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Some asymptotic problems for non-convex discrete systems

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Atque eadem magni refert primordia saepe
cum quibus et quali positura contineantur
et quos inter se dent motus accipiantque;
namque eadem caelum mare terras flumina solem
constituunt, eadem fruges arbusta animantis,
verum aliis alioque modo commixta moventur.

De Rerum Natura **I** 817-822
Lucretius

And often it is of great matter with what others those first-beginnings are bound up, and in what position, and what movements they mutually give and receive; for the same build up sky, sea, earth, rivers, sun, the same too crops, trees, living creatures, but only when mingled with different things and moving in different ways.

Translation of Cyril Bailey
http://lf-oll.s3.amazonaws.com/titles/2242/Lucretius_1496_EBk_v6.0.pdf

Abstract

In the present work we treat from a variational point of view some discrete-to-continuum passage problems for energies defined on lattices. We use the notion of Γ-convergence, of equivalence by Γ-convergence, of Homogenization by blow-up and the Minimizing Movements technique, that will be recalled in the first chapter.

In the second chapter, inspired by Gay-Berne type energies for Liquid Crystals, we consider energies depending on orientation and position. Assuming periodic and finite range interactions we apply a homogenization by blow-up technique. We prove a homogenization theorem, where the key feature is that the homogenization formula takes into account sharp conditions on the discrete averages, i.e., we prove two geometric lemmas allowing us to choose test functions in the homogenization formula exactly satisfying an average constraint for the orientation variable, despite the lack of convexity of the constraint. The result stated in a general setting is also applied to Gay-Berne type energies and we give an explicit construction in the 1D setting.

In the third chapter we study chirality transitions in frustrated ferromagnetic spin chains, in view of a possible connection with the theory of Liquid Crystals. We consider non-convex discrete systems with nearest-neighbor ferromagnetic and next-to-nearest neighbor antiferromagnetic interactions governed by a parameter α describing the competition between the two types of interactions. The description of chirality transitions has been addressed in Cicalese and Solombrino (2015)[40] for $\alpha \approx 4$. We extend their analysis to any $\alpha \geq 0$ in the framework of surface energies for discrete systems. We link our result to the gradient theory of phase transitions, by showing a uniform equivalence by Γ-convergence with Modica-Mortola type functionals for the value of the parameter α in $[0, 4]$.

In the fourth chapter we present an asymptotic description of local minimization problems, and of quasistatic and dynamic evolutions of discrete one-dimensional scaled Perona-Malik functionals. The scaling is chosen in such a way that these energies Γ-converge to the Mumford-Shah functional by a result by Morini and Negri (2003)[57]. We show that Γ-convergence does not preserve the pattern of local minima and we propose the construction of "equivalent theories" which keep the simplified form of the Γ-limit but maintain the pattern of local minima. However the Mumford-Shah continuum approximation still provides a good description of quasistatic and gradient-flow type evolutions, while it must be suitably corrected to maintain the pattern of local minima and to account for long-time evolution.

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Contents

Notation

Basic notation

(v_1, v_2)	standard scalar product between $v_1, v_2 \in \mathbb{R}^m$
$\#(A)$	the number of elements of set A
$\langle u \rangle_\Omega$	the average of the function u over a set Ω
$\lfloor x \rfloor$	integer part of $x \in \mathbb{R}$
$\mathcal{M}^{m\times n}$	the $m \times n$ matrix space
$\mathcal{S}^{m-1}$	standard unit sphere of $\mathbb{R}^m$
Ω	open bounded set in $\mathbb{R}^m$
$\|A\|$	Lebesgue measure of a set A
$\|v\|$	usual euclidean norm
$\{e_1, \ldots, e_m\}$	standard orthonormal basis of $\mathbb{R}^m$
$B^m_\rho(z)$	unit m-dimensional ball with radius ρ and center z
I	the open interval $(0,1)$
Q_T	the open cube $(0,T)^m$

Measure theory

$\frac{d\mu}{d\nu} / \frac{d\mu}{dx}$	Radon-Nikodym derivative with respect to a measure $\nu/\mathcal{L}^m$
$\mathcal{H}^m$	m-dimensional Hausdorff measure
$\mathcal{L}^m$	m-dimensional Lebesgue measure
μ^a/μ^s	the absolutely continuous / singular part of a measure μ
μ^j/μ^c	the jump/Cantor part of a measure μ
$\|\mu\|$	total variation of μ

$S(u)$	set of discontinuity points of a function u

Functions space

$BV(\Omega, \mathbb{R}^m)$	space of $\mathbb{R}^m$-valued functions of bounded variation
$C_0(\Omega.\mathbb{R}^m)$	closure, in the sup norm, of $C_c(\Omega, \mathbb{R}^m)$
$C_c(\Omega, \mathbb{R}^m)$	space of $\mathbb{R}^m$-valued functions with compact support in Ω
$C_c^\infty(\Omega, \mathbb{R}^m)$	space of $\mathbb{R}^m$-valued infinitely differentiable functions with compact support in Ω
$PC_{loc}(\mathbb{R})$	set of locally piecewise functions on $\mathbb{R}$
$SBV(\Omega, \mathbb{R}^m)$	space of $\mathbb{R}^m$-valued functions of special bounded variation

Introduction

The modelization of natural phenomena from a mathematical point of view has always been extremely challenging, involving different fields of mathematics and stimulating the birth of new approaches and new mathematical theories. Among those, we will focus on the study of discrete systems from a variational point of view, since it has proven to be a powerful tool to understand a large class of events in many applied fields, e.g., chemical interactions, atomistic models, computer vision models, biological systems.

Since the end of the past century, the study of lattice problems through discrete systems has been fundamental also for giving a theoretical justification of continuum theories. We recall here some seminal works of Chambolle (1992)[41] and Chambolle (1995)[42] in the field of computer vision, and of Braides, Dal Maso and Garroni (1999)[26] in the field of computational mechanics. In [42, 41] the Mumford-Shah functional is derived as the approximation of the Blake-Zisserman weak-membrane discrete energy and it's outlined the link between variational's method of segmentation and Perona-Malik algorithm for image restoration. In [26] considering an atomistic model with pairwise interactions and whose interaction potential is a convex-concave function, it's showed that it can approximate continua allowing softening and fracture.

Since then, many works have focused on a systematic analysis of discrete systems and lattice problems. Among the most important we want to cite Blanc, Le Bris and Lions (2002)[17], in which, starting from the hypothesis that the macroscopic displacement is equal to the microscopic one, there is proven that a class of continuum mechanic models can be seen as limit of molecular models. In two different works of Braides and Gelli (2002)[30, 31] it is showed under different hypotheses (without convexity hypotheses on the energy densities in [30] or considering long-range interactions in [31]) that discrete energies are approximated by continuum energies with bulk and interfacial energy density. At the same time, Friesecke and Theil (2002)[53] analyze the validity of the Cauchy-Born hypothesis in a two-dimensional cubic lattice in which nearest-neighbors and elements on the diagonal are interacting via harmonic springs. Later a work of E and Ming (2007)[66] focused on the relation between atomistic and continuum models in elastic regime, i.e., they prove that under some stability conditions, the Cauchy-Born rule is always valid for elastically deformed crystals.

Other works of interest for this thesis are Morini and Negri (2003)[57], in which it is showed that the Mumford-Shah is an approximation of a discrete family of scaled Perona-Malik energies [59], and Braides, Lew and Ortiz (2006)[32] where discrete energies with Lennard-Jones type potential are approximated by the Mumford-Shah functional.

In many of this approaches the term "approximation" is intended in the sense of Γ-convergence [20, 45]: under suitable assumptions instead of a family of global minimum problems for a sequence of discrete energies, we can compute an "effective" minimum problem involving a continuum energy, called the Γ-limit (for further details on Γ-convergence see the next chapter).

The general approach of a discrete-to-continuum problem involves the definition of a lattice and of a spacing parameter ε. Then we take into account families of scaled energies $\{E_\varepsilon\}_\varepsilon$ whose domain are discrete functions defined on the nodes of a scaled lattice. Up to an identification of such discrete functions with a continuous interpolation, the domain of the functionals E_ε can be seen as a subspace of a space of continuous functions. Defining now a suitable discrete-to-continuum convergence, we can find an "effective" continuum energy via the Γ-convergence technique.

We remark that the choice of the scaling can lead to different limit theories: for example Alicandro and Cicalese (2004)[1] considered a family of discrete energy with a bulk scaling

$$E_\varepsilon(u) = \sum_{\substack{\alpha,\beta\in\varepsilon\mathbb{Z}^m \\ [\alpha,\beta]\subset\Omega}} \varepsilon^m g_\varepsilon\left(\alpha,\beta,\frac{u(\alpha)-u(\beta)}{\varepsilon}\right)$$

and assuming superlinear growth conditions on the energy density, they showed the approximation by a continuous integral energy

$$E(u) = \int_\Omega g(x,\nabla u)\,dx.$$

Later Alicandro,Cicalese and Gloria (2008)[2] made use of another scaling, a statistical scaling, to prove that energies of the form

$$E_\varepsilon(u) = \sum_{\alpha,\beta\in\varepsilon\mathbb{Z}^m\cap\Omega} \varepsilon^m g_\varepsilon(\alpha,\beta,u(\alpha),u(\beta))$$

under general assumptions on the energy densities, Γ-converge to continuum integral energies

$$E(u) = \int_\Omega g(x,u(x))\,dx.$$

In some cases the description given by the Γ-limit $E(u)$ can be too coarse, for this reason we may have to iterate the Γ-limit procedure, i.e., we can find some $k > 0$ such that

$$E_\varepsilon^{(k)} = \frac{E_\varepsilon - \min E}{\varepsilon^k} \qquad \Gamma\text{-converge to} \qquad E^{(k)}$$

and

$$\min E_\varepsilon^{(k)} \to \min E^{(k)}.$$

So we can introduce the so-called development by Γ-convergence [35], which formally reads:

$$E_\varepsilon = E + \varepsilon^k E^{(k)} + o(\varepsilon^k).$$

Among the many mathematical contributions already cited in the fields of Γ-convergence and Homogenization problems, we will make use also of the works of Fonseca and Müller (1992)[52], Braides, Defranceschi and Vitali (1996)[28], Braides and Defranceschi (1998)[27], Braides, Maslennikov and Sigalotti (2008)[33], Braides and Chiadó Piat (2008)[22].

In the following chapters we will analyze different features of Γ-convergence technique, having particular attention on discrete models and lattice problems. In the first two chapters we give particular attention to Liquid Crystals models [46, 65]: here we recall that Braides, Cicalese and Solombrino (2015)[24] showed that, under standard coerciveness and growth assumptions, a discrete spin energy in which is supposed a head-to-tail symmetry is asymptotically approximated by a continuum energy whose domain is the space of de Gennes Q-tensors. Moreover Cicalese and Solombrino (2015)[40] proved that in a frustrated ferromagnetic spin chains, close to the ferromagnet/helimagnet transition point, chiral ground states emerge (see also Cicalese, Ruf and Solombrino (2016)[39] for the case of $\mathcal{S}^2$-valued spins chains). In the next section we propose a quick overview on the main features of Liquid Crystals, giving particular attention to the description of Gay-Berne energies.

Liquid Crystals Liquid Crystals are materials which share properties peculiar of liquid and solid phase: for instance a Liquid Crystal flows like a liquid, but its molecules show some degrees of orientation. So we can think of Liquid Crystals as anisotropic fluids. Those peculiarities make Liquid Crystals highly interesting for the many uses in LCD industries or in electronic devices and more recently also in Life Sciences [67].
The width of possible applications makes the study of Liquid Crystals the perfect example of interdisciplinary field of research, involving physicists, chemists, engineers and mathematicians.
Depending on the temperature or on the concentration in a solvent, we can distinguish a lot of mesophases in Liquid Crystals [46, 67]: the principal ones are

(a) **nematic**: there are rod-like molecules which flow freely like in a liquid, but tend to align themselves along a preferred direction. This means that we have an anisotropic liquid with long-range orientational order.

(b) **cholesteric**: like in the nematic phase, we have rod-like molecules which flow freely, but in this phase the molecules organize themselves into layers and the director field rotates between the layers around an axis, forming a helical profile. As a consequence a molecule cannot be superimposed on its mirror image via any proper rotation or translation (this property is often called *chirality*).

(c) **smectic**: in a smectic phase the molecules are divided in layers and they tend to align themselves with each other inside each layer. In this configuration the molecules show also a positional ordering.

The most widely studied Liquid Crystals are the nematics, for which, during the years, different models have been proposed. In the following we will expose one of the most recent: the Gay-Berne potential for Liquid Crystals.

(a) nematic (b) cholesteric (c) smectic

Gay-Berne energies A first prototypical model of a Gay-Berne energy made is appearance in a work of Berne and Pechukas (1971)[15], in which there was a proposal of an overlap potential where also the shape and the geometrical properties of the molecules were taken into account. Starting from these considerations, a later work of Gay and Berne (1981)[14] modified the previous overlap potential to obtain the actual GB potential that we will describe below. Such refinements avoids mathematical problems due to particular geometric configurations of the molecules [14] and result to be very good for numerical simulations [10, 68, 12]. Consider two particles i and j which can be represented by an ellipsoid of revolution and suppose that they are interacting anisotropically. We denote their orientations by unit vectors u_i and u_j, with $\mathbf{r}_{ij}$ the intermolecular vector between the centers and with $r_{ij} = |\mathbf{r}_{ij}|$. Moreover $\hat{\mathbf{r}}_{ij}$ will be the unit vector along the intermolecular vector. We introduce the following quantities: by σ_s and σ_e we denote the parameters reflecting the breadth and the length of the particles, while the parameter χ is defined as

$$\chi := \frac{k^2-1}{k^2+1} \quad \text{where} \quad k = \frac{\sigma_e}{\sigma_s}.$$

We can then introduce a function σ which reflects the geometrical properties of the particles:

$$\sigma(u_i, u_j, \hat{\mathbf{r}}_{ij}) := \sigma_s \left\{ 1 - \frac{\chi}{2} \left[\frac{((u_i, \hat{\mathbf{r}}_{ij}) + (u_j, \hat{\mathbf{r}}_{ij}))^2}{1 + \chi(u_i, u_j)} + \frac{((u_i, \hat{\mathbf{r}}_{ij}) - (u_j, \hat{\mathbf{r}}_{ij}))^2}{1 - \chi(u_i, u_j)} \right] \right\}^{-1/2},$$

and a function η which reflects the anisotropy in the attractive forces:

$$\eta(u_i, u_j, \hat{\mathbf{r}}_{ij}) := \eta_0 \, \eta'^{\mu}(u_i, u_j, \hat{\mathbf{r}}_{ij}) \eta^{\nu}(u_i, u_j).$$

Here χ' is used to adjust the ratio of side-by-side and end-to end well depths

$$\chi' := \frac{k'^{1/\mu} - 1}{k'^{1/\mu} + 1} \quad \text{where} \quad k' = \frac{\eta_s}{\eta_e},$$

while

$$\eta(u_i, u_j) := \left[1 - \chi^2 (u_i, u_j)^2\right]^{-1/2}$$

$$\eta'(u_i, u_j, \hat{\mathbf{r}}_{ij}) = 1 - \frac{\chi'}{2} \left[\frac{((u_i, \hat{\mathbf{r}}_{ij}) + (u_j, \hat{\mathbf{r}}_{ij}))^2}{1 + \chi'(u_i, u_j)} + \frac{((u_i, \hat{\mathbf{r}}_{ij}) - (u_j, \hat{\mathbf{r}}_{ij}))^2}{1 - \chi'(u_i, u_j)} \right].$$

The quantities η_s and η_e are the values desired for the strength parameter for a side-by-side and an end-to-end configurations. The other parameters μ and ν can be chosen to adjust the shape of the potential. Now the GB potential is expressed by:

$$U(u_i, u_j, \mathbf{r}_{ij}) := \tag{1}$$
$$4\eta(u_i, u_j, \hat{\mathbf{r}}_{ij}) \left\{ \left[\frac{\sigma_s}{r_{ij} - \sigma(u_i, u_j, \hat{\mathbf{r}}_{ij}) + \sigma_s} \right]^{12} - \left[\frac{\sigma_s}{r_{ij} - \sigma(u_i, u_j, \hat{\mathbf{r}}_{ij}) + \sigma_s} \right]^{6} \right\}$$

It's worth to notice that σ_s and σ_e are the separations at which the attractive and repulsive terms in the potential cancel when the molecules are in the side-by-side and end-to-end configuration [10], so that $\sigma(u_i, u_j, \hat{\mathbf{r}}_{ij})$ is, in a good approximation, the distance at which two ellipsoid of major and minor axes σ_s and σ_e and relative orientation u_i, u_j and $\hat{\mathbf{r}}_{ij}$ touch [14].

We observe that the potential (1) is symmetric in the first two variable:

$$U(u_i, u_j, \mathbf{r}_{ij}) = U(-u_i, u_j, \mathbf{r}_{ij}) = U(-u_i, -u_j, \mathbf{r}_{ij}) = \\ U(u_i, -u_j, \mathbf{r}_{ij}) = U(u_j, u_i, \mathbf{r}_{ji}),$$

symmetry that reflects the analogue property of nematic liquid crystal molecules. We suggest [10] and [14] for more details on the model and the numerical simulations.

During the last two decades the study of discrete systems has driven the attention of many scientists, mainly due to the success of the description of macroscopical phenomena through a discrete analysis of microscopical settings, but also for a theoretical derivation of continuum mechanical energies. Alicandro and Cicalese (2004)[1] and Alicandro, Cicalese and Gloria (2008)[2] provide quite general results in the study of the passage from discrete to continuum systems: considering a cubic lattice $\varepsilon\mathbb{Z}^m \cap \Omega$ over a fixed open set $\Omega \subset \mathbb{R}^m$ and a discrete energy

$$E_\varepsilon(u) = \sum_{\alpha,\beta \in \varepsilon\mathbb{Z}^m \cap \Omega} \varepsilon^m g_\varepsilon(\alpha, \beta, u(\alpha), u(\beta)),$$

with domain on discrete functions $u : \varepsilon\mathbb{Z}^m \cap \Omega \to \mathbb{R}^N$, they showed that under suitable growth and coerciveness hypotheses, the above energy can be continuously approximated in the sense of Γ-convergence by functional of the following form:

$$E(u) = \int_\Omega g(x, u(x))\, dx,$$

where g is a Carathéodory function, convex in the second variable and satisfying some growth hypotheses. However the above energies do not include molecular models where particles are interacting through a potential including both orientation and position variables, i.e., a potential of Gay-Berne type or its approximations.

In Chapter 2 we analyze the asymptotic behaviour of a family of discrete energies with Gay-Berne type energy density.

We observe that the natural setting to analyze the behaviour of a Liquid Crystals particle is the three-dimensional space, in which the position of the

particle is described by a vector $w \in \mathbb{R}^3$ and its orientation is a vector $u \in \mathcal{S}^2$. Given two particles α and β and denoted with u_α and u_β the respective orientations and with w_α and w_β their position, having in mind potential of the form (1), we focus our analysis on energies whose energy density is a function of both u_α, u_β and the distance $\zeta_{\alpha\beta} = w_\alpha - w_\beta$.
Generalizing to arbitrary dimension we consider an open bounded domain $\Omega \subset \mathbb{R}^m$ and we introduce an energy density

$$G : \mathbb{Z}^m \times \mathbb{Z}^m \times \mathcal{S}^{N-1} \times \mathcal{S}^{N-1} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$$

so that $G^\xi(\alpha, u, v, \zeta) = G(\alpha, \alpha + \xi, u, v, \zeta)$ represents the free energy of two molecules oriented as u and v at a distance ζ, occupying the sites α and $\alpha + \xi$ in the reference lattice. Hence we approach the analysis of the asymptotic behaviour of discrete energies over a cubic lattice $\mathbb{Z}_\varepsilon(\Omega) = \{\alpha \in \varepsilon\mathbb{Z}^m : (\alpha + [0,\varepsilon)^m) \cap \Omega \neq \emptyset\}$: let $R > 0$ be a cut-off parameter representing the relevant range of the interactions, we can define a family of functionals

$$E_\varepsilon(u, w) = \sum_{\xi \in \mathbb{Z}^m} \sum_{\alpha \in R^\xi_\varepsilon(\Omega)} \varepsilon^m G^\xi \left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha + \varepsilon\xi), \frac{w(\alpha + \varepsilon\xi) - w(\alpha)}{\varepsilon|\xi|} \right)$$

where $R^\xi_\varepsilon(\Omega) := \{\alpha \in \mathbb{Z}_\varepsilon(\Omega) \ : \ \alpha, \alpha + \varepsilon\xi \in \Omega\}$, while $u : \mathbb{Z}_\varepsilon(\Omega) \to \mathcal{S}^{N-1}$ and $w : \mathbb{Z}_\varepsilon(\Omega) \to \mathbb{R}^n$ are discrete functions describing respectively the particles orientation and position.

In order to understand the asymptotic behaviuor of the above energies as $\varepsilon \to 0$, we make use of the Fonseca-Müller blow-up technique [33, 52]: with that we are able to show the following partial result, i.e., for functions u with $\|u\|_\infty < 1$, the Γ-limit of the sequence E_ε is a continuous functional

$$E_0(u, w) = \int_\Omega G_{hom}(u, \nabla w)\, dx.$$

The function G_{hom} is defined via a homogenization formula

$$G_{hom}(z, M) = \lim_{T \to \infty} \frac{1}{T^m} \inf \left\{ \mathcal{E}_T(u, w) : \langle u \rangle^{d,1}_{Q_T} = z,\ w(\alpha) = (M, \alpha) \text{ on } \partial Q_T \right\}$$

and

$$\mathcal{E}_T(u, w) = \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\beta \in R^\xi_1(Q_T)} G^\xi \left(\beta, u(\beta), u(\beta + \xi), D^\xi_1 w(\beta) \right).$$

The main feature of the homogenization formula above is that test functions u satisfy a non-convex sharp constraint on the average. This result is a first step in the extension of both the homogenization theorem in [1], where no dependence on u is present, and that of [2], which instead deals with u only (for further details see Theorem 10 and Theorem 12 in the next chapter). We remark that the sharp condition in homogenization formula is not straightforward since $\mathcal{S}^{N-1}$ is not a convex set.

We also study the case of energies depending only on orientation:

$$E_\varepsilon(u) = \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\alpha \in R^\xi_\varepsilon(\Omega)} \varepsilon^m G^\xi \left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha + \varepsilon\xi) \right)$$

showing that Γ-converge, again for functions u with $\|u\|_\infty < 1$, to

$$E_0(u) = \int_\Omega G_{hom}(u)\,dx,$$

where G_{hom} is defined again via a homogenization formula.

We conclude applying such results to discrete functionals with Gay-Berne type energy densities: in a general setting we can state a result similar to the ones already treated.

In the one dimensional case we consider $V^E : B^1 \times \mathbb{R}$ an effective potential derived from the one dimensional version of the Gay-Berne potential and we are able to show an explicit construction, independent from the previous results. Considering the following family of discrete functionals

$$E_\varepsilon(z,w) = \sum_{i=1}^{N_\varepsilon} \varepsilon V^E\left(z_i, \frac{w_i - w_{i-1}}{\varepsilon}\right)$$

where $N_\varepsilon = \lfloor 1/\varepsilon \rfloor$, $z_i = z(\varepsilon i)$ is the average orientation between i-th and the $(i-1)$-th particle and $w_i = w(\varepsilon i)$ is the position of the i-th particle, we get that the above energies Γ-converge to a functional of the form

$$E(z,w) = \int_0^1 Co\Big(V^E(z,w')\Big)\,dx,$$

where with $Co(F)$ we denote the convex envelope of a function F. This is a partial result which holds for $\|z\|_\infty < 1$.

In Chapter 3, inspired by the recent results of Cicalese and Solombrino (2015)[40], we analyze the asymptotic behaviour of a discrete one-dimensional model in which a frustration term is present. We show that this kind of competition favors the presence of a chiral symmetry, i.e., the angle between two nearest neighbor molecular is constant [6, 54, 48]. Moreover we show a variational equivalence with problems in gradient theory of phase transitions, i.e., we show the uniform equivalence by Γ-convergence with Modica-Mortola type functionals.

On the one-dimensional torus $[0,1]$ let consider the lattice $\frac{1}{n}\mathbb{Z} \cap [0,1]$, $n \in \mathbb{N}$ and let $u : \frac{1}{n}\mathbb{Z} \cap [0,1] \to S^1$, $u^i = u(\frac{i}{n})$ be a vectorial spin variable on which we assume periodic boundary conditions $(u^0, u^1) = (u^n, u^{n+1})$. The discrete energy of a given state of the system is given by

$$E_n^\alpha(u) = -\alpha \sum_{i=0}^{n-1} (u^i, u^{i+1}) + \sum_{i=0}^{n-1} (u^i, u^{i+2}) - n m_\alpha, \tag{2}$$

where $\alpha \geq 0$ is the *frustration parameter* and

$$m_\alpha = \begin{cases} -\left(\frac{\alpha^2}{8} + 1\right) & \text{if } \alpha \in [0,4], \\ -\alpha + 1 & \text{if } \alpha \in [4, +\infty). \end{cases}$$

While the first term of the energy (2) is ferromagnetic and favors the alignment of neighboring spins, the second, being antiferromagnetic, frustrates it as it favors antipodal next-to-nearest neighboring spins. As a result, the frustration of the system depends on the parameter α. In order to characterize the ground states of this system and their dependence on the value of α, we rewrite the energy in term the discrete scalar variable θ defined as

$$\theta^i = \chi[u^i, u^{i+1}] \arccos((u^i, u^{i+1})),$$
$$\chi[u^i, u^{i+1}] = \mathrm{sign}(u_1^i u_2^{i+1} - u_2^i u_1^{i+1}).$$

The energies now read

$$E_n^\alpha(\theta) = -\frac{\alpha}{2}\sum_{i=0}^{n-1}\Big(\cos\theta^i + \cos\theta^{i+1}\Big) + \sum_{i=0}^{n-1}\cos(\theta^i + \theta^{i+1}) - nm_\alpha. \tag{3}$$

Following the approach by Braides and Cicalese (2007)[23] for lattice systems of the form (3) we are able to show that:

- if $\alpha \geq 4$ the nearest neighbors prefer to stay aligned (ferromagnetic order);
- if $0 \leq \alpha \leq 4$, instead, the minimal configurations of E_n^α are $\theta^i = \theta^{i+1} \in \{\pm\theta_\alpha\}$ with $\theta_\alpha = \arccos(\alpha/4)$; that is, the angle between pairs of nearest neighbors u^i, u^{i+1} and u^{i+1}, u^{i+2} is constant and depending on the particular value of α (helimagnetic order).

In this last case, the two possible choices for θ_α correspond to either clockwise or counterclockwise spin rotations, or, in other words, to a positive or a negative chirality (see Fig. 1). Such a degeneracy is known in literature as *chirality symmetry*.

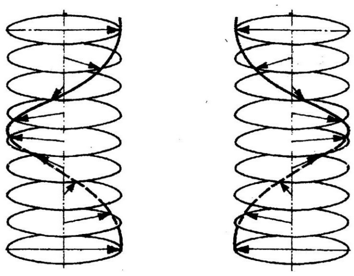


Figure 1: A schematic representation of the ground states of the spin system for $0 \leq \alpha < 4$ for clockwise (on the left) and counterclockwise (on the right) chirality (picture taken from [48]).

The asymptotic behaviour of energies E_n^α as $n \to \infty$ for fixed α reflects such different regimes for the ground states. If $\alpha \geq 4$ the limit is trivially finite (and equal to zero) only on the constant function $\theta \equiv 0$, while if $0 \leq \alpha < 4$ it is finite on functions with bounded variation taking only the two values $\{\pm\theta_\alpha\}$ and it counts the number of chirality transitions; more precisely,

$$\Gamma - \lim_{n\to+\infty} E_n^\alpha(\theta) = C_\alpha \#(S(\theta)),$$

where $C_\alpha = C(\alpha)$ is the cost of each chirality transition (see (3.18) for the mathematical definition). More precisely the value C_α represents the energy of an interface which is obtained by means of a 'discrete optimal-profile problem' connecting the two constant (minimal) states $\pm\theta_\alpha$. Moreover it is continuous as a function of α in the interval $[0,4)$ and can be defined to be equal 0 for $\alpha \geq 4$. Then we state a result already present in [40] in a slight different form, useful for the sequel. More precisely, we prove that, if we choose as order parameter the "flat" angular variable

$$v = \frac{\theta}{\theta_\alpha},$$

the Γ-limit with respect to the strong topology of L^1 of the scaled energies

$$F_n^\alpha(v) := \mu_\alpha E_n^\alpha(v) = \frac{8E_n^\alpha(v)}{\sqrt{2}(4-\alpha)^{3/2}},$$

give an analogous result to that of [40], i.e., we show that, within this scaling, several regimes are possible depending on the value

$$l := \lim_n \frac{\sqrt{2}}{4n(4-\alpha)^{1/2}}.$$

Motivated by the particular form of the result by Cicalese and Solombrino and in the spirit of Braides and Truskinovsky (2008)[35] we give a link between such energies (seen as a 'parametrized' family of functionals) and the gradient theory of phase transitions in the framework of the *equivalence* by Γ-convergence. More precisely, we show the *uniform* equivalence by Γ-convergence on $[0,4]$ of the energies $F_n^\alpha(v)$ with the "Modica-Mortola type" functionals given by

$$G_n^\alpha(v) = \mu_\alpha\Big(\lambda_{n,\alpha}\int_I \Big(v^2-1\Big)^2\,dt + \frac{M_\alpha^2}{\lambda_{n,\alpha}}\int_I (\dot{v})^2\,dt\Big),\ v \in W^{1,2}_{|per|}(I),$$

where $\lambda_{n,\alpha} = 2n\theta_\alpha^4$ and $M_\alpha = 3C_\alpha/8$.

The value $\alpha_0 = 4$ is a *singular point*, since the Γ-limit of G_n^α will depend on choice of the particular sequence $\alpha_n \to \alpha_0^- = 4^-$. Each $\alpha_0 \in [0,4)$, instead, is a *regular point*; i.e., it is not singular. As a consequence we deduce the uniform equivalence of the energies $E_n^\alpha(\theta)$ for $\alpha \in [0,4)$ with the family

$$H_n^\alpha(\theta) = \frac{\lambda_{n,\alpha}}{\theta_\alpha^4}\int_I \Big(\theta^2(t)-\theta_\alpha^2\Big)^2\,dt + \frac{M_\alpha^2}{\lambda_{n,\alpha}\theta_\alpha^2}\int_I (\dot{\theta}(t))^2\,dt,\ \theta \in W^{1,2}_{|per|}(I),$$

whose potentials $\mathcal{W}_\alpha(\theta) := (\theta^2-\theta_\alpha^2)^2$ have the wells at the minimal angles $\theta = \pm\theta_\alpha$.

In Chapter 4 we deal with a one dimensional scaled Perona-Malik functional. Starting from a Γ-convergence result by Morini and Negri (2003)[57], we analyze how much the approximation by Γ-convergence can be extended beyond the global minimization standpoint. It is known that Γ-convergence cannot be easily extended as a theory to the analysis of the behaviour of local minima or to a dynamical setting beyond, essentially, the "trivial" case of convex energies [21, 25].

However, several recent examples suggest that for problems with concentration some quasistatic and dynamic models are compatible with Γ-convergence (such as for Ginzburg-Landau [61] or for Lennard-Jones [29] energies). We consider a one-dimensional system of N sites with nearest-neighbour interactions. Let $\varepsilon = 1/N$ denote the *spacing parameter* and let $u := (u_0, \ldots, u_N)$ be a function defined on the lattice $\varepsilon\mathbb{Z} \cap [0,1]$ where we denote with $u_i = u(\varepsilon i)$. When taking ε as a parameter, we also denote $N = N_\varepsilon$.

We define the *scaled one-dimensional Perona-Malik* functional as

$$F_\varepsilon(u) := \sum_{i=1}^{N_\varepsilon} \frac{1}{|\log\varepsilon|} \log\left(1 + |\log\varepsilon| \frac{|u_i - u_{i-1}|^2}{\varepsilon}\right).$$

Morini and Negri (2003)[57], showed that as $\varepsilon \to 0$, F_ε Γ-converge to the Mumford-Shah functional M_s

$$M_s(u) = \int_0^1 |u'|^2 + \#(S(u))$$

defined on piecewise H^1-function, where $S(u)$ is the set of discontinuity points of u. When local minimization is taken into account, we show that indeed for some classes of problems the pattern of local minima of F_ε differ from that of M_s (see Figure 2). The computation of the Γ-limit can nevertheless be used as a starting point for the construction of "equivalent theories", which keep the simplified form of the Γ-limit but maintain the pattern of local minima. In our case we prove the Γ-equivalence of energies of the form

$$G_\varepsilon(u) = \int_0^1 |u'|^2 dx + \sum_{x \in S(u)} \frac{1}{|\log\varepsilon|} g\left(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}} |u^+ - u^-|\right) \tag{4}$$

with g a concave function with $g'(0) = 1$ and $g(w) \sim 2\log w$ for w large. Such functionals possess the same pattern of local minima as F_ε, which is instead lost

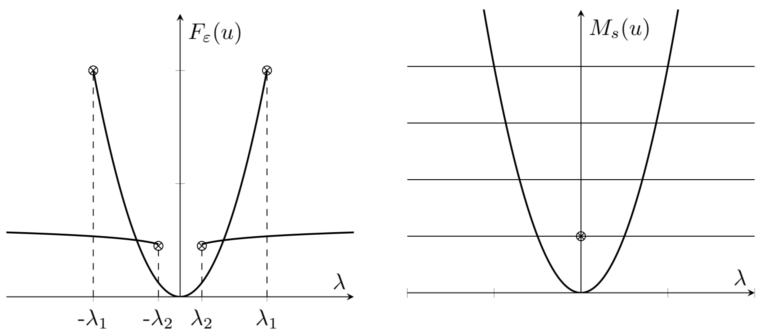


Figure 2: Perona-Malik and Mumford-Shah local minima under Dirichlet boundary conditions $u(0^-) = 0$, $u(1^+) = \lambda$. Here $\lambda_2 = 2\sqrt{\frac{1-\varepsilon}{|\log\varepsilon|}}$ and $\lambda_1 = \frac{1}{\sqrt{\varepsilon|\log\varepsilon|}}$.

in the pattern of the local minima of the Mumford-Shah functional. Then we compare quasistatic motion (also sometimes denoted as variational evolution) for F_ε with that of the Mumford-Shah functional and we show that in the limit the first converge to the latter under Dirichlet boundary conditions. We adopt as the definition of quasistatic motion that of a limit of equilibrium problems involving energy and dissipation with varying boundary conditions. To this end we propose a "dissipation principle", formalizing the assumption that the concave part of the Perona-Malik energies corresponds to a fracture energy which cannot decrease during a variational evolution. To conclude we focus on the dynamical analysis of functionals F_ε. We closely follow a result by Braides, Defranceschi and Vitali (2014)[29] on minimizing movements for Lennard-Jones systems, showing that the gradient-flow type dynamics of F_ε converge to the corresponding dynamics for the Mumford-Shah functional under some hypotheses on the initial data. The main technical difficulty here, with respect to [29], is that the analysis cannot be subdivided into separate computations corresponding to the convex and concave parts of the energy densities, but a finer argument by Morini and Negri must be used that allows to construct interpolations which can be treated as in [29]. Following an observation already included in [21] we remark that the dynamical analysis cannot be carried on to long-time scalings, for which the corrected equivalent energies (4) give a better description.

Chapter 1

Preliminary results

In this first chapter we recall some definitions and results already known, to make the content of the next chapters more readable. In particular we recall the notion of Γ-convergence and some useful properties, then the definition of equivalence by means of Γ-convergence. We recall the main features of homogenization by blow-up and of the technique of minimizing movements. A section is devoted to recall some important results in Measure Theory. We will also give an overview on the spaces BV and SBV. In the end we state the discrete-to-continuum convergence and we review some important integral representation theorems.

1.1 Γ-convergence

Here we recall only the main features of Γ-convergence. For a complete treatise we suggest [20, 45].

The main scope of Γ-convergence is the description of the asymptotic behaviour of families of minimum problems: under suitable assumptions, this means that instead of a family of global minimum problems of a sequence $\{F_n\}_n$, we want to compute an "effective" minimum problem involving its "Γ-limit". This notion will be made clear below.

Let X be a metric space equipped with a distance d. Let $F_n : X \to \bar{\mathbb{R}}$ be a family of functionals, then

Definition 1. A sequence $F_n : X \to \bar{\mathbb{R}}$ Γ-converges in X to $F : X \to \bar{\mathbb{R}}$ if for all $u \in X$ we have:

(i) (lim inf inequality) for every sequence $u_n \to u$ it holds

$$F(u) \leq \liminf_{n\to\infty} F_n(u_n);$$

(ii) (lim sup inequality) there exists a sequence $u_n \to u$ such that

$$F(u) \geq \limsup_{n\to\infty} F_n(u_n).$$

The functional F is called the Γ-limit of $\{F_n\}_n$ and we write $F = \Gamma\text{-}\lim F_n$.

Since for every u_n satisfying (ii) we have also that

$$F(u) \leq \liminf_{n\to\infty} F_n(u_n) \leq \limsup_{n\to\infty} F_n(u_n) \leq F(u),$$

often (ii) is substituted by

(ii)' (existence of a recovery sequence) there exists a sequence $u_n \to u$ such that

$$F(u) = \lim_{n\to\infty} F_n(u_n).$$

In the next chapters we will use also families of functionals indexed by continuous parameters, so we introduce the following definition:

Definition 2. Let $\varepsilon \to 0$ and let $F_\varepsilon : X \to \bar{\mathbb{R}}$ be a sequence of functionals, then F_ε Γ-converges to F if for all sequences $\varepsilon_n \to 0$ we have $\Gamma - \lim_n F_{\varepsilon_n} = F$.

The main reason to introduce this kind of convergence is to detect informations on the behaviour of minima. To this end we make some assumptions on the sequence $\{F_n\}_n$.

Definition 3. A functional $F : X \to \bar{\mathbb{R}}$ is said to be coercive if for every $t \in \mathbb{R}$ the set $\{F \leq t\}$ is precompact. A functional $F : X \to \bar{\mathbb{R}}$ is said to be mildly coercive if there exists a non-empty compact set $K \subset X$ such that $\inf_X F = \inf_K F$. A sequence $\{F_n\}_n$ is said to be equi-mildly coercive if there exists a non-empty compact set $K \subset X$ such that for all $n \in \mathbb{N}$ it holds $\inf_X F_n = \inf_K F_n$.

We can now state the Fundamental Theorem of Γ-convergence [20].

Theorem 1. *Let $K \subset X$ a compact subset such that $\inf_n F_n = \inf_K F_n$ for every $n \in \mathbb{N}$, then if F_n Γ-converges to F it holds that*

(i) F admits minimum and $\min_X F = \lim_{n\to\infty} \inf_X F_n$;

(ii) if $\{u_n\}_n$ is a precompact sequence such that $\lim_{n\to\infty} F_n(u_n) = \lim_{n\to\infty} \inf_X F_n$ then every limit of a subsequence of $\{u_n\}_n$ is a minimum point for F.

Remark 1. The above Theorem state the convergence of global minima. This is not true for local minima: Γ-convergence does not imply convergence of local minimizers.

In the sequel we also make use of the following functionals:

Definition 4. Let $F_n : X \to \bar{\mathbb{R}}$ and let $u \in X$, then the Γ-lower limit and the Γ-upper limit of a sequence $\{F_n\}_n$ at a point u are respectively the quantities:

$$\Gamma - \liminf_n F_n(u) = \inf\{\liminf_n F_n(u_n) : u_n \to u\},$$
$$\Gamma - \limsup_n F_n(u) = \inf\{\limsup_n F_n(u_n) : u_n \to u\}.$$

Those functionals exist for every $u \in X$, moreover it holds

Proposition 1. *The Γ-upper and lower limit of a sequence $\{F_n\}_n$ are lower semicontinuous functionals.*

More properties of Γ-convergence are stated below:

Proposition 2 (Compactness). *Let (X,d) be a separable metric space and let* $\{F_n\}_n$ *be a sequence of functionals, then there is a subsequence* $\{F_{n_j}\}_j$ *such that the* $\Gamma - \lim_j F_{n_j}$ *exists for all* $u \in X$.

Proposition 3 (Urysohn property of Γ-convergence). *A functional* $F : X \to \bar{\mathbb{R}}$ *is such that* $F = \Gamma - \lim_n F_n$ *if and only if for every sequence* $\{F_{n_j}\}_j$ *there exists a further subsequence which* Γ*-converges to* F.

1.1.1 Equivalence by Γ-convergence

Definition 5 (Γ-equivalence). Let $\{F_n\}_n$ and $\{G_n\}_n$ be sequences of functionals on a separable metric space X. We say that they are equivalent by Γ-convergence (or Γ-equivalent) if there exists a sequence $\{m_n\}_n$ of real numbers such that, if $\{F_{n_k} - m_{n_k}\}_k$ and $\{G_{n_k} - m_{n_k}\}_k$ are Γ-converging sequences, their Γ-limits coincide and are proper (i.e., not identically $+\infty$ and not taking the value $-\infty$).

We now recall some definitions about Γ-equivalence for families of parametrized functionals, uniform equivalence, regular and singular points, as introduced by Braides and Truskinovsky (2008)[35].

Definition 6. Let $\mathcal{A}$ be a set of parameters. Two families of parametrized functionals F_n^α and G_n^α are *equivalent at scale 1 at* $\alpha_0 \in \mathcal{A}$ if $F_n^{\alpha_0}$ and $G_n^{\alpha_0}$ are equivalent at scale 1, i.e.

$$\Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha_0} = \Gamma\text{-}\lim_{n\to+\infty} G_n^{\alpha_0} \tag{1.1}$$

and these Γ-limits are non-trivial.

Definition 7 (uniform Γ-equivalence). Let $\mathcal{A}$ be a set of parameters. Two families of parametrized functionals F_n^α and G_n^α are *uniformly equivalent at scale 1 at* $\alpha_0 \in \mathcal{A}$ if for all $n \to +\infty$, $\alpha_n \to \alpha_0$ we have, up to subsequences,

$$\Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha_n} = \Gamma\text{-}\lim_{n\to+\infty} G_n^{\alpha_n} \tag{1.2}$$

and these Γ-limits are non-trivial. They are uniformly equivalent on $\mathcal{A}$ if they are uniformly equivalent at all $\alpha_0 \in \mathcal{A}$.

