, y)| = |(y, \infty)| = \lambda$$

So we create a descending chain of sets of size λ

$$Y = Y_0 \supseteq Y_1 \supseteq Y_3 \supseteq ...... \supseteq Y_n \supseteq ....$$

which produces, in the way just described, a strictly increasing sequence $(y_n)_{n\in\omega}$ such that $A = \{y_n : n \in \omega\}$ is obviously ω-ordered.
It is, of course, λ-full (on the right) by construction.
$\dashv$

Remark 2.1.3
First we observe that, in the $\lambda^{\leftrightarrows}$-case, because of the choice of y_α is free (in the subset of size λ), in Y we can find many $\lambda^{\leftrightarrows}$-ordered subsets: in fact given one of these, any subset of size λ is also $\lambda^{\leftrightarrows}$-ordered.

In the $\omega^{\leftrightarrows}$-case, the λ-full sets we built have a stronger property:

Definition ($\bullet$)
We say that a set $Y \subset X$ is *completely λ-full* if

$$\forall y, y' \in Y \text{ such that } y < y' \Big(|(y, y')_X| = \lambda \Big)$$

In the proof just presented we showed the existence of at least one set with the desired properties by building it explicitly.
If, instead of “final” sets, we had choosen y_i’s in “initial” sets, we would have obtained a strictly decreasing sequence (and so a $\omega^{\leftarrow}$-ordered set) λ-full on the left.

Not only, we can find "a lots" of other sets $\omega^{\leftrightarrows}$-ordered and λ-full: starting with an Y of size λ, we consider the y's which progressively divides Y in sets of size λ; then it is built a binary tree (see figure below) whose more external branches correspond to the two sequences of orders ω and $\omega^{\leftarrow}$ respectively in the following way:

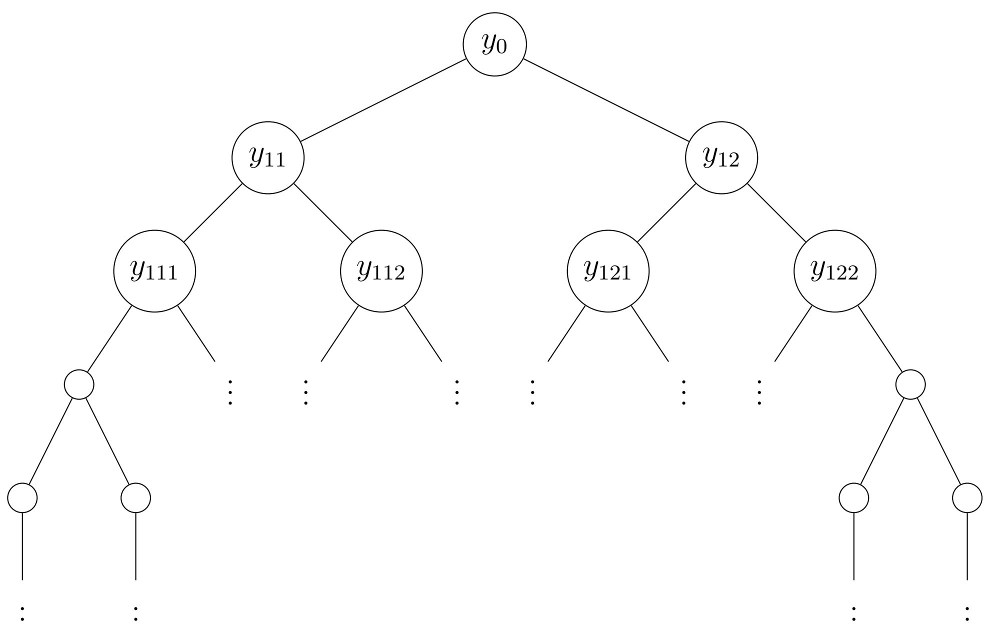
$$y_0 < y_{12} := y_1 < y_{122} := y_2 < ....$$

and

$$y_0 > y_{11} := y_1' > y_{111} := y_2' > ....$$

Inside the tree we can then choose branches whose nodes constitute sequences eventually increasing or eventually decreasing, that is sequences which became monotone after a finite number of nodes; from these we can extract sub-sequences of order ω or $\omega^{\leftarrow}$ erasing initial segments composed by disordered nodes.

