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Advanced energy methods in fluid-mechanics

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If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is. (John von Neumann)

Non c'è matematica che tenga nel descrivere la complessità dei rapporti umani, in questi miei tre anni di vita. Eppure di umanità ne ho ricevuta a bizzeffe, in questi miei tre anni di matematica - *sì, sì, sono di più, ma qui parliamo degli ultimi tre.* Questa "sezione" di ringraziamenti, di così difficile redazione, inizia, in virtù della mia arguzia (per ora lasciate ogni speranza, tornerò ad essere una persona seria a pagina 3), con alcune *scuse* anticipate: ne sono profondamente dispiaciuta, ma si tratterà di una parte pesantemente incompleta. L'incipit non è certo una *scusa - lo è eccome* - ma, d'altra parte, l'ansatz in questa faccenda consiste nel pensare che le persone interessate abbiano già capito tutto quello che mi passa per la testa in questo momento, molto prima di leggere anche solo la prima di queste tre parole. E, insomma, siccome qui non si tratta di matematica, approssimiamo un po' di qua e di là e consideriamolo d'ora in poi un bel teorema. Tra le persone che hanno già capito tutto di cui sopra, ma a cui è bene ricordare le cose, vista la moltitudine di impegni quotidiani, c'è Roberto Natalini. Ecco, già qui mi trovo parecchio in difficoltà con le definizioni - *"Ah, pensa te, e coi teoremi?", si potrebbe pensare* - ma, davvero, "direttore di tesi, advisor, tutor" è parecchio riduttivo, per quanto non si possa che dirne di positivo a riguardo. Beh, quindi grazie Roberto, per avermi fatto conoscere la bellezza di questo lavoro, per avermi insegnato più o meno tutto quello che oggi so fare, per avermi sostenuto nei momenti di debolezza, per avermi aiutato a vedere delle possibilità, e per aver creduto in me molto prima che iniziassi a farlo anch'io, come solo un ottimo relatore e un grande amico sa fare.

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La mia famiglia, gli amici e tutte le persone che mi sono state e mi sono vicine sanno bene di aver svolto un ruolo fondamentale.

Contents

Introduction

The aim of this thesis is to present some existence, uniqueness and long-time behavior results on the solutions to different kinds of fluid-dynamics models, achieved by means of energy methods combined with the Fourier setting and, in some cases, with paradifferential techniques.
In particular, here we investigate two main topics concerning fluid-dynamics: first we consider multiphase models arising from *mixture theory*, and then we focus on a semilinear hyperbolic approximation to more classical hydrodynamic equations, namely the incompressible Navier-Stokes equations.
Multiphase models of an arbitrary number of constituents arising from mixture theory, [65, 19, 20, 56, 57], present a wide range of applications, mainly in biological fields, as tumor growth and vasculogenesis [3], biological tissues and porous media [32]. Mixture theory models have been used to describe flows through biological tissues since the sixties. In this context, the most general model takes into account few but essential constituents, such as cells, extracellular matrix and liquid, but it can be generalized to an arbitrary number of sub-populations of cells, and several components of the extracellular matrix. These models are based on three main assumptions, [3].

- First, we assume that the components of the extracellular matrix constitute an intricate network such that they all move together.
- Besides, the pressure gradient and the interaction forces involving the liquid are much smaller than the other ones.
- The third assumption consists in assuming that cells mechanically respond to the compression coming from the surrounding cells.

Starting from the ideas of mixture theory, these models are composed by balance equations which essentially represent mass and momentum conservation. Although mixture models are largely diffused, up to now the analytical theory has been mainly developed in one space dimension, see for instance [38, 72, 79], and [41], while some results about linear stability and numerical approximations were considered in [35].
As a matter of facts, our starting point in the analytical study of mixture theory was the biofilms system presented in [27]. Although it has been adapted for modeling these particular gel-like biological structures, called *biofilms*, by extending the role of the physical coefficients and the source terms, this system can be seen as a general multiphase model arising from mixture theory. A complete analytical study of the one dimensional model presented in [27] is given here in Chapter 5. In more space dimensions, there are many other difficulties which will be explained in details in the following. Thus, without loss of generality, we will consider the simpler case of a fluid

composed by two phases, a solid component denoted by B (for instance, Bacteria), and a liquid one L, which comes from mass and momentum conservation and reads:

$$\begin{cases} \partial_t B + \nabla \cdot (B\mathbf{v}_S) = \Gamma_B, \\ \partial_t L + \nabla \cdot (L\mathbf{v}_L) = \Gamma_L = -\Gamma_B, \\ \partial_t (B\mathbf{v}_S) + \nabla \cdot (B\mathbf{v}_S \otimes \mathbf{v}_S) + \gamma \nabla B + B\nabla P = (M - \Gamma_L)\mathbf{v}_L - M\mathbf{v}_S, \\ \partial_t (L\mathbf{v}_L) + \nabla \cdot (L\mathbf{v}_L \otimes \mathbf{v}_L) + L\nabla P = -(M - \Gamma_L)\mathbf{v}_L + M\mathbf{v}_S, \\ \nabla \cdot (B\mathbf{v}_S + L\mathbf{v}_L) = 0, \end{cases} \tag{0.0.1}$$

where $\mathbf{v}_S, \mathbf{v}_L$ are the solid and liquid phase velocities, Γ_B, Γ_L are the source terms, γ, M are experimental constants and P is the hydrostatic pressure. We will give more details on the physical formulation in the following. Model (0.0.1) is a quasilinear system whose hyperbolic part is given by the incompressible Euler equations. It is of intermediate type between an incompressible system, since the average velocity $B\mathbf{v}_S + L\mathbf{v}_L$ is divergence free, and a compressible system, since the presence of ∇B, i.e. the gradient of the compressible pressure term like the isentropic Euler equations. A key role is played here by the inertial terms in the momentum equations. At some point, in the general framework of mixture theory models, they are usually neglected in order to simplify the analysis, see for instance [32]. We choose to keep the inertial terms and study the fully hyperbolic problem. In fact, the inertial terms guarantee the hyperbolicity of the system and the finite speed of propagation of the front. However, this choice, together with the compressible and incompressible nature of the system, introduce some analytical difficulties, which we will overcome by studying an intermediate model in Chapter 6, and applying energy methods combined with paradifferential techniques in Chapter 7. At the best of our knowledge, this is the first existence and uniqueness result for local smooth solutions to mixture models in more than one space dimension.

The second part of this thesis is focused on another completely different fluid-dynamics model, which is included in the framework of the BGK - Bhatnagar, Gross and Krook - approximations for hydrodynamic equations. These models were introduced by Bhatnagar, Gross and Krook in the fifties as a continuous velocities simplification of the Boltzmann equation. Precisely, that simplification occurs in terms of the collision operator. Taking inspiration from the hydrodynamic limits of the scaled Boltzmann equation, they have been intended as kinetic approximations to some hydrodynamic systems. A general mathematical framework for BGK models was given in [16]. In the last years, BGK models have been extended to the case of systems with discrete velocities. This additional simplification provides semilinear hyperbolic approximations for quasilinear systems, with applications to hydrodynamic equations. In the context of semilinear approximations, it is worth mentioning the relaxation method and, among several examples, we refer to the *Jin-Xin* approach introduced in [43]. However, a more detailed presentation of the relaxation method can be found in [54], and it will be discussed in the following. An important point to highlight is the intimate connection between the relaxation method and the BGK approximation with discrete velocities. Precisely, in one space dimension, the Jin-Xin approximation

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \lambda^2 \partial_x u = \frac{1}{\varepsilon}(f(u) - v), \end{cases} \tag{0.0.2}$$

for hyperbolic conservation laws

$$\partial_t u + \partial_x f(u) = 0, \tag{0.0.3}$$

is equivalent, on behalf of a linear change of variables, to a two constant velocities BGK model, see [59]. This is the reason why, at the beginning of the last part of the thesis, we will focus on the diffusive Jin-Xin system, as a simplified version of the BGK approximation for the Navier-Stokes equations which will be considered. In particular, global existence, global in time convergence in the diffusive limit, and long-time behavior for the solutions to the diffusive Jin-Xin system will be proved. These novel results are based on the study of the Green function of the system and the energy method.
In the last part of the thesis, we consider the following BGK model for hydrodynamic equations, from [23, 18]:

$$\partial_t f_i + \frac{\lambda_i}{\varepsilon} \cdot \nabla_x f_i = \frac{1}{\tau\varepsilon^2}(M_i(U) - f_i), \tag{0.0.4}$$

where $U = \sum_i f_i = (\rho, \varepsilon\rho\mathbf{u})$, ρ is the macroscopic approximating density, and $\mathbf{u}$ the velocity field, namely the moments associated with the isentropic Euler equations. This is a discrete velocities BGK model in the sense that $\lambda_i, i = 1, \cdots, L \geq D+1$, are a finite number of constant vectors. Moreover, as it is usual in the context of BGK approximations, see [16], $M_i(U)$ are the Maxwellian functions. Under some compatibility conditions which will be explained later on, this BGK model formally approximates the isentropic Euler equations in the hyperbolic limit τ going to zero, [23], and the incompressible Navier-Stokes equations for ε which goes to zero, [23]. Actually, the hyperbolic limit of this system was also rigorously proved in [68]. Here we study the diffusive limit from an analytical point of view. We state local in time existence, uniqueness and convergence theorems of the smooth solutions to the BGK model to the smooth solutions to the Navier-Stokes equations, which are the first analytical results on this diffusive BGK approximation. We will discuss on the details below. In the future, we aim to extend these results by applying the Green function method for the diffusive Jin-Xin system mentioned before to this BGK model.
The connection between the two main parts of the thesis is represented by the mathematical tools which have been used here in order to investigate these different systems: the multiphase model on one hand, and the BGK approximation on the other one. The key role is actually played by the classical energy method, here applied through suitable symmetrizers, first combined with paradifferential techniques, and then with dissipative properties and analysis of the Green function of the problem.
We end this paragraph by introducing an example on the energy method and the related symmetrization techniques, which represent the main connection between the different physical systems presented here. Consider the homogeneous wave equation with constant velocity $c = 1$ in one dimension in space, as follows:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x u = 0, \end{cases}$$

with initial data $(u(0,x), v(0,x)) = (u_0, v_0)$, and take the $L^2(\mathbb{R})$ scalar product $(\cdot,\cdot)$ with the unknown vector $(u\ v)$. Denoting by $\|\cdot\|$ the associated norm, since $u, v \in L^2(\mathbb{R})$

implies that $\int_{\mathbb{R}} (u \cdot v)_x \, dx = 0$, the resulting expression is the following:

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|v\|^2) = 0,$$

i.e.

$$\|u(t)\|^2 + \|v(t)\|^2 = \|u_0\|^2 + \|v_0\|^2.$$

Now, let be

$$\partial_t \mathbf{u} + A\partial_x \mathbf{u} = 0, \tag{0.0.5}$$

a *symmetric system*, i.e. $A = A^T$. Taking the scalar product with $\mathbf{u}$,

$$(A\partial_x \mathbf{u}, \mathbf{u}) = \frac{1}{2}\int_{\mathbb{R}} \partial_x (A\mathbf{u} \cdot \mathbf{u}) \, dx = 0,$$

thanks to the symmetry property. Thus,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 = 0.$$

Symmetry is not a necessary condition to get energy estimates, but, on the other hand, there are some non-symmetric systems which do not admit energy estimates as the ones before. For instance, we mention the *Cauchy-Riemann system*, see [8]. Another less restrictive condition which allows to obtain energy estimates is the so-called *symmetrizability*, i.e.: given a system as in (0.0.5), where A is not necessarily symmetric, there exists a symmetric strictly positive matrix A_0 such that $A_0 A = (A_0 A)^T$. This way, taking the scalar product of the symmetrized system

$$A_0 \partial_t \mathbf{u} + A_0 A \partial_x \mathbf{u} = 0$$

with $\mathbf{u}$, we get the desired energy estimate, since $\int (A_0 A\mathbf{u} \cdot \mathbf{u})_x \, dx = 0$. Actually, in this constant coefficient case, symmetrizability is the optimal condition in order to apply the energy method. For an extended discussion on this topic we refer to Chapter 2.

In the following, we provide a more detailed description of the results collected in this thesis.

