Dipartimento di Matematica

Corso di Dottorato in Matematica, XXIX Ciclo

Tesi di Dottorato

Ancient Solutions of Curvature Flows

Author

Susanna Risa

Supervisor

Prof. Carlo Sinestrari

Coordinator

Prof. Andrea Braides

Anno Accademico 2016-2017

Acknowledgements

During my PhD I experienced fully how important and fruitful sharing is when doing math. I would like to thank everyone who helped this thesis happen and had a role in my growth during these years.

I am deeply grateful to my supervisor, Professor Carlo Sinestrari, who has guided me with inestimable dedication and patience. His unique clarity, perspicacity and kindness had a major impact in my evolution as a mathematician and as a human being, teaching me how to extract the most even from my frequent mistakes and tirelessly stimulating, and directing, my enthusiasm on this subject.

I also express my appreciation to the anonymous referees and to the members of the PhD committee, Professor Carlo Mantegazza and Professor Barbara Nelli, who accepted to evaluate this thesis.

I am thankful to the coordinator of the PhD School, Andrea Braides, and to all the secretary staff, especially Simonetta De Nicola, Solange Barcaccia and Luisa Montoro, for taking care of all the concrete and sometimes confusing details of academic daily life. I could also benefit of the efforts of Giancarlo Baglioni and Mauro Moscetti from the Calculation Center, who keep our server, and everything gravitating around it, alive.

Together with research, I also started to learn, and value, what it means to interact with younger students. Thanks to Professor Teresa D'Aprile and Professor Michael McQuillan for the opportunity to know from different perspectives what teaching is like.

This is a good occasion to show gratitude to my family: to my parents, whose support I feel much more than I am usually able to convey; to my irreplaceable *sorellona* Flaminia, who reminds me there is someone "who is just as strange as me", assists me constantly in unearthing my defects and qualities and helps me survive the toughest times, having fun meanwhile!

Thanks to all my fellow PhD colleagues in Tor Vergata! Especially to my friend and officemate Martina, the perfect complement to my sometimes-too-blunt simplicity. These years would have been unspeakably duller, more flat and impervious without her fellowship, advices and aid. Also, to Viola, who "enjoyed" the frenzy of these last months, right after her arrival!

To my friends: Veronica, who is "always here", no matter how many kilometers divide us; Alessio, who makes my nerd soul feel at home; Nicola, whose willpower I will never cease to admire; Daniele, for the soothing and utterly inspiring reflections, the rivers and the puns (especially the puns); Raffa and Vale, with whom I shared most of my university life before the PhD; the "teachers" Michaela and Paula, who laid the basis of my education; everyone I didn't explicitly list, I'm grateful for you all!

A special thanks goes to Silvio and all my fellow *compagni d'arme* of the Accademia Romana d'Armi: an amazing group, whose teachings, influence and all the *assai belle cose* have improved my life (and work, even!) way beyond historical fencing.

Thanks to everyone I met in Turin, Pisa, Levico, Oberwolfach, Florence, Trieste and Regensburg, in particular to Julia, for our kindred expectations and fears about our work, and Alessandra, who gave me the chance of facing my very first talk; thanks also to my "academic brother and sister" Giuseppe and Maria Chiara, to Professor Paolo Piccinni and to Professor Stefano Marchiafava.

To end, I would like to mention the staff in Copynet, who concretely printed this thesis, Barbara, who never complained about the ever-growing stacks of papers and origami on my desk, and the teams in "Bar Rosso" and "Bar Giallo" for that boost to the mood in the morning that coffee alone just can't give.

Thank you, everyone.

Contents

Introduction **1**

1 Preliminaries **7**

1.1 Geometry of Immersions 7
1.2 Convex hypersurfaces in $\mathbb{R}^{n+1}$ 10
1.3 Starshaped Hypersurfaces in $\mathbb{R}^{n+1}$ 11

2 Pinched ancient solutions of higher codimensional MCF **13**

2.1 Few properties of the Mean Curvature Flow 15
2.2 The rigidity theorem for higher codimensional MCF 17
2.3 High codimension Mean Curvature Flow in the sphere 23

3 Ancient solutions of contractive fully nonlinear flows **29**

3.1 A general estimate 32
3.2 Flows with degree of homogeneity 1 34
3.2.1 Ancient solutions with a diameter bound 36
3.3 Flows by powers of the Gauss curvature 39
3.4 General flows with high degree of homogeneity 44
3.5 A convex non-pinched counterexample 48

4 Ancient solutions of expansive curvature flows **55**

4.1 The k-th isoperimetric ratio 57
4.2 Proof of Theorem 12 60

Bibliography **64**

Introduction

Over the last thirty-five years, the study of deformations of manifolds driven by parabolic partial differential equations has undergone a sharp expansion. Since the introduction of the Ricci Flow in 3 dimensions by Hamilton [35] and of the Mean Curvature Flow of hypersurfaces by Huisken [42], a multitude of generalizations has been developed, as curvature flows are powerful tools to derive geometric results (e.g. classification theorems or isoperimetric inequalities) and, in turn, their analysis benefits of improvements in geometric knowledge.

We consider extrinsic curvature flows, defined as time-dependent families of immersions $\varphi : M \times [T_1, T_2) \to N$ satisfying a parabolic system of the form

$$\frac{\partial \varphi}{\partial t}(x,t) = V(x,t). \tag{0.0.1}$$

The velocity $V(\cdot, t)$ at each fixed time is a normal section of the immersed submanifold, related to its extrinsic curvature; (0.0.1) is thus a second order system of differential equations for the immersion. This thesis focuses on the properties of a particular class of solutions for curvature flows, called ancient solutions.

A solution of a parabolic partial differential equation is ancient if it is defined on an interval of times $(-\infty, T)$, for $T \in \mathbb{R}$. It is immediate to observe that the notion in the context of curvature flows is not void: as intuition might suggest, standard immersed spheres in Euclidean space move by homotheties, since the system (0.0.1) reduces to an ordinary equation for the evolving radius; thus, they provide the simplest example of a nontrivial (i.e. non-stationary) ancient solution. If the radius of a sphere decreases in time, we will call the flow contractive, expansive if it increases. The most examined flow of the form (0.0.1) is undoubtedly Mean Curvature Flow (MCF) of hypersurfaces, obtained by choosing $V = \mathbf{H} = -H\nu$, the mean curvature vector in codimension 1. By the classical result of Huisken [42], convex closed n-dimensional hypersurfaces immersed in $\mathbb{R}^{n+1}$ shrink into a "round point" in finite time under MCF. We recall that every compact smooth flow becomes singular in finite time due to the avoidance principle (see Theorem 2), by comparison with enclosing spheres. A non-convex hypersurface does not disappear entirely onto a point at the singular time. Different singularity profiles arise, according to the rate of explosion of the curvature:

$$\text{Type I singularity: } \max_{M_t} |h| \leq \frac{C}{\sqrt{T-t}} \qquad \exists C \in \mathbb{R}, \forall t \in [0,T)$$
$$\text{Type II singularity: } \lim_{t\to T} \max_{M_t} |h|\sqrt{T-t} = +\infty$$

where h is the second fundamental form of the immersion; we note that the opposite bound $\max\limits_{M_t} |h| \geq \frac{\overline{C}}{\sqrt{T-t}}$ always holds, for some constant $\overline{C}$.

Ancient solutions have a role in the analysis of singularities, since they appear as limit flows of a the following blow-up procedure for Type I mean convex singularities. This technique is roughly equivalent to "zooming in" towards the singular point in spacetime; ancient solutions, thus, describe the asymptotic shape of the flow. We will describe the procedure using the argument in [47]. Let $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$ be a mean convex immersion evolving by MCF having a Type I singularity at $t = T$. Consider a sequence of points $(x_k, t_k) \in M \times [0,T)$ such that, as $k \to +\infty$, $t_k \to T$ and $\max\limits_{M_{t_k}} H(x, t_k) = H(x_k, t_k)$. A sequence of flows $\widetilde{\varphi}$ is defined by parabolic rescaling as follows:

$$\widetilde{\varphi}_k(x,\tau) = \lambda_k \left(\varphi(x, \frac{\tau}{\lambda_k^2} + t_k) - \varphi(x_k, t_k) \right)$$

with the choice of $\lambda_k = H(x_k, t_k)$; each flow is defined in $[\alpha_k, \omega_k]$, with $\alpha_k = -\lambda_k^2 t_k$, $\omega_k = \lambda_k^2(T - t_k)$. Thanks to parabolic invariance, these flows also move by mean curvature; the family is precompact, thus they admit a subsequence converging to a limit flow φ_∞. Due to the assumptions on the Type of the singularity, φ_∞ is defined on $(-\infty, \omega_\infty)$ for $\omega_\infty < +\infty$ and it is thus an ancient solution.

If the starting geometry is suitably restricted, possible limit ancient solutions for Mean Curvature Flow of hypersurfaces are classified. In particular, a compact parabolic blow-up of a Type I mean-convex hypersurface is necessarily a self-similar shrinking sphere ([45, 46]); convexity of the limit also holds without assuming compactness. The statement above was proved in [45] and it follows from a monotonicity formula. Huisken defined a strictly decreasing quantity by integrating along the flow the backward heat kernel, thus introducing a functional whose critical points satisfy the equation $H + \langle \varphi, \nu \rangle = 0$. This equation characterizes immersions evolving by homothety.

In general, the existence of monotonic quantities has an important role in classification processes for ancient solutions of curvature flows, as an infinite amount of time forces those functionals to converge to their critical points, which are usually geometrically rigid.

Huisken and Sinestrari in 2014 ([48], see also the independent work [40]) have provided sufficient conditions for a compact convex ancient solution of 1-codimensional MCF in Euclidean space and in $\mathbb{S}^{n+1}$ to be a shrinking sphere and a spherical cap or an equator respectively. Recalling the structure of limit solutions, uniform convexity is a natural assumption in this class; in addition, a classification result without restrictive assumptions on the curvature seems hopeless. Simply weakening uniform strict convexity, Haslhofer and Hershkovits [40] described the construction of a family of rotationally symmetric examples of ancient solutions that are not homotetical. A self-shrinker that is not a shrinking sphere also exists: Angenent's shrinking torus [9]. Very recently, Bourni, Langford and Tinaglia have defined an ancient 2-dimensional collapsed solution in dimension 2 [15].

This thesis aims to generalize the statement of the Theorem in [48], proving rigidity results for compact convex ancient solutions for flows different from 1-codimensional MCF. Though the details of the techniques used in the various cases are rather different, there are nodal points common to most proofs: identification of monotone quantities that characterize spheres and the ascertainment of parabolic regularity to ensure precompactness and deduce the behaviour of the whole solution from that of a subsequent limit at $-\infty$ (we remark that sphericity is invariant forward in time).

Outline of the thesis

After recalling shortly the geometry of submanifolds and convex parametrizations in the preliminary first chapter, we will consider Mean Curvature Flow of submanifolds in higher codimension. General convergence results for MCF in this setting are much scarcer than for hypersurfaces, as the complexity of the normal bundle reduces the number of suitable scalar quantities for the description of the evolution; analytical properties of the equations are also more involved. Ben Andrews and Charles Baker [5] proved convergence to a round point for submanifolds of Euclidean space under a pinching condition for the second fundamental tensor of the form $|h|^2 \leq C|\mathbf{H}|^2$, for a constant C. Directly inspired by the argument in [48], we deduce an estimate interior in time from the evolution equation of an integral quantity that vanishes on the sphere. The proof also adapts a technique by Hamilton [37] to bound the intrinsic diameter of the evolving submanifold and conclude an estimate on the volume using Bishop-Gromov's Theorem. In the following section, we demonstrate an analogous result for submanifolds of the spheres $\mathbb{S}^{n+k}_K$ of constant curvature $K > 0$; the argument is similar but simpler, as positive ambient curvature helps convergence of the flow and increases rigidity.

In the last two chapters we consider fully nonlinear homogeneous and isotropic flows of hypersurfaces. Chapter 3 regards pinched convex ancient solutions of contractive flows. We preliminarly prove a general estimate on the inner and outer radii of the evolving surface and a bound from above on the speed, based on a well-established technique first introduced by Tso in [70]. Then, we show spherical rigidity of ancient solutions under a pinching condition $\lambda_n \leq C\lambda_1$ on the principal curvatures for the class of 1-homogeneous flows analysed by Andrews in [2]; we also show that pinching can be weakened if a suitable bound on the diameter holds. Both proofs make crucial use of a Harnack Inequality ([50, 3]) to derive bounds on the velocity and uniform parabolicity of the system. Krylov-Safonov theory (see, for example, [50]) and Schauder estimates for parabolic differential equations then grant precompactness of the solution and allow to conclude with the aid of monotone quantities.

The second part of the chapter deals with flows having higher homogeneity; we first examine pinched ancient solutions for the class of Gaussian Curvature Flows. As in the previous section, the conclusion follows thanks to a family of monotone nonincreasing entropies introduced by Andrews, Guan and Ni in the general case ([6], see also [34]), characterizing the spheres. The issue of parabolicity is delicate, since uniform lower bounds for the speed do not seem to follow from the Harnack inequality ([38, 25]). We thus adapt a technique for degenerate flows introduced by Schulze [65] and readapted by Cabezas-Rivas and Sinestrari [19] in the context of volume-preserving flows; after rescaling, the equation for the speed $\widetilde{K}^\beta$ can be rearranged to be of porous medium type and Hölder estimates follow from a Theorem of DiBenedetto and Friedman [27]; we can then deduce that $\widetilde{K}$ cannot vanish at $-\infty$. We then discuss flows with high homogeneity and a stronger pinching as in [7]; in this case, we can exploit arguments similar to those already examined for the Mean Curvature Flow. The chapter ends with a generalization of the example of a nonspherical ancient solution introduced by

Haslhofer and Hershkovits [40]. We show how their construction also yields a solution with the same properties for a class of nonlinear flows. We need some new arguments in the proof, as we lack a strong convergence theorem as the Global Convergence Theorem of Haslhofer and Kleiner [41]; this is in turn related to the lack of invariance for two-sided noncollapsing and of an analogous of Huisken's monotonicity formula.

The final chapter regards expansive flows. Evolution of hypersurfaces under the p-homogeneous inverse flows we will consider has been examined only recently in full generality for $p > 1$, mainly by Gerhardt ([33], see also [62] and the recent work by [49]), though partial results had already been obtained in dimension 2, see for example [64]. We will prove monotonicity of a certain isoperimetric quantity, the k-th isoperimetric ratio,

$$I_k(\Omega) = \frac{\left(\int H_{k-1} d\mu\right)^{n+1}}{|\Omega|^{n+1-k}},$$

that is minimized by spheres, for flows with speed $\frac{1}{H_k^{\frac{p}{k}}}$ where H_k is the k-th mean curvature. Under a pinching assumption and a restriction on the rate of blow-up at infinity for curvature, we prove that convex ancient solutions are automatically spheres also in this case.

Chapter 1

Preliminaries

In this preliminary chapter, we will recall the basic relations founding the geometry of immersed submanifolds in Euclidean space and in other Riemannian manifolds, and fix the notation that will be used throughout the thesis. The following material is standard and well-known; good references are [30, 53, 69, 21]

1.1 Geometry of Immersions

Throughout the following, M^n will be a differentiable closed manifold of dimension n and $(N^{n+k}, \overline{g})$ a Riemannian manifold of dimension $n+k$ (we will often omit the explicit notation of the dimension); differentiable means C^∞, unless otherwise stated. We will work with immersed submanifolds, defined as the the images $\varphi(M)$ of differentiable maps $\varphi : M \to N$ having injective (thus of maximal rank) differential $d\varphi_x : T_xM \to T_{\varphi(x)}\varphi(M)$ at every $x \in M$; k is the codimension of the submanifold. In general, we do not require that φ is an embedding, so the topology of the submanifold might be different from the induced topology as a subset of $\mathbb{R}^{n+k}$ (e.g. a priori there might be self-intersections). Quantities defined on the submanifold will be denoted by latin indices and quantities defined on the ambient manifold by greek indices. We will consider isometric (or Riemannian) immersions, defining a metric g on M as the pullback metric $\varphi^*\overline{g}$, so there holds $g(v, w) = \overline{g}(d\varphi_x(v), d\varphi_x(w))$ for all $v, w \in T_xM$ and for all $x \in M$. We will denote the components of g by g_{ij} and those of the inverse g^{-1} by g^{ij}; choosing coordinates $\{x^i\}_{i=1}^n$ on M, g_{ij} can be computed as $g_{ij} = \overline{g}\left(\frac{\partial\varphi}{\partial x^i}, \frac{\partial\varphi}{\partial x^j}\right)$. In the last sentences we have used (and will continue to do so without further notice) the identifications of M with $\varphi(M)$ and of T_xM with $d\varphi_x(T_xM) \simeq T_{\varphi(x)}\varphi(M)$. The induced

Riemannian volume form will be denoted by $d\mu$.

The tangent space of the ambient manifold along M splits into the direct sum of the tangent space of the submanifold and its orthogonal complement. Recalling the identification above, $T_{\varphi(x)}N = T_{\varphi(x)}M \oplus N_xM$, with $N_xM = \left\{w \in T_{\varphi(x)}N \mid \overline{g}(w,v) = 0,\ \forall v \in T_{\varphi(x)}M\right\}$ the normal space at x. The decomposition also holds for bundles: $TN = TM \oplus NM$, where $NM = \bigcup\limits_{x\in M} N_xM$ is the normal bundle of M; it allows to define different geometric operators by the restriction of the ambient metric connection $\overline{\nabla}$ to $TM \oplus TN$. We will denote sections of a vector bundle B by $C^\infty(B)$. For X, Y tangent vector fields on M, we have the Gauss' formula

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.1.1}$$

stating that the tangent component of $\overline{\nabla}_X Y$ is the Levi-Civita connection ∇ of the induced metric on M, while the normal component is the second fundamental form h, a 2-covariant tensor field vith values in the normal bundle that encodes the information about the extrinsic curvature of the submanifold. In coordinates, if $\{e_i\}_{i=1}^n$ is a basis for T_xM and $\{\nu_\alpha\}_{\alpha=1}^k$ is a basis for N_xM, we will use the following notation:

$$\begin{aligned}\nabla_{e_i} e_j &= \Gamma_{ij}^k e_k\\ h(e_i, e_j) &= h_{ij\alpha}\nu_\alpha;\end{aligned}$$

we will always assume Einstein convention of summing over repeated indices holds; in the higher codimensional case, following the notation in [12], the repeated indices need not be of different variance. The Weingarten formula

$$\overline{\nabla}_X \xi = -W_\xi(X) + \nabla^\perp_X \xi, \qquad \forall X \in C^\infty(TM),\ \xi \in C^\infty(NM) \tag{1.1.2}$$

describes the covariant derivatives of normal vectors. $\nabla^\perp = \pi^\perp(\overline{\nabla})$ is the projection on the normal space of the ambient connection (providing a metric connection on the normal space with respect to the metric $g^\perp$); W_ξ is a selfadjoint operator on the tangent space called the Weingarten map (or shape operator) in the direction ξ. The tensor h and the family of W_ξ are linked by the fundamental Weingarten relation:

$$g^\perp(\xi, h(X,Y)) = g(X, W_\xi(Y)) \qquad \forall X, Y \in C^\infty(TM),\ \xi \in C^\infty(NM) \tag{1.1.3}$$

that in particular states that the Weingarten operator can be constructed by contraction of the second fundamental tensor with the metric.