Or we could be in a case similar to case "λ" with a complete disordered set: in this case we just use the same division with subsets $\mathcal{I}$ and $\mathcal{J}$, so we find two subsequences, one ω-ordered and the other $\omega^{\leftarrow}$-ordered.



Now we can extend last part of last result for every countable ordinal:

Corollary 2.1.4
Let X be as in Theorem 2.1.2, and suppose that $(**)$.
Then X contains $\alpha^{\leftrightarrows}$-ordered λ-full subsets for all countable α.

Proof
We know from the proof of case (2) of Theorem 2.1.2 that, by using just $(**)$, we can build an $\omega^{\leftrightarrows}$-ordered λ-full subset inside any subset of size λ of X.
We start by fixing inside X a subset Y of size λ and we build a set λ-full on the right.

For the 0 step we build an ω-ordered λ-full (on the right) set S_ω in $(-\infty, y]_Y$, where y is one of those points which divide Y in two parts of size λ.

For the successor step we do the following: we suppose we can build, by using just $(**)$, a λ-full α-ordered subset in any subset of size λ of X, and we suppose we built such an S_α in $(-\infty, y_\alpha]_Y$, where again y_α is one of those points which divide Y in two parts of size λ.
Now, $|(y_\alpha, \infty)_Y| = \lambda$, so there is an $y' \in (y_\alpha, \infty)_Y$ such that

$$|(y_\alpha, y')_Y| = |(y', \infty)_Y| = \lambda$$

So by using $(**)$ we can build a copy S'_α of S_α inside $(y_\alpha, y')_Y$.
We define $y_{\alpha+1} := y'$ and

$$S_{\alpha+1} := S'_\alpha \cup \{y_{\alpha+1}\}$$

For the limit step $\beta = \sup_{\alpha<\beta} \alpha$ we suppose we have α-ordered λ-full subsets S_α as above, $\forall \alpha < \beta$, so we define

$$S_\beta := \bigcup_{\alpha<\beta} S_\alpha$$

We can do this because
1) every S_α is well-ordered with type α and
2) every element of S_α is less than every element of $S_{\alpha+1}$, in fact $S_\alpha \subseteq (-\infty, y_\alpha]_Y$ and $S_{\alpha+1} \subseteq (y_\alpha, y']_Y$, and of course every element of $(-\infty, y_\alpha]_Y$ is smaller than every element of $(y_\alpha, y']_Y$.

So the union for limit steps guarantees that we obtain a well ordered set.

Those subsets are also λ-full by construction (and in fact completely λ-full).

The anti-well-ordered case is analogous by replacing final intervals with initial ones.
$\dashv$

So, from Theorem 2.1.4 and using the same method of case (1) of the proof of Theorem 2.1.2, it follows next result:

Corollary 2.1.5
Let λ be an infinite regular cardinal.
Let X be a LO set such that $|X| = \lambda$.
Then X has a $\lambda^{\leftrightarrows}$-ordered subset or, for every countable α, a λ-full $\alpha^{\leftrightarrows}$-ordered subset.
$\dashv$

Remark 2.1.6 (suggested by P. Lipparini)
If $\lambda = \mathfrak{c}$ (the size of the continuum) there are LOTS of cardinality λ which do not contain any strictly monotone sequence of cardinality κ with $\omega < \kappa \leq \mathfrak{c}$ (that is subset $(> \omega)^{\leftrightarrows}$-ordered), and, a fortiori, no one λ-full.
In fact $X = \mathbb{R}$ can not contain any $\omega^{\leftrightarrows}$-ordered subsets. So, if $\lambda = \mathfrak{c}$, Corollary 2.1.5 is the best result we can obtain.