Part I: Models in mixture theory In the first part, we start by presenting a complete analytical study in one space dimension of the multiphase model proposed in [27],

$$\begin{cases}
\partial_t B + \nabla \cdot (B\mathbf{v}_S) = \Gamma_B, \\
\partial_t E + \nabla \cdot (E\mathbf{v}_S) = \Gamma_E, \\
\partial_t D + \nabla \cdot (D\mathbf{v}_S) = \Gamma_D, \\
\partial_t L + \nabla \cdot (L\mathbf{v}_L) = \Gamma_L, \\
\partial_t((1-L)\mathbf{v}_S) + \nabla \cdot ((1-L)\mathbf{v}_S \otimes \mathbf{v}_S) = -(1-L)\nabla P - \gamma\nabla(1-L) \\
\quad + (M - \Gamma_L)\mathbf{v}_L - M\mathbf{v}_S; \\
\partial_t(L\mathbf{v}_L) + \nabla \cdot (L\mathbf{v}_L \otimes \mathbf{v}_L) = -L\nabla P - (M - \Gamma_L)\mathbf{v}_L + M\mathbf{v}_S, \\
\Gamma_B + \Gamma_D + \Gamma_E + \Gamma_L = 0, \\
\nabla \cdot ((1-L)\mathbf{v}_S + L\mathbf{v}_L) = 0,
\end{cases} \tag{0.0.6}$$

composed by four different phases: Bacteria, Dead bacteria, Extracellular polymeric matrix, and Liquid. The incompressibility condition in one space dimension, i.e. the vanishing spatial derivative of the average velocity in the last equation of system (0.0.6), allows to solve for the hydrostatic pressure P. Thus, we get a symmetrizable quasilinear hyperbolic system and the classical theory in [52] provides a local existence and uniqueness result for smooth solutions. We investigate the long-time behavior of this 1D system by using the Nishida approach presented in [61] and developed in [39], i.e. we define the functional

$$N_l^2(t) := sup_{\ 0\leq\tau\leq t}\ ||\mathbf{w}(\tau)||^2_{H^l(\mathbb{R})} + \int_0^t ||\mathbf{w}(\tau)||^2_{H^l(\mathbb{R})}\, d\tau, \tag{0.0.7}$$

for $l = 0, 1, 2$ and, by using the dissipative property of the system, we prove that, starting from initial data close enough to the equilibrium point, the solutions are global in time and they asymptotically converge towards the equilibrium. This result is based on [12]. This procedure does not work in the case of more than one space dimension, since in that case we cannot solve for P, and we have to deal with the hydrostatic pressure of the Euler type. In more space dimensions, even in the divergence free variable $\mathbf{w} = (1-L)\mathbf{v}_S + L\mathbf{v}_L$, the theory of symmetrizable hyperbolic systems does not apply and there are several problems. Without loss of generality, we consider the two-phase system (0.0.1).

- First, dropping the pressure P, the remaining system is symmetrizable in the sense of Friedrichs, and so the classical symmetrizer provides an energy functional which should allow us to apply the energy method. The difficulty here is that the scalar product induced by the classical symmetrizer does not preserve the orthogonality of the gradient of the incompressible pressure with respect to the divergence free averaged velocity $\mathbf{w}$. Therefore, we cannot get rid of the incompressible pressure, unlike the case of energy estimates in the Sobolev spaces for the incompressible Euler equations, see for instance [9].

- Furthermore, it is not obvious how to get these estimates by using the elliptic equation associated with the pressure P, as in [76]. In fact, because of the gradient of the compressible pressure $\gamma\nabla(1-L)$, and the inertial terms of the momentum equation, our hydrostatic pressure P does not possess enough regularity in space to close the estimates.

In order to overcome these difficulties, in Part II we consider a model of a compressible-incompressible fluid in several space dimensions, which can be seen as a one phase reduction of the two phase model (0.0.1) and the more general biofilms system (0.0.6). This single phase model contains the gradient of the incompressible hydrostatic pressure, and also a compressible one, which depends on the density and the velocity of the fluid itself. In fact, the coexistence of compressible and incompressible pressure terms is one of the main features of system (0.0.1) and (0.0.6). We prove the local well-posedness of this simplified model by applying three different methods, all of them based on approximating equations. This part is based on [14]. The first approximation is obtained by applying the *Leray projector* $\mathbb{P}$, i.e. the projector onto the space of the divergence free vector field, and mollifiers, to the compressible-incompressible model, see [9] for an application of this technique. In the second case, we define a continuous version of the *Chorin-Temam projection method*, [75], which is a singular perturbation system and requires well-prepared initial data.

The last approximation is an application of the artificial compressibility method, [75]. The local in time convergence of the smooth solutions to these approximations to the smooth solutions to the one phase system in the general multidimensional case is proved by using energy methods and paradifferential calculus. Following this direction, in Part II we are able to prove the convergence to system (0.0.1) of an adapted version of one of the approximations used for the compressible-incompressible fluid discussed before, made by the composition of some smoothing operators and the Leray projector $\mathbb{P}$. The main idea is as follows. First, we apply the projector onto the space of the vectors such that the averaged velocity $\mathbf{w}$ is divergence free. Then, we consider the paradifferential operator associated with the projected system (0.0.1), we notice that its highest order part is a strongly hyperbolic operator of the first order, and therefore it is possible to construct a *Lax symmetrizer* for it. The construction of this symmetrizer is essentially based on the techniques developed in [55], which are combined to some ideas in [34]. We point out that the main idea here is to symmetrize the whole projected operator, rather than just to use the symmetrizer of the hyperbolic part of (0.0.1). Using paradifferential calculus, we are able to establish some uniform energy estimates and the convergence of this method to the unique local solution to (0.0.1), as well as in the case of more general models deriving from mixture theory, both in two space dimensions. This result is based on [13].

Research perspective In three space dimensions, system (0.0.1) and the more general multiphase models deriving from mixture theory present some additional analytical difficulties with respect to the two dimensional case. Precisely, in $\mathbb{R}^3$ the principal symbol of the projected system loses its property of strong hyperbolicity, as we will show later, and so we are not able to construct a Lax symmetrizer, according to the definition given in [55]. However, in order to prove the well-posedness of the three dimensional model, which numerically works well, as it is shown in [27], we could try to apply some recent works by *Métivier* et al. that are based on a weaker notion of symmetrizability.
Moreover, since our analytical study on system (0.0.1) has been made on the whole space $\mathbb{R}^d$, $d = 1, 2$, it would be interesting to investigate the case of a general bounded domain, with homogeneous Neumann boundary conditions for the volume ratios B, D, E, L, and no-flux boundary conditions for the velocities $\mathbf{v}_S, \mathbf{v}_L$, which are the natural boundary conditions used for the numerical tests, see [27]. It is important to notice that in this case the boundary is characteristic, and so the classical theory does not apply in a standard way.

Part II: BGK approximation for hydrodynamic equations In the framework of fluid-dynamics systems, we consider a vector BGK - Bhatnagar, Gross, Krook - model for hydrodynamic equations presented in [23], which is a singular perturbation approximation inspired by the hydrodynamic limits of the Boltzmann equation (see [5, 6, 24]) on one hand, and the relaxation approximation for the incompressible Navier-Stokes equations, see for instance [22], on the other one. Unlike the Lattice Boltzmann schemes, which are scalar velocities model of kinetic equations widely used in computational physics, the vector BGK approximations associate every scalar velocity with one vector of unknowns. This structure provides nice analytical properties, in particular the natural compatibility with a mathematical entropy, which guarantees stability. The vector BGK model for hydrodynamic equations introduced in [23] is given by the following semilinear hyperbolic system:

$$\partial_t f_i + \frac{\lambda_i}{\varepsilon} \cdot \nabla_x f_i = \frac{1}{\varepsilon^2 \tau}(M_i(\rho, \varepsilon\rho\mathbf{u}) - f_i), \tag{0.0.8}$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^D$, $i = 1, \cdots, L$, and $L \geq D + 1$. Moreover,

$$\begin{aligned} &f_i(t,x) = (f_i^0, f_i^1, \cdots, f_i^D) : \mathbb{R}^+ \times \mathbb{R}^D \to \mathbb{R}^{D+1}, \\ &M_i(t,x) = (M_i^0, M_i^1, \cdots, M_i^D) : \mathbb{R}^+ \times \mathbb{R}^D \to \mathbb{R}^{D+1}, \\ &\lambda_i = (\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{iD}) \in \mathbb{R}^D, \end{aligned} \tag{0.0.9}$$

and

$$(\rho, \varepsilon\rho\mathbf{u}) = \sum_{i=1}^{L} f_i \in \mathcal{U} \subset \mathbb{R}^{D+1}.$$

Under some consistency conditions, this model formally converges to the isentropic Euler equations in the hyperbolic limit $\tau \to 0$, and to the incompressible Navier-Stokes equations in the parabolic limit $\varepsilon \to 0$. Moreover, assuming some additional hypothesis on the Maxwellian functions M_i, in [78] and [68] the rigourous convergence, in the hyperbolic limit, of the solutions to system (0.0.8) to the solutions to the isentropic Euler equations was proved. On the diffusive limit in ε, the convergence, at the formal level, to the solutions to the Navier-Stokes equations, and a zero order uniform energy estimate were given in [23]. More precisely, assuming that, in a suitable functional space,

$$\rho^\varepsilon \to \hat{\rho}, \qquad \mathbf{u}^\varepsilon \to \hat{\mathbf{u}}, \qquad \text{and} \qquad \frac{\rho^\varepsilon - \bar{\rho}}{\varepsilon^2} \to \hat{P},$$

under some consistency conditions of the BGK approximation with respect to the Navier-Stokes equations, see [23], it can be shown that formally the couple $(\hat{\mathbf{u}}, \hat{P})$ is a solution to the incompressible Navier-Stokes equations. Here we improve the results on the diffusive limit as follows. We provide a rigorous proof of this convergence in the Sobolev spaces. In this context, we consider a five velocities vector BGK model on the two dimensional torus for simplicity reasons. We briefly explain the main ideas. Rather than the entropy function associated with the BGK approximation, whose explicit expression is not known, we use a constant right symmetrizer Σ, weighted with respect to the singular parameter ε, which provides some dissipative properties for the singular linear part of the source term. More precisely, the symmetrized system in the unknown $\mathbf{W} = C(f_1, \cdots, f_L)$, with C a constant matrix, which reads

$$\Sigma\partial_t\mathbf{W} + B_1\Sigma\partial_x\mathbf{W} + B_2\Sigma\partial_y\mathbf{W} = -L\Sigma\mathbf{W} + N(\mathbf{W}), \tag{0.0.10}$$

where $B_1\Sigma, B_2\Sigma$ are symmetric, $-L\Sigma$ is negative definite and singular in ε, and N is the nonlinear part of the source term, is conservative-dissipative, according to the definition given in [15]. Besides, the weights, in terms of the diffusive parameter ε, of the symmetrizer, allow us to control the remaining nonlinear part N of the source. This way, we are able to perform uniform energy estimates in the Sobolev spaces and to get the convergence by compactness. This chapter is based on [11]. This kind of convergence result of smooth solutions is local in time, and it holds for an interval of time which depends on the Sobolev norm of the initial data. In the context of semilinear relaxation approximations, here we consider a singular parabolic scaling to the *Jin-Xin* approximation for conservation laws, see [43]. The system reads

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \varepsilon^2 \partial_t v + \lambda^2 \partial_x u = f(u) - v. \end{cases} \tag{0.0.11}$$

In the one dimensional case, this system can be written as a very simple BGK model. Thus, we study its smooth solutions and, by using an approach based on the Green function, in the spirit of [15], we prove their global existence and we also investigate their asymptotic behavior in the singular perturbation limit. We obtain, indeed, sharp decay estimates in time to the solution to system (9.0.1) in the Sobolev spaces, which are uniform with respect to the singular parameter. This provides the convergence to the limit nonlinear parabolic equation

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ v = f(u) - \lambda^2 \partial_x u, \end{cases} \tag{0.0.12}$$

both asymptotically in time, and in the vanishing ε-limit. To this end, we perform an crucial change of variables that highlights the dissipative property of the Jin-Xin system, and provides a faster decay of the dissipative variable with respect to the conservative one, which allows to close the estimates. Next, a deep investigation on the Green function of the linearized system (9.0.1) and the related spectral analysis is provided, since explicit expressions are needed in order to deal with the singular parameter ε. The dissipative property of the diffusive Jin-Xin system, together with the uniform decay estimates discussed above, and the Green function analysis combined with the Duhamel formula provide our main result. This work can be seen as an intermediate step in order to extend the results on the solutions to the BGK model for Navier-Stokes, and it is based on [10].

Research perspective Here we proved the convergence of the solutions to the vector BGK model to the solutions to the incompressible Navier-Stokes equations on the two dimensional torus $\mathbb{T}^2$. It could be worth extending these results to the whole space and to a general bounded domain with suitable boundary conditions, but new ideas are needed to approach these cases. Rather than the more classical kinetic entropy method, here our main tool was the use of a constant right symmetrizer, which provides the conservative-dissipative form introduced in [15], and allows us to get higher order energy estimates. Nevertheless, we do not have an estimate for the rate of convergence, in terms of the difference $||\mathbf{u}^\varepsilon - \mathbf{u}^{NS}||_s$, with $\mathbf{u}^\varepsilon, \mathbf{u}^{NS}$ the velocity fields associated with the BGK system and the Navier-Stokes equations respectively. Finally, since we obtained a local existence result for general initial data in the Sobolev spaces, it would be interesting to investigate the possibility to get a global in time result, with some assumptions of smallness on the initial data. Taking inspiration from the study of the Green function of the singular diffusive Jin-Xin system presented here, the idea is to adapt this technique to the vector BGK model for Navier-Stokes.