Definition 8 (regular point). $\alpha_0 \in \mathcal{A}$ is a *regular point* if for all $n \to +\infty$, $\alpha_n \to \alpha_0$ we have, up to a subsequence,

$$\Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha_n} = \Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha_0}. \tag{1.3}$$

Definition 9 (singular point). $\alpha_0 \in \mathcal{A}$ is a *singular point* if it is not regular; that is, if for all $n \to +\infty$, there exist $\alpha'_n \to \alpha_0$, $\alpha''_n \to \alpha_0$ such that (up to subsequences)

$$\Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha'_n} \neq \Gamma\text{-}\lim_{n\to+\infty} F_n^{\alpha''_n}. \tag{1.4}$$

1.2 Measure Theory

Here we give a quick overview on the main features of Measure Theory that are used in the rest of this work. For further details we suggest [4].

Let X be a non empty set and $\mathcal{E}$ be a σ-algebra, then the pair $(X, \mathcal{E})$ is called a measure space. We say that $E \subset X$ is σ-finite with respect to a positive measure μ if it is the union of an increasing sequence of sets with finite measure. If X itself is σ-finite then we say μ to be σ-finite.

Definition 10. Let $\mu : \mathcal{E} \to \mathbb{R}^m$ be a measure on $(X, \mathcal{E})$, then for every $E \in \mathcal{E}$ we denote with $|\mu|$ its total variation as follows

$$|\mu|(E) = \sup \left\{ \sum_{h=0}^{\infty} |\mu(E_h)| : E_h \in \mathcal{E} \text{ pairwise disjoint}, \ E = \bigcup_{h=0}^{\infty} E_h \right\}.$$

Definition 11. Let μ be a positive measure and ν be a real or vector measure on the measure space $(X, \mathcal{E})$. We say that ν is absolutely continuous with respect to μ and we write $\nu << \mu$, if for every $B \in \mathcal{E}$ it holds:

$$\mu(B) = 0 \Longrightarrow |\nu|(B) = 0.$$

Moreover, if μ, ν are positive measure, we say that they are mutually singular, and we write $\nu \perp \mu$, if there exists $E \in \mathcal{E}$ such that $\mu(E) = 0$ and $\nu(X \backslash E) = 0$; if μ or ν are real or vector valued, we say that they are mutually singular if $|\mu|$ and $|\nu|$ are so.

Theorem 2 (Radon-Nikodym)**.** *Let μ, ν as in the previous Definition, and assume that μ is σ-finite. Then there is a unique pair of $\mathbb{R}^m$ valued measures ν^a, ν^s such that $\nu^a << \mu$, $\nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$. Moreover there is a unique function $f \in [L^1(X, |\mu|)]^m$ such that $\nu^a = f\mu$. The function is called density of ν with respect to μ and it is denoted by $d\nu/d\mu$.*

Theorem 3 (Besicovitch Derivation Theorem)**.** *Let μ be a positive Radon measure in an open set $\Omega \subset \mathbb{R}^N$ and ν an $\mathbb{R}^m$-valued Radon measure. Then for μ-a.e. x in the support of μ the limit*

$$f(x) = \lim_{\rho \to 0} \frac{\nu(B_\rho(x))}{\mu(B_\rho(x))}$$

exists in $\mathbb{R}^m$ and moreover the Radon-Nikodym decomposition of ν is given by $\nu = f\mu + \nu^s$.

It's easy to observe that $f(x)$ defined as above coincide with the Radon-Nikodym derivative of ν with respect to μ, i.e., $f(x) = d\nu/d\mu(x)$.

Definition 12. Let μ be a finite Radon measure and $\{\mu_n\}$ be a sequence of finite Radon measure, then μ_n weakly* converges to μ, and we write $\mu_n \rightharpoonup^* \mu$, if

$$\lim_{n \to \infty} \int_X u \, d\mu_n = \int_X u \, d\mu$$

for every $u \in C_0(X)$.

Theorem 4 (Weak* compactness)**.** *Let $\{\mu_n\}_n$ be a sequence of finite Radon measures on a locally compact separable matrix space X, such that $\sup\{|\mu_n|(X) : n \in \mathbb{N}\} < \infty$, then it has weakly* converging subsequence. Moreover the map $\mu \to |\mu|(X)$ is lower semicontinuous with respect to the weak* convergence.*

1.3 Functions of Bounded Variation

Here we want to collect some important results in the field of Bounded Variation functions. For more information about the topic we suggest [4].

Let $\Omega \subset \mathbb{R}^m$, then a function $u \in L^1(\Omega, \mathbb{R}^N)$ is called of bounded variation if its distributional derivative is a finite Radon measure:

$$\int_\Omega u \,\mathrm{div}\varphi \, dx = -\sum_{i=1}^N \int_\Omega \varphi_i d\, D_i u \qquad \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}^N).$$

Where $Du = (D_1 u, \ldots, D_N u)$ is a Radon measure. Equivalently a function u is in $BV(\Omega, \mathbb{R}^N)$ if and only if its variation $|Du|(\Omega)$ is finite, where

$$|Du|(\Omega) = \sup\left\{ \int_\Omega u \,\mathrm{div}\varphi \, dx \,:\varphi \in C_c(\Omega, \mathbb{R}^N),\ \|\varphi\|_\infty \leq 1 \right\}.$$

If $u \in BV(\Omega)$ the map $u \to |Du|(\Omega)$ is a lower semicontinuous map with respect to the $L^1_{loc}(\Omega)$ topology. Moreover the space $BV(\Omega)$ is a Banach space if endowed with the norm

$$\|u\|_{BV} = \int_\Omega |u| \, dx + |Du|(\Omega),$$

and it holds the following compactness results:

Theorem 5. *Let $u_n \subset BV(\Omega)$ such that*

$$\sup\left\{ \int_A |u_n| \, dx + |Du|(A) :\ n \in \mathbb{N},\ A \subset\subset \Omega \right\} < \infty,$$

then there exists a subsequence $\{u_{n_k}\}_k$ converging in L^1 to $u \in BV(\Omega)$.

Definition 13. Let $u \in BV(\Omega)$ and $\{u_n\}_n \in BV(\Omega)$. We say that $\{u_n\}_n$ weakly* converges in $BV(\Omega)$ to u if $u_n \to u$ in $L^1(\Omega)$ and $Du_n \rightharpoonup^* Du$ in Ω, which means:

$$\lim_{n\to\infty} \int_\Omega \varphi d Du_n \, dx = \int_\Omega \varphi dDu \qquad \forall \varphi \in C_0(\Omega).$$

Now we want to state some approximation results:

Theorem 6 (Approximation by smooth functions)**.** *Let $u \in L^1(\Omega)$, then $u \in BV(\Omega)$ if and only if there exists a sequence $\{u_n\}_n \subset C^\infty(\Omega)$ converging to u in L^1 such that*

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n| \, dx < \infty.$$

For any function in $u \in BV(\Omega)$ we will call $\bar{u}(x)$ its approximate limit if at $x \in \Omega$ it holds

$$\lim_{\rho\to 0} \int_{B_\rho(x)} |u(y) - \bar{u}(x)| \, dy = 0.$$

The set of points where this property does not hold is called approximate discontinuity set and denoted by $S(u)$. We remark that $S(u)$ is a $\mathcal{L}^m$ negligible set, where $\mathcal{L}^m$ is the standard m-dimensional Lebesgue measure.

We will call $x \in \Omega$ an approximate jump point of u if there exist $u^+ : S(u) \to \mathbb{R}^N$, $u^- : S(u) \to R^N$ and $\nu : S(u) \to \mathcal{S}^{N-1}$ such that

$$\lim_{\rho\to 0} \int_{B_\rho^+(x,\nu)} |u(y) - u^+(x)|\, dy = 0 \quad \lim_{\rho\to 0} \int_{B_\rho^-(x,\nu)} |u(y) - u^-(x)|\, dy = 0,$$

where $B^+_{\rho,\nu}(x) = \{y \in B_\rho(x) : (y - x, \nu) > 0\}$ and $B^-_{\rho,\nu}(x) = \{y \in B_\rho(x) : (y - x, \nu) < 0\}$. The triplet $(u^+(x), u^-(x), \nu(x))$ is uniquely determined up to exchange u^+ and u^- and to change sign to $\nu(x)$.
A function $u \in BV(\Omega)$ is said to be approximately differentiable at $x \in \Omega$ if there exists a matrix $\nabla u(x) \in \mathcal{M}^{m\times N}$ such that

$$\lim_{\rho\to 0} \int_{B_\rho(x)} \frac{|u(y) - \bar{u}(x) - \nabla u(x)(y - x)|}{\rho}\, dy = 0.$$

Recalling now the Radon-Nikodym decomposition for measure, we can state the following theorem:

Theorem 7. *Let $u \in BV(\Omega)$, then its distributional derivative Du can be decomposed respect to $\mathcal{L}^m$ as*

$$Du = D^a u + D^j u + D^c u,$$

where

$$D^a u = \nabla u \mathcal{L}^m, \qquad D^a u << \mathcal{L}^m,$$
$$D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{m-1}_{\lfloor S(u)}, \qquad D^j u << \mathcal{H}^{m-1}_{\lfloor S(u)}.$$

A useful subset of BV function is the so-called SBV space.

Definition 14. A function $u \in BV(\Omega)$ is said to be a special function of bounded variation if the Cantor part of its derivative $D^c u$ is zero. This means that for $u \in SBV(\Omega)$ it holds

$$Du = D^a u + D^j u = \nabla u \mathcal{L}^m + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{m-1}_{\lfloor S(u)}.$$

Also for the space SBV we can state a compactness result.

Theorem 8 (Compactness of SBV)**.** *Let $\{u_n\}_n \subset SBV(\Omega)$ and let $\varphi : [0, \infty) \to [0, \infty]$ and $\theta : (0, \infty) \to (0, \infty]$ be such that*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty \qquad \lim_{t\to 0} \frac{\theta(t)}{t} = \infty.$$

If it holds

$$\sup_n \left\{ \int_\Omega \varphi(|\nabla u_n|)\, dx + \int_{S(u_n)} |u_n^+ - u_n^-| d\mathcal{H}^{m-1} + \|u_n\|_\infty \right\} < \infty,$$

then there exists a subsequence $\{u_{n_k}\}_k$ weakly converging in $BV(\Omega)$ to $u \in SBV(\Omega)$.*

1.4 Discrete-to-continuum convergence

In many application it is useful to consider functionals defined on discrete functions or, from another point of view, to approximate an integral energy with a discrete system.
Let Ω be an open bounded domain of $\mathbb{R}^m$ and on Ω let consider a cubic lattice $\varepsilon\mathbb{Z}^m \cap \Omega$, where $\varepsilon > 0$ is called spacing parameter. Let $u : \varepsilon\mathbb{Z}^m \cap \Omega \to \mathbb{R}^N$ be a discrete function, considering a piecewise constant interpolation $\tilde{u}$ of the function u, we define the following space

$$\mathcal{A}_\varepsilon(\Omega) = \left\{\tilde{u} : \Omega \to \mathbb{R}^N \ : \ \tilde{u}(t) \equiv u(\alpha) \text{ if } t \in \alpha + [0,\varepsilon)^m, \text{ for } \alpha \in \varepsilon\mathbb{Z}^m \cap \Omega\right\}.$$

With a little abuse of notation we can identify each u with its piecewise constant interpolation $\tilde{u}$. Let $E_\varepsilon(u) : \mathcal{A}_\varepsilon(\Omega) \to [0,+\infty]$ be a functional of the following form:

$$E_\varepsilon(u) = \sum_{\alpha,\beta \in \varepsilon\mathbb{Z}^m \cap \Omega} g_\varepsilon(\alpha, \beta, u(\alpha), u(\beta)).$$

Varying $\varepsilon > 0$, each functional E_ε is defined on a different functions space $\mathcal{A}_\varepsilon$, so it is useful to identify each $\mathcal{A}_\varepsilon$ as a subspace of a common functions space of Ω, for example $L^1(\Omega, \mathbb{R}^N)$. With such identification, when we write that a sequence $\{u_\varepsilon\}_\varepsilon$ is converging to some u (and we denote it by $u_\varepsilon \to u$), we mean that $\{\tilde{u}_\varepsilon\}_\varepsilon$ is converging to u in $L^1(\Omega, \mathbb{R}^N)$ (or $L^1_{loc}(\mathbb{R}^m, \mathbb{R}^N)$). Moreover the energies E_ε now can be seen as functionals $E_\varepsilon : L^1(\Omega, \mathbb{R}^N) \to [0,+\infty]$ and they read

$$E_\varepsilon(u) = \begin{cases} \displaystyle\sum_{\alpha,\beta \in \varepsilon\mathbb{Z}^m \cap \Omega} g_\varepsilon(\alpha, \beta, u(\alpha), u(\beta)) & \text{if} \quad u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5}$$

1.4.1 Integral Representation

In this section we recall two general representation result due to Alicandro and Cicalese (2004)[1] and Alicandro,Cicalese and Gloria (2008)[2]. In [2] they consider energies of the form (1.5). After a rescaling given by

$$f^\xi_\varepsilon(\alpha, u, v) = \varepsilon^{-m} g_\varepsilon(\alpha, \alpha + \varepsilon\xi, u, v)$$

we can rewrite E_ε as a functional $E_\varepsilon : L^\infty(\Omega, \mathbb{R}^N) \to \mathbb{R}$ of the form

$$E_\varepsilon(u) = \begin{cases} \displaystyle\sum_{\xi \in \mathbb{Z}^m} \sum_{\alpha \in R^\xi_\varepsilon(\Omega)} \varepsilon^m f^\xi_\varepsilon(\alpha, u(\alpha), u(\alpha + \varepsilon\xi)) & \text{if} \quad u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where $R^\xi_\varepsilon(\Omega) = \{\alpha \in \varepsilon\mathbb{Z}^m \cap \Omega : \alpha, \alpha + \varepsilon\xi \in \Omega\}$. Let $K \subset \mathbb{R}^N$ be a bounded set. Let $f^\xi_\varepsilon : (\varepsilon\mathbb{Z}^m \cap \Omega) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be as above and such that

(H1) $f^\xi_\varepsilon(\alpha, u, v) = +\infty$ if $(u, v) \notin K \times K$ for every $\alpha \in \varepsilon\mathbb{Z}^m \cap \Omega$, $\xi \in \mathbb{Z}^m$ and $\varepsilon > 0$.

(H2) There exists $C^\xi_{\varepsilon,\alpha} \geq 0$ such that $|f^\xi_\varepsilon(\alpha, u, v)| \leq C^\xi_{\varepsilon,\alpha}$ for all $(u, v) \in K \times K$.

The constants $\{C^\xi_{\varepsilon,\alpha}\}$ satisfy

(H3) $\limsup_{\varepsilon\to 0} \sup_{\alpha\in\varepsilon\mathbb{Z}^m\cap\Omega} \sum_{\xi\in\mathbb{Z}^m} C^\xi_{\varepsilon,\alpha} < \infty$.

(H4) For all $\delta > 0$ there exists $M_\delta > 0$ such that

$$\limsup_{\varepsilon\to 0} \sup_{\alpha\in\varepsilon\mathbb{Z}^m\cap\Omega} \sum_{|\xi|\geq M_\delta} C^\xi_{\varepsilon,\alpha} \leq \delta.$$

Theorem 9. *Let E_ε be as above and suppose that $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy $(H1)$-$(H2)$ and let $(H3)$-$(H4)$ hold. Then for every sequence converging to zero, there exists a subsequence $\{\varepsilon_{n_k}\}_k$ and a Carathéodory function $f : \Omega\times\bar{K} \to \mathbb{R}$ convex in the second variable such that E_ε Γ-converge with respect to the weak* convergence of L^∞ to the functional $E : L^\infty(\Omega,\mathbb{R}^N) \to \mathbb{R}\cup\{+\infty\}$*

$$E(u) = \begin{cases} \displaystyle\int_\Omega f(x,u(x))dx & \textit{if} \quad u \in L^\infty(\Omega,\bar{K}) \\ +\infty & \textit{otherwise.} \end{cases}$$

where $\bar{K}$ is the convex hull of K in $\mathbb{R}^N$.

An interesting Γ-convergence result holds true if the functions f^ξ_ε are obtained by a rescaling of a periodic function. Let $f^\xi : \mathbb{Z}^m\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ be such that $f^\xi(\cdot,u,v)$ is Q_k-periodic for any $\xi\in\mathbb{Z}^m$ and $(u,v)\in\mathbb{R}^N\times\mathbb{R}^N$. Let f^ξ satisfies the hypotheses:

(H5) $f^\xi(\beta,u,v) = +\infty$ if $(u,v)\notin K\times K$ for every $\beta\in\mathbb{Z}^m\cap\Omega$ and $\xi\in\mathbb{Z}^m$.

(H6) For all $\beta\in\mathbb{Z}^m\cap\Omega$ and $\xi\in\mathbb{Z}^m$ there exists $C^\xi \geq 0$ such that $|f^\xi(\beta,u,v)| \leq C^\xi$ for all $(u,v)\in K\times K$ and $\sum_\xi C^\xi < \infty$.

If $f^\xi_\varepsilon(\alpha,u,v) = f^\xi\left(\frac{\alpha}{\varepsilon},u,v\right)$ we have that

Theorem 10. *Let $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy $(H5)$-$(H6)$. Then E_ε Γ-converge with respect to the weak* topology of L^∞ to*

$$E_0(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(u(x))dx & \textit{if} \quad u \in L^\infty(\Omega,\bar{K}) \\ +\infty & \textit{otherwise.} \end{cases}$$

The function f_{hom} is given by the homogenization formula

$$f_{hom}(z) = \lim_{\rho\to 0}\lim_{h\to+\infty}\frac{1}{h^m}\inf\left\{\sum_{\xi\in\mathbb{Z}^m}\sum_{\beta\in R^\xi_1(Q_h)} f^\xi(\beta,v(\beta),v(\beta+\xi)), \langle v\rangle^d_{Q_h}\in\bar{B}_\rho(z)\right\}.$$

In [1] they consider a particular case of the above energies (1.5), given by

$$E_\varepsilon(u) = \sum_{\substack{\alpha,\beta\in\varepsilon\mathbb{Z}^m\cap\Omega \\ [\alpha,\beta]\subset\Omega}} g_\varepsilon(\alpha,\beta,u(\alpha)-u(\beta))$$

Up to an identification of the discrete functions u with their piecewise constant interpolation, and after a rescaling given by

$$f^\xi_\varepsilon(\alpha,z) = \varepsilon^{-m} g_\varepsilon(\alpha,\alpha+\varepsilon\xi,\varepsilon|\xi|z),$$

we can rewrite E_ε as a functional $E_\varepsilon : L^p(\Omega, \mathbb{R}^N) \to [0, +\infty]$ defined in the following way

$$E_\varepsilon(u) = \begin{cases} \displaystyle\sum_{\xi \in \mathbb{Z}^m} \sum_{\alpha \in R^\xi_\varepsilon(\Omega)} \varepsilon^m f^\xi_\varepsilon(\alpha, D^\xi_\varepsilon u(\alpha)) & \text{if} \quad u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where $D^\xi_\varepsilon u(\alpha) = \frac{u(\alpha+\varepsilon\xi)-u(\alpha)}{\varepsilon|\xi|}$. Let $f^\xi_\varepsilon : (\varepsilon\mathbb{Z}^m \cap \Omega) \times \mathbb{R}^N \to [0, +\infty]$ as above and such that

(H7) $f^{e_i}_\varepsilon(\alpha, z) \geq c_1(|z|^p - 1)$ for every $(\alpha, z) \in (\varepsilon\mathbb{Z}^m \cap \Omega) \times \mathbb{R}^N$ and $i \in \{1, \dots, N\}$, $c_1 > 0$.

(H8) $f^\xi_\varepsilon(\alpha, z) \leq C^\xi_\varepsilon(|z|^p + 1)$ for every $(\alpha, z) \in (\varepsilon\mathbb{Z}^m \cap \Omega) \times \mathbb{R}^N$ and $\xi \in \mathbb{Z}^m$.

Let the constants $\{C^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy :

(H9) $\limsup_{\varepsilon\to 0^+} \sup_{\xi\in\mathbb{Z}^m} C^\xi_\varepsilon < +\infty$.

(H10) For every $\delta > 0$ there exists a $M_\delta > 0$ such that

$$\limsup_{\varepsilon\to 0^+} \sum_{|\xi|>M_\delta} C^\xi_\varepsilon < \delta.$$

Theorem 11. *Let $\{f^\xi_\varepsilon\}_{\varepsilon,\xi}$ satisfy $(H7)-(H8)$ and let $(H9)-(H10)$ hold. Then for every sequence $\{\varepsilon_n\}_n$ of vanishing parameters, there exists a subsequence $\{\varepsilon_{n_k}\}_k$ and a Carathéodory function quasi-convex in the second variable $f : \Omega \times \mathbb{R}^{m\times N}$ satisfying*

$$c(|M|^p - 1) \leq f(x, M) \leq C(|M|^p + 1),$$

with $0 < c < C$ such that $E_{\varepsilon_{n_k}}$ Γ-converge with respect to the L^p-topology to the functional $F : L^p(\Omega, \mathbb{R}^N) \to [0, +\infty]$ defined as

$$E(u) = \begin{cases} \displaystyle\int_\Omega f(x, \nabla u)\, dx & \text{if} \quad u \in W^{1,p}(\Omega, \mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

If the functions f^ξ_ε are obtained by a rescaling of a periodic function it is possible to state a result similar to Theorem 10. Let $f^\xi : \mathbb{Z}^m \times \mathbb{R}^N \to [0, +\infty)$ be such that $f^\xi(\cdot, \zeta)$ is $\mathcal{R}_k$-periodic for any $\xi \in \mathbb{Z}^m$ and $z \in \mathbb{R}^N$, where $\mathcal{R}_k = (0, k_1) \times (0, k_2) \times \cdots \times (0, k_m))$ and $k = (k_1, \dots, k_m)$. Let f^ξ satisfies the hypotheses:

(H11) $f^{e_i}(\beta, z) \geq c_1(|z|^p - 1)$ for every $i \in \{1, \dots, m\}$;

(H12) $f^\xi(\beta, z) \leq C^\xi(|z|^p + 1)$;

(H13) $\displaystyle\sum_{\xi\in\mathbb{Z}^m} C^\xi < +\infty$.

If $f^\xi_\varepsilon(\alpha, z) = f^\xi\left(\dfrac{\alpha}{\varepsilon}, z\right)$, we have that

Theorem 12. *Let* $\{f_\varepsilon^\xi\}_{\varepsilon,\xi}$ *satisfy* $(H11)-(H12)$ *and let* $(H13)$ *hold, then* E_ε Γ-*converge with respect to* L^p-*topology to the functional* $E_0 : L^p(\Omega,\mathbb{R}^N) \to [0,+\infty]$ *defined as*

$$E_0(u) = \begin{cases} \displaystyle\int_\Omega f_{hom}(\nabla u)\,dx & \text{if} \quad u \in W^{1,p}(\Omega,\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}$$

The function $f_{hom} : \mathcal{M}^{m\times N} \to [0,+\infty)$ *is given by the homogenization formula:*

$$f_{hom}(M) = \lim_{h\to\infty} \frac{1}{h^m} \min\left\{ \sum_{\zeta\in\mathbb{Z}^m} \sum_{\beta\in R_1^\xi(Q_h)} f^\xi(\beta, D_1^\xi v(\beta)) \quad v \in \mathcal{A}_{1,M}(Q_h) \right\},$$

with $\mathcal{A}_{1,M}(Q_h) = \{\zeta \in \mathcal{A}_\varepsilon(\mathbb{R}^m) \,:\, \zeta(\alpha) = (M,\alpha)\,\text{if}\,(\alpha + [-1,1]^m) \cap Q_h^c \neq \emptyset\}$.

1.5 Blow-up technique for Homogenization Problems

In this section we want to highlight the main steps of the blow-up technique. For a complete overview on this topic we suggest [33, 52].
Let Ω be an open bounded set of $\mathbb{R}^m$ and $u : \varepsilon\mathbb{Z}^m \cap \Omega \to \mathbb{R}^N$ and $v : \varepsilon\mathbb{Z}^m \cap \Omega \to \mathbb{R}^N$ be discrete functions. Let $f^\xi : \mathbb{Z}^m \times \mathbb{R}^N \times \mathbb{R}^N \to [0,+\infty)$ be such that $f^\xi(\cdot,u,v)$ is $\mathcal{R}_k$-periodic for any $\xi \in \mathbb{Z}^m$ and $u,v \in \mathbb{R}^N$, where $\mathcal{R}_k$ is defined as in the previous section. Up to identify u and v with their piecewise constant interpolations, we can define the energies $E_\varepsilon : L^1(\Omega,\mathbb{R}^N) \to [0,+\infty]$ as

$$E_\varepsilon(u) = \begin{cases} \displaystyle\sum_{\xi\in\mathbb{Z}^m} \sum_{\alpha\in R_\varepsilon^\xi(\Omega)} f^\xi\left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha+\varepsilon\xi)\right) & \text{if} \qquad u \in \mathcal{A}_\varepsilon(\Omega) \\ +\infty & \qquad\qquad \text{otherwise.} \end{cases}$$

To every energy E_ε we can associate the measure

$$\mu_\varepsilon(A) = \sum_{\xi\in\mathbb{Z}^m} \sum_{\alpha\in R_\varepsilon^\xi(\Omega)} \varepsilon^m f^\xi\left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha+\varepsilon\xi)\right) \delta_{\alpha+\frac{\varepsilon}{2}\xi}(A),$$

then we can divide the blow-up argument in five steps.

Definition of a limit measure: Let $u_\varepsilon \to u_0$ and suppose that $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < \infty$ is finite. The equiboundedness of the energies implies, up to subsequences, the weak* convergence of the measures μ_ε to a limit measure $\mu = \lim_\varepsilon \mu_\varepsilon$. Now let $\mathcal{L}^m$ be the canonical m-dimensional Lebesgue measure, then we can consider the Radon-Nikodym decomposition the measure μ with respect to $\mathcal{L}^m$:

$$\mu = \frac{d\mu}{dx}\mathcal{L}^m + \mu^s.$$

Local Analysis Let $x_0 \in \Omega$ be a Lebesgue point for μ with respect to $\mathcal{L}^m$ then it holds

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho\to 0^+} \frac{\mu(Q_\rho(x_0))}{\mathcal{L}^m(Q_\rho(x_0))} = \lim_{\rho\to 0^+} \frac{1}{\rho^m}\mu(Q_\rho(x_0)).$$

The Besicovitch theorem states that $\mathcal{L}^m$-almost every $x \in \Omega$ is a Lebesgue point for μ with respect to $\mathcal{L}^m$.

Blow-up By a diagonalization argument we can choose $\varepsilon_j \to 0$ and $\rho_j \to 0$ such that

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho_j \to 0^+} \frac{1}{\rho_j^m} E_{\varepsilon_j}(u_{\varepsilon_j}, Q_{\rho_j}(x_0))$$

By a proper rescaling of the functional E_{ε_j} we can define a new functionals

$$G_{\varepsilon_j}(\zeta, Q_1(x_0)) = \frac{1}{\rho_j^m} E_{\varepsilon_j}(u, Q_{\rho_j}(x_0)).$$

Here ζ is defined through a rescaling of the function u, a rescaling suggested by the requirement that the ζ_{ε_j} linked to the u_{ε_j} are converging to a meaningful ζ_0, which depends only on u_0 and x_0.

Local estimates We modify ζ_{ε_j} so that they satisfy the same boundary conditions of ζ_0. Then we are sure that

$$G_{\varepsilon_j}(\zeta_{\varepsilon_j}, Q_1(x_0)) \geq \inf\{G_{\varepsilon_j}(\zeta, Q_1(x_0) \,:\, \zeta = \zeta_0 \text{ on } \partial Q_1(x_0))\}.$$

Defining now

$$\varphi(x_0) = \liminf_j \inf\{G_{\varepsilon_j}(\zeta, Q_1(x_0) \,:\, \zeta = \zeta_0 \text{ on } \partial Q_1(x_0))\},$$

we get

$$\frac{d\mu}{dx}(x_0) \geq \varphi(x_0),$$

and we observe that the above inequality gives rise to a formula of homogenization type.

Global estimates We reach the conclusion integrating the local estimates we get above.

1.6 Minimizing Movements

Here we summarize some notion that are used in the next chapters. More informations about minimizing movements can be found in [21, 29].

Let X be a separable Hilbert space and let $F : X \to [0, +\infty]$ be a coercive and lower semicontinuous functional. Given $u_0 \in X$ and $\tau > 0$ we can define recursively $u_k \in X$ as a minimizer for the problem

$$\min \left\{ F(u) + \frac{1}{2\tau} \|u - u_{k-1}\|^2 \right\},$$

and the piecewise constant trajectory $u^\tau : [0, +\infty] \to X$ given by $u^\tau(t) = u_{\lfloor t/\tau \rfloor}$.

Definition 15. A minimizing movement for F from u_0 is any limit of a subsequence u^{τ_j} uniform on compact sets of $[0, +\infty)$.

Proposition 4. *Let $F : X \to [0, +\infty]$ be a coercive and lower semicontinuous functional. Given $u_0 \in X$ there exists a minimizing movement u for F from u_0. In particular $u \in C^{1/2}([0, +\infty); X)$.*

We remark that for general functional F can exist more than one minimizing movement, i.e., we can have different minimizing movements depending on the time step τ.
We want to generalize the notion of minimizing movements for a functional to that of minimizing movements along a sequence of functionals. So let $\varepsilon > 0$ and let $\{F_\varepsilon\}_\varepsilon$ be a sequence of functionals equicoercive and lower semicontinuous. Let $u_0^\varepsilon \to u_0$ such that $F_\varepsilon(u_0^\varepsilon) \le C < +\infty$ and $\tau_\varepsilon > 0$ be a vanishing sequence of parameters as $\varepsilon \to 0$. Then fixed $\varepsilon > 0$ we can define recursively u_k^ε as a minimizer for the problem

$$\min\left\{F_\varepsilon(u) + \frac{1}{2\tau_\varepsilon}\|u - u_{k-1}^\varepsilon\|^2\right\},$$

and the piecewise constant trajectory $u^\varepsilon : [0, +\infty) \to X$ given by $u^\varepsilon(t) = u_{\lfloor t/\tau_\varepsilon \rfloor}$.

Definition 16. A minimizing movement for $\{F_\varepsilon\}_\varepsilon$ from u_0^ε is any limit of a subsequence $\{u^{\varepsilon_j}\}_j$ uniform on compact sets of $[0, +\infty)$.

Proposition 5. *Let $\{F_\varepsilon\}_\varepsilon$ be a sequence of functionals as above then for every $k \in N$ the following properties hold:*

(i) $F_\varepsilon(u^k) \le F_\varepsilon(u^{k-1})$;

(ii) $\|u^k\|_\infty \le \|u^{k-1}\|_\infty \le \|u_0^\varepsilon\|_\infty$.

The behaviour of minimizing movements depends on the choice of the $\varepsilon - \tau$ regimes. The following Theorem holds.

Theorem 13. *Let $\{F_\varepsilon\}_\varepsilon$ be an equicoercive sequence of non-negative lower semicontinuous functionals on a separable Hilbert space X. Let F_ε Γ-converges to a limit functional F and let $u_\varepsilon \to u_0$, then*

(i) There exists a choice $\varepsilon = \varepsilon(\tau)$ such that every minimizing movement along F_ε, with time step τ and with initial data u_ε is a minimizing movement for F from u_0 on $[0, T]$ for all T.

(ii) There exists a choice $\tau = \tau(\varepsilon)$ such that every minimizing movement along F_ε, with time step τ and with initial data u_ε is a limit of a sequence of minimizing movements for F_ε from u_ε, where ε is fixed, on $[0, T]$ for all T.

Chapter 2

Energies depending on orientation and position

The content of this chapter is based on a joint work with Andrea Braides [36].

2.1 Introduction

A motivation for the analysis in the present chapter is in the study of molecular models where particles are interacting through a potential including both orientation and position variables. In particular we have in mind potentials of Gay-Berne type in models of Liquid Crystals [14, 10, 47, 65, 67]. In that context a molecule of a Liquid Crystal is thought of as an ellipsoid with a preferred axis, whose position is identified with a vector $w \in \mathbb{R}^3$ and whose orientation is a vector $u \in \mathcal{S}^2$. Given α and β two such particles, the interaction energy will depend on their distance $\zeta_{\alpha\beta} = w_\alpha - w_\beta$ but also on their orientations u_α, u_β.

We restrict to a lattice model where all particles are considered as occupying the sites of a regular (cubic) lattice in the reference configuration. We introduce an energy density $G : \mathbb{Z}^m \times \mathbb{Z}^m \times \mathcal{S}^{N-1} \times \mathcal{S}^{N-1} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, so that

$$G^\xi(\alpha, u, v, \zeta) = G(\alpha, \alpha + \xi, u, v, \zeta)$$

represents the free energy of two molecules oriented as u and v at distance ζ, occupying the sites α and $\alpha + \xi$ in the reference lattice. Note that we have included a dependence on α to allow for a microstructure at the lattice level, but the energy density is meaningful also in the homogeneous case, independent of α. The function G is assumed to be lower semicontinuous and to satisfy some superlinear polynomial growth condition in the last variable. We note that this assumption actually rules out the Gay-Berne potential, which has a Lennard-Jones type behaviour at infinity so that our analysis can be applied in that context only if voids and macroscopic discontinuities are assumed to have a negligible impact.

We introduce a small scaling parameter $\varepsilon > 0$, $\mathbb{Z}_\varepsilon(\Omega) := \{\alpha \in \varepsilon\mathbb{Z}^m : (\alpha + [0, \varepsilon)^m) \cap \Omega \neq \emptyset\}$ and a cut-off parameter $R > 0$ representing the relevant range of the interactions (which we assume to be finite), we define the family of scaled functionals

$$E_\varepsilon(u,w)=\sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\le R}}\sum_{\alpha\in R_\varepsilon^\xi(\Omega)}\varepsilon^m G^\xi\left(\frac{\alpha}{\varepsilon},u(\alpha),u(\alpha+\varepsilon\xi),D_\varepsilon^\xi w(\alpha)\right)$$

defined on pairs $u:\mathbb{Z}_\varepsilon(\Omega)\to\mathcal{S}^{N-1}$ and $w:\mathbb{Z}_\varepsilon(\Omega)\to\mathbb{R}^n$, while

$$R_\varepsilon^\xi(\Omega):=\{\alpha\in\mathbb{Z}_\varepsilon(\Omega)\ :\ \alpha,\,\alpha+\varepsilon\xi\in\Omega\},\ D_\varepsilon^\xi w(\alpha):=\frac{w(\alpha+\varepsilon\xi)-w(\alpha)}{\varepsilon|\xi|}.$$

We remark that, due to technical difficulties, we will provide only a partial result, i.e., the Γ-convergence is proven only for function $u\in L^\infty(\Omega,B_1^N)$ such that $\|u\|_\infty<1$ (see Remark 4).

Extending functions defined on $\mathbb{Z}_\varepsilon(\Omega)$ as their piecewise-constant interpolations, we may define a discrete-to-continuum convergence $(u_\varepsilon,w_\varepsilon)$ to (u,w). The assumptions on G ensure that u takes values in the unit ball and w is a Sobolev function. We can then perform an asymptotic analysis using the notation of Γ-convergence: in particular we make use of the Fonseca-Müller blow-up technique [33, 52] to compute the lim inf inequality. Let $\varepsilon_j\to 0$ and let $(u_{\varepsilon_j},w_{\varepsilon_j})\to(u,w)$ be such that $\sup_j E_{\varepsilon_j}(u_{\varepsilon_j},w_{\varepsilon_j})<\infty$, then we can define a sequence of measures

$$\mu_j(A)=\sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\le R}}\sum_{\alpha\in R_{\varepsilon_j}^\xi(\Omega)}\varepsilon_j^m G^\xi\left(\frac{\alpha}{\varepsilon_j},u_{\varepsilon_j}(\alpha),u_{\varepsilon_j}(\alpha+\varepsilon\xi),D_{\varepsilon_j}^\xi w_{\varepsilon_j}(\alpha)\right)\delta_{\alpha+\frac{\varepsilon_j}{2}\xi}(A)$$

which, thanks to the equiboundedness of the above energies, weak* converges in the sense of measure to a limit measure μ.
Then we can perform a local analysis, i.e, for every Lebesgue point $x_0\in\Omega$ for μ with respect to the m-dimensional Lebesgue measure, we can apply the blow-up. This means that we rescale and modify $(u_{\varepsilon_j},w_{\varepsilon_j})$ in such a way that they satisfy proper average and boundary conditions. While for w_{ε_j} we can apply a standard cut-off argument, to modify the sequence u_{ε_j} we need to prove two geometrical lemmas. The first one is a simple observation that each point in the unit ball in $\mathbb{R}^N$ with $N>1$ can be written exactly as the average of k vectors in S^{N-1} for all $k\ge 2$, while the second one allows to modify sequences $(u_{\varepsilon_j},w_{\varepsilon_j})$ satisfying an asymptotic condition on the discrete average of u_{ε_j} with a sequence $(\widetilde{u}_\varepsilon,\widetilde{w}_\varepsilon)$ satisfying a sharp one and with the same energy E_ε. This is done by changing the values of u_{ε_j} in an asymptotically negligible percentage of nodes using the first lemma.

Minimizing on all the functions satisfying the same boundary and average conditions, then integrating the local estimates so obtained, we reach the conclusion, i.e., we show that the Γ-limit of the sequence E_ε, for functions $u\in L^\infty(\Omega,B_1^N)$ such that $\|u\|_\infty<1$ and $w\in W^{1,p}(\Omega,\mathbb{R}^n)$, is a continuous functional

$$E_0(u,w)=\int_\Omega G_{hom}(u,\nabla w)\,dx,$$

where the function G_{hom} is defined via a homogenization formula

$$G_{hom}(z,M)=\lim_{T\to\infty}\frac{1}{T^m}\inf\Big\{\mathcal{E}_T(u,w):\langle u\rangle_{Q_T}^{d,1}=z,\ w(\alpha)=(M,\alpha)\text{ on }\partial Q_T\Big\},$$

where $Q_T = (0,T)^m$ and

$$\mathcal{E}_T(u,w) = \sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\le R}} \sum_{\beta\in R_1^\xi(Q_T)} G^\xi\left(\beta, u(\beta), u(\beta+\xi), D_1^\xi w(\beta)\right).$$

This result is a first step in the extension of both the homogenization theorem by Alicandro and Cicalese (2004)[1], where no dependence on u is present, and that of Alicandro, Cicalese and Gloria (2008)[2] which instead deals with u only (see Theorem 12 and Theorem 10 of the previous chapter). The main feature of the homogenization formula above, which makes it different from the ones just recalled, is that test functions u satisfy the non-convex constraint on the average. This is a non-trivial fact, and its proof is the main technical point of the work.

It is worth to notice that the same argument holds in the case of energies depending only on orientation; i.e., of the form

$$E_\varepsilon(u) = \sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\le R}} \sum_{\alpha\in R_\varepsilon^\xi(\Omega)} \varepsilon^m G^\xi\left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha+\varepsilon\xi)\right),$$

providing a simplified formula with respect to the one in [2].

We conclude applying such results to discrete functionals with Gay-Berne type energy densities: in a general setting we can state a result similar to the ones already treated (see Theorem 16), in the one dimensional case we are able to show an explicit construction (see Theorem 17) which is independent from the previous results (see Remark 6). Let $V^E : B^1 \times \mathbb{R}$ be an effective potential derived from the one dimensional version of the Gay-Berne potential, then if we consider the discrete functional

$$E_\varepsilon(z,w) = \sum_{i=1}^{N_\varepsilon} \varepsilon V^E\left(z_i, \frac{w_i - w_{i-1}}{\varepsilon}\right)$$

where $N_\varepsilon = \lfloor 1/\varepsilon \rfloor$, $z_i = z(\varepsilon i)$ is the average orientation between i-th and the $(i-1)$-th particle and $w_i = w(\varepsilon i)$ is the position of the i-th particle, we get that the above energies Γ-converge to a functional of the form

$$E(z,w) = \int_0^1 Co\Big(V^E(z,w')\Big)\,dx,$$

where with $Co(F)$ we denote the convex envelope of a function F. This result is only a partial result, since the Γ-convergence is proven only on functions z with $\|z\|_\infty < 1$.

2.2 Setting of the problem

Let $m, n \ge 1$, $N \ge 2$ be fixed. We denote by $\{e_1, \dots, e_m\}$ the standard basis of $\mathbb{R}^m$. Given two vectors $v_1, v_2 \in \mathbb{R}^n$, by (v_1, v_2) we denote their scalar product. If $v \in \mathbb{R}^m$, we use $|v|$ for the usual euclidean norm. $\mathcal{S}^{N-1}$ is the standard unit sphere of $\mathbb{R}^N$ and B_1^N the closed unit ball of $\mathbb{R}^N$. $\mathcal{M}^{m\times n}$ is the space of all $m \times n$-matrices. If $P \in \mathcal{M}^{m\times n}$ and $Q \in \mathcal{M}^{n\times l}$, then (P,Q) will also denote the standard row by column product. If $x \in \mathbb{R}$, its integer part is denoted by $\lfloor x \rfloor$. We also set $Q_T = (0,T)^m$ and $\mathcal{B}(\Omega)$ as the family of all open subsets of Ω. If A is an open bounded set, given a function $u : A \to \mathbb{R}^N$ we denote its average over A as

$$\langle u \rangle_A = \frac{1}{|A|} \int_A u(x)\, dx.$$

2.2.1 Discrete functions

Let $\Omega \subset \mathbb{R}^m$ be an open bounded domain with Lipschitz boundary, and let $\varepsilon > 0$ be the spacing parameter of the cubic lattice $\varepsilon\mathbb{Z}^m$. We define the set $\mathbb{Z}_\varepsilon(\Omega) := \{\alpha \in \varepsilon\mathbb{Z}^m : (\alpha + [0,\varepsilon)^m) \cap \Omega \neq \emptyset\}$ and we will consider pairs of discrete functions $u : \mathbb{Z}_\varepsilon(\Omega) \to \mathcal{S}^{N-1}$ and $w : \mathbb{Z}_\varepsilon(\Omega) \to \mathbb{R}^n$ defined on the lattice.