Here we used just the order density of $\mathbb{Q}$ in $\mathbb{R}$: an ordered set A is *order-dense* if for all a and a' in A such that $a < a'$, there is $a'' \in A$ such that $a < a'' < a'$. An ordered set A is *order dense in a set* B if for all a and a' in A such that $a < a'$, there is a $b \in B$ such that $a < b < a'$.
So we can deduce that a κ-separable set does not contain a $(> \kappa)^{\leftrightarrows}$-sequence, where we call a set X *κ-separable* if X contains an order-dense subset of size κ (so ω-separability is the usual one).

If a LOTS X has cardinality $> 2^{\kappa}$ it can not be κ-separable: let us suppose that X contains an order dense subset B of size κ.
Then the function

$$f : X \longrightarrow \mathcal{P}(B)$$

which maps x in $\{y \in B : y < x\}$, is an injection of X into the power set of B, and so $|X| \leq 2^{\kappa}$.

Then it follows that, if the LOTS has cardinality $> 2^\kappa$, then it could have a κ^+-ordered subset. We will see this is the case and we will show that is possible to build well-ordered subsets which are also λ-full.

Now, separability apart, Remark 2.1.6 can be generalized by

Proposition 2.1.7
if $\lambda = 2^\kappa$ there is at least one LOTS X which does not contain subsets $(> \kappa)^{\leftrightarrows}$-ordered and so no one λ-full.

Whose proof follows directly from

Lemma 2.1.8 (Theorem 5.4 of [12])
Let T be the ordered set $\langle \{0,1\}^\kappa, \leq_{lex} \rangle$, with $\kappa \geq \omega$;
if $T' \subset T$ is a well (or anti well)-order, it has cardinality at most κ.
$\dashv$

In the next theorem (Proposition 7.2 of ([36])), the last of this section, we present an important connection among λ-full well-orders, the notion of gap and $[\lambda, \lambda]$-compactness:

Theorem 2.1.9
Let $\kappa \leq \lambda$ be regular cardinals. Then

(i) Every LO set of cardinality λ has a subset $\lambda^{\leftrightarrows}$-ordered, or a subset $\kappa^{\leftrightarrows}$-ordered λ-full

$$\Leftrightarrow$$

(ii) Every LOTS which has no κ-gap or λ-gap is $[\lambda, \lambda]$-compact.

Proof
(i)$\Rightarrow$ (ii): let X be a LOTS and let Z be a subset of X of cardinality λ.
By applying (i) to Z, we suppose first that there is a $Y \subseteq Z$ which is κ-ordered (risp. $\kappa^{\leftarrow}$-ordered) and λ-full in Z. Because of X has no κ-gap, X contains $y = \sup(Y)$ (risp. $y = \inf(Y)$) and $y \in \bar{Y}$. So every neighbourhood of y contains λ elements of Z because Y is λ-full in Z; in particular neighbourhood contains λ elements of X (because $Z \subseteq X$). So y is a c.a.p. in X

for Z.
If Y was just $\lambda^{\leftrightarrows}$-ordered, again, because of X has no λ-gap, it follows that X contains $y = \sup(Y)$ (or $y = \inf(Y)$) e $y \in \bar{Y}$.
By the arbitrariness of Z, thesis follows from the Alexandroff-Urysohn Theorem.

(ii)$\Rightarrow$ (i): we show the contrapositive $\neg$ (i)$\Rightarrow$ $\neg$ (ii); so there is a LO set L of cardinality λ without $\lambda^{\leftrightarrows}$-ordered and λ-full $\kappa^{\leftrightarrows}$-ordered subsets. From Corollary 2.1.5 it follows necessarily that $\kappa > \gamma$ for every countable γ, so in particular $\kappa > \omega$.
We extend L to another LO set X: we fill every gap with both components of type κ by adding an element x in the middle: in this way x becomes the maximum of A or the minimum of B.
If the gap has left type κ and different right type, we add in the middle a copy of ω, viceversa we fill it with a copy of $\omega^{\leftarrow}$.
In this way we pass from a κ-gap (A, B) to a (A', B)-gap with $cf(A') = \omega$ (or to a (A, B') gap with $ci(B') = \omega$) and we canceled every κ-gap.
Now we have to cancel into X every c.a.p. of L (if not, for Alexandroff-Uryshon Theorem X could be $[\lambda, \lambda]$-compact).
Now we prove the following observation in order to complete the proof of Theorem 2.1.9:

Let L be a LO set of cardinality λ. A point $l \in L$ is a c.a.p. if and only if

$$(l, \infty) \text{ is } \lambda\text{-full on the left or } (-\infty, l) \text{ is } \lambda\text{-full on the right (or both)}$$

In fact l is a c.a.p. iff for every neighbourhood U_l we have that $|U_l \cap L| = |L| = \lambda$.
It follows that $\forall b \in (l, \infty)\ \ |(l, b)| = \lambda$ or $\forall b' \in (-\infty, l)\ \ |(b', l)| = \lambda$ (or both).
If we are in the first case, it follows that

$$\forall b \in (l, \infty) \qquad \left| \bigcup_{l' \in (l,b)} (l', b) \right| = \lambda$$

and so (l, ∞) is λ-full on the left.
If we are in the second case it follows analogously that $(-\infty, l)$ is λ-full on the right.
On the other hand, if $Y = (l, \infty)$ is λ-full on the left, for every neighborhood

U_l of l, $U_l \cap Y$ is λ-full on the left, so $|U_l \cap Y| = \lambda$ and l is a c.a.p..
$\dashv$

In this last proof we omitted the subscripts "L" because the set X was not involved at all: everything is inside L.
Now, to accomplish the proof of Theorem 2.1.9, we substitute l with a copy of $\mathbb{Z}$: then every $(a, b) \subseteq \mathbb{Z}$ can have cardinality at most countable; same for their unions, and so we do not have λ-fullness.
At the end, it follows from Alexandroff-Uryshon Theorem that, because of there is in X a subset of cardinality λ which has no c.a.p., X cannot be $[\lambda, \lambda]$-compact and (ii) fails.
$\dashv$

Remark 2.1.10
We want to stress a fact about the construction just done: this is composed by two operations: (a) destruction of every κ-gap and (b) elimination of c.a. points.
We want to convince that these two operation are coherent: the first does not add c.a.p., in fact the added element could be (or contain) a c.a.p. just if, into a κ-gap, the cofinal (or coinitial) set was λ-full too, which is avoided by hypothesis.
Moreover the second operation does not add κ-gap or λ-gap (these last were not present in L since the beginning because in this case L would have ad at least $\lambda^{\leftrightarrows}$-ordered subset, against the hypothesis) because a new gap could not be "broken" in some point of $\mathbb{Z}$ (if yes, then A admits maximum and B a minimum, so (A, B) is not a gap), so either $A \supset \mathbb{Z}$ or $B \supset \mathbb{Z}$ and so the type of the gap could turn to be ω, but not κ nor λ, being the starting type different from κ and λ.

2.2 λ-Fullness beyond the Countable

It is quite natural to ask about λ-full $\omega_{\alpha}^{\leftrightarrows}$-ordered subsets with $\alpha \geq 1$.
Here we show they exist for appropriate λ's.
The existence of well-ordered subsets has been studied quite extensively by Hausdorff and Uryshon, who proved the following:

Theorem 2.2.1 (Theorem 2.9' of section 6.2 of [23], proved by Urysohn in [48])
If X is a LO set of size $> 2^{\aleph_\alpha}$, then it contains a $\omega_{\alpha+1}^{\leftrightarrows}$-ordered subsets.
$\dashv$

Now, it is clear that last theorem does not say anything about the existence of λ-full well-ordered sets, but we will use this result, both technically and methodologically, in our next theorem:

Theorem 2.2.2
Let X be a LO set such that $|X| = \lambda > 2^{\aleph_\alpha}$ for λ regular, and let us suppose that $(**)$ holds.
Then X contains an $\omega_{\alpha+1}^{\leftrightarrows}$-ordered λ-full subset.