The second fundamental form also allows to recover the intrinsic curvature of the manifold M with respect to the induced metric connection, using the following Gauss identity:

$$R(X,Y,Z,V) = \overline{R}(X,Y,Z,V) + h(X,Z)h(Y,V) - h(X,V)h(Y,Z) \quad (1.1.4)$$

where R, $\overline{R}$ are the Riemann curvature tensors of M and N respectively and X, Y, Z, V are tangent vector fields. The curvature of the normal bundle $R^{\perp}$ is described by the Ricci equation

$$g^{\perp}(R^{\perp}(X,Y)\xi,\eta) = \overline{g}(\overline{R}(X,Y)\xi,\eta) + g([W_{\xi}, W_{\eta}]X, Y) \quad (1.1.5)$$

for tangent fields X, Y and normal fields ξ, η. The last fundamental equation of submanifolds, called the Codazzi equation, relates the derivatives of the second fundamental tensor with the normal component of R:

$$(R(X,Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y)h(X,Z). \quad (1.1.6)$$

We also recall that if the ambient manifold N is $\mathbb{R}^{n+k}$ or any sphere $\mathbb{S}^{n+k}(r)$ with the standard Riemannian metric, the Gauss, Ricci and Codazzi equations simply reduce to:

$$\begin{aligned} R(X,Y,Z,V) &= K(\overline{g}(X,Z)\overline{g}(Y,V) - \overline{g}(X,V)\overline{g}(Y,Z)) && (1.1.7)\\ &\quad + h(X,Z)h(Y,V) - h(X,V)h(Y,Z) && (1.1.8)\\ \overline{g}(R^{\perp}(X,Y)\xi,\eta) &= \overline{g}([W_{\xi}, W_{\eta}]X, Y) && (1.1.9)\\ (\overline{\nabla}_X h)(Y,Z) &= (\overline{\nabla}_Y)h(X,Z) && (1.1.10) \end{aligned}$$

due to the expression of the tensor $\overline{R}$ for manifolds of constant sectional curvature K.

The trace of the second fundamental tensor with respect to g provides a normal vector $\mathbf{H} = H^{\alpha}\nu_{\alpha} = g^{ij}h_{ij\alpha}\nu_{\alpha}$, called the mean curvature vector. A point x is an umbilic (or umbilical point) if $h(u,v) = g(u,v)\mathbf{H}$ for all $u, v \in T_xM$. A submanifold whose points are all umbilics is totally umbilical, while if $|h|$ is identically zero the submanifold is called totally geodesic.

If the codimension k is one, the normal bundle has fibers of dimension 1 and the choice of a normal outer unit vector ν at each point provides a basis for NM (we observe that we can choose ν globally and continuously only if the submanifold is orientable). There is just one Weingarten operator $-W_{\nu}$ (the minus sign is conventional) that will be simply denoted as W. As the second fundamental tensor in this setting has values in $\mathbb{R}\nu$, we will simply identify it with a field of real-valued symmetric bilinear forms on TM that we will still denote h, as there is no possibility of confusion. As a selfadjoint operator, at each $x \in M$ the Weingarten map W admits a diagonal basis; the eigenvalues $\lambda_1, \dots, \lambda_n$ are called principal curvatures of the immersion. The mean curvature vector can be written in the same way as $\mathbf{H} = -H\nu$, where H is a real-valued function on M, and it is given at each point by the trace of the matrix second fundamental form; we will simply call it the mean curvature. Thanks to the Weingarten relation, H is given as the sum of the principal curvatures; we underline how other relevant geometric quantities can be expressed in codimension 1 using principal curvatures:

$$\begin{aligned} |h|^2 =& \sum_{i=1}^{n} \lambda_i^2; \\ scal =& 2 \sum_{1 \le i < j \le n} \lambda_i \lambda_j; \\ K =& \prod_{i=1}^{n} \lambda_i, \end{aligned}$$

where $scal$ is the intrinsic scalar curvature and $K = \frac{\det h_{ij}}{\det g_{ij}}$ is the Gaussian curvature.

1.2 Convex hypersurfaces in $\mathbb{R}^{n+1}$

Convex bodies in Euclidean space have well established properties, and the description of their geometry can be tackled from several points of view and using various parametrizations. An interested reader may consult [63, 61, 56] for a good introduction to convex geometry; we will only state few fundamental and well-known facts that are relevant in this thesis.

A body $\Omega \subset \mathbb{R}^{n+1}$ is convex if it contains every segment between its points; we will only consider closed sets whose interior is not empty. A support halfspace of Ω is a halfspace $E \supset \Omega$ such that $\Omega \cap \partial E \neq \emptyset$. Ω is the intersection of all its support halfspaces; the boundary $\partial E = H$ is called a support hyperplane for Ω at each point in $\Omega \cap H$. If $\partial\Omega$ is smooth, there is a unique support hyperplane at each $x \in \partial\Omega$ and it coincides with the tangent space $T_x(\partial\Omega)$. The support function with respect to $x_0 \in \Omega$ is a function from the sphere defined as $u_{x_0}(z) = \sup\limits_{x \in \Omega} \langle x - x_0, z \rangle = \sup\limits_{x \in \partial\Omega} \langle x - x_0, z \rangle$ and it gives the Euclidean distance to x_0 of the support hyperplane with unit normal z; Ω is then characterized as $\Omega = \bigcap_{z \in \mathbb{S}^n} \left\{ y \in \mathbb{R}^{n+1} \mid \langle z, y \rangle \leq u_{x_0}(z) \right\}$ for arbitrary $x_0 \in \Omega$. A support function allows to parametrize $\partial\Omega$ as an immersion from the sphere, defining $\psi_{x_0}(z) = u_{x_0}(z) z + \overline{\nabla} u_{x_0} + x_0$, where $\overline{\nabla}$ is the Levi-Civita connection of $\mathbb{S}^n$.

An immersion $\varphi : M^n \to \mathbb{R}^{n+1}$ is called convex (strictly convex) if the principal curvatures λ_i are nonnegative (positive) for all i. If φ is an embedding, then $\varphi(M)$ is the boundary of a convex body Ω in $\mathbb{R}^{n+1}$ according to the definition in the previous paragraph (intuitively, the submanifold bends away from the outer normal vector at each point). The Gauss map $\nu : M \to \mathbb{S}^n$ associating to $x \in M$ the unit normal vector $\nu(x)$ is then a diffeomorphism, as the differential is given by the Weingarten map and it is invertible; we can use ν^{-1} to reparametrize M using the support function, as $\varphi(\nu^{-1}(z))$ is the point realizing the maximum in the definition of u (we will assume $0 \in \Omega$ and refer to $u = u_0$, $\psi = \psi_0$ without loss of generality). Thus, we have $u(z) = \left\langle z, \varphi(\nu^{-1}(z)) \right\rangle$ and $\varphi(x) = \psi(\nu(x))$. In these coordinates, there holds:

$$b_i^j = (h^{-1})_i^j = \overline{g}^{ik} \left(\overline{\nabla}_k \overline{\nabla}_j u + u \overline{g}_{ik} \right) \tag{1.2.1}$$

so the eigenvalues of b are the radii of curvature, inverse of the principal curvatures. We will often not distinguish between this parametrization as defined on M or on the sphere after the application of the Gauss map.

1.3 Starshaped Hypersurfaces in $\mathbb{R}^{n+1}$

A subset A in $\mathbb{R}^{n+1}$ is starshaped if there exists a point $x_0 \in A$, the center, such that A contains all the segments starting from x_0 to any other point in A or equivalently, if every line L passing though x_0 has connected intersection with A (and thus the intersection between a ray from x_0 and ∂A is a singleton). Obviously, a convex subset is starshaped with respect to any point. If A is a closed smooth starshaped set, the boundary can be parametrized as the graph of the radial function r from the sphere centered in x_0 to the real line; for notational simplicity we will assume again the hypersurface is starshaped with respect to the origin. We can thus describe ∂A as $\partial A = \left\{ (z, r(z)) \mid z \in \mathbb{S}^n \right\}$, where polar coordinates have been introduced on $\mathbb{R}^{n+1}$. If we already have a parametrization $\varphi : M \to \mathbb{R}^{n+1}$ for a starshaped hypersurface, we can associate to each $x \in M$ a unique direction $z_x = \frac{\varphi(x)}{|\varphi(x)|}$ and the radial function is defined by the relation $r(z_x) z_x = \varphi(x)$.

The induced metric on M in this parametrization and its inverse can be computed as

$$g_{ij} = \overline{\nabla}_i r \overline{\nabla}_j r + r^2 \sigma_{ij}, \tag{1.3.1}$$

$$g^{ij} = \frac{1}{r^2}\left(\sigma^{ij} - \frac{\overline{\nabla}_i r \overline{\nabla}_j r}{r^2 + |\overline{\nabla} r|_S^2}\right), \tag{1.3.2}$$

where $\overline{\nabla}$, σ_{ij} and $|\cdot|_S$ are the standard Levi-Civita connection, metric and induced norm on the sphere respectively. The second fundamental form is given by:

$$h_{ij} = \frac{r^2\sigma_{ij} + 2\overline{\nabla}_i r \overline{\nabla}_j r - r\overline{\nabla}_{ij} r}{(r^2 + |\overline{\nabla} r|^2)^{\frac{1}{2}}}.$$

The quantity $v = \frac{1}{r}(r^2 + |\overline{\nabla} r|^2)^{\frac{1}{2}} = \sqrt{1 + |\overline{\nabla}\log r|^2}$ is the ratio between the radial function and the support function with respect to 0 and it quantifies the starshapedness of the hypersurface.

Remark. As last remark, we underline constants C throughout the thesis might vary from one line to another.

Chapter 2

Pinched ancient solutions of higher codimensional MCF

As we discussed in the introduction, the aim of this thesis is to discuss sufficient conditions on the curvature that ensure an ancient compact solution of a certain geometric evolution system is "the simplest possible solution", namely a sphere. We recall that an ancient solution is a solution defined for an interval of times unbounded in the past, so that intuitively the diffusive and regularizing properties of the equation have acted on the surface for infinitely long.

In this chapter, we will prove rigidity results for ancient solutions of Mean Curvature Flow (MCF)

$$\frac{\partial \varphi}{\partial t}(x,t) = \mathbf{H}(x,t) \qquad (x,t) \in M \times [T_0, T), \tag{2.0.1}$$

where the velocity $\mathbf{H}$ is the mean curvature vector. Good references on the Mean Curvature Flow in codimension 1 are [60, 55, 28]; for the higher codimensional case the interested reader can consult, for example, the survey by Smoczyk [68]. M is assumed to be a closed submanifold of dimension at least two; we will discuss both time-dependent immersion φ of $M^n \times [T_0, T)$ into $\mathbb{R}^{n+k}$ and time-dependent immersions into $\mathbb{S}^{n+k}$. We suppose $T = 0$ without loss of generality.

The main theorem we aim to prove is the following:

Theorem 1. *Let $M_t = \varphi(M,t)$ be a smooth closed ancient solution of* (2.0.1) *in $\mathbb{R}^{n+k}$, with $n, k \geq 2$. Suppose that, for all $t \in (-\infty, 0)$ we have $|H|^2 > 0$*

and $|h|^2 \leq C_0|H|^2$, *with* C_0 *a constant satisfying*

$$C_0 < \begin{cases} \dfrac{1}{n-1} & n \geq 4 \\ \dfrac{4}{3n} & n = 2, 3. \end{cases} \tag{2.0.2}$$

Suppose furthermore that the norm of the second fundamental form is uniformly bounded away from the singularity, so there exists $h_0 > 0$ *such that* $|h|^2 \leq h_0$ *in* $(-\infty, -1)$. *Then* M_t *is a family of shrinking spheres.*

The result above was inspired by an analogous theorem for hypersurfaces proved by Gerhard Huisken and Carlo Sinestrari in 2014 [48]. The authors provide some equivalent conditions for a convex closed ancient solution immersed in Euclidean space to be a homotetically shrinking sphere:

Theorem (Huisken-Sinestrari, [48]). *Let* M_t *be a smooth convex closed ancient solution of MCF in codimension* 1 *defined in Euclidean space. The following conditions are equivalent:*

1. M_t *is a shrinking sphere;*
2. *the second fundamental form satisfies a pinching condition* $h_{ij} \geq \epsilon H g_{ij}$ *for some* $\epsilon > 0$ *uniformly on* $(-\infty, 0)$;
3. $diam(M_t) \leq C_1(1 + \sqrt{-t})$ *on* $(-\infty, 0)$, *for some* $C_1 > 0$;
4. *there exists a constant* $C_2 > 0$ *such that* $\rho_+ \leq C\rho_-$ *on* $(-\infty, 0)$;
5. *there exists a constant* $C_3 > 0$ *such that* $\max H(\cdot, t) \leq C_3 \min H(\cdot, t)$ *on* $(-\infty, 0)$;
6. M_t *satisfies the reverse isoperimetric inequality* $|M_t|^{n+1} \leq C_4|\Omega_t|^n$ *for some* $C_4 > 0$, *where* Ω_t *is the (convex) region enclosed by* M_t;
7. M_t *is of Type I (where we mean of Type I "at* $-\infty$*", so* $\limsup_{t \to -\infty} \sqrt{-t} \max H(\cdot, t) < +\infty$*).*

The condition considered in our result, and the first condition in the Theorem above, is pinching of the second fundamental form. Both in codimension one and for higher codimensional closed submanifolds it guarantees that under (2.0.1) the flow will shrink the submanifold entirely into a round point, thanks to the results of Huisken [42] and Andrews-Baker [5]. In particular, in codimension 1 it is automatically satisfied by any strictly convex hypersurface on any compact interval of times, and in all codimensions it is preserved by the flow. The last assertion is proved forward in time by the maximum principle for parabolic PDEs and its extension to tensors by Hamilton [35]; as parabolic problems are ill-posed backwards, we cannot extend the technique directly to our case and we need to assume the condition a priori on the solution. The final point is called "round" as an appropriate rescaling of the solution, constructed to mantain the area $|M_t|$ constant, converges to a standard sphere, showing the regularizing effect of the flow on the curvature. The assumption in the high codimensional case was also hinted by a previous result by Okumura [58], which states that a submanifold satisfying (2.0.2) and with parallel mean curvature vector is diffeomorphic to a sphere; the condition $|\nabla^\perp \mathbf{H}| = 0$ is in general not preserved by MCF. We underline how most of the other hypotheses in Theorem 2 have no counterpart in high codimensional geometry; for example, convexity or enclosed/enclosing spheres are not well-defined and the second fundamental tensor has no associated meaningful eigenvalue (each $(h_{ij\alpha})_{ij}$ is symmetric for any fixed α, but in general there is no basis $\{e_1, \ldots, e_n, \nu_1, \ldots, \nu_k\}$ of $\mathbb{R}^{n+k}$ diagonalizing simultaneously all the $(h_{ij\alpha})_{ij}$).

Independently, Haslhofer and Hershkovits [40] proved a characterization of ancient hypersurfaces similar to Theorem 2. In their work, the assumption of uniform α-noncollapsing is essential: they supposed the existence of interior and exterior tangent balls of radius $\frac{\alpha}{H(x)}$ for a fixed α and at every $x \in M$. This is an invariant property for mean convex evolutions, but it is not clear how it could be generalized to higher codimensional flows. They also provided the example of a non-uniformly convex ovaloid, an ancient nonspherical solution; we will give more details on this in the next chapter, where the construction will be generalized to a class of 1-homogeneous flows of hypersurfaces.

In the following section, we provide an introduction to (higher codimensional) MCF. We will then prove Theorem 1 and that analogous conclusions hold for the evolution of a pinched submanifold in a sphere. We remark that similar results have been independently obtained recently by Lynch and Nguyen [54].

2.1 Few properties of the Mean Curvature Flow

Mean Curvature Flow can be regarded as the simplest flow of submanifolds having a geometric character (i.e. invariant under tangential reparametrizations and isometries of the ambient space) and a variational origin; as the variation of the area is given by

$$\delta|M| = -\int_M \mathbf{H}\cdot X d\mu$$

for any submanifold M and any variational field X, the evolution can be considered as the gradient of such functional and thus the surface flows in the direction minimizing the area in the most efficient way.

MCF is also analogous to a heat equation for the immersion, as in Euclidean space thanks to the Gauss-Weingarten relations (1.1.1),(1.1.2) there holds:

$$\Delta\varphi = g^{ij}\nabla_i\nabla_j\varphi = g^{ij}h_{ij} = \mathbf{H}.$$

The system is only weakly parabolic: its symbol is degenerate in tangential directions due to symmetry under reparametrization of the submanifold.

The evolution by mean curvature was first inspected (in codimension 1) from a measure-theoretical point of view by Brakke [16]. As anticipated before, Huisken in 1984 [42] applied the methods of parabolic differential systems to the flow, inspired by the results of Richard Hamilton on Ricci Flow [35], an evolution equation for a Riemannian metric on a manifold that shares relevant similarities with MCF. In his influential work, Huisken proved that convex compact hypersurfaces immersed in Euclidean space converge to a round point in finite time under (2.0.1); that is, there exists a time T such that $\lim\limits_{t\to T}\sup_{M_t}|h|^2 = +\infty$ and $\lim\limits_{t\to T}\rho_+ = \lim\limits_{t\to T}\rho_- = 0$ and in addition, the expression $\dfrac{1}{H^2}\left(|h|^2 - \frac{1}{n}H^2\right)$ converges to 0 as $t\to T$. As $|h|^2 - \frac{1}{n}H^2 = \dfrac{1}{n}\sum\limits_{i<j}(\lambda_i - \lambda_j)^2$, this implies that, suitably rescaling in order to keep the surface area fixed, the hypersurfaces get rounder as the principal curvatures approach each other and converge to the sphere in infinite time. Another important property satisfied by MCF in codimension 1 is the avoidance principle (or comparison principle), that follows from an application of the parabolic maximum principle:

Theorem 2 (Avoidance Principle)**.** *Let M_t, M'_t be two compact hypersurfaces evolving by Mean Curvature Flow. If they are initially disjoint, they remain disjoint as long as they are both smooth.*

The case of codimension higher than one in full generality was studied after several years; the complexity of the normal bundle prevents from isolating suitable simple scalar quantities for the description of the evolution, and the associated equations are also more involved. For this reason, the first studied examples had strong additional assumptions on the evolving submanifold: they were either the evolution of a Lagrangian initial immersion (see e.g. [67, 73, 57]), or had the structure of a graph of a smooth function from a space form into a space form of the same type (for example [71]).

The first general results using a pinching assumption were developed in the PhD thesis of Charles Baker [12], and in the related article [5], in 2012; the starting hypothesis is the same bound on the second fundamental form in terms of the norm of the mean curvature vector appearing in our Theorem 1. Under the assumption, the submanifold evolves into a round point in finite time and becomes spherical in the process. We underline that one of the differences between the case of hypersurfaces and of general submanifolds is the lack of an avoidance principle in the latter; flows of closed submanifold still lose smoothness in finite time, thanks to the maximum principle applied to the function $|\varphi|^2 + 2nt$, see for example [68]. Baker also partially classified a subclass of Type I singularities (the definition is the same as in the 1-codimensional case, with respect to the tensorial norm of h), parabolically rescaling by remaining time. He chose a fixed limit point as center of the rescalings and a scale factor $\lambda_k = \frac{1}{\sqrt{2(T-t_k)}}$, for a sequence t_k converging to the singular time T. By Huisken's monotonicity formula, that also holds in higher codimension, and by compactness arguments for immersions, he proved that under his pinching assumption the sequence of rescaled flows admits a (subsequential) ancient limit and it must be a sphere as for hypersurfaces.

The following sections of this chapter will only regard immersions having codimension higher than one: the full mean curvature vector will be denoted with H instead of $\mathbf{H}$, as it should not be a source of confusion.