Plan of the thesis This thesis is organized as follows.
Part I is devoted to the presentation of the backgrounds on modeling of multiphase fluids on one hand, and the general analytical theory on hyperbolic systems on the

other one. In particular, besides the notions of hyperbolicity and symmetrizability and the related tools, we present a short review on paradifferential calculus in Chapter 2. Pseudo and paradifferential tools and their strict relation with the energy method will be important ingredients in the following. At the end of Chapter 2, we also provide a local well-posedness result for first order paradifferential systems in Theorem 2.4.2. More precisely, in this context we describe a standard way to construct a Lax-symmetrizer for a system that satisfies some reasonable assumptions. The procedure presented here comes from some ideas in [34] and [55]. In [34], the author shows an energy method which is similar to the one presented here, but there are some differences which require a different approach. In fact, [34] is the study of a particular form of singular perturbation approximations, a sort of paradifferential version of the singular approximation by *Kleinerman* and *Majda* in [47], and some points of that theory are really based on the particular structure of those systems. On the other hand, in [55] the energy estimates are obtained by assuming the existence of a symmetrizer as in Definition 2.4.1 below. What it is worth pointing out is that the general ideas of the symmetrization technique presented here are well-known, but, to the author's knowledge, they are not collected and written in this form in the literature, though they are really useful for a lot of applications.
At the end of Part I we provide a presentation of dissipative hyperbolic systems. The symmetrizers theory, rather than the more classical entropy approach, is used in order to discuss these notions. To highlight the role of the dissipative mechanisms, in Chapter 3 also the *Shizuta-Kawashima condition* is presented.

Part II, which is focused on multiphase models in *mixture theory*, is divided in three chapters. The first one is devoted to the proof of existence, uniqueness and asymptotic convergence to the equilibrium of the smooth solutions to the Cauchy problem in one space dimension, with small initial data, associated with the mixture theory model presented at the beginning of this part, and it is based on [12]. In the second part, i.e. Chapter 6, we investigate the multidimensional case by studying an intermediate model, which contains most of the analytical difficulties of the general d-dimensional multiphase model. We prove the well-posedness of this intermediate model with three different methods: the first one makes use of the *Leray* projector and related paradifferential tools; besides, we show the convergence of both a continuous version of the Chorin-Temam projection method, viewed as a singular perturbation approximation, and the so-called *artificial compressibility method*, see [75]. These results are based on [14]. At the end of this part, i.e. Chapter 7, we consider the two dimensional version of the two-phase model and also the four phases model presented at the beginning. More precisely, we prove the convergence of one approximation to the two dimensional two-phase system, made by the composition of some smoothing operators and the *Leray* projector, see [9] for different applications of this technique. Thus, we get the local well-posedness of this multiphase model in two space dimensions. The proof is based on the energy method combined with paradifferential tools. Besides, the problems related to the three dimensional case in space are discussed at the end of this chapter. These results are based on [13].

Part III consists of three chapters. In the first one we present the BGK approach and its connection with the relaxation method. In Chapter 9, we consider the parabolic scaled version of the Jin-Xin approximation for scalar conservation laws introduced in [43]. By studying the Green function associated with the system, we get some uniform energy estimates which provide global existence and global in time convergence to the limit system for small initial data, besides the analysis of the long time behavior of these solutions. The study of the Jin-Xin model under the diffusion scalling comes from [10]. The last chapter of this part is devoted to the presentation of the BGK approximation for hydrodynamic equations introduced in [23]. By using the symmetrizers theory combined with the dissipative property of the singular approximation BGK system, we prove its convergence to the solutions to the incompressible Navier-Stokes equations on the two dimensional torus, for a finite interval of time. This result is based on [11].

Part I

Backgrounds: hyperbolicity, multiphase models and dissipation

Chapter 1

Multiphase models

The aim of this chapter is to recall briefly the general setting, the main assumptions, and the physical derivation of models arising from *mixture theory*, see [65, 19, 20, 56, 57]. Consider a mixture of N constituents. In the following, we write the equations of balance of mass

$$\rho_n(\partial_t \phi_n + \nabla \cdot (\phi_n v_n)) = \Gamma_n,$$

momentum

$$\rho_n(\partial_t(\phi_n v_n) + \nabla \cdot (\phi_n v_n \otimes v_n)) = \nabla \cdot \tilde{T}_n + m_n + \Gamma_n v_n,$$

and energy

$$\rho_n(\partial_t(\phi_n E_n) + \nabla \cdot (\phi_n E_n v_n)) = tr(\tilde{T}_n L_n) - \nabla \cdot q_n + \Gamma_n E_n,$$

for the n^{th} constituent, $n = 1, \cdots, N$, where

- ρ_n is the density of the n^{th} phase, assumed to be the same constant for every phase,
- $\phi_n(t,x)$ is the volume fraction of each constituent,
- $v_n(t,x)$ is the specific velocity,
- $\Gamma_n(t,x)$ is the mass exchange rate between different phases,
- $\tilde{T}^n(t,x)$ is the partial stress tensor,
- $m_n(t,x)$ is the interaction force, which is related to interactions between different phases across the interfaces,
- E_n is the specific internal energy,
- q_n is the partial heat flux vector,
- $L_n = \nabla v_n$ is the velocity gradient.

There is also an hypothesis on the volume fractions,

$$\sum_{n=1}^{N} \phi_n(t,x) = 1, \qquad (1.0.1)$$

which means that the mixture is saturated, and no space is left. Besides, we assume the conservation of the total mass, which is given by the following constraint

$$\sum_{n=1}^{N} \Gamma_n(t,x) = 0, \tag{1.0.2}$$

the conservation of the total momentum, i.e.

$$\sum_{n=1}^{N} m_n + \Gamma_n v_n = 0, \tag{1.0.3}$$

and the conservation of the total energy

$$\sum_{n=1}^{N} \Gamma_n(E_n + v_n \cdot v_n/2) + v_n \cdot m_n = 0. \tag{1.0.4}$$

Now, summing the mass balance equations for $n = 1, \cdots, N$, we get the following divergence free constraint on the averaged velocity of the mixture

$$\sum_{n=1}^{N} \nabla \cdot (\phi_n v_n) = 0, \tag{1.0.5}$$

which is an incompressibility condition for the whole mixture. Furthermore, it will be useful to consider the equations for the mixture as a whole

$$\begin{aligned} &\partial_t \rho_m + \nabla \cdot (\rho_m v_m) = 0, \\ &\rho_m(\partial_t v_m + v_m \cdot \nabla v_m) = \nabla \cdot T_m, \\ &\rho_m(\partial_t E_m + v_m \cdot \nabla E_m) = tr(\sum_{n=1}^{N} \tilde{T}_n L_n) - \nabla \cdot q_m - \sum_{n=1}^{N} v_n \cdot m_n, \end{aligned} \tag{1.0.6}$$

where

- $\rho_m = \sum_{n=1}^{N} \rho_n \phi_n$ is the density of the mixture,
- $v_m = \sum_{n=1}^{N} \dfrac{\rho_n \phi_n v_n}{\rho_m}$ is the velocity of the mixture,
- $T_m = \sum_{n=1}^{N} \tilde{T}_n - \rho_n \phi_n z_n \otimes z_n$ is the tensor for the mixture as a whole and $z_n = v_n - v_m$ is the diffusion velocity,
- $E_m = \sum_{n=1}^{N} \dfrac{\rho_n \phi_n E_n}{\rho_m}$ is the "inner part" of the energy density of the mixture,
- $q_m = \sum_{n=1}^{N} (q_n + \rho_n \phi_n E_n z_n)$ is the heat flux for the mixture.

The equations above have been obtained summing the N mass and momentum equations and using the constraints above.
Following [32], we derive the general form of a mixture model, by considering a binary mixture of a solid phase S and a liquid one L. Let us focus on the interaction forces m_S, m_L, where, hereafter, the suffixes S, L refer to the solid and the liquid phase

respectively. Since we are in the isothermal case, assuming that the interaction forces depend on the growth terms and the relative velocity and using (1.0.2), (1.0.3), then

$$\begin{aligned} m_S &= P\nabla\phi_S + M(v_L - v_S) + \frac{\Gamma_S}{2}(v_L - v_S), \\ m_L &= P\nabla\phi_L + M(v_S - v_L) + \frac{\Gamma_L}{2}(v_S - v_L), \end{aligned} \tag{1.0.7}$$

where P is a pressure term, while M is an experimental constant. In (1.0.7), the second terms represent the drag forces, while the third ones are the corrections with respect to the Darcy law. Besides, it can be checked, see [32], that the relations before are compatible with the Clausius-Duhem inequality, which, according to [66], represents the second law of thermodynamics.
Now, we notice that (1.0.5) in the case of a binary mixture reads

$$\phi_S tr(L_S) + \phi_L tr(L_L) + (v_S - v_L) \cdot \nabla\phi_S = 0.$$

This equation indicates an indeterminacy, which is represented in the entropy inequality by introducing a scalar multiplier λ. At this point, one needs restrictions, see again [32] for more details, and then particular expressions for the Helmholtz free energy for the solid and the liquid phase are derived. Finally, additional physical assumptions provide the following expressions of the stress tensors:

$$\begin{aligned} \tilde{T}_S &= -\phi_S P I + \rho_S \phi_S T_S, \\ \tilde{T}_L &= -\phi_L P I, \end{aligned} \tag{1.0.8}$$

where T_S is the excess stress tensor for the solid and P is the hydrostatic pressure. Let us notice that the physical assumptions lead to an expression for the liquid stress tensor that does not contain anything else except the hydrostatic pressure.

Chapter 2

Hyperbolic systems and paradifferential calculus

This chapter is dedicated to an introduction for the Cauchy problem for multidimensional hyperbolic systems, which involves the use of energy estimates. First, following [69, 8], we will provide some definitions and technical tools on the key concept of *hyperbolicity*. In the context of energy methods, we will introduce the role of the so-called *classical* or *Friedrich symmetrizer* for an hyperbolic system. Furthermore, following [1, 55, 8, 25, 74, 4], we will show how *paradifferential symmetrizers* and the related paradifferential calculus can be used in order to get energy estimates. We point out that this survey is far from being complete. Here we try to collect the results that will be useful in the following. For a further discussion we refer to the works cited above.

2.1 Hyperbolic systems

Consider a $N \times N$ system

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} A_j(t, x, \mathbf{u}) \partial_{x_j} \mathbf{u} = G(t, x, \mathbf{u}) \tag{2.1.1}$$

in d space dimensions, where t is the time variable and the unknown $u \in \mathbb{R}^N$ depends on the space-time variable $(t, x) \in \mathbb{R} \times \mathbb{R}^d$. Here, the A_j are $N \times N$ matrices, and G is a source term. Expression (2.1.1) is the most general form of a multidimensional first order system. However, for our purpose it is sufficient to restrict the attention to the case of *quasilinear* systems, where the matrices A_j and the source G depend explicitly just on the unknown variable $\mathbf{u}(t, x)$. Thus, here the most general form of first order systems under consideration is given by

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} A_j(\mathbf{u}) \partial_{x_j} \mathbf{u} = G(\mathbf{u}). \tag{2.1.2}$$

It is worth recalling that, for smooth solutions, a system of *conservation laws*

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} \partial_{x_j} F_j(\mathbf{u}) = G(\mathbf{u}), \tag{2.1.3}$$

is equivalent to (2.1.2), where $A_j(\mathbf{u}) = \nabla_{\mathbf{u}} F_j(\mathbf{u})$. Systems of conservation laws have several physical applications, such as the Euler equations of fluid dynamics, vehicular and pedestrian traffic, see [28] for a further discussion.
In the following, our setting will be represented by the Sobolev spaces $H^s(\mathbb{R}^d)$, at least with $s > \frac{d}{2}$. Then, the definitions and the results below will be set in $H^s(\mathbb{R}^d)$.
First, let us consider the Cauchy problem for a linear constant coefficients system

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} A_j \partial_{x_j} \mathbf{u} = f(t,x), \qquad \mathbf{u}(0,x) = \mathbf{u}_0, \tag{2.1.4}$$

where A_j are $N \times N$ constant matrices, and we assume $G(t,x) = f(t,x)$. If $\mathbf{u}(t,\cdot)$ belongs to $C([0,T], H^s(\mathbb{R}^d))$ for a fixed $T > 0$, we can apply the Fourier transform to the previous equation, obtaining

$$\partial_t \hat{\mathbf{u}} + iA(\xi)\hat{\mathbf{u}} = \hat{f}, \qquad \hat{\mathbf{u}}(0,\xi) = \hat{\mathbf{u}}_0,$$

where

$$A(\xi) = \sum_{j=1}^{d} \xi_j A_j. \tag{2.1.5}$$

Thus, the solution is given by

$$\hat{\mathbf{u}}(t,\xi) = e^{-itA(\xi)}\hat{\mathbf{u}}_0(\xi) + \int_0^t e^{-i(t-s)A(\xi)} \hat{f}(s,\xi)\, ds. \tag{2.1.6}$$

Now, the point is to find conditions in order to have a tempered distribution in ξ on the right hand side of equation (2.1.6), so obtaining the solution $\mathbf{u}$ by applying the inversion formula of the Fourier transform. This fact depends on the behavior of $e^{-tA(\xi)}$ for $|\xi| \to \infty$.
The following lemma from [55] follows immediately.

Lemma 2.1.1. *If the eigenvalues of $A(\xi)$ are real for any $\xi \in \mathbb{R}^d$, then the exponential $e^{-iA(\xi)}$ presents at most a polynomial growth for $|\xi| \to \infty$.*

Thus, we introduce the definition of hyperbolic systems.