For $\xi \in \mathbb{Z}^m$, we define

$$R_\varepsilon^\xi(\Omega) := \{\alpha \in \mathbb{Z}_\varepsilon(\Omega) \ : \ \alpha,\, \alpha + \varepsilon\xi \in \Omega\}, \qquad D_\varepsilon^\xi w(\alpha) := \frac{w(\alpha + \varepsilon\xi) - w(\alpha)}{\varepsilon|\xi|},$$

while the "discrete" average of a function $v : \mathbb{Z}_\varepsilon(A) \to \mathcal{S}^{N-1}$ over an open bounded domain A will be denoted by

$$\langle v \rangle_A^{d,\varepsilon} = \frac{1}{\#(\mathbb{Z}_\varepsilon(A))} \sum_{\alpha \in \mathbb{Z}_\varepsilon(A)} v(\alpha).$$

2.2.2 Discrete energies

Given a function $G : \mathbb{Z}^m \times \mathbb{Z}^m \times \mathcal{S}^{N-1} \times \mathcal{S}^{N-1} \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ we assume that

1. the function satisfies a growth hypothesis

$$C_1(|\zeta|^p - 1) \leq G(\alpha, \beta, u, v, \zeta) \leq C_2(|\zeta|^p + 1) \quad \text{with } 0 < C_1 < C_2 \tag{2.1}$$

for every $(\alpha, \beta, u, v, \zeta) \in \mathbb{Z}^m \times \mathbb{Z}^m \times \mathcal{S}^{N-1} \times \mathcal{S}^{N-1} \times \mathbb{R}^n$;

2. the function is periodic in the space variables, i.e., there exists $l \in \mathbb{N}$ such that

$$G(\cdot, \cdot, u, v, \zeta) \text{ is } \ Q_l \ \text{ periodic.} \tag{2.2}$$

Then we introduce the following notation:

$$G^\xi\,(\alpha, u, v, \zeta) = G(\alpha, \alpha + \xi, u, v, \zeta). \tag{2.3}$$

Since we restrict ourselves to finite interactions on the lattice, we introduce a cut-off parameter $R > 0$. The energies we treat are of the form

$$E_\varepsilon(u, w; \Omega) = \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \leq R}} \sum_{\alpha \in R_\varepsilon^\xi(\Omega)} \varepsilon^m G^\xi\left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha + \varepsilon\xi), D_\varepsilon^\xi w(\alpha)\right), \tag{2.4}$$

defined for $u : \mathbb{Z}_\varepsilon(\Omega) \to \mathcal{S}^{N-1}, \ w : \mathbb{Z}_\varepsilon(\Omega) \to \mathbb{R}^n$.

2.2.3 Discrete-to-continuum convergence

In what follows we identify each discrete function u with its piecewise constant extension $\tilde{u}$ defined by $\tilde{u}(t) = u(\alpha)$ if $t \in \alpha + [0, \varepsilon)^m$. We introduce the sets:

$$\mathcal{A}_\varepsilon(\Omega; \mathcal{S}^{N-1}) := \Big\{ \tilde{u} : \mathbb{R}^m \to \mathcal{S}^{N-1} \ : \ \tilde{u}(t) \equiv u(\alpha) \text{ if } t \in \alpha + [0,\varepsilon)^m, \text{for } \alpha \in \mathbb{Z}_\varepsilon(\Omega) \Big\},$$

$$\mathcal{A}_\varepsilon(\Omega; \mathbb{R}^n) := \Big\{ \tilde{w} : \mathbb{R}^m \to \mathbb{R}^n : \ \tilde{w}(t) \equiv w(\alpha) \text{ if } t \in \alpha + [0,\varepsilon)^m, \text{ for } \alpha \in \mathbb{Z}_\varepsilon(\Omega) \Big\}.$$

If no confusion is possible, we will simply write u instead of $\tilde{u}$ and w instead of $\tilde{w}$. If $\varepsilon = 1$ we will simply write $\mathcal{A}(\Omega; \cdot) = \mathcal{A}_1(\Omega; \cdot)$.

With that identification we may see the functionals in (2.4) as defined on a subset of $L^\infty(\Omega, \mathcal{S}^{N-1}) \times L^p(\Omega, \mathbb{R}^n)$ and consider their extensions on that Lebesgue space. With an abuse of notation we do not rename such functionals and we set $E_\varepsilon : L^\infty(\Omega, \mathcal{S}^{N-1}) \times L^p(\Omega, \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ as

$$E_\varepsilon(u, w; \Omega) = \tag{2.5}$$
$$\begin{cases} \displaystyle\sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\alpha \in R^\xi_\varepsilon(\Omega)} \varepsilon^m G^\xi \left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha + \varepsilon\xi), D^\xi_\varepsilon w(\alpha) \right) & \text{if} \quad \begin{array}{l} w \in \mathcal{A}_\varepsilon(\Omega; \mathbb{R}^n), \\ u \in \mathcal{A}_\varepsilon(\Omega; \mathcal{S}^{N-1}) \end{array} \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\varepsilon_j \to 0$ and let $\{u_j\}$ be a sequence of functions $u_j : \mathbb{Z}_{\varepsilon_j}(\Omega) \to \mathcal{S}^{N-1}$, we will say that $\{u_j\}$ converges to a function u if $\tilde{u}_j$ is converging to u weakly* in L^∞. Let $\{w_j\}$ be a sequence of functions $w_j : \mathbb{Z}_{\varepsilon_j}(\Omega) \to \mathbb{R}^n$, we will say that w_j converges to w if $\tilde{w}_j$ is converging to w in L^1_{loc}. Then we will say that the functionals defined in (2.4) Γ-converge to E_0 if E_ε defined in (2.5) Γ-converge to E_0 with respect to that convergence.

The choice of the convergence is justified by the following compactness result. Its proof follows from the compactness result in [1], after noting that $\{u_j\}$ play no role once the energy is equibounded.

Proposition 6 (compactness)**.** *Let $\varepsilon_j \to 0$ be a sequence of vanishing parameters, let $\{w_j\}$ be a bounded sequence in $L^p(\Omega, \mathbb{R}^n)$, and $\{u_j\}$ in $L^\infty(\Omega, \mathcal{S}^{N-1})$ such that $\sup_{\varepsilon_j} E_{\varepsilon_j}(u_j, w_j; \Omega) < +\infty$, then, up to subsequences, there exists $w \in W^{1,p}(\Omega, \mathbb{R}^n)$ such that $w_j \to w$ in $L^p(\Omega, \mathbb{R}^n)$.*

2.2.4 Statement of the convergence theorem

Here we analyze the behaviour of the energies (2.5) by characterizing their Γ-limit in the context of variational homogenization (see for further details [21, 20, 27]). In particular we will apply the Fonseca-Müller homogenization by blow-up technique [33, 52].
The main result we want to prove is

Theorem 14. *Let E_ε be the energies defined in (2.5)and suppose that (2.1) and (2.2) hold. Then, for functions $u \in L^\infty(\Omega, B^N_1)$ such that $\|u\|_\infty < 1$ and $w \in W^{1,p}(\Omega, \mathbb{R}^n)$, E_ε Γ-converge as $\varepsilon \to 0$ with respect to the convergence above to the homogenized functional*

$$E_0(u, w) = \int_\Omega G_{hom}(u, \nabla w)\, dx. \tag{2.6}$$

The function G_{hom} is given by the following asymptotic formula

$$G_{hom}(z, M) = \lim_{T\to\infty} \frac{1}{T^m} \inf\Big\{E_1(u, \zeta; Q_T) : \langle u\rangle^{d,1}_{Q_T} = z, \zeta \in \mathcal{A}^R_M(Q_T)\Big\} \quad (2.7)$$

and

$$A^R_M(Q_T) := \Big\{\zeta : \mathbb{Z}_1(Q_T) \to \mathbb{R}^n : \zeta(\alpha) = (M, \alpha) \text{ if } (\alpha+[R, R]^m)\cap(\mathbb{R}^m \backslash Q_T) \neq \emptyset\Big\}.$$

2.2.5 Energies depending on orientation

As a particular case of Theorem 14 (or, more precisely, as a small variation of its proof) we can treat energies depending only on orientation. Given a function $G : \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{S}^{N-1} \times \mathcal{S}^{N-1} \to \mathbb{R} \cup \{+\infty\}$ we assume that

1. the function satisfies the bound

$$\sup\big\{|G(\alpha, \beta, u, v)| : \alpha, \beta \in \mathbb{R}^m, u, v \in \mathcal{S}^{N-1}\big\} < \infty; \quad (2.8)$$

2. the function is periodic in the space variables, i.e., there exists $l \in \mathbb{N}$ such that

$$G(\cdot, \cdot, u, v) \text{ is } Q_l \text{ periodic.} \quad (2.9)$$

Analogously to the first section, given $\xi \in \mathbb{R}^m$, we introduce the notation

$$G^\xi(\alpha, u, v) = G(\alpha, \alpha + \varepsilon\xi, u, v).$$

Up to identify each function u and w with its piecewise constant extension, we can consider energies $E_\varepsilon : L^\infty(\Omega, \mathcal{S}^{N-1}) \to \mathbb{R} \cup \{+\infty\}$ of the following form:

$$E_\varepsilon(u; \Omega) = \quad (2.10)$$
$$\begin{cases} \displaystyle\sum_{\substack{\xi\in\mathbb{Z}^m \\ |\xi|\leq R}} \sum_{\alpha\in R^\xi_\varepsilon(\Omega)} \varepsilon^m G^\xi\left(\frac{\alpha}{\varepsilon}, u(\alpha), u(\alpha + \varepsilon\xi)\right) & \text{if } u \in \mathcal{A}_\varepsilon(\Omega; \mathcal{S}^{N-1}), \\ +\infty & \text{otherwise.} \end{cases}$$

We have that

Theorem 15. *Let E_ε be the energy defined in (2.10) and suppose that (2.8) and (2.9) hold. Then, for functions $u \in L^\infty(\Omega, B^N_1)$ such that $\|u\|_\infty < 1$, E_ε Γ-converge to the homogenized functional*

$$E_0(u) = \int_\Omega G_{hom}(u)\, dx. \quad (2.11)$$

The function G_{hom} is given by the following asymptotic formula

$$G_{hom}(z) = \lim_{T\to\infty} \frac{1}{T^m} \inf\Big\{E_1(u; Q_T) : \langle u\rangle^{d,1}_{Q_T} = z\Big\}. \quad (2.12)$$

This is a partial improvement with respect to Theorem 5.3 in [2] (see also Theorem 10 in the previous chapter), where G_{hom} is characterized imposing a weaker constraint on the average of u. In our case we can match the average conditions thanks to a geometric lemma valid for $\mathcal{S}^{N-1}$, as shown in the next section.

2.3 Two geometric lemmas

In this section we provide two general lemmas. The first is a simple observation on the characterization of sums of vectors in $\mathcal{S}^{N-1}$, while the second one allows to satisfy conditions on the average of discrete functions with values in $\mathcal{S}^{N-1}$.

Lemma 1. *Let $u \in B_k^N$ a vector in the ball centred in the origin and with radius $k \geq 2$, then u can be written as the sum of k vectors on $\mathcal{S}^{N-1}$:*

$$u = \sum_{i=1}^{k} u_i \qquad u_i \in \mathcal{S}^{N-1}.$$

Equivalently, given $u \in B_1^N$ and $k \geq 2$, u can be written as the average of k vectors on $\mathcal{S}^{N-1}$:

$$u = \frac{1}{k}\sum_{i=1}^{k} u_i \qquad u_i \in \mathcal{S}^{N-1}.$$

Proof. We proceed by induction on k.
Let $k = 2$ and let $u \in B_2^N$. If $u = 0$ then it can be written as the sum of two opposite vectors in $\mathcal{S}^{N-1}$. Otherwise we consider a vector $v \in (u+\mathcal{S}^{N-1}) \cap \mathcal{S}^{N-1}$: we observe in fact that $(u + \mathcal{S}^{N-1}) \cap \mathcal{S}^{N-1} \neq \emptyset$. In particular we can write $u = v + (u - v)$: $v \in \mathcal{S}^{N-1}$ by construction, while $v \in (u + \mathcal{S}^{N-1}) \Rightarrow v - u \in \mathcal{S}^{N-1} \Rightarrow u - v \in \mathcal{S}^{N-1}$, so denoting $u_1 = v$ and $u_2 = (u - v)$ the first step of induction is proven. Let the thesis be true for $k = d - 1$, we show that it holds for $k = d$: let $u \in B_d^N$ and consider a $v \in (u + \mathcal{S}^{N-1}) \cap B_{d-1}^N$. Again this set is not empty and we can write $u = v + (u - v)$. Since $v \in (u + \mathcal{S}^{N-1}) \Rightarrow v - u \in \mathcal{S}^{N-1} \Rightarrow u - v \in \mathcal{S}^{N-1}$. Moreover $v \in B_{d-1}^N$ and this means that we can apply the induction hypothesis: $v = u_1 + \cdots + u_{d-1}$. Denoting $u_d = u - v$ we have the thesis. □

Lemma 2. *Let $A \subset \mathbb{R}^m$ be an open bounded domain. Let $\delta_j > 0$ be a spacing parameter and $u_j : \mathbb{Z}_{\delta_j}(A) \to \mathcal{S}^{N-1}$ be a sequence of discrete functions. Suppose that $u_j \rightharpoonup^* u$ in $L^\infty(A, B_1^N)$, and that the average*

$$\langle u \rangle_A = \frac{1}{|A|}\int_A u(x)\,dx$$

is such that $|\langle u \rangle_A| < 1$. Then for all j there exist $\tilde{u}_j$ such that

1. *the discrete average $\langle \tilde{u}_j \rangle_A^{d,\delta_j} = \dfrac{1}{\#(\mathbb{Z}_{\delta_j}(A))} \displaystyle\sum_{i \in \mathbb{Z}_{\delta_j}(A)} \tilde{u}_j(i)$ is equal to $\langle u \rangle_A$;*

2. *the function $\tilde{u}_j$ is such that $\#\{i : \tilde{u}_j(i) \neq u_j(i)\} = o(\#(\mathbb{Z}_{\delta_j}(A)))_{j\to+\infty}$.*

Proof. To simplify the notation we set $\mathbb{Z}_j(A) = \mathbb{Z}_{\delta_j}(A)$, $\langle u_j \rangle_A^d = \langle u_j \rangle_A^{d,\delta_j}$ and $u_j^i = u_j(i)$.

Note that, by the weak convergence of u_j,

$$\eta_j := |\langle u_j \rangle_A^d - \langle u \rangle_A| = o(1) \tag{2.13}$$

as $j \to +\infty$. We will treat the case that $\eta_j \neq 0$ since otherwise we simply take $\tilde{u}_j = u_j$.

Since $\langle u_j \rangle_A^d \to \langle u \rangle_A$, by the hypothesis that $|\langle u \rangle_A| < 1$ we may suppose that

$$|\langle u_j \rangle_A^d| \leq 1 - 2b \tag{2.14}$$

for all j, for some $b \in (0, 1/2)$.

Claim: setting $B = b/(4-2b)$, for every $i \in \mathbb{Z}_j(A)$ there exist at least $B\,\#(\mathbb{Z}_j(A))$ indices $l \in \mathbb{Z}_j(A)$ such that $(u_j^i, u_j^l) \leq 1 - b$.

Indeed, otherwise there exists at least one index i for which the set

$$\mathcal{A}_b := \big\{ l \in \mathbb{Z}_j(A) : (u_j^i, u_j^l) > 1 - b,\ l \neq i \big\} \tag{2.15}$$

is such that $\#(\mathcal{A}_b) \geq (1-B)\#(\mathbb{Z}_j(A))$ and we have

$$\begin{aligned}
|\langle u_j \rangle_A^d| &\geq (\langle u_j \rangle_A^d, u_j^i) = \frac{1}{\#(\mathbb{Z}_j(A))} \sum_{l \in \mathbb{Z}_j(A)} (u_j^l, u_j^i) \\
&= \frac{1}{\#(\mathbb{Z}_j(A))} \sum_{l \in \mathcal{A}_b} (u_j^l, u_j^i) + \frac{1}{\#(\mathbb{Z}_j(A))} \sum_{l \in \mathbb{Z}_j(A) \setminus \mathcal{A}_b} (u_j^l, u_j^i) \\
&\geq \frac{1}{\#(\mathbb{Z}_j(A))} \left(\#(\mathcal{A}_b)(1-b) - (\#(\mathbb{Z}_j(A)) - \#(\mathcal{A}_b))\right) \\
&= \frac{1}{\#(\mathbb{Z}_j(A))} \Big((2-b)\#(\mathcal{A}_b) - \#(\mathbb{Z}_j(A))\Big) \\
&\geq (2-b)(1-B) - 1 = 1 - \frac{3}{2}b > |\langle u_j \rangle_A^d|,
\end{aligned}$$

where we have used (2.14) in the last estimate. We then obtain a contradiction, thus proving the claim.

By the Claim above, there exist $\lfloor (B/2)\#(\mathbb{Z}_j(A)) \rfloor$ pairs of indices (i_s, l_s) with $\{i_s, l_s\} \cap \{i_r, l_r\} = \emptyset$ if $r \neq s$ and

$$(u_j^{i_s}, u_j^{l_s}) \leq 1 - b. \tag{2.16}$$

Since $\eta_j \to 0$, with fixed $c > 0$ we may suppose that

$$B\,\#(\mathbb{Z}_j(A)) > 2 \left\lfloor \frac{\eta_j}{c} \#(\mathbb{Z}_j(A)) \right\rfloor + 1 \tag{2.17}$$

for all j.

We now set

$$P_j = \left\lfloor \frac{\eta_j}{c} \#(\mathbb{Z}_j(A)) \right\rfloor + 1, \tag{2.18}$$

so that by (2.17) there exist pairs (i_s, l_s) as above, with $s \in I_j := \{1, \dots, P_j\}$. Note that $P_j = o(\#(\mathbb{Z}_j(A)))_{j \to +\infty}$.

If for fixed j we define the vector

$$w = \sum_{i \in \mathbb{Z}_j(A)} u_j^i - \#(\mathbb{Z}_j(A))\langle u \rangle_A - \sum_{s \in I_j} (u_j^{i_s} + u_j^{l_s}),$$

then we have

$$\begin{aligned} |w| &\leq \#(\mathbb{Z}_j(A))|\langle u_j\rangle^d - \langle u\rangle_A| + \sum_{s\in I_j} |u_j^{i_s} + u_j^{l_s}| \\ &\leq \#(\mathbb{Z}_j(A))\eta_j + \sum_{s\in I_j} \sqrt{2 + 2(u_j^{i_s}, u_j^{l_s})} \\ &\leq \#(\mathbb{Z}_j(A))\eta_j + P_j\sqrt{4-2b}. \end{aligned}$$

Since $\#(\mathbb{Z}_j(A))\eta_j < c\,P_j$ by (2.18), we then have $|w| \leq c\,P_j + P_j\sqrt{4-2b}$. We finally choose $c > 0$ such that $\sqrt{4-2b} < 2-c$, so that

$$|w| < 2P_j.$$

By Lemma 1, applied with $u = -w$ and $k = 2P_j$, there exists a set of $2P_j$ vectors in $\mathcal{S}^{N-1}$, that we may label as

$$\{\overline{u}_j^{i_s}, \overline{u}_j^{l_s} : s \in I_j\},$$

such that

$$\sum_{s\in I_j} (\overline{u}_j^{i_s} + \overline{u}_j^{l_s}) = -w. \tag{2.19}$$

If we now define $\tilde{u}_j$ by setting

$$\tilde{u}_j^i = \begin{cases} \overline{u}_j^i & \text{if } i \in \{i_s, l_s : s \in I_j\} \\ u_j^i & \text{otherwise,} \end{cases} \tag{2.20}$$

we have

$$\begin{aligned} \langle \tilde{u}_j\rangle_A^d &= \frac{1}{\#(\mathbb{Z}_j(A))}\Big(\sum_{i\in\mathbb{Z}_j(A)} u_j^i - \sum_{s\in I_j}(u_j^{i_s} + u_j^{l_s}) + \sum_{s\in I_j}(\overline{u}_j^{i_s} + \overline{u}_j^{l_s})\Big) \\ &= \frac{1}{\#(\mathbb{Z}_j(A))}\Big(\sum_{i\in\mathbb{Z}_j(A)} u_j^i - \sum_{s\in I_j}(u_j^{i_s} + u_j^{l_s}) - w\Big) \\ &= \langle u\rangle_A, \end{aligned}$$

and

$$\#\{i : \tilde{u}_j(i) \neq u_j(i)\} = 2P_j = o(\#(\mathbb{Z}_j(A)))_{j\to+\infty}. \tag{2.21}$$

Hence the second claim of the Theorem hold and the proof is concluded. ☐

Remark 2. The assumption $|\langle u\rangle_A| < 1$ in Lemma 2 is motivated by the following fact: if $|\langle u\rangle_A| = 1$, we can find sequences $u_j \rightharpoonup^* u$, $u_j \neq u$, such that $|\langle u_j\rangle_A^{d,\delta_j}| = 1$ for every j. In this case, to have $\langle u_j\rangle_A^{d,\delta_j} = \langle u\rangle_A$, we should change the function u_j in every point.

2.4 The homogenisation formula

In this section we prove that the homogenisation formula characterizing G_{hom} in Theorem 14 is well defined, and derive some properties of that function. To that end we consider spaces of functions subject to boundary conditions defined by

$$A_M^{R,\varepsilon}(Q_T) := \Big\{\zeta \in \mathcal{A}_\varepsilon(\Omega;\mathbb{R}^n) : \zeta(\alpha) = (M,\alpha) \text{ if } (\alpha + [-\varepsilon R, \varepsilon R]^m) \cap (\mathbb{R}^m \setminus Q_T) \neq \emptyset\Big\}.$$

If $\varepsilon = 1$ we will simply write $A_M^R(Q_T) = A_M^{R,1}(Q_T)$.

In order to comply to boundary conditions we will make use of the following result, whose proof can be found, for example, in [1](see Theorem 3.10 therein).

Lemma 3. *Let $A \subset \mathbb{R}^m$ be an open bounded set with Lipschitz boundary. Let $\varepsilon_j \to 0$ be a vanishing sequence of parameters, let $\{u_j\}$ be in $L^\infty(A, B_1^N)$ and $\{w_j\}$ be in $W^{1,p}(A, \mathbb{R}^n)$ such that $\sup_{\varepsilon_j} E_{\varepsilon_j}(u_j, w_j; A) < C < +\infty$. If $w_j \to w$ in L^p with $w(x) = (M, x)$ for some $M \in \mathcal{M}^{m\times n}$, then there exists another sequence of functions $\tilde{w}_j \to w$ in L^p such that $\tilde{w}_j \in A_M^{R,\varepsilon_j}(A)$ and*

$$E_{\varepsilon_j}(u_j, \tilde{w}_j; A) \le E_{\varepsilon_j}(u_j, w_j; A) + o(1) \quad \text{for} \quad j \to +\infty.$$

Furthermore, for each fixed $\rho > 0$ we can find a $\bar{j} > 0$ such that for any $j \ge \bar{j}$ we have $\tilde{w}_j$ such that $\tilde{w}_j(x) = w_j(x)$ if $\operatorname{dist}(x, \partial A) \ge \rho$.

Proposition 7. *Let G be a function satisfying (2.1) and (2.2) and let G^ξ be defined as in (2.3). For all $T > 0$ consider an arbitrary $x_T \in \mathbb{R}^m$, then the limit*

$$\lim_{T\to\infty} \frac{1}{T^m} \inf\left\{ E_1(u, \zeta; x_T + Q_T) : \ \langle u\rangle^{d,1}_{x_T+Q_T} = z,\ \zeta \in \mathcal{A}_M^R(x_T + Q_T) \right\} \tag{2.22}$$

exists for all $z \in B_1^N$ and $M \in \mathcal{M}^{m\times n}$.

Proof. Let $z \in B_1^N$ and $M \in \mathcal{M}^{m\times n}$ be fixed. In the following we will assume G to be 1-periodic (which means that in (2.2) we consider $l = 1$) and $x_T = 0$, since the general case can be derived from arguments already present for example in [1] and [2] and showing only heavier notations. Let $t > 0$ and consider the function

$$g_t(z, M) = \frac{1}{t^m} \inf\left\{ E_1(u, \zeta; Q_t) : \ \langle u\rangle^{d,1}_{Q_t} = z,\ \zeta \in \mathcal{A}_M^R(Q_t) \right\}. \tag{2.23}$$

In the rest of the proof we will denote with $g_t = g_t(z, M)$ and for $s > t$ also $g_s = g_s(z, M)$. Let u_t and ζ_t be two test functions for g_t such that

$$\frac{1}{t^m} E_1(u_t, \zeta_t; Q_t) \le g_t + \frac{1}{t}, \tag{2.24}$$

then for every $s > t$ we want to prove that $g_s < g_t$. We introduce the following notation:

$$I := \left\{0, \dots, \left\lfloor \frac{s}{t} \right\rfloor - 1\right\}^m.$$

We can construct two test functions for g_s in the following way:

$$u_s(\beta) = \begin{cases} u_t(\beta - ti) & \text{if} \quad \beta \in ti + Q_t \quad i \in I \\ \bar{u}(\beta) & \text{otherwise,} \end{cases}$$

$$\zeta_s(\beta) = \begin{cases} \zeta_t(\beta - ti) + (M, ti) & \text{if} \quad \beta \in ti + Q_t \quad i \in I \\ (M, \beta) & \text{otherwise,} \end{cases}$$

where $\bar{u}$ is a $\mathcal{S}^{N-1}$-valued function such that $\langle u_s\rangle^{d,1}_{Q_s} = z$. We can choose such $\bar{u}$ thanks to Lemma 1: let define

$$\mathbb{Z}(Q_s) = \mathbb{Z}^m \cap Q_s,$$
$$Q_{s,t} = \Big(\bigcup_{i\in I} (ti + Q_t) \cap \mathbb{Z}(Q_s)\Big).$$

We want $\bar{u}$ to be such that it holds

$$\sum_{\beta\in\mathbb{Z}(Q_s)} u_s(\beta) = z\#(\mathbb{Z}(Q_s)).$$

Equivalently

$$\sum_{\substack{\beta\in Q_{s,t}\\ \beta\in ti+Q_t}} u_t(\beta - ti) + \sum_{\beta\in\mathbb{Z}(Q_s)\setminus Q_{s,t}} \bar{u}(\beta) = z\#(\mathbb{Z}(Q_s)),$$

which means

$$\sum_{\beta\in\mathbb{Z}(Q_s)\setminus Q_{s,t}} \bar{u}(\beta) = z\Big(\#(\mathbb{Z}(Q_s)) - \#(Q_{s,t})\Big). \tag{2.25}$$

On the left side of (2.25) we are summing $\#(\mathbb{Z}(Q_s)) - \#(Q_{s,t})$ vectors in $\mathcal{S}^{N-1}$ while on the right side we have a vector which belongs to a ball whose radius is at most $\#(\mathbb{Z}(Q_s)) - \#(Q_{s,t})$.

If $|z| < 1$, thanks to Lemma 1 we know that it is possible to choose the values of $\bar{u}$ in such a way that the relation (2.25) is satisfied.

If $|z| = 1$, we simply set $\bar{u}(\beta) \equiv z$, and again (2.25) is satisfied.

Moreover we observe that

$$R_1^\xi(Q_s) \subseteq \left(\bigcup_{i\in I} R_1^\xi(ti+Q_t)\right) \cup \left(R_1^\xi\Big(Q_s\setminus\bigcup_{i\in I}(ti+Q_t)\Big)\right) \cup \\ \left(\bigcup_{i\in I}(ti + (\{0,\dots,t+R\}^N\setminus\{0,\dots,t-R\}^N))\right)$$

and if β belongs to one of the last two set of indices, then $D_1^\xi\zeta_s(\beta) = M(\xi/|\xi|)$. Recalling now (2.1), for some $\bar{C} > 0$ big enough, we have that

$$g_s \le \frac{1}{s^m} E_1(u_s,\zeta_s;Q_s) \le \left\lfloor\frac{s}{t}\right\rfloor^m \frac{1}{s^m} E_1(u_t,\zeta_t;Q_t) \\ + \frac{1}{s^m}\bar{C}|M|^p\left(s^m - \left\lfloor\frac{s}{t}\right\rfloor^m t^m + \left\lfloor\frac{s}{t}\right\rfloor^m ((t+R)^m - (t-R)^m)\right).$$

Using now (2.24) we get

$$g_s \le \left\lfloor\frac{s}{t}\right\rfloor^m \frac{t^m}{s^m}\left(gt + \frac{1}{t}\right) \\ + \frac{1}{s^m}\bar{C}|M|^p\left(s^m - \left\lfloor\frac{s}{t}\right\rfloor^m t^m + \left\lfloor\frac{s}{t}\right\rfloor^m ((t+R)^m - (t-R)^m)\right). \tag{2.26}$$

Letting now $s \to +\infty$ and then $t \to +\infty$, we have that

$$\limsup_s g_s(z,M) \le \liminf_t g_t(z,M),$$

which concludes the proof. □

Proposition 8. *The function G_{hom} as defined in (2.7) satisfies the following properties:*

1. *$G_{hom}(\cdot, M)$ is convex for every $M \in \mathcal{M}^{m\times n}$;*

2. *$G_{hom}(z, \cdot)$ is continuous for every $z \in B_1^N$.*

Proof. We prove the convexity: let $M \in \mathcal{M}^{m\times n}$ be fixed, we want to show that for every $0 \le t \le 1$ and for every $z_1, z_2 \in B_1^N$ it holds:

$$G_{hom}(tz_1 + (1-t)z_2, M) \le tG_{hom}(z_1, M) + (1-t)G_{hom}(z_2, M). \tag{2.27}$$

Let $k \in \mathbb{N}$ be fixed; having in mind (2.2) and thanks to Proposition 7, we can choose $k \in l\mathbb{N}$. We define

$$g_k(z, M) = \frac{1}{k^m} \inf\Big\{E_1(u, \zeta; Q_k) : \ \langle u\rangle^{d,1}_{Q_k} = z,\ \zeta \in \mathcal{A}^R_M(Q_k)\Big\}. \tag{2.28}$$

In the following we will denote $g_k^1 = g_k(z_1, M)$, $g_k^2 = g_k(z_2, M)$.

Then let (u_k^1, ζ_k^1) and (u_k^2, ζ_k^2) be pairs of functions such that

$$\frac{1}{k^m} E_1(u_k^1, \zeta_k^1; Q_k) \le g_k^1 + \frac{1}{k}, \tag{2.29}$$

$$\frac{1}{k^m} E_1(u_k^2, \zeta_k^2; Q_k) \le g_k^2 + \frac{1}{k}. \tag{2.30}$$

Let $h > k$ be such that $h/k \in \mathbb{N}$.

Let denote $g_h = g_h(tz_1 + (1-t)z_2, M)$, we define the following test functions for g_h:

$$u_h(\beta) = \begin{cases} u_k^1(\beta - ki) & \text{if} \quad \beta \in ki + Q_k \quad i \in \left\{0, \dots, \frac{h}{k} - 1\right\}^{m-1} \times \left\{0, \dots, \left\lfloor \frac{h}{k} t\right\rfloor - 1\right\} \\ u_k^2(\beta - ki) & \text{if} \quad \beta \in ki + Q_k \quad i \in \left\{0, \dots, \frac{h}{k} - 1\right\}^{m-1} \times \left\{\frac{h}{k} - \left\lfloor \frac{h(1-t)}{k}\right\rfloor, \dots, \frac{h}{k} - 1\right\} \\ \bar{u}(\beta) & \text{otherwise,} \end{cases}$$

$$\zeta_h(\beta) = \begin{cases} \zeta_k^1(\beta - ki) + k(M, i) & \text{if } \beta \in ki + Q_k,\ i \in \left\{0, \dots, \frac{h}{k} - 1\right\}^{m-1} \times \left\{0, \dots, \left\lfloor \frac{h}{k} t\right\rfloor - 1\right\} \\ \zeta_k^2(\beta - ki) + k(M, i) & \text{if } \beta \in ki + Q_k,\ i \in \left\{0, \dots, \frac{h}{k} - 1\right\}^{m-1} \times \left\{\frac{h}{k} - \left\lfloor \frac{h(1-t)}{k}\right\rfloor, \dots, \frac{h}{k} - 1\right\} \\ (M, \beta) & \text{otherwise.} \end{cases}$$

Reasoning as in Proposition 7, thanks to Lemma 1, we can choose the values of $\bar{u}$ such that $\langle u_s\rangle^{d,1}_{Q_s} = tz_1 + (1-t)z_2$.

By (2.1) and (2.2), for some $\bar{C} > 0$ big enough, we get

$$\begin{aligned} g_h &\le \frac{1}{h^m} E_1(u_h, \zeta_h; Q_h) \\ &\le \frac{1}{h^m}\left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h}{k} t\right\rfloor E_1(u_k^1, \zeta_k^1; Q_k) + \frac{1}{h^m}\left(\frac{h}{k}\right)^{m-1} \left\lfloor \frac{h(1-t)}{k}\right\rfloor E_1(u_k^2, \zeta_k^2; Q_k) \\ &+ \frac{1}{h^m}\bar{C}|M|^p \left(h^m - \left(\frac{h}{k}\right)^{m-1}\left(\left\lfloor \frac{h}{k} t\right\rfloor + \left\lfloor \frac{h(1-t)}{k}\right\rfloor\right) k^m\right) \\ &+ \frac{1}{h^m}\bar{C}|M|^p \left(\frac{h}{k}\right)^m ((k+R)^m - (k-R)^m). \end{aligned}$$

Then, thanks to (2.29) and (2.30), we can rewrite the above relation as

$$g_h \leq \frac{k^m}{h^m}\left(\frac{h}{k}\right)^{m-1}\left\lfloor\frac{h}{k}t\right\rfloor\left(g_k^1+\frac{1}{k}\right)+\frac{k^m}{h^m}\left(\frac{h}{k}\right)^{m-1}\left\lfloor\frac{h(1-t)}{k}\right\rfloor\left(g_k^2+\frac{1}{k}\right)$$
$$+\frac{1}{h^m}\bar{C}|M|^p\left(h^m-\left(\frac{h}{k}\right)^{m-1}\left(\left\lfloor\frac{h}{k}t\right\rfloor+\left\lfloor\frac{h(1-t)}{k}\right\rfloor\right)k^m\right)$$
$$+\frac{1}{h^m}\bar{C}|M|^p\left(\frac{h}{k}\right)^m((k+R)^m-(k-R)^m).$$

Letting $h \to +\infty$ and then $k \to +\infty$, we can conclude the proof.

Now we prove the continuity of $G_{hom}(z,\cdot)$ for any fixed $z \in B_1^N$. Let $A \in \mathcal{M}^{m\times n}$ and $B = (b_{ij}e_i \otimes e_j)_{i=1,\dots,m}^{j=1,\dots,n} \in \mathcal{M}^{m\times n}$, where $\{e_i \otimes e_j\}_{i=1,\dots,m}^{j=1,\dots,n}$ is a reference system for $\mathcal{M}^{m\times n}$.
We observe that the continuity of G_{hom} is straightforward once we prove the convexity along each direction $e_i \otimes e_j$: in this case G_{hom} would be locally Lipschitz along the directions $e_i \otimes e_j$ and hence, also by the growth conditions (2.1), locally Lipschitz.

We prove the convexity along $e_1 \otimes e_1$, being the others identical, i.e., for $0 \leq t \leq 1$ we prove

$$G_{hom}(z, tA+(1-t)(A+e_1\otimes e_1)) \leq$$
$$tG_{hom}(z,A)+(1-t)G_{hom}(z,A+e_1\otimes e_1).$$

Let $k \in \mathbb{N}$ be fixed and g_k be as in (2.28), then we denote $g_k^A = g_k(z,A)$ and $g_k^B = g_k(z, A+e_1\otimes e_1)$. Let (u_k^A, ζ_k^A) and (u_k^B,ζ_k^B) be pairs of functions such that

$$\frac{1}{k^m}E_1(u_k^A,\zeta_k^A;Q_k) \leq g_k^A+\frac{1}{k}, \tag{2.31}$$
$$\frac{1}{k^m}E_1(u_k^B,\zeta_k^B;Q_k) \leq g_k^B+\frac{1}{k}. \tag{2.32}$$

To make the content of the next constructions more readable we treat the case $t \in \mathbb{Q}$ and having in mind (2.2), thanks to Proposition 7, we can consider $k \in l\mathbb{N}$, the general case showing only heavier notations. Let $h > k$ be such that $t(h/k) \in \mathbb{N}$, we denote with $g_h = g_h(z, tA+(1-t)(A+e_1\otimes e_1))$. We consider the functions

$$u_h(\beta) =$$
$$\begin{cases} u_k^A(\beta-ki) & \text{if} \quad \beta \in ki+Q_k \quad i \in \left\{0,\dots,\frac{h}{k}t-1\right\}\times\left\{0,\dots,\frac{h}{k}-1\right\}^{m-1} \\ u_k^B(\beta-ki) & \text{if} \quad \beta \in ki+Q_k \quad i \in \left\{\frac{h}{k}t,\dots,\frac{h}{k}-1\right\}\times\left\{0,\dots,\frac{h}{k}-1\right\}^{m-1} \end{cases}$$

$$\zeta_h(\beta) =$$
$$\begin{cases} \zeta_k^A(\beta-ki)+k(A,i) \text{ if } \beta \in ki+Q_k,\ i \in \left\{0,\dots,\frac{h}{k}t-1\right\}\times\left\{0,\dots,\frac{h}{k}-1\right\}^{m-1} \\ \zeta_k^B(\beta-ki)+k(A,i)+(ki-th)e_1 \\ \qquad\qquad \text{if } \beta \in ki+Q_k,\ i \in \left\{\frac{h}{k}t,\dots,\frac{h}{k}-1\right\}\times\left\{0,\dots,\frac{h}{k}-1\right\}^{m-1}. \end{cases}$$

The function ζ_h is h-periodic, then we extend it on all $\mathbb{R}^m$ by periodicity. Let $\varepsilon_j > 0$ be a sequence of vanishing parameters, and consider the scaled function $\zeta_j(\alpha) := \varepsilon_j \zeta_h(\alpha/\varepsilon_j)$. For such function, we can prove that ζ_j converges to the affine function given by $(tA + (1-t)(A + e_1 \otimes e_1), x)$ on $L^p(Q_1, \mathbb{R}^n)$, so we can apply Lemma 3 to ζ_j, i.e., there exists a function $\tilde{\zeta}_j$ still converging to $(tA + (1-t)(A + e_1 \otimes e_1), x)$ and satisfying the correct boundary conditions. Scaling back $\tilde{\zeta}_j$, we have a test function $\tilde{\zeta}_h$ for g_h.
By construction, u_h is such that $\langle u_h \rangle^{d,1}_{Q_h} = z$ and so it is a test function for g_h. Then

$$g_h \leq \frac{1}{h^m} E_1(u_h, \tilde{\zeta}_h; Q_h) \leq \frac{1}{h^m} E_1(u_h, \zeta_h; Q_h).$$

By (2.1) and (2.2), for some $\bar{C} > 0$ big enough, we have

$$\begin{aligned} g_h \leq & \frac{1}{h^m}\left(\frac{h}{k}\right)^m tE_1(u_k^A, \zeta_k^A; Q_k) + \frac{1}{h^m}\left(\frac{h}{k}\right)^m (1-t)E_1(u_k^B, \zeta_k^B; Q_k) \\ & + \frac{1}{h^m}\bar{C}|tA + (1-t)(A + e_1 \otimes e_1)|^p \left(\left(\frac{h}{k}\right)^m ((k+R)^m - (k-R)^m)\right). \end{aligned}$$

Thanks to (2.31) and (2.32) we get

$$\begin{aligned} g_h \leq & \frac{k^m}{h^m}\left(\frac{h}{k}\right)^m t\left(g_k^A + \frac{1}{k}\right) + \frac{k^m}{h^m}\left(\frac{h}{k}\right)^m (1-t)\left(g_k^B + \frac{1}{k}\right) \\ & + \frac{1}{h^m}\bar{C}|tA + (1-t)(A + e_1 \otimes e_1)|^p \left(\left(\frac{h}{k}\right)^m ((k+R)^m - (k-R)^m)\right). \end{aligned}$$

Letting $h \to +\infty$ and then $k \to +\infty$, we can conclude the proof. □

***Remark* 3.** Let G_{hom} be defined as in (2.7). Then, if G_{hom} is defined for $|z| < 1$, from the above Proposition we can conclude that $G_{hom}(z, M)$ is jointly continuous in z and M (see for example Theorem 10.7 in [60]). Moreover $G_{hom}(z, M)$ satisfies a p-growth condition in the last variable: let $T > 0$ be fixed and $g_T(z, M)$ be defined as in (2.23), then for each $T > 0$ fixed, we can choose as test functions $w(\alpha) = (M, \alpha)$ and a u_T such that $\langle u_T \rangle^{d,1}_{Q_T} = z$. With few simple calculations the claim now follows.

2.5 Proof of the main theorem

***Remark* 4.** The proof of Theorem 14 holds only for functions $u \in L^\infty(\Omega, B_1^N)$ such that $\|u\|_\infty < 1$. Such restriction is due to the following technical difficulties: a key point in the liminf inequality is Lemma 2, that can be applied only if u satisfies $\|u\|_\infty < 1$ (see Remark 2). For general functions we can still prove the liminf inequality without Lemma 2, if in 2.6, instead of G_{hom}, we consider its lower-semicontinuous envelope, but still we cannot reach the limsup inequality. In fact in limsup inequality we apply a density argument, which is valid only if the limit functional E_0 in (2.6) is continuous. The continuity of E_0 is ensured once we have an estimate from above and the continuity of G_{hom}. However the function $G_{hom}(z, M)$, as already observed in Remark 3, is continuous only if z is in the interior of the unit ball.

In the proof of Theorem 14 we will make use of Lemma 2 to satisfy the average constraint. The proof can be performed also without such Lemma (see Remark 4), but we want to show how the Lemma works in changing the average condition. Moreover, since we want to be sure that changing the value of u_j in some points will not affect too much the energy (2.5), which also depends on w_j, we need to give an estimate on how much we pay for that operation, as in the following remark.

Remark 5. Let $K > 0$, and $\varepsilon_j \to 0$ be a vanishing sequence of parameters. Let $E_j = E_{\varepsilon_j}$ be as in (2.5). We show that if the energies E_j are bounded, then we can exhibit a bound also for the points $\alpha \in R^{\xi}_{\varepsilon_j}(\Omega)$ where the absolute value of the "discrete" gradient $|D^{\xi}_{\varepsilon_j} w(\alpha)|$ is greater than K. Let $\mathcal{B}_K$ be the following set:

$$\mathcal{B}_K := \left\{\alpha \in R^{\xi}_{\varepsilon_j}(\Omega) : \exists\, \xi \in \mathbb{Z}^m, |\xi| \le R \,:\, |D^{\xi}_{\varepsilon_j} w(\alpha)| \ge K\right\}$$

and let $u \in \mathcal{A}_{\varepsilon_j}(\Omega; \mathcal{S}^{N-1})$, $w \in \mathcal{A}_{\varepsilon_j}(\Omega; \mathbb{R}^n)$ be such that $\sup_j E_j(u, v; \Omega) \le \tilde{C} < +\infty$. Recalling (2.1) we have that

$$\begin{aligned} &\tilde{C} \ge E_j(u, v; \Omega) \ge \varepsilon_j^m C_1 (K^p - 1) \#(\mathcal{B}_K) \Rightarrow \\ &\#(\mathcal{B}_K) \le \frac{\tilde{C}}{C_1 \varepsilon_j^m (K^p - 1)} \le \frac{\bar{C}}{K^p} \#(\mathbb{Z}_{\varepsilon_j}(\Omega)) \end{aligned} \tag{2.33}$$

for some $\bar{C} > 0$ big enough.