2.2 The rigidity theorem for higher codimensional MCF

The proof of Theorem 1 is built over some estimates for high codimensional immersions shown in Charles Baker's PhD thesis and it is inspired by the proof of the first equivalence in Huisken and Sinestrari's Theorem. In [5], Andrews and Baker have proved that the pinching inequality (2.0.2) is invariant under the flow, and that submanifolds satisfying this condition at an initial time T_0 evolve into a "round point" in finite time. We recall the evolution equations for the relevant geometric quantities derived in that paper:

$$\frac{\partial g_{ij}}{\partial t} = -2H \cdot h_{ij} \tag{2.2.1}$$

$$\frac{\partial H}{\partial t} = \Delta H + H \cdot h_{pq} h_{pq} \tag{2.2.2}$$

$$\frac{\partial |h|^2}{\partial t} = \Delta |h|^2 - 2|\nabla h|^2 + 2R_1 \tag{2.2.3}$$

$$\frac{\partial |H|^2}{\partial t} = \Delta |H|^2 - 2|\nabla H|^2 + 2R_2 \tag{2.2.4}$$

where R_1 and R_2 are two terms related to the curvature of the normal bundle and are expressed in coordinates as:

$$R_1 = \sum_{\alpha,\beta} \left(\sum_{i,j} h_{ij\alpha} h_{ij\beta} \right)^2 + \sum_{i,j,\alpha,\beta} \left(\sum_p h_{ip\alpha} h_{jp\beta} - h_{jp\alpha} h_{ip\beta} \right)^2 \tag{2.2.5}$$

$$R_2 = \sum_{i,j} \left(\sum_{\alpha} H_\alpha h_{ij\alpha} \right)^2. \tag{2.2.6}$$

As in [5, 48], for fixed small $\sigma > 0$ we consider the function

$$f_\sigma = \frac{|h|^2 - \frac{1}{n}|H|^2}{|H|^{2(1-\sigma)}} \tag{2.2.7}$$

and we observe that, for any σ, f_σ vanishes at $x \in M$ if and only if x is an umbilical point. Therefore, if $f_\sigma = 0$ everywhere on M_t for some t, then M_t is a totally umbilical submanifold, hence an n-dimensional sphere in $\mathbb{R}^{n+k}$. As spheres evolve by homothetic shrinking, f_σ will remain zero for all subsequent times. Thus, to obtain Theorem 1, it is enough to show that f is identically zero on some time interval $(-\infty, T_1]$, with $T_1 < 0$. To this purpose, we prove the following estimate.

Proposition 3. *Under the hypotheses of Theorem 1, there are constants $\alpha, \beta > 0$ depending only on n, C_0 and $C = C(C_0, n, h_0) > 0$ such that, for all $[T_0, T_1] \subset (-\infty, -1)$ and for all $p > \alpha$, $\sigma \leq \frac{\beta}{\sqrt{p}}$, $\sigma p > n$, we have*

$$\left(\int_{M_t} f_\sigma^p \right)^{\frac{1}{\sigma p}} \leq \frac{C}{|T_0|^{1-\frac{n}{\sigma p}} - |t|^{1-\frac{n}{\sigma p}}} \qquad \forall t \in (T_0, T_1]. \tag{2.2.8}$$

The proposition immediately implies Theorem 1. Indeed, sending T_0 to $-\infty$ in (2.2.8), we obtain that f_σ^p is zero for every $t < T_1$ for suitable values of σ and p, thus for all $t \in (-\infty, 1)$. M_t is then a family of shrinking spheres.

We now prove Proposition 3.

Proof. The first part of the proof follows the strategy of [48] together with the estimates of [5]. If we set

$$\epsilon_\nabla = \frac{3}{n+2} - C_0,$$

where C_0 is the constant in our pinching assumption (2.0.2), then ϵ_∇ is positive and Proposition 13 in [5] states that

$$\begin{aligned} \frac{d}{dt}\int_{M_t} f_\sigma^p \, d\mu_t \leq & -\frac{p(p-1)}{2}\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 \, d\mu_t \\ & - p\epsilon_\nabla \int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2 \, d\mu_t + 2p\sigma \int_{M_t} |H|^2 f_\sigma^p \, d\mu_t \end{aligned} \tag{2.2.9}$$

for any $p \geq \max\left\{2, \frac{8}{\epsilon_\nabla + 1}\right\}$.

In addition, Proposition 12 of the same paper shows that there exists a constant ϵ_0 depending only on C_0 and n such that

$$\begin{aligned} \int_{M_t} |H|^2 f_\sigma^p \, d\mu_t \leq & \frac{p\eta + 4}{\epsilon_0}\int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}}|\nabla H|^2 \, d\mu_t \\ & + \frac{p-1}{\epsilon_0 \eta}\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 \, d\mu_t \end{aligned}$$

for all $p \geq 2$, $\eta > 0$. If we fix $\eta = \frac{8\sigma}{\epsilon_0}$ and we take any p, σ such that $p > \frac{16}{\epsilon_\nabla}$,

$\sigma \leq \frac{\epsilon_0}{8}\sqrt{\frac{\epsilon_\nabla}{p}}$, we obtain

$$
\begin{aligned}
4p\sigma \int_{M_t} |H|^2 f_\sigma^p \, d\mu_t \leq & \left(\frac{32\sigma^2 p}{\epsilon_0^2} + \frac{16\sigma}{\epsilon_0} \right) p \int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} |\nabla H|^2 \, d\mu_t \\
& + \frac{p(p-1)}{2} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu_t \\
\leq & \left(\frac{\epsilon_\nabla}{2} + \frac{\epsilon_\nabla}{2} \right) p \int_{M_t} \frac{f_\sigma^{p-1}}{|H|^{2(1-\sigma)}} |\nabla H|^2 \, d\mu_t \\
& + \frac{p(p-1)}{2} \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu_t,
\end{aligned}
$$

so that (2.2.9) implies

$$
\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu_t \;\leq\; -2p\sigma \int_{M_t} |H|^2 f_\sigma^p \, d\mu_t \tag{2.2.10}
$$

for all $p > \frac{16}{\epsilon_\nabla}$, $\sigma \leq \frac{\epsilon_0}{8}\sqrt{\frac{\epsilon_\nabla}{p}}$. Thanks to the definition and our pinching assumption, we have

$$
0 \leq f_\sigma \leq \left(C_0 - \frac{1}{n} \right) |H|^{2\sigma} \leq |H|^{2\sigma},
$$

so we obtain

$$
\begin{aligned}
\frac{d}{dt} \int_{M_t} f_\sigma^p \, d\mu_t & \leq -2p\sigma \int_{M_t} f_\sigma^{p+\frac{1}{\sigma}} \, d\mu_t \\
& \leq -2p\sigma \left(\int_{M_t} f_\sigma^p d\mu_t \right)^{1+\frac{1}{\sigma p}} \cdot |M_t|^{-\frac{1}{\sigma p}}
\end{aligned} \tag{2.2.11}
$$

using Hölder's inequality, where $|M_t|$ is the volume of M_t.

Claim. *There exists a constant $C = C(n, h_0)$ such that*

$$
|M_t| \leq C|t|^n, \text{ for all } t \leq -1. \tag{2.2.12}
$$

Once the claim is proved, the statement of the proposition follows easily. In fact, setting $\psi(t) = \int_{M_t} f_\sigma^p d\mu_t$, and using (2.2.11), we have

$$
\frac{d}{dt}\psi^{-\frac{1}{\sigma p}} = -\frac{1}{\sigma p}\psi^{-\left(\frac{1}{\sigma p}+1\right)} \frac{d}{dt}\psi \geq C(|t|)^{-\frac{n}{\sigma p}}.
$$

As $\psi(t) \neq 0$ implies $\psi(s) \neq 0$ for $s < t$, we obtain, integrating on a time interval $(T_0, t]$, with $t \leq T_1$,

$$\psi^{-\frac{1}{\sigma p}}(t) \geq \psi^{-\frac{1}{\sigma p}}(T_0) + C\int_{|t|}^{|T_0|} \tau^{-\frac{n}{\sigma p}} d\tau > C\int_{|t|}^{|T_0|} \tau^{-\frac{n}{\sigma p}} d\tau$$
$$> C\left(|T_0|^{1-\frac{n}{\sigma p}} - |t|^{1-\frac{n}{\sigma p}}\right),$$

as $\sigma p > n$.

It remains to prove claim (2.2.12). For this part, we cannot adapt the technique of [48], which only applies to hypersurfaces, and we use a different argument.

We first recall a result by Chen [20], which gives a lower bound for the minimum of sectional curvatures $K_\pi(p)$ under our pinching assumption:

$$\min_{\pi \subset Gr(2,T_pM)} K_\pi(p) \geq \frac{1}{2}\left(\frac{1}{n-1} - C_0\right)|H|^2(p).$$

In particular, this implies that all the evolving submanifolds have nonnegative Ricci curvature, so by Bishop-Gromov's Theorem we can bound the volume of balls of arbitrary radius in M_t with the volume of balls in $\mathbb{R}^n$.

Theorem 4 (Bishop-Gromov)**.** *Let M be a complete Riemannian manifold with Ricci curvature Ric bounded from below, $Ric_{ij} \geq (n-1)Kg_{ij}$ for some constant $K \in \mathbb{R}$. Let $B(r)$ be a geodesic ball of radius r in M. Then*

$$Vol(B(r)) \leq Vol(B^k(r)),$$

where $B^k(r)$ is geodesic ball of radius r in the space form of the same dimension and of constant sectional curvature K.

In particular, if d_t is the intrinsic diameter of M_t, we have

$$|M_t| \leq Vol(B_{d_t}) = \theta d_t^n, \tag{2.2.13}$$

where θ is a constant only depending on C_0 and n.

To bound the intrinsic diameter, we estimate the change of the distance between two points during the evolution, using a technique inspired by the one of [36, §17] for the Ricci Flow.

Let P and Q be two fixed points in M and let γ be a curve from P and Q. The length of γ with respect to the evolving metric varies with time along the flow, and we denote it by $L[\gamma_t]$. The evolution equation (2.2.1) for the metric gives

$$\frac{d}{dt}L[\gamma_t] = -\int_\gamma H^\alpha h_{ij\alpha}\gamma'^i\gamma'^j ds,$$

where γ is parametrized by arclength at time t and γ' is the tangent vector. By the same proof as in [36, Lemma 17.3], we can estimate the time derivative of the distance from P and Q as follows. Let Ξ denote the set of all distance minimizing geodesics between P and Q at time t; Ξ is a compact set. Then

$$-\sup_{\gamma\in\Xi}\int_{\gamma_t} H^\alpha h_{ij\alpha}{\gamma'}^i{\gamma'}^j ds \le \frac{d}{dt}d(P,Q,t) \le -\inf_{\gamma\in\Xi}\int_{\gamma_t} H^\alpha h_{ij\alpha}{\gamma'}^i{\gamma'}^j ds. \tag{2.2.14}$$

We have $|H^\alpha h_{ij\alpha}{\gamma'}^i{\gamma'}^j| \le |H||h| \le C_0|H|^2$. As estimate (2.2) implies, for the Ricci curvature,

$$Ric(\gamma',\gamma') \ge (n-1)\frac{1}{2}\left(\frac{1}{n-1} - C_0\right)|H|^2,$$

we also have

$$H^\alpha h_{ij\alpha}{\gamma'}^i{\gamma'}^j \le C Ric(\gamma',\gamma')$$

along γ, with C a constant depending only on C_0 and n.

We now recall Theorem 17.4 from [37], which states the following. If the Ricci curvature of a n-dimensional manifold is nonnegative, for any curve γ of length L and for any $v \in (0, \frac{L}{2})$,

$$\int_v^{L-v} Ric(\gamma',\gamma')\,ds \le \frac{2(n-1)}{v}.$$

We observe that the norm of the Ricci tensor of the submanifold is uniformly bounded away from the singular time, as a consequence of the boundedness of the second fundamental form and the Gauss equations. Thus, there is $R > 0$ such that $|Ric_{ij}(x,t)| \le R$ for $t \le -1$. Then (2.2.14) implies

$$\begin{aligned}-\frac{d}{dt}d(P,Q,t) &\le \int_0^v CR\,ds + \frac{2(n-1)}{v} + \int_{L-v}^L CR\,ds\\ &\le 2CRv + \frac{2(n-1)}{v}.\end{aligned}$$

Since our aim is to estimate the diameter of M_t for large negative times, it is not restrictive to assume that $d(P,Q,t) > 2$. Then we can choose $v = 1$ to obtain

$$-\frac{d}{dt}\,d(P,Q,t)\ \le 2CR + 2(n-1) =: C'$$

and integrating on $[t,-1]$ for an arbitrary $t < -1$, we get

$$d(P,Q,t) \le d(P,Q,-1) + C'(|t|-1),$$

which implies

$$d_t \leq d_{-1} + C'(|t| - 1) \leq C''|t| \qquad \forall t \leq -1. \tag{2.2.15}$$

Using (2.2.13), we conclude that $|M_t| \leq C''|t|^n$, proving our claim (2.2.12) and the proposition. $\square$

2.3 High codimension Mean Curvature Flow in the sphere

Mean Curvature Flow of hypersurfaces in ambient manifolds different than Euclidean space was investigated soon after the former, for example in [43, 44]. In particular, the reaction terms in the evolution equation for the curvature favour convergence if the evolution develops in a positively curved ambient space; an assumption weaker than strict convexity is enough for a starting hypersurface to contract to a point or converge to a totally geodesic submanifold. Charles Baker generalized Huisken's work using a pinching assumption in [12]; recently, a similar result has been obtained by Pipoli and Sinestrari for submanifolds of the Complex Projective Space [59].

In [48], Huisken and Sinestrari also proved a rigidity theorem for ancient hypersurfaces of the sphere, where the condition of positivity of the mean curvature is relaxed to $H \geq 0$ and the ancient solution can be either a totally geodesic hypersurface or a spherical cap.
We want to characterize the same phenomenon on spheres $\mathbb{S}^{n+k}_K$ of constant sectional curvature K, showing that any ancient solution of the Mean Curvature Flow that satisfies a uniform pinching condition on the sphere is a family of shrinking totally umbilical submanifolds; we will refer to them with the term "spherical caps". We also prove that we can relax the condition to admit points with $|H| = 0$ and the ancient solution will be either a spherical cap or a totally geodesic submanifold.

Theorem 5. *Let M^n_t be a closed ancient solution of* (2.0.1) *in $\mathbb{S}^{n+k}_K$.*

1. *If, for all $t \in (-\infty, 0)$, $0 < |h|^2 \leq \frac{4}{3n}|H|^2$ holds, then M_t is a shrinking spherical cap.*

2. If, for all $t \in (-\infty, 0)$, we have $|h|^2 \leq \alpha |H|^2 + \beta K$, with

$$\begin{cases} \alpha = \dfrac{1}{n-1} & \beta = 2 & n \geq 4 \\[2mm] \alpha = \dfrac{4}{9} & \beta = \dfrac{3}{2} & n = 3 \\[2mm] \alpha = \dfrac{2}{4-\beta} & \beta < \dfrac{12}{13} & n = 2, \end{cases}$$

then M_t is either a shrinking spherical cap or a totally geodesic submanifold.

Proof. The evolution equations for metric and curvature in a spherical ambient manifold, see [12], are given by:

$$\frac{\partial |H|^2}{\partial t} = \Delta |H|^2 - 2|\nabla H|^2 + 2nK|H|^2 + 2R_2 \tag{2.3.1}$$

$$\frac{\partial |h|^2}{\partial t} = \Delta |h|^2 - 2|\nabla h|^2 + 2R_1 + 4K|H|^2 - 2nK|h|^2, \tag{2.3.2}$$

where R_1 and R_2 are as in (2.2.5) and (2.2.6). To prove the first statement, let us define

$$f_0 = \frac{|h|^2 - \frac{1}{n}|H|^2}{|H|^2}.$$

Using the evolution equations above, we find

$$\begin{aligned} \frac{\partial f_0}{\partial t} =& \Delta f_0 + \frac{4H}{|H|^2} \langle \nabla |H|, \nabla f_0 \rangle - \frac{2}{|H|^2} \left[|\nabla h|^2 - \left(\frac{1}{n} + f_0 \right) |\nabla H|^2 \right] \\ &+ \frac{2}{|H|^2} \left[R_1 - \left(\frac{1}{n} + f_0 \right) R_2 \right] - 4nK f_0. \end{aligned} \tag{2.3.3}$$

By our pinching assumption, we have $\frac{1}{n} + f_0 \leq \frac{4}{3n} < \frac{3}{n+2}$. Then the gradient terms give a nonpositive contribution, as it is well known, see e.g. [12], that

$$|\nabla h|^2 \geq \frac{3}{n+2} |\nabla H|^2. \tag{2.3.4}$$

In order to analyze the reaction terms, we recall some more notation from [12]. By assumption, $|H|^2 > 0$ everywhere, so we can choose an adapted orthonormal basis $\left\{ \{e_i\}_{i=1}^n, \{\nu_\alpha\}_{\alpha=1}^k \right\}$ for the sphere, such that $\{\nu_\alpha\}_{\alpha=1}^k$ is a frame for the normal space of the submanifold with $\nu_1 = \frac{H}{|H|}$, while, if we write $\mathring{h} = h - \frac{1}{n} H \otimes g = \sum\limits_{\alpha=1}^{k} \mathring{h}_\alpha$, the frame $\{e_i\}_{i=1}^n$ is tangent and diagonalizes $\mathring{h}_1$. In addition, we denote the norm of $\mathring{h}$ in the other directions with $|\mathring{h}_-|^2$, so that $|\mathring{h}|^2 = |\mathring{h}_1|^2 + |\mathring{h}_-|^2$.

With this choice we have

$$R_2 = |\mathring{h}|^2 |H|^2 + \frac{1}{n} |H|^4, \tag{2.3.5}$$

and the following estimate holds, proved in [12, §5.2],

$$R_1 - \frac{1}{n} R_2 \leq |\mathring{h}_1|^4 + \frac{1}{n} |\mathring{h}_1|^2 |H|^2 + 4 |\mathring{h}_1|^2 |\mathring{h}_-|^2 + \frac{3}{2} |\mathring{h}_-|^4. \tag{2.3.6}$$

From this, we obtain

$$\begin{aligned} 2\left[R_1 - \left(\frac{1}{n} + f_0\right) R_2\right] \leq{}& 2|\mathring{h}_1|^4 + \left(\frac{2}{n} - 2f_0\right) |\mathring{h}_1|^2 |H|^2 - \frac{2}{n} f_0 |H|^4 \\ &+ 8|\mathring{h}_1|^2 |\mathring{h}_-|^2 + 3|\mathring{h}_-|^4. \end{aligned}$$

Using the definition of f_0 and substituting $|\mathring{h}|^2 = |\mathring{h}_1|^2 + |\mathring{h}_-|^2$, some terms simplify and the right hand side can be rewritten as

$$|\mathring{h}_-|^2 \left(6|\mathring{h}_1|^2 - \frac{2}{n}|H|^2 + 3|\mathring{h}_-|^2\right).$$

Thus, using our pinching assumption, we conclude

$$\begin{aligned} 2R_1 - 2\left(\frac{1}{n} + f_0\right) R_2 &\leq |\mathring{h}_-|^2 \left(6|\mathring{h}_1|^2 - \frac{2}{n}|H|^2 + 3|\mathring{h}_-|^2\right) \\ &\leq 2|\mathring{h}_-|^2 \left(3|h|^2 - \frac{4}{n}|H|^2\right) \leq 0. \end{aligned} \tag{2.3.7}$$

So we can proceed as in the first case of [48]; we have

$$\frac{\partial f_0}{\partial t} \leq \Delta f_0 + \frac{4H}{|H|^2} \langle \nabla f_0, \nabla H \rangle - 4nK f_0. \tag{2.3.8}$$

If there existed $t_1 < 0$ such that $f_0(t_1)$ is not identically zero on M_{t_1}, we could apply the maximum principle to get

$$0 < \max_{M_{t_1}} f_0 \leq e^{-4nK(t_1 - t)} \max_{M_t} f_0 \quad \forall t < t_1 \tag{2.3.9}$$

and the function would explode for $t \to -\infty$, contradicting our assumption $f_0 \leq \frac{4}{3n} - \frac{1}{n}$. So f_0 is identically zero for all times and M_t is a family of totally umbilical submanifolds of the sphere.

For the second case, we do not require $|H|^2 \neq 0$; we want to show that M_t is a shrinking spherical cap or an equator.