Definition 2.1.1. *System (2.1.4) is* hyperbolic *if the matrix $A(\xi)$ in (2.1.5) has real eigenvalues for any $\xi \in \mathbb{R}^d$.*

From the Plancherel theorem, a necessary and sufficient condition for the well-posedness of the Cauchy problem associated with (2.1.4) in $H^s(\mathbb{R}^d)$ is the following estimate, where, hereafter, $\|\cdot\|_s$ indicates the Sobolev norm:

$$\sup_{t\in(0,T)} \|\mathbf{u}(t)\|_s \le C_T \|\mathbf{u}_0\|_s,$$

for a constant C_T depending on T. We introduce the following definition, see [55].

Definition 2.1.2. *The linear system with constant coefficients (2.1.4) is strongly hyperbolic if there exists a constant C such that, for any $\xi \in \mathbb{R}^d$,*

$$\sup_{\xi\in\mathbb{R}^d} \|e^{-iA(\xi)}\| \le C,$$

or, equivalently, both these two conditions are satisfied:

- *for any $\xi \in \mathbb{R}^d$ the eigenvalues of $A(\xi)$ are real and semisimple;*
- *there exists a constant C such that, for any $\xi \in \mathbb{R}^d$, the norm of the eigenprojectors of $A(\xi)$ is bounded by C.*

Thus, from [55] we have the following theorem.

Theorem 2.1.1. *If system (2.1.4) is strongly hyperbolic, then, for all $\boldsymbol{u}_0 \in H^s(\mathbb{R}^d)$ and $f \in L^1([0,T], H^s(\mathbb{R}^d))$, there exists a unique $\mathbf{u}$ which solves the Cauchy problem for (2.1.4).*

A particular class of *strongly hyperbolic systems* is represented by *symmetric* and *symmetrizable hyperbolic systems.*

Definition 2.1.3. *System (2.1.4) is* symmetric hyperbolic *if A_j is self-adjoint for $j = 1, \cdots, d$.*

Definition 2.1.4. *System (2.1.4) is symmetrizable if there exists a self-adjoint matrix A_0, positive definite, such that $A_0 A_j$ is self-adjoint for $j = 1, \cdots, d$. In this case, A_0 is a* classical symmetrizer *or* Friedrich symmetrizer *for the system.*

The following theorem holds for symmetrizable hyperbolic systems, [55].

Theorem 2.1.2. *If the system is hyperbolic symmetrizable, then it is strongly hyperbolic.*

There is also another subclass of strongly hyperbolic systems, which is represented by *hyperbolic systems with constant multiplicities* and, in particular, *strictly hyperbolic systems.*

Definition 2.1.5. *System (2.1.4) is* hyperbolic with constant multiplicity *if, for all $\xi \neq 0$, $A(\xi)$ has only real and semisimple eigenvalues with constant multiplicities.*

Definition 2.1.6. *System (2.1.4) is* strictly hyperbolic *if, for all $\xi \neq 0$, $A(\xi)$ has N distinct real eigenvalues.*

From the previous definitions and [55], we have the following lemma.

Lemma 2.1.2. *Hyperbolic systems with constant multiplicity, and, in particular, strictly hyperbolic systems, are strongly hyperbolic.*

Now, we return to the case of quasilinear systems in (2.1.2). Consider the symbol

$$A(\mathbf{u}, \xi) = \sum_{j=1}^{d} \xi_j A_j(\mathbf{u}). \tag{2.1.7}$$

Then the previous definitions on hyperbolicity apply to the linearized system

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} A_j(\bar{\mathbf{u}}) \partial_{x_j} \mathbf{u} = G(\mathbf{u}), \tag{2.1.8}$$

for any $\bar{\mathbf{u}}$ belonging to the domain under consideration. However, the strong hyperbolicity of the linearized system above is not enough in order to guarantee the well-posedness of the system. In this context, let us provide the following definition, see [8].

Definition 2.1.7. *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^N$. The quasilinear system (2.1.2) is* Friedrichs symmetrizable *in $\mathcal{U}$ if there exists a C^∞ symmetric matrix $A_0(\boldsymbol{u})$, positive definite, such that $A_0(\boldsymbol{u})A_j(\boldsymbol{u})$ is symmetric for $j = 1, \cdots, d$, and for all $\boldsymbol{u} \in \mathcal{U}$.*

For instance, the isentropic Euler equations in terms of the pressure p and the velocity $\mathbf{v}$, namely

$$\begin{cases} \partial_t p + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0, \\ \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p = 0, \\ \rho = p^{1/\gamma} e^{-S_0/\gamma}, \end{cases}$$

where γ, S_0 are positive constants, can be written as a symmetrizable hyperbolic system in the sense of Friedrichs. In this case, the system is symmetrized by the diagonal matrix

$$A_0(\mathbf{u}) = diag(1, \gamma p \rho I_d),$$

where $\mathbf{u} = (p, \mathbf{v})$, I_d is the d-dimensional identity matrix. This definition leads to the following local well-posedness theorem, see [52].

Theorem 2.1.3. *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^N$. Assume that $A_j(\boldsymbol{u}), G(\boldsymbol{u})$ in (2.1.2) are C^∞ functions of the unknown $\boldsymbol{u}$, and that system (2.1.2) is Friedrichs symmetrizable. Consider the Cauchy problem associated with (2.1.2) and initial data $\boldsymbol{u}_0 \in H^s(\mathbb{R}^d)$, with $s > \frac{d}{2} + 1$. There exists a $T > 0$ such that there is a unique classical solution to this problem $\boldsymbol{u} \in C^1([0,T] \times \mathbb{R}^d) \cap C([0,T], H^s(\mathbb{R}^d)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^d))$.*

Furthermore, there exists also another more extended version of *symmetrizability*. In order to deal with, we need to introduce some preliminary tools on pseudo and paradifferential calculus.

2.2 Pseudo and paradifferential tools

Following [1, 55, 74, 4], here we aim to provide a self-contained description of the main tools on pseudo and paradiffential calculus. We limit ourselves to the arguments that will be useful in the following, while, for an accurate discussion, we refer to [1, 55, 74, 4].

2.2.1 An introduction

The spatial Fourier transform $\hat{p}(\xi)$ of $p(x) \in \mathcal{S}(\mathbb{R}^d)$ is given by

$$\hat{p}(\xi) = \int_{\mathbb{R}^d} p(x) e^{-ix\cdot\xi}\, dx.$$

Similarly, the inversion formula reads

$$p(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{p}(\xi) e^{ix\cdot\xi}\, d\xi.$$

Consider a multi-index with $\alpha = (\alpha_j)_{j=1,\cdots,d}$ and take the α derivative

$$D^\alpha p(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \xi^\alpha \hat{p}(\xi) e^{ix\cdot\xi}\, d\xi,$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, and $D_j = \frac{1}{i}\frac{\partial}{\partial_{x_j}}$. We associate the notation $p(D) = D^\alpha$, to the integral operator above. In a standard way, $p(D)$ is said to be a differential operator with symbol $p(\xi) = \xi^\alpha$. More generally, a differential operator can also depend on the spatial variable x. For instance, we can write a differential operator of order k in the following form:

$$p(x, D) = \sum_{|\alpha| \le k} a_k(x) D^\alpha,$$

for some smooth coefficients $a_\alpha(x)$. In this case, we say that the symbol associated with the operator above is

$$p(x, \xi) = \sum_{|\alpha| \le k} a_\alpha(x) \xi^\alpha,$$

and it acts on $u \in \mathcal{S}$ (the Schwartz space) in the following way:

$$p(x, D)u(x) = \mathcal{F}^{-1}_{\xi \to x}(p(x, \xi)\hat{u}(\xi)) = (2\pi)^{-d} \int_{\mathbb{R}^d} p(x, \xi)\hat{u}(\xi) e^{ix\cdot\xi}\, d\xi.$$

However, differential operators are restricted to polynomial symbols. We can consider a more general class of symbols by introducing the notion of *pseudodifferential operators*. We look for a class of *admissible* functions $p(x, \xi)$, which means that

- $p(x, \xi)$ has a polynomial type behavior with respect to ξ, i.e.

  $$|\partial_\xi^\alpha p(x, \xi)| \le c_\alpha \Lambda(\xi)^{m-|\alpha|},$$

  for some m,

- the variation of $p(x, \xi)$ with respect to x must be weak enough such that the difference between the amplitude $p(x, \xi)$ and the phase $e^{ix\cdot\xi}$ in the integral above is preserved, i.e.

  $$|\partial_x^\alpha p(x, \xi)| \le c_\alpha \Lambda(\xi)^m,$$

where, hereafter, we set

$$\Lambda(\xi) = (1 + |\xi|^2)^{1/2}. \tag{2.2.9}$$

This qualitative description leads to the following definition.

Definition 2.2.1. *Let $m \in \mathbb{R}$. Let $S^m = S^m(\mathbb{R}^d \times \mathbb{R}^d)$ be the set of $p(x, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that, for all α, β,*

$$|\partial_x^\beta \partial_\xi^\alpha p(x, \xi)| \le c_{\alpha,\beta} \Lambda(\xi)^{m-|\alpha|}.$$

An element of S^m is called a symbol of order m. We also denote infinitely smooth symbols with $S^{-\infty} = \cap_m S^m$. The related operator acting on $u \in \mathcal{S}$ is

$$p(x, D_x)u(x) = \mathcal{F}^{-1}_{\xi \to x}(p(x, \xi)\hat{u}(\xi)) = \int e^{ix\cdot\xi} p(x, \xi)\hat{u}(\xi)\, d\xi. \tag{2.2.10}$$

Example 2.2.1. *Let $p(\xi)$ an homogeneous function of degree m, i.e., for all $\lambda > 0$, $p(\lambda\xi) = \lambda^m p(\xi)$ which is C^∞ for $\xi \ne 0$. If $\chi \in C_0^\infty(\mathbb{R}^d)$, with*

$$\chi(\xi) = \begin{cases} 1 & for\, |\xi| \le 1, \\ 0 & for\, |\xi| > 1, \end{cases}$$

then $\tilde{p}(\xi) = (1 - \chi(\xi))p(\xi)$ is a symbol of order m.

Example 2.2.2. *Let $\phi \in \mathcal{S}$. Then $\phi(\xi)$ is a symbol of order $-\infty$.*

Example 2.2.3. *The function $p(x,\xi) = e^{ix\cdot\xi}$ is not a symbol.*

We have the following elementary properties:

- if $p \in S^m$, then $\partial_x^\beta \partial_\xi^\alpha p \in S^{m-|\alpha|}$;
- if $p \in S^{m'}$ and $q \in S^{m''}$, then the composition $pq \in S^{m'+m''}$;
- if $p \in S^m$, then $p \in \mathcal{S}'(\mathbb{R}^{2d})$, i.e. the space of tempered distributions.

Now we introduce the notion of *asymptotic sum*, which will be useful in the context of adjoint and composition operators.

Definition 2.2.2. *Let $p_j \in S^{m_j}$, $j \in \mathbb{N}$, for a decreasing sequence $m_j \to -\infty$. We write*

$$p \sim \sum p_j,$$

i.e. an asymptotic sum in the sense of the behavior for $|\xi| \to +\infty$, which means that, for all $k \geq 0$,

$$p - \sum_{j=0}^{k} p_j \in S^{m_{k+1}}.$$

The previous definition is based on the following lemma, see [1].

Lemma 2.2.1. *(Borel) Let $(b_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers. There exists $f(x) \in C^\infty(\mathbb{R})$ such that, $\forall j$ $f^{(j)}(0) = b_j$, then it holds $f(x) \sim \sum b_j \frac{x_j}{j!}$ when $x \to 0$.*

Thus, we have:

Proposition 2.2.1. *There exists $p \in S^{m_0}$ such that $p \sim \sum p_j$.*

2.2.2 Pseudodifferential operators in $\mathcal{S}$ and $\mathcal{S}'$

Here we provide some results on pseudodifferential operators in the Schwartz space $\mathcal{S}$ and the space of tempered distributions $\mathcal{S}'$.

Proposition 2.2.2. *If $p(x,D) \in S^m$ and $u \in \mathcal{S}$, then*

$$p(x,D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi$$

defines a function of $\mathcal{S}$.

We introduce the *Schwartz kernel* associated with a pseudodifferential operator acting on the Schwartz space $\mathcal{S}$. Let $p(x,\xi) \in S^{-\infty}$. For $u \in \mathcal{S}$,

$$\begin{aligned} P(x,D)u(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} u(y)\, dy \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} p(x,\xi)\, d\xi. \end{aligned}$$

Thus,

$$K(x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} p(x,\xi)\, d\xi$$

is the *Schwartz kernel* associated with the operator $p(x,D)$. If $p(x,D) \in S^m$, we extend the previous formula as follows:

$$K(x,y) = (2\pi)^{-d}(\mathcal{F}_{\xi\to x}p)(x, y-x),$$

where $\mathcal{F}_{\xi\to x}p$ is the Fourier transform of p with respect to the variable ξ in $\mathcal{S}'(\mathbb{R}^{2n})$, and, by the inversion formula, one yields

$$p(x,\xi) = \mathcal{F}_{y\to\xi}[K(x, x-y)],$$

meaning that there is a bijection in $\mathcal{S}'(\mathbb{R}^{2d})$ between symbols and operator kernels. This is useful for what follows.