Consider Lemma 2 with $A = \Omega$ and $\delta_j = \varepsilon_j$. Recalling (2.17), (2.18) and (2.21), we require K to be such that

$$\frac{\bar{C}}{K^p} \#(\mathbb{Z}_{\varepsilon_j}(\Omega)) < B\#(\mathbb{Z}_{\varepsilon_j}(\Omega)) - 2\left(\left\lfloor \frac{\eta_j}{c} \#(\mathbb{Z}_{\varepsilon_j}(\Omega))\right\rfloor + 1\right). \tag{2.34}$$

Denoting with $\mathcal{B}^c_K = \mathbb{Z}_{\varepsilon_j}(\Omega) \setminus \mathcal{B}_K$, we can then modify u_j in a function $\tilde{u}_j$ such that Lemma 2 hold and $\{\alpha : \tilde{u}_j(\alpha) \ne u_j(\alpha)\} \subset \mathcal{B}^c_K$. This means that we can change the values of u_j where $|D^{\xi}_{\varepsilon_j} w(\alpha)| < K$.

Proof. (Γ*-liminf inequality*) . Let $\varepsilon_j \to 0$ be a vanishing sequence of parameters, and let $\{u_j\}$ and $\{w_j\}$ as in Proposition 6, then $u_j \rightharpoonup^* u$ with $u \in L^\infty(\Omega, B_1^N)$ and $w_j \to w$ in $L^p(\Omega, \mathbb{R}^n)$ with $w \in W^{1,p}(\Omega, \mathbb{R}^n)$. Now, let $u \in L^\infty(\Omega, B_1^N)$ be such that $\|u\|_\infty < 1$, and let $u_j \rightharpoonup^* u$. For all $A \in \mathcal{B}(\mathbb{R}^n)$ we can define the following measures:

$$\mu_j(A) = \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\alpha \in R^{\xi}_{\varepsilon_j}(\Omega)} \varepsilon_j^m G^{\xi}\left(\frac{\alpha}{\varepsilon_j}, u_j(\alpha), u_j(\alpha + \varepsilon_j \xi), D^{\xi}_{\varepsilon_j} w_j(\alpha)\right) \delta_{\alpha + \frac{\varepsilon_j}{2}\xi}(A)$$

where $\delta(\cdot)$ denotes the usual Dirac delta. Since the measures are equibounded, by the weak* compactness of measures there exists a limit measure μ on Ω such that, up to subsequences, $\mu_j \rightharpoonup^* \mu$.

Let consider the limit measure μ and its Radon-Nikodym decomposition with respect to the m-dimensional Lebesgue measure $\mathcal{L}^m$:

$$\mu = \frac{d\mu}{dx} d\mathcal{L}^m + \mu^s, \qquad \mu^s \perp \mathcal{L}^m,$$

the Besicovitch Derivation Theorem [4] states that almost every point in Ω with respect to $\mathcal{L}^m$ is a Lebesgue point for μ. So, let $x_0 \in \mathbb{Z}_{\varepsilon_j}(\Omega)$ be a Lebesgue point for μ and let $Q_\rho(x_0) = x_0 + (-\rho/2, \rho/2)^m$, it holds

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho\to 0^+} \frac{\mu(Q_\rho(x_0))}{\mathcal{L}^m(Q_\rho(x_0))} = \lim_{\rho\to 0^+} \frac{1}{\rho^m}\mu(Q_\rho(x_0)). \tag{2.35}$$

Recalling that

$$\mu(Q_\rho(x_0)) = \lim_j \mu_j(Q_\rho(x_0)), \tag{2.36}$$

by a diagonalization argument on (2.35) and (2.36) we can extract a subsequence $\{\rho_j\}$ such that it holds

$$\frac{d\mu}{dx}(x_0) = \lim_{j\to+\infty} \frac{1}{\rho_j^m}\mu_j(Q_{\rho_j}(x_0)).$$

This means that

$$\frac{d\mu}{dx}(x_0) = \tag{2.37}$$
$$\lim_{j\to+\infty}\left(\frac{\varepsilon_j}{\rho_j}\right)^m \sum_{\substack{\xi\in\mathbb{Z}^m\\ |\xi|\le R}} \sum_{\alpha\in R^\xi_{\varepsilon_j}(\Omega)} G^\xi\left(\frac{\alpha}{\varepsilon_j}, u_j(\alpha), u_j(\alpha+\varepsilon_j\xi), D^\xi_{\varepsilon_j} w_j(\alpha)\right)\delta_{\alpha+\frac{\varepsilon_j}{2}\xi}(Q_{\rho_j}(x_0)).$$

Now we want to modify $\{w_j\}$ in order to obtain a new sequence $\{\tilde{w}_j\}$ such that:

i) up to subsequences $\tilde{w}_j \to (\nabla w(x_0), x)$ in $L^p(Q_1(0), \mathbb{R}^n)$;

ii) $\tilde{w}_j \in \mathcal{A}^{R,\varepsilon_j}_{\nabla w(x_0)}(Q_1(0))$.

Firstly we define the scaled sequence

$$w_j^{\rho_j}(\gamma) = \frac{w_j(x_0+\rho_j\gamma) - w(x_0)}{\rho_j} \qquad \gamma\in\mathbb{Z}_{\varepsilon_j/\rho_j}(\Omega)$$

and we consider its piecewise constant approximation $w_j^{\rho_j} \in \mathcal{A}_{\varepsilon_j/\rho_j}(\Omega;\mathbb{R}^n)$. Simple calculations show that $D^\xi_{\varepsilon_j/\rho_j} w_j^{\rho_j} = D^\xi_\varepsilon w_j(\alpha+\rho_j\gamma)$.

So (2.37) now reads

$$\frac{d\mu}{dx}(x_0) = \lim_{j\to+\infty}\left(\frac{\varepsilon_j}{\rho_j}\right)^m \delta_{\alpha+\frac{\varepsilon_j}{2}\xi}(Q_{\rho_j}(x_0))\cdot$$
$$\cdot\sum_{\substack{\xi\in\mathbb{Z}^m\\ |\xi|\le R}} \sum_{\alpha\in R^\xi_{\varepsilon_j}(\Omega)} G^\xi\left(\frac{\alpha}{\varepsilon_j}, u_j(\alpha), u_j(\alpha+\varepsilon_j\xi), D^\xi_{\varepsilon_j/\rho_j} w_j^{\rho_j}\left(\frac{\alpha-x_0}{\rho_j}\right)\right).$$

Calling $\gamma = \frac{\alpha-x_0}{\rho_j}$ and making a substitution we have:

$$\frac{d\mu}{dx}(x_0) = \lim_{j\to+\infty}\left(\frac{\varepsilon_j}{\rho_j}\right)^m \delta_{\gamma+\frac{\varepsilon_j}{2\rho_j}\xi}(Q_1(0))\cdot \tag{2.38}$$
$$\cdot\sum_{\substack{\xi\in\mathbb{Z}^m\\ |\xi|\le R}} \sum_{\gamma\in R^\xi_{\frac{\varepsilon_j}{\rho_j}}(\Omega)} G^\xi\left(\frac{x_0+\rho_j\gamma}{\varepsilon_j}, u_j(x_0+\rho_j\gamma), u_j(x_0+\rho_j\gamma+\varepsilon_j\xi), D^\xi_{\varepsilon_j/\rho_j} w_j^{\rho_j}(\gamma)\right).$$

Now we observe that $w_j^{\rho_j} \to (\nabla w(x_0), x)$ in $L^p(Q_1(0), \mathbb{R}^n)$, so we apply Lemma 3 to $\{w_j^{\rho_j}\}$ with $A = Q_1(0)$, i.e., there exists a sequence $\{\tilde{w}_j\}$ such that both i) and ii) are satisfied. Moreover by Lemma 3, the sequence $\{\tilde{w}_j\}$ is such that

$$\left(\frac{\varepsilon_j}{\rho_j}\right)^m \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\gamma \in R^{\xi}_{\frac{\varepsilon_j}{\rho_j}}(Q_1)} G^{\xi}\left(\frac{x_0 + \rho_j \gamma}{\varepsilon_j}, u_j(x_0 + \rho_j \gamma), u_j(x_0 + \rho_j \gamma + \varepsilon_j \xi), D^{\xi}_{\varepsilon_j/\rho_j} \tilde{w}_j(\gamma)\right) \le$$
$$\left(\frac{\varepsilon_j}{\rho_j}\right)^m \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\gamma \in R^{\xi}_{\frac{\varepsilon_j}{\rho_j}}(Q_1)} G^{\xi}\left(\frac{x_0 + \rho_j \gamma}{\varepsilon_j}, u_j(x_0 + \rho_j \gamma), u_j(x_0 + \rho_j \gamma + \varepsilon_j \xi), D^{\xi}_{\varepsilon_j/\rho_j} w_j^{\rho_j}(\gamma)\right) + o(1). \tag{2.39}$$

Let K be chosen to satisfy (2.34). Using Lemma 2 with $A = Q_1(0)$ and $\delta_j = \varepsilon_j/\rho_j$, we change u_j in a function $\tilde{u}_j$ still weakly* converging to u and satisfying $\langle \tilde{u}_j \rangle^{d, \varepsilon_j/\rho_j}_{Q_1(0)} \equiv \langle u \rangle_{Q_1(0)}$.

By (2.1), Lemma 2 and Remark 5, recalling (2.21), modifying u_j effects the total energy of

$$\left(\frac{\varepsilon_j}{\rho_j}\right)^m \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\gamma \in R^{\xi}_{\frac{\varepsilon_j}{\rho_j}}(Q_1)} G^{\xi}\left(\frac{x_0 + \rho_j \gamma}{\varepsilon_j}, u_j(x_0 + \rho_j \gamma), u_j(x_0 + \rho_j \gamma + \varepsilon_j \xi), D^{\xi}_{\varepsilon_j/\rho_j} \tilde{w}_j(\gamma)\right)$$
$$\le \left(\frac{\varepsilon_j}{\rho_j}\right)^m \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\gamma \in R^{\xi}_{\frac{\varepsilon_j}{\rho_j}}(Q_1)} G^{\xi}\left(\frac{x_0 + \rho_j \gamma}{\varepsilon_j}, \tilde{u}_j(x_0 + \rho_j \gamma), \tilde{u}_j(x_0 + \rho_j \gamma + \varepsilon_j \xi), D^{\xi}_{\varepsilon_j/\rho_j} \tilde{w}_j(\gamma)\right)$$
$$+ 2\left(\frac{\varepsilon_j}{\rho_j}\right)^m P_j C_2(K^2 + 1)$$
$$\le \left(\frac{\varepsilon_j}{\rho_j}\right)^m \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\gamma \in R^{\xi}_{\frac{\varepsilon_j}{\rho_j}}(Q_1)} G^{\xi}\left(\frac{x_0 + \rho_j \gamma}{\varepsilon_j}, \tilde{u}_j(x_0 + \rho_j \gamma), \tilde{u}_j(x_0 + \rho_j \gamma + \varepsilon_j \xi), D^{\xi}_{\varepsilon_j/\rho_j} \tilde{w}_j(\gamma)\right)$$
$$+ o(1).$$

By (2.38) and (2.39), the above relation implies also that

$$\frac{d\mu}{dx}(x_0) \ge \liminf_{j \to +\infty} \left(\frac{\varepsilon_j}{\rho_j}\right)^m \delta_{\gamma + \frac{\varepsilon_j}{2\rho_j}\xi}(Q_1(0)) \cdot \tag{2.40}$$
$$\cdot \sum_{\substack{\xi \in \mathbb{Z}^m \\ |\xi| \le R}} \sum_{\gamma \in R^{\xi}_{\frac{\varepsilon_j}{\rho_j}}(\Omega)} G^{\xi}\left(\frac{x_0 + \rho_j \gamma}{\varepsilon_j}, \tilde{u}_j(x_0 + \rho_j \gamma), \tilde{u}_j(x_0 + \rho_j \gamma + \varepsilon_j \xi), D^{\xi}_{\varepsilon_j/\rho_j} \tilde{w}_j(\gamma)\right).$$

Calling now

$$\beta = \frac{x_0 + \rho_j \gamma}{\varepsilon_j}, \quad v_j(\beta) = \tilde{u}_j(\varepsilon_j \beta), \quad \zeta_j(\beta) = \frac{\rho_j}{\varepsilon_j} w_j\left(\frac{\varepsilon_j \beta - x_0}{\rho_j}\right),$$

we have that

$$\frac{d\mu}{dx}(x_0) \geq \liminf_{j\to+\infty} \left(\frac{\varepsilon_j}{\rho_j}\right)^m \sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\leq R}} \sum_{\beta\in R_1^\xi(\Omega)} G^\xi\left(\beta, v_j(\beta), v_j(\beta+\xi), D_1^\xi \zeta_j(\beta)\right) \delta_{\beta+\frac{1}{2}\xi}\left(Q_{\frac{\rho_j}{\varepsilon_j}}\left(\frac{x_0}{\varepsilon_j}\right)\right). \tag{2.41}$$

Replacing ρ_j/ε_j with T and x_0/ε_j with x_T we get

$$\frac{d\mu}{dx}(x_0) \geq \liminf_{T\to+\infty} \frac{1}{T^m} \inf\Bigg\{ \sum_{\substack{\xi\in\mathbb{Z}^m\\|\xi|\leq R}} \sum_{\beta\in R_1^\xi(\Omega)} G^\xi\left(\beta, v(\beta), v(\beta+\xi), D_1^\xi \zeta(\beta)\right) \delta_{\beta+\frac{1}{2}\xi}\left(Q_T(x_T)\right) : \langle v\rangle^{d,1}_{Q_T(x_T)} = \langle u\rangle_{Q_T(x_T)},\, \zeta\in\mathcal{A}^R_{\nabla w(x_0)}(Q_T(x_T))\Bigg\}. \tag{2.42}$$

By (2.42) we have that for $\mathcal{L}^m$-almost every $x_0\in\Omega$ it holds

$$\frac{d\mu}{dx}(x_0) \geq G_{hom}(u(x_0), \nabla w(x_0)).$$

Integrating now on Ω we can conclude that

$$\mu(\Omega) \geq \int_\Omega G_{hom}(u(x), \nabla w(x)\, dx$$

and since μ_j is converging weakly in the sense of measures to μ, we get

$$\liminf_j E_j(u_j, w_j; \Omega) = \liminf_j \mu_j(\Omega) \geq \mu(\Omega) \geq \int_\Omega G_{hom}(u(x), \nabla w(x))\, dx = E_0(u, w). \tag{2.43}$$

(Γ-*limsup inequality*) (i) Let $\mathfrak{T}$ be a triangulation by m-simplices on Ω. We begin considering as target functions $u:\Omega\to B_1^N$, $\|u\|_\infty<1$, constant on each $\tau\in\mathfrak{T}$, and $w:\Omega\to\mathbb{R}^m$ affine on each $\tau\in\mathfrak{T}$: this means that on each element τ there exists a $M_\tau\in\mathcal{M}^{m\times n}$ and a $c_\tau\in\mathbb{R}^n$ such that $w(x)=(M_\tau,x)+c_\tau$.

For the sake of simplicity we will consider $c_\tau=0$, being the case $c_\tau\neq 0$ identical but with heavier notations. We will define the function u_j and w_j on a single simplex τ, being the construction identical for each $\tau\in\mathfrak{T}$. Recalling (2.2), let $k\in l\mathbb{N}$ and $g_k(u,M_\tau)$ be defined as in (2.28), then there exist two functions $u_k:\mathbb{Z}^m\cap Q_k\to\mathcal{S}^{N-1}$ and $\zeta_k:\mathbb{Z}^m\cap Q_k\to\mathbb{R}^m$ such that

$$\frac{1}{k^m}E_1(u_k,\zeta_k;Q_k) \leq g_k + \frac{1}{k}; \tag{2.44}$$

we can extend the functions u_k and ζ_k on all $\mathbb{R}^m$ in the following way:

$$v(\beta) = u_k(\beta - ki), \qquad \zeta(\beta) = \zeta_k(\beta - ki) + k(M_\tau, i) \quad \text{if } \beta \in ki + Q_k.$$

Let $\varepsilon_j > 0$ be a vanishing sequence of parameters, we define $v_j : \varepsilon_j\mathbb{Z}^m \to \mathcal{S}^{N-1}$ and $\zeta_j : \varepsilon_j\mathbb{Z}^m \to \mathbb{R}^n$ rescaling v and ζ defined above:

$$v_j(\alpha) := v\left(\frac{\alpha}{\varepsilon_j}\right), \qquad \zeta_j(\alpha) = \varepsilon_j\zeta\left(\frac{\alpha}{\varepsilon_j}\right).$$

Let $u_j^\tau : \varepsilon_j\mathbb{Z}^m \to \mathcal{S}^{N-1}$ and $w_j^\tau : \varepsilon_j\mathbb{Z}^m \to \mathbb{R}^n$ be as follows

$$u_j^\tau(\alpha) = \begin{cases} v_j(\alpha) & \text{if } \alpha + [0,\varepsilon_j)^m \in \tau \\ \bar{u} & \text{otherwise,} \end{cases} \tag{2.45}$$
$$w_j^\tau(\alpha) = \begin{cases} \zeta_j(\alpha) & \text{if } \alpha + [0,\varepsilon_j)^m \in \tau \\ (M_\tau, \alpha) & \text{otherwise,} \end{cases}$$

where $\bar{u} \in \mathcal{S}^{N-1}$ is an arbitrary unitary vector. It holds $u_j^\tau \to u$ and $w_j^\tau \to w$ as $j \to \infty$ on τ. We also observe that, considering w_j restricted on τ, thanks to Lemma 3, we can find another sequence, that with a little abuse we still call w_j^τ, such that $w_j^\tau \in \mathcal{A}_{M_\tau}^{R,\varepsilon_j}(\tau)$.

We can repeat such construction on each $\tau \in \mathfrak{T}$ and then define the functions $u_j : \mathbb{Z}_{\varepsilon_j}(\Omega) \to \mathcal{S}^{N-1}$ and $w_j : \mathbb{Z}_{\varepsilon_j}(\Omega) \to \mathbb{R}^n$ such that

$$u_j(\alpha) = u_j^\tau(\alpha), \qquad w_j(\alpha) = w_j^\tau(\alpha) \qquad \text{if} \quad \alpha \in \mathbb{Z}_{\varepsilon_j}(\tau).$$

By construction, we have that $(u_j, w_j) \to (u, w)$ in Ω for $j \to \infty$.
For each $\tau \in \mathfrak{T}$, let $C(\tau, R)$ be a constant depending on the hypersurface of the m-simplex τ and on the range of the interactions we are considering. Recalling (2.1), for some $\bar{C} > 0$ big enough it holds

$$E_j(u_j, w_j; \Omega)\Big|_\tau \le E_j(u_j, w_j; \tau) + \bar{C}\, C(\tau, R)\varepsilon_j^m |M_\tau|^p.$$

By the definition of u_j and w_j, again thanks to (2.1) and (2.2), we get

$$\begin{aligned} E_j(u_j, w_j; \Omega)\Big|_\tau &\le \sum_{\substack{\xi\in\mathbb{Z}^m \\ |\xi|\le R}} \sum_{\alpha\in R^\xi_{\varepsilon_j}(\tau)} \varepsilon_j^m G^\xi\left(\frac{\alpha}{\varepsilon_j}, v\left(\frac{\alpha}{\varepsilon_j}\right), v\left(\frac{\alpha+\varepsilon_j\xi}{\varepsilon_j}\right), D^\xi_{\varepsilon_j}\varepsilon_j\zeta\left(\frac{\alpha}{\varepsilon_j}\right)\right) \\ &\quad + \varepsilon_j^m \bar{C}|M_\tau|^p C(\tau, R) \\ &\le \frac{|\tau|}{k^m} \sum_{\substack{\xi\in\mathbb{Z}^m \\ |\xi|\le R}} \sum_{\beta\in R^\xi_1(Q_k)} G^\xi\left(\beta, u_k(\beta), u_k(\beta+\xi), D^\xi_1\zeta_k(\beta)\right) \\ &\quad + \bar{C}M_\tau|^p\left(\frac{|\tau|}{k^m}((k+R)^m - (k-R)^m) + \varepsilon_j^m C(\tau, R)\right). \end{aligned}$$

Using (2.44) we conclude with

$$\begin{aligned} &E_j(u_j, w_j; \Omega)\Big|_\tau \le \\ &|\tau|\left(g_k + \frac{1}{k}\right) + \bar{C}|M_\tau|^p\left(\frac{|\tau|}{k^m}((k+R)^m - (k-R)^m) + \varepsilon_j^m C(\tau, R)\right). \end{aligned} \tag{2.46}$$

Letting now $j \to \infty$ and then $k \to \infty$ we have that for any $\tau \in \mathfrak{T}$ it holds

$$\limsup_j E_j(u_j, w_j; \Omega)\Big|_\tau \le E_0(u, w)\Big|_\tau + o(1).$$

(ii) We can extend the previous proof to a generic piecewise constant function $u(x)$ and a piecewise affine function

$$w(x) = \sum_{k=1}^{d} \chi_{\Omega_k}((M_k, x) + c_k),$$

where $d \in \mathbb{N}$, $M_k \in \mathcal{M}^{m\times n}$, $c_k \in \mathbb{R}$ and $\cup_{k=1}^d \Omega_k = \Omega$, $\Omega_k \cap \Omega_j = \emptyset$ if $k \neq j$. In fact for each $w_k = (M_k, x) + c_k$ we can consider a triangulation on Ω_k and repeat the same construction of point (i).

(iii) Let now $u \in L^\infty(\Omega, B_1^N)$, $\|u\|_\infty < 1$, and $w \in W^{1,p}(\Omega, \mathbb{R}^n)$, then we can find a sequence of piecewise constant function $u_k \rightharpoonup^* u$ in $L^\infty(\Omega, B_1^N)$ and a piecewise affine function $w_k \to w$ strongly in $W^{1,p}(\Omega, \mathbb{R}^n)$. We observe that by Proposition 8 and Remark 3 G_{hom} is a continuous function which satisfies a polynomial growth condition of order p, so that the functional E_0 defined in (2.6) is continuous. Then by the lower semicontinuity of $\Gamma - \limsup_j E_j$ we have :

$$\Gamma - \limsup_{j\to\infty} E_j(u, w; \Omega) \le \liminf_{k\to\infty}\left(\Gamma - \limsup_{j\to\infty} E_j(u_k, w_k; \Omega)\right) \le \liminf_{k\to\infty} E_0(u_k, w_k) = E_0(u, w).$$

□

2.6 Gay-Berne energies

In the previous sections we considered a general class of discrete energies whose potential is a function of both orientation and position, in view of a possible application to energies with Gay-Berne type potential. We recall (see the Introduction for a brief overview on the topic or [10, 14, 11]) that given two particles i and j with orientations u_i and u_j and intermolecular vector $\mathbf{r}_{ij}$, the Gay-Berne potential is expressed by:

$$U(u_i, u_j, \mathbf{r}_{ij}) := 4\eta(u_i, u_j, \hat{\mathbf{r}}_{ij})\left\{\left[\frac{\sigma_s}{r_{ij} - \sigma(u_i, u_j, \hat{\mathbf{r}}_{ij}) + \sigma_s}\right]^{12} - \left[\frac{\sigma_s}{r_{ij} - \sigma(u_i, u_j, \hat{\mathbf{r}}_{ij}) + \sigma_s}\right]^{6}\right\}. \tag{2.47}$$

We can recognize that the potential U has a structure similar to the well known atomistic Lennard-Jones potential

$$J(r) = 4a\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right] \tag{2.48}$$

where r denotes the distance between the particles, a is the depth of the potential well and σ is the distance at which the interparticle potential is zero [67, 29]. Moreover the Lennard-Jones potential satisfies the property that:

- there exists a r_0 such that $J(r)$ is convex on $(0, r_0)$ and concave on $(r_0, +\infty)$.

In the following we will assume that each interaction does not exceed the convexity threshold in (2.47) given by the shifted Lennard-Jones part of the potential and that it is not too close to 0. Since the Lennard-Jones potential is bounded and convex on each closed interval $[r_1, r_2] \subset (0, r_0)$, we can consider a function $F : (0, +\infty) \to \mathbb{R}$ such that $F(r) = J(r)$ on $[r_1, r_2]$ and such that

$$c_1(r^p - 1) \leq F(r) \leq c_2(r^p + 1) \qquad c_2 > c_1 > 0. \tag{2.49}$$

Now we can define the approximated potential $V : \mathcal{S}^2 \times \mathcal{S}^2 \times \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$V(u_i, u_j, \mathbf{r}_{ij}) := 4\eta(u_i, u_j, \hat{\mathbf{r}}_{ij})F(r - \sigma(u_i, u_j, \hat{\mathbf{r}}_{ij}) + \sigma_s). \tag{2.50}$$

Since both σ and η are positive bounded functions, we can assume that also V satisfies a p-growth estimate similar to (2.49).

2.6.1 Mathematical Model and Results

The natural setting to study the behaviour of Liquid Crystals is the three-dimensional space, where the position of a particles is identified by a vector in $\mathbb{R}^3$ and its orientation by a vector in $\mathcal{S}^2$. So we restrict to an open bounded domain $\Omega \subset \mathbb{R}^3$. On Ω we consider the cubic lattice $\varepsilon\mathbb{Z}^3 \cap \Omega$ where $\varepsilon > 0$ is the spacing parameter and $\mathbb{Z}_\varepsilon(\Omega)$ defined as in Section 2.2.1, where we assume $m = 3$. Let $u : \mathbb{Z}_\varepsilon(\Omega) \to \mathcal{S}^2$ and $w : \mathbb{Z}_\varepsilon(\Omega) \to \mathbb{R}^3$ be two functions on the lattice, then with $u(\alpha)$ and $w(\alpha)$ we denote respectively the orientation and the position of the α-th molecule, while the difference $w(\alpha) - w(\beta)$ represents the intermolecular vector between the α-th molecule and the β-th molecule.

The interaction between the α-th and the β-th molecule is then described by the function $V(u(\alpha), u(\beta), w(\beta) - w(\alpha))$ defined as in (2.50). In three dimension the potential V is the analog of the function G in arbitrary dimension: in fact we observe that the potential V satisfies the hypotheses (2.1) and (2.2), since there is no dependence on the spatial variable.

In analogy with the previous section, with a little abuse of notation, we denote with u and w also their piecewise constant extension. Moreover we make use of the sets $\mathcal{A}_\varepsilon(\Omega; \mathcal{S}^{N-1})$ and $\mathcal{A}_\varepsilon(\Omega; \mathbb{R}^n)$, assuming $m = N = n = 3$.

With such identification, the energy of the system is given by the functional $E_\varepsilon : L^\infty(\Omega, \mathcal{S}^2) \times L^p(\Omega, \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$E_\varepsilon(u, w; \Omega) = \tag{2.51}$$
$$\begin{cases} \displaystyle\sum_{\substack{\xi \in \mathbb{Z}^3 \\ |\xi| \leq \sqrt{2}}} \sum_{\alpha \in R_\varepsilon^\xi(\Omega)} \varepsilon^3 V\left(u(\alpha), u(\alpha + \varepsilon\xi), D_\varepsilon^\xi w(\alpha)\right) & \text{if } \begin{array}{l} w \in \mathcal{A}_\varepsilon(\Omega; \mathbb{R}^3), \\ u \in \mathcal{A}_\varepsilon(\Omega; \mathcal{S}^2) \end{array} \\ +\infty & \text{otherwise.} \end{cases}$$

As a consequence of Theorem 14, we are able to state the following Theorem.

Theorem 16. *Let $\varepsilon \to 0$ a sequence of vanishing positive parameters. Let E_ε be the energy defined in (2.51) and suppose that (2.1) and (2.2) hold for the* *potential V defined in (2.50). Then, for functions $u \in L^\infty(\Omega, B^1)$ such that $\|u\|_\infty < 1$ and $w \in W^{1,p}(\Omega, \mathbb{R}^3)$, E_ε Γ-converge to a functional*

$$E_0(u,w) = \int_\Omega V_{hom}(u, \nabla w)\,dx, \tag{2.52}$$

where the function V_{hom} is given by the following asymptotic formula

$$V_{hom}(z,M) = \lim_{T\to\infty} \frac{1}{T^3} \inf\Big\{ E_1(u,\zeta;Q_T) : \ \langle u\rangle^{d,1}_{Q_T} = z,\, \zeta \in \mathcal{A}^{\sqrt{2}}_M(Q_T) \Big\}. \tag{2.53}$$

In the one dimensional case we show an explicit characterization of the limit functional E_0, giving a proof independent from Theorem 14.

2.6.2 A Chain Model

In this section we consider a chain of N particles with nearest-neighbors interactions where all the particles centers are forced to stay on a line. Let $I = (0,1)$ be the unitary open interval, $\varepsilon > 0$ be a positive parameter and $N = N_\varepsilon = \lfloor 1/\varepsilon \rfloor$. On the one dimensional lattice $I \cap \varepsilon\mathbb{Z}$. we define the following functions:

$$u : I \cap \varepsilon\mathbb{Z} \to \mathcal{S}^1 \qquad \text{where we denote } u_i = u(\varepsilon i),$$
$$w : I \cap \varepsilon\mathbb{Z} \to \mathbb{R} \qquad \text{where we denote } w_i = w(\varepsilon i).$$

Each u_i describes the orientation of the i-th molecule on the unit sphere of $\mathbb{R}$, while w_i describes the position of the i-th molecule on the line.

With this notation, $\mathbf{r}_{il} \equiv e_1$ for every i and l. Moreover we assume, without loss of generality, that $w_i > w_l$ if $i > l$. We introduce the following "*effective*" functions:

$$\sigma_E(z) := \left\{ \sigma(u,v) : \frac{u+v}{2} = z \quad \text{and} \quad u,v \in \mathcal{S}^1_+ \right\},$$
$$\eta_E(z) := \left\{ \eta(u,v) : \frac{u+v}{2} = z \quad \text{and} \quad u,v \in \mathcal{S}^1_+ \right\},$$

where $\mathcal{S}^1_+$ is the upper half of the one-dimensional sphere in which we exclude the vector $-e_1$. We observe that both the σ_E and η_E are scalar functions defined on the set $B^1_+ = \{\frac{u+v}{2} : u,v \in \mathcal{S}^1_+\}$: in Figure 2.1 such vectors are in the upper half of the unitary ball, with the exclusion of those vectors inside the two half circumferences with radius 1/2.

With this notation we are able to introduce an "*effective* " potential as:

$$V^E(z_i, w_i - w_{i-1}) = 4\eta_E(z_i) F(w_i - w_{i-1} - (\sigma_E(z_i) - \sigma_s)), \tag{2.54}$$

where F is the same of (2.50). The energy of the system is given by

$$E_\varepsilon(z,w) = \sum_{i=1}^{N_\varepsilon} \varepsilon V^E\left(z_i, \frac{w_i - w_{i-1}}{\varepsilon}\right). \tag{2.55}$$

With a little abuse of notation we will identify the function z with its piecewise constant interpolation $z \in \mathcal{A}_\varepsilon(I; B^1_+)$, where we are assuming $m = 1$, and

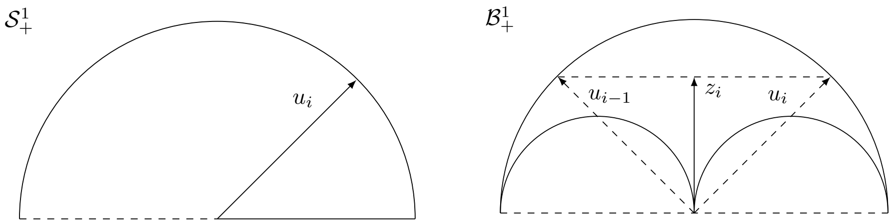


Figure 2.1: The domain S^1_+ of the directions u_i and the domain B^1_+ of the *effective* functions

the function w with its piecewise affine interpolation, so that we introduce the following set:

$$\mathcal{D}_\varepsilon(I) := \left\{ w : I \to \mathbb{R} \ : \ w(x) = w_{i-1} + \frac{w_i - w_{i-1}}{\varepsilon}(x - \varepsilon(i-1)) \right. \\ \left. \text{if} \quad x \in \varepsilon[i-1, i),\, i = 1, \dots, N_\varepsilon \right\}.$$

We can now rewrite (2.55) as:

$$E_\varepsilon(z, w) = \begin{cases} \displaystyle\sum_{i=1}^{N_\varepsilon} \varepsilon V^E \left(z_i, \frac{w_i - w_{i-1}}{\varepsilon} \right) & \text{if} \quad z \in \mathcal{A}_\varepsilon(I; B^1_+),\, w \in \mathcal{D}_\varepsilon(I) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.56}$$

We want to prove the following Γ-convergence result.

Theorem 17. *Let $\varepsilon \to 0$ be a vanishing sequence of parameters. Let E_ε be the energies defined in (2.56) and V^E be as in (2.54). Then, for functions $z \in L^\infty(I, B^+)$ such that $\|z\|_\infty < 1$ and $w \in H^1(I, \mathbb{R})$, E_ε Γ-converge to the functional*

$$E(z, w) = \int_0^1 \mathrm{Co}\Big(V^E(z, w')\Big)\, dx, \tag{2.57}$$

where $\mathrm{Co}(F(z, w'))$ is the convex envelope of a function F with respect to both the variables and B^+ is the ball where the antipodal vectors are identified.

Remark 6. We observe that the proof of Theorem 17 is independent from that of Theorem 14, but still it holds only for function z with $\|z\|_\infty < 1$. The reason is similar to that explained in Remark 3: a convex function is continuous, actually locally Lipschitz, only in the interior points of its domain. Since we use a density argument for the limsup inequality, we need to restrict ourself to the interior points of the unitary ball.

Proof. Let $\varepsilon_j \to 0$ be a sequence of vanishing parameters. In the following we will denote $z_j = z_{\varepsilon_j}$ and $w_j = w_{\varepsilon_j}$. Let $\{z_j\} \subset L^\infty(I, B^1_+)$ and $\{w_j\} \subset L^p(I, \mathbb{R})$ such that $\sup_j E_j(z_j, w_j) < C < +\infty$, then $z_n \rightharpoonup^* z$ in $L^\infty(I, B^+)$ and $w_j \to w$

in $L^p(I,\mathbb{R})$, with $w \in W^{1,p}$. Now the Γ-liminf inequality is straightforward from the lower semi-continuity of the function V^E and the definition of convex envelope:

$$\liminf_{j\to\infty} E_j(z_j,w_j) \geq \liminf_{j\to\infty} \int_0^1 \mathrm{Co}\Big(V^E(z_j,w_j')\Big)dx \geq \int_0^1 \mathrm{Co}\Big(V^E(z,w')\Big)dx. \quad (2.58)$$

For the Γ-limsup inequality: let $z \in L^\infty(I,B^+)$ and $w \in H^1(I,\mathbb{R})$ be fixed, we want to construct a sequence $(z_j,w_j) \to (z,w)$ such that $\limsup_j E_j(z_j,w_j) \geq E(z,w)$.
Let consider as target functions w piecewise affine and z piecewise constant. By Carathéodory Theorem we have that

$$\mathrm{Co}(V^E(z,w')) = \sum_{k=0}^{3} \lambda_k V^E(z_k,w_k') \quad \text{such that}$$
$$\sum_{k=0}^{3}\lambda_k = 1, \quad \sum_{k=0}^{3}\lambda_k(z_k,w_k') = (z,w'),\ z_k \in B^1_+,\ w_k' \in \mathbb{R}.$$

Let $T > 0$ be fixed and such that for $x \in [0,T]$ it holds $w(x) = Mx$ with $M > 0$ and $z(x) = \bar{z}$ with $\bar{z} \in B^+$. Let $\eta_j >> \varepsilon_j$ be such that $T >> \eta_j$ and $\lim_{j\to\infty} \varepsilon_j/\eta_j = 0$.

We consider the interval $[0,\eta_j]$ and we divide it in four subintervals, each one of amplitude $\lambda_k\eta_j$ (we denote each of these intervals with $I(\lambda_k\eta_j)$), where the λ_k-s are those of the Carathéodory Theorem. We observe that, since $z_k \in B^1_+$, there exist $u_k^1, u_k^2 \in \mathcal{S}^1_+$ such that $z_k = \frac{u_k^1+u_k^2}{2}$. So on $[0,\eta_j]$ we define the following function

$$u_j^i = \begin{cases} u_k^1 & \text{if} \quad i \text{ even and } i \in I(\lambda_k\eta_j) \\ u_k^2 & \text{if} \quad i \text{ odd and } i \in I(\lambda_k\eta_j) \end{cases}$$

and then we extend periodically in all $[0,T]$.
We can now consider the discrete function $z_j : [0,T]\cap\varepsilon\mathbb{Z} \to B^1_+$ defined as

$$z_j^i = \frac{u_j^i + u_j^{i-1}}{2}$$

and its piecewise-constant extension on $[0,T]$.

Let $w_j : [0,T]\cap\varepsilon\mathbb{Z} \to \mathbb{R}$ be the discrete function such that $w_j^i = M\,\varepsilon_j i$ and let $w_j(x) \in \mathcal{D}_{\varepsilon_j}(I)$ be its piecewise-affine interpolation. Let $\varepsilon_j \to 0$, then simple computations prove that $(z_j,w_j) \to (z,w)$. Moreover it holds

$$\begin{aligned}\limsup_{j\to+\infty} E_j(z_j,w_j)\Big|_{[0,T]} &= \limsup_{j\to+\infty} \int_0^T V^E(z_j,w_j')dx \\ &= \limsup_{j\to+\infty} \lfloor T/\eta_j\rfloor \int_0^{\eta_j} V^E(z_j,w_j')dx + \int_{\eta_j\lfloor T/\eta_j\rfloor}^T V^E(z_j,w_j')dx \\ &= \limsup_{j\to+\infty} \lfloor T/\eta_j\rfloor \int_0^{\eta_j} V^E(z_j,w_j')dx + O(\eta_j). \end{aligned} \quad (2.59)$$

Now we focus on

$$\limsup_{j\to+\infty}\lfloor T/\eta_j\rfloor\int_0^{\eta_j} V^E(z_j,w_j')dx=\limsup_{j\to+\infty}\lfloor T/\eta_j\rfloor\Big(\sum_{k=0}^{3}\int_{I(\lambda_k\eta_j)}V^E(z_j,w_j')dx\Big)$$
$$=\limsup_{j\to+\infty}\eta_j\lfloor T/\eta_j\rfloor\Big(\sum_{k=0}^{3}\lambda_k V^E(z_k,M)\Big)=\limsup_{j\to+\infty}\eta_j\lfloor T/\eta_j\rfloor \mathrm{Co}(V^E(z,w'))\Big|_{[0,T]}.$$

Inserting now the above result in (2.59), we have :

$$\limsup_{j\to+\infty}E_j(z_j,w_j)\Big|_{[0,T]}=\limsup_{j\to+\infty}\eta_j\lfloor T/\eta_j\rfloor \mathrm{Co}(V^E(z,w'))\Big|_{[0,T]}+O(\eta_j)$$
$$=T\,\mathrm{Co}(V^E(z,w'))\Big|_{[0,T]}=\int_0^T \mathrm{Co}(V^E(z,w'))dx.$$

The same construction can be repeated on each interval in which z is constant and w is piecewise affine.
The general result is obtained by density for every $w\in H^1(I,\mathbb{R})$ and $z\in L^\infty(I,B^+)$. $\square$

Chapter 3

Chirality transitions in frustrated ferromagnetic spin chains

The content of this chapter is based on a joint work with Giovanni Scilla [63].

3.1 Introduction

The phenomenon of frustration arises from the competition between different interactions, in a continuous or discrete physical system, that favor incompatible ground states. It occurs, for instance, in the liquid-crystalline phases of *chiral* molecules: a chiral molecule cannot be superimposed on its mirror image through any proper rotation or translation. The main effect of chirality is that chiral molecules do not align themselves parallel to their neghbors but tend to form a characteristic angle with them (see, e.g., [6, 54, 48]).

Edge-sharing chains of cuprates, instead, provide an example of frustrated lattice systems, where the frustration results from the competition between ferromagnetic (F) nearest-neighbour (NN) and antiferromagnetic (AF) next-nearest-neighbour (NNN) interactions (see, e.g., [49]).

In this chapter we study the asymptotic properties of a one-dimensional frustrated spin system at zero temperature via Γ-convergence (see [20] and [45]), focusing also on the variational equivalence with problems in gradient theory of phase transitions (see, e.g., [20, 19] for a simple introduction to the topic). Our contribution has been inspired by the recent results about the variational discrete-to-continuum analysis of such systems provided by Cicalese and Solombrino (2015)[40] in the vicinity of the so called "helimagnet/ferromagnet transition point", exhibiting at a suitable scale different scenarios not detected by a first-order Γ-limit. Indeed, the Γ-convergence approach provides a rigorous way of deriving a continuum limit for discrete systems as the number of interacting particles is increasing. However, the Γ-limit does not always capture the main features of the discrete model and in some cases more refined approximations are needed (see, e.g., [23, 40, 64, 21]). This motivated the derivation of the *uniformly* Γ*-equivalent theories*, introduced by Braides and Truskinovsky

(2008)[35] for a wide class of discrete systems and developed, e.g., in the framework of fracture mechanics, by Scardia, Schlömerkemper and Zanini (2011)[62] for one-dimensional chains of atoms with Lennard-Jones interactions between nearest-neighbours. The following results can also be seen as a first step in the analysis of chirality transitions in more complicated physical systems like as chiral liquid crystals. A discrete-to-continuum analysis via Γ-convergence of some problems in liquid crystals has been recently treated, e.g., by Braides, Cicalese and Solombrino (2015)[24], but this promising research field is still largely unexplored.

We consider the so-called F-AF spin chain model, where the state of the system is described by an S^1-valued spin variable $u = (u^i)$ parameterized over the points of the set $\frac{1}{n}\mathbb{Z}\cap[0,1]$, $n \in \mathbb{N}$. The energy of a given state of the system is

$$E_n^\alpha(u) = -\alpha \sum_{i=0}^{n-1}(u^i, u^{i+1}) + \sum_{i=0}^{n-1}(u^i, u^{i+2}) - nm_\alpha, \tag{3.1}$$

with *periodic* boundary conditions $(u^0, u^1) = (u^n, u^{n+1})$, where $\alpha \geq 0$ is the *frustration parameter*, $(\cdot,\cdot)$ denotes the scalar product between vectors in $\mathbb{R}^2$and m_α are constants depending on α (see (3.10) for the precise definition).