Proof
We start by considering the following relation on X:

$$a \ \rho \ b \quad \Leftrightarrow \quad \Big(a \leq b \ \ \wedge \ \ |[a,b]| < \lambda\Big) \ \ \vee \ \ \Big(b \leq a \ \ \wedge \ \ |[b,a]| < \lambda\Big)$$

ρ is an equivalence relation.
Let us prove that the cardinality of the quotient X/ρ is λ: in fact let us suppose there is a class H with λ elements; then there is a $h \in H$ such that

$$|(-\infty, h)_H| = |(h, \infty)_H| = \lambda$$

and so, for $(**)$, in $(-\infty, h)$ there is a h' such that

$$|(h', h)_H| = \lambda \Rightarrow |[h', h]_X| = \lambda$$

meaning that h and h' are not in relation, against the hypothesis that they belong to the same class.
So no one class has λ elements and so, for regularity of λ, the classes must

be λ.

Now we choose a representative $\bar{x}_i$ for every class K_i, and we define

$$X^* := \{\bar{x}_i \in K_i, \forall K_i \in X/\rho\}$$

Evidently $X^* \subseteq X$ and it inherits the total order of X, so it is a LO set; its cardinality is λ and morevover

$$\forall x, y \in X^*, x < y \ \Big(|(x,y)_X| = \lambda\Big)$$

by construction.
For Theorem 2.2.1, X^* admits an $\omega_{\alpha+1}^{\leftrightarrows}$-ordered subset Z; now, every element of Z is an element of X^*, so for every couple of elements $z_i < z_j \in Z$ we have that

$$|(z_i, z_j)_X| = \lambda$$

and so Z is completely λ-full and then λ-full.
$\dashv$

Remark 2.2.3
For what we just proved, it follows that, under $(**)$, every subset of size λ of X contains an $\omega_{\alpha+1}^{\leftrightarrows}$-ordered λ-full subset, so, by using the same recursion of the proof of Corollary 2.1.4, X contains a λ-full $\gamma^{\leftrightarrows}$-ordered subset for every γ of cardinality $\aleph_{\alpha+1}$: we start with an $\omega_{\alpha+1}^{\leftrightarrows}$-ordered λ-full subset and, by using hypothesis $(**)$ as in Corollary 2.1.4, we build bigger (from the point of view of the order) λ-full subsets.

Last theorem together with the result in case (1) of the proof of Theorem 2.1.2 yield to the following

Corollary 2.2.4
Let X be a LO set s.t. $|X| = \lambda > 2^{\aleph_\alpha}$ for λ regular.
Then X admits an $\omega_{\alpha+1}^{\leftrightarrows}$-ordered subset λ-full or a $\lambda^{\leftrightarrows}$-ordered one.
$\dashv$

Chapter 3

Weakly Compact Cardinals and λ-Full Well-Ordered Sets

In this section we give the notion of *weakly compact cardinal*, which will be central in what follows.
Then we state a connection among weakly compact cardinals, λ-full well-ordered sets and $\mathfrak{H}$-compactness of LOTS.

Definition 3.1
Let X be an infinite set, and let be $[X]^n = \{A \subset X : |A| = n\}$.
The following **Ramsey Theorem** (see, for example, [14] ch. 2, paragraph 8) guarantees the existence of a particular subset of X:
If X is an infinite set, and $[X]^n$ is partitioned into a finite number of sets

$$[X]^n = Y_0 \cup .... \cup Y_m$$

then there is an infinite subset $Y \subset X$ and an $i \leq m$ such that $[Y]^n \subset Y_i$.
$\dashv$

The set Y as above is called *homogeneous* for the partition.

Now we can recall the *large cardinal* (see [14], [28] or [29] for this topic) notion that will be soon used:

Definition 3.2
A cardinal $\kappa > \omega$ is said to be *weakly compact* (w.c. so on) if for every set X of order type κ and $f : [X]^2 \longrightarrow \{0, 1\}$, there is $Y \subset X$, of order type κ,

homogeneous for f.