Adjoint operator Let $p(x,D)$ be an operator acting on $\mathcal{S}$. By definition, the adjoint $p^*(x,D) : \mathcal{S}' \to \mathcal{S}'$ is such that, $\forall u \in \mathcal{S}', \forall v \in \mathcal{S}$,

$$(p(x,D)u, v) = (u, p^*(x,D)v).$$

Then, if $p^*(x,D)$ exists, it is unique, thanks to a density argument. Clearly, if $u \in \mathcal{S}'$ and $v \in \mathcal{S}$ as before, then

$$(u,v) = \langle u, \bar{v}\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket, and so

$$(p(x,D)u, v) = \langle p(x,D)u, \bar{v}\rangle = (u, p^*(x,D)v) = \langle u, \overline{p^*(x,D)v}\rangle.$$

If $p(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ is a differential operator with slowly increasing coefficients, it is easy to see that

$$(p(x,D)u, v) = (u, p^*(x,D)v),$$

where $p^*v = \sum_{|\alpha|\le m} D^\alpha(\bar{a}_\alpha v)$. More generally,

$$\begin{aligned}
\langle K(x,y)u(y), v(x)\rangle = \langle p(x,D)u, v\rangle &= \langle u, \overline{p^*(x,D)\bar{v}}\rangle \\
&= \overline{\langle \bar{u}, p^*(x,D)\bar{v}\rangle} \\
&= \overline{\langle K^*(x,y)\bar{v}(x), \bar{u}(y)\rangle},
\end{aligned}$$

namely

$$K^*(x,y) = \overline{K(x,y)} = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-i(x-y)\cdot\xi}\overline{p(x,\xi)}\, d\xi,$$

and

$$\begin{aligned}
p^*(x,\xi) &= \int_{\mathbb{R}^d} K^*(x, x-y) e^{-iy\cdot\xi}\, dy \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{iy\cdot(\eta-\xi)}\bar{p}(x-y,\eta)\, dy d\eta \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-iy\cdot\eta}\bar{p}(x-y,\xi-\eta)\, dy d\eta.
\end{aligned}$$

The previous considerations lead to the following theorem, see [1].

Theorem 2.2.1. *Let $p(x,\xi) \in S^m$, then $p^*(x,\xi) \in S^m$ and*

$$p^*(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \bar{p}(x,\xi).$$

Composition of operators Let $p_1(x,D)$ and $p_2(x,D)$ be two different pseudodifferential operators. For $u \in \mathcal{S}$,

$$p_1(x,D)p_2(x,D)u(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}p_1(x,\xi)\widehat{p_2(x,\xi)u}(\xi)\,d\xi,$$

where

$$\widehat{p_2(x,\xi)u}(\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-iy\cdot(\xi-\eta)}p_2(y,\eta)\hat{u}(\eta)\,d\xi d\eta dy.$$

This means that, at least formally,

$$q(x,\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-i(x-y)(\xi-\eta)}p_1(x,\eta)p_2(y,\xi)\,dyd\eta. \tag{2.2.11}$$

As in the adjoint before, here the composition q is a convolution in (y,η), for (x,ξ) fixed. Thus, we have the following composition theorem, see [1].

Theorem 2.2.2. *Let $p_1(x,\xi) \in S^{m_1}, p_2(x,\xi) \in S^{m_2}$. Then,*

$$p_1(x,D)p_2(x,D) = q(x,D),$$

with $q(x,\xi) = p_1(x,\xi)\circ p_2(x,\xi) \in S^{m_1+m_2}$ in (2.2.2), and

$$q \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha p_1 D_x^\alpha p_2.$$

The following corollary, proved in [1], will be extremely useful in Section 7.

Corollary 2.2.1. *Let $p_1(x,\xi) \in S^{m_1}, p_2(x,\xi) \in S^{m_2}$ be symbols of two different pseudodifferential operators. The commutator*

$$[p_1,p_2] = p_1p_2 - p_2p_1 \tag{2.2.12}$$

is an operator of order m_1+m_2-1.

Remark 2.2.1. *Let us point out that the statement of Corollary 2.2.1 follows directly from the asymptotic sums given by the composition theorem 2.2.2, and it holds for scalar-valued symbols. Actually, in more general cases, for instance in the case of matrix-valued symbols, i.e.*

$$p_1(x,\xi) \in (S^{m_1}(\mathbb{R}^d\times\mathbb{R}^d))^{n\times m},\ p_2(x,\xi) \in (S^{m_2}(\mathbb{R}^d\times\mathbb{R}^d))^{m\times k},$$

the commutator operator $[p_1,p_2]$ has order m_1+m_2-1 if the highest order terms of the asymptotic sums associated with the compositions p_1p_2 and p_2p_1, i.e. $p_1(x,\xi)p_2(x,\xi)$ and $p_2(x,\xi)p_1(x,\xi)$, commute with respect to the standard matrix product. Otherwise, the order of the commutator is m_1+m_2.

Action on Sobolev spaces From [1], we have the following theorems.

Theorem 2.2.3. *If $p(x,\xi) \in S^0$, then it defines an endomorphism of L^2.*

Theorem 2.2.4. *If $p(x,\xi) \in S^m$, then, for s real and $u \in H^s$,*

$$p : H^s \to H^{s-m}, \quad \|p(x,D)u\|_{s-m} \leq C\|u\|_s,$$

for a constant value C.

Homogeneous symbols A symbol $p(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d - \{0\})$, satisfying Definition 2.2.1 and homogeneous of degree m in ξ, is almost a symbol of a pseudifferential operator of degree m, in the sense that the singularity at the origin has to be removed. To this end, it is customary to introduce a cut-off function $\chi(\xi) \in C^\infty$, vanishing in a neighbourhood of the origin and such that $\chi(\xi) = 1$ for $|\xi| \geq 1$. This way, defining

$$\tilde{p}(x,\xi) = p(x,\xi)\chi(\xi),$$

we get a symbol associated with a pseudodifferential operator of order m. Any other regularized symbols differ from $\tilde{p}$ by an infinitely smooth symbol belonging to $S^{-\infty}$. The class of *homogeneous symbols* of degree m will be denoted by $\dot{S}^m$. From [8], we have the following inequality for positive symbols.

Theorem 2.2.5. *(Gårding inequality) If $p \in S^m$ ($p \in \dot{S}^m$) such that, for a positive constant C,*

$$p(x,\xi) + p(x,\xi)^* \geq C\Lambda(\xi)^m I_d,$$

for all $x \in \mathbb{R}^d$ and $|\xi|$ large, where $p(x,\xi)^*$ is the conjugate transpose of $p(x,\xi)$ in the sense of matrices, and I_d is the d-dimensional identity matrix, then

$$Re(p(x,D)u,u) \geq c_1\|u\|^2_{m/2} - c_2\|u\|^2_{m/2-1},$$

for some constants c_1, c_2 and $u \in H^{m/2}$.

Symbols satisfying spectral properties We just mention a special class of symbols satisfying the following definition.

Definition 2.2.3. *Consider $p(x,\xi) \in S^m$ such that*

$$\exists \delta > 0 : \mathcal{F}_{x\to\eta} p(\eta,\xi) = 0 \quad \textit{for} \quad |\eta| > \delta\Lambda(\xi),$$

i.e.

$$supp(\mathcal{F}_{x\to\eta} p(\eta,\cdot)) \subset B(0,\delta\Lambda(\xi)),$$

where $B(0,\delta\Lambda(\xi))$ is the ball of center 0 and radius $\delta\Lambda(\xi)$. This class of symbols will be denoted by Σ^m.

These symbols with spectral localization will be useful in the context of paradifferential calculus.
Notice that in Definition 2.2.1, we require the function $p(x,\xi)$ to be infinitely smooth in x, so the previous results are based on this assumption, which is clearly too much restrictive if one wants to apply the pseudodifferential theory to get some results on a given nonlinear problem. Thus, we need to relax this hypothesis, or, in other words, we need to apply a regularization procedure to the functions which are not smooth enough in x to be considered symbols. We aim to explain this in the following subsection.

2.2.3 Littlewood-Paley theory and dyadic decomposition

The main idea under the so-called *dyadic decomposition* is well explained in [25]. It is a regularization procedure which consists in localizing the frequencies (with respect to the spatial variable x of a function $p(x,\xi)$, after taking its Fourier transform) by considering a decomposition of the frequency space into annuli of size 2^q, where q is a natural number. This kind of decomposition regularizes $p(x,\cdot) \in \mathcal{S}'$ thanks to the behavior of the tempered distributions with respect to differentiation, when their Fourier transform in compactly supported (Paley-Wienier-Schwartz theorem). In particular, from [25] we have the following result.

Theorem 2.2.6. *Let $\lambda_1 < \lambda_2$ be two positive constants and take $u \in L^2$. There exists a positive constant C such that, for all integer k and, for a constant value $r > 0$,*

$$\text{if} \quad \text{supp}\,\hat{u} \in B(0, \lambda_1 r), \quad \text{then} \quad \sup_{|\alpha|=k} \|\partial^\alpha u\|_0 \leq C^k r^k \|u\|_0,$$
$$\text{if} \quad \text{supp}\,\hat{u} \in C(0, \lambda_1 r, \lambda_2 r), \quad \text{then} \quad C^{-k} r^k \|u\|_0 \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_0 \leq C^k r^k \|u\|_0,$$

where $C(0, \lambda_1 r, \lambda_2 r)$ is the ring of center 0, short radius λ_1 and long radius λ_2.

Thus, the idea here is to write a function $p(x,\xi)$ with limited smoothness in x as a sum of (smooth) symbols, obtained, thanks to the Paley-Wienier-Schwartz theorem, by localizing the frequencies as the spectral localization of Σ in Definition 2.2.3, and a remainder of lower order.
Now, take $\psi(\xi) \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \psi \leq 1$, so that

$$\psi(\xi) = \begin{cases} 1, & |\xi| \leq 1/2, \\ 0, & |\xi| \geq 1. \end{cases} \tag{2.2.13}$$

Introduce $\phi(\xi) = \psi(\xi/2) - \psi(\xi)$, which is supported in $2^{-1} \leq |\xi| \leq 2$, and, for all ξ, setting $\phi_k(\xi) := \phi(2^{-k}\xi)$,

$$\psi(\xi) + \sum_{k=0}^{\infty} \phi_k(\xi) = 1.$$

Now, let

$$S_0 u = \Delta_{-1} u = \psi(D) u, \qquad \Delta_k u = \phi_k(D) u.$$

Then

$$u = S_0 u + \sum_{k=0}^{\infty} \Delta_k u,$$

where $\textit{supp}\,\phi_k(\xi) \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$. Besides, we denote by

$$S_p u = \sum_{k=-1}^{p-1} \Delta_k u$$

the partial sums. Notice that, by convention, $\Delta_p = 0$ for $p \leq -2$ and $S_p = 0$ for $p \leq -1$. From [25], we have the following propositions.

Proposition 2.2.3. *For s real, let $u \in H^s(\mathbb{R}^d)$ $(\mathcal{S}'(\mathbb{R}^d))$. Then*

$$\lim_{p \to \infty} S_p u = u,$$

in the sense of convergence for tempered distributions.

Proposition 2.2.4. *There exists a constant C such that, for s real and $u \in H^s(\mathbb{R}^d)$,*

$$C^{-1}\|u\|_s^2 \leq \sum_{q\geq -1} 2^{2qs}\|\Delta_q u\|_0^2 \leq C\|u\|_s^2.$$

This last result allows us to define the norm

$$|u|_s := \left(\sum_{q\geq -1} 2^{2qs}\|\Delta_q u\|_0^2\right)^{1/2}, \tag{2.2.14}$$

which is equivalent to the usual H^s-norm. Moreover, the following is a useful theorem from [25].

Theorem 2.2.7. *Let $(u_q)_{q\geq -1}$ be a sequence in $\mathcal{S}'(\mathbb{R}^d)$ such that $\operatorname{supp} u_0 \in B(0,R)$ for a fixed $R > 0$, and $\operatorname{supp} u_q \in 2^q B(0,R)$. If the sequence $\delta_q = (2^{qs}\|u_q\|_0)$ is square integrable, then*

$$u = \sum_{q\geq -1} u_q \in H^s(\mathbb{R}^d), \qquad \text{and} \qquad \|u\|_s^2 \leq C\left(\sum_{q\geq -1}\delta_q^2\right)^{1/2},$$

for a positive constant C.

2.3 Paradifferential operators

Paradifferential calculus comes from the regularization procedure applied to symbols $p(x,\xi)$ with limited regularity in the spatial variable x. It is customary to say that, after smoothing the symbols, one obtains symbols associated with paradifferential operators. We begin with the simplest case of symbols independent of ξ.

Paraproduct Given two tempered distribution u, v, we write

$$u = \sum_{p\geq -1}\Delta_p u, \qquad v = \sum_{q\geq -1}\Delta_q v,$$

and, formally, this yields

$$uv = \sum_{p,q\geq -1}\Delta_p u\Delta_q v.$$

This product can be written as

$$uv = T_u v + T_v u + R(u,v),$$

where the first term concerns the high frequencies of u compared with low frequencies of v, the second addend represents the high frequencies of v against the low ones of u, while the last term is made by the frequencies of u and v of the same size. Here, the *paraproduct of v by u* is

$$T_u v = \sum_{p\geq 2}\Delta_p u\sum_{q=-1}^{p-3}\Delta_q v = \sum_{q\geq 2} S_{q-2} u\Delta_q v,$$

and the remainder is given by

$$R(u,v) = \sum_{|p-q|\le 2} \Delta_p u \Delta_q v.$$

When it is well-defined, the remainder term is the smoothest one. Precisely, the regularity of $R(u,v)$ is almost the one of u plus the one of v (see [8], Theorem C.9). Moreover, $T_u v$ is a typical example of paradifferential operator of order 0.