The first term of the energy (3.1) is ferromagnetic and favors the alignment of NN spins, while the second, being antiferromagnetic, frustrates it as it favors antipodal NNN spins. Consequently, the frustration of the system depends on the parameter α. In order to characterize the ground states of this system and their dependence on the value of α, we first associate to each pair of nearest neighbours u^i, u^{i+1} the corresponding oriented central angle $\theta^i \in [-\pi, \pi)$. Then, by the periodicity assumption, we may reread the energies in terms of this scalar variable as

$$E_n^\alpha(\theta) = -\frac{\alpha}{2} \sum_{i=0}^{n-1}\Big(\cos\theta^i + \cos\theta^{i+1}\Big) + \sum_{i=0}^{n-1}\cos(\theta^i + \theta^{i+1}) - nm_\alpha, \tag{3.2}$$

and follow the approach by Braides and Cicalese (2007)[23] for lattice systems of the form (3.2). Indeed, by "minimizing out" for each fixed i the nearest neighbours interactions, we are led to the definition of the *effective potential* W_α (equation (3.14)) such that

$$E_n^\alpha(\theta) \geq \sum_{i=0}^{n-1} W_\alpha(\theta^i), \tag{3.3}$$

where W_α is convex with minimum at $\theta = \theta_\alpha = 0$ if $\alpha \geq 4$, while it is a double-well potential with wells at $\theta = \pm\theta_\alpha$ if $0 \leq \alpha \leq 4$ (see Fig. 3.2). Since the inequality in (3.3) is strict if $\theta^i \neq \theta^{i+1}$ or $\theta^i \neq \pm\theta_\alpha$, we deduce that if $\alpha \geq 4$ the nearest neighbours prefer to stay aligned (ferromagnetic order); if $0 \leq \alpha \leq 4$, instead, the minimal configurations of E_n^α are $\theta^i = \theta^{i+1} \in \{\pm\theta_\alpha\}$; that is, the angle between pairs of nearest neighbours u^i, u^{i+1} and u^{i+1}, u^{i+2} is constant and depending on the particular value of α (helimagnetic order). The two possible choices for θ_α (a degeneracy known in literature as *chirality symmetry*) correspond to either clockwise or counterclockwise spin rotations, or, equivalently, to a positive or a negative chirality (see Fig. 3.1).

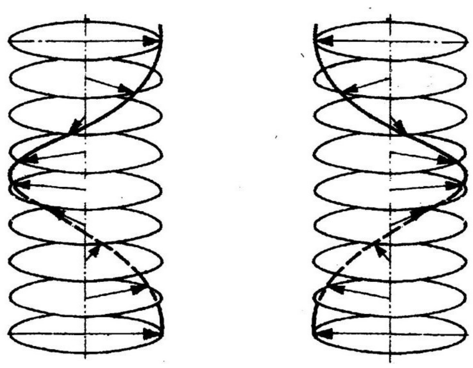


Figure 3.1: A schematic representation of the ground states of the spin system for $0 \leq \alpha < 4$ for clockwise (on the left) and counterclockwise (on the right) chirality (picture taken from [48]).

The asymptotic behaviour of energies E_n^α as $n \to \infty$ and for fixed α (Theorem 18) reflects such different regimes for the ground states. If $\alpha \geq 4$ the limit is trivially finite (and equal to zero) only on the constant function $\theta \equiv 0$, while if $0 \leq \alpha < 4$ it is finite on functions with bounded variation taking only the two values $\{\pm\theta_\alpha\}$ and it counts the number of chirality transitions. More precisely,

$$\Gamma - \lim_{n\to+\infty} E_n^\alpha(\theta) = C_\alpha \#(S(\theta)),$$

where $S(\theta)$ is the jump set of function θ and $C_\alpha = C(\alpha)$ is the cost of each chirality transition. The value C_α (see Section 3.3.1) represents the energy of an interface which is obtained by means of a 'discrete optimal-profile problem' connecting the two constant (minimal) states $\pm\theta_\alpha$. It is continuous as a function of α on the interval $[0, 4)$ (as shown by Proposition 10) and can be defined to be equal to 0 for $\alpha \geq 4$. Moreover, $C_\alpha \to 0$ as $\alpha \to 4$ and (compare with [50] and Remark 8)

$$C_\alpha \sim \frac{\sqrt{2}}{3}(4-\alpha)^{3/2}, \text{ as } \alpha \to 4^-. \tag{3.4}$$

In a recent paper [40], Cicalese and Solombrino investigated the asymptotic behaviour of this system close to the ferromagnet/helimagnet transition point; that is, they found the correct scaling (heuristically suggested by (3.4)) to detect the symmetry breaking and to compute the asymptotic behaviour of the scaled energy describing this phenomenon as α is close to 4. They let the parameter α depend on n and be close to 4 from below; i.e., they rewrite energies (3.2) in terms of $4 - \alpha_n$, with $4 - \alpha_n \to 0$ as $n \to \infty$.

We state their result in a slight different form, useful for the sequel. More precisely, we prove in Theorem 19 that an analogous result can be obtained if we choose as order parameter the "flat" angular variable

$$v = \frac{\theta}{\theta_\alpha},$$

which is equivalent to the variable considered in [40] in the regime of small angles. We compute the Γ-limit F^0 as $n \to \infty$, $\alpha = \alpha_n \to 4$ with respect to the strong L^1-topology of the scaled energies

$$F_n^{\alpha_n}(v) := \frac{E_n^{\alpha_n}(v)}{\mu_{\alpha_n}} = \frac{8E_n^{\alpha_n}(v)}{\sqrt{2}(4-\alpha_n)^{3/2}}, \tag{3.5}$$

and show that, within this scaling, several regimes are possible depending on the value

$$l := \lim_n \frac{\sqrt{2}}{4n(4-\alpha_n)^{1/2}}.$$

Namely, if $l=0$ then $F^0(v) = \frac{8}{3}\#(S(v))$, $v \in BV(I,\{\pm 1\})$, if $l=+\infty$ then F^0 is finite (and equal to zero) only on constant functions, while in the intermediate case $l \in (0,+\infty)$ we get

$$F^0(v) = \frac{1}{l}\int_I \Big(v^2(t)-1\Big)^2\,dt + l\int_I (\dot{v}(t))^2\,dt,\ v \in W^{1,2}_{|per|}(I),$$

where $I=(0,1)$, $BV(I,\{\pm 1\})$ is the space of functions of bounded variation defined on I and taking the values $\{\pm 1\}$, and $W^{1,2}_{|per|}(I) = \{v \in W^{1,2}(I) : |v(0)| = |v(1)|\}$.

Motivated by the particular form of this result and in the spirit of Braides and Truskinovsky (2008)[35], with Theorem 21 we find a variational link between such energies (seen as a 'parametrized' family of functionals) and the gradient theory of phase transitions, in the framework of the *equivalence* by Γ-convergence. Roughly speaking, two families of functionals are equivalent by Γ-convergence if they have the same Γ-limit (see Definition 17 and the subsequent ones for the rigorous definitions useful in this framework). More precisely, we show the *uniform* equivalence by Γ-convergence on $[0,4]$ of the energies $F_n^{\alpha}(v)$ defined in (3.5) with the "Modica-Mortola type" functionals given by

$$G_n^{\alpha}(v) = \frac{1}{\mu_\alpha}\Big[\lambda_{n,\alpha}\int_I \Big(v^2-1\Big)^2\,dt + \frac{M_\alpha^2}{\lambda_{n,\alpha}}\int_I (\dot{v})^2\,dt\Big],\ v \in W^{1,2}_{|per|}(I),$$

where $\lambda_{n,\alpha} = 2n\theta_\alpha^4$ and $M_\alpha = 3C_\alpha/8$.

The value $\alpha_0 = 4$ is a *singular point*, since the Γ-limit of G_n^α will depend on choice of the particular sequence $\alpha_n \to \alpha_0^- = 4^-$. Each $\alpha_0 \in [0,4)$, instead, is a *regular point*; i.e., it is not singular. As a consequence of Theorem 21, we deduce (see Corollary 1) the uniform equivalence of the energies $E_n^\alpha(\theta)$ for $\alpha \in [0,4)$ with the family

$$H_n^{\alpha}(\theta) = \frac{\lambda_{n,\alpha}}{\theta_\alpha^4}\int_I \Big(\theta^2(t)-\theta_\alpha^2\Big)^2\,dt + \frac{M_\alpha^2}{\lambda_{n,\alpha}\theta_\alpha^2}\int_I (\dot{\theta}(t))^2\,dt,\ \theta \in W^{1,2}_{|per|}(I),$$

whose potentials $\mathcal{W}_\alpha(\theta) := (\theta^2-\theta_\alpha^2)^2$ have the wells located at the minimal angles $\theta = \pm\theta_\alpha$.

As a final remark, we would like to observe that our result can be useful also to analyze more general problems of interest for the applied community. For instance, a natural extension would be the case of S^2-valued spins, that has been recently investigated by Cicalese, Ruf and Solombrino (2016)[39] in the vicinity of the transition point. In that paper, the authors modify the energies penalizing the distance of the S^2 field from a finite number of copies of S^1 and prove the emergence of non-trivial chirality transitions. However, even in the case of values in S^1, the Villain Helical XY-model studied there could be attacked with our approach, at least in the regime of "strong" ferromagnetic interaction considered therein by the authors.

3.2 Setting of the problem

Preliminarily, we fix some notation that will be used throughout. We denote by $I = (0,1)$ and by $\lambda_n = \frac{1}{n}, n \in \mathbb{N}$ a positive parameter. Given $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the integer part of x. The symbol S^1 stands for the standard unit sphere of $\mathbb{R}^2$. Given a vector $v \in \mathbb{R}^2$ with components v_1 and v_2 with respect to the canonical basis of $\mathbb{R}^2$, we will use the notation $v = (v_1|v_2)$. Given two vectors $v, w \in \mathbb{R}^2$ we will denote by (v, w) their scalar product. Here and in the following, $\mathcal{U}_n(I)$ will be the space of the functions $w : \lambda_n\mathbb{Z} \cap [0,1] \to S^1$, $\Theta_n(I)$ the space of the functions $\varphi : \lambda_n\mathbb{Z} \cap [0,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ and we use the notation $w^i = w(i\lambda_n)$, $\varphi^i = \varphi(i\lambda_n)$; $\bar{\mathcal{U}}_n(I)$ will denote the subspace of those $w \in \mathcal{U}_n(I)$ satisfying the following *periodic boundary condition*

$$(w^1, w^0) = (w^{n+1}, w^n). \tag{3.6}$$

Analogously, $\bar{\Theta}_n(I)$ will denote the subspace of those $\varphi \in \Theta_n(I)$ such that $\varphi^0 = \varphi^n$.

We will identify each lattice function $w \in \bar{\mathcal{U}}_n(I)$ with its piecewise-constant interpolation belonging to the class

$$\mathcal{C}_n(I) = \{w : \mathbb{R} \to S^1 : \ w(t) = w(\lambda_n i) \text{ if } t \in (i, i+1)\lambda_n, \ i \in \{0, 1, \ldots, n-1\}\},$$

while the symbol $\mathcal{D}_n(I)$ will denote the analogous space for functions $\varphi \in \bar{\Theta}_n(I)$.

Given a pair of vectors $v = (v_1|v_2), w = (w_1|w_2) \in S^1$, we define the function $\chi[v, w] : S^1 \times S^1 \to \{\pm 1\}$ as

$$\chi[v, w] = \operatorname{sign}(v_1 w_2 - v_2 w_1), \tag{3.7}$$

with the convention that $\operatorname{sign}(0) = -1$, and the corresponding oriented central angle $\theta \in [-\pi, \pi)$ by

$$\theta = \chi[v, w] \arccos((v, w)). \tag{3.8}$$

The positivity of the determinant in (3.7) represents the counterclockwise ordering of the vectors v and w.

3.2.1 The model energies E_n^α

We consider the energy of a given state u of the F-AF spin chain model, defined as

$$E_n^\alpha(u) = P_n^\alpha(u) - n m_\alpha = -\alpha \sum_{i=0}^{n-1} (u^i, u^{i+1}) + \sum_{i=0}^{n-1} (u^i, u^{i+2}) - n m_\alpha, \tag{3.9}$$

where $u \in \mathcal{C}_n(I)$, $\alpha \geq 0$ and (see [40, Proposition 3.2])

$$m_\alpha = \frac{1}{n} \min_{u \in L^\infty(I, S^1)} P_n^\alpha(u) = \begin{cases} -\left(\frac{\alpha^2}{8} + 1\right) & \text{if } \alpha \in [0, 4], \\ -\alpha + 1 & \text{if } \alpha \in [4, +\infty). \end{cases} \tag{3.10}$$

First we note that, thanks to the periodicity assumption (3.6), we can write the energies (3.9) equivalently in the form

$$E_n^\alpha(u) = -\frac{\alpha}{2} \sum_{i=0}^{n-1} \Big[(u^i, u^{i+1}) + (u^{i+1}, u^{i+2})\Big] + \sum_{i=0}^{n-1} (u^i, u^{i+2}) - n m_\alpha. \tag{3.11}$$

Now we associate to each pair of neighbouring spins u^i, u^{i+1} the corresponding oriented central angle θ^i defined as in (3.8), and taking θ^i as (scalar) order parameter, the energies (3.11) can be rewritten as

$$E_n^\alpha(\theta) = -\frac{\alpha}{2}\sum_{i=0}^{n-1}\Big(\cos\theta^i + \cos\theta^{i+1}\Big) + \sum_{i=0}^{n-1}\cos(\theta^i + \theta^{i+1}) - nm_\alpha, \qquad (3.12)$$

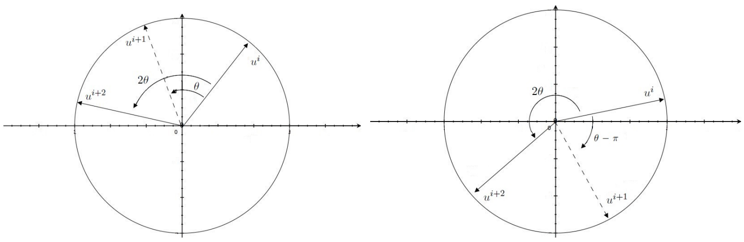
where $\theta \in \mathcal{D}_n(I)$.

3.2.2 Ground states of E_n^α

In this section, we focus on the ground states of the energies E_n^α. We will show the emergence of *chiral* ground states for $\alpha \in [0,4]$ by means of a double-minimization technique introduced by Braides and Cicalese (2007)[23] for lattice systems of the form (3.12). Following their approach, for each $i = 0, 1, \dots, n-1$ we fix the next-to-nearest neighbour interactions $\theta^i + \theta^{i+1} = 2\theta$ and solve the minimum problem

$$\min_{\theta^i \in [-\pi,\pi)}\Big\{-\frac{\alpha}{2}\Big[\cos\theta^i + \cos(2\theta - \theta^i)\Big] + \cos 2\theta - m_\alpha\Big\}. \qquad (3.13)$$

By a direct computation, we find that the unique minimizers in (3.13) for $\alpha \neq 0$ are $\theta^i = \theta^{i+1} = \theta$ if $\theta \in (-\pi/2, \pi/2)$, and $\theta^i = \theta^{i+1} = \theta - \pi$ if $|\theta| \in (\pi/2, \pi)$, while for $\alpha = 0$ we have $\theta^i = \theta^{i+1} = \pm\pi/2$. The following picture shows that, up to a reparametrization, θ and $\theta - \pi$ actually represent the same minimizer.



(a) The angle between NN is θ (b) The angle between NN is $\theta - \pi$

Without loosing generality we will assume up to the end that $\theta^i = \theta^{i+1} = \theta \in J$, $J := [-\pi/2, \pi/2]$ and correspondingly we define the *effective potential* as

$$W_\alpha(\theta) = \cos 2\theta - \alpha\cos\theta - m_\alpha. \qquad (3.14)$$

The potential W_α is thus obtained by integrating out the effect of nearest-neighbour interactions optimizing over atomic-scale oscillations, and its properties strongly depend on the value α. Indeed, if $0 \le \alpha < 4$ then W_α is a "double-well" potential, while if $\alpha \ge 4$ the potential is convex (see Fig. 3.2). Moreover,

$$\arg\min W_\alpha(\theta) = \begin{cases} \{\pm\theta_\alpha\} := \{\pm\arccos(\frac{\alpha}{4})\}, & \text{if } \alpha \in [0,4], \\ \{0\}, & \text{if } \alpha \in [4, +\infty), \end{cases} \qquad (3.15)$$

We note that by the definition of W_α and (3.13) we get

$$E_n^\alpha(\theta) \geq \sum_{i=0}^{n-1} W_\alpha(\theta^i), \tag{3.16}$$

the inequality being strict if $\theta^i \neq \theta^{i+1}$ or $\theta^i \neq \pm\theta_\alpha$. In particular, $E_n^\alpha(\theta) \geq 0$.

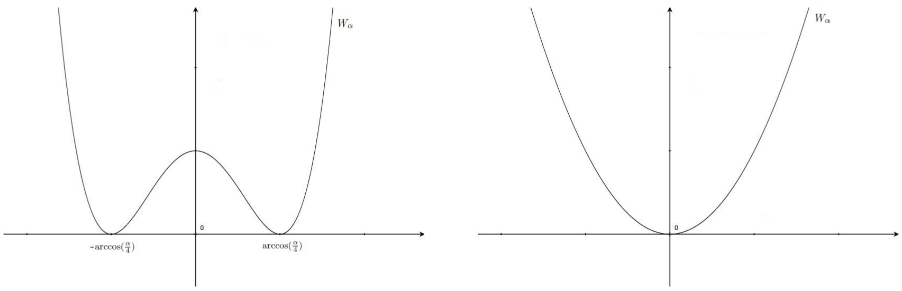


Figure 3.2: The potential W_α for $0 \leq \alpha < 4$ (on the left) and for $\alpha \geq 4$ (on the right).

Thus, the minimization procedure leading to the definition of W_α (and then to inequality (3.16)) allows us to deduce some information about the ground states of the energies E_n^α from the properties of this potential. More precisely, if $\alpha \leq 4$ the minimal configurations of E_n^α are $\theta^i = \theta^{i+1} \in \{\pm\theta_\alpha\}$; that is, the angle between pairs of nearest neighbours u^i, u^{i+1} and u^{i+1}, u^{i+2} is constant and depending on the particular value of α. If $\alpha \geq 4$, instead, the nearest neighbours prefer to stay aligned ($-\theta_\alpha = +\theta_\alpha = 0$).

Let be $\theta \in \mathcal{D}_n(I)$. We may regard the energies E_n^α as defined on a subset of $L^\infty(I, J)$ and consider their extension on $L^\infty(I, J)$. With a slight abuse of notation, we set $E_n^\alpha : L^\infty(I, J) \to [0, +\infty]$ as

$$E_n^\alpha(\theta) = \begin{cases} -\dfrac{\alpha}{2}\displaystyle\sum_{i=0}^{n-1}(\cos\theta^i + \cos\theta^{i+1}) + \sum_{i=0}^{n-1}\cos(\theta^i + \theta^{i+1}) - nm_\alpha, & \text{if } \theta \in \mathcal{D}_n(I), \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.17}$$

3.3 Limit behaviour of E_n^α with fixed α

Our first result is the explicit computation of the Γ-limit, as $n \to \infty$, of the energies E_n^α with fixed $\alpha \in [0, +\infty)$. As we will show with Theorem 18, the asymptotic behaviour of the energies E_n^α reflects the different regimes for the ground states outlined in Section 3.2.2. Indeed, the limit is non-trivial only in the helimagnetic regime ($0 \leq \alpha < 4$), representing the energy the system spends on the scale 1 for a finite number of chirality transitions from $-\theta_\alpha$ to θ_α.

3.3.1 Crease transition energies

The cost C_α of each chirality transition can be characterized as the energy of an interface which is obtained by means of a 'discrete optimal-profile problem' connecting the two constant (minimal) states $\pm\theta_\alpha$.

Let $\alpha \in [0, 4)$. According to [23, Section 2.2], we define the *crease transition energy* between $-\theta_\alpha$ and θ_α as

$$\begin{aligned} C_\alpha &:= C(-\theta_\alpha, \theta_\alpha) \\ &= \inf_{N\in\mathbb{N}} \min\Big\{ \sum_{i=-\infty}^{+\infty} \Big[\cos(\theta^i + \theta^{i+1}) - \frac{\alpha}{2}(\cos\theta^i + \cos\theta^{i+1}) - m_\alpha\Big] : \\ &\qquad\qquad \theta : \mathbb{Z} \to [-\pi/2, \pi/2],\, \theta^i = \operatorname{sign}(i)\theta_\alpha, \text{ if } |i| \geq N \Big\}. \end{aligned} \tag{3.18}$$

We note that the infinite sums in (3.18) are well defined, since they involve only non negative terms and, actually, for fixed N they are finite sums, since the summands are 0 for $i \geq N$ and $i \leq -N-1$. Moreover, it follows by the definition a useful symmetry property of the crease energy; that is,

$$C(-\theta_\alpha, \theta_\alpha) = C(\theta_\alpha, -\theta_\alpha). \tag{3.19}$$

Now we prove that the optimal test function in (3.18) is constantly equal to $\pm\theta_\alpha$ only for $N \to +\infty$, thus relaxing the boundary condition as a condition at infinity in the definition of C_α. We notice that an analogous property of crease energies has been showed by Braides and Solci (2016)[34] for a one-dimensional system of Lennard-Jones nearest and next-to-nearest neighbour interactions.

Proposition 9. *The infimum in (3.18) is obtained for $N \to \infty$; that is,*

$$\begin{aligned} C_\alpha = \inf\Big\{ \sum_{i=-\infty}^{+\infty} \Big[\cos(\theta^i + \theta^{i+1}) - \frac{\alpha}{2}(\cos\theta^i + \cos\theta^{i+1}) - m_\alpha\Big] : \\ \theta : \mathbb{Z} \to [-\pi/2, \pi/2],\, \lim_{i\to\pm\infty} \operatorname{sign}(i)\theta^i = \theta_\alpha \Big\}. \end{aligned} \tag{3.20}$$

Moreover, $C_\alpha > 0$.

Proof. Let θ^i be a test function for the problem (3.20) and denote by $\widetilde{C}_\alpha$ the infimum in (3.20). With fixed $\eta > 0$, let N_η be such that $|\theta^i - \operatorname{sign}(i)\theta_\alpha| < \eta$ for $|i| \geq N_\eta$, and define

$$\theta^i_\eta = \begin{cases} \theta^i, & \text{if } |i| \leq N_\eta \\ \operatorname{sign}(i)\theta_\alpha, & \text{if } |i| > N_\eta. \end{cases}$$

We then have

$$
\begin{aligned}
&\sum_{i=-\infty}^{+\infty}\Big[-\frac{\alpha}{2}(\cos\theta_\eta^i+\cos\theta_\eta^{i+1})+\cos(\theta_\eta^i+\theta_\eta^{i+1})-m_\alpha\Big]\\
&=\sum_{i=-N_\eta-1}^{N_\eta}\Big[-\frac{\alpha}{2}(\cos\theta_\eta^i+\cos\theta_\eta^{i+1})+\cos(\theta_\eta^i+\theta_\eta^{i+1})-m_\alpha\Big]\\
&=\sum_{i=-N_\eta}^{N_\eta-1}\Big[-\frac{\alpha}{2}(\cos\theta^i+\cos\theta^{i+1})+\cos(\theta^i+\theta^{i+1})-m_\alpha\Big]\\
&\quad-\frac{\alpha}{2}(\cos\theta^{N_\eta}+\cos\theta_\alpha)+\cos(\theta^{N_\eta}+\theta_\alpha)-m_\alpha\\
&\quad-\frac{\alpha}{2}(\cos\theta_\alpha+\cos\theta^{-N_\eta})+\cos(\theta^{-N_\eta}-\theta_\alpha)-m_\alpha\\
&\le\sum_{i=-\infty}^{+\infty}\Big[-\frac{\alpha}{2}(\cos\theta^i+\cos\theta^{i+1})+\cos(\theta^i+\theta^{i+1})-m_\alpha\Big]+2\omega(\eta)
\end{aligned}
$$

where

$$
\omega(\eta):=\max\Big\{-\frac{\alpha}{2}(\cos\theta+\cos\theta_\alpha)+\cos(\theta+\theta_\alpha)-m_\alpha:\,|\theta-\theta_\alpha|\le\eta\Big\} \qquad (3.21)
$$

is infinitesimal as $\eta\to 0$. This shows that the value C_α defined in (3.18) is less or equal than $\widetilde{C}_\alpha$. Then we are done, the converse inequality being trivial since any test function for problem (3.18) is a test function for problem (3.20). The estimate $C_\alpha>0$ easily follows from (3.16) and the fact that $\pm\theta_\alpha$ are the unique minimizers of W_α. □

Remark 7. In the ferromagnetic regime $\alpha\ge 4$, we may define $C_\alpha=0$ consistently with (3.18), where now $m_\alpha=-\alpha+1$. Indeed, being $\theta_\alpha=-\theta_\alpha=0$, we can choose $\theta\equiv 0$ as a test function in (3.18) thus obtaining the estimate $C_\alpha\le 0$.

It will be useful in the sequel the following continuity property of C_α with respect to the frustration parameter α.

Proposition 10 (Continuity)**.** *The crease energy C_α defined as before is continuous in $[0,4)$; i.e., for any $\bar\alpha\in[0,4)$ and any sequence α_j such that $0\le\alpha_j<4$, $\alpha_j\to\bar\alpha$ it results $C_{\alpha_j}\to C_{\bar\alpha}$.*

Proof. Let us fix $\eta>0$ and let $\alpha,\alpha'\in[0,4)$ be such that if $|\alpha-\alpha'|<\delta(\eta)$ for a suitable $\delta(\eta)>0$, then $|\theta_\alpha-\theta_{\alpha'}|<\eta/2$.
From the definition of $C_{\alpha'}$ as in (3.20), there exists a function $\theta:\mathbb{Z}\to[-\pi/2,\pi/2]$ such that $\sum_{i\in\mathbb{Z}}\mathcal{E}^{i,\alpha'}(\theta)<C_{\alpha'}+\eta$, where we have set

$$
\mathcal{E}^{i,\alpha'}(\theta):=-\frac{\alpha'}{2}\left(\cos\theta^i+\cos\theta^{i+1}\right)+\cos(\theta^i+\theta^{i+1})+\frac{(\alpha')^2}{8}+1,
$$

and $\lim_{i\to\pm\infty}\operatorname{sign}(i)\theta^i=\theta_{\alpha'}$. This means that there exist two indices $h_1(\eta),h_2(\eta)\in\mathbb{N}$ such that $|\theta^i-\theta_{\alpha'}|<\eta/2$ for every $i>h_2(\eta)$ and $|\theta^i-(-\theta_{\alpha'})|<\eta/2$ for every $i<-h_1(\eta)$.

Setting $\bar{h} = \bar{h}(\eta) := \max\{h_1, h_2\}$ and $K_{\bar{h}} := \{i \in \mathbb{Z} : |i| \leq \bar{h}\}$, we observe that for every $i \notin K_{\bar{h}}$ it also holds that $|\text{sign}(i)\theta^i - \theta_\alpha| < \eta$.

Now we modify θ in order to obtain a test function for the problem defining C_α by setting

$$\tilde{\theta}^i = \begin{cases} \theta^i, & \text{if } i \in K_{\bar{h}} \\ \text{sign}(i)\theta_\alpha, & \text{otherwise.} \end{cases} \tag{3.22}$$

We then have

$$\begin{aligned} C_\alpha &\leq \sum_{i\in\mathbb{Z}} \mathcal{E}^{i,\alpha}(\tilde{\theta}) = \sum_{|i|<\bar{h}} \mathcal{E}^{i,\alpha}(\theta) + \mathcal{E}^{-\bar{h},\alpha}(\tilde{\theta}) + \mathcal{E}^{\bar{h},\alpha}(\tilde{\theta}) \\ &\leq \sum_{i\in\mathbb{Z}} \mathcal{E}^{i,\alpha'}(\theta) + 2|\alpha-\alpha'|\#(K_{\bar{h}}) + \left[\mathcal{E}^{-\bar{h},\alpha}(\tilde{\theta}) - \mathcal{E}^{-\bar{h},\alpha'}(\theta)\right] \\ &+ \left[\mathcal{E}^{\bar{h},\alpha}(\tilde{\theta}) - \mathcal{E}^{\bar{h},\alpha'}(\theta)\right], \end{aligned} \tag{3.23}$$

where in the second inequality we used the estimate

$$\sum_{|i|<\bar{h}} \left|\mathcal{E}^{i,\alpha}(\theta) - \mathcal{E}^{i,\alpha'}(\theta)\right| \leq 2|\alpha-\alpha'|\#(K_{\bar{h}}).$$

Each of the last two terms in (3.23) can be estimated in the same way, so we make an explicit computation only for the latter. We have

$$\left|\mathcal{E}^{\bar{h},\alpha}(\tilde{\theta}) - \mathcal{E}^{\bar{h},\alpha'}(\theta)\right| \leq \left|(\cos(\theta^{\bar{h}} + \theta_\alpha) - \cos(\theta^{\bar{h}} + \theta^{\bar{h}+1})\right| + \left|\frac{\alpha^2 - (\alpha')^2}{8}\right| + \\ \left|\frac{\alpha}{2}(\cos\theta^{\bar{h}} + \cos\theta_\alpha) - \frac{\alpha'}{2}(\cos\theta^{\bar{h}} + \cos\theta^{\bar{h}+1})\right| \leq \eta + 2|\alpha-\alpha'|.$$

Collecting all the previous estimates and inserting them into (3.23) we obtain

$$\begin{aligned} C_\alpha &\leq C_{\alpha'} + \eta + 2|\alpha-\alpha'|\#(K_{\bar{h}}) + \eta + 2|\alpha-\alpha'| \\ &\leq C_{\alpha'} + 2\eta + 2|\alpha-\alpha'|(1 + \#(K_{\bar{h}})). \end{aligned}$$

Choosing now $\gamma \geq 4\eta$ and α and α' such that

$$|\alpha-\alpha'| \leq \min\left\{\delta(\eta), \frac{\gamma}{4(1+\#(K_{\bar{h}}))}\right\} =: \sigma(\gamma,\eta),$$

we finally obtain $C_\alpha \leq C_{\alpha'} + \gamma$.

If we change the role of α and α', we get an analogous estimate for $C_{\alpha'}$. Hence, we conclude that for every $\gamma \geq 4\eta$, there exists $\sigma(\gamma,\eta) > 0$ such that if $|\alpha-\alpha'| < \sigma(\gamma,\eta)$ then $|C_\alpha - C_{\alpha'}| \leq \gamma$. Since the choice of η was arbitrary, the assertion immediately follows. □

3.3.2 Compactness and Γ-convergence results

The following compactness result states that sequences θ_n with equibounded energy E_n^α converge to a limit function θ which has a finite number of jumps and takes the values $\{\pm\theta_\alpha\}$ almost everywhere if $0 \leq \alpha < 4$, while if $\alpha \geq 4$ the limit function is identically 0.

Proposition 11 (Compactness). *Let $E_n^\alpha : L^\infty(I,J) \to [0,+\infty]$ be the energies defined by (3.17). If $\{\theta_n\}$ is a sequence of functions such that*

$$\sup_n E_n^\alpha(\theta_n) < +\infty, \tag{3.24}$$

then we have two cases:
(i) if $0 \le \alpha < 4$ there exists a set $S \subset (0,1)$ with $\#(S) < +\infty$ such that, up to subsequences, θ_n converges to θ in $L^1_{loc}((0,1)\backslash S)$, where θ is a piecewise constant function and $\theta(0+) = \theta(1+)$. Moreover, $\theta(t) \in \{\pm\theta_\alpha\}$ for a.e. $t \in (0,1)$ and $S(\theta) \subseteq S$;
(ii) if $\alpha \ge 4$ then the limit function θ is identically 0.

Proof. (i) We first note that $-\alpha\cos\theta \ge |p_\alpha\theta| - \alpha - 1$ for $\theta \in [-\pi/2, \pi/2]$ and a constant p_α depending on α, so that

$$\begin{aligned} C > C\lambda_n \ge \lambda_n E_n^\alpha(\theta_n) \ge & |p_\alpha| \sum_{i=0}^{n-1} \lambda_n |\theta_n^i| - (\alpha+1)n\lambda_n + \lambda_n - n\lambda_n + \frac{\alpha^2}{8} + 1 \\ \ge & |p_\alpha| \sum_{i=0}^{n-1} \lambda_n |\theta_n^i| - \alpha - 1, \end{aligned}$$

from which we deduce that

$$\int_0^1 |\theta_n(t)|\, dt < +\infty. \tag{3.25}$$

From the equiboundedness assumption (3.24) there exists a constant $C > 0$ such that

$$\sup_n \sum_{i=0}^{n-1} \mathcal{E}_n^i(\theta_n) \le C < +\infty, \tag{3.26}$$

where we have set

$$\mathcal{E}_n^i(\theta_n) = \cos(\theta_n^i + \theta_n^{i+1}) - \frac{\alpha}{2}(\cos\theta_n^i + \cos\theta_n^{i+1}) - m_\alpha. \tag{3.27}$$

Now, if for every fixed $\eta > 0$ we define

$$I_n(\eta) := \{i \in \{0, 1, \dots, n-1\} : \mathcal{E}_n^i(\theta_n) > \eta\},$$

then (3.26) implies the existence of a uniform constant $C(\eta)$ such that

$$\sup_n \#(I_n(\eta)) \le C(\eta) < +\infty. \tag{3.28}$$

Let $i \in \{0, 1, \dots, n-1\}$ be such that $i \notin I_n(\eta)$; that is,

$$\mathcal{E}_n^i(\theta_n) = \cos(\theta_n^i + \theta_n^{i+1}) - \frac{\alpha}{2}(\cos\theta_n^i + \cos\theta_n^{i+1}) - m_\alpha \le \eta.$$

Let $\sigma = \sigma(\eta) > 0$ be defined such that if

$$0 \le \cos(\theta_1 + \theta_2) - \frac{\alpha}{2}(\cos\theta_1 + \cos\theta_2) - m_\alpha \le \eta, \quad \theta_1 + \theta_2 = 2\theta,\ \theta \in \{\pm\theta_\alpha\},$$

then

$$|\theta_1 - \theta| + |\theta_2 - \theta| \le \sigma(\eta).$$

As a consequence, if $i \notin I_n(\eta)$ we deduce the existence of $\theta \in \{\pm\theta_\alpha\}$ such that

$$|\theta_n^i - \theta| \leq \sigma \text{ and } |\theta_n^{i+1} - \theta| \leq \sigma.$$

Hence, up to a finite number of indices i, both θ_n^i and its nearest neighbour θ_n^{i+1} are close to the same minimal angle $\pm\theta_\alpha$. Namely, there exists a finite number of indices $0 = i_0 < i_1 < \cdots < i_{N_n} = n-1$ such that for all $k = 1, 2, \dots, N_n$ we can find $\theta_{k,n} \in \{\pm\theta_\alpha\}$ satisfying for all $i \in \{i_{k-1}+1, i_{k-1}+2, \dots, i_k - 1\}$ the aforementioned closeness property

$$|\theta_n^i - \theta_{k,n}| \leq \sigma \quad \text{and} \quad |\theta_n^{i+1} - \theta_{k,n}| \leq \sigma. \tag{3.29}$$

Now, let $\{i_{j_r}\}$, $r = 1, \dots, M_n$ be the maximal subset of $0 = i_0 < i_1 < \cdots < i_{N_n} = n-1$ defined by the requirement that if $\theta_{j_r,n} = \pm\theta_\alpha$ then $\theta_{j_{r+1},n} = \mp\theta_\alpha$; this means that $\{i_{j_1}, \dots, i_{j_{M_n}}\} \subseteq I_n(\eta)$. Hence, there exists $C(\eta) > 0$ such that $\sum_{i=0}^{n-1} \mathcal{E}_n^i(\theta_n) \geq C(\eta) M_n$ and then $E_n^\alpha(\theta_n) \geq C(\eta) M_n$, so that from (3.24) M_n are equibounded. Thus, up to further subsequences, we can assume that $M_n = M$ and that for every $r = 1, \dots, M$, $t_{i_{j_r}}^n = \lambda_n i_{j_r} \to t_r$ for some $t_r \in [0,1]$ and $\theta_{j_r,n} = \theta_r$. Set $S = \{t_1, \dots t_M\}$ and, for fixed $\delta > 0$, $S_\delta = \bigcup_r (t_r - \delta, t_r + \delta)$. Then, by identifying θ_n with its piecewise constant interpolation, from (3.29) and for n large enough we get

$$\sup_{t \in (0,1) \setminus S_\delta} |\theta_n(t) - \theta_r| \leq \sigma.$$

The previous estimates, together with (3.25) ensure that $\{\theta_n\}$ is an equicontinuous and equibounded sequence in $(0,1) \setminus S_\delta$. Thus, thanks to the arbitrariness of δ, up to passing to a further subsequence (not relabelled), θ_n converges in $L^\infty_{loc}((0,1) \setminus S)$ (and in $L^1_{loc}((0,1) \setminus S)$) to a function θ such that $\theta(t) \in \{\pm\theta_\alpha\}$ for a.e. $t \in (0,1)$. Moreover, $S(\theta) \subseteq S$. Finally, by the periodicity assumption (3.6), we have $\theta_n^0 = \theta_n^n$ from which passing to the limit as $n \to \infty$ we conclude that $\theta(0+) = \theta(1+)$.
(ii) The proof of (ii) requires minor changes in the argument above, so we will omit it. □

Now we can state and prove the Γ-convergence result.

Theorem 18. *(i) Let $\alpha \in [0, 4)$. Then E_n^α Γ-converges with respect to the L^1_{loc}-topology to*

$$E^\alpha(\theta) = \begin{cases} C_\alpha \#(S(\theta) \cap [0,1)), & \text{if } \theta \in PC_{loc}(\mathbb{R}),\ \theta \in \{\pm\theta_\alpha\} \\ & \theta \text{ is 1-periodic} \\ +\infty, & \text{otherwise} \end{cases} \tag{3.30}$$

on $L^1_{loc}(\mathbb{R})$, where $C_\alpha = C(-\theta_\alpha, \theta_\alpha)$ is given by (3.18) and $PC_{loc}(\mathbb{R})$ denotes the space of locally piecewise constant functions on $\mathbb{R}$.
(ii) Let $\alpha \in [4, +\infty)$. Then E_n^α Γ-converges with respect to the L^1_{loc}-topology to

$$E^\alpha(\theta) = \begin{cases} 0, & \text{if } \theta = 0 \\ +\infty, & \text{otherwise} \end{cases} \tag{3.31}$$

on $L^1_{loc}(\mathbb{R})$.

Proof. (i) **Liminf inequality.** We may assume, without loss of generality, that θ is left-continuous at each jump. Let $\theta_n \to \theta$ in $L^1(0,1)$ be such that $E_n^\alpha(\theta_n) < +\infty$. Then, from Proposition 11 there exist $N \in \mathbb{N}$, $\bar{\theta}_1, \dots, \bar{\theta}_N \in \{\pm\theta_\alpha\}$ and $0 = s_0 < s_1 < \cdots < s_N = 1$, $\{s_j\} = \{t_k\}$ (the set of indices may be different if $s_k = s_{k+1}$ for some k) such that

$$\theta^n_{j_k} \to \bar{\theta}_j, \quad \text{on the interval } (s_{j-1}, s_j),\ j \in \{1, \dots, N\}. \tag{3.32}$$

For $l \in \{0, 1, \dots, N\}$, let $\{k_n^l\}_n$ be a sequence of indices such that $k_n^0 = 0$,

$$\lim_n \lambda_n k_n^l = s_l,$$

and let $\{h_n^l\}_n$ be another sequence of indices such that $h_n^0 = 0$,

$$\lim_n \lambda_n h_n^l = \frac{s_l + s_{l-1}}{2}.$$

To get the Γ-liminf inequality, we rewrite the energy as follows:

$$E_n^\alpha(\theta_n) = \sum_{j=1}^{N-1} E_n^\alpha(\theta_n, h_n^j, h_n^{j+1}) + r_n, \tag{3.33}$$

where we have set

$$E_n^\alpha(\theta_n, h_n^j, h_n^{j+1}) = \sum_{i=h_n^j}^{h_n^{j+1}-1} \Big[\cos(\theta_n^i + \theta_n^{i+1}) - \frac{\alpha}{2}(\cos\theta_n^i + \cos\theta_n^{i+1}) - m_\alpha\Big],$$

$m_\alpha = -\frac{\alpha^2}{8} - 1$ and

$$r_n = \sum_{i=0}^{h_n^1 - 1} \mathcal{E}_n^i(\theta_n) + \sum_{i=h_n^N+1}^{n-1} \mathcal{E}_n^i(\theta_n),$$

with $r_n > 0$ and $\mathcal{E}_n^i(\theta_n)$ as in (3.27). Defining for $j \in \{1, 2, \dots, N-1\}$

$$\tilde{\theta}_n^i = \begin{cases} \bar{\theta}_j, & \text{if } i \le h_n^j - k_n^j - 1, \\ \theta_n^{i+k_n^j}, & \text{if } h_n^j - k_n^j \le i \le h_n^{j+1} - k_n^j - 1, \\ \bar{\theta}_{j+1}, & \text{if } i \ge h_n^{j+1} - k_n^j, \end{cases} \tag{3.34}$$

we have that $\tilde{\theta}_n^i$ is a test function for the minimum problem defining $C(\theta(s_j-), \theta(s_j+))$ as in (3.18), where $\theta(s_j-) = \bar{\theta}_j$ and $\theta(s_j+) = \bar{\theta}_{j+1}$.