Next lemma establishes a connection between weakly compact cardinals and linear orders with appropriate sub-well-orders. Moreover this connection turns to be an equivalence (see [14], chp 7, section 3).

Lemma 3.3 (see [14], chp. 7, Theorem 3.6)
If $\kappa > \aleph_0$ is a regular cardinal then:

κ weakly compact $\Rightarrow$ every linear order of cardinality κ has a $\kappa^{\leftrightarrows}$-ordered subset.

Proof:
Let $X_<$ be a linear order with $|X| = \kappa$, and let $<_\kappa$ be an ordering of X of type κ.
Then define

$$f : [X]^2 \longrightarrow \{0, 1\}$$

in the following way:

$$x_1 <_\kappa x_2 \Rightarrow f(x_1, x_2) = \begin{cases} 0 \text{ if } x_1 < x_1 \\ 1 \text{ if } x_2 < x_1 \end{cases}$$

By definition of weakly compact cardinal there is a homogeneous set $Y \subseteq X$ of size κ.
Then, by the definiton of our f, we have two cases:
1) on Y, $<_\kappa$ agrees with $<$ and so Y is κ-ordered, or
2) $<_\kappa$ agrees with the reverse of $<$, and Y is $\kappa^{\leftarrow}$-ordered.
$\dashv$

We notice that, thanks to Theorem 2.1.2, every LO set of cardinality $\aleph_0$ has an $\omega^{\leftrightarrows}$-ordered subset.
Next theorem is the first link between w.c. cardinals and property $\mathbf{P}(\mathfrak{H})$, and is a sort of root for our last result:

Theorem 3.4 (Lipparini, see [36])
Let $\mathfrak{H}$ be a class of regular cardinals. The following holds:
1) if $\mathbf{P}(\mathfrak{H})$ holds, then $\inf \mathfrak{H}$ is ω or a weakly compact cardinal.
2) if $\inf \mathfrak{H} = \omega$ then $\mathbf{P}(\mathfrak{H})$ holds.

Proof
1) Let be $\lambda = \inf \mathfrak{H}$ and we suppose $\lambda \neq \omega$ (conversely proof is complete).
We suppose that λ was not w.c.; so there is a total order L of cardinality λ which has no subsets $\lambda^{\leftrightarrows}$-ordered.
Let us consider the LOTS $X = L \times \mathbb{Z}$ with the lexicographic order.
X has no gap of type λ because $\lambda > \omega$; moreover it has no gap of type $\kappa \in \mathfrak{H} \setminus \{\lambda\}$, because $|X| = \lambda = \inf \mathfrak{H} < \kappa$.
From $\mathbf{P}(\mathfrak{H})$ it follows that X is $[\lambda, \lambda]$-compact, which is absurd, being X a space of cardinality λ with discrete topology.

2) Let us suppose that a LOTS X has no gap of type in $\mathfrak{H}$. From Theorem 2.1.2, for every $\lambda \in \mathfrak{H}$, condition (i) of Theorem 2.1.9 is satisfied when $\kappa = \omega$.
From (i)$\Rightarrow$(ii), X is $[\lambda, \lambda]$-compact.
And this is true for every $\lambda \in \mathfrak{H}$, so X is $\mathfrak{H}$-compact.
The other implication in $\mathbf{P}(\mathfrak{H})$ follows from Remark 1.2.7.
$\dashv$

With these elements we want to prove the following statement:

Theorem ($\natural$)
If $\inf \mathfrak{H}$ *is a w.c. cardinal, then* $\mathbf{P}(\mathfrak{H})$ *holds.*

A proof of ($\natural$), together with Theorem 3.4, would characterize completely $\mathbf{P}(\mathfrak{H})$:

Theorem ($\flat$)
$\mathbf{P}(\mathfrak{H})$ *holds if and only if* $\inf \mathfrak{H}$ *is* ω *or a w.c. cardinal.*

In order to prove Theorem ($\natural$) we first prove the following

Theorem 3.6
If X is a LO set of regular cardinality $\lambda \geq \mu$, with μ w.c., then X admits a λ-full $\mu^{\leftrightarrows}$-ordered subset or a $\lambda^{\leftrightarrows}$-ordered one.