Proposition 2.3.1. *If $u \in L^\infty$, for all s real and $v \in H^s$, there exists $C > 0$ such that*

$$\|T_u v\|_s \le C\|u\|_\infty \|v\|_s.$$

For a proof see [8, 55], where they also prove the following error estimate.

Proposition 2.3.2. *For $s > 0$, there exists $C > 0$ such that, for $u, v \in L^\infty \cap H^s$,*

$$\|uv - T_u v\|_s \le C\|u\|_\infty \|v\|_s.$$

Besides, another useful result is proved in [55].

Proposition 2.3.3. *If $v \in L^\infty$ and $\nabla u \in H^{s-1}$, with $s > 0$, there exists a constant $C > 0$ such that*

$$\|uv - T_u v\|_s \le C\|\nabla u\|_{s-1} \|v\|_\infty.$$

Symbols with limited spatial smoothness Now, let us return to the more general case of a function $p(x,\xi)$, depending on both x, ξ and with limited regularity in x. We state the following definition, see [55] for a further discussion.

Definition 2.3.1. *Let $m \in \mathbb{R}$ and $\mathcal{B} \in \mathcal{S}'$ be a Banach space. We denote by $\Gamma^m_{\mathcal{B}}$ the space of distributions $p(x,\xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$ that are C^∞ in ξ, and such that, for all multi-index $\alpha \in \mathbb{N}^d$, there exists a constant C_α such that*

$$\|\partial^\alpha_\xi p(\cdot,\xi)\|_{\mathcal{B}} \le \Lambda(\xi)^{m-|\alpha|}.$$

Moreover, $\Sigma^m_{\mathcal{B}}$ is the subclass of functions $\sigma(x,\xi)$ such that there exists $0 < \delta < 1$:

$$supp\, \mathcal{F}_{x\to\eta}\sigma(\eta,\xi) \subset B(0, \delta\Lambda(\xi)).$$

Remark 2.3.1. *For our purpose, we refer to the space $\mathcal{B} = H^s, s > \frac{d}{2}+1$. As remarked in [55], if $\mathcal{B} \subset L^\infty$, then $\Gamma^m_{\mathcal{B}} \subset \Gamma^m_0$ and $\Sigma^m_{\mathcal{B}} \subset \Sigma^m_0$, where the subscript 0 refers to the space L^∞ equipped with the standard norm. Notice that the functions of $\Sigma^m_{\mathcal{B}}$ are such that their spatial Fourier transform is compactly supported, then these functions are C^∞ in x, and so they already are symbols. As a matter of facts, the smoothing procedure for non regular symbols $\Gamma^m_{\mathcal{B}}$ works by associating any function $p(x,\xi) \in \Gamma^m_{\mathcal{B}}$ with a symbol $\sigma(x,\xi) \in \Sigma^m_{\mathcal{B}}$.*

The smoothing procedure makes use of an admissible cut-off function to truncate the spatial Fourier transform of functions in $\Gamma^m_{\mathcal{B}}$.

Definition 2.3.2. *$\chi(\eta,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is an admissible cut-off function if there exist $\varepsilon_1, \varepsilon_2$, $0 < \varepsilon_1 < \varepsilon_2 < 1$ such that*

$$\chi(\eta,\xi) = \begin{cases} 1 & \text{if} \quad |\eta| \leq \varepsilon_1(1+|\xi|), \\ 0 & \text{if} \quad |\eta| \geq \varepsilon_2(1+|\xi|), \end{cases}$$

and, for all $(\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there exists $C_{\alpha,\beta}$ such that, for all (η,ξ)

$$|\partial_\eta^\alpha \partial_\xi^\beta \chi(\eta,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|-|\beta|}.$$

Example 2.3.1. *If ϕ, ψ are as in the Littlewood-Paley decomposition above, then an admissible cut-off function is*

$$\chi(\eta,\xi) = \sum_{p\geq 0} \psi(2^{2-p}\eta)\phi(2^{-p}\xi) = \sum_{p\geq 0} \psi_{p-2}(\eta)\phi_p(\xi).$$

The following proposition, see [8, 55], gives the smoothing procedure.

Proposition 2.3.4. *Let χ be an admissible cut-off function. For any $p(x,\xi) \in \Gamma^m_{\mathcal{B}}$, define*

$$\sigma_p(x,\xi) = \mathcal{F}^{-1}_{\eta\to x}(\chi(x,\xi)) \star_x p(x,\xi),$$

where $\star_x$ indicates the convolution operator with respect to the variable x. Then $\sigma_p(x,\xi) \in \Sigma^m_{\mathcal{B}} \subset \Sigma^m_0$. Moreover, if $p(x,\xi)$ is at least Lipschitz in x, i.e. $p(x,\xi) \in \Gamma^m_1$, where the subscript 1 refers to the space $W^{1,\infty}$ in x, then the remainder $p(x,\xi) - \sigma_p(x,\xi) \in \Gamma^{m-1}_0$.

It can be shown, see [8, 55], that the smoothing procedure does not depend on the particular choice of the admissible function. Now, we can define the paradifferential operator associated with $p(x,\xi) \in \Gamma^m_{\mathcal{B}}$, which is just the pseudodifferential one that comes from $\sigma_p(x,\xi)$.

Definition 2.3.3. *For $p(x,\xi) \in \Gamma^m_{\mathcal{B}} \subset \Gamma^m_0$, given an admissible cut-off function $\chi(\eta,\xi)$, the paradifferential operator T_p is defined by*

$$T_p u(x) = (2\pi)^{-d} \int_{\mathbb{R}}^d e^{i\xi\cdot x} \sigma_p(x,\xi)\hat{u}(\xi)\, d\xi.$$

Now, since $\sigma_p \in \Sigma^m_0 \subset S^m$, the previous results on pseudodifferential operators apply here and they also extend to the case of matrix-valued symbols.

2.4 Hyperbolic systems and symbolic calculus

Now, we are ready to go back to the notion of symmetrizability for hyperbolic systems. Consider again the quasilinear system in (2.1.2). From the previous section, we can now associate a symbol,

$$A(\mathbf{u},\xi) = \sum_{j=1}^d \xi_j A_j(\mathbf{u}), \tag{2.4.15}$$

to the first order operator in (2.1.2). Since the symbol also depends on the unknown $\mathbf{u}(x) \in H^s(\mathbb{R}^d), s > \frac{d}{2} + 1$, we need to apply the smoothing procedure in order to get

the paradifferential operator $T_{iA(\mathbf{u},\xi)}$ associated with (2.4.15) and do calculations. Nevertheless, as we have seen, operations with pseudo or paradifferential operators reduce to calculations with the related symbols, in the spirit of the so-called *symbolic calculus*.
First of all, from [55], we have the following useful error estimate.

Lemma 2.4.1. *Let $s > \frac{d}{2} + 1$ and $u \in C([0,T], H^1(\mathbb{R}^d))$, Then, for $j = 1, \cdots, d$, there exists a constant C such that*

$$\|[A_j(\boldsymbol{u}) - T_{A_j(\boldsymbol{u})}]\partial_{x_j}\boldsymbol{u}\|_s \leq C\|\nabla_x A_j(\boldsymbol{u})\|_{s-1}\|\boldsymbol{u}\|_s.$$

Here we extend the previous definition of symmetrizability.

Definition 2.4.1. *System (2.1.2) is said to be* Lax-*symmetrizable if there exists a matrix $S(\boldsymbol{u}, \xi)$, homogeneous of degree 0 in ξ, with entries C^∞ in $(\boldsymbol{u}, \xi)$ when $\xi \neq 0$ and such that:*

- *$S(\boldsymbol{u}, \xi)$ is self-adjoint and positive definite,*
- *$S(\boldsymbol{u}, \xi)A(\boldsymbol{u}, \xi)$ is self-adjoint.*

Notice that the symbolic symmetrizer $S(\mathbf{u}, \xi)$ can be singular in ξ. This requires to apply the regularization procedure discussed previously in paragraph *Homogeneous symbols*. We have the following well-posedness result (see [55]).

Theorem 2.4.1. *Let $s > \frac{d}{2} + 1$, and consider the Cauchy problem associated with a Lax-symmetrizable system (2.1.2), and initial data $\boldsymbol{u}_0 \in H^s(\mathbb{R}^d)$. Then, there exists a time $T > 0$ such that this problem has a unique solution $\boldsymbol{u} \in C([0,T], H^s(\mathbb{R}^d))$.*

Construction of symbolic symmetrizers However, it is not so simple to find a symmetrizer in practice. Here we describe a standard way to construct a Lax-symmetrizer for a system that satisfies the assumptions below. The procedure presented here comes from some ideas in [34] and [55]. In [34], the author shows an energy method which is similar to the one presented here, but there are some differences which require a different approach. In fact, [34] is the study of a particular form of singular perturbation approximations, a sort of paradifferential version of the singular approximation by *Kleinerman* and *Majda* in [47], and some points of that theory are really based on the particular structure of those systems. On the other hand, in [55] energy estimates are obtained by assuming the existence of a symmetrizer as in Definition 2.4.1. What it is worth pointing out is that the general ideas of the symmetrization technique presented here are well-known, but, to the author's knowledge, they are not collected and written in this form in the literature, though they are really useful for a lot of applications.
Here we consider the most general case of a hyperbolic quasilinear system of the first order

$$\partial_t \mathbf{u} + B(\mathbf{u}, D)\mathbf{u} = G(\mathbf{u}), \tag{2.4.16}$$

where $B(\mathbf{u}, D)$ is "almost" a paradifferential operator of the first order, with matrix-valued symbol of degree 1 in ξ, given by $B(\mathbf{u}, \xi) \in \mathcal{M}^{N\times N}$, and $\mathbf{u}(t,x)$ belonging to a subdomain of $\mathbb{R}^N$, with $(t,x) \in \mathbb{R} \times \mathbb{R}^d$. The word *almost* refers to the fact that, since $B(\mathbf{u}, \xi)$ depends on the spatial variable x through $\mathbf{u}(t,x)$, we need to process the regularization procedure described in section *Paradifferential operators* in order to apply the theory developed before. System (2.4.16) also covers the case of $B(\mathbf{u},\xi)=iA(\mathbf{u},\xi)$, where $A(\mathbf{u},\xi)$ is the symbol in (2.4.15), associated with a classical quasilinear hyperbolic system as in (2.1.2). However, we will discuss this and another important case at the end of this section. Let us list here the assumptions that we require on system (2.4.16).

- The matrix-valued symbol $B(\mathbf{u},\xi)\in\mathcal{M}^{N\times N}$ in (2.4.16) is diagonalizable with purely imaginary eigenvalues, namely

$$B(\mathbf{u},\xi)=V(\mathbf{u},\xi)iD(\mathbf{u},\xi)V^{-1}(\mathbf{u},\xi),$$

  where $V(\mathbf{u},\xi)$ is the matrix with the eigenvectors on the columns, while $iD(\mathbf{u},\xi)$ is the diagonal matrix of the eigenvalues;

- the smoothed version, through an admissible cut-off function as in Proposition 2.3.4, of the matrix-valued symbol $iD(\mathbf{u},\xi)+(iD(\mathbf{u},\xi))^*$, where $(iD(\mathbf{u},\xi))^*$ is the adjoint in the sense of matrices, is a bounded symbol belonging to $(S^0(\mathbb{R}^d))^{N\times N}$ for $\xi\in\mathbb{R}^d-\{0\}$;

- $V(\mathbf{u},\xi)$ and $V^{-1}(\mathbf{u},\xi)$, which is the inverse in the sense of matrices, are bounded matrix-valued symbols for $\xi\in\mathbb{R}^d-\{0\}$.

The standard symmetrization procedure takes inspiration from the following *formal* observation. Let $S(\mathbf{u},\xi):=(V^{-1}(\mathbf{u},\xi))^*V^{-1}(\mathbf{u},\xi)$, where the adjoint and the inversion are intended to be in the sense of matrices. *Symbolically* and *formally*, setting $(\cdot,\cdot)_0$ be the standard L^2 product, for $\mathbf{u}\in\mathbb{R}^N$ we write

$$\begin{aligned}(S(\mathbf{u},\xi)B(\mathbf{u},\xi)\mathbf{u},\mathbf{u})_0&=((V^{-1}(\mathbf{u},\xi))^*V^{-1}(\mathbf{u},\xi)B(\mathbf{u},\xi)\mathbf{u},\mathbf{u})_0\\&=(V^{-1}(\mathbf{u},\xi)B(\mathbf{u},\xi)\mathbf{u},V^{-1}(\mathbf{u},\xi)\mathbf{u})_0\\&=(V^{-1}(\mathbf{u},\xi)V(\mathbf{u},\xi)iD(\mathbf{u},\xi)V^{-1}(\mathbf{u},\xi)\mathbf{u},V^{-1}(\mathbf{u},\xi)\mathbf{u})_0\\&=(iD(\mathbf{u},\xi)V^{-1}(\mathbf{u},\xi)\mathbf{u},V^{-1}(\mathbf{u},\xi)\mathbf{u})_0\\&=(iD(\mathbf{u},\xi)\mathbf{w},\mathbf{w})_0,\end{aligned}$$

where $\mathbf{w}=V^{-1}(\mathbf{u},\xi)\mathbf{u}$. This formal calculation shows that $S(\mathbf{u},\xi)$ is “almost” a symbolic symmetrizer for the system, in the sense that it symmetrizes the matrix-valued symbol $A(\mathbf{u},\xi)$ in the sense of matrices, but we still have to deal with the following two problems:

1. the singularity of $S(\mathbf{u},\xi)$ in $\xi=0$;
2. the positivity of the paradifferential operator associated with the symbol $S(\mathbf{u},\xi)$ with respect to the scalar product of H^s. Indeed, according to the *Gårding inequality* in (2.2.5), this does not follow immediately from the positivity, in the sense of matrices, of the matrix-valued symbol $S(\mathbf{u},\xi)$, which is guaranteed by the definition of S itself.