We can follow a computation similar to par. 5.3 in [12] and consider a perturbed pinching function $f = \frac{|\mathring{h}|^2}{a|H|^2+bK}$ where $a = \alpha - \frac{1}{n}$ and the constant b is given by:

$$b = \begin{cases} \dfrac{11}{10} & n \geq 4 \\ \dfrac{33}{40} & n = 3 \\ \dfrac{24\beta}{13(4-\beta)} & n = 2. \end{cases}$$

The function f satisfies the equation

$$\begin{aligned} \frac{\partial}{\partial t} f = \Delta f &+ \frac{2a}{a|H|^2+bK} \left\langle \nabla_i |H|^2, \nabla_i f \right\rangle \\ &- \frac{2}{a|H|^2+bK} \left(|\nabla h|^2 - \frac{1}{n}|\nabla H|^2 - \frac{a|\mathring{h}|^2}{a|H|^2+bK}|\nabla H|^2 \right) \\ &+ \frac{2}{a|H|^2+bK} \left(R_1 - \frac{1}{n}R_2 - nK|\mathring{h}|^2 - \frac{aR_2|\mathring{h}|^2}{a|H|^2+bK} - \frac{anK|\mathring{h}|^2|H|^2}{a|H|^2+bK} \right). \end{aligned} \tag{2.3.10}$$

Using (2.3.4) and $b \leq \beta$, we can estimate the gradient terms as follows:

$$\begin{aligned} &- \left(|\nabla h|^2 - \frac{1}{n}|\nabla H|^2 - \frac{a|\mathring{h}|^2}{a|H|^2+bK}|\nabla H|^2 \right) \\ &\leq - \left(\frac{2(n-1)}{n(n+2)} - \frac{a^2|H|^2 + a\beta K}{a|H|^2+bK} \right) |\nabla H|^2 \\ &\leq - \left(\frac{2(n-1)}{n(n+2)} - \frac{a\beta}{b} \right) |\nabla H|^2, \end{aligned}$$

and this expression is negative for the chosen values of b.

At the points with $|H|^2 \neq 0$, we can employ the adapted frame to analyse the reaction terms in (2.3.10). We set

$$\tilde{R} = \left(a|H|^2 + bK\right) \left(R_1 - \frac{1}{n}R_2 - nK|\mathring{h}|^2 \right) - aR_2|\mathring{h}|^2 - anK|\mathring{h}|^2|H|^2$$

and we use (2.3.5)–(2.3.6) to obtain

$$\begin{aligned}\tilde{R} \leq a|H|^2 &\left(3|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{3}{2}|\mathring{h}_-|^4 - \frac{1}{n}|\mathring{h}_-|^2|H|^2\right)\\ &+ bK\left(|\mathring{h}_1|^4 + \frac{1}{n}|\mathring{h}_1|^2|H|^2 + 4|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{3}{2}|\mathring{h}_-|^4\right)\\ &- nK|\mathring{h}|^2(2a|H|^2 + bK).\end{aligned}$$

We use a Peter-Paul inequality to estimate

$$|\mathring{h}_1|^4 + 4|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{3}{2}|\mathring{h}_-|^4 \leq \frac{5}{3}|\mathring{h}_1|^4 + \frac{10}{3}|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{5}{3}|\mathring{h}_-|^4 = \frac{5}{3}|\mathring{h}|^4.$$

Thanks to our assumptions, we have $|H|^2 \geq \frac{|\mathring{h}|^2 - \beta K}{a}$, $a \leq \frac{1}{3n}$ and $b \leq \beta$. Using this, we find

$$\begin{aligned}\tilde{R} &\leq a|H|^2|\mathring{h}_-|^2\left(3|\mathring{h}|^2 - \frac{1}{na}\left(|\mathring{h}|^2 - \beta K\right)\right)\\ &\quad + \frac{5}{3}bK|\mathring{h}|^4 + \frac{bK}{n}|\mathring{h}_1|^2|H|^2 - nK|\mathring{h}|^2(2a|H|^2 + bK)\\ &\leq |H|^2\left(\frac{\beta K}{n}|\mathring{h}|^2 - 2anK|\mathring{h}|^2\right) + \frac{5}{3}bK|\mathring{h}|^4 - nbK^2|\mathring{h}|^2.\end{aligned}$$

Since our choice of the constants implies that $\beta/n < 2an$, we can use again $|H|^2 \geq \frac{|\mathring{h}|^2 - \beta K}{a}$ and $\beta \geq b$ to find that, for any small $\epsilon > 0$

$$\begin{aligned}\tilde{R} \leq &- \epsilon aK|H|^2|\mathring{h}|^2 - \frac{1}{a}(|\mathring{h}|^2 - \beta K)\left(2anK - \epsilon aK - \frac{\beta K}{n}\right)|\mathring{h}|^2\\ &+ \frac{5}{3}bK|\mathring{h}|^4 - nbK^2|\mathring{h}|^2\\ \leq &- \left[2n - \epsilon - \frac{\beta}{na} - \frac{5}{3}b\right]K|\mathring{h}|^4 - \left[\frac{\beta^2}{na} + nb - 2n\beta\right]K^2|\mathring{h}|^2\\ &- \epsilon K(a|H|^2 + bK)|\mathring{h}|^2.\end{aligned}$$

With our choice of a, β, b, we can check that both expressions in square parentheses are negative if ϵ is suitably small. The above estimates, together with (2.3.10), imply the inequality

$$\frac{\partial f}{\partial t} \leq \Delta f + \frac{2a}{a|H|^2 + bK}\left\langle \nabla_i|H|^2, \nabla_i f\right\rangle - 2\epsilon Kf.$$

It remains to consider points where $H = 0$. In this case we have $f = \frac{|h|^2}{b}$ and the reaction terms in (2.3.10) become

$$\frac{2}{b}\{R_1 - nK|h|^2\}.$$

As in [12, Lemma 5.1], $R_1 \leq \frac{3}{2}|h|^4$ holds whenever $H = 0$, so that

$$2R_1 - 2nK|h|^2 \leq (3\beta - 2n)\, K|h|^2 = (3\beta - 2n)\, Kbf.$$

Since $\beta < \frac{2n}{3}$, we conclude that the reaction terms are bounded above by a negative multiple of f also at these points. Then, as in the first part of the theorem, we can apply the maximum principle and find a contradiction unless $f \equiv 0$. Since we are allowing $H = 0$, the solution can be either a shrinking spherical cap or a totally geodesic submanifold. $\square$

Chapter 3

Ancient solutions of contractive fully nonlinear flows

In this chapter, we examine ancient solutions of fully nonlinear second-order contractive curvature flows of compact hypersurface immersions into $\mathbb{R}^{n+1}$. The evolution for an immersion φ from a n-dimensional manifold M has the form:

$$\frac{\partial \varphi}{\partial t}(x,t) = -f(\lambda(W(x,t)))\nu(x,t) = -F(W(x,t))\nu(x,t) \tag{3.0.1}$$

where λ is the function that associates to a self-adjoint operator the n-ple $\lambda = (\lambda_1, \ldots, \lambda_n)$ of its ordered eigenvalues. The class of speeds f we consider must satisfy some properties to ensure that (3.0.1) is well-defined, so we will assume throughout:

(H1) $f : \Gamma \to \mathbb{R}$ is a symmetric smooth function, homogeneous of degree α for some $\alpha > 0$, defined on an open symmetric cone $\Gamma \subset \mathbb{R}^n$ containing the positive cone Γ_+;

(H2) f satisfies

$$\dot{f}^i = \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma \text{ for all } i = 1, \ldots, n.$$

The first hypothesis assures that the speed only depends on the values of the principal curvatures and not on their order; to simplify the notation, we will assume $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. It is well known (see, for example, [32, 51]) that the function F is also smooth as a function of the components of W under the assumptions above. The derivatives of F with respect to the curvature are denoted by dots as follows:

$$\frac{d}{ds}F(A+sB)|_{s=0} = \dot{F}^{ij}|_A B_{ij}$$

$$\frac{d^2}{ds^2}F(A+sB)|_{s=0} = \ddot{F}^{ij,kl}|_A B_{ij}B_{kl}$$

for any A, B symmetric matrices such that $\lambda(A)$ and $\lambda(B)$ belong to Γ. Assumption (H2) ensures that $\dot{F}$ is positive definite and that (3.0.1) is a (weakly) parabolic system, since the relation

$$\dot{f}^i_a \delta^{ij} = \dot{F}^{ij}_A$$

holds for any diagonal A, with $a = \lambda(A)$. We denote by $\mathcal{L}$ the elliptic operator on $C^\infty(M)$ defined as $\mathcal{L} = \dot{F}^{ij}\nabla_i\nabla_j$. We assume that the flow is isotropic, so invariant under isometries of the ambient Euclidean space. The system is is also invariant under reparametrizations of the submanifold; as for Mean Curvature Flow, this translates into degeneracy in directions tangential to M. Assumption (H1) also implies invariance under parabolic rescalings. We remark that homogeneity and monotonicity imply that f is strictly positive on Γ_+, as the Euler relation

$$\frac{\partial G}{\partial \lambda_i}(\lambda)\lambda_i = \alpha G(\lambda)\,. \tag{3.0.2}$$

holds for any α-homogeneous function G.
Some examples of speeds f that satisfy the properties above are:

1. the k-th mean curvatures H_k (for $k = 1, \dots, n$). We recall that H_k is defined as the k-th elementary symmetric polynomial computed on the principal curvatures of the immersion:

$$H_k(\lambda) = \sum_{1 \le j_1 < j_2 < \dots < j_k \le n} \lambda_{j_1}\lambda_{j_2}\dots\lambda_{j_k}$$

   (we remark that some authors define H_k multiplying by a normalization factor $\binom{n}{k}^{-1}$). Each H_k is homogeneous of degree k; $H_1 = H$, $H_2 = \frac{1}{2}scal$, while H_n coincides with the Gaussian curvature K;

2. roots or powers $f = H_k^{\frac{a}{b}}$ (for a,b positive integers) and ratios $f = \frac{H_l}{H_m}$ $(0 < m < l \in \mathbb{N})$ of the k-th mean curvatures;

3. power means $\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i^q\right)^{\frac{1}{q}}$ for integer $q \neq 0$;

4. positive linear combinations, geometric means and images of homogeneous functions of the above.

These functions also enjoy convexity/concavity properties and are covered by our results; a good reference is [4].

This class of flows can be regarded as a natural generalization of the Mean Curvature Flow, which obviously corresponds to the choice of $f(\lambda) = H_1 = \sum_{i=1}^{n} \lambda_i$. The first results were developed in the Eighties by Tso and Chow ([70], [23], [24]) for speeds with an established geometric meaning, namely positive roots of the Gaussian curvature, or the square root of the scalar curvature. In all cases a closed convex hypersurface converges to a point in finite time, but roundness of the rescaled limits was only proved for velocities of homogeneity 1. The latter was demonstrated only recently by Brendle, Choi and Daskalopoulos ([22, 17]) for flows involving powers of K; we will provide more details in a specific section. The "round point" behaviour was later established by Ben Andrews in 1994 ([2]) for speeds within a large class of 1-homogeneous functions with convexity or concavity properties. Results for homogeneity different from 1 are harder to obtain, since the analysis of the evolution equations associated to the system is more intricated; for instance, reaction terms with the wrong sign may prevent from proving invariance of pinching conditions easily. Known examples exist of evolutions that do not deform a convex hypersurface into a sphere: the rescaled solutions of (3.0.1) with $f = K^{\frac{1}{n+2}}$ converge to ellipsoids. In all cases described, a key step of the proof of asymptotic roundness was the identification of a monotonic quantity characterizing the sphere: suitable quotients of curvature functions of homogeneity one appear in [2] for 1-homogeneous speeds, while an entropy was introduced for the Gauss Curvature Flows in [34, 6]. The choice of the right paramtetrization is also particularly relevant in the investigation of the evolving geometric quantities of the submanifold; some estimates are easier to obtain exploiting fully the convexity of the hypersurface, by switching to polar coordinates or the support function. These techniques are also crucial for our case of study.

We consider convex ancient solutions of (3.0.1) satisfying a uniform pinching condition on the principal curvatures expressed in the following form: there exists $C_1 > 0$ such that

$$\lambda_n(x,t) < C_1 \lambda_1(x,t), \qquad \text{for all } t \in (-\infty, 0), \text{ for all } x \in M_t. \tag{3.0.3}$$

Our results for this chapter will be rigidity theorems for compact, convex, pinched hypersurfaces under several flows. We will need some additional assumptions on the subcone where the speed is positive or some convexity/concavity condition, that will be described in each case.

We first present an estimate on the speed and the diameter of a pinched ancient solution which holds under very general assumptions on F. To this purpose, we recall some definitions. If M is a n-dimensional embedded submanifold of $\mathbb{R}^{n+1}$ bounding a convex body Ω, the inner and outer radii of M are defined respectively as

$$\rho_- = \sup\left\{r \mid B_r(y) \subset \Omega \text{ for some } y \in \mathbb{R}^{n+1}\right\}$$
$$\rho_+ = \inf\left\{r \mid \Omega \subset B_r(y) \text{ for some } y \in \mathbb{R}^{n+1}\right\}.$$

Along our flow, these quantities depend on time and will be denoted by $\rho_\pm(t)$.

By a result in [2], if the pinching condition (3.0.3) holds, then there also exists $\bar{C}_1 = \bar{C}_1(C_1, n)$ such that

$$\rho_+(t) \leq \bar{C}_1 \rho_-(t), \qquad \forall\, t < 0. \tag{3.0.4}$$

3.1 A general estimate

We show that the pinching condition (3.0.3) implies strong bounds on the inner and outer radii and on the speed of an ancient solution. The result holds for general positively homogeneous speeds.

Theorem 6. *Let $M_t = \varphi(M,t)$ be a compact convex ancient solution of the flow (3.0.1) defined for $t \in (-\infty, 0)$ and shrinking to a point as $t \to 0$. Suppose that the pinching condition (3.0.3) is satisfied for some $C_1 > 0$. Then there exist constants C_2, C_3 such that for all $t \in (-\infty, -1)$:*

$$C_2^{-1} \rho_+(t) \leq |t|^{\frac{1}{\alpha+1}} \leq C_2 \rho_-(t)\,, \tag{3.1.1}$$

$$\sup F(\cdot, t) \leq C_3 |t|^{-\frac{\alpha}{\alpha+1}}\,. \tag{3.1.2}$$

Proof. For simplicity, we assume that f is normalized in order to satisfy $f(1,1,\ldots,1) = 1$. Then the spherical solution of (3.0.1) that shrinks to a point at time $t = 0$ has a radius given by $(\alpha+1)|t|^{\frac{1}{\alpha+1}}$. By comparison, we deduce

$$\rho_-(t) \leq (\alpha+1)|t|^{\frac{1}{\alpha+1}} \leq \rho_+(t), \qquad \forall\, t < 0. \tag{3.1.3}$$

Combining these inequalities with (3.0.4), we immediately obtain (3.1.1).

To bound $\sup F$ from above, we use a well-known technique first introduced in [70]. We fix any $t_0 < 0$ and we call z_0 the center of a ball realizing $\rho_-(t_0)$. We denote by $u(x,t) = \langle \varphi(x,t) - x_0, \nu(x,t) \rangle$ the support function centered at x_0 and defined on M. By convexity, we have $u(\cdot, t_0) \geq \rho_-(t_0)$ and due to the shrinking nature of the flow the inequality holds for all times $t \leq t_0$. Hence, the function

$$q(x,t) = \frac{F(x,t)}{2u(x,t) - \rho_-(t_0)}$$

is well defined for $t \in (-\infty, t_0)$. We have the evolution equation, see e.g. [8]:

$$\left(\frac{\partial}{\partial t} - \mathcal{L} \right) q = \frac{4}{2u - \rho_-(t_0)} \dot{F}^{ij} \nabla_i u \nabla_j q + \frac{F((1+\alpha)F - \rho_-(t_0)\dot{F}^{ij} h_{ik} h_j^k)}{(2u - \rho_-(t_0))^2}.$$

The pinching condition allows to estimate

$$\dot{F}^{ij} h_{ik} h_j^k = \sum_{i=1}^n \frac{\partial f}{\partial \lambda_i} \lambda_i^2 \geq \lambda_1 \sum_{i=1}^n \frac{\partial f}{\partial \lambda_i} \lambda_i = \alpha \lambda_1 F.$$

Let us set $|\lambda| = \sqrt{\sum_i \lambda_i^2}$. Define

$$\overline{f} = \max \{ f(\lambda_1, \ldots, \lambda_n) \ : \ 0 < \lambda_n \leq C_1 \lambda_1, \ |\lambda| = 1 \}.$$

By homogeneity, we have

$$f(\lambda_1, \ldots, \lambda_n) \leq \overline{f} |\lambda|^\alpha \leq \overline{f} (\sqrt{n} C_1 \lambda_1)^\alpha, \tag{3.1.4}$$

for all $(\lambda_1, \ldots, \lambda_n) \in \Gamma_+$ such that $\lambda_n \leq C_1 \lambda_1$. It follows

$$\dot{F}^{ij} h_{ik} h_j^k \geq C F^{1+\frac{1}{\alpha}}, \tag{3.1.5}$$

where we denote by C any constant only depending on n, C_1. If we define $Q(t) = \sup_{M_t} q(x,t)$, we obtain the inequality

$$\frac{d}{dt} Q \leq Q^2 (1 + \alpha - C \rho_-(t_0) F^{\frac{1}{\alpha}}) \leq Q^2 (1 + \alpha - C \rho_-(t_0)^{1+\frac{1}{\alpha}} Q^{\frac{1}{\alpha}}). \tag{3.1.6}$$

It follows that $Q(t) \geq \psi(t)$ for all $t \leq t_0$, where ψ is the solution to the equation

$$\frac{d}{dt} \psi = \psi^2 (1 + \alpha - C \rho_-(t_0)^{1+\frac{1}{\alpha}} \psi^{\frac{1}{\alpha}}), \qquad t \leq t_0, \tag{3.1.7}$$

with final datum $\psi(t_0) = Q(t_0)$. It is easily seen that, if $\psi(t_0)$ is such that the left hand side is negative, that is, if

$$Q(t_0)^{\frac{1}{\alpha}} > \frac{1+\alpha}{C\rho_-(t_0)^{1+\frac{1}{\alpha}}}, \tag{3.1.8}$$

then $\psi(t)$ is decreasing for all $t < t_0$ and blows up at a finite time strictly smaller than t_0, since the right hand side is superlinear in ψ. On the other hand, $Q(t)$ is defined for all $t \in (-\infty, t_0]$ and we obtain a contradiction. It follows that (3.1.8) cannot hold and that the reverse inequality is satisfied. This implies, by the definition of F and by estimate (3.1.1),

$$\max F(\cdot, t_0) \leq 2 \max u(\cdot, t_0) Q(t_0) \leq C \frac{\rho_+(t_0)}{\rho_-(t_0)^{\alpha+1}} \leq C|t|^{-\frac{\alpha}{\alpha+1}}.$$

Since $t_0 < 0$ is arbitrary, this completes the proof of the inequality in (3.1.2). $\square$

In the next section, we will show how the above result can be used to prove that pinched ancient solutions are spheres when the degree of homogeneity is one.

Remark 1. In the previous theorem, the pinching hypothesis (2.0.2) can be replaced by assuming a priori that the solution satisfies a bound of the form (3.0.4), and that the α-root of the speed function f is inverse concave on the positive cone, that is, the function

$$(\rho_1, \ldots, \rho_n) \to f^{-\frac{1}{\alpha}}\left(\rho_1^{-1}, \ldots, \rho_n^{-1}\right)$$

is concave. In fact, in this case property (3.1.5) holds even without the pinching assumption, see Lemma 5 in [8], and the same proof applies.