Proof
If μ is w.c., from Lemma 3.3 we know that every LO of size bigger or equal

than μ w.c. has a subset $\mu^{\leftrightarrows}$-ordered.
Then we could be in case (1) of the proof of Theorem 2.1.2 and we obtain a $\lambda^{\leftrightarrows}$-ordered set (which is a $\mu^{\leftrightarrows}$-ordered one if $\lambda = \mu$).
Or we are in case (2) of the proof of Theorem 2.1.2, then with the same construction used in the proof of Theorem 2.2.2 we obtain a subset still $\mu^{\leftrightarrows}$-ordered and completely λ-full, so λ-full.
$\dashv$

Now we can give the

Proof of Teorem ($\natural$)
We have to prove that, if $\inf \mathfrak{H}$ is a w.c. cardinal, then for every LOTS X, X is $\mathfrak{H}$-compact iff X has no gaps of type in $\mathfrak{H}$.
Theorem 3.6 says that condition (i) of Theorem 2.1.9 is satisfied when $\kappa = \mu$ w.c., exactly as Theorem 2.1.2 implied it for $\kappa = \omega$.
So, by applying (i) as in proof of Theorem 3.4(2) and thanks to Remark 1.2.7, it follows the thesis.
$\dashv$

3.1 The GO-spaces Case

Here we want briefly extend the results of last section to GO-spaces.

Definition 3.1.1
Given an ordered set $(X, <)$, a subset A of X is *order-convex* if

$$\forall x, y \in A((z \in X \wedge x < z < y) \Rightarrow z \in A)$$

Definition 3.1.2
A *generalized ordered space* (GO-space or GOTS) is a linearly ordered set with a T_2 topology having a base of order-convex sets.

Examples of GO-spaces are the space of all coutable ordinals $[0, \omega_1)$ with the natural order topology, the *Sorgenfrey line* ($\mathbb{R}$ with the topology generated by the sets $[a, b)$) and the *Michael line* ($\mathbb{R}$ with the topology generated by the sets $\{U \cup K : U$ open in $\mathbb{R}$ and $K \subset \mathbb{R} \setminus \mathbb{Q}\}$).

As we said in the introduction, Čech showed that GO-spaces are precisely the subspaces of LOTS, so the formers generalize the latters.
In fact, since X is a LOTS and $Y \subseteq X$, then the subset topology on Y induced by the topology on X can be finer then the order topology on Y given by the restriction of the order on X.

Moreover:

Theorem 3.1.3 (commonly accredited to Čech, see [6])
There is a canonical construction that produces, for any GO-space X, a LOTS X^* that contains X as a closed subspace.
$\dashv$

Remark 3.1.4 (see [6], [37], [41])
It is not unusual that a GO-space X has a property if and only if the LOTS X^* has the same property, for example metrizability and Lindelöf property.
It is not unusual but not necessary: the Sorgenfrey line S is a *perfect space* (every closed subset of S is the countable intersection of open sets), but S^* is not.
Anyway, the space $(\mathbb{R} \times \{0, 1\}, <_{Lex})$ is a perfect LOTS which contains S as

a subspace.
It happens also that statements involving LOTS are equivalent to statements involving GO-spaces. See again Bennet and Lutzer in [6] for examples about this.
A particular example, for what concerns us, is the characterization of w.c. cardinals via the notion of LOTS that we mentioned in the introduction: it is equivalent to the one with GO-spaces with no gaps nor *pseudo-gaps* (see next definition) of type κ, in place of LOTS.