For n large enough and any $\sigma > 0$, we then find that each summand in (3.33) can be estimated from below as

$$
\begin{aligned}
E_n^\alpha(\theta_n, h_n^j, h_n^{j+1}) &= \sum_{i=h_n^j}^{k_n^j} \mathcal{E}_n^i(\theta_n) + \sum_{i=k_n^j}^{h_n^{j+1}-1} \mathcal{E}_n^i(\theta_n) \\
&= \sum_{l=h_n^j-k_n^j}^{0} \mathcal{E}_n^{l+k_n^j}(\theta_n) + \sum_{l=1}^{h_n^{j+1}-k_n^j-1} \mathcal{E}_n^{l+k_n^j}(\theta_n) \\
&= \sum_{l=h_n^j-k_n^j}^{h_n^{j+1}-k_n^j-1} \mathcal{E}_n^l(\tilde\theta_n) \\
&= \sum_{l\in\mathbb{Z}} \mathcal{E}_n^l(\tilde\theta_n) - \sum_{l\le h_n^j-k_n^j-1} \mathcal{E}_n^l(\tilde\theta_n) - \sum_{l\ge h_n^{j+1}-k_n^j} \mathcal{E}_n^l(\tilde\theta_n) \\
&= \sum_{l\in\mathbb{Z}} \mathcal{E}_n^l(\tilde\theta_n) - \mathcal{E}_n^{h_n^j-k_n^j-2}(\tilde\theta_n) \\
&\ge \sum_{i\in\mathbb{Z}} \Big[\cos(\tilde\theta_n^i + \tilde\theta_n^{i+1}) - \frac{\alpha}{2}(\cos\tilde\theta_n^i + \cos\tilde\theta_n^{i+1}) - m_\alpha\Big] - \omega(\sigma) \\
&\ge C(\bar\theta_j, \bar\theta_{j+1}) - \omega(\sigma),
\end{aligned}
\tag{3.35}
$$

where $\omega : [0,+\infty) \to [0,+\infty)$ is a suitable continuous function, $\omega(0) = 0$. Finally, since for every $j \in \{1,\dots,N-1\}$ it holds $C(\bar\theta_j, \bar\theta_{j+1}) = C_\alpha$ by (3.19), combining (3.35) with (3.33) and passing to the liminf as $n \to +\infty$ we get the liminf inequality

$$
\begin{aligned}
\liminf_{n\to\infty} E_n^\alpha(\theta_n) &\ge \liminf_{n\to\infty} \Big[(N-1)C_\alpha - (N-1)\omega(\sigma)\Big] \\
&= C_\alpha \#(S(\theta)\cap[0,1)),
\end{aligned}
\tag{3.36}
$$

where the latter equality follows by the arbitrariness of σ.
Limsup inequality. Let θ be such that $E^\alpha(\theta) < +\infty$. Then there exist $M \in \mathbb{N}$, $\bar\theta_1, \dots, \bar\theta_M \in \{\pm\theta_\alpha\}$ and $0 = t_0 < t_1 < \dots < t_M = 1$ such that $\#(S(\theta)) = M-1$ and

$$
\theta(t) = \bar\theta_j, \quad t \in (t_{j-1}, t_j),\ j \in \{1,2,\dots,M\}. \tag{3.37}
$$

Fixed $\eta > 0$, from the definition of $C(\theta(t_j-), \theta(t_j+))$ for $j \in \{1,2,\dots,M-1\}$ we can find functions $\psi_{j,j+1} : \mathbb{Z} \to [-\pi/2, \pi/2]$, such that

$$
\psi_{j,j+1}^i = \begin{cases} \bar\theta_j & \text{for } i \le -N_j, \\ \bar\theta_{j+1} & \text{for } i \ge N_j, \end{cases} \tag{3.38}
$$

and

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}} \Big[\cos(\psi_{j,j+1}^i + \psi_{j,j+1}^{i+1}) - \frac{\alpha}{2}(\cos\psi_{j,j+1}^i + \cos\psi_{j,j+1}^{i+1}) - m_\alpha\Big] \\
&\le C(\theta(t_j-), \theta(t_j+)) + \eta = C_\alpha + \eta,
\end{aligned}
\tag{3.39}
$$

where the latter equality follows again by (3.19). Note that in (3.38) we may assume $N = N_j$ independent of j, up to choose $N = \max_{1\le j\le M-1}\{N_j\}$.

We define a recovery sequence $\tilde{\theta}_n$ by means of a translation argument involving functions $\psi_{j,j+1}$, $j \in \{1, \dots, M-1\}$, that will allow us to estimate the energy contribution from above with (3.39) in a suitable neighbourhood of each jump point t_j. Namely, for every j, we set $\tilde{\theta}_n^i = \psi_{j,j+1}^{i-\lfloor t_j n \rfloor}$ if $i \in \{\lfloor t_j n \rfloor - N, \dots, \lfloor t_j n \rfloor + N\}$, while if $i \in \{\lfloor t_j n \rfloor + N, \dots, \lfloor t_{j+1} n \rfloor - N\}$, we define $\tilde{\theta}_n^i$ to be constantly equal to $\bar{\theta}_{j+1}$, according to (3.38). This definition can be summarized as follows:

$$\tilde{\theta}_n^i = \begin{cases} \bar{\theta}_1 & \text{if } 0 \le i \le \lfloor t_1 n \rfloor - N \\ \psi_{j,j+1}^{i-\lfloor t_j n \rfloor} & \text{if } \lfloor t_j n \rfloor - N \le i \le \lfloor t_j n \rfloor + N, \\ \bar{\theta}_{j+1} & \text{if } \lfloor t_j n \rfloor + N \le i \le \lfloor t_{j+1} n \rfloor - N, \quad j \in \{1, \dots, M-1\} \\ \bar{\theta}_M & \text{if } n - N \le i \le n-1. \end{cases} \tag{3.40}$$

We note that the corresponding $\tilde{\theta}_n \in \mathcal{D}_n(I)$ satisfy $\tilde{\theta}_n \to \theta$ in L^∞ and (here we use the simplified notation for the energies as in (3.27))

$$\begin{aligned} E_n^\alpha(\tilde{\theta}_n) = \sum_{i=0}^{n-1} \mathcal{E}_n^i(\tilde{\theta}_n) &= \sum_{i=0}^{\lfloor t_1 n \rfloor - N - 1} \mathcal{E}_n^i(\tilde{\theta}_n) + \sum_{j=1}^{M-1} \left(\sum_{i=\lfloor t_j n \rfloor - N}^{\lfloor t_j n \rfloor + N - 1} \mathcal{E}_n^i(\tilde{\theta}_n) \right) \\ &+ \sum_{j=1}^{M-1} \left(\sum_{i=\lfloor t_j n \rfloor + N}^{\lfloor t_{j+1} n \rfloor - N - 1} \mathcal{E}_n^i(\tilde{\theta}_n) \right) + \sum_{i=n-N}^{n-1} \mathcal{E}_n^i(\tilde{\theta}_n) \\ &= \sum_{j=1}^{M-1} \left(\sum_{i=\lfloor t_j n \rfloor - N}^{\lfloor t_j n \rfloor + N - 1} \mathcal{E}_n^i(\psi_{j,j+1}^{i-\lfloor t_j n \rfloor}) \right) = \sum_{j=1}^{M-1} \sum_{i \in \mathbb{Z}} \mathcal{E}_n^i(\psi_{j,j+1}^i) \\ &\le (M-1)(C_\alpha + \eta), \end{aligned}$$

whence, by the arbitrariness of η, we deduce that

$$\limsup_{n \to +\infty} E_n^\alpha(\tilde{\theta}_n) \le (M-1)(C_\alpha + \eta) = C_\alpha \#(S(\theta) \cap [0,1)). \tag{3.41}$$

Thus, (3.41) shows that the lower bound (3.36) is sharp, and this concludes the proof of (i).
(ii) In this case the proof is immediate. Indeed, for any $\theta_n \to 0$, from $E_n^\alpha(\theta_n) \ge 0$ we have in particular that

$$\liminf_{n \to +\infty} E_n^\alpha(\theta_n) \ge 0.$$

As a recovery sequence, we can choose $\theta_n \equiv 0$, for which we obtain $\lim_{n \to +\infty} E_n^\alpha(\theta_n) = 0$. □

3.4 Limit behaviour near the transition point $\alpha = 4$

The description of the limit as $n \to +\infty$ of the energies E_n^α with fixed α, carried out in the previous section, has a gap for $\alpha = 4$. Indeed, the crease energy C_α jumps from a strictly positive value (corresponding to $0 \le \alpha < 4$)

to 0 (when $\alpha \geq 4$). Note also that the explicit value of C_α defined implicitly by (3.18) is not known in literature. This suggests to focus near the transition point $\alpha = 4$, let the parameter α depend on n and be close to 4 from below; that is, replace α by $4 - \alpha_n$, $\alpha_n \to 4^-$.

Such analysis is the main content of a recent paper by Cicalese and Solombrino (2015)[40], where they find suitable scaling and order parameter to compute the energy the system spends in a transition between two states with different chirality when $\alpha \simeq 4$. Moreover, they show the dependence of the limit on the particular sequence $\alpha_n \to 4^-$ and the existence of different regimes. Our aim is to show that their result can be retrieved also correspondingly to a different choice of the order parameter in the energies. First of all, we write the energies (3.17) in terms of $4 - \alpha$ as

$$E_n^\alpha(\theta) = [4-(4-\alpha)]\sum_{i=0}^{n-1}(1-\cos\theta^i) - \sum_{i=0}^{n-1}[1-\cos(\theta^i+\theta^{i+1})] + n\frac{(4-\alpha)^2}{8}, \tag{3.42}$$

and when necessary, we may think also the quantities W_α, C_α, etc. to be functions of $4-\alpha$. Note that if we choose as a test function in (3.18) $\theta^{i,\alpha} = \operatorname{sign}(i)\arccos(\alpha/4)$ then we obtain a first rough estimate

$$0 < C_\alpha \leq (4-\alpha) - \frac{(4-\alpha)^2}{8},$$

showing in particular that $C_\alpha \to 0$ as $\alpha \to 4^-$.

The following proposition (compare with [40, Proposition 4.3]) characterizes the angles between neighbours for an equibounded (in energy) sequence of spins as the frustration parameter approaches the critical value from below.

Proposition 12. *If $\{\theta_n\}$ is a sequence such that*

$$\sup_n E_n^{\alpha_n}(\theta_n) \leq C(4-\alpha_n)^{3/2}, \tag{3.43}$$

then $\theta_n^i \to 0$ as $\alpha_n \to 4^-$ uniformly with respect to $i \in \{0, 1, \ldots, n-1\}$.

Proof. The claim follows immediately from the estimate

$$0 \leq 2\Big(\cos\theta_n^i - \frac{\alpha_n}{4}\Big)^2 \leq \sum_{i=0}^{n-1} W_{\alpha_n}(\theta_n^i) \leq E_n^{\alpha_n}(\theta_n) \leq C(4-\alpha_n)^{3/2} \tag{3.44}$$

valid for all $i \in \{0, 1, \ldots, n-1\}$. □

We introduce a new order parameter

$$v_n^i = \frac{\theta_n^i}{\theta_{\alpha_n}} \tag{3.45}$$

and reformulate the Γ-convergence result by Cicalese and Solombrino (2015)([40, Theorem 4.2]) in terms of this new variable. However, it is worth noting that the "flat" angular parameter v_n^i is equivalent with their variable z_n^i in the regime of "small angles", i.e., as $\alpha_n \to 4^-$, $\theta_n^i \to 0$, since in this case

$$\theta_{\alpha_n} = \arccos\left(1 - \frac{(4-\alpha_n)}{4}\right) \simeq \frac{\sqrt{4-\alpha_n}}{\sqrt{2}}$$

and

$$z_n^i = \frac{2\sqrt{2}}{\sqrt{4-\alpha_n}} \sin\left(\frac{\theta_n^i}{2}\right) \simeq \frac{\sqrt{2}\theta_n^i}{\sqrt{4-\alpha_n}}.$$

The change of variables (3.45) associates to any given $\theta_n \in \mathcal{D}_n(I)$ a piecewise-constant function $v_n \in \widetilde{\mathcal{D}}_n(I)$ where

$$\widetilde{\mathcal{D}}_n(I) := \Big\{ v : [0,1) \to \mathbb{R} : \, v(t) = v_n^i \text{ if } t \in \lambda_n(i + [0,1)), \, i \in \{0,1,\dots,n-1\} \Big\},$$

with v_n as in (3.45). With a slight abuse of notation, we regard $E_n^{\alpha_n}$ as a functional defined on $v \in L^1(I,\mathbb{R})$ by

$$E_n^{\alpha_n}(v) = \begin{cases} E_n^{\alpha_n}(\theta), & \text{if } v \in \widetilde{\mathcal{D}}_n(I) \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.46}$$

and correspondingly we define the scaled energies

$$F_n^{\alpha_n}(v) := \frac{8E_n^{\alpha_n}(v)}{\sqrt{2}(4-\alpha_n)^{3/2}}. \tag{3.47}$$

Theorem 19 (Cicalese and Solombrino (2015)[40]). *Let $F_n^{\alpha_n} : L^1(I,\mathbb{R}) \to [0,+\infty]$ be the functional in (3.47). Assume that there exists $l := \lim_n \sqrt{2}\lambda_n/4(4-\alpha_n)^{1/2}$. Then $F^0(v) := \Gamma - \lim_n F_n^{\alpha_n}(v)$ with respect to the $L^1(I)$ convergence is given by:*

(i) if $l = 0$,

$$F^0(v) := \begin{cases} \frac{8}{3}\#(S(v)) & \text{if } v \in BV(I,\{\pm 1\}), \\ +\infty & \text{otherwise;} \end{cases} \tag{3.48}$$

(ii) if $l \in (0,+\infty)$,

$$F^0(v) := \begin{cases} \frac{1}{l}\int_I \Big(v^2(t) - 1\Big)^2 dt + l\int_I (\dot{v}(t))^2\, dt & \text{if } v \in W^{1,2}_{|per|}(I), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.49}$$

where we have set $W^{1,2}_{|per|}(I) := \{v \in W^{1,2}(I) : |v(0)| = |v(1)|\}$;

(iii) if $l = +\infty$,

$$F^0(v) := \begin{cases} 0 & \text{if } v = const., \\ +\infty & \text{otherwise.} \end{cases} \tag{3.50}$$

Proof. In order to simplify the notation, we put $\varepsilon_n := 4 - \alpha_n \to 0$ as $n \to \infty$. Let $\{v_n\}$ be a sequence in $\widetilde{\mathcal{D}}_n(I)$ such that $\sup_n \dfrac{E_n^{\varepsilon_n}(v_n)}{\varepsilon_n^{3/2}} \leq C < \infty$. As remarked before, correspondingly, there exists a sequence $\{\theta_n\}$ in $\mathcal{D}_n(I)$ such that $\sup_n \dfrac{E_n^{\varepsilon_n}(\theta_n)}{\varepsilon_n^{3/2}} \leq C < \infty$, satisfying $\theta_n^i \to 0$ uniformly with respect to i by Proposition 12.

From the estimates contained in the proof of [40, Theorem 4.2] we get

$$E_n^{\varepsilon_n}(\theta_n) \geq 8\sum_{i=0}^{n-1}\Big[\sin^2\Big(\frac{\theta_n^i}{2}\Big) - \frac{\varepsilon_n}{8}\Big]^2 + 2(1-\gamma_n)\sum_{i=0}^{n-1}\Big[\sin\Big(\frac{\theta_n^{i+1}}{2}\Big) - \sin\Big(\frac{\theta_n^i}{2}\Big)\Big]^2,$$

for some $\gamma_n \to 0$. Since $\sin\theta \simeq \theta$ as $\theta \to 0$, we may improve the estimate obtaining

$$E_n^{\varepsilon_n}(\theta_n) \geq 8(1-\gamma_n')\sum_{i=0}^{n-1}\Big[\Big(\frac{\theta_n^i}{2}\Big)^2 - \frac{\varepsilon_n}{8}\Big]^2 + \frac{(1-\gamma_n'')}{2}\sum_{i=0}^{n-1}\Big[\Big(\frac{\theta_n^{i+1}}{2}\Big) - \Big(\frac{\theta_n^i}{2}\Big)\Big]^2,$$

for suitable $\gamma_n', \gamma_n'' \to 0$. In terms of the new order parameter v_n^i defined by (3.45) the previous inequality now reads

$$\begin{aligned} E_n^{\varepsilon_n}(\theta_n) \geq{}& \frac{2\theta_{\varepsilon_n}^4}{\lambda_n}(1-\gamma_n')\sum_{i=0}^{n-1}\lambda_n\Big[(v_n^i)^2 - \frac{\varepsilon_n}{2\theta_{\varepsilon_n}^2}\Big]^2 \\ &+ \frac{\theta_{\varepsilon_n}^2\lambda_n}{8}(1-\gamma_n'')\sum_{i=0}^{n-1}\lambda_n\Big(\frac{v_n^{i+1}-v_n^i}{\lambda_n}\Big)^2, \end{aligned}$$

where $\lambda_n = \frac{1}{n}$. If we multiply both the sides by $8/\sqrt{2}\varepsilon_n^{3/2}$, since $\dfrac{\varepsilon_n}{2\theta_{\varepsilon_n}^2} \to 1$ we get

$$\begin{aligned} \frac{8E_n^{\varepsilon_n}(\theta_n)}{\sqrt{2}\varepsilon_n^{3/2}} \geq{}& \frac{8\sqrt{2}\theta_{\varepsilon_n}^4}{\lambda_n\varepsilon_n^{3/2}}(1-\gamma_n')\sum_{i=0}^{n-1}\lambda_n\Big[(v_n^i)^2 - 1\Big]^2 \\ &+ \frac{\sqrt{2}\theta_{\varepsilon_n}^2\lambda_n}{2\varepsilon_n^{3/2}}(1-\gamma_n'')\sum_{i=0}^{n-1}\lambda_n\Big(\frac{v_n^{i+1}-v_n^i}{\lambda_n}\Big)^2. \end{aligned}$$

Since $\theta_{\varepsilon_n} = \arccos(1-\frac{\varepsilon_n}{4})$ and $\theta_{\varepsilon_n} \simeq \frac{\sqrt{\varepsilon_n}}{\sqrt{2}}$ as $\varepsilon_n \to 0$, we note that $\dfrac{8\sqrt{2}\theta_{\varepsilon_n}^4}{\lambda_n\varepsilon_n^{3/2}} \simeq \dfrac{4\sqrt{\varepsilon_n}}{\sqrt{2}\lambda_n}$ and $\dfrac{\sqrt{2}\theta_{\varepsilon_n}^2\lambda_n}{2\varepsilon_n^{3/2}} \simeq \dfrac{\sqrt{2}\lambda_n}{4\sqrt{\varepsilon_n}}$ as $n \to \infty$. Thus, we finally get

$$\begin{aligned} \frac{8E_n^{\varepsilon_n}(\theta_n)}{\sqrt{2}\varepsilon_n^{3/2}} \geq{}& \frac{4\sqrt{\varepsilon_n}}{\sqrt{2}\lambda_n}(1-\tilde{\gamma}_n')\sum_{i=0}^{n-1}\lambda_n\Big[(v_n^i)^2 - 1\Big]^2 \\ &+ \frac{\sqrt{2}\lambda_n}{4\sqrt{\varepsilon_n}}(1-\tilde{\gamma}_n'')\sum_{i=0}^{n-1}\lambda_n\Big(\frac{v_n^{i+1}-v_n^i}{\lambda_n}\Big)^2, \end{aligned} \tag{3.51}$$

for suitable $\tilde{\gamma}_n', \tilde{\gamma}_n'' \to 0$. The estimate (3.51) implies the liminf inequality both in case (i) and (ii) as remarked in [40], and the limsup inequality can be obtained in both cases by the constructive argument contained therein, so we will omit the proof. □

Remark 8. (asymptotic behaviour of C_α). As remarked before, $C_\alpha \to 0$ as $\alpha \to 4^-$. However, we may use Theorem 19(i) to refine this estimate and determine the right order of C_{α_n} with respect to $4-\alpha_n$ as $\alpha_n \to 4$.

In the regime $\lambda_n << (4-\alpha_n)^{1/2}$ we can compute the limit of energies $F_n^{\alpha_n}(v)$ first as $n \to \infty$ while keeping $\alpha_n \equiv \alpha_0 \neq 4$ fixed, and then the limit as $\alpha_0 \to 4^-$. Thanks to Theorem 18 and the continuity result ensured by Proposition 10, we obtain

$$F^{\alpha_0}(v) := \Gamma - \lim_{n\to+\infty} \frac{8E_n^{\alpha_n}(v)}{\sqrt{2}(4-\alpha_n)^{3/2}} = \frac{8C_{\alpha_0}}{\sqrt{2}(4-\alpha_0)^{3/2}} \#(S(v)), \tag{3.52}$$

whence, by means of Theorem 19(i), we get

$$F^0(v) := \Gamma - \lim_{\alpha_0\to 4} F^{\alpha_0}(v) = \frac{8}{3}\#(S(v)). \tag{3.53}$$

The convergence of minimum problems as $\alpha \to 4^-$ finally gives

$$\lim_{\alpha\to 4^-} \frac{3C_\alpha}{\sqrt{2}(4-\alpha)^{3/2}} = 1. \tag{3.54}$$

Thus, near the ferromagnet-helimagnet transition point, the energy C_α coincides with the energy $E_{dw} \propto (4-\alpha)^{3/2}$ for the excitation of a chiral domain wall separating two domains of opposite chirality, which is a well-known universal low-temperature property of frustrated classical spin chains (see, e.g., Dmitriev and Krivnov (2011)[50]).

3.5 A link with the gradient theory of phase transitions

In this section we show that the variational asymptotic behaviour of the energies F_n^α for any $\alpha \in [0,4]$, both in the case of fixed α (Theorem 18) and in the case $\alpha \simeq 4$ (Theorem 19), is the same as that of a parametrized family of Modica-Mortola type functionals, thus providing an interesting connection between frustrated lattice spin systems and the gradient theory of phase transitions (see also [23, Section 6]).

In order to do that in the framework of the *equivalence by* Γ*-convergence*, we recall some definitions about Γ-equivalence for families of parametrized functionals, uniform equivalence, regular and singular points, as introduced by Braides and Truskinovsky (2008)[35].

Definition 17 (Γ-equivalence)**.** Let $\mathcal{A}$ be a set of parameters. Two families of parametrized functionals F_n^α and G_n^α are *equivalent at scale 1 at* $\alpha_0 \in \mathcal{A}$ if $F_n^{\alpha_0}$ and $G_n^{\alpha_0}$ are equivalent at scale 1, i.e.,

$$\Gamma - \lim_{n\to+\infty} F_n^{\alpha_0} = \Gamma - \lim_{n\to+\infty} G_n^{\alpha_0} \tag{3.55}$$

and these Γ-limits are non-trivial.

Definition 18 (uniform Γ-equivalence)**.** Let $\mathcal{A}$ be a set of parameters. Two families of parametrized functionals F_n^α and G_n^α are *uniformly equivalent at scale 1 at* $\alpha_0 \in \mathcal{A}$ if for all $\alpha_n \to \alpha_0$ we have, up to subsequences,

$$\Gamma - \lim_{n\to+\infty} F_n^{\alpha_n} = \Gamma - \lim_{n\to+\infty} G_n^{\alpha_n} \tag{3.56}$$

and these Γ-limits are non-trivial. They are uniformly equivalent on $\mathcal{A}$ if they are uniformly equivalent at all $\alpha_0 \in \mathcal{A}$.

Definition 19 (regular point)**.** $\alpha_0 \in \mathcal{A}$ is a *regular point* if for all $\alpha_n \to \alpha_0$ we have, up to a subsequence,

$$\Gamma - \lim_{n\to+\infty} F_n^{\alpha_n} = \Gamma - \lim_{n\to+\infty} F_n^{\alpha_0}. \tag{3.57}$$

Definition 20 (singular point)**.** $\alpha_0 \in \mathcal{A}$ is a *singular point* if it is not regular; that is, if there exist $\alpha_n' \to \alpha_0$, $\alpha_n'' \to \alpha_0$ such that (up to subsequences)

$$\Gamma - \lim_{n\to+\infty} F_n^{\alpha_n'} \neq \Gamma - \lim_{n\to+\infty} F_n^{\alpha_n''}. \tag{3.58}$$

According to the previous definitions, each $0 \leq \alpha_0 < 4$ is a regular point for F_n^α, since, as already observed in Remark 8, for any sequence $\alpha_n \to \alpha_0$, we have

$$F^{\alpha_0}(v) := \Gamma - \lim_{n\to+\infty} F_n^{\alpha_n}(v) = \frac{8C_{\alpha_0}}{\sqrt{2}(4-\alpha_0)^{3/2}} \#(S(v)).$$

As a consequence of Theorem 19, instead, $\alpha_0 = 4$ is a singular point for F_n^α.

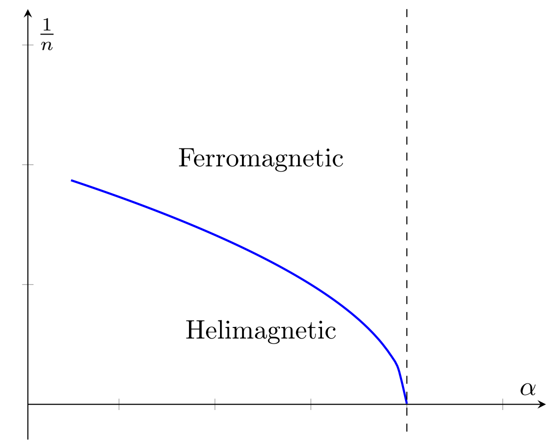


Figure 3.3: The $\frac{1}{n} - \alpha$ space. In blue the failure curve $\frac{1}{n} = (4-\alpha)^{1/2}$.

The behaviour of the system close to the transition point $\alpha = 4$ can be pictured in the $\frac{1}{n}$–α plane (see Fig. 3.3), where the crossover line $\frac{1}{n} = (4-\alpha)^{1/2}$ separates a zone where there is *helimagnetic order* ($\frac{1}{n} << (4-\alpha)^{1/2}$) from one where we have *ferromagnetic order* ($\frac{1}{n} >> (4-\alpha)^{1/2}$).

For our purposes, it is useful to recall the well known Γ-convergence result in gradient theory of phase transitions due to Modica and Mortola (1977)[56]. Let $\Omega \subset \mathbb{R}$ be an open set, $u : \Omega \to \mathbb{R}$ and $W = W(u)$ a non-convex energy such that $W \geq 0$, $W(u) \geq c(u^2 - 1)$ and $W = 0$ if and only if $u = a, b$. W is called a *double-well* potential. Let $C > 0$ and consider the energies

$$F_n(u) = n \int_\Omega W(u)\,dx + \frac{C^2}{n} \int_\Omega (\dot{u})^2\,dx, \quad u \in W^{1,2}(\Omega). \tag{3.59}$$

Theorem 20 (Modica-Mortola's theorem)**.** *The functionals F_n above Γ-converge as $n \to \infty$ and with respect to the $L^1(\Omega)$ convergence to the functional*

$$F_\infty(u) = \begin{cases} C \cdot c_W \#(S(u) \cap \Omega), & \text{if } u \in \{a, b\} \text{ a.e.} \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.60}$$

where $c_W := 2\int_a^b \sqrt{W(s)}\,ds$.

The following theorem states the announced uniform equivalence by Γ-convergence of energies F_n^α with parametrized Modica-Mortola type functionals.

Theorem 21. *Setting* $\lambda_{n,\alpha} := 2n\theta_\alpha^4$, $M_\alpha := 3C_\alpha/8$, $\mu_\alpha := \frac{\sqrt{2}(4-\alpha)^{3/2}}{8}$, *the energies*

$$G_n^\alpha(v) = \begin{cases} \dfrac{1}{\mu_\alpha}\Big[\lambda_{n,\alpha}\displaystyle\int_I \Big(v^2-1\Big)^2 dt + \dfrac{M_\alpha^2}{\lambda_{n,\alpha}}\displaystyle\int_I (\dot v)^2\,dt\Big], & \text{if } v \in W^{1,2}_{|per|}(I), \\ +\infty, & \text{otherwise in } L^1_{loc}(\mathbb{R}), \end{cases}$$

and $F_n^\alpha(v) := \frac{1}{\mu_\alpha}E_n^\alpha(v)$ *are uniformly equivalent by Γ-convergence on* $[0,4]$. *Moreover,*

(i) *each* $\alpha_0 \in [0,4)$ *is a regular point;*

(ii) $\alpha_0 = 4$ *is a singular point.*

Proof. As before, we put $\varepsilon_n := 4-\alpha_n$, so that $\varepsilon_n \geq 0$.
(i) Let $\varepsilon_n \to \varepsilon_0 \neq 0$ and $\{v_n\}$ be a sequence with equibounded energy. Correspondingly, by (3.45) we may find a sequence $\{\theta_n\}$ such that $v_n = \theta_n/\theta_{\varepsilon_n}$. The continuity of the energies G_n^ε with respect to the parameter $\varepsilon = \varepsilon_n$, ensured also by Proposition 10, allows us to consider, without loss of generality, the energies

$$G_n^{\varepsilon_0}(v) = \frac{8}{\sqrt{2}\varepsilon_0^{3/2}}\Big[2n\theta_{\varepsilon_0}^4\int_I \Big(v^2-1\Big)^2 dt + \frac{1}{2n\theta_{\varepsilon_0}^4}\Big(\frac{3C_{\varepsilon_0}}{8}\Big)^2\int_I (\dot v)^2\,dt\Big],$$

where $v = \theta/\theta_{\varepsilon_0}$. After simplifying the constants, we may rewrite the energies in terms of θ as

$$G_n^{\varepsilon_0}(\theta) = \frac{8}{\sqrt{2}\varepsilon_0^{3/2}}\Big[2n\int_I \Big(\theta^2-\theta_{\varepsilon_0}^2\Big)^2 dt + \frac{1}{2n}\Big(\frac{3C_{\varepsilon_0}}{8\theta_{\varepsilon_0}^3}\Big)^2\int_I (\dot\theta)^2\,dt\Big].$$

In order to compute the Γ-limit as $n \to \infty$ of the energies $G_n^{\varepsilon_0}$, we may apply the Γ-convergence result by Modica and Mortola (Theorem 20), thus obtaining

$$G^{\varepsilon_0}(\theta) := \Gamma - \lim_{n\to+\infty} G_n^{\varepsilon_0}(\theta) = \frac{8C_{\varepsilon_0}}{\sqrt{2}\varepsilon_0^{3/2}}\#(S(\theta)),$$

since

$$c_W = 2\int_{-\theta_{\varepsilon_0}}^{\theta_{\varepsilon_0}} |\theta^2-\theta_{\varepsilon_0}^2|\,d\theta = \frac{8}{3}\theta_{\varepsilon_0}^3.$$

This result coincides with (3.52), once we remark that $\#(S(v)) = \#(S(\theta))$.
(ii) Let $\varepsilon_n \to 0$ and $v \in W^{1,2}_{loc}(\mathbb{R})$. The estimates contained in the proof of Theorem 19 and equation (3.54) allow us to rewrite the functional $G_n^{\varepsilon_n}$ as

$$G_n^{\varepsilon_n}(v) = \frac{4n\sqrt{\varepsilon_n}}{\sqrt{2}}(1+\eta_n)\int_I \Big(v^2-1\Big)^2 dt + \frac{\sqrt{2}}{4n\sqrt{\varepsilon_n}}(1+\eta_n')\int_I (\dot v)^2\,dt,$$

for suitable sequences $\eta_n, \eta'_n \to 0$. In order to simplify the notation, we put

$$K_n := \frac{\sqrt{2}}{4n\sqrt{\varepsilon_n}},$$

and we write

$$G_n^{\varepsilon_n}(v) = \frac{1}{K_n}(1+\eta_n)\int_I \Big(v^2-1\Big)^2\,dt + K_n(1+\eta'_n)\int_I (\dot v)^2\,dt.$$

We distinguish between three cases:
(a) $K_n \to 0$. In this case, we apply again Theorem 20 (with $C = 1$), thus obtaining

$$G^0(v) := \Gamma - \lim_{n\to+\infty} G_n^{\varepsilon_n}(v) = \frac{8}{3}\#(S(v)), \tag{3.61}$$

since $c_W = 2\int_{-1}^{1}|v^2-1|\,dv = \frac{8}{3}$.
(b) $K_n \to l \in (0,+\infty)$. A sequence v_n with equibounded energy is weakly compact in $W^{1,2}_{|per|}(I)$, then by lower semicontinuity in $W^{1,2}_{|per|}(I)$ we get

$$\liminf_n G_n^{\varepsilon_n}(v_n) \geq \frac{1}{l}\int_I \Big(v^2-1\Big)^2\,dt + l\int_I (\dot v)^2\,dt.$$

In order to obtain the limsup inequality, we can argue by density considering $v_n \in W^{1,2}_{|per|}(I)\cap C^\infty(\bar I)$, $v_n \to v$ such that

$$\lim_n G_n^{\varepsilon_n}(v_n) = \frac{1}{l}\int_I \Big(v^2-1\Big)^2\,dt + l\int_I (\dot v)^2\,dt.$$

(c) $K_n \to +\infty$. Let v be a constant function, and consider the constant sequence $v_n \equiv v$. Trivially,

$$\liminf_n G_n^{\varepsilon_n}(v_n) = \liminf_n \Big(\frac{1}{K_n}(1+\eta_n)\int_I \Big(v_n^2-1\Big)^2\,dt\Big) \geq 0,$$

and

$$\lim_n G_n^{\varepsilon_n}(v_n) = 0.$$

□

The proof of point (i) of Theorem 21 permits us to deduce an equivalence result also for the energies $E_n^\alpha(\theta)$ defined in (3.17) with Modica Mortola type functionals whose potentials $\mathcal{W}_\alpha(\theta) := (\theta^2-\theta_\alpha^2)^2$ have the wells located at the minimal angles $\theta = \pm\theta_\alpha$. It can be stated as follows.

Corollary 1. *Let α be a positive number, $\alpha \in [0,4)$. The energies $E_n^\alpha(\theta)$ and the family of functionals $H_n^\alpha(\theta)$ defined on $L^1_{loc}(\mathbb{R})$ as*

$$H_n^\alpha(\theta) = \begin{cases} \dfrac{\lambda_{n,\alpha}}{\theta_\alpha^4}\displaystyle\int_I \Big(\theta^2(t)-\theta_\alpha^2\Big)^2\,dt + \frac{M_\alpha^2}{\lambda_{n,\alpha}\theta_\alpha^2}\int_I (\dot\theta(t))^2\,dt, & \text{if } \theta \in W^{1,2}_{|per|}(I),\\ +\infty, & \text{otherwise,}\end{cases}$$

are uniformly equivalent by Γ-convergence on $[0,4)$.

Chapter 4

Static, quasistatic and dynamic analysis for scaled Perona-Malik functionals

The content of this chapter is based on a joint work with Andrea Braides [37].

4.1 Introduction

The Perona-Malik *anisotropic diffusion* technique in Image Processing [59] is formally based on a gradient-flow dynamics related to the non-convex energy

$$F_{PM}(u) = \int_{\Omega} \log\Big(1 + \frac{1}{K^2}|\nabla u|^2\Big)\,dx, \tag{4.1}$$

where u represents the output signal or picture defined on Ω and K a tuning parameter. In the convexity domain of the energy function; i.e., if $|\nabla u| \leq K$ the effect of the gradient flow is supposed to smoothen the initial data, while on discontinuity sets where $|\nabla u| = +\infty$ the gradient of the energy is formally zero and no motion is expected. In reality, such a gradient flow is ill-posed and even in dimension one we may have strong solutions only for some certain classes of initial data, or weak solutions which develop complex microstructure. However, the anisotropic diffusion technique is always applied in a discrete or semi-discrete context, where energy F_{PM} is only a (formal) continuum approximation of some discrete energy defined on some space of finite elements or in a finite-difference context. Indeed, it is well known that convex-concave energies, which give ill-posed problems if simply extended to continuum energies by replacing finite differences by gradients, can be approximated to well-posed problems in a properly defined passage discrete-to-continuum. In a static framework, the prototype of this argument dates back to the analysis of Chambolle (1992)[42], who showed that the Blake-Zisserman weak-membrane discrete energy (involving truncated quadratic potentials) [16] can be approximated by the Mumford-Shah functional [58]. The latter functional (together with its anisotropic variants) is a common continuum approximation of a class of lattice energies with convex-concave energy functions, which also comprises atomistic energies such as Lennard-Jones

ones [32] and the discretized version of the Perona-Malik functional itself as showed by Morini and Negri (2003)[57] (see also [20] Section 11.5). The approximation of these lattice energies is performed by considering the lattice spacing ε as a small parameter and suitably scaling the energies. In the case of the Perona-Malik discretized energy on the cubic lattice $\varepsilon\mathbb{Z}^n$ the scaled functionals

$$F_\varepsilon(u) := \sum_{i,j} \frac{\varepsilon^{n-1}}{|\log\varepsilon|} \log\left(1 + |\log\varepsilon| \frac{|u_i - u_j|^2}{\varepsilon}\right)$$

(u_i denotes the value of u at $i \in \varepsilon\mathbb{Z}^n$ and the sum is performed on nearest neighbours) Γ-converge to a Mumford-Shah energy, with an anisotropic surface energy density in dimension larger than one [57, 20]. This means that the solutions to global minimization problems involving F_ε, identified with their piecewise-constant interpolations, converge as ε tends to zero to the solutions to the corresponding global minimization problems involving the Mumford-Shah functional. Examples of such global minimum problems comprise problems in Image Processing with an additional lower-order fidelity term (typically an L^2-distance of u from the input datum u_0).

In this chapter we analyze how much this approximation procedure can be extended beyond the global-minimization standpoint by examining the one-dimensional case. It is known that Γ-convergence cannot be easily extended as a theory to the analysis of the behaviour of local minima or to a dynamical setting beyond, essentially, the "trivial" case of convex energies [21, 25]. However, several recent examples suggest that for problems with concentration some quasistatic and dynamic models are compatible with Γ-convergence (such as for Ginzburg-Landau [61] or for Lennard-Jones [29] energies). We show that this holds also for some one-dimensional quasistatic and dynamic problems obtained as minimizing movements along the family F_ε [21]. They coincide, up to some technical point that will be explained below, with the corresponding problems related to the one-dimensional Mumford-Shah functional

$$M_s(u) = \int_0^1 |u'|^2 + \#(S(u)) \tag{4.2}$$

defined on piecewise H^1-function, where $S(u)$ is the set of discontinuity points of u. We note the difficulty of the extension to dimension larger or equal than two, for which a characterization of minimizing movements for the Mumford-Shah functional is still lacking [3].

In the quasistatic case, our analysis relies on a modeling assumption, that amounts to considering as dissipated the energy beyond the convexity threshold. Again, we show that the Mumford-Shah energy is an approximation of F_ε also in that framework. For an analysis of the quasistatic case in dimension larger than one within its application to Fracture Mechanics we refer to [18]. As for the dynamic case, gradient-flow type evolutions for the Mumford-Shah functional have not been analyzed completely. However, under some assumptions on the initial data, it may be proved (see e.g. [21]) that in dimension one the resulting minimizing movement consists in an evolution satisfying the heat equation with Neumann boundary conditions on a fixed jump set, until the "first collision time"; i.e., until the first time when we have a decrease in the cardinality of the jump set. This can be proved to hold also for the minimizing movements for F_ε. We do not address the behaviour of this motion at and after the collision times, and a characterization of the minimizing movements for all time is largely open and beyond the scopes of this chapter.

When local minimization is taken into account, we show that indeed for some classes of problems the pattern of local minima of F_ε differ from that of M_s. The computation of the Γ-limit can nevertheless be used as a starting point for the construction of "equivalent theories", which keep the simplified form of the Γ-limit but maintain the pattern of local minima. This process has been formalized in [35]. In our case we prove the Γ-equivalence of energies of the form

$$G_\varepsilon(u) = \int_0^1 |u'|^2 dx + \sum_{x\in S(u)} \frac{1}{|\log\varepsilon|} g\left(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}|u^+ - u^-|\right) \tag{4.3}$$

with g a concave function with $g'(0) = 1$ and $g(w) \sim 2\log w$ for w large. Another case in which the Mumford-Shah functional is not a good approximation of the scaled Perona-Malik ones is for the long-time behaviour of gradient-flow dynamics, as we briefly illustrate in the final section.

The content of the chapter is organized as follows. In Section 4.2 we introduce the scaled Perona-Malik functionals F_ε and recall the result by Morini and Negri (2003)[57] describing their Γ-limit as a Mumford-Shah functional. Section 4.3 contains a description of local minima for F_ε subjected to Dirichlet boundary conditions, highlighting their dependence on the small parameter ε and the role of the convex and concave parts of the energy densities of F_ε. This analysis is then used to exhibit an "equivalent" sequence of energies defined, as the Mumford-Shah functional, on piecewise H^1-functions, in which the interfacial energy density is an explicit concave function suitably scaled. Such functionals possess the same pattern of local minima as F_ε, which is instead lost in the pattern of the local minima of the Mumford-Shah functional. In Section 4.4 we propose a "dissipation principle", formalizing the assumption that the concave part of the Perona-Malik energies corresponds to a fracture energy which cannot decrease during a variational evolution. We show that the corresponding boundary-displacement driven evolutions converge to the corresponding evolution for the Mumford-Shah energy with an increasing fracture-site assumption. In Section 4.5 we closely follow a result by Braides, Defranceschi and Vitali (2014)[29] on minimizing movements for Lennard-Jones systems, showing that the gradient-flow type dynamics of F_ε converge to the corresponding dynamics for the Mumford-Shah functional under some hypotheses on the initial data. The main technical difficulty here, with respect to [29], is that the analysis cannot be subdivided into separate computations corresponding to the convex and concave parts of the energy densities, but a finer argument by Morini and Negri must be used that allows to construct interpolations which can be treated as in [29]. In the final part of Section 4.5 we follow an observation already included in [21] to remark that the dynamical analysis cannot be carried on to long-time scalings, for which the corrected equivalent energies proposed in Section 4.3 give a better description.

4.2 The scaled Perona-Malik functional

We consider a one-dimensional system of N sites with nearest-neighbors interactions. Let $\varepsilon = 1/N$ denote the *spacing parameter* and let $u := (u_0, \dots, u_N)$

be a function defined on the lattice $I_\varepsilon = \varepsilon\mathbb{Z}\cap[0,1]$. Here and in the following we denote with $u_i = u(\varepsilon i)$. When taking ε as a parameter, we also denote $N = N_\varepsilon$.

We define the *scaled one-dimensional Perona-Malik* functional as

$$F_\varepsilon(u) := \sum_{i=1}^{N_\varepsilon} \frac{1}{|\log\varepsilon|}\log\left(1 + |\log\varepsilon|\frac{|u_i - u_{i-1}|^2}{\varepsilon}\right). \tag{4.4}$$

The behaviour of global minimum problems involving F_ε as $\varepsilon \to 0$ can be described through the computation of their Γ-limit. To that end, we define the *discrete-to-continuum convergence* $u_\varepsilon \to u$ as the L^1-convergence of the piecewise-constant interpolations $u_\varepsilon(x) = (u_\varepsilon)_{\lfloor x/\varepsilon\rfloor}$ to u.

Theorem 22 (Morini and Negri (2003)[57])**.** *The domain of the* Γ*-limit of the functionals* F_ε *as* $\varepsilon \to 0$ *is the space of piecewise-*H^1 *functions on which it coincides with the Mumford-Shah functional* M_s *defined in* (4.2)*.*

With the application of the Mumford-Shah functional to Fracture Mechanics in mind, by this result the Perona-Malik energy can be interpreted in terms of a mass-spring model approximation of Griffith brittle-fracture theory. We will then refer in what follows to the quantities $u_i - u_{i-1}$ (or w_i in the notation introduced below) as "spring elongations".

As a consequence of Theorem 22 we easily deduce that minimum problems of the form

$$\min\Big\{F_\varepsilon(u) + \alpha\sum_{i=0}^{N_\varepsilon}\varepsilon|u_i - u_i^0|^2\Big\}$$

converge (both as minimum value and minimizers are concerned) to the minimum problem

$$\min\Big\{M_s(u) + \alpha\int_0^1 |u - u^0|^2\,dx\Big\},$$

provided, e.g., that $u^0 \in L^\infty(0,1)$ is such that the interpolations $\{u_i^0\}$ converge to u^0 [57].