3.2 Flows with degree of homogeneity 1

Let us now restrict to the case where the speed is homogeneous of degree one. We show here that the estimates of the previous section allow us to give a quick proof of the result that ancient pinched solutions are shrinking spheres. We remark that the result has also been proved independently by Langford and Lynch in [52] using a different approach.

We will require either convexity or concavity of the speed, since this will allow us to apply Krylov-Safonov's regularity theory for fully nonlinear parabolic equations [50] to the equations associated with our flow. An easy case of a convex speed is $f = |\lambda|$, while classical concave examples are $f = H_k^{1/k}$ or $f = H_k/H_{k-1}$.

Theorem 7. *Let $M_t = \varphi(M,t)$ be an ancient solution of the flow* (3.0.1), *with f satisfying* (H1)–(H2) *for $\alpha = 1$. Suppose in addition that f is either convex or concave on the positive cone, and that M_t satisfies* (3.0.3). *Then M_t is a family of shrinking spheres.*

Proof. We first observe that, under our hypotheses, our solution shrinks to a point at the singular time by the results of [2]. In that paper, additional assumptions are made in order to obtain preservation of pinching, but we do not need them here because we are assuming pinching a priori; we observe that the estimates of Theorem 6 apply. We adapt to our setting the procedure of §7 in [2], with the difference that we want to prove convergence to a spherical profile backwards in time rather than forward; we will omit the details that are entirely analogous to [2].

As usual, we assume that the singular time is $t = 0$. We consider a rescaling of the solution, choosing a new time variable $\tau = -\frac{1}{2}\log(-t)$, so that $\tau \in (-\infty, +\infty)$, and define immersions $\widetilde{\varphi}_\tau = (-2t)^{-\frac{1}{2}}(\varphi_t - p)$, where p is the limit point of the original system; quantities pertaining to rescaled solutions will be denoted with a tilde. By Theorem 6, $\tilde{\rho}_\pm$ and $\sup \widetilde{F}$ are bounded from above and below uniformly for all τ. Then, as in Lemma 7.7 in [2], we can write the rescaled solution as a spherical graph and apply Krylov-Safonov's Harnack inequality to show that $\min \widetilde{F}$ is bounded away from zero, ensuring that the curvatures of the rescaled solution stay in a compact subset of the positive cone and that the flow is uniformly parabolic.

The argument is as follows: as in the proof of Theorem 6, we fix $\tau_0 \in \mathbb{R}$; let $\widetilde{r} : \mathbb{S}^n_{\rho_-(\tau_0)} \to \mathbb{R}$ be the radial function of M parametrized as a spherical graph on $\mathbb{S}^n_{\rho_-(\tau_0)}$. The estimates in Theorem 6 allow to conclude

$$\frac{1}{2C_2} \le \widetilde{\rho_-}(\tau_0) \le \widetilde{r} \le 2\widetilde{\rho}_+ \le 2C_2$$

on a time interval $[\tau_0, \tau_0 + \delta]$, where $\delta = \delta(C_2)$ is independent of our choice of τ_0. As the hypersurface is convex and the support function in the same interval satisfies $\langle \widetilde{\varphi}(x,\tau), \widetilde{\nu} \rangle \ge \widetilde{\rho}_-(\tau_0)$, we have that $|\overline{\nabla}\widetilde{r}| \le C$ holds for the derivative of the radius. The evolution equation for the rescaled speed satisfies:

$$\frac{\partial \widetilde{F}}{\partial \tau} - \widetilde{\mathcal{L}}\widetilde{F} = \dot{\widetilde{F}}(\widetilde{W}^2) - \widetilde{F} - \frac{1}{r}\dot{F}\widetilde{g}^{ij}(\overline{\nabla}_i \widetilde{F} \overline{\nabla}_j \widetilde{r} + \overline{\nabla}_i \widetilde{r} \overline{\nabla}_j \widetilde{F}) \tag{3.2.1}$$

where $\overline{\nabla}$ is the connection on the sphere $\mathbb{S}^n_{\rho_-(\tau_0)}$. The coefficients of $\widetilde{F}$ and $\overline{\nabla}\widetilde{F}$ are bounded and we can apply Krylov-Safonov Harnack inequality: $\inf \widetilde{F}$

is bounded from below on the whole subinterval $[\tau_0, \tau_0 + \delta]$. As δ does not depend on τ_0, the bound holds on $(-\infty, +\infty)$.

Then, as in Lemma 7.9 in [2], we can apply Krylov-Safonov's regularity results to show that the support function $\widetilde{u}$ of the rescaled immersion is uniformly bounded in C^k for all k. We can thus find sequences of times τ_k going to $-\infty$ along which $\widetilde{u}$ converges to some limit $\widetilde{u}_{-\infty}$ that is the support function of a convex, compact hypersurface.

To study the structure of the possible limits, we consider a suitable zero-homogeneous function of the curvatures whose integral is monotone along the rescaled flow. We recall for instance the procedure for a concave F. In this case we define $\eta = \frac{|\widetilde{W}|}{\widetilde{F}}$, and we denote by $\eta_0 = \eta(1, \ldots, 1)$. It is easy to check that $\eta \geq \eta_0$, with equality only at umbilical points. Then [2, Lemma 7.9] gives the following estimate, for suitably large $p > 0$:

$$\frac{d}{d\tau} \int_M \widetilde{K}(\eta^p - \eta_0^p) d\widetilde{\mu} \leq -C_5 \int_M \widetilde{K} \frac{|\widetilde{\nabla}\widetilde{W}|^2}{|\widetilde{W}|^2} d\widetilde{\mu} \qquad (3.2.2)$$

where $\widetilde{K}$ is the rescaled Gauss curvature. We can then integrate the inequality above on an interval $[\tau_0, \tau_1]$ and send τ_0 to $-\infty$: as both $\eta^p - \eta_0^p$ and $\widetilde{K}$ are bounded, positive functions, the left-hand side remains bounded, and we conclude that

$$\int_{-\infty}^0 \left(\int_M \widetilde{K} \frac{|\widetilde{\nabla}\widetilde{W}|^2}{|\widetilde{W}|^2} d\widetilde{\mu} \right) d\tau < +\infty.$$

So we can choose a sequence of times $\{\tau_k\}$ such that $|\widetilde{\nabla}\widetilde{W}(\cdot, \tau_k)|_{L^2(M)} \to 0$; by possibly taking a further subsequence, the corresponding support functions $\tilde{u}(\cdot, \tau_k)$ converge smoothly to the support function of a standard sphere. Therefore, along this subsequence, we have

$$\int_M \widetilde{K}(\eta^p - \eta_0^p) d\widetilde{\mu} \to 0.$$

Since by (3.2.2) the left hand side is positive and nonincreasing, it must be identically zero for all τ. Then the solution is a family of spheres, as claimed. $\square$

3.2.1 Ancient solutions with a diameter bound

In this section, we prove that a convex ancient solution with a weaker pinching hypothesis and a time-dependent diameter bound is a shrinking sphere, provided the speed of the flow satisfies some concavity conditions and it is defined on a suitable convex cone. The result parallels an equivalence obtained by Huisken and Sinestrari for the Mean Curvature Flow in [48] and it relies on a generalized Harnack estimate obtained by Ben Andrews in [3].

Theorem 8. *Let M_t be an ancient solution of* (3.0.1)*, with the speed f satisfying (H1)–(H2) and* 1*-homogeneous. Suppose f is defined on an open cone such that $\{\lambda = (\lambda_1, \ldots, \lambda_n) |\, \lambda_1 \geq 0,\, \lambda_1 + \lambda_2 > 0\} \subset \Gamma_2$ and $\Gamma_2 \supset \Gamma_+$; suppose furthermore that f is convex or concave and inverse-concave. If $\lambda_n \leq \overline{C}\lambda_2$ for some constant $\overline{C}$ uniformly in $(-\infty, 0)$ and there exists $C_1 > 0$ such that*

$$diam(M_t) \leq C_1(1 + \sqrt{-t}) \quad \forall t \in (-\infty, 0), \tag{3.2.3}$$

then M_t is a family of shrinking spheres.

We recall that a function is inverse-concave if f_*, defined by

$$f_*(\lambda_1^{-1}, \ldots, \lambda_n^{-1}) = f(\lambda_1, \ldots, \lambda_n)^{-1} \tag{3.2.4}$$

is a concave function.

To prove Theorem 8, we first need to show that the estimate (3.2.3) is equivalent to a two-sided time-dependent bound on the speed F.

Lemma 9. *Let M_t be a convex ancient solution of* (3.0.1)*, with F inverse-concave. The diameter of M satisfies* (3.2.3) *if and only if there exist two positive constants C_2, C_3 such that*

$$\frac{C_2}{\sqrt{-t}} \leq F \leq \frac{C_3}{\sqrt{-t}} \quad \forall t \in (-\infty, 0) \tag{3.2.5}$$

Proof. The proof of the Lemma is essentially the same as in [48]; we will sketch it here for completeness. If (3.2.5) holds, (3.2.3) follows from the inequalities:

$$\begin{aligned} |\varphi(x_1, t) - \varphi(x_2, t)| &\leq \int_t^0 F(x_1, \tau)d\tau + \int_t^0 F(x_2, \tau)d\tau \\ &\leq 2C_3 \int_t^0 \frac{d\tau}{\sqrt{-\tau}} = 4C_3\sqrt{-t} \qquad\qquad \forall x_1, x_2 \in M. \end{aligned}$$

To prove the opposite implication, we recall the Harnack inequality 2 from Theorem 5.21 in [3], which implies for an ancient solution, for $x_1, x_2 \in M$ and $t_1 < t_2 < 0$

$$F(x_1, t_1) \leq F(x_2, t_2) \exp\left(\frac{C diam_I^2(M_t)}{4(t_2 - t_1)}\right).$$

If we choose $t_1 = t$, $t_2 = \frac{t}{2}$, since by comparison $\min_{M_t} F \leq \frac{\sqrt{n}}{\sqrt{-2t}}$, we obtain $\max_{M_t} F \leq e^{\widetilde{C}} \frac{\sqrt{n}}{\sqrt{-2t}}$. Setting $t_1 = 2t$, $t_2 = t$ instead, from $\max_{M_t} \geq \frac{1}{\sqrt{-2t}}$, there also holds $\min_{M_t} F \geq \frac{e^{-\widetilde{C}}}{2\sqrt{-t}}$ and, thus, (3.2.5). □

We can now prove Theorem 8

Proof. We can restrict to the case where the inequality $\lambda_n \leq C\lambda_1$ is not satisfied, else the theorem follows from the previous section. If it does not hold, then there is a sequence of points in spacetime (p_k, t_k) such that $t_k \to -\infty$ and $\frac{\lambda_n}{\lambda_1}(p_k, t_k) \to +\infty$. We can rescale the flow for $t \in [2t_k, t_k]$ by factors $\frac{1}{\sqrt{-t_k}}$ in space and $\frac{1}{-t_k}$ in time to obtain a sequence of smooth flows in $[-2, -1]$. By a standard compactness argument, the latter admits a subsequence converging to a solution of the flow in $[-\frac{3}{2}, -1]$, since the diameter and the speed are bounded by the previous lemma and the bound, together with the pinching assumption, in turn implies an estimate on H. In fact, let $\overline{\Gamma}$ be the pinching subcone $\overline{\Gamma} = \{\lambda \in \Gamma_2 | \lambda_n \leq \overline{C}\lambda_2\}$; as $\partial\Gamma_2 \cap \partial\overline{\Gamma} = \{0\}$, $\overline{\Gamma} \cap \mathbb{S}^n$ has compact closure in $\Gamma_2 \cap \mathbb{S}^n$. As F is positive on Γ_2, $\frac{F}{H} \geq C_4 > 0$ on $\overline{\Gamma} \cap \mathbb{S}^n$ and then on the whole $\overline{\Gamma}$, as $\frac{F}{H}$ is homogeneous of degree zero. Bounds on the derivatives of F are then assured by Krylov's estimates in [50]. This solution has a point with $\lambda_1 = 0$; the Splitting Theorem in [14] implies that $\lambda_1 = 0$ on the whole submanifold, which is thus the product of a compact manifold times a flat factor, contradicting boundedness of the diameter. □

Remark 2. For the same class of velocities and under the same convexity hypotheses of Theorem 8, the conclusion holds if we change the assumption of uniform two-pinching with any of the following (C is a constant):

- uniform pinching for the radii of the submanifold,
  $\rho_+ \leq C\rho_-$ along the flow;
- a reverse isoperimetric inequality of the form $Vol(\Omega_t)^n \geq C|M_t|^{n+1}$;
- M_t is a Type I solution, so $\max_{M_t}(h_i^j)\sqrt{-t} \leq C$ on $(-\infty, T_1)$, for some T_1.

as all these conditions imply the bound (3.2.3) (see [48]). We can also conclude that an ancient solution of the same kind which is not a family of shrinking spheres must admit a family of rescaled flows converging to a translating soliton, thanks to the Harnack inequalities and the results of [51] (see Section 5.4 and Appendix B).

3.3 Flows by powers of the Gauss curvature

We will now analyse a precise class of flows of variable homogeneity, namely

$$\frac{\partial \varphi}{\partial t}(x,t) = -K^{\beta}(x,t)\nu(x,t) \tag{3.3.1}$$

where K is the Gauss curvature of the submanifold and $\beta > 0$.

Many authors in the last decades have investigated the singular behaviour of these flows. The first one, in the case $\beta = 1$, was Firey [29], who proved that a compact convex hypersurface with spherical symmetry shrinks to a round point in finite time, and conjectured that the same property holds without symmetry assumptions. After various partial results through the decades, the conjecture was proved for a general $\beta > 1/(n+2)$ by combining the results of the papers [6, 22, 17], where the reader can also find more detailed references.

In this section we consider an ancient compact convex solution of (3.3.1) defined in $(-\infty, 0)$. We translate the coordinates if necessary, so that the solution shrinks to the origin as $t \to 0$. Following [6, 34], we consider the rescaled flow $\widetilde{\varphi}(\cdot,\tau) = e^{\tau}\varphi(\cdot, t(\tau))$, where t and τ are related by

$$\tau(t) = \frac{1}{n+1}\log\left(\frac{|B(1)|}{|\Omega_t|}\right).$$

Here $|B(1)|$ is the volume of the unit ball, which we can also write as $|B(1)| = (n+1)^{-1}\omega_n$, with $\omega_n = |\mathbb{S}^n|$. In this way, the volume of the rescaled enclosed region $\widetilde{\Omega}_\tau$ is constant and equal to $|B(1)|$. In addition, the flow is defined for $\tau \in (-\infty, \infty)$ and satisfies the equation

$$\frac{\partial \widetilde{\varphi}}{\partial \tau}(x,\tau) = -\frac{\tilde{K}^{\beta}(x,t)}{\omega_n^{-1}\int_{\mathbb{S}^n}\tilde{K}^{\beta-1}d\theta}\nu(x,\tau) + \widetilde{\varphi}(x,\tau). \tag{3.3.2}$$

An important feature of these flows is the existence of monotone integral quantities, called entropies, see e.g. [25, 29]. Here we will use the one considered in [6], which is defined as follows. Let M be any convex embedded hypersurface in $\mathbb{R}^{n+1}$ and let Ω be the convex body enclosed by M. The entropy functional $\mathcal{E}_\beta(\Omega)$ is defined for each $\beta > 0$ as

$$\mathcal{E}_\beta(\Omega) = \sup_{z\in\Omega}\mathcal{E}_\beta(\Omega, z),$$

where

$$\mathcal{E}_\beta(\Omega,z) = \begin{cases} \dfrac{1}{\omega_n}\displaystyle\int_{\mathbb{S}^n}\log u_z d\theta & \beta = 1 \\ \dfrac{\beta}{\beta-1}\log\left(\dfrac{1}{\omega_n}\displaystyle\int_{\mathbb{S}^n}u_z^{1-\frac{1}{\beta}}d\theta\right) & \beta \neq 1. \end{cases}$$

Here, u_z is the support function of Ω with respect to the center $z \in \Omega$ and it is defined on the sphere $\mathbb{S}^n$ with the aid of the (inverse) Gauss map. For simplicity of notation, we will use in the following the same symbol for u_z considered as a function on $\mathbb{S}^n$ and as a function defined on M. In [6], it is proved that for each Ω, there exists a unique point $e \in \Omega$, called entropy point, such that the supremum in the definition is attained.

The main property of the entropy is monotonicity along the solutions of the rescaled flow (3.3.2). In fact, we have the inequality (see [6], Theorem 3.1):

$$\frac{d}{d\tau}\mathcal{E}_\beta(\widetilde{\Omega}_\tau) \leq -\left[\frac{\int_{\mathbb{S}^n} f^{1+\frac{1}{\alpha}} d\sigma_\tau \cdot \int_{\mathbb{S}^n} d\sigma_\tau}{\int_{\mathbb{S}^n} f^{\frac{1}{\alpha}} d\sigma_\tau \cdot \int_{\mathbb{S}^n} f d\sigma_\tau} - 1\right], \tag{3.3.3}$$

where $f = \frac{\widetilde{K}^\beta}{u_{e(\tau)}}$, $d\sigma_\tau = \frac{u_{e(\tau)}}{\widetilde{K}} d\theta$, and $u_{e(\tau)}$ is the support function of the rescaled solution at the entropy point $e(\tau)$ of $\widetilde{\Omega}_\tau$.

By the Hölder inequality, the right hand side of (3.3.3) is nonpositive, and it is strictly negative unless f is constant. The manifolds with constant f are the stationary solutions of (3.3.2), which correspond to the homothetically shrinking solitons of (3.3.1). Using this property, it was showed in [6] that, for every $\beta > \frac{1}{n+2}$, convex hypersurfaces evolve into a singularity which is a soliton under rescaling. It was finally proved in [22, 17] that the only soliton is the sphere, thus proving Firey's conjecture.

Our result for flows by powers of the Gauss curvature is the following:

Theorem 10. *Let M_t be an ancient closed strictly convex solution of* (3.3.1) *with $\beta > \frac{1}{n+2}$. If there exists $C > 0$ such that $\frac{\lambda_n}{\lambda_1} \leq C$ on $(-\infty, 0)$, then M_t is a family of shrinking spheres.*

Proof. By our pinching assumption, the solution satisfies the conclusions of Theorem 6. Let us consider the rescaled flow (3.3.2). By construction, the domains $\widetilde{\Omega}_\tau$ contain the origin for all times. It is easy to check that estimate (3.1.2) translates into a uniform upper bound on the Gauss curvature $\widetilde{K}$ of the rescaled hypersurfaces, as $\widetilde{K} = e^{-n\tau}K$. By pinching, each principal curvature is also bounded. In addition, by (3.1.1), the inner and outer radius are bounded from both sides by positive constants uniformly in time. Thanks to these bounds, we know from Lemma 4.4 of [6] that the entropy point of $\widetilde{\Omega}_\tau$ satisfies $\operatorname{dist}(e(\tau), \partial\widetilde{\Omega}_\tau) \geq \varepsilon_0$, for some ε_0 independent of τ. Therefore, we have the estimates

$$\frac{1}{C} \leq \widetilde{u}_{e(\tau)} \leq C, \qquad \widetilde{K} \leq C \tag{3.3.4}$$

on $\widetilde{M}_\tau$, where we denote by C any large positive constant independent of τ. It follows that the entropy $\mathcal{E}_\beta(\widetilde{\Omega}_\tau)$ is also bounded from above for all τ. Since $\mathcal{E}_\beta(\widetilde{\Omega}_\tau)$ is monotone decreasing, it converges to some finite value $\mathcal{E}_{-\infty}$ as $\tau \to -\infty$. To conclude the proof, we need to show that $\widetilde{M}_\tau$ converges to a stationary point of the entropy as $\tau \to -\infty$.