Definition 3.1.5
An ordered pair (A, B), where A and B are open subsets of a GO-space X, both non-empty, with $A \cup B = X$ and $a < b$ for every $a \in A, b \in B$, is a *pseudo-gap* for X if either A has a maximum and B has no minimum, or A has no maximum and B has a minimum.
If B has a minimum the *type* of (A, B) is the cofinality of A, while if A has a maximum the type is the coinitiality of B.
A pseudo-gap can occur just in GO-spaces that are not LOTS because of the fact that A and B must be open sets.

Now, Theorem 2.1.2 holds also for GO-spaces because about X we used just its linear order and its cardinality.
In Theorem 2.1.9, by considering a GO-space in place of a LOTS, and by adding the non-existence of pseudo-gaps, we have that:

$(i) \Leftrightarrow$ ($\sharp$: For every GO-space X, if X has no gaps nor pseudo-gap of type κ or λ, then it is $[\lambda, \lambda]$-compact).

The proof follows from $(i) \Rightarrow \sharp \Rightarrow (ii) \Rightarrow (i)$, where the first and the third implications are proved in 2.1.9, and the second one is trivial (by definitions of GOTS and LOTS).

So we can replace LOTS with GO-space (by adding nonexistence of pseudo-gaps of the same type too) also in 3.4(2).

We introduce the *generalized* $\mathbf{P}(\mathfrak{H})$

$\mathbf{GP}(\mathfrak{H})$: For every GO-space X, X is $\mathfrak{H}$-compact $\Leftrightarrow$ X has no gaps nor pseudo-gap of type in $\mathfrak{H}$

and we conclude, exactly as for Theorem ($\natural$), that

$$\inf \mathfrak{H} \text{ is a w.c. cardinal (or } \omega) \Leftrightarrow \mathbf{GP}(\mathfrak{H}) \text{ holds.}$$

So we can summarize the results obtained in the following theorem:

Theorem 3.1.6
Let $\mathfrak{H}$ be a class of infinite regular cardinals such that $\omega < \kappa = \inf \mathfrak{H}$ and let X be a GO-space (equivalently LOTS). Then the following are equivalent:
a) κ is a weakly compat cardinal
b) X is $\mathfrak{H}$-compact $\Leftrightarrow$ X has no gaps nor pseudo-gap of type in $\mathfrak{H}$

$\dashv$

We recall the well known characterization of w.c. cardinals given in the introduction:

$$\text{a regular cardinal } \kappa > \omega \text{ is weakly compact}$$

$$\Leftrightarrow$$

for every GOTS (equivalently LOTS) $[\kappa, \kappa]$-compactness is equivalent to having no gaps nor pseudo-gaps of type κ

This characterization can be seen as a consequence of 3.1.6 by choosing $\mathfrak{H} = \{\kappa\}$

Last Remark
We want to conclude this work with two considerations.
The first one: we worked only with regular cardinals. For $\mathfrak{H}$ this choice has been motivated, as we said in section 1.2, by two positive examples that led us to this choice.
For the cardinalities of linearly ordered spaces the restriction to regular ones has been motivated mainly by the use Alexandroff-Uryshon Theorem and Theorem 2.1.9.
And this resctriction allowed us to prove Theorem 2.1.2, which is a sort of "0 step" of our results.
But we want to stress the fact that neither $\mathbf{P}(\mathfrak{H})$ nor $\mathbf{GP}(\mathfrak{H})$ are restricted to spaces of regular cardinalities.
We do not know, so far, whether it is possible to work with singular cardinals.

The second consideration: we used the Axiom of Choice. As it is common in set theory, it is natural to ask what about our results is possible to save - or is still meaningfull - in absence of AC.
In this context, as well known, we lose many equivalences, and crucial (for this work) notions have a different behaviour as, for example, not all sets may be well-ordered and successor cardinals do not need to be regular.
About this we mention, as an example of possible difficulties without AC, the result by Gitik ([18]) who proved the consistency, with ZF, of the following negation of AC: "every uncountable cardinal is singular" (Gitik proved it from the consistency of "there is a proper class of strongly compact cardinals").

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