Remark 2.4.1. *We point out that one could have to deal with the singularity in $\xi=0$ also in the case of a symbolic symmetrizer. In fact, $S(\mathbf{u},\xi)$ in Definition 2.4.1 is a matrix-valued symbol* homogeneous *of degree zero and, as we have seen, homogeneous* *symbols can have a singularity in $\xi = 0$. In paragraph* Homogeneous symbols*, we mentioned a standard regularization procedure near to $\xi = 0$ that allows to solve this problem. Moreover, also the Gårding inequality has to be handled here. In the case of matrix-valued symbols, these kinds of processes are not so standard. In the following, we will deal with this problem for systems satisfying the assumption above. For a general symmetrizable system in the sense of Definition 2.4.1, this part is presented in details in [55].*

Taking inspiration from [55], we define the following matrix-valued symbol:

$$W(\mathbf{u}, \xi) := (1 - \theta_\lambda(\xi))V^{-1}(\mathbf{u}, \xi), \tag{2.4.17}$$

where

$$\theta_\lambda(\xi)Id = \theta(\lambda^{-1}\xi)Id$$

for any fixed positive parameter λ and for any $\theta(\xi) \in C_c^\infty(\mathbb{R}^d)$, such that

$$\begin{cases} \theta = 1 \quad \text{for } |\xi| \leq 1, \\ 0 \leq \theta \leq 1 \quad \text{for } 1 < |\xi| < 2, \\ \theta = 0 \quad \text{for } |\xi| \geq 2. \end{cases}$$

We define the symbol:

$$S(\mathbf{u}, \xi) = W^*(\mathbf{u}, \xi)W(\mathbf{u}, \xi) + \theta_\lambda^2(\xi)Id, \tag{2.4.18}$$

where W^* is the adjoint in the sense of matrices. Since we are working in the Sobolev spaces and matrix (2.4.18) depends on x through u, we need to apply the regularization technique described in paragraph *Paradifferential operators* in order to use symbolic calculus tools. This leads to the following paradifferential symmetrizer:

$$\Sigma := (T_W)^*T_W + \theta_\lambda^2(D)Id, \tag{2.4.19}$$

where $(T_W)^*$ is the adjoint operator. By definition, Σ is self-adjoint, and

$$(\Sigma\mathbf{u}, \mathbf{u})_0 = \|T_W\mathbf{u}\|_0^2 + \|\theta_\lambda(D)\mathbf{u}\|_0^2,$$

for any $\mathbf{u} \in L^2(\mathbb{R}^d)$. This norm is equivalent to the L^2-norm, as it is proved in the following Lemma.

Lemma 2.4.2. *There exist constant values $\bar{c}, \underline{c}$ such that, for every $\boldsymbol{u} \in L^2(\mathbb{R}^2)$, we have*

$$\underline{c}\|\boldsymbol{u}\|_0^2 \leq (\Sigma\boldsymbol{u}, \boldsymbol{u})_0 \leq \bar{c}\|\boldsymbol{u}\|_0^2.$$

Proof. Since Σ in (2.4.19) is an operator of order 0, the right side inequality follows directly from paradifferential properties (see [55]). We focus on the left one.
Let $W_1 := (1 - \theta(\xi))V^{-1}(\mathbf{u}, \xi)$ and $W_2 := (1 - \theta(\xi))V(\mathbf{u}, \xi)$. By construction,

$$W_2W_1 = (1 - \theta(\xi))^2Id.$$

Notice that, for $\lambda \geq 2$,

$$(1 - \theta_\lambda(\xi))(1 - \theta(\xi)) = (1 - \theta_\lambda(\xi)).$$

Definition (2.4.17) yields

$$(1-\theta(\xi))(1-\theta_\lambda(\xi))V^{-1}(\mathbf{u},\xi)$$

$$= (1-\theta_\lambda(\xi))V^{-1}(\mathbf{u},\xi) =: W,$$

and

$$W_2 W = (1-\theta(\xi))(1-\theta_\lambda(\xi))Id = (1-\theta_\lambda(\xi))Id.$$

From the composition theorem in Section 2, we have

$$T_{W_2}T_W = (Id+R)(1-\theta_\lambda(D_x)),$$

where R is a remainder of order less than or equal to -1. In particular,

$$\|(1-\theta_\lambda(D_x))\mathbf{u}\|_0 \le c(W_2)\|T_W\mathbf{u}\|_0 + c(R)\|(1-\theta_\lambda(D_x))\mathbf{u}\|_{H^{-1}},$$

for every $\mathbf{u}\in L^2(\mathbb{R}^2)$. Now, recalling that $\Lambda(\xi) = (1-\Delta(\xi))^{\frac{1}{2}}$, where $\Delta(\xi)$ is the symbol of the Laplace operator,

$$\|(1-\theta_\lambda(D_x))\mathbf{u}\|_{H^{-1}} = \|(1-\theta_\lambda(\xi))\Lambda^{-1}(\xi)\hat{\mathbf{u}}\|_0. \tag{2.4.20}$$

From the definition of $\theta_\lambda(\xi)$, we also have $(1-\theta_\lambda(\xi))\Lambda(\xi)^{-1} = \dfrac{(1-\theta_\lambda(\xi))}{(1+|\xi|^2)^{1/2}} \le \dfrac{1}{\lambda}$, and, from (2.4.20),

$$\|(1-\theta_\lambda(D_x))\mathbf{u}\|_{H^{-1}} \le \frac{1}{\lambda}\|(1-\theta_\lambda(D_x))\mathbf{u}\|_0.$$

This gives

$$\|(1-\theta_\lambda(D_x))\mathbf{u}\|_0 \le c(W_2)\|T_W\mathbf{u}\|_0 + \frac{c(R)}{\lambda}\|(1-\theta_\lambda(D_x))\mathbf{u}\|_0,$$

then we can choose the parameter $\lambda \ge 2$ big enough such that $\frac{c(R)}{\lambda} < 1$. This way,

$$\|(1-\theta_\lambda(D_x))\mathbf{u}\|_0 \le c(W_2)\|T_W\mathbf{u}\|_0.$$

Squaring, we have

$$\|\mathbf{u}\|_0^2 \le c(W_2)\|T_W\mathbf{u}\|_0^2 + \|\theta_\lambda\mathbf{u}\|_0^2.$$

□

Now, we are ready to get energy estimates in the Sobolev spaces $H^s(\mathbb{R}^d)$, $s > \frac{d}{2}+1$. We apply the operator $\Lambda^s(D)$, whose symbol is $\Lambda^s(\xi) = (1+|\xi|^2)^{s/2}$ and we take the time derivative:

$$\frac{d}{dt}(\Sigma\Lambda^s\mathbf{u},\Lambda^s\mathbf{u})_0 = (\partial_t\Sigma\Lambda^s\mathbf{u},\Lambda^s\mathbf{u})_0 + 2Re(\Sigma\Lambda^s\partial_t\mathbf{u},\Lambda^s\mathbf{u})_0.$$

The first term of the right hand side,

$$\partial_t\Sigma = (T_{\partial_t W})^* T_W + (T_W)^* T_{\partial_t W},$$

is an operator of order 0, depending on $\partial_t \mathbf{u} = -B(\mathbf{u}, D)\mathbf{u} + G(\mathbf{u})$. Thus,

$$|(\partial_t \Sigma \Lambda^s \mathbf{u}, \Lambda^s \mathbf{u})_0| \leq c(|\partial_t \mathbf{u}|_\infty)\|\mathbf{u}\|_s^2 \leq c(|\mathbf{u}|_\infty, |\partial_{x_j} \mathbf{u}|_\infty)\|\mathbf{u}\|_s^2 \leq c(\|\mathbf{u}\|_s)\|\mathbf{u}\|_s^2,$$

where inequalities above follow from Lemma 2.4.2 and the Sobolev embedding theorem. Besides,

$$(\Sigma \Lambda^s \partial_t \mathbf{u}, \Lambda^s \mathbf{u})_0 = -(\Sigma \Lambda^s T_{B(\mathbf{u},\xi)} \mathbf{u}, \Lambda^s \mathbf{u})_0 + (\Sigma \Lambda^s T_{G(\mathbf{u})}, \Lambda^s \mathbf{u})_0 + Q,$$

where

$$Q = (\Lambda^s [T_{B(\mathbf{u},\xi)} - B(\mathbf{u}, D)]\mathbf{u}, \Lambda^s \mathbf{u})_0 - (\Lambda^s [T_{G(\mathbf{u})} - G(\mathbf{u})], \Lambda^s \mathbf{u})_0.$$

From Section 2,

$$|Q| \leq c(||\mathbf{u}||_s)||\mathbf{u}||_s$$

and, from the composition theorem in Section 2,

$$|(\Sigma \Lambda^s T_{G(\mathbf{u})}, \Lambda^s \mathbf{u})_0| \leq c(\|\mathbf{u}\|_s)\|\mathbf{u}\|_s.$$

It remains to deal with

$$\begin{aligned} Re(\Sigma \Lambda^s T_{B(\mathbf{u},\xi)} \mathbf{u}, \Lambda^s \mathbf{u})_0 = {} & Re(\Sigma T_{B(\mathbf{u},\xi)} \Lambda^s \mathbf{u}, \Lambda^s \mathbf{u})_0 \\ & + Re(\Sigma [\Lambda^s, T_{B(\mathbf{u},\xi)}]\mathbf{u}, \Lambda^s \mathbf{u})_0. \end{aligned}$$

The composition theorem in Section 2 states that the commutator symbol

$$\begin{aligned} [\Lambda^s, T_{B(\mathbf{u},\xi)}] &= \sum_{|\alpha| \geq 0} \partial_\xi^\alpha \Lambda^s(\xi) D_x^\alpha B(\mathbf{u}, \xi) - \partial_\xi^\alpha B(\mathbf{u}, \xi) D_x^\alpha \Lambda^s(\xi) \\ &= \sum_{|\alpha| \geq 1} \partial_\xi^\alpha \Lambda^s(\xi) D_x^\alpha B(\mathbf{u}, \xi) - \partial_\xi^\alpha B(\mathbf{u}, \xi) D_x^\alpha \Lambda^s(\xi), \end{aligned}$$

due to the fact that $\Lambda^s(\xi) = \Lambda^s(\xi) Id$ is diagonal, and so the term of degree 0

$$\Lambda^s(\xi) B(\mathbf{u}, \xi) - B(\mathbf{u}, \xi) \Lambda^s(\xi) = 0.$$

This means that operator $[\Lambda^s, T_{B(\mathbf{u},\xi)}]$ has order s, i.e., by the Sobolev embedding theorem,

$$|Re(\Sigma [\Lambda^s, T_{B(\mathbf{u},\xi)}]\mathbf{u}, \Lambda^s \mathbf{u})_0| \leq c(\|\mathbf{u}\|_s)\|\mathbf{u}\|_s^2.$$

Finally, we have to deal with

$$Re(\Sigma T_{B(\mathbf{u},\xi)} \Lambda^s \mathbf{u}, \Lambda^s \mathbf{u})_0.$$

From the adjoint and composition theorems in Section 2, and by definition (2.4.18), the matrix-valued symbol of degree 1 in the expansion of $\Sigma T_{iB(\mathbf{u})}$ is given by

$$(V^{-1}(\mathbf{u}, \xi))^* V^{-1}(\mathbf{u}, \xi)(1 - \theta_\lambda(\xi))^2 B(\mathbf{u}, \xi) + \theta_\lambda(\xi)^2 B(\mathbf{u}, \xi).$$

By construction, as mentioned before,

$$B(\mathbf{u}, \xi) = V(\mathbf{u}, \xi) i D(\mathbf{u}, \xi) V^{-1}(\mathbf{u}, \xi).$$

We define

$$N := (V^{-1}(\mathbf{u},\xi))^* iD(\mathbf{u},\xi)V^{-1}(\mathbf{u},\xi)(1-\theta_\lambda(\xi))^2.$$