Using the property that

$$\int_{\mathbb{S}^n} d\sigma_\tau = \int_{\widetilde{M}_\tau} \widetilde{u}_{e(\tau)} d\widetilde{\mu} = (n+1)\mathrm{Vol}(\widetilde{\Omega}_\tau) = \omega_n,$$

we can rewrite formula (3.3.3) as

$$\frac{d}{d\tau}\mathcal{E}_\beta(\widetilde{\Omega}_\tau) \leq -\left[\frac{\int_{\widetilde{M}_\tau} f^{1+\frac{1}{\beta}} d\nu}{\int_{\widetilde{M}_\tau} f^{\frac{1}{\beta}} d\nu \cdot \int_{\widetilde{M}_\tau} f d\nu} - 1\right],$$

where $d\nu := \omega_n^{-1}\widetilde{u}_{e(\tau)} d\widetilde{\mu}$ is a probability measure on $\widetilde{M}_\tau$. Then, as in Proposition 4.3 in [66], we can use the refinement of Jensen's inequality

$$\int_{\widetilde{M}_t} \left(\frac{f}{\bar{\bar{f}}}\right)^{1+\frac{1}{\beta}} d\nu \geq 1 + \frac{\beta+1}{2\beta}\int_{\widetilde{M}_t}\left(\frac{f}{\bar{\bar{f}}} - 1\right)^2 d\nu, \tag{3.3.5}$$

together with

$$\int_{\widetilde{M}_t} f^{1+\frac{1}{\beta}} d\nu \geq \left(\int_{\widetilde{M}_t} f^{1+\frac{1}{\beta}} d\nu\right)^{\frac{1}{1+\frac{1}{\beta}}} \int_{\widetilde{M}_t} f^{\frac{1}{\beta}} d\nu,$$

to estimate the right-hand side and deduce that, for any $\varepsilon_1 > 0$ there exists $\varepsilon_2 > 0$ such that

$$\int_{\widetilde{M}_\tau} \left(f - \bar{f}\right)^2 d\nu \geq \varepsilon_1 \implies \frac{d}{d\tau}\mathcal{E}_\beta(\widetilde{\Omega}_\tau) \leq -\varepsilon_2; \tag{3.3.6}$$

in the previous lines we have set

$$\bar{f} = \int_{\widetilde{M}_\tau} f d\nu = \frac{1}{\omega_n}\int_{\widetilde{M}_\tau} \widetilde{K}^\beta d\widetilde{\mu}.$$

We want to use (3.3.6) to show that f converges to a constant as $\tau \to -\infty$. To do this, we need some uniform control on the regularity of the solution; we begin by estimating $\bar{f}$. Since the area of $\widetilde{M}_\tau$ is bounded from both sides by convexity and the bounds on the radii, an upper bound on $\bar{f}$ holds in view of (3.3.4). To find a lower bound, we argue as follows: if $\beta \geq 1$, we can use the Hölder inequality and the bound on $|\widetilde{M}_\tau|$ to conclude

$$\bar{f} \geq \frac{1}{\omega_n}\left(\int_{\widetilde{M}_\tau} \widetilde{K} d\widetilde{\mu}\right)^\beta |\widetilde{M}_\tau|^{1-\beta} = \left(\frac{|\widetilde{M}_\tau|}{\omega_n}\right)^{1-\beta} \geq \frac{1}{C},$$

while if $\beta < 1$, we can write

$$\bar{f} \geq \frac{1}{\omega_n}\frac{1}{(\sup \widetilde{K})^{1-\beta}}\int_{\widetilde{M}_\tau} \widetilde{K} d\widetilde{\mu} = \frac{1}{(\sup \widetilde{K})^{1-\beta}} \geq \frac{1}{C}.$$

Observe that we lack a lower bound on $\widetilde{K}$, and that the methods of [34, 6] to obtain such a bound do not seem to work in the backward limit $\tau \to -\infty$. This means that we do not know yet whether our problem remains uniformly parabolic as time decreases. We then follow a strategy introduced by Schulze in [65] and later used in [19, 66], which exploits the theory for degenerate or singular parabolic equations. In our case, we can do computations similar to [19, §7.2], and find that the speed K^β satisfies an equation of porous medium type, to which the Hölder regularity results from [26, 27] can be applied. For clarity, we recall briefly the theorem and the procedure for $\beta = 1$.

Theorem (Di Benedetto, Friedman [27]). *Let $v \in C^2(B_r \times [T_1, T_2])$ be a nonnegative solution of the degenerate parabolic equation*

$$\frac{\partial v}{\partial \tau} = D_i\left(a^{ij}(x,t,Dv)D_j v^d\right) + b(x,t,v,Dv), \tag{3.3.7}$$

with $B_r \subset \mathbb{R}^n$ the ball of radius r centered in the origin, $d > 1$ (D denotes derivation with respect to the coordinates). Suppose

$$m|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq M|\xi|^2 \tag{3.3.8}$$

for two constants m, M on $B_r \times [T_1, T_2]$. Furthermore let c_1, c_2, N be such that

$$|b| \leq c_1|Dv| + c_2 \tag{3.3.9}$$

$$\sup_{T_1<\tau<T_2} \|v(\cdot,\tau)\|_{L^2(B_r)} + \|Dv^d\|^2_{L^2(B_r\times[T_1,T_2])} \leq N. \tag{3.3.10}$$

Then for any $T_1 < \delta < T_2$ and $0 < r_1 < r$, we have

$$\|v\|_{C^\alpha(B_{r_1}\times[\delta,T_2])} \leq C \tag{3.3.11}$$

for suitable $C > 0$ and $\alpha \in (0,1)$.

In order to apply the theorem above, we rewrite the evolution equation for $\widetilde{K}$,

$$\frac{\partial \widetilde{K}}{\partial \tau} = \widetilde{\mathcal{L}}\widetilde{K} + \widetilde{H}\widetilde{K}^2 - n\widetilde{K}. \tag{3.3.12}$$

as in (3.3.7) with $a^{ij} = \frac{1}{d}\widetilde{K}^{\frac{(1-n)}{n}} \frac{\partial \widetilde{K}}{\partial \widetilde{h}_{ij}}$, $b = \widetilde{\Gamma}^j_{jl} \frac{\partial \widetilde{K}}{\partial \widetilde{h}_{ij}} D_i \widetilde{K} + \widetilde{H}\widetilde{K}^2 - n\widetilde{K}$ and $d = 1 + \frac{(n-1)}{n}$.

Due to the pinching condition and convexity, $\frac{\partial \widetilde{K}}{\partial \widetilde{h}_{ij}}$ is equivalent as a quadratic form to $\widetilde{H}^{n-1}\widetilde{g}^{ij}$ This implies that a^{ij} is equivalent to $\frac{1}{d}\widetilde{g}^{ij}$, thus (3.3.8) is satisfied. Moreover,

$$|b| \leq \frac{1}{d}\left|\widetilde{K}^{\frac{(1-n)}{n}} \widetilde{\Gamma}^j_{jl} \frac{\partial \widetilde{K}}{\partial \widetilde{h}_{ij}} D_i \widetilde{K}^d\right| + c_2 \leq \frac{1}{d}\left|C\widetilde{\Gamma}^j_{jl}\widetilde{g}^{ij} D_i \widetilde{K}^d\right| + c_2 \leq c_1 |D\widetilde{K}^d| + c_2$$

follows by representing the hypersurface locally as a graph on its tangent plane (see [19], Cor.5.3) and from the uniform estimate from above on $\widetilde{K}$. To obtain a bound on the L^2 norm of $D\widetilde{K}$, we observe we can integrate by parts to obtain:

$$\int_{\widetilde{M}_\tau} |\widetilde{\nabla}\widetilde{K}^d|^2 d\widetilde{\mu}_\tau = -dC \int_{\widetilde{M}_\tau} \widetilde{K}^d \widetilde{\mathcal{L}}\widetilde{K} d\widetilde{\mu}_\tau. \tag{3.3.13}$$

From (3.3.12) then follows

$$\int_{\widetilde{M}_\tau} |\widetilde{\nabla}\widetilde{K}^d|^2 d\widetilde{\mu}_\tau \leq \frac{d}{dt} C \int_{\widetilde{M}_\tau} \widetilde{K}^{d+1} d\widetilde{\mu}_\tau + C_1 \tag{3.3.14}$$

and integrating on $[T_1, T_2] \subset (-\infty, 0)$ we establish the required estimate on the L^2 norm $\|Dv^d\|^2_{L^2(B_r \times [T_1,T_2])} \leq C_-$, with C_- depending on the length $T_2 - T_1$ of the time interval, but otherwise independent of T_1, T_2. We can thus apply (3.3) to find that there exist $\alpha \in (0,1)$ and $\eta > 0$ such that, for any $x_0 \in M$ and $\tau_0 \in \mathbb{R}$, the parabolic α-Hölder norm of $\widetilde{K}^\beta$ on $B_\eta(x_0) \times (\tau_0 - \eta, \tau_0 + \eta)$ is bounded by some C independent of (x_0, τ_0).

Now we are able to prove a lower bound on $\widetilde{K}$ as $\tau \to -\infty$. In fact, suppose that $\widetilde{K}(x_0, \tau_0) = \delta_0$ at some (x_0, τ_0), with $\delta_0 > 0$ suitably small depending on the constants C of the previous estimates. Then, the bounds from below on u_e and $\bar{f}$ imply that $|f(x_0, \tau_0) - \bar{f}(\tau_0)|$ is far from zero. By Hölder continuity, the same holds for $(x, \tau) \in B_\eta(x_0) \times (\tau_0 - \eta, \tau_0 + \eta)$. It follows that $\int_{\widetilde{M}_\tau} \left(f - \bar{f}\right)^2 d\nu \geq \varepsilon_1$ for $\tau \in [\tau_0 - \eta, \tau_0 + \eta]$, where ϵ_1, η do not depend on τ_0. In view of (3.3.6) and of the boundedness of the entropy, this can only occur on a finite number of intervals. We deduce that $\widetilde{K}(\cdot,\tau) > \delta_0$ for all $\tau << 0$.

As we have shown that the rescaled Gaussian curvature is uniformly bounded from both sides, we deduce from the pinching condition that each principal curvature is bounded between two positive constants. This implies that the equation is uniformly parabolic with bounded coefficients, and we have uniform estimates on all the derivatives of the solution from Krylov-Safonov and Schauder theory. Standard arguments ensure precompactness for the family $\widetilde{M}_\tau$, so that every sequence $\widetilde{M}_{\tau_k}$ with $\tau_k \to -\infty$ as $k \to +\infty$ admits a subsequence converging in C^∞ to a limit $\widetilde{M}_{-\infty}$.

We now want to show that the right-hand side of (3.3.3) must vanish on $\widetilde{M}_{-\infty}$. To this purpose, we need some continuity with respect to τ of the function $\widetilde{u}_{e(\tau)}$. In Lemma 4.3 of [6] it was proved that, for convex bodies Ω satisfying uniform bounds on the inner and outer radii, the entropy point is a continuous function of Ω with respect to the Hausdorff distance. Since the speed of our flow is bounded, we deduce that for any $\epsilon > 0$ there exists $\eta > 0$ such that

$$||e(\tau) - e_{-\infty}|| \leq \epsilon, \qquad \forall \tau \in [\tau_k - \eta, \tau_k + \eta],$$

for all k sufficiently large, where $e_{-\infty}$ is the entropy point of $\widetilde{\Omega}_{-\infty}$. Therefore $\widetilde{u}_{e(\tau)}$ is uniformly close to $u_{-\infty}$. By the regularity of $\widetilde{K}$, we find that the right hand side of (3.3.3) is uniformly close to the same expression computed on $\widetilde{M}_{-\infty}$ for $\tau \in [\tau_k - \eta, \tau_k + \eta]$. Therefore, if the right-hand side of (3.3.3) is nonzero on $\widetilde{M}_{-\infty}$, it is also uniformly negative for τ in a set of infinite measure, in contradiction with the boundedness of the entropy.

We conclude that the right-hand side of (3.3.3) vanishes on $\widetilde{M}_{-\infty}$. As already recalled, it is proved in [17, 22], that the only convex hypersurface with this property is the sphere. This implies that the whole flow converges to a sphere as $\tau \to \infty$. On the other hand, it is known that the entropy attains its minimum value on the sphere among all convex bodies with fixed volume. By monotonicity, the entropy must remain constant on the flow $\widetilde{M}_\tau$, and $\widetilde{M}_\tau$ is a sphere for every τ. $\square$

3.4 General flows with high degree of homogeneity

It is already evident in the special case of evolution by Gaussian Curvature that flows with high degree of homogeneity have in general more complicate analytic properties than in the 1-homogeneous case. Various authors have proved convergence of convex hypersurfaces to a spherical profile for certain specific speeds with homogeneity $\alpha > 1$, see [1, 19, 23, 24, 65], under the requirement that the initial datum satisfies a suitably strong pinching condition. A general result of this form has been obtained in [7], where a large class of speeds is considered, with no structural assumptions such as convexity or concavity. Here we show that an ancient solution which satisfies the same pinching requirement as in [7] is necessarily a shrinking sphere.

We consider a speed $f(\lambda)$ satisfying (H1)–(H2) for some $\alpha > 1$; for simplicity, we assume the normalization $f(1,\dots,1) = n$. By the smoothness and homogeneity of F, there exists $\mu > 0$ such that, for any matrices A, B:

$$|\ddot{F}^{kl,rs}(A)B_{kl}B_{rs}| \le \mu H^{\alpha-2}|B|^2. \tag{3.4.1}$$

Since by symmetry $\dot{F}|_{HI} = \alpha H^{\alpha-1} I$, where I is the identity matrix, it is easy to deduce

$$\begin{aligned}(\alpha H^{\alpha-1} - \mu H^{\alpha-2}|\mathring{h}|)I \le& \dot{F} \le (\alpha H^{\alpha-1} + \mu H^{\alpha-2}|\mathring{h}|)I, &(3.4.2)\\ H^{\alpha} - \frac{\mu}{2\alpha}H^{\alpha-2}|\mathring{h}|^2 \le& F \le H^{\alpha} + \frac{\mu}{2\alpha}H^{\alpha-2}|\mathring{h}|^2 &(3.4.3)\end{aligned}$$

(we recall that $\mathring{h} = h^2 - \frac{1}{n}H^2$).

Theorem 11. *If $M_t = \varphi(M,t)$, $t \in (-\infty, 0)$ is an ancient solution of (3.0.1), with F homogeneous of degree $\alpha > 1$ such that $|\mathring{h}|^2 \le \epsilon H^2$ holds for all times $t \in (-\infty, 0)$ for a suitable $\epsilon \in (0, \frac{1}{n(n-1)})$ depending only on μ, α, n, then it is a family of shrinking spheres.*

Proof. This time, similarly to the mean curvature case, we study the function

$$f_\sigma = \frac{|\mathring{h}|^2}{H^{2-\sigma}} \tag{3.4.4}$$

and we want to show that it is identically zero for all negative times, for a suitable choice of $\sigma \in (0,1)$. We observe that, under our hypothesis, $f_0 \le \epsilon$. If ϵ is small enough, we obtain from (3.4.3)

$$F \ge \frac{\alpha}{2}H^{\alpha}. \tag{3.4.5}$$

The evolution equations for the relevant quantities are computed in [7]:

$$\frac{\partial}{\partial t}H = \mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla_i h_{rs} + \dot{F}^{kl}h_{km}h_l^m H + (1-\alpha)F|h|^2 \tag{3.4.6}$$

$$\begin{aligned}\frac{\partial}{\partial t}G =& \mathcal{L}G + [\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}]\nabla_i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl}h_{km}h_l^m\dot{G}^{ij}h_{ij}\\ &+ (1-\alpha)F\dot{G}^{ij}h_{im}h_j^m\end{aligned} \tag{3.4.7}$$

if G is any smooth symmetric function of the eigenvalues of the Weingarten operator. Using these, we compute the evolution equation for f_σ:

$$\begin{aligned}
\frac{\partial}{\partial t} f_\sigma = \mathcal{L} f_\sigma &+ \frac{(2-\sigma)}{H} \dot{F}^{ij} \left[\nabla_i H \nabla_j f_\sigma + \nabla_j H \nabla_i f_\sigma\right] \qquad (3.4.8) \\
&+ \frac{1}{H^{2-\sigma}} \left[2 \left(h^{ij} - \frac{1}{n} H g^{ij} - \frac{2-\sigma}{2} f_0 H g^{ij} \right) \ddot{F}^{kl,rs} \right] \nabla_i h_{kl} \nabla_j h_{rs} \\
&- \frac{(1-\sigma)(\sigma-2)}{H^{2-\sigma}} \dot{F}^{ij} \nabla_i H \nabla_j H f_0 \\
&- \frac{2}{H^{2-\sigma}} \dot{F}^{ij} \left[\nabla_i h_{kl} \nabla_j h^{kl} - \frac{1}{n} \nabla_i H \nabla_j H \right] \\
&+ \sigma f_\sigma \dot{F}^{kl} h_{km} h_l^m + \frac{2(1-\alpha)}{n H^{2-\sigma}} F(nC - H|h|^2) - \frac{(1-\alpha)(2-\sigma)}{H} F |h|^2 f_\sigma,
\end{aligned}$$

where $C = \sum_i \lambda_i^3$. We estimate the terms in second row, using $f_0 \leq \epsilon$, $0 < \sigma < 1$ and the bound (3.4.1):

$$\begin{aligned}
&\left| 2 \left(h^{ij} - \frac{1}{n} H g^{ij} - \frac{2-\sigma}{2} f_0 H g^{ij} \right) \ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs} \right| \qquad (3.4.9) \\
&\leq 2 |\ddot{F}| \, |\nabla h|^2 H \sqrt{\epsilon} \left(1 + \sqrt{n\epsilon}\right) \\
&\leq 2 \mu H^{\alpha-1} |\nabla h|^2 \sqrt{\epsilon} \left(1 + \sqrt{n\epsilon}\right).
\end{aligned}$$

The term in the third row satisfies

$$\begin{aligned}
&\left| \dot{F}^{ij} (1-\sigma)(\sigma-2) f_0 \nabla_i H \nabla_j H \right| \leq 2\epsilon |\dot{F}| |\nabla H|^2 \\
&\leq 2\epsilon \left(\alpha + \mu\sqrt{\epsilon}\right) H^{\alpha-1} \frac{n+2}{3} |\nabla h|^2. \qquad (3.4.10)
\end{aligned}$$

The next term gives a negative contribution. In fact, it was shown in the proof of Theorem 5.1 in [7] that

$$-2\dot{F}^{ij} \left[\nabla_i h_{kl} \nabla_j h^{kl} - \frac{1}{n} \nabla_i H \nabla_j H \right] \leq -\frac{4(n-1)}{3n} H^{\alpha-1} (\alpha - \mu\sqrt{\epsilon}) |\nabla h|^2. \qquad (3.4.11)$$

We observe that all positive terms occurring in the above estimates can be made arbitrarily small by choosing a small $\epsilon > 0$, and the total contribution of the second, third and fourth row of (3.4.8) is nonpositive due to the negative term with the α factor in (3.4.11).