The heuristic explanation of why the scaling in (4.4) gives the Mumford-Shah functional is as follows. If the difference quotient $\varepsilon(u_i - u_{i-1})$ is bounded then

$$|\log\varepsilon|\frac{|u_i - u_{i-1}|^2}{\varepsilon} << 1$$

so that

$$\frac{1}{|\log\varepsilon|}\log\left(1 + |\log\varepsilon|\frac{|u_i - u_{i-1}|^2}{\varepsilon}\right) \sim \frac{|u_i - u_{i-1}|^2}{\varepsilon} = \varepsilon\Big|\frac{u_i - u_{i-1}}{\varepsilon}\Big|^2,$$

which gives a discretization of the Dirichlet integral. Conversely, if at an index i we have $|u_i - u_{i-1}|^2 \sim c > 0$ (corresponding to a jump point in the limit) then

$$\frac{1}{|\log\varepsilon|}\log\left(1 + |\log\varepsilon|\frac{|u_i - u_{i-1}|^2}{\varepsilon}\right) \sim \frac{1}{|\log\varepsilon|}\log\left(1 + |\log\varepsilon|\frac{c}{\varepsilon}\right) \sim 1.$$

The actual proof of Theorem 22 is technically complex since the analysis of the two possible behaviours of discrete functions (as Dirichlet integral or as jump

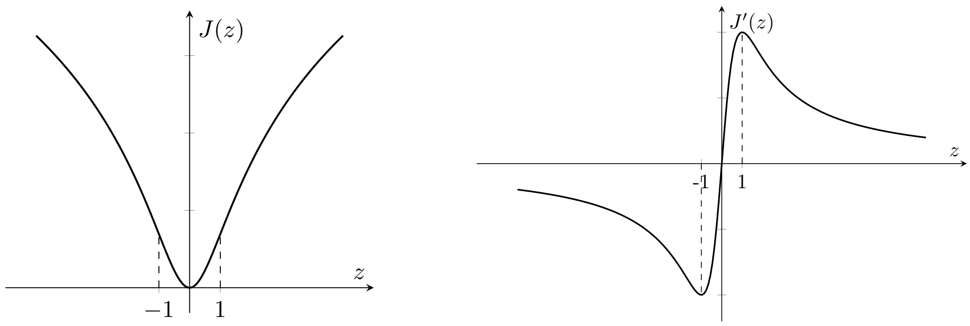


Figure 4.1: The potential J and its derivative

points) does not correspond exactly to examining the difference quotients above or below the inflection points (contrary to what can be done for the Blake-Zisserman truncated quadratic potentials considered by Chambolle (1992)[42]).

We find it convenient to rewrite (4.4) in terms of the function (see Fig. 4.1):

$$J(z) = \log(1 + z^2).$$

The energy then reads

$$F_\varepsilon(u) = \sum_{i=1}^{N_\varepsilon} \frac{1}{|\log \varepsilon|} J\Big(\sqrt{\frac{|\log \varepsilon|}{\varepsilon}}(u_i - u_{i-1})\Big). \tag{4.5}$$

Note that the function J satisfies:

- it is an even function;
- it is monotone increasing in $[0, +\infty)$ and monotone decreasing in $(-\infty, 0]$;
- it is convex in the interval $[-1, 1]$ and is concave on $[1, +\infty)$;
- there exists a constant $c > 0$ such that $J(z) \geq c\, z^2$ for all $|z| \leq 1$.

4.3 Analysis of local minima

Γ-convergence does not describe the behaviour of local minimum problems. In this section we compute energies defined on piecewise-H^1 functions which are close to F_ε in the sense of Γ-convergence and maintain the pattern of local minima. A description of local minimizers for a discrete Perona-Malik functional can also be found in [8]. We give here a different proof in which we highlight the dependence on the rescaling, which differs from that of [8] and is closer to the analysis of local minima for Lennard-Jones type interactions [26]. The analysis of the role of the convex and concave parts of the energy in this proof will be used to justify the choice of "equivalent" energy densities in the following section.

4.3.1 Local minima for F_ε with Dirichlet boundary conditions

We first characterize local minima of (4.4) with Dirichlet boundary conditions $u_0 = 0$ and $u_N = \lambda$ for $\lambda \in \mathbb{R}$.

The stationarity conditions for (4.5) read

$$J'\left(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}(u_i - u_{i-1})\right) = \sigma \tag{4.6}$$

for some $\sigma \in \mathbb{R}$.

From now on we simplify the notation introducing the scaled variable

$$w_i := \sqrt{\frac{|\log\varepsilon|}{\varepsilon}}(u_i - u_{i-1}). \tag{4.7}$$

Theorem 23. *Let F_ε be as in (4.4) and let u_ε satisfy the Dirichlet boundary conditions $u_0 = 0$ and $u_N = \lambda$ for $\lambda \in \mathbb{R}$, then u_ε is a local minimizer for F_ε if and only if one of the two following conditions holds:*

1. *all w_i are equal and $|\lambda| < \dfrac{1}{\sqrt{\varepsilon|\log\varepsilon|}}$. By (4.7) in this case $|w_i| < 1$ for every $i = 1, \dots, N$;*
2. *(4.6) is satisfied, all w_i are equal and $|w_i| < 1$ except for one, which satisfies $|w_i| > 1$, and $|\lambda| > 2\sqrt{\dfrac{1-\varepsilon}{|\log\varepsilon|}}$.*

Proof. First, we observe that we can limit our analysis to the case $w_i \geq 0$ for all $i = 1, \dots, N$. In fact, $J(z)$ is even; moreover, for $w_i \geq 0$ also $J'(w_i) \geq 0$, so that if $w_i \geq 0$ for some i then by (4.6) all the remaining w_j must be non-negative. In the same way we can handle the case with $w_i < 0$.

Now we characterize local minimizers in some different cases.

Suppose that $w_i < 1$ for all i. Then the monotonicity of J' in $[0, 1]$ implies $w_i = w_j$ for all $i, j = 1, \dots, N$. In particular, thanks to the boundary conditions, we have that

$$\frac{w_i}{\sqrt{\varepsilon|\log\varepsilon|}} = \lambda \qquad \Longrightarrow \qquad \lambda < \frac{1}{\sqrt{\varepsilon|\log\varepsilon|}}. \tag{4.8}$$

This is a local minimum. In fact, for $w_i < 1$ the function is strictly convex, which means that $J''(w_i) > 0$ for all i. We observe that when $\lambda = 0$ the only solution is the trivial one ($w_i \equiv 0$), which is a global minimum.

Not more than one index can satisfy $w_i > 1$. Indeed, suppose that there exist two indices $j \neq k$ such that $w_j = w_k (= \overline{w}) > 1$ and consider a perturbed configuration $(v_0, \dots, v_N)$ such that, denoting

$$\tilde{v}_i := (v_i - v_{i-1})\sqrt{\frac{|\log\varepsilon|}{\varepsilon}},$$

we have

- $\tilde{v}_i = w_i$ for all $i \neq j, k$;

• $\tilde{v}_j = \overline{w} + s$ and $\tilde{v}_k = \overline{w} - s$.
We observe that the difference between the energies of the configurations is

$$f(s) = J(\overline{w} + s) + J(\overline{w} - s) - 2J(\overline{w})$$

and it is such that $f'(0) = 0$ and $f''(0) = 2J''(\overline{w}) < 0$. This means that we have a local maximum in $s = 0$, so that we cannot have a local minimum for such configuration.

In case $w_i = 1$ for some i then we have $w_j = 1$ for all j, with the consequence that $\lambda = \frac{1}{\sqrt{\varepsilon |\log \varepsilon|}}$. This is not a local minimum. Indeed, consider the perturbed energy

$$f(t) = (N-1)J\left(1 - \frac{t}{N-1}\right) + J(1+t). \tag{4.9}$$

Then we observe that

$$f'(0) = 0, \qquad f''(0) = 0, \qquad f'''(0) = J'''(1)\left(1 - \frac{1}{(N-1)^2}\right) < 0,$$

so that 0 is not a minimum for f.

Finally, we take into account the case with only one index exceeding the convexity threshold. Suppose that there exists an index i such that

$$\sqrt{\frac{|\log \varepsilon|}{\varepsilon}}(u_i - u_{i-1}) = \sqrt{\frac{|\log \varepsilon|}{\varepsilon}} w > 1.$$

Then, thanks to the boundary conditions, we can rewrite the energy of the system as

$$\tilde{F}_\varepsilon(w) = (N-1)J\left(\sqrt{\frac{|\log \varepsilon|}{\varepsilon}}\left(\frac{\lambda - w}{N-1}\right)\right) + J\left(\sqrt{\frac{|\log \varepsilon|}{\varepsilon}} w\right),$$

defined on the domain

$$A = \left\{ w \in \mathbb{R}^+ : w \geq \max\left\{ \sqrt{\frac{\varepsilon}{|\log \varepsilon|}}, \lambda - (N-1)\sqrt{\frac{\varepsilon}{|\log \varepsilon|}} \right\} \right\}. \tag{4.10}$$

To compute the values of w we impose that $\tilde{F}'_\varepsilon(w) = 0$. This gives

$$J'\left(\sqrt{\frac{|\log \varepsilon|}{\varepsilon}}\left(\frac{\lambda - w}{N-1}\right)\right) - J'\left(\sqrt{\frac{|\log \varepsilon|}{\varepsilon}} w\right) = 0.$$

We then obtain three solutions: $w = \varepsilon\lambda$ (and hence $w_i \equiv w_0$, which is not a local minimum) and

$$w_{1,2} = \frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} - \frac{1 - \varepsilon}{|\log \varepsilon|}}. \tag{4.11}$$

We observe that the solutions in (4.11) are both positive for $\lambda > 2\sqrt{\frac{1-\varepsilon}{|\log\varepsilon|}}$ but we need to check for which λ they belong to A. We get that

$$w_1 = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} - \frac{1-\varepsilon}{|\log\varepsilon|}} \in A \quad \Longleftrightarrow \quad \lambda \geq 2\sqrt{\frac{1-\varepsilon}{|\log\varepsilon|}} \tag{4.12}$$

$$w_2 = \frac{\lambda}{2} - \sqrt{\frac{\lambda^2}{4} - \frac{1-\varepsilon}{|\log\varepsilon|}} \in A \quad \Longleftrightarrow \quad 2\sqrt{\frac{1-\varepsilon}{|\log\varepsilon|}} \leq \lambda \leq \frac{1}{\sqrt{|\log\varepsilon|}}. \tag{4.13}$$

Since we are interested in local minima, we have to verify that $\tilde{F}''_\varepsilon(w_i) > 0$, which means

$$\tilde{F}''_\varepsilon(w_i) = \frac{1}{(N-1)} J''\left(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\left(\frac{\lambda - w_i}{N-1}\right)\right) + J''\left(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}} w_i\right) > 0.$$

Simple computations show that

$$\tilde{F}''_\varepsilon(w_1) > 0 \Longleftrightarrow \lambda > 2\sqrt{\frac{1-\varepsilon}{|\log\varepsilon|}}$$

$$\tilde{F}''_\varepsilon(w_2) < 0$$

So, we can finally state that, when an index exceeds the convexity threshold, there exist only a local minimum for $\lambda > 2\sqrt{\frac{1-\varepsilon}{|\log\varepsilon|}}$. □

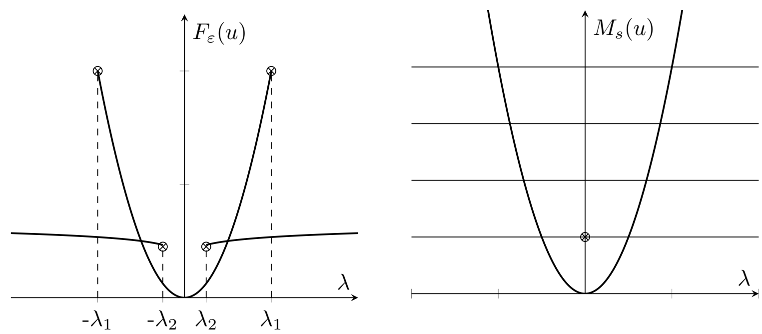


Figure 4.2: Perona-Malik and Mumford-Shah local minima. Here $\lambda_2 = 2\sqrt{\frac{1-\varepsilon}{|\log\varepsilon|}}$ and $\lambda_1 = \frac{1}{\sqrt{\varepsilon|\log\varepsilon|}}$.

Remark 9 (local and global minima for the Mumford-Shah functional). (a) We note that the pattern of local minima for the functional M_s subjected to the boundary conditions $u(0-) = 0$ and $u(1+) = \lambda$ differs from that of F_ε. Indeed, we have

- the function $u_\lambda(x) = \lambda x$, corresponding to the energy $M_s(u_\lambda) = \lambda^2$;
- all functions u with $u' = 0$, for which we have $M_s(u) = \#(S(u))$.

Note that for $\lambda = 0$ we cannot have a local minimum of the second type with $M_s(u) = 1$, while for all other λ we have no restriction on the number of jumps. A description of the energy of local minima in dependence of λ and compared with those of Perona-Malik energies is pictured in Fig. 4.2.

(b) from the analysis above we trivially have that the (global) minimum energy in dependence of the boundary datum λ is $\min\{\lambda^2, 1\}$, achieved on the linear function u_λ if $|\lambda| \leq 1$ and on any function $u(x) = \lambda\chi_{[x_0,+\infty)}$ jumping in x_0 if $|\lambda| \geq 1$.

4.3.2 Γ-equivalence

In this section we propose a "correction" to the Γ-limit of F_ε. In place of M_s we want to compute functions G_ε such that

- G_ε maintain the structure of M_s; i.e., they are defined on piecewise-H^1 functions and can be written as the sum of the Dirichlet integral and an energy defined on the jump set $S(u)$;
- the structure of local minima of G_ε is the same as that of F_ε;
- G_ε and F_ε are "equivalent" with respect to Γ-convergence.

We recall the general definition of Γ-equivalence [21].

Definition 21. Let $\{F_\varepsilon\}$ and $\{G_\varepsilon\}$ be sequences of functionals on a separable metric space X. We say that they are equivalent by Γ-convergence (or Γ-equivalent) if there exists a sequence $\{m_\varepsilon\}$ of real numbers such that, if $\left\{F_{\varepsilon_j} - m_{\varepsilon_j}\right\}$ and $\left\{G_{\varepsilon_j} - m_{\varepsilon_j}\right\}$ are Γ-converging sequences, their Γ-limits coincide and are proper (i.e., not identically $+\infty$ and not taking the value $-\infty$).

In our case this definition simplifies: we may consider $m_\varepsilon \equiv 0$ and we look for functionals which Γ-converge to M_s. We look for G_ε of the form

$$G_\varepsilon(u) = \int_0^1 |u'|^2 dt + \sum_{t \in S(u)} g_\varepsilon(|u^+ - u^-|) \tag{4.14}$$

with boundary conditions $u^-(0) = 0$, $u^+(1) = \lambda$. Furthermore, the scaling argument in the definition of F_ε suggests to look for g_ε with the same scaling of the Perona-Malik functional, so that

$$g_\varepsilon(z) = \frac{1}{|\log \varepsilon|} g\left(\sqrt{\frac{|\log \varepsilon|}{\varepsilon}}\, z\right).$$

In order to have functionals with the same structure of local minimizers as Perona-Malik's, we have to require that, if u is a minimizer, then $\#(S(u)) \leq 1$. This condition surely holds when g is concave: in fact if z_1,z_2 are two points in $S(u)$, then the function $t \to g(z_1 + t) + g(z_2 - t)$ is still concave. To ensure the Γ-convergence of G_ε to M_s we impose that

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} g\left(\sqrt{\frac{|\log \varepsilon|}{\varepsilon}} z\right) = 1.$$

This is ensured by the condition

$$\lim_{w\to+\infty}\frac{g(w)}{2\log(w)}=1.$$

Finally we require that $g(0)=0$ and that the slope of g_ε in the origin be $1/\sqrt{\varepsilon|\log\varepsilon|}$, which is the value after which it is energetically convenient to introduce a fracture (further details for this argument can be found in [26]). This means that

$$\frac{1}{|\log\varepsilon|}g'\left(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}w\right)\Bigg|_{w=0}=\frac{1}{\sqrt{\varepsilon|\log\varepsilon|}},$$

which gives $g'(0)=1$.

Summarizing, we have proved the following.

Theorem 24. *The functionals G_ε above are Γ-equivalent to F_ε and maintain the same patter of local minima, provided that*

- $g:[0,+\infty)\to[0,+\infty)$ *is concave;*
- $g(0)=0$*,* $g'(0)=1$ *and* $\lim_{w\to+\infty}\frac{g(w)}{2\log(w)}=1$*.*

4.4 Quasi-static motion

In this section we compare quasistatic motion (also sometimes denoted as variational evolution) for F_ε with that of the Mumford-Shah functional and we show that in the limit the first converge to the latter under Dirichlet boundary conditions. We adopt as the definition of quasistatic motion that of a limit of equilibrium problems involving energy and dissipation with varying boundary conditions. For more general definitions and related discussion we refer to [55, 18, 21].

We consider a sufficiently regular function $h:[0,+\infty)\to\mathbb{R}$ with $h(0)=0$ and boundary conditions $u_0=0$, $u_N=h(t)$, which means that the function h is describing the position of the endpoint of the N-th spring.

Remark 10 (quasistatic motion of the Mumford-Shah functional with increasing fracture). In the framework of Fracture Mechanics, the Dirichlet integral in the Mumford-Shah functional is interpreted as an elastic energy and the jump term as a dissipation term necessary to create a crack. The *dissipation principle* underlying crack motion is that, once a crack is created, this jump term cannot be recovered. If we apply time-dependent boundary conditions $u(0,t)=0$ and $u(1,t)=h(t)$ then a solution is given by

$$u(x,t)=\begin{cases}h(t)x & \text{if } t\le t_h\\ h(t)\chi_{[x_0,1]} & \text{if } t>t_h,\end{cases}$$

where $t_h=\inf\{t:h(t)>1\}$ and x_0 in any given point in $[0,1]$. With this definition the crack site $K(t)=\bigcup_{s\le t}S(u(\cdot,s))$ is non-decreasing with t and $u(\cdot,t)$ is a global minimizer of the Mumford-Shah energy on $(0,1)\setminus K(t)$.

In the case of the Perona-Malik functional we do not have a distinction between elastic and fracture parts of the energy. We then assume the following dissipation principle, where those two parts are identified with the convex and concave regions of the energy function, respectively.

Dissipation Principle: if for some i the spring elongation w_i in (4.7) overcomes the convexity threshold then the energy $J(w_i)$ cannot be recovered.

The above statement will be made more precise in the following section. In analogy with the case of the quasi-static motion of the Mumford-Shah functional, this principle will be translated into modifying the total energy on indices i for which w_i has overcome the convexity threshold during the evolution process.

The quasistatic motion of F_ε can be defined through a time-discrete approximation as follows.

We fix a *time step* $\tau > 0$, and for all $k \in \mathbb{N}$ we consider the boundary conditions $u_0 = 0$ and $u_N = h(k\tau)$, and the related "time-parameterized" minimum problems subjected to the Dissipation Principle stated above. With this process we obtain a discrete-time orbit, which we extend to continuous time by setting $u^\tau(t) = u_{\lfloor t/\tau \rfloor}$. The limit of such u^τ for $\tau \to 0$ defines the quasistatic motion of F_ε.

We now analyze the properties of the corresponding solutions. The analogous procedure for M_s produces the solutions for the quasistatic motion of M_s as in Remark 10.

4.4.1 Analysis of discrete quasi-static motion.

In this section we prove the following result.

Theorem 25. *Let h be a continuous piecewise-monotone function with $h(0) = 0$. Then, the quasi-static evolution for the rescaled Perona-Malik functional subject to the Dissipation Principle above and Dirichlet boundary conditions $u_0 = 0$, $u_N = h(t)$, in the limit for $\varepsilon \to 0$, gives a corresponding quasi-static evolution for the Mumford-Shah functional with increasing fracture-site condition.*

Proof. For simplicity of notation we will consider the case when $h \geq 0$.

Note preliminarily that by the convergence of the global minima of F_ε to those of M_s subjected to Dirichlet boundary conditions, we deduce the existence of a threshold $\tilde{h}_\varepsilon$ beyond which it is energetically convenient that one elongation w_i lies in the concavity domain of J. Note that $\tilde{h}_\varepsilon \to 1$ as $\varepsilon \to 0$.

We can point out different behaviours as follows

• If for all $k' < k$ we have $|h(k'\tau)| \leq \tilde{h}_\varepsilon$ then the dissipation principle is not enforced and the corresponding solution u^k is the only minimizer for the energy (4.5) corresponding to the interpolation of the linear function $h(k\tau)x$. Its energy is

$$F_\varepsilon(u^k) = \frac{1}{\varepsilon|\log\varepsilon|} J\left(h(k\tau)\sqrt{\varepsilon|\log\varepsilon|}\right). \tag{4.15}$$

• Now suppose that at some k' we have $|h(k'\tau)| > \tilde{h}_\varepsilon$. It is not restrictive to suppose that $h(k'\tau) > 0$ (the negative case being treated symmetrically). Then, there exists an index i such that the w_i corresponding to the solution exceeds $\tilde{h}_\varepsilon$. Without losing generality we will suppose $i = N$.

We have two possibilities.

1. If $h((k'+1)\tau) > h(k'\tau)$ then we have to minimize

$$(N-1)J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\Big(\frac{u_{N-1}}{N-1}\Big)\Big) + J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}(u_N - u_{N-1})\Big)$$

under the boundary conditions $u_0 = 0$, $u_N = h(k\tau)$. Note that we have used the convexity of J to simplify the contribution of the first $N-1$ interactions. Indeed, in this case their common elongation is

$$u_i - u_{i-1} = \frac{u_{N-1} - u_0}{N-1} = \frac{u_{N-1}}{N-1}.$$

The previous considerations (see Section 3) show that there exists a unique minimizer for $h(k\tau) > 2\sqrt{(1-\varepsilon)/|\log\varepsilon|}$ and, denoted

$$w := \frac{h(k\tau)}{2} + \sqrt{\frac{h(k\tau)^2}{4} - \frac{1-\varepsilon}{|\log\varepsilon|}},$$

the energy reads

$$F_\varepsilon(k) = \frac{1}{|\log\varepsilon|}\Big((N-1)J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\Big(\frac{h(k\tau)-w}{N-1}\Big)\Big) + J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}w\Big)\Big).$$

We can iterate this process as long as $k \mapsto h(k\tau)$ increases. Note that in this case the application of the Dissipation Principle does not change the minimum problems since the load on the last spring, $J(\sqrt{\frac{\varepsilon}{|\log\varepsilon|}}w)$, is increasing with k.

2. The function $k \mapsto h(\tau k)$ has a local maximum at $\bar{k}$. In this case the Dissipation Principle does force a change in the minimization problem. As long as $h(\tau k) \leq h(\tau\bar{k})$ we have to minimize

$$\begin{aligned}&(N-1)J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\Big(\frac{u_{N-1}}{N-1}\Big)\Big)+\\ &+\underbrace{\Big(J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}(u_N-u_{N-1})\Big)\vee J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}(u_N^{\bar{k}}-u_{N-1}^{\bar{k}})\Big)\Big)}_{*}\end{aligned} \tag{4.16}$$

with boundary conditions $u_0 = 0$ and $u_N = h(k\tau)$ ($k > \bar{k}$, otherwise we are in the same assumption of Step 1). Considering the part $(*)$ in (4.16) for the last term ensures that the energy spent for the elongation of the N-th spring is not reabsorbed.

We denote

$$\bar{w} := u_N^{\bar{k}} - u_{N-1}^{\bar{k}}, \qquad w_k := u_N - u_{N-1}$$
$$z_k = \frac{u_{N-1}}{N-1} = \frac{h(k\tau) - w_k}{N-1}$$

and rewrite (4.16) as

$$\min\Big\{(N-1)J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}z_k\Big)+\Big(J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}w_k\Big)\vee J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\bar{w}\Big)\Big)\Big\}. \tag{4.17}$$

In particular, we note that, when $w_k < \bar{w}$, minimizing (4.17) is equivalent to minimize the contribution of the first $N-1$ springs. Due to the convexity of the function J on the $N-1$ springs, the energy reaches the minimum value for the minimum value of z_k. Now observing that

$$z_k = \frac{h(k\tau) - w_k}{N-1} \geq \frac{h(k\tau) - \bar{w}}{N-1}$$

the minimum is reached for

$$z_k = \frac{h(k\tau) - \bar{w}}{N-1},$$

so that the energy reads

$$F_\varepsilon(k) = \begin{cases} \dfrac{1}{|\log\varepsilon|}\Big((N-1)J\Big(\sqrt{\tfrac{|\log\varepsilon|}{\varepsilon}}\Big(\tfrac{h(k\tau)-\bar{w}}{N-1}\Big)\Big) \\ \qquad\qquad\qquad\qquad + J\Big(\sqrt{\tfrac{|\log\varepsilon|}{\varepsilon}}\bar{w}\Big)\Big) & \text{if } h(k\tau) > \bar{w} \\ \dfrac{1}{|\log\varepsilon|}J\Big(\sqrt{\tfrac{|\log\varepsilon|}{\varepsilon}}\bar{w}\Big) & \text{otherwise.} \end{cases}$$

This description holds as long as $|h(k\tau)| \leq h(\bar{k}\tau)$, after which we return to case 1 above. □

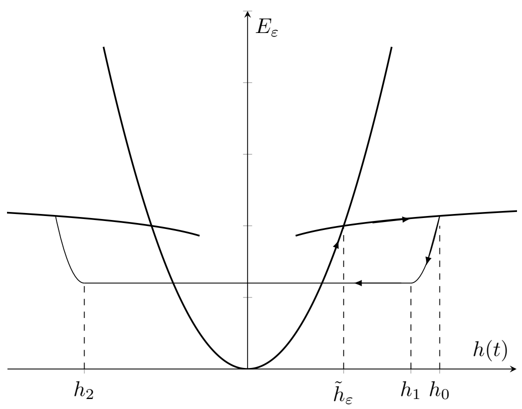


Figure 4.3: Schematic representation of the energy of a quasistatic evolution as described in Remark 11, where we denote $h_i = h(t_i)$

Remark 11 (Comparison with the Mumford-Shah quasistatic motion). We only treat a case with piecewise-constant h, the general case being reduced to that one by a reparameterization.

We can test the quasistatic behaviour of the Perona-Malik system at fixed ε with $h(t) = t_0 - |t - t_0|$, and $t_0 > 1$, so that $t_0 > \tilde{h}_\varepsilon$ for ε small enough. If $E^k_{\tau,\varepsilon}$

denotes the minimal energy at fixed ε and τ, by the description above we have

$$E^k_{\tau,\varepsilon} = \begin{cases} \frac{1}{\varepsilon|\log\varepsilon|} J\left(h(k\tau)\sqrt{\varepsilon|\log\varepsilon|}\right) & \text{if } k\tau \le \tilde{h}_\varepsilon \\ \frac{1}{|\log\varepsilon|}\Big((N-1)J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\Big(\frac{h(k\tau)-w}{N-1}\Big)\Big) & \\ \qquad + J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}w\Big)\Big) & \text{if } \tilde{h}_\varepsilon < k\tau \le t_0 \\ \frac{1}{|\log\varepsilon|}\Big((N-1)J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\Big(\frac{h(k\tau)-\bar{w}}{N-1}\Big)\Big) & \\ \qquad + J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\bar{w}\Big)\Big) & \text{if } t_0 < k\tau \le t_1 \\ \frac{1}{|\log\varepsilon|} J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\bar{w}\Big) & \text{if } t_1 < k\tau \le t_2 \\ \frac{1}{|\log\varepsilon|}\Big((N-1)J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\Big(\frac{h(k\tau)+\bar{w}}{N-1}\Big)\Big) & \\ \qquad + J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\bar{w}\Big)\Big) & \text{if } k\tau > t_2\,, \end{cases}$$

where $t_1, t_2 > t_0$ satisfy $h(t_1) = \bar{w}$ and $h(t_2) = -\bar{w}$. The values $E^k_{\tau,\varepsilon}$ in dependence of $h(t)$ lie in the curves pictured in Fig. 4.3.

As $\tau \to 0$ and $\varepsilon \to 0$ the piecewise-constant functions defined by $E_{\tau,\varepsilon}(t) = E^{\lfloor t/\tau \rfloor}_{\tau,\varepsilon}$ converge to E given by

$$E(t) = \begin{cases} |h(t)|^2 & \text{if } t \le 1 \\ 1 & \text{if } t > 1. \end{cases} \tag{4.18}$$

In fact for $k\tau \le \tilde{h}_\varepsilon$ the energy $E^k_{\tau,\varepsilon}$ reads

$$\begin{aligned} \frac{1}{\varepsilon|\log\varepsilon|} J\left(h(k\tau)\sqrt{\varepsilon|\log\varepsilon|}\right) &= \frac{1}{\varepsilon|\log\varepsilon|}\log\left(1 + h^2(k\tau)\varepsilon|\log\varepsilon|\right) \\ &\sim \frac{1}{\varepsilon|\log\varepsilon|} h^2(k\tau)\varepsilon|\log\varepsilon| \to h^2(t). \end{aligned}$$

For $k\tau > \tilde{h}_\varepsilon$ the contribution of the $N-1$ springs in the convex part vanishes. We make the computation only for one contribution, the others being analogous:

$$\begin{aligned} & \frac{1}{|\log\varepsilon|}(N-1)J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\Big(\frac{h(k\tau)-w}{N-1}\Big)\Big) \\ = \; & \frac{1-\varepsilon}{\varepsilon|\log\varepsilon|}\log\left(1 + \frac{|\log\varepsilon|}{\varepsilon}\left(\frac{\varepsilon}{1-\varepsilon}\right)^2 (h(k\tau)-w)^2\right) \\ \sim \; & \frac{1}{1-\varepsilon}(h(k\tau)-w)^2 \\ = \; & \frac{1}{1-\varepsilon}\left(\frac{h^2(k\tau)}{2} - \frac{1-\varepsilon}{|\log\varepsilon|} - h(k\tau)\sqrt{\frac{h^2(k\tau)}{4} - \frac{1-\varepsilon}{|\log\varepsilon|}}\right) \to 0. \end{aligned}$$

Finally, the spring in the non-convex part gives a constant contribution:

$$\begin{aligned}\frac{1}{|\log\varepsilon|}J\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}w\Big) &= \frac{1}{|\log\varepsilon|}\log\left(1+\frac{|\log\varepsilon|}{\varepsilon}w^2\right)\\ &\sim \frac{1}{|\log\varepsilon|}\left(\log\left(\frac{|\log\varepsilon|}{\varepsilon}\right)+\log\left(w^2\right)\right)\\ &= \frac{1}{|\log\varepsilon|}\left(\log(|\log\varepsilon|)+|\log(\varepsilon)|+\log\left(w^2\right)\right)\to 1.\end{aligned}$$

The energy $E(t)$ corresponds to the energy of the quasistatic motion of the Mumford-Shah functional with increasing fracture.

4.5 Dynamic analysis

We will make use of the method of *minimizing movements* along the sequence of functionals F_ε at a scale $\tau = \tau_\varepsilon \to 0$ [21]. With varying τ, minimizing movements describe the possible gradient-flow type evolutions along F_ε. When $\varepsilon \to 0$ fast enough with respect to τ then we obtain a minimizing movement for the Γ-limit of F_ε; i.e., in our case for the Mumford-Shah functional (for further properties of minimizing movements for a single energy we refer to [5]). In analogy with the result of Braides *et al.* (2014)[29] we will prove that the restriction that $\varepsilon \to 0$ fast enough may be removed, so that we may regard the Mumford-Shah functional as a dynamic approximation for F_ε.

The computation of the minimizing movements of the Mumford-Shah functional starting from arbitrary sequences of initial data has not been carried out, to our knowledge. However, if the initial datum is a fixed piecewise-H^1 function u_0 on $(0,1)$ with discontinuity set S_0, then the minimizing movements are characterized as solving the heat equation with Neumann boundary conditions and initial datum u_0 in each interval of $(0,1)\setminus S_0$, until the first time T_0 when solutions of two neighbouring intervals have the same value at the common endpoint (*first collision time*). A short proof under some simplifying assumptions can be found in [21]. In the following theorem we will consider such a minimizing movement, which is the analog of that considered in [29]. Since we consider discrete initial data u^0_ε varying with ε we define their "discrete jump set" as the subset of $\varepsilon\mathbb{Z}$ indexed by

$$I^j_\varepsilon(u^0_\varepsilon) = \Big\{ i\in\mathbb{Z},\, 0\le i\le \frac{1}{\varepsilon}-1 : \frac{|(u^0_\varepsilon)_{i+1}-(u^0_\varepsilon)_i|}{\varepsilon} > \frac{1}{\sqrt{\varepsilon|\log\varepsilon|}} \Big\}, \qquad (4.19)$$

and make the assumption that this set converges to the jump set of the initial datum.

Theorem 26. *Let u^0_ε be initial data that satisfy*

$$\sup\{|(u^0_\varepsilon)_i| \ : 0\le i\le N,\ \varepsilon>0\}<\infty. \qquad (4.20)$$

$$F_\varepsilon(u^0_\varepsilon)\le M \quad \text{for some } M>0 \quad \text{and for every } \varepsilon>0, \qquad (4.21)$$

and converge to a piecewise-H^1 function u_0. Furthermore, we suppose that

$$S := \Big\{x\in\mathbb{R} : x=\lim_{\varepsilon\to 0}\varepsilon i_\varepsilon \text{ for some } i_\varepsilon\in I^j_\varepsilon(u^0_\varepsilon)\Big\} = S(u^0). \qquad (4.22)$$

Let u_ε be a minimizing movement for the scaled Perona-Malik functional F_ε defined in (4.5) at scale $\tau = \tau_\varepsilon$, for which we assume the technical hypothesis $4\tau_\varepsilon < \varepsilon^2$, and with initial datum u^0_ε. Then, u_ε converges in $L^\infty((0,T_0);L^2(0,1))$ as $\varepsilon \to 0$ to a minimizing movement of the Mumford-Shah functional with initial datum u_0, where $T_0 > 0$ is the first collision time for the Mumford-Shah evolution as defined above.

The theorem will be obtained as a consequence of a series of propositions in the next sections. The line of proof closely follows that in [29], where the properties of the evolving jump set and the characterization of the motion as the heat equation are obtained, respectively, by examining the Euler equations when difference quotients are above or below the inflection points of the energy density. Here the proof is more complex since inflection points do not characterize the Γ-limit and a technical modification of minimum points is needed following a construction by Morini and Negri (see the proof of Lemma 5 below).

In the following it is useful to express (4.5) in the following way: we define $f_\varepsilon : \mathbb{R} \to \mathbb{R}^+$ by

$$f_\varepsilon(u) = \frac{1}{\varepsilon|\log\varepsilon|} J\Big(\sqrt{\varepsilon|\log\varepsilon|}u\Big)$$

and rewrite $F_\varepsilon(u)$ as

$$F_\varepsilon(u) = \sum_{i=1}^{\mathbb{N}_\varepsilon} \varepsilon\, f_\varepsilon\Big(\frac{u_i - u_{i-1}}{\varepsilon}\Big). \tag{4.23}$$

Remark 12. If $u : I_\varepsilon \to \mathbb{R}$ then, with a little abuse of notation, we will denote with u also the piecewise-constant extension defined by $u(x) = u_{\lfloor x/\varepsilon \rfloor}$.

4.5.1 A compactness results

The paper [29] analyzes the dynamic behaviour for functionals similar to (4.23), up to a scaling factor, but with $\widetilde{J}(z) = \min\{z^2, 1\}$ (Blake and Zisserman potential), which has a convex-concave form similar to the Perona-Malik potential. A crucial argument in that paper is an observation by Chambolle (1992)[42] which allows to identify each function u with a function v such that $F_\varepsilon(u) = M_s(v)$ and v is ε-close in L^1-norm to u. In this way the coerciveness properties for the Mumford-Shah functional imply compactness properties for sequences with equibounded energy.

The Chambolle argument simply identifies indices i such that $(u_i - u_{i-1})/\varepsilon$ is not in the 'convexity region' for the corresponding f_ε with jump points of v. This argument is not possible in our case. Indeed, let $\varepsilon_n \to 0$ be a vanishing sequence of indices and let $u_n : I_{\varepsilon_n} \to \mathbb{R}$ be such that

- $\{u_n\}$ is a sequence of equibounded functions;
- $\sup_n F_{\varepsilon_n}(u_n) < M$, $M > 0$ a constant.

We define the set

$$I^j_{\varepsilon_n}(u_n) = \Big\{ i \in \mathbb{Z},\, 0 \le i \le N_n - 1 : \frac{|(u_n)_{i+1} - (u_n)_i|}{\varepsilon_n} > \frac{1}{\sqrt{\varepsilon_n|\log\varepsilon_n|}} \Big\},$$

when there is no possible confusion we will simply denote $I_n^j = I_{\varepsilon_n}^j(u_n)$, and we consider the Chambolle interpolation

$$w_n(x) := \begin{cases} (u_n)_i & \text{if } i := \lfloor x/\varepsilon_n \rfloor \in I_n^j \quad \text{or} \quad i = N_n \\ (1-\lambda)(u_n)_i + \lambda (u_n)_{i+1} & \text{otherwise} \quad (\lambda := x/\varepsilon_n - \lfloor x/\varepsilon_n \rfloor)\,. \end{cases} \tag{4.24}$$

Then the set of jump points of $w_n(x)$ may not be bounded as $n \to \infty$, since the only a priori estimate we may have is

$$\begin{aligned} M \quad &\geq \quad F_{\varepsilon_n}(u_n) \geq \sum_{i \in I_n^j} \varepsilon_n f_{\varepsilon_n}\Big(\frac{|(u_n)_{i+1} - (u_n)_i|}{\varepsilon_n}\Big) \\ &\geq \quad \sum_{i \in I_n^j} \varepsilon_n f_{\varepsilon_n}\Big(\frac{1}{\sqrt{\varepsilon_n |\log \varepsilon_n|}}\Big) \geq \frac{\log(2)}{|\log \varepsilon_n|} \#\,(I_n^j), \end{aligned} \tag{4.25}$$

so that

$$\#\,(S(w_n)) \leq \#\,(I_n^j) \leq C |\log \varepsilon_n|. \tag{4.26}$$

In order to avoid this obstacle we need to modify the previous sequence. To that end it is useful to briefly recall a result due to Morini and Negri (2003)[57].

Lemma 4. *Let $p(\varepsilon) > 0$ be such that $\lim_{\varepsilon \to 0^+} p(\varepsilon) = 0$ and*

$$\lim_{\varepsilon \to 0^+} \Big(p(\varepsilon) |\log(\varepsilon)| - \log(|\log \varepsilon|) \Big) = +\infty,$$

let $c_\varepsilon := \varepsilon^{p(\varepsilon)}$, then it holds that

$$\lim_{\varepsilon \to 0^+} c_\varepsilon |\log(\varepsilon)| = 0 \tag{4.27}$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\log(\varepsilon)|} J\Big(\sqrt{\frac{|\log \varepsilon|}{\varepsilon}} c_\varepsilon \Big) = 1. \tag{4.28}$$

We are now able to prove the following

Lemma 5. *Let $\varepsilon_n \to 0$ be a sequence of vanishing indices, $\{u_n\}$ be an equi-bounded sequence of functions $u_n : I_{\varepsilon_n} \to \mathbb{R}$ such that $\sup_n F_{\varepsilon_n}(u_n) \leq M$ for a constant $M > 0$. Let w_n be as in (4.24), then, up to a subsequence, there exists a function $u \in SBV([0,1])$ such that $w_n \to u$ and $w_n' \rightharpoonup u'$ in $L^2(0,1)$.*

Proof. Step 1: we modify the sequence $\{u_n\}$.
We define $b_\varepsilon := (\varepsilon |\log \varepsilon|)^{1/4}$. Denoted $b_n = b_{\varepsilon_n}$ and $c_n = c_{\varepsilon_n}$, we define the following sets

$$\begin{aligned} I_{\varepsilon_n}^1(u_n) :&= \Big\{ i \in \mathbb{Z},\ 0 \leq i \leq N_n - 1 : \frac{b_n}{\sqrt{\varepsilon_n |\log \varepsilon_n|}} \leq \frac{|(u_n)_{i+1} - (u_n)_i|}{\varepsilon_n} \leq \frac{c_n}{\varepsilon_n} \Big\} \\ &= \{i_1, \ldots, i_{m_n}\}, \end{aligned}$$

with $i_k < i_{k+1}$, where $m_n := \#(I_{\varepsilon_n}^1(u_n))$, and we denote the corresponding points on the lattice as $\{x_n^1, \ldots, x_n^{m_n}\} = \{\varepsilon_n i_1, \ldots, \varepsilon_n i_{m_n}\}$

The sequence $\{u_n\}$ may be modified into a sequence $\{\tilde{u}_n\}$ such that

1. $I^1_{\varepsilon_n}(\tilde{u}_n)$ is empty;
2. $\|\tilde{u}_n - u_n\|_1 \to 0$;
3. $F_{\varepsilon_n}(\tilde{u}_n) \le F_{\varepsilon_n}(u_n)$.

To that end we define by induction the following sequence

$$v_n^0 \equiv u_n$$
$$v_n^{k+1}(t) := \begin{cases} v_n^k(t) & \text{if } t < x_n^{k+1} + \varepsilon_n \\ v_n^k(t) - [v_n^k(x_n^{k+1} + \varepsilon_n) - v_n^k(x_n^{k+1})] & \text{if } t \ge x_n^{k+1} + \varepsilon_n \end{cases}$$

for $k = 0, \dots, m_n - 1$, and then we set $\tilde{u}_n := v_n^{m_n}$. This sequence satisfies all our requests (see [57]).

Moreover it is worth noting that $(\tilde{u}_n)_{i+1} - (\tilde{u}_n)_i = (u_n)_{i+1} - (u_n)_i$ for indices in $I_{\varepsilon_n} \backslash I^1_{\varepsilon_n}(u_n)$.