To estimate the terms in the last row, we rewrite them as

$$\sigma f_\sigma \left(\dot{F} h_{km} h_l^m + \frac{(1-\alpha)}{H} F|h|^2 \right) + \frac{2(1-\alpha)}{H} F \left(\frac{nC - H|h|^2}{nH^{1-\sigma}} - |h|^2 f_\sigma \right).$$

The second part will give a negative contribution. To see this, we first apply Lemma 2.3 in [7] to obtain

$$nC - (1+nf_0)H|h|^2 \geq f_0(1+nf_0)(1-\sqrt{n(n-1)f_0})H^3 \geq \frac{1}{2} f_0 H^3.$$

Then

$$\frac{nC - H|h|^2}{nH^{1-\sigma}} - |h|^2 f_\sigma \geq \frac{nf_\sigma H|h|^2 + \frac{1}{2} f_\sigma H^3}{nH} - |h|^2 f_\sigma = f_\sigma \frac{H^2}{2n}, \qquad (3.4.12)$$

and we conclude, using $\alpha > 1$ and (3.4.5),

$$\frac{2(1-\alpha)}{H} F \left(\frac{nC - H|h|^2}{nH^{1-\sigma}} - F|h|^2 f_\sigma \right) \leq (1-\alpha) f_\sigma \frac{H^{\alpha+1}}{2n}. \qquad (3.4.13)$$

At the same time, we have

$$\begin{aligned} \sigma f_\sigma \left(\dot{F} h_{km} h_l^m + \frac{(1-\alpha)}{H} F|h|^2 \right) &\leq \sigma f_\sigma (\dot{F} h_{km} h_l^m) \\ &\leq \sigma f_\sigma \left(\alpha H^{\alpha-1} + \mu H^{\alpha-2} |\mathring{h}| \right) |h|^2 \leq \sigma f_\sigma (\alpha+\mu) H^{\alpha+1}. \qquad (3.4.14) \end{aligned}$$

The above estimates and (3.4.8) imply that, choosing $\sigma < \frac{\alpha-1}{4n(\alpha+\mu)}$, we have

$$\frac{\partial}{\partial t} f_\sigma \leq \mathcal{L} f_\sigma + \frac{2(2-\sigma)}{H} \dot{F}^{ij} \nabla_i H \nabla_j f_\sigma + \frac{(1-\alpha)}{4n} f_\sigma H^{\alpha+1}.$$

Since $f_\sigma \leq H^\sigma$, if we set $\psi(t) = \max_{M_t} f_\sigma$, we find

$$\frac{d}{dt} \psi \leq -\frac{(\alpha-1)}{4n} \psi^{1+\frac{\alpha+1}{\sigma}}.$$

An easy comparison argument, similar to the one in Proposition 3, shows that $\psi(t)$ cannot be defined for all $t \in (-\infty, 0)$ unless it is identically zero. Therefore, $|\mathring{h}|^2 \equiv 0$ on our solution and the assertion is proved. □

3.5 A convex non-pinched counterexample

We conclude our analysis of convex compact ancient solutions for nonlinear curvature flows of contractive type by partially generalizing the example of Angenent Ovaloids for Mean Curvature Flow, which was discussed in detail by Haslhofer and Hershkovits [40]. The authors constructed a family of solutions that are not uniformly convex in time and proved that they are not shrinking spheres; by Huisken's convergence theorem in [42], due to convexity these hypersurfaces contract to round points at $T = 0$, while for $t \to -\infty$, after parabolic rescaling, they are asymptotic to generalized round cylinders. The "tips" also converge by a suitable blow-down to the translating bowl [39]. This counterexample was inspired by a remark by White [74] and was anticipated by a formal asymptotic study by Angenent [10]. We follow the same procedure to construct an analogous convex nonspherical ancient solution for (3.0.1); we are not able to deduce the asymptotics from our construction, since the result in [40] relies on the powerful Global Convergence Theorem for Mean Curvature Flow by Haslhofer and Kleiner ([41]) and on Huisken's monotonicity formula and both have no extension to our case. The Global Convergence Theorem in particular depends essentially on the α-noncollapsing property of solutions of MCF, and this condition is not invariant in general under evolution via fully nonlinear flows (for an exhaustive account of one-sided noncollapsing, see [51]). As in [40], let $M_0^l \subset \mathbb{R}^{n+1}$, for $l \in \mathbb{N}$ be the family of compact hypersurfaces defined by capping the truncated cylinder $\mathbb{S}^{n-1} \times [-l, -l]$ at a scale (independent of l) of length one and in such a way that the result is rotationally symmetric and convex. The sequence obtained is also uniformly 2-convex, namely $\lambda_1^l + \lambda_2^l \geq CH^l$ for some constant C independent of l (the index l obviously identifies geometric quantities relative to the l-th submanifold in the sequence and its evolution). We will obtain the ancient solution as limit (after suitable rescaling) of a subsequence of flows whose initial data are given by the M_0^l.
We assume f satisfies (H1)–(H2), is 1-homogeneous and it is defined on a convex symmetric cone $\Gamma \subset \mathbb{R}^n$, such that $\Gamma_+ \subset \Gamma$ and

$$\Gamma_+^{n-1} = \{\lambda = (0, \lambda_2, \dots, \lambda_n) \,|\, \lambda_i > 0,\ i = 2, \dots, n\} \subset \Gamma.$$

We will also suppose one of the following conditions holds:

1. $n = 2$;

2. f is convex;

3. f is concave and inverse concave up to the boundary of the positive cone, meaning the function

$$f_*(\lambda_2, \ldots, \lambda_n) := f\left(0, \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right)^{-1}$$

   is also smooth and concave as a function on Γ_+^{n-1}. We require two additional conditions on f in this case:

   (a) $\lim\limits_{\lambda \to \partial\Gamma} f(\lambda) = 0$

   (b) $\lim\limits_{\mu \to +\infty} f(1, \ldots, 1, \mu) = +\infty$.

Most of the additional assumptions are needed to fix issues related to invariance of convexity: we need to ensure that compact weakly convex hypersurfaces become immediately strictly convex, that strict convexity is preserved and that there is convergence to a round point in finite time; we also need Hölder estimates of the second order for the solution.
To address the first two problems, we invoke the Splitting Theorems of Langford [51] and Bourni-Langford [14], that hold in general in dimension $n = 2$, for convex speed functions and for inverse-concave functions if $n \geq 2$. Convergence to round points of convex hypersurfaces is also granted for $n = 2$ in full generality, for convex speeds ([2]) or inverse-concave speeds which are also concave, or with $f_* \to 0$ as $\lambda \to \partial\Gamma_+$ ([8]), a condition equivalent to (3b) above for axially symmetric submanifolds. Hölder estimates follow from parabolic theory, once established second order bounds and parabolicity of the equation, if the speed is either convex or concave, or, again, if $n = 2$. We will thus require f convex or concave and inverse-concave, unless the dimension is equal to 2.

In what follows, we will denote by M_t^l the evolution of M_0^l by (3.0.1). We observe that all the starting M_0^l are enclosed by the standard cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ and enclose the standard sphere $\mathbb{S}^n$ of radius 1: by avoidance, the extinction time for all the flows of the sequence must be comparable to one. We will denote by x the coordinates on $\mathbb{R}^{n+1}$ and define $a^l(t) = \max\limits_{x \in M_t^l} |x_{n+1}|$ and $b^l(t) = \max\limits_{x \in M_t^l} \left(\sum_{i=1}^{n-1} x_i^2\right)^{\frac{1}{2}}$; due to rotational symmetry and convexity, they are equivalent to the outer and the inner radius respectively, and the spheres realizing the radii are centered in 0. Since the hypersurfaces are becoming spherical we can parabolically rescale the sequence of flows with a

scaling factor that depends on l and perform a translation in time to ensure that each one is defined on a time interval $[-T_l, 0)$, $\dfrac{a^l(-1)}{b^l(-1)} = 2$ and $\dfrac{a^l(t)}{b^l(t)} \geq 2$ for every $t < -1$ and for every l. From this point, we will always work on the rescaled flows; we use the same notation as it should not be a source of confusion.
First, we want to obtain uniform spatial bounds and prove the lifespan of each solution is becoming longer as l increases. The avoidance principle grants a two-sided bound on the diameter at time -1 which is uniform in l. Then $diam(M^l_{-1}) \geq c$ must hold or we could enclose the hypersurfaces of the sequence, from some index l, in a sphere with a small radius, that shrinks in less than unit time. As the hypersurfaces are convex, their diameter is comparable to a^l and as the ratio $\dfrac{a(-1)}{b(-1)}$ is fixed, the opposite inequality $diam(M^l_{-1}) \leq C$ holds, or $b(-1)$ would be too large and we could enclose (from a certain l) a sphere too big to cease to exist in unit time.
We also have that b^l is not increasing in time by comparison with enclosing cylinders, so $b^l(t) \geq c$ in $[-T^l, -1]$. Then a^l, and thus $\dfrac{a^l}{b^l}$, need a uniform time period of at least $\dfrac{c^2}{32n}$ to be halved by the same argument as in [40]: for $t_0 < -1$ and for every l, a sphere of radius $\frac{b^l(t_0)}{4}$ can be enclosed in $M^l_{t_0}$ at distance $\frac{a^l(t_0)}{2}$ from the origin, since $\frac{a^l(t_0)}{b^l(t_0)} \geq 2$ for every $t_0 < -1$, and then the comparison principle applies. This also implies $T^l \to -\infty$ as l goes to $+\infty$, as $a^l(T^l)$ explodes.

We need to prove that the diameter and the speed have bounds independent of l on each compact subset of times (we can obviously discard a finite number of undefined terms of the sequence). Let $K \subset (-\infty, 0)$ be a compact subinterval of times; then the uniform two-sided bound $c_K \leq diam(M^l_t) \leq C_K$ holds for some constants depending only on K. We can assume $K \subset (-\infty, -1)$ without loss of generality, as the bounds follow immediately in $(-1, 0)$ from the condition in $t = -1$ and the smoothness of the hypersurface; the lower bound is also trivial from what we have already shown. If a uniform upper bound did not hold, then there would exist a subsequence M^{l_k} such that $\max\limits_{t \in K} diam(M^{l_k}_t) \to +\infty$ as $k \to +\infty$; for convexity, a^{l_k} would also explode. Then a contradiction would follow from the fixed rate of shrinking for a and the uniform upper bound at $t = -1$. Thus, both a and b have two-sided bounds uniform in l on each compact subinterval of times.

In order to obtain l-uniform bounds on the speed and the curvatures, we describe the submanifolds as $\left\{(\psi^l(x^l_{n+1}(t),t)\omega, x^l_{n+1}(t))\right\}$, with $\psi(\cdot,t)$ smooth profile functions and $\omega \in \mathbb{S}^{n-1} \subset \{x_{n+1}=0\}$; we will denote by a prime mark the derivatives with respect to x_{n+1}. Omitting the explicit dependence on t and the index l, the value λ of the $n-1$ principal curvatures in the radial directions and the axial curvature μ are computed as:

$$\lambda = \frac{1}{\psi\left[1+(\psi')^2\right]^{\frac{1}{2}}} \tag{3.5.1}$$

$$\mu = \frac{-\psi''}{\left[1+(\psi')^2\right]^{\frac{3}{2}}}; \tag{3.5.2}$$

we observe that the convexity of the surfaces implies $-\psi'' \geq 0$. The evolution of the profile functions is given by:

$$\frac{\partial\psi}{\partial t} = -\frac{1}{\psi} f\left(1,\ldots,1,-\frac{\psi''\psi}{1+(\psi')^2}\right); \tag{3.5.3}$$

the nonnegative quantity $R = -\frac{\psi''\psi}{1+(\psi')^2}$ is the pinching function $\frac{\mu}{\lambda}$; with a small abuse of notation, we will denote $f(R)$ the function $f(1,\ldots,1,R)$. Inspired by the argument in [11], we show that $R \leq 1$ along the flow, for all l, so the axial curvature cannot exceed the radial one. We already know $\max R^l \geq 1$, since the symmetries of the hypersurfaces force $R=1$ at the tips for all l and t. Since ψ is smooth, we have:

$$\frac{\partial R}{\partial t} = \frac{-(\partial_t\psi)''\psi - \psi''\partial_t\psi}{1+(\psi')^2} + \frac{2\psi''\psi'\psi(\partial_t\psi)'}{[1+(\psi')^2]^2}$$

We compute the spatial derivatives of (3.5.3):

$$(\partial_t\psi)' = \frac{-(f(R))'}{\psi} + \frac{\psi' f(R)}{\psi^2}$$

$$(\partial_t\psi)'' = -\frac{(f(R))''}{\psi} + \frac{\psi'' f(R)}{\psi^2} + \frac{2\psi'(f(R))'}{\psi^2} - \frac{2(\psi')^2 f(R)}{\psi^3}$$

and $(f(R))' = \dot{f}R'$, $(f(R))'' = \ddot{f}(R')^2 + \dot{f}R''$. From the equations above we obtain

$$\frac{\partial R}{\partial t} = \frac{\dot{f}R'' + \ddot{f}R''}{1+(\psi')^2} - \frac{2\psi'(1-R)}{\psi(1+(\psi')^2)}\left(R'\dot{f} - \frac{f\psi'}{\psi}\right) \tag{3.5.4}$$

At a spatial maximum, $R' = 0$ and $R'' \leq 0$, so there holds

$$\frac{\partial(\max R)}{\partial t} \leq \frac{2(\psi')^2 f(R)}{\psi^2(1+(\psi')^2)}(1-R)$$

If $\max R^l$ was greater than 1 on any $M^l_{t_0}$, the equation above would imply $\max R^l > 1$ on M^l_t for all $-T^l \leq t \leq t_0$, which is clearly in contradiction with our choice of initial data; so $R \leq 1$ and $\mu \leq \lambda$ for all times on the sequence of flows. Corollary 2 of Theorem 14 in [8] provides a l-uniform lower bound on the speed depending only on the compact K and the bound on a^l; the assumptions of the Corollary are equivalent, in the axially symmetric setting, to $f(0, 1, \dots, 1) > 0$ and (3b) (or they follow trivially from $f \geq H$ if f is convex). We observe that $f(\lambda, \dots, \lambda, \mu) \geq c$ implies the lower bound $\lambda \geq c$ thanks to homogeneity and $R \leq 1$. A l-uniform upper bound on the speed can be obtained as in Theorem 8 of [8], since inverse-concavity grants $\dot{F}(W^2) \geq F^2$, as in Lemma 5 in the same article (a convex speed is inverse-concave on the positive cone). Due to our assumptions, we obtain uniform upper bounds for λ and μ, that thus remain in the same compact subset $\Gamma_K \subset \Gamma$ for every l along the flow. As $\dot{F}$ is zero-homogeneous, there holds $b_K Id \leq \dot{F} \leq B_K Id$ where B_K, b_K are the maximum and minimum respectively of $\dot{F}$ on $\mathbb{S}^n \cap \Gamma_K$; the system is then l-uniformly parabolic on each K.

Now we consider the polar coordinate representation, so M^l_t is described as a graph on $\mathbb{S}^n$ for $t \in K$; we have radial functions $r^l : \mathbb{S}^n \times [-T^l, 0) \to \mathbb{R}$, and $M^l_l = graph(r^l(\cdot, t))$. From what we have shown, the radial functions are uniformly bounded from both sides on each subinterval of times and so are the support functions centered in the origin $u^l = \dfrac{r^l}{\sqrt{(r^l)^2 + |\overline{\nabla} r^l|^2}}$.

In these coordinates, the second fundamental form is related to the radial function as:

$$h_{ij} = \frac{r}{v}(\sigma_{ij} + \eta_i \eta_j - \eta_{ij}) \tag{3.5.5}$$

where $\eta = \log r$, σ_{ij} is the standard metric on $\mathbb{S}^n$ and the subscripts stand for covariant derivatives with respect to the Levi-Civita connection of the sphere. Since r, h_{ij} and the derivatives η_i are bounded on K, the second derivatives of r are also bounded on K independently of l. The last two bounds show the parabolicity of the system and of the related equation for the radial function (derived in [2], see also [31]):

$$\frac{\partial r^l}{\partial t} = -v^l \widetilde{F^l}(W^l) \tag{3.5.6}$$

where $v^l = \dfrac{\sqrt{(r^l)^2 + |\overline{\nabla} r^l|^2}}{r^l}$ and $\widetilde{F^l}$ is the function associated to F^l and defined on the negative cone such that $\widetilde{F^l}(A) = F^l(-A)$. The above PDE is parabolic since in polar coordinates W is expressed as

$$h^i_j = \frac{1}{rv}\left\{\delta^i_j + \left[-\sigma^{ik} + \frac{\eta^i f^k}{v^2}\right]\eta_{jk}\right\}$$

(indices are raised using the spherical metric tensor).
Krylov-Safonov theory gives then Hölder estimates of order 2 for the solution of (3.0.1) under our assumptions on the speed and standard arguments provide uniform bounds on the derivatives of all orders on each K.

As all the bounds are uniform in l, the sequence of radial functions u^l admits a converging subsequence on each compact subinterval of times. By choosing a sequence $t_i \to -\infty$ of times, we construct the family of nested compact sets $K_i = [t_i, -\frac{1}{i}]$; we can then obtain a limit radial function r^∞ defined on $(-\infty, 0)$ by precompactness on the sets K_i and a diagonal argument. We construct the desired ancient solution M^∞_t as spherical graph of the function r^∞: as r^∞ is the limit of a subsequence which is bounded on each K_i, the diameter of the submanifold is also bounded at each time and the evolving hypersurface is compact; the same holds for (strict) convexity, recalling (3.5.5). This solution cannot be a shrinking sphere, as the ratio $\dfrac{a^\infty}{b^\infty}$ is equal to 2 at $t = -1$.

Chapter 4

Ancient solutions of expansive curvature flows

In this last chapter we will analyse ancient solutions of inverse flows by curvature functions. In general, inverse curvature flows are geometric flows whose speeds are given by the reciprocal of curvature functions:

$$\frac{\partial \varphi}{\partial t}(x,t) = \frac{\nu}{F}(x,t) \tag{4.0.1}$$

for $\varphi : M \times [T_1, T_2] \to \mathbb{R}^{n+1}$ a time-dependent immersion, where F is a p-homogeneous function of the principal curvatures satisfying properties (H1)–(H2) described in the previous chapter. In order for the system to be parabolic, the speed must be directed as the outer normal vector, so the evolution has an expanding character; geometric isotropy properties also hold.

Expansive flows (4.0.1) having homogeneity equal to one are a boundary case dividing two categories of evolution with a complementary behaviour, as described in [33]. The totally umbilic spheres in Euclidean space evolve by homotetic expansion; the evolution equation of the radius $R(t)$ under a speed of homogeneity $p \neq 1$ starting from time 0 is given by:

$$R(t) = \left(\frac{(1-p)}{n^p} t + R(0)^{1-p} \right)^{\frac{1}{1-p}}, \tag{4.0.2}$$

while for $p = 1$ the dilation rate is exponential and the spherical radius is $R(t) = R(0)e^{\frac{t}{n}}$. It is thus evident that for $p \in (0, 1]$ the flow of spheres exists for all times $t > 0$, while the solution blows up in finite time if $p > 1$, so there exists $T > 0$ such that

$$\limsup_{t \to T} |\varphi(x,t)| = +\infty \, .$$

In [32, 33] Gerhardt proved convergence of suitably rescaled solutions of (4.0.1) for a concave F to round spheres for all homogeneities, provided the starting submanifold is starshaped if $p \in (0,1]$ or strictly convex if $p > 1$ (for $p = 1$, Urbas [72] proved independently the same result). He also assumed the speed vanishes on $\partial\Gamma_+$ in the last case.

We specifically consider general flows of the form

$$\frac{\partial \varphi}{\partial t}(x,t) = \frac{\nu}{H_k^{\frac{p}{k}}}(x,t) \tag{4.0.3}$$

where H_k is the k-th Mean Curvature and $p > 1$. We will prove rigidity results for this class of speeds akin to those proved for contractive flows; we underline that the usual definition and properties of ancientness are not well defined in the range $(0,1)$, as from (4.0.2) one deduces that the radius of a sphere blows up in finite time in the past. For $p = 1$ the spheres provide an example of eternal solution. We establish the following result:

Theorem 12. *Let M_t be a convex closed ancient solution of* (4.0.3) *with $p > 1$ such that the following conditions are satisfied for all times:*

1. *the principal curvatures are uniformly pinched: there exists $C \geq 1$ such that $\frac{\lambda_n}{\lambda_1} \leq C$ holds for all $t \in (-\infty, 0)$;*
2. *M_t is uniformly starshaped with respect to 0: $u(x,t) = \langle \varphi(x,t), \nu(x,t) \rangle \geq \epsilon |\varphi(x,t)|$ for some constant $\epsilon \leq 1$;*
3. *the second fundamental form has polynomial growth: there exists $s > 0$ such that $\lim_{t \to -\infty} \frac{\max H_k^{\frac{1}{k}}(t)}{|t|^s} = 0$.*

Then M_t is a shrinking sphere.

The conditions on the solution are more restrictive than those in [33], but we observe that we need to assume strict convexity: the k-th mean curvatures do not belong to the class directly covered in the results by Gerhardt, since they do not vanish approaching the boundary of the positive cone, if $k \neq n$, and convexity is generally not preserved under an inverse flow by a concave function.

We will prove Theorem 12 by an argument similar to the one we used for Gaussian Curvature Flow in the previous chapter. We will introduce a monotonic quantity to state that the sphericity of the submanifold is actually increasing along the flow, and, intuitively, it is "increasing fast enough to have already reached a critical point from the beginning", on an infinitely long interval of time.

4.1 The k-th isoperimetric ratio

The quantity in question is the k-th isoperimetric ratio of the convex body Ω_t enclosed by M_t (we will often omit the subscript t):

$$I_k(\Omega) = \frac{\left(\int H_{k-1} d\mu\right)^{n+1}}{|\Omega|^{n+1-k}} \tag{4.1.1}$$

which is invariant under rescaling. We will show the ratio is decreasing along the flow; this will allow us to conclude, since the critical points of the functional I_k are standard spheres, as soon as we establish precompactness for the solution and that $\liminf_{t \to -\infty} \frac{d}{dt} I_k(\Omega_t) = 0$.

Lemma 13. *If M_t is a solution of* (4.0.3)*, then $I_k(\Omega_t)$ is nonincreasing.*

Proof. We will prove that the derivative of the isoperimetric ratio is negative. We will use the following notation (as in Burago-Zalgaller [18], with an ambient space of dimension $n+1$):

$$V_{n-i}(M) = \frac{1}{(n+1)\binom{n}{i}} \int_M H_i d\mu = \frac{1}{(n+1)\binom{n}{i}} |M_{n-i}| d\mu \qquad i = 1, \dots, n$$

so that

$$V_n = \frac{1}{n+1}|M|$$
$$I_k(\Omega) = \frac{|M_{n-k+1}|^{n+1}}{V_{n+1}^{n-k+1}}$$

and we also fix $V_{n+1} = |\Omega|$, $V_0 = \omega_{n+1} = |B^n(1)|$.

We have, using the same computations as in Bertini-Sinestrari [13]:

$$\begin{aligned}
\frac{\partial |M_{n-k+1}|}{\partial t} &= \int \partial_t H_k d\mu + \int H_k \partial_t d\mu = \\
&= \int \frac{\partial H_{k-1}}{\partial h_i^j} \frac{\partial h_i^j}{\partial t} d\mu + \int H_{k-1} \frac{H}{H_k^{\frac{p}{k}}} d\mu \\
&= \int \frac{\partial H_{k-1}}{\partial h_i^j} \left[\nabla^j \nabla_i \left(-H_k^{\frac{p}{k}} \right) - H_k^{\frac{p}{k}} h_m^j h_i^m \right] d\mu \\
&+ \int H_{k-1} \frac{H}{H_k^{\frac{p}{k}}} d\mu
\end{aligned}$$

As

$$\begin{aligned}
&\frac{\partial H_{k-1}}{\partial h_i^j} h_m^j h_i^m = H H_{k-1} - k H_k; \\
&\nabla^i \frac{\partial H_k}{\partial h_i^j} = 0,
\end{aligned}$$

integrating by parts we have:

$$\frac{d|M_{n-k+1}|}{dt} = k \int \frac{H_k}{H_k^{p/k}} d\mu \tag{4.1.2}$$

So we can compute

$$\begin{aligned}
\frac{dI^k(\Omega)}{dt} = \frac{1}{|\Omega|^{(n+1-k)^2}} \Bigg(&(n+1)k |M_{n-k+1}|^n |\Omega|^{n-k+1} \int \frac{H_k}{H_k^{\frac{p}{k}}} d\mu \\
&- (n-k+1) |M_{n-k+1}|^{n+1} |\Omega|^{n-k} \int \frac{1}{H_k^{\frac{p}{k}}} d\mu \Bigg)
\end{aligned}$$

In the following computation we will temporarily neglect the denominator of the previous equation, since it is always positive and it does not affect the computation of the sign. We will estimate the integrals in the numerator using Jensen's inequality; if $p > k$ we have:

$$\fint \left(\frac{1}{H_k}\right)^{\frac{p}{k}} =$$

$$\left[\fint \left(\left(\frac{1}{H_k}\right)^{\frac{p-k}{k}}\right)^{\frac{p}{p-k}} d\mu\right]^{\frac{p-k}{p}} \cdot \left[\fint \left(\frac{1}{H_k}\right)^{\frac{p}{k}} d\mu\right]^{\frac{k}{p}}$$

$$\geq \fint \left(\frac{1}{H_k}\right)^{\frac{p-k}{k}} d\mu \cdot \fint \frac{1}{H_k} d\mu$$

which gives

$$\frac{dI^k(\Omega)}{dt} \leq \fint \left(\frac{1}{H_k}\right)^{\frac{p}{k}-1} d\mu \Big[k(n+1)|M^n_{n-k+1}|\Omega|^{n-k+1}|M| - (n+1-k)|M_{n-k+1}|^{n+1}|\Omega|^{n-k}|M| \fint \frac{1}{H_k} d\mu\Big]$$

In terms of the normalized mixed volumes:

$$\frac{dI^k(\Omega)}{dt} \leq \int \left(\frac{1}{H_k}\right)^{\frac{p}{k}-1} d\mu (n+1)^{n+1} \binom{n}{k-1}^n \Big[kV^n_{n-k+1}V^{n-k+1}_{n+1} - (n-k+1)\binom{n}{k-1} V^{n+1}_{n-k+1} V^{n-k}_{n+1} \fint \left(\frac{1}{H_k}\right) d\mu\Big]$$

As

$$\binom{n}{k-1}\binom{n}{k}^{-1} = \frac{k}{n-k+1}$$

and

$$\fint \left(\frac{1}{H_k}\right) d\mu \geq \left(\fint H_k d\mu\right)^{-1} = \frac{V_n}{\binom{n}{k} V_{n-k}},$$

we have

$$\frac{dI^k(\Omega)}{dt} \leq \int \left(\frac{1}{H_k}\right)^{\frac{p}{k}-1} d\mu (n+1)^{n+1} \binom{n}{k-1}^n k V^n_{n-k+1} V^{n-k}_{n+1} \frac{1}{V_{n-k}} \cdot [V_{n-k}V_{n+1} - V_{n-k+1}V_n]$$

and the last expression is nonpositive thanks to repeated applications of the Newton inequalities: $V_i^2 \geq V_{i-1}V_{i+1}$. If $k > p$ we can use the same procedure but we apply Jensen's inequality to:

$$\fint \left(\frac{1}{H_k}\right)^{\frac{p}{k}-1} d\mu \leq \left(\fint \left(\left(\frac{1}{H_k}\right)^{\frac{p}{k}}\right)^{\frac{p-k}{p}} d\mu\right)^{\frac{p}{p-k}} \cdot \left(\fint \left(\frac{1}{H_k}\right)^{\frac{p}{k}-1} d\mu\right)^{\frac{-k}{p-k}}$$

obtaining

$$\fint \left(\frac{1}{H_k}\right)^{\frac{p}{k}-1} d\mu \leq \fint \left(\frac{1}{H_k}\right)^{\frac{p}{k}} d\mu \cdot \fint H_k d\mu$$

and then substituting as above; for $k = p$ we just need

$$\fint \frac{1}{H_k} d\mu \geq \left(\fint H_k d\mu\right)^{-1}.$$

The k-isoperimetric ratio is thus nonincreasing along the flow. We underline that the relevant quantity $[V_{n-k}V_{n+1} - V_{n-k+1}V_n]$ is zero if and only if the hypersurface is a standard sphere. $\square$

4.2 Proof of Theorem 12

Following Gerhardt [33], we assume the solution is parametrized as a graph over $\mathbb{S}^n$, so we have $M_t = graph(r(\cdot, t))$. The functions $\Theta_{t_0}(t, r)$ denote the radius at time t of the evolving sphere with radius r at the initial time t_0. We can immediately bound from above the inner and the outer radii of the solution as the hypersurface is pinched, the flow has an expansive character and the following avoidance principle holds.

Theorem (Avoidance Principle for expansive flows, [33])**.** *If M_{t_0} is such that there exist two constants r_1, r_2 with $r_1 < r(\cdot, t_0) < r_2$, then*

$$\Theta_{t_0}(t, r_1) < r(\cdot, t)$$
$$r(\cdot, t) < \Theta_{t_0}(t, r_2)$$

and each inequality is valid as long as both functions are finite.

This principle implies that the flow remains enclosed in a spherical shell in any compact subset $[t_0, t_1] \subset (-\infty, 0)$.

For $t \in (-\infty, -1)$, $\rho_-(t)$ and $\rho_+(t)$ are both finite and strictly greater than 0. Due to avoidance and the fact that the blow-up time only depends on the initial radius, $\lim_{t \to -\infty} \rho_-(t) = 0$ must hold, or we could include a sphere of a fixed diameter in the evolving submanifold at an arbitrarily small time in the past, and this would blow up in the interior before the submanifold.

The pinching condition allows to conclude immediately that $\rho_+(t)$ also converges to zero at negative infinity. Denoting by $\Theta(t) = \left\{\frac{1-p}{n^p}t\right\}^{\frac{1}{1-p}}$ the radius of the shrinking ancient sphere blowing up at $T = 0$ (which is assumed to have "starting radius 0 at $-\infty$"), there exist two constants such that

$$c_1\Theta(t) \leq \rho_-(t) \leq C\rho_+(t) \leq c_2\Theta(t)$$

along the flow. Using the avoidance principle again, we obtain

$$\min_{\mathbb{S}^n} r(\cdot,t) \leq \Theta(t) \leq \max_{\mathbb{S}^n} r(\cdot,t),$$

immediately implying $\max_{\mathbb{S}^n} r \leq c\Theta(t)$ by comparison with the circumradius. As the solution is uniformly starshaped by assumption, we have

$$v = \frac{r}{u} = \sqrt{1 + |\overline{\nabla}\log r|^2_{\mathbb{S}^n}} \leq \frac{1}{\epsilon}, \tag{4.2.1}$$

so the oscillation of each $r(\cdot,t)$ as a function on the sphere is bounded from above indepentently of t and there holds the opposite inequality $\min_{\mathbb{S}^n} r(\cdot,t) \geq c\max_{\mathbb{S}^n} r(\cdot,t) \geq c\Theta(t)$.

The inequality (4.2.1) with the estimates above also yield uniform C^1 bounds on r. We need to prove C^2 bounds from above and thus deduce uniform parabolicity for (4.0.3), in order to trigger Krylov-Safonov and Schauder theory and obtain compactness at $-\infty$. To estimate the speed of the flow from above, we employ again Tso's technique as in the previous chapter.

Lemma 14. *Let M_t be a closed convex ancient solution such that the assumptions of Theorem 12 hold. Then $H_k^{\frac{p}{k}}(\cdot,t) \geq \frac{C}{\Theta(t)}$ for $t \in (-\infty, 0)$, for a constant C.*

Proof. We choose $t_0 \in (-\infty,-1)$; eventually translating the hypersurface we can assume the enclosing sphere of radius $\rho_+(t_0)$ is centered in 0. We consider the function:

$$q(x,t_0) = \frac{1}{H_k^{\frac{p}{k}}(2\rho_+(t_0) - u)}.$$

The evolution equation for q reads (see, for example, [64]):

$$\begin{aligned}\frac{\partial q}{\partial t} &= \mathcal{L}q - \frac{2}{2\rho_+ - u}\langle\nabla q, \nabla u\rangle_k \\ &\quad + q^2\left[(1-p) + \frac{2\rho_+ p}{kH_k}(HH_k - (k+1)H_{k+1})\right]\end{aligned} \tag{4.2.2}$$

where $\mathcal{L}$ and $\langle\cdot,\cdot\rangle_k$ are the elliptic operator and the scalar product induced by contracting the connection with the derivative of the speed as in the previous chapter. To derive (4.2.2) we also used the identity

$$\frac{\partial H_k}{\partial h_{ij}} h_{il} h_j^l = H H_k - (k+1) H_{k+1}.$$

As in the contractive case, the equation for $Q(t) = \sup_{x \in M_t} q(x,t)$ then yields:

$$\frac{dQ}{dt} \leq Q^2 [(1-p) + \frac{2\rho_+ p}{k H_k}(H H_k - (k+1) H_{k+1})] \tag{4.2.3}$$

and comparison with the corresponding ODE as in (6) forces the term in square brackets to be nonnegative on $(-\infty,-1)$. We have:

$$\begin{aligned} 0 &\leq (1-p) + \frac{2\rho_+ p}{k H_k}(H H_k - (k+1) H_{k+1}) = \\ &= -(p-1) + \frac{2\rho_+ p H}{k}\left[1 - \frac{(k+1) H_{k+1}}{H_k H}\right] \\ &\leq -(p-1) + \frac{2\rho_+ p H}{k} \end{aligned}$$

as $\frac{(k+1)H_{k+1}}{H_k H}$ belongs to $[0,1]$. So we have the estimate on H:

$$H \leq \frac{(p-1)k}{2\rho_+ p}.$$

The lower estimate on ρ_+ and the pinching condition then allow to conclude $\lambda_1 > \frac{C}{\Theta}$ for a uniform C. Obviously, this implies $H_k^{-\frac{p}{k}} \leq C\Theta^p$. □

We remark that the assumption of polynomial growth of the speed at infinity was not needed to prove this Lemma.

To conclude, we need to establish a bound on principal curvatures from above. We consider the rescaled immersion $\widetilde{\varphi}(x,t) = \varphi(x,t)\Theta^{-1}(t)$; the associated geometric quantities obviously satisfy $\widetilde{u} = \Theta^{-1}u$, $\widetilde{r} = \Theta^{-1}r$, $\widetilde{h}_j^i = \Theta h_j^i$, $\widetilde{g}_{ij} = \Theta^{-2} g_{ij}$, $\widetilde{g}^{ij} = \Theta^2 g^{ij}$ (we are denoting rescaled quantities by a tilde as usual). From the paragraph above, the support function and the radial function are both uniformly bounded by constants and $\widetilde{H}_k^{\frac{1}{k}} \geq C$.

Lemma 15. *In the assumptions of Theorem 12, there exists a constant C such that $\widetilde{H_k}^{\frac{p}{k}} \leq C$.*

Proof. We define the family of functions $w = \log(\widetilde{H_k}^{\frac{p}{k}}) + \lambda\widetilde{r}$ where λ is a parameter to be defined later. As in [49], we will estimate the derivative of this function to prove an upper bound on the principal curvatures directly on the rescaled immersion. This is where we need the additional assumption on the speed; we state the argument for $F = H_k^{\frac{p}{k}}$, but we underline that it is valid for any concave p-homogeneous F with the same assumptions on the growth at $-\infty$.

The evolution equation for w gives:

$$\frac{\partial w}{\partial t} = \mathcal{L}w + \frac{\Theta^{p-1}}{n^p}\left(p - \frac{\lambda}{2}\widetilde{r}\right) + \lambda\Theta^{p-1}\left(\frac{(p+1)}{v\widetilde{H}_k^{\frac{p}{k}}} - \frac{\widetilde{r}}{2n^p}\right)$$

Let $\widetilde{H_k}^{-\frac{p}{k}} = \dfrac{e^{\lambda\widetilde{r}}}{e^w}$ If $\lambda > \lambda_0 = \frac{4p}{\min\widetilde{r}}$, the first bracket is negative and we can conclude for $W(t) = \max\limits_{\widetilde{M}_t} w(x,t)$:

$$\frac{dW}{dt} \leq -\frac{\lambda\Theta^{p-1}}{4n^p} + \frac{\lambda\Theta^{p-1}}{e^W}\left((p+1)e^{\lambda\widetilde{r}} - \frac{(\min\widetilde{r})e^W}{2n^p}\right) \tag{4.2.4}$$

We suppose by contradiction that $\limsup\limits_{t\to-\infty} \max\widetilde{H}_k^{\frac{p}{k}} = +\infty$. This obviously implies the same for the function W and $\limsup\limits_{t\to-\infty} W = +\infty$ holds for any $\lambda > \lambda_0$. Then, there exists a sequence of times going to $-\infty$ such that W explodes along the sequence; we can thus find a $t_0(\lambda)$ such that, at t_0, $\dfrac{\min\widetilde{r}}{2n^p}e^w > (p+1)e^{\lambda\max\widetilde{r}}$ and (4.2.4) implies $\dfrac{dW}{dt} < 0$. As W is decreasing forward in time, again (4.2.4) also implies $\dfrac{dW}{dt} < 0$ and $W(t) > W(t_0)$ on the whole halfline $(-\infty, t_0)$. Using the explicit expression of Θ, we have:

$$\frac{dW}{dt} \leq -\frac{\lambda\Theta^{p-1}}{4n^p} = -\frac{\lambda}{4(p-1)|t|}.$$

We can now integrate on a fixed subinterval $[t, t_0]$, obtaining

$$W(t) - W(t_0) \geq \frac{\lambda}{4(p-1)}(\log|t| - \log|t_0|)$$

for any $\lambda > \lambda_0$; then, for any λ, it holds

$$\liminf_{t\to-\infty} \frac{W(t)}{\log|t|} \geq \frac{\lambda}{4(p-1)}.$$

But we observe

$$\begin{aligned}\liminf_{t\to-\infty}\frac{W(t)}{\log|t|} &= \liminf_{t\to-\infty}\frac{\max(\log\widetilde{H_k}^{\frac{p}{k}}+\lambda\widetilde{r})}{\log|t|}\\ &= \liminf_{t\to-\infty}\frac{\max(\log\widetilde{H_k}^{\frac{p}{k}})}{\log|t|}\leq\liminf_{t\to-\infty}\frac{\log C|t|^{sp}}{\log|t|}=sp\end{aligned}$$

holds for arbitrary $\lambda \geq \lambda_0$ thanks to our assumptions. We obtain a contradiction by choosing $\lambda > 4sp(p-1)$. $\square$

We have demonstrated that the rescaled inverse speed $\widetilde{H}_k^{\frac{p}{k}}$ is uniformly bounded from above; by uniform pinching, the rescaled system is uniformly parabolic, as the derivative of $\widetilde{H}_k$ with respect to the curvature is a polynomial of homogeneity $k-1$ in the $\widetilde{\lambda}_i$. Thus, we can apply parabolic theory and conclude as in the case of contractive Gaussian Curvature Flow. We observe that the isoperimetric ratio of the rescaled solution $I^k(\widetilde{\Omega})$ is uniformly bounded and thus the same holds for $I^k(\Omega)$ due to the scaling invariance of this functional. For compactness, every sequence $\widetilde{M_{t_j}}$, with $t_j \to -\infty$ admits a subsequence converging to a limit $\widetilde{M_\infty}$. Uniform C^2 estimates hold, so there exists an $\epsilon > 0$ such that $I_k(\widetilde{\Omega}_t)$ and its derivatives are uniformly close to their value on $\widetilde{M_{-\infty}}$ for $t \in [t_j - \epsilon, t_j + \epsilon]$. The derivative $\frac{d}{dt}I^k(\widetilde{\Omega}_\infty)$ must then be zero, or it would be uniformly negative on a set of infinite measure, contradicting the boundedness of the ratio. Thus $[V_{n-k}V_{n+1} - V_{n-k+1}V_n]$ vanishes asymptotically, and we already remarked this implies the limit is a sphere. $I^k(\Omega_{t_j})$ thus converges to the spherical value, assumed on the limit. By monotonicity, the ratio must be constant and equal to its minimum on the rescaled flow, so $\widetilde{M}_t$ (and M_t) is a sphere for every time.

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