Step 2: We define a new interpolation.
We now consider the following sets

$$I^2_{\varepsilon_n} := \Big\{ i \in \mathbb{Z},\ 0 \le i \le N_n - 1 : \frac{|(\tilde{u}_n)_{i+1} - (\tilde{u}_n)_i|}{\varepsilon_n} \le \frac{b_n}{\sqrt{\varepsilon_n |\log \varepsilon_n|}} \Big\},$$
$$I^3_{\varepsilon_n} := \Big\{ i \in \mathbb{Z},\ 0 \le i \le N_n - 1 : \frac{|(\tilde{u}_n)_{i+1} - (\tilde{u}_n)_i|}{\varepsilon_n} \ge \frac{c_n}{\varepsilon_n} \Big\},$$

and the extension $\tilde{w}_n$ of the function $\tilde{u}_n$ on $[0,1]$ such that $\tilde{w}_n$ is the affine interpolation of $\tilde{u}_n$ on $I^2_{\varepsilon_n}$ and it is piecewise-constant on $I^3_{\varepsilon_n}$.

$$\tilde{w}_n(x) := \begin{cases} (\tilde{u}_n)_i & \text{if } i := \lfloor x/\varepsilon_n \rfloor \in I^3_{\varepsilon_n} \quad \text{or} \quad i = N_n \\ (1-\lambda)(\tilde{u}_n)_i + \lambda (\tilde{u}_n)_{i+1} & \text{otherwise} \quad (\lambda := x/\varepsilon_n - \lfloor x/\varepsilon_n \rfloor)\,. \end{cases} \tag{4.29}$$

We remark that $I_{\varepsilon_n} = I^2_{\varepsilon_n} \cup I^3_{\varepsilon_n}$, so that $\tilde{w}_n$ is defined for all x.

We note that $\tilde{w}_n$ still converge to u_n in L^1. Moreover it can be proved (see [57]) that for a fixed $\delta << 1$ there exists an $\bar{\varepsilon}$ such that for $\varepsilon_n \le \bar{\varepsilon}$ it holds

$$F_{\varepsilon_n}(u_n) \ge (1-\delta)\Big(\int_0^1 |\tilde{w}_n'|^2\, dx + \mathcal{H}^0(S(\tilde{w}_n)) \Big), \tag{4.30}$$

where $S(\tilde{w}_n)$ is the set of jump points of $\tilde{w}_n$.

Collecting (4.29) and (4.30), and using the coerciveness properties of the Mumford-Shah energy [4], we have that, up to a subsequence, there exists a function $u \in SBV([0,1])$ such that $\tilde{w}_n \to u$, $\tilde{w}_n' \rightharpoonup u'$ in $L^2(0,1)$. Moreover, $D^j \tilde{w}_n \rightharpoonup D^j u$ weakly-* in the sense of measures.

Step 3: We extend the convergence to $\{w_n\}$.
In fact the previous results for $\{\tilde{w}_n\}$ imply that the sequence $\{u_n\}$ converges to u in $L^2(0,1)$. Indeed, $\{u_n\}$ are equibounded and $\|\tilde{w}_n - u_n\|_1 \to 0$ for construction, so that there exists a subsequence in $L^\infty(0,1)$ which converges to u a.e. The result follows now from Lebesgue's dominated convergence theorem.

Now, the L^2-convergence of u_n implies the L^2-convergence of w_n defined in (4.24). Indeed, recalling Remark 12 it holds that for every $x \in [0,1]$

$$|w_n(x) - u_n(x)| \le \sqrt{\frac{\varepsilon_n}{|\log \varepsilon_n|}}. \tag{4.31}$$

Moreover a simple computation as follows shows that $\{w_n'\}$ is equibounded in $L^2(0,1)$:

$$\begin{aligned} M &> F_{\varepsilon_n}(u_n) \geq \sum_{i \notin I_n^j(w_n)} \varepsilon_n f_{\varepsilon_n}\Big(\frac{(u_n)_{i+1}-(u_n)_i}{\varepsilon_n}\Big) \\ &= \sum_{i \notin I_n^j(w_n)} \frac{1}{|\log \varepsilon_n|} J\left(\sqrt{\frac{|\log \varepsilon_n|}{\varepsilon_n}}\,((u_n)_{i+1}-(u_n)_i)\right) \\ &\geq \sum_{i \notin I_n^j(w_n)} \varepsilon_n \Big(\frac{(u_n)_{i+1}-(u_n)_i}{\varepsilon_n}\Big)^2 \geq \int_0^1 |w_n'|^2(x)\,dx. \qquad (4.32) \end{aligned}$$

Hence, there exists a subsequence weakly converging in $L^2(0,1)$. By an integration by parts argument, up to subsequence, we have $w_n' \rightharpoonup u'$ in $L^2(0,1)$. □

The behaviour of points which are above the convexity threshold is of particular interest and it is described in the following lemma.

Lemma 6. *Let $\{u_n\}$ be as in Lemma 5 and w_n as in (4.24), then, up to subsequence, for every $\bar{x} \in S(u)$ there exists a sequence $\{x^n\}$ converging to $\bar{x}$ such that*

$$x^n \in S(w_n) \quad \textit{and} \quad \lim_{n\to+\infty} |w_n^+(x^n) - w_n^-(x^n)| > \gamma > 0. \qquad (4.33)$$

Proof. We observe that $\{\tilde{w}_n\}$ satisfies the hypotheses of Lemma 2.4 in [29] which is an analogous of Lemma 6: indeed in [29] Lemma 6 is proved for functions which satisfy a Mumford-Shah estimate, as the $\{\tilde{w}_n\}$ in (4.30). We obtain then that for every $\bar{x} \in S(u)$ there exists a sequence $\{x^n\}$ converging to $\bar{x}$ such that

$$x^n \in S(\tilde{w}_n) \quad \text{and} \quad \lim_{n\to+\infty} |\tilde{w}_n^+(x^n) - \tilde{w}_n^-(x^n)| > \gamma > 0. \qquad (4.34)$$

Lemma 6 is now proved once we observe that $S(\tilde{w}_n) \subseteq S(w_n)$ and that for those points $w_n^+(x^n) - w_n^-(x^n) = \tilde{w}_n^+(x^n) - \tilde{w}_n^-(x^n)$. □

Remark 13. For points in $S(w_n)\backslash S(\tilde{w}_n)$ it holds that

$$\begin{aligned} &\lim_{n\to+\infty} \sum_{x \in S(w_n)\backslash S(\tilde{w}_n)} |w_n^+(x^n) - w_n^-(x^n)| \leq \lim_{n\to+\infty} c_n \#\left(S(w_n)\backslash S(\tilde{w}_n)\right) \\ &\leq \lim_{n\to+\infty} c_n \#\left(I_n^j\right) \leq \lim_{n\to+\infty} K c_n |\log \varepsilon_n| = 0, \end{aligned}$$

where in the last inequality we use (4.25).

4.5.2 Minimizing Movements

With fixed ε and $\tau = \tau_\varepsilon$, from an initial state $u_0^\varepsilon : I_\varepsilon \to \mathbb{R}$, we define the sequence $u^k := u^k_{\varepsilon,\tau}$ such that u^k is a minimizer of

$$v \to F_\varepsilon(v) + \frac{1}{2\tau} \sum_{i=0}^{N_\varepsilon} \varepsilon |v_i - u_i^{k-1}|^2 \quad \forall\, v : I_\varepsilon \to \mathbb{R}. \qquad (4.35)$$

We define the piecewise-constant extension $u_{\varepsilon,\tau} : [0,1] \times [0,+\infty) \to \mathbb{R}$ as

$$u_{\varepsilon,\tau}(x,t) = (u^k_{\varepsilon,\tau})_i \quad \text{with } k = \lfloor t/\tau \rfloor \quad \text{and} \quad i = \lfloor x/\varepsilon \rfloor, \tag{4.36}$$

and take the limit (upon extraction of a subsequence) for both parameters $\varepsilon \to 0$ and $\tau \to 0$. A limit u is called a *minimizing movement along F_ε at scale $\tau = \tau_\varepsilon$*. Observe that in general the limit will depend on the choice of ε and τ.

We now state some properties of minimizing movements along a sequence (see [21], [5]).

Proposition 13. *Let F_ε be as in (4.23) and u^k be defined as above. Then for every $k \in \mathbb{N}$ it holds that*

$$\begin{aligned}
&1) \quad F_\varepsilon(u^k) \leq F_\varepsilon(u^{k-1});\\
&2) \quad \sum_{i=0}^{N_\varepsilon} \varepsilon |u^k_i - u^{k-1}_i|^2 \leq 2\tau[F_\varepsilon(u^{k-1}) - F_\varepsilon(u^k)];\\
&3) \quad \|u^k\|_\infty \leq \|u^{k-1}\|_\infty \leq \|u^0_\varepsilon\|_\infty.
\end{aligned}$$

From (4.23) and (4.35) we obtain the following optimality conditions.

Proposition 14. *Let $\{u^k\}_k$ be a sequence of minimizer of (4.35). Then we have*

$$-f'_\varepsilon\Big(\frac{u^k_1 - u^k_0}{\varepsilon}\Big) + \frac{\varepsilon}{\tau}(u^k_0 - u^{k-1}_0) = 0,$$

$$f'_\varepsilon\Big(\frac{u^k_i - u^k_{i-1}}{\varepsilon}\Big) - f'_\varepsilon\Big(\frac{u^k_{i+1} - u^k_i}{\varepsilon}\Big) + \frac{\varepsilon}{\tau}(u^k_i - u^{k-1}_i) = 0,$$

$$f'_\varepsilon\Big(\frac{u^k_N - u^k_{N-1}}{\varepsilon}\Big) + \frac{\varepsilon}{\tau}(u^k_N - u^{k-1}_N) = 0.$$

Under hypotheses (4.20) and (4.21) it is possible to prove the following result.

Theorem 27. *Let $\{\varepsilon_n\}, \{\tau_n\} \to 0$. Let $v_n = u_{\varepsilon_n,\tau_n}$ be defined as in (4.36) and consider $\tilde{v}_n$ its extension by linear interpolation as in (4.24). Then there exist a subsequence of $\{v_n\}$ and a function $u \in C^{1/2}([0,+\infty]; L^2(0,1))$ such that*

1) $v_n \to u$, $\tilde{v}_n \to u$ in $L^\infty([0,T]; L^2(0,1))$ and a.e. in $(0,1) \times (0,T)$ for every $T \geq 0$;

2) for all $t \geq 0$ the function $u(\cdot,t)$ is piecewise-$H^1(0,1)$ and $(\tilde{v}_n)_x(\cdot,t) \rightharpoonup u_x(\cdot,t)$ in $L^2(0,1)$;

3) for every $\bar{x} \in S(u(\cdot,t))$ there exist a subsequence $\{v_{n_h}\}$ (which can also depend on t) and a sequence $(x^h)_h$ converging to $\bar{x}$ such that $x^h \in S(\tilde{v}_{n_h})$.

Proof. We will only give a brief sketch of the proof since it follows strictly the one in [29].

For fixed $t \geq 0$ the equiboundedness of initial data guarantees that also $F_{\varepsilon_n}(v_n(\cdot,t))$ is bounded, so that we are in the hypotheses of Lemma 5, that can be applied with $u_n = v_n(\cdot,t)$. Recalling (4.31), this shows that up to subsequences, $\tilde{v}_n(\cdot,t)$ is converging in $L^2(0,1)$ to a piecewise-$H^1(0,1)$ function $u(\cdot,t)$ and also $(\tilde{v}_n)_x(\cdot,t)$ is weakly converging in $L^2(0,1)$ to $u_x(\cdot,t)$. Now, from of Proposition 13(2) and the Cauchy-Schwarz inequality (see for example [21], [29]), we get

$$\|v_n(\cdot,t) - v_n(\cdot,s)\|_2 \le C\sqrt{t-s-\tau_n} \tag{4.37}$$

that in the limit becomes

$$\|u(\cdot,t) - u(\cdot,s)\|_2 \le C\sqrt{t-s} \tag{4.38}$$

with C independent from both t and s. So that the limit function u belongs to $C^{1/2}([0,+\infty];L^2(0,1))$.

We prove the convergence in $L^\infty([0,T];L^2(0,1))$: for $T>0$ fixed, consider $M\in\mathbb{N}$ and $t_j = jT/M$ for $j=0,\dots,M$. Then for every $t\in[0,T]$ there exists a $j=0,\dots,M$ such that $t_{j-1}<t<t_j$, so we have that

$$\begin{aligned}\|v_n(\cdot,t)-u(\cdot,t)\|_2 &\le \|v_n(\cdot,t)-v_n(\cdot,t_{j-1})\|_2 + \|v_n(\cdot,t_{j-1}) - u(\cdot,t_{j-1})\|_2\\ &\quad + \|u(\cdot,t_{j-1}) - u(\cdot,t)\|_2\\ &\le 2C\sqrt{t-t_j+\tau_n} + \|v_n(\cdot,t_{j-1}) - u(\cdot,t_{j-1})\|_2. \end{aligned}\tag{4.39}$$

Since $v_n(\cdot,t)$ is a converging sequence to $u(\cdot,t)$, for $\overline{n} >> 1$ it is possible to find an $\eta << 1$ such that $\|v_n(\cdot,t_{j-1}) - u(\cdot,t_{j-1})\|_2 \le \eta$ for all $n\ge\overline{n}$. Finally, we have that

$$\sup_{t\in[0,T]} \|v_n(\cdot,t)-u(\cdot,t)\|_2 \le 2C\sqrt{(T/M)+\tau_n} + \eta$$

for all $n\ge\overline{n}$, which means

$$\limsup_{n\to+\infty}\sup_{t\in[0,T]} \|v_n(\cdot,t)-u(\cdot,t)\|_2 \le 2C\sqrt{(T/M)+\tau_n} + \eta$$

and the claims now follows from the arbitrariness of M and η.

We conclude observing that claim 3 follows from Lemma 6. □

4.5.3 Computation of the limit equation

Consider now two sequences of indices $\{\varepsilon_n\}\to 0$, $\{\tau_n\}\to 0$ (to simplify the notation from now on we will write ε instead of ε_n and similarly τ instead of τ_n). We define the function

$$\phi_n(x,t) := f'_\varepsilon\Big(\frac{(u^k_{\varepsilon,\tau})_{i+1} - (u^k_{\varepsilon,\tau})_i}{\varepsilon}\Big) \quad \text{if } i = \lfloor x/\varepsilon\rfloor \text{ and } k=\lfloor t/\tau\rfloor. \tag{4.40}$$

Proposition 15. *If ϕ_n is defined in (4.40), then for every $t\ge 0$ we have $\phi_n(\cdot,t)\rightharpoonup 2u_x(\cdot,t)$ in $L^2(0,1)$.*

Moreover, for every $T>0$ the sequence $\{\phi_n(\cdot,t)\}$ is uniformly bounded in $L^2(0,1)$ with respect to $t\in[0,T]$ and $u_x\in L^2((0,1)\times(0,T))$.

Proof. Let $t\ge 0$ fixed, $v_n := v_{\varepsilon_n}$ and $\tilde{v}_n := \tilde{v}_{\varepsilon_n}$ be as defined in Theorem 27. Consider the function

$$\chi_n(x) = \begin{cases} 1 & \text{if } \quad x\in\bigcup_{i\in I^j_\varepsilon(v_n(\cdot,t))}\varepsilon[i,i+1)\\ 0 & \text{otherwise}\end{cases}$$

and the decomposition $\phi_n(\cdot,t) = \chi_n\phi_n(\cdot,t) + (1-\chi_n)\phi_n(\cdot,t)$. From (4.25) and Proposition 13 we get that

$$\int_0^1 |\chi_n\phi_n(x,t)|^2 dx = \sum_{i\in I_\varepsilon^j} \varepsilon|\phi_n(i\varepsilon,t)|^2 \leq \varepsilon\#\,(I_\varepsilon^j) f_\varepsilon'\Big(\frac{1}{\sqrt{\varepsilon|\log\varepsilon|}}\Big)^2 \leq M,$$

which means that the sequence is L^2-bounded. Moreover,

$$\int_0^1 |\chi_n\phi_n(x,t)|dx \leq \varepsilon\#\,(I_\varepsilon^j) f_\varepsilon'\Big(\frac{1}{\sqrt{\varepsilon|\log\varepsilon|}}\Big) \leq M\sqrt{\varepsilon|\log\varepsilon|} \to 0. \tag{4.41}$$

This proves that $\chi_n\phi_n(x,t) \rightharpoonup 0$ in $L^2(0,1)$.

We now obtain a similar result for $(1-\chi_n)\phi_n(\cdot,t)$. First at all we observe that

$$(\tilde{v}_n)_x(x,t) = \begin{cases} \dfrac{(u_{\varepsilon,\tau}^k)_{i+1} - (u_{\varepsilon,\tau}^k)_i}{\varepsilon} & x\in[i\varepsilon,(i+1)\varepsilon) \text{ and } i\notin I_\varepsilon^j \\ 0 & \text{otherwise} \end{cases} \tag{4.42}$$

This means that $(1-\chi_n)\phi_n(x,t) = f_\varepsilon'((\tilde{v}_n)_x(x,t))$. Using a Taylor expansion of f_ε' in a neighbourhood of the origin we get

$$f_\varepsilon'(\tilde{v}_n)_x(x,t)) = f_\varepsilon'(0) + f_\varepsilon''(0)(\tilde{v}_n)_x(x,t) + \frac{1}{2}f_\varepsilon'''(\xi_n)((\tilde{v}_n)_x(x,t))^2$$

for some $\xi_n \in [0,(\tilde{v}_n)_x(x,t)]$, so that

$$f_\varepsilon'((\tilde{v}_n)_x(x,t)) = 2(\tilde{v}_n)_x(x,t) + \frac{1}{2}\sqrt{\varepsilon|\log\varepsilon|}J'''(\sqrt{\varepsilon|\log\varepsilon|}\xi_n)((\tilde{v}_n)_x(x,t))^2. \tag{4.43}$$

Moreover, recalling (4.42), we have

$$-\frac{1}{\sqrt{\varepsilon|\log\varepsilon|}} \leq (\tilde{v}_n)_x(x,t) \leq \frac{1}{\sqrt{\varepsilon|\log\varepsilon|}},$$

so that the sequence $\sqrt{\varepsilon|\log\varepsilon|}(\tilde{v}_n)_x(x,t)$ is bounded, as is $J'''(\sqrt{\varepsilon|\log\varepsilon|}\xi_n)$. From this, it follows that there exists a constant $C>0$ such that

$$|f_\varepsilon'((\tilde{v}_n)_x(x,t))| \leq C|(\tilde{v}_n)_x(x,t)|.$$

Now, fix $T>0$ and $t\in[0,T]$. The estimate in (4.32) is easily adapted to show that $(\tilde{v}_n)_x(x,t)$ is bounded in $L^2(0,1)$ and from the above inequality this implies that also $f_\varepsilon'((\tilde{v}_n)_x(x,t))$ is bounded in the same space. So there exists at least a subsequence weakly converging in $L^2(0,1)$. We will show that the entire sequence is weakly convergent, i.e.

$$f_\varepsilon'((\tilde{v}_n)_x(x,t)) \rightharpoonup 2u_x(x,t) \quad \text{in} \quad L^2(0,1).$$

Recalling now Theorem 27, we observe that in (4.43) the right-hand side is weakly converging to $2u_x(x,t)$ in $L^1(0,1)$. Indeed, notice that $J'''(0)=0$ and $(\tilde{v}_n)_x(x,t)$ is equibounded in $L^2(0,1)$. Hence, we can conclude that $\phi_n(x,t) \rightharpoonup 2u_x(x,t)$ in $L^2(0,1)$. Finally we have that

- $\chi_n\phi_n$ is uniformly bounded in $L^2(0,1)$;
- $(1-\chi_n)\phi_n$ is itself uniformly bounded because it is $f_\varepsilon'((\tilde{v}_n)_x(x,t))$.

This means that also $\phi_n(x,t)$ is uniformly bounded in $L^2(0,1)$. □

We can improve the result above. In particular, we may deduce which boundary conditions are satisfied by the weak-limit of $\phi_n(x,t)$. To that end, in the following, we extend definition (4.36) by setting

$$(u^k_{\varepsilon,\tau})_i = \begin{cases} (u^k_{\varepsilon,\tau})_0 & \text{if } \ i \in \mathbb{Z},\ i < 0 \\ (u^k_{\varepsilon,\tau})_N & \text{if } \ i \in \mathbb{Z},\ i > N. \end{cases} \tag{4.44}$$

Theorem 28. *Consider a sequence of functions v_n as defined in Theorem 27. Let u be its strong limit in $L^2(0,1)$, then*

1) $u_x(\cdot,t) \in H^1(0,1)$ for almost every $t \geq 0$ and $(u_x)_x \in L^2((0,1)\times(0,T))$ for every $T > 0$;

2) for almost every $t \geq 0$ the function u satisfies the boundary conditions $u_x(0,t) = u_x(1,t) = 0$ and $u_x(\cdot,t) = 0$ on $S(u(\cdot,t))$.

Proof. Let $\widetilde{\phi}_n$ be the linear interpolation of the function ϕ_n defined in (4.40). Our first claim is that $\widetilde{\phi}_n \rightharpoonup 2u_x(x,t)$ in $H^1(0,1)$.

We recall that, from Proposition 13, it holds

$$\sum_{i=0}^{N} \varepsilon |(u^k_{\varepsilon,\tau})_i - (u^{k-1}_{\varepsilon,\tau})_i|^2 \leq 2\tau[F_\varepsilon(u^{k-1}_{\varepsilon,\tau}) - F_\varepsilon(u^k_{\varepsilon,\tau})],$$

so that, fixing $T > 0$ and denoting $N_\tau = \lfloor T/\tau \rfloor$, we have

$$\sum_{k=1}^{N_\tau} \sum_{i=0}^{N_\varepsilon} \tau\varepsilon |(u^k_{\varepsilon,\tau})_i - (u^{k-1}_{\varepsilon,\tau})_i|^2 \leq 2\tau^2 F_\varepsilon(u^0_\varepsilon) \leq 2\tau^2 M.$$

Using the optimality conditions in Proposition 14 and the extension (4.44), we get

$$\sum_{k=1}^{N_\tau} \tau \sum_{i\in\mathbb{Z}} \varepsilon\tau^2 \Big[\frac{1}{\varepsilon}\Big(f'_\varepsilon\big(\frac{(u^k_{\varepsilon,\tau})_{i+1} - (u^k_{\varepsilon,\tau})_i}{\varepsilon}\big) - f'_\varepsilon\big(\frac{(u^k_{\varepsilon,\tau})_i - (u^k_{\varepsilon,\tau})_{i-1}}{\varepsilon}\big)\Big)\Big]^2 \leq 2\tau^2 M\,.$$

Taking the extension by linear interpolation $\widetilde{\phi}_n$ on I_ε into account, we rewrite the previous estimate as

$$\sum_{k=1}^{N_\tau} \tau \int_{\mathbb{R}} [(\widetilde{\phi}_n)_x(x,k\tau)]^2\,dx \leq 2M,$$

so that for $\delta > 0$ and $\tau < \delta$ we obtain

$$\int_\delta^T dt \int_{\mathbb{R}} [(\widetilde{\phi}_n)_x(x,k\tau)]^2\,dx \leq 2M$$

and

$$\liminf_{n\to+\infty} \int_\delta^T dt \int_{\mathbb{R}} [(\widetilde{\phi}_n)_x(x,k\tau)]^2\,dx \leq 2M.$$

By Fatou's Lemma

$$\int_\delta^T \Big(\liminf_{n\to+\infty} \int_{\mathbb{R}} [(\widetilde{\phi}_n)_x(x,k\tau)]^2\,dx\Big) dt \leq 2M;$$

in particular this means that

$$\liminf_{n\to+\infty}\int_{\mathbb{R}}[(\widetilde{\phi}_n)_x(x,k\tau)]^2\,dx<\infty \qquad \text{for a.e. } t\in[\delta,T]. \tag{4.45}$$

Let t be such that the previous inequality holds and consider a subsequence $(\widetilde{\phi}_{n_k})_x(x,k\tau)$ such that

$$\liminf_{k\to+\infty}\int_{\mathbb{R}}[(\widetilde{\phi}_{n_k})_x(x,k\tau)]^2\,dx=\liminf_{n\to+\infty}\int_{\mathbb{R}}[(\widetilde{\phi}_n)_x(x,k\tau)]^2\,dx.$$

Then there exists C independent of k such that

$$\int_{\mathbb{R}}[(\widetilde{\phi}_{n_k})_x(x,k\tau)]^2\,dx\le C. \tag{4.46}$$

We recall that, by Proposition 15, in $L^2(0,1)$ we have

$$\phi_n(\cdot,t)\rightharpoonup\phi(\cdot,t)=\begin{cases}2u_x(\cdot,t) & \text{in } (0,1)\\ 0 & \text{otherwise.}\end{cases}$$

The same result also holds for $\widetilde{\phi}_{n_k}(\cdot,t)$ observing that from (4.46) we get

$$\sum_{i\in\mathbb{Z}:\varepsilon i\in I_\varepsilon}\varepsilon|\phi_{n_k}((i+1)\varepsilon,t)-\phi_{n_k}(i\varepsilon,t)|^2\le\varepsilon^2C.$$

Moreover, the L^2-weak-convergence of $\widetilde{\phi}_{n_k}$, the boundedness proved in (4.46) and an integration by parts argument show that, for any open interval $I\subset[0,1]$, $\phi\in H^1(I)$ and

$$(\widetilde{\phi}_{n_k})_x(\cdot,t)\rightharpoonup\phi_x(\cdot,t)\quad\text{in } L^2(0,1). \tag{4.47}$$

The above observation and convergence result prove that $u_x\in H^1(0,1)$ with $u_x(0,t)=u_x(1,t)=0$ for almost every $t\ge 0$.

Now, we want to show that $u_{xx}\in L^2((0,1)\times(0,T))$: since

$$\int_0^1[\phi_x(x,t)]^2\,dx\le\liminf_{k\to+\infty}\int_{\mathbb{R}}[(\widetilde{\phi}_{n_k})_x(x,t)]^2\,dx=\liminf_{n\to+\infty}\int_{\mathbb{R}}[(\widetilde{\phi}_n)_x(x,t)]^2\,dx$$

then for every $\delta>0$

$$\int_\delta^T dt\int_0^1[\phi_x(x,t)]^2\,dx\le\int_\delta^T\Big(\liminf_{n\to+\infty}\int_{\mathbb{R}}[(\widetilde{\phi}_n)_x(x,t)]^2\,dx\Big)dt\le 2M.$$

To conclude we want to understand which values $u_x(\cdot,t)$ attains on $S(u)$. Let t be such that (4.46) still holds. Observing that $H^1(0,1)\subset\subset C([0,1])$, we have

$$\widetilde{\phi}_{n_k}(x,t)\to 2u_x(\cdot,t)\qquad\text{in } C([0,1]).$$

Now if $\bar{x}\in S(u)$, thanks to Lemma 6 we know that there exists a sequence $\{x^n\}$ converging to x and such that for every n

$$x^n\in S(\tilde{v}_n(\cdot,t))\quad\text{ and } |\tilde{v}_n^+(x^n,t)-\tilde{v}_n^-(x^n,t)|\ge\gamma>0.$$

We observe that, if $x^n = i_n\varepsilon \in S(\tilde v_n(\cdot,t))$, then $(i_n-1) \in I^j_\varepsilon(v_n(\cdot,t))$, so that we can write

$$|\widetilde{\phi}_n((i_n-1)\varepsilon,t)| = |\phi_n((i_n-1)\varepsilon,t)| = f'_\varepsilon\Big(\frac{|\tilde v_n^+(x^n,t)-\tilde v_n^-(x^n,t)|}{\varepsilon}\Big)$$
$$\leq f'_\varepsilon\Big(\frac{\gamma}{\varepsilon}\Big) = \frac{1}{\sqrt{\varepsilon|\log\varepsilon|}}J'\Big(\sqrt{\frac{|\log\varepsilon|}{\varepsilon}}\gamma\Big).$$

Recalling that $J'(z) = \frac{2z}{1+z^2}$, a simple calculation shows that

$$|\widetilde{\phi}_n((i_n-1)\varepsilon,t)| \leq \frac{2\gamma}{\varepsilon+\gamma^2|\log\varepsilon|} \to 0 \text{ if } \varepsilon_n \to 0.$$

Now the uniform convergence of $\widetilde{\phi}_n(\cdot,t)$ to $2u_x(\cdot,t)$ implies that $u_x(\bar x,t) = 0$. □

We conclude this section collecting all the previous results to obtain the limit equation satisfied by a minimizing movement of functional (4.4) (see the analog result in [29]).

Theorem 29. *Let $\{u^0_\varepsilon\}_\varepsilon$ be a sequence of functions which satisfies (4.20) and (4.21) and let $v_n = u_{\varepsilon_n,\tau_n}$ be a sequence converging to u as in Theorem 27. Then we have*

$$u_t = 2\,u_{xx} \tag{4.48}$$

in the distributional sense in $(0,1)\times(0,+\infty)$. Moreover

$$u(\cdot,0) = u^0 \quad \text{a.e. in } (0,1)$$
$$u_x(\cdot,t) = 0 \quad \text{on } S(u(\cdot,t))\cup\{0,1\} \quad \text{for a.e. } t \geq 0,$$

where u^0 is the a.e.-limit of the sequence $\{u^0_\varepsilon\}_\varepsilon$.

Proof. We give a quick sketch of the computation of the limit equation. The main tools we are going to use are the optimality conditions in Proposition 14 and the following summation formula:

$$\sum_{i=0}^{M-1} a_i(b_{i+1}-b_i) = a_M b_M - a_0 b_0 - \sum_{i=0}^{M-1}(a_{i+1}-a_i)b_i. \tag{4.49}$$

Now let $T>0$ be fixed and consider $M_\tau = \lfloor T/\tau \rfloor$, then consider a function $\varphi \in C^\infty_0$ and denote $\varphi^k_i = \varphi(i\varepsilon, k\tau)$ with $i,k \in \mathbb{Z}$. Now we have

$$\sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1} \varepsilon\tau(u^k_{\varepsilon,\tau})_i \frac{\varphi^{k+1}i - \varphi^k_i}{\tau} =$$
$$\sum_{i=0}^{N_\varepsilon}\varepsilon\left((u^{M_\tau}_{\varepsilon,\tau})_i\varphi^{M_\tau}_i - (u^0_{\varepsilon,\tau})_i\varphi^0_i - \sum_{k=0}^{M_\tau-1}\Big((u^{k+1}_{\varepsilon,\tau})_i - (u^k_{\varepsilon,\tau})_i\Big)\varphi^k_i\right).$$

Since φ has compact support, we have $\varphi_i^0 = \varphi_i^{M_\tau} = 0$ if τ is sufficiently small. Then we apply the optimality conditions and again the (4.49):

$$-\sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1} \varepsilon\Big((u_{\varepsilon,\tau}^{k+1})_i - (u_{\varepsilon,\tau}^{k})_i\Big)\varphi_i^k =$$
$$-\tau\sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1}\left(f_\varepsilon'\left(\frac{(u_{\varepsilon,\tau}^{k+1})_{i+1} - (u_{\varepsilon,\tau}^{k+1})_i}{\varepsilon}\right) - f_\varepsilon'\left(\frac{(u_{\varepsilon,\tau}^{k+1})_{i} - (u_{\varepsilon,\tau}^{k+1})_{i-1}}{\varepsilon}\right)\right)\varphi_i^{k+1} =$$
$$\tau\sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1}\left(\varphi_{i+1}^{k+1} - \varphi_i^{k+1}\right) f_\varepsilon'\left(\frac{(u_{\varepsilon,\tau}^{k+1})_{i+1} - (u_{\varepsilon,\tau}^{k+1})_i}{\varepsilon}\right),$$

since $\varphi_0^k = \varphi_{N_\varepsilon}^k = 0$ if ε is sufficiently small. In the end we have the following equality

$$\sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1} \varepsilon\tau(u_{\varepsilon,\tau}^k)_i\frac{\varphi_i^{k+1} - \varphi_i^k}{\tau} = \sum_{i=0}^{N_\varepsilon}\sum_{k=0}^{M_\tau-1} \varepsilon\tau\frac{\varphi_{i+1}^{k+1} - \varphi_i^{k+1}}{\varepsilon} f_\varepsilon'\left(\frac{(u_{\varepsilon,\tau}^{k+1})_{i+1} - (u_{\varepsilon,\tau}^{k+1})_i}{\varepsilon}\right). \tag{4.50}$$

We define the following piecewise constant functions:

$$\varphi_{\varepsilon,\tau}^{(0,1)}(x,t) \equiv \frac{\varphi_i^{k+1} - \varphi_i^k}{\tau}, \quad \varphi_{\varepsilon,\tau}^{(1,0)}(x,t) \equiv \frac{\varphi_{i+1}^{k} - \varphi_i^{k+1}}{\varepsilon}, \quad x = \lfloor x/\varepsilon\rfloor,\ k = \lfloor t/\tau\rfloor,$$

then, recalling (4.40), and thanks to the compactness of the support of φ we can rewrite (4.50) as:

$$\int_0^1\int_0^T u_{\varepsilon,\tau}(x,t)\varphi_{\varepsilon,\tau}^{(0,1)}(x,t)\,dxdt = \int_0^1 dx\int_0^T \varphi_{\varepsilon,\tau}^{(1,0)}(x,t)\phi_n(x,t)\,dt.$$

Now, letting $n \to +\infty$ and recalling Proposition 15, we have that

$$\int_0^1\int_0^T u(x,t)\varphi_t(x,t)\,dxdt = 2\int_0^1\int_0^T \varphi_t(x,t)u_x(x,t)\,dxdt.$$

Now the thesis is straightforward. □

4.5.4 Evolution of the singular set and conclusion

Let u_ε^0 be an initial datum satisfying (4.20) and (4.21), and consider the sequence $\{u_{\varepsilon,\tau}^k\}$ of minimizers of functional (4.35). In this section we want to understand the behaviour of the set of singular points $I_\varepsilon^j(u_{\varepsilon,\tau}^k)$ with respect to k.

We simplify the notation introducing

$$u_i^k := (u_{\varepsilon,\tau}^k)_i, \qquad v_i^k := \frac{u_{i+1}^k - u_i^k}{\varepsilon}.$$

Using the optimality condition for $0 < i < N-1$, we observe that

$$\begin{aligned} v_i^{k+1} - v_i^k &= \frac{1}{\varepsilon}\Big(u_{i+1}^{k+1} - u_i^{k+1} - u_{i+1}^k + u_i^k\Big) \\ &= \frac{1}{\varepsilon}\Big(u_{i+1}^{k+1} - u_{i+1}^k\Big) - \frac{1}{\varepsilon}\Big(u_i^{k+1} - u_i^k\Big) \\ &= \frac{\tau}{\varepsilon^2}\Big[f_\varepsilon'\Big(\frac{u_{i+2}^{k+1} - u_{i+1}^{k+1}}{\varepsilon}\Big) + f_\varepsilon'\Big(\frac{u_i^{k+1} - u_{i-1}^{k+1}}{\varepsilon}\Big) - 2f_\varepsilon'\Big(\frac{u_{i+1}^{k+1} - u_i^{k+1}}{\varepsilon}\Big)\Big]. \end{aligned}$$

Rewriting these terms we obtain

$$(v_i^{k+1} - v_i^k) + 2\frac{\tau}{\varepsilon^2} f_\varepsilon'\Big(\frac{u_{i+1}^{k+1} - u_i^{k+1}}{\varepsilon}\Big) \leq 2\frac{\tau}{\varepsilon^2}\max f_\varepsilon'. \tag{4.51}$$

Observe that this estimate still holds for $i = 0$ or $i = N-1$. In fact, in those cases we have

$$(v_i^{k+1} - v_i^k) + 2\frac{\tau}{\varepsilon^2} f_\varepsilon'\Big(\frac{u_{i+1}^{k+1} - u_i^{k+1}}{\varepsilon}\Big) \leq \frac{\tau}{\varepsilon^2}\max f_\varepsilon'\,.$$

Consider now the function $h(z) = 2\frac{\tau}{\varepsilon^2} f_\varepsilon'(z)$, so that (4.51) reads

$$(v_i^{k+1} - v_i^k) + h(v_i^{k+1}) \leq \max h. \tag{4.52}$$

The key point is the following lemma (see Lemma 4.2 in [29]) which allows to use stationary solutions as barriers in difference equations .

Lemma 7. *Let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $L < 1$. Let $\{a_k\}$ be a sequence of real numbers and $C \in \mathbb{R}$ such that*

$$a_{k+1} - a_k + h(a_{k+1}) \leq h(C) \quad \textit{for all } k \geq 0.$$

Then, if $a_0 \leq C$, it holds $a_k \leq C$ for all $k \geq 0$.

First we prove that h is a Lipschitz function with $L < 1$:

$$h(z_1) - h(z_2) = 2\frac{\tau}{\varepsilon^2}\Big(f_\varepsilon'(z_1) - f_\varepsilon'(z_2)\Big) \leq 2\frac{\tau}{\varepsilon^2} f_\varepsilon''(\xi)(z_1 - z_2) \qquad \xi \in (z_1, z_2)$$

so we have to require that

$$2\frac{\tau}{\varepsilon^2}\max f_\varepsilon'' = 2\frac{\tau}{\varepsilon^2}\max J'' = 4\frac{\tau}{\varepsilon^2} < 1. \tag{4.53}$$

When this condition is satisfied, choosing $C = 1/\sqrt{\varepsilon|\log\varepsilon|}$, we have the following result.

Proposition 16. *If (4.53) holds, then $I_\varepsilon^j(u_{\varepsilon,\tau}^{k+1}) \subseteq I_\varepsilon^j(u_{\varepsilon,\tau}^k)$ for every $k \geq 0$.*

Conclusion of the proof of Theorem 26. By Proposition 16 the sets $S(\tilde{v}_n(\cdot,t))$ are non-increasing in t, so that they are all contained in $S(\tilde{v}_n(\cdot,0))$. Note that $\tilde{v}_n(\cdot,0)$ is the interpolation of the initial datum u_0^ε, so that the limit of the sets $S(\tilde{v}_n(\cdot,0))$ is the set S as defined in (4.22). Since each point in $S(u(\cdot,t))$ is a limit of points in $S(\tilde{v}_n(\cdot,t))$, we deduce that $S(u(\cdot,t)) \subset S$, and hence $S(u(\cdot,t)) \subset S(u^0)$ by the equality $S = S(u^0)$ in assumption (4.22). This implies that there exists $T_0 > 0$ such that $S(u(\cdot,t)) = S(u^0)$ for $t \in [0,T_0]$. Indeed, otherwise we would have a sequence $t_j \to 0^+$ such that $\#(S(u(\cdot,t_j))) < \#(S(u^0))$, so that, up to a subsequence, $u(\cdot,t_j)$ would be equibounded in $H^1(a,b)$ for some subinterval (a,b) of $(0,1)$ and $S(u^0)\cap(a,b) \neq \emptyset$, which gives a contradiction since $u(\cdot,t_j) \to u^0$ in L^2. Recalling Theorem 29, this proves that a minimizing movement along the functionals (4.5) satisfies the heat equation with Neumann boundary conditions on $(0,1)\backslash S(u^0)$ for $t \in [0,T_0]$. This result is the same for the minimizing movement of Mumford-Shah functional [21], and it holds until the first collision time, which means it is valid for every $t \in [0,\bar t)$, where $\bar t > 0$ is the first time such that $\#(S(u(\cdot,\bar t^+))) < \#(S(u(\cdot,\bar t^-))) = \#(S(u^0))$. Indeed, if $T_0 < \bar t$ we may consider T_0 as an initial time and argue as above. $\square$

4.5.5 Long-time behaviour

In this section we conclude with an example showing that, in parallel with the study of local minima, some corrections to the Mumford-Shah energy must be made to capture long-time behaviour of the gradient-flow of our Perona-Malik energies. Indeed, we may have minimizing movements converging to a trivial motion (this happens, e.g., in the case of an initial datum which is a local minimum of the Mumford-Shah energy), but which, for a suitable time scaling, converge to a non-trivial evolution.

Long-time dynamics can be defined by introducing a time-scaling parameter $\lambda > 0$, and applying a recursive minimizing scheme to the scaled energies. Fixed an initial datum x_0 we define recursively x_k as a minimizer for the minimum problem

$$\min\left\{\frac{1}{\lambda}F_\varepsilon(x) + \frac{1}{2\tau}\|x - x_{k-1}\|^2\right\}. \tag{4.54}$$

Note that in this statement we are regarding all three parameters as varying; in particular, we may think of $\tau = \tau_\varepsilon$ and $\lambda = \lambda_\varepsilon$ as depending on ε. In the terminology of [35], the minimizing movement is related to an expansion at order λ of F_ε. Equivalently the same minimum problem can be written as

$$\min\left\{F_\varepsilon(x) + \frac{\lambda}{2\tau}\|x - x_{k-1}\|^2\right\}. \tag{4.55}$$

so that x_k can be seen as produced by a minimizing movements scheme with time step $\eta = \tau/\lambda$. Now, if u^η is a discretization over the lattice of time-step η, we have

$$u^\tau(t) := x_{\lfloor t/\tau\rfloor} = x_{\lfloor t/\lambda\eta\rfloor} = u^\eta\Big(\frac{t}{\lambda}\Big).$$

This shows that the introduction of the constant parameter λ is equivalent to a scaling in time.

In order to show that for some time scaling $\lambda = \lambda_\varepsilon$ the sequence F_ε is not equivalent to M_s we consider an initial datum u_0 which is a local minimum for M_s, so that the corresponding motion is trivial at all scales: $u(t) = u_0$ for all t. We then exhibit some λ such that the recursive minimization scheme above gives a non-trivial limit evolution.

We consider additional constraints on the domain of F_ε by limiting the test function to local minimizers of M_s with prescribed boundary conditions. More precisely,

• the initial datum u_0 is a a piecewise-constant function with $S(u_0) = \{x_0, x_1\}$ and $0 < x_0 < x_1 < 1$;

• competing functions are non-negative piecewise-constant functions with $S(u_k) \subseteq S(u_0)$;

• boundary conditions read $u(0^-) = 0$ and $u(1^+) = 1$.

We may describe the minimizers u_k by a direct computation using the minimality conditions: if z_k is the constant value of u_k on the interval (x_0, x_1), then z_k solves the equation

$$(x_1 - x_0)\frac{z_k - z_{k-1}}{\tau} = -\frac{2}{\lambda}\Big(\frac{z_k}{\varepsilon + |\log\varepsilon| z_k^2} + \frac{z_k - 1}{\varepsilon + |\log\varepsilon|(z_k - 1)^2}\Big).$$

In order to obtain a non-trivial limit as $\varepsilon, \tau \to 0$,we may choose the scaling

$$\lambda = \frac{1}{|\log\varepsilon|}. \tag{4.56}$$

With such a time-scaling, in the limit we get an equation for $z(t)$ of the form

$$z' = -\frac{2}{(x_1 - x_0)} \cdot \frac{1 - 2z}{z(1 - z)}. \tag{4.57}$$

Hence, if the initial datum has the value $z_0 \neq 1/2$ in the interval (x_0, x_1), the motion is not trivial. We refer to [21] for further examples.

Remark 14 (equivalent energies for long-time motion)*.* Note that the time-scaled minimizing-movement scheme along the functionals G_ε in Section 4.3.2 for λ as in (4.56) gives the same limit equation (4.57) for the computation above, provided that also $g'(w) \sim \frac{2}{w}$ as $w \to +\infty$. This suggests that G_ε may be considered as a finer approximation, in the sense of expansions by Γ-convergence as defined by Braides-Truskinovsky carrying on the equivalence to long-time behaviours.

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