This way,

$$Re((V^{-1}(\mathbf{u},\xi))^* V^{-1}(\mathbf{u},\xi)B(\mathbf{u},\xi)(1-\theta_\lambda(\xi))^2) = N + N^* = 0, \tag{2.4.21}$$

where, again, the adjoint is intended to be in the sense of matrices. From the adjoint and the composition theorems in Section 2, the symbol associated with the paradifferential operator $Re(\Sigma T_{iB(\mathbf{u},\xi)})$ has order less than or equal to 0 with respect to ξ, thanks to the vanishing term of degree 1 in ξ in (2.4.21) in the asymptotic expansion related to the composition. Thus,

$$|Re(\Sigma T_{B(\mathbf{u},\xi)}\Lambda^s\mathbf{u},\Lambda^s\mathbf{u})| \leq c(\|\mathbf{u}\|_s)\|\mathbf{u}\|_s^2.$$

Moreover,

$$\begin{aligned}|Re(i\theta_\lambda(D)B(\mathbf{u},\xi)\Lambda^s\mathbf{u},\Lambda^s\mathbf{u})| &\leq \|\theta_\lambda(D)B(\mathbf{u},\xi)\Lambda^s\mathbf{u}\|_0\|\Lambda^s\mathbf{u}\|_0\\ &\leq \sqrt{1+4\lambda^2}\|\theta_\lambda(D)B(\mathbf{u},\xi)\Lambda^{s-1}\mathbf{u}\|_0\|\Lambda^s\mathbf{u}\|_0\\ &\leq c(\|\mathbf{u}\|_s)\|\mathbf{u}\|_s^2.\end{aligned}$$

Putting them all together,

$$|Re(\Sigma T_{B(\mathbf{u},\xi)}\Lambda^s\mathbf{u},\Lambda^s\mathbf{u})_0| \leq c(\|\mathbf{u}\|_s)\|\mathbf{u}\|_s^2,$$

and so

$$\frac{d}{dt}(\Sigma\Lambda^s\mathbf{u},\Lambda^s\mathbf{u}) \leq c(\|\mathbf{u}\|_s)\|\mathbf{u}\|_s^2. \tag{2.4.22}$$

Energy estimate (2.4.22) implies, through a standard argument, for instance see [55], local existence and uniqueness of the smooth solution to the Cauchy problem related to system (2.4.16). Precisely, we state as follows.

Theorem 2.4.2. *Let $s > \frac{d}{2}+1$. There exists a unique solution $\boldsymbol{u} \in C([0,T], H^s(\mathbb{R}^d))$, $T > 0$, with initial data $\boldsymbol{u}_0 \in H^s(\mathbb{R}^d)$, to system*

$$\partial_t \boldsymbol{u} + B(\boldsymbol{u}, D)\boldsymbol{u} = G(\boldsymbol{u}),$$

satisfying the following assumptions:

- *the matrix-valued symbol of the first order $B(\boldsymbol{u},\xi) \in \mathcal{M}^{N\times N}$ is diagonalizable with purely imaginary eigenvalues, namely*

$$B(\boldsymbol{u},\xi) = V(\boldsymbol{u},\xi)iD(\boldsymbol{u},\xi)V^{-1}(\boldsymbol{u},\xi);$$

- *the smoothed version of $iD(\boldsymbol{u},\xi) + (iD(\boldsymbol{u},\xi))^*$ is a bounded symbol belonging to $(S^0(\mathbb{R}^d))^{N\times N}$ for $\xi \in \mathbb{R}^d - \{0\}$;*
- *$V(\boldsymbol{u},\xi)$ and $V^{-1}(\boldsymbol{u},\xi)$ are bounded matrix-valued symbols for $\xi \in \mathbb{R}^d - \{0\}$.*

Though the assumptions listed above are more restrictive than symmetrizability in Definition 2.4.1, they provide a standard procedure to construct an explicit symbolic symmetrizer, also in the case of a quasilinear first order system in (2.1.2). Thus, it is worth mentioning the following result.

Corollary 2.4.1. *Let $s > \frac{d}{2} + 1$. There exists a unique solution $\boldsymbol{u} \in C([0,T], H^s(\mathbb{R}^d))$, $T > 0$, to system*

$$\partial_t \boldsymbol{u} + \sum_{j=1}^{d} A_j(\boldsymbol{u})\partial_{x_j}\boldsymbol{u} = G(\boldsymbol{u}),$$

with initial data $\boldsymbol{u}_0 \in H^s(\mathbb{R}^d)$, if the matrix-valued symbol

$$A(\boldsymbol{u}, \xi) = \sum_{j=1}^{d} \xi_j A_j(\boldsymbol{u})$$

satisfies the assumptions of Theorem 2.4.2.

We end this section with an important observation. Theorem 2.4.2 applies also when $B(\mathbf{u}, \xi) = B_1(\mathbf{u}, \xi)B_2(\mathbf{u}, \xi)$, i.e. the first order operator is given by the composition of two or an arbitrary number of operators. In this case, in order to apply the construction before, the assumptions in Theorem 2.4.2 have to be satisfied by the matrix-valued product symbol $B_1(\mathbf{u}, \xi)B_2(\mathbf{u}, \xi)$, which is the highest degree term in ξ in the asymptotic sum related to the composition operator. For our purpose, it is interesting to consider the *projected* system:

$$\partial_t \mathbf{u} + \mathbf{P}\sum_{j=1}^{d} A_j(\mathbf{u})\partial_{x_j}\mathbf{u} = \mathbf{P}G(\mathbf{u}),$$

where $\mathbf{P}$ is any projector onto the space of $\mathbf{u}$ satisfying some properties. We ask for $\mathbf{P}$ to be an operator of degree less than or equal to 0 in ξ.
For instance, in the following we will deal with the *Leray projector*, namely the projector onto the space of the divergence free vector field. The Leray projector is homogeneous of degree 0 in ξ, then it also requires the regularization procedure for homogeneous symbols near $\xi = 0$, see Section 2. The following result will be useful in sections 6 and 7.

Corollary 2.4.2. *Let $s > \frac{d}{2} + 1$, and let $\mathbb{P}$ be the Leray projector, i.e. the projector onto the space of the divergence free vector field. There exists a unique solution $\boldsymbol{u} \in C([0,T], H^s(\mathbb{R}^d))$, $T > 0$, to system*

$$\partial_t \boldsymbol{u} + \mathbb{P}\sum_{j=1}^{d} A_j(\boldsymbol{u})\partial_{x_j}\boldsymbol{u} = \mathbb{P}G(\boldsymbol{u}),$$

with initial data $\boldsymbol{u}_0 \in H^s(\mathbb{R}^d)$, if the matrix-valued symbol

$$\mathbb{P}(\xi)A(\boldsymbol{u}, \xi) = \mathbb{P}(\xi)\sum_{j=1}^{d} \xi_j A_j(\boldsymbol{u})$$

satisfies the assumptions of Theorem 2.4.2.

Chapter 3

The role of the dissipative mechanism

According to the general theory on system of balance laws, see [28], system

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} A_j(\mathbf{u})\partial_{x_j}\mathbf{u} = G(\mathbf{u}), \tag{3.0.1}$$

with initial data $\mathbf{u}(0,x) = \mathbf{u}_0(x)$, where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $\mathbf{u} \in \mathbb{R}^N$, has a unique local smooth solution if the initial data are smooth enough. In the general case, also for smooth initial data, classical solutions may break down in finite time. Nevertheless, in some cases, dissipative properties of the source term $G(\mathbf{u})$ can prevent the formation of singularities, at least for initial data small enough, in the sense of the norm of the Sobolev spaces. A typical example is the Euler equations with damping, see [40, 61] for the one dimensional case, and [71] for the three dimensional one. It is worth recalling that a very useful ingredient in the framework of dissipative hyperbolic systems is represented by the mathematical *entropy* associated to the system, whose definition in given in [39], and the related *entropy dissipation condition*. We refer to [39, 15] for a detailed discussion. Here we do not explore this theory, since in the following we do not have an explicit entropy function for the systems that we are going to study. On the other hand, we will consider the more general class of *symmetrizable hyperbolic systems*, and the dissipative properties in the following will refer to these kinds of structures.

Assumption 3.0.1. *Let (3.0.1) be a symmetrizable system with symmetrizer $A_0(\boldsymbol{u})$, i.e., according to Chapter 2, $A_0(\boldsymbol{u})$ is symmetric and positive definite, and*

$$A_0(\boldsymbol{u})\partial_t \boldsymbol{u} + \sum_{j=1}^{d} \tilde{A}_j(\boldsymbol{u})\partial_{x_j}\boldsymbol{u} = \tilde{G}(\boldsymbol{u}) \tag{3.0.2}$$

with $\tilde{A}_j(\boldsymbol{u}) = (\tilde{A}_j(\boldsymbol{u}))^T$ for $j = 1, \cdots, d$, where $\tilde{A}_j(\boldsymbol{u}) = A_0(\boldsymbol{u})A_j(\boldsymbol{u})$, and $\tilde{G}(\boldsymbol{u}) = A_0(\boldsymbol{u})G(\boldsymbol{u})$.

The first very strong condition which prevents shock singularities is called *totally dissipative property*.

Definition 3.0.1. *System (3.0.1) is said to be* totally dissipative *if there exists an $N \times N$ matrix $D(\boldsymbol{u})$ such that:*

- $D(\boldsymbol{u})$ *is strictly negative definite;*
- $\tilde{G}(\boldsymbol{u}) = D(\boldsymbol{u})\boldsymbol{u}$.

With some modifications due to the particular context, this is essentially the dissipative property satisfied by the one-dimensional hyperbolic system in Chapter 5. However, this strong dissipative condition is verified just by very few physical systems. Another more reasonable property is the *partially dissipative condition.*

Definition 3.0.2. *System (3.0.1) is said to be* partially dissipative *if*

$$G(\boldsymbol{u}) = \begin{pmatrix} 0 \\ q(\boldsymbol{u}) \end{pmatrix},$$

where $0 \in \mathbb{R}^{N_1}, q(\boldsymbol{u}) \in \mathbb{R}^{N_2}$, *with* $N_1 + N_2 = N$, *and*

$$\tilde{G}(\boldsymbol{u}) = B(\boldsymbol{u})\boldsymbol{u} = \begin{pmatrix} 0 & 0 \\ D_1(\boldsymbol{u}) & D_2(\boldsymbol{u}) \end{pmatrix} \boldsymbol{u},$$

where $\frac{1}{2}(B(\boldsymbol{u}) + B(\boldsymbol{u})^T)$ *has* N_2 *strictly negative eigenvalues.*

However, partial dissipation is not enough to prevent the formation of singularities, see [39] for a discussion on *partially dissipative hyperbolic systems.* We need to impose another supplementary condition, which comes from the approach by *Shizuta* and *Kawashima*, see [46, 70]. Although there are many equivalent formulations, the so-called *Shizuta-Kawashima condition* for system (3.0.1) reads:

$$ker\, G'(\bar{\mathbf{u}}) \cap \left\{ eigenspaces\ of\ \sum_{j=1}^{d} A_j(\bar{\mathbf{u}})\xi_j \right\} = \{0\},$$

for every $\xi \in \mathbb{R} - \{0\}$, and every $\bar{\mathbf{u}}$ belonging to a subdomain of $\mathbb{R}^N$, with $G(\bar{\mathbf{u}}) = 0$. This condition, which is satisfied by many physical systems, allows to prove a global existence result for smooth solutions, for initial data that are small perturbations of the equilibrium point, i.e. $\bar{\mathbf{u}}$ such that $G(\bar{\mathbf{u}}) = 0$, see [39] for the one dimensional case, and [78] for the general multidimensional one.
Assuming that $\bar{\mathbf{u}} = 0$ is an equilibrium point with $G(0) = 0$, let us consider the linearized version of system (3.0.1) with source $G(\mathbf{u}) = (0 \quad q(\mathbf{u}))^T$, i.e.

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} A_j \partial_{x_j} \mathbf{u} = B\mathbf{u}, \tag{3.0.3}$$

where $A_j = A_j'(0)$, and $B = G'(0)$. In this context, we introduce the definition of *Conservative-Dissipative (C-D) form* for a hyperbolic system.

Definition 3.0.3. *Let us assume that system (3.0.3) is symmetric, i.e.* $A_j = A_j^T$ *for* $j = 1, \cdots, d$. *It is said to be in* conservative-dissipative (C-D) form *if there exists a strictly negative definite* $N_2 \times N_2$ *matrix* D *such that*

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}. \tag{3.0.4}$$

This definition implies that dissipation, represented by D, only acts on the last N_2 components of $\mathbf{u}$. We denote by $\mathbf{u}_1$, the so-called *conservative variable*, the first N_1 components of $\mathbf{u}$, and by $\mathbf{u}_2$, the *dissipative variable*, the last N_2 components. Thanks to the (C-D) structure, the dissipation acts on the dissipative variable $\mathbf{u}_2$, and then it propagates somehow to the conservative one. Indeed, very sharp energy estimates can be obtained by using the (C-D) form, see [15]. Thus, the (C-D) form results to be really important to highlight the dissipative nature of a hyperbolic system. In particular, the (C-D) form will be useful in Chapter 10.
The (C-D) form allows to establish an equivalent formulation of the (SK) condition, see [15].

Theorem 3.0.1. *Under condition (C-D), the Shizuta-Kawashima assumption (SK) is equivalent to the following:*

- *there exists a matrix* $K = K(\xi) \in \mathbb{R}^{N_1 \times N_2}$ *such that, for every* $\xi \in \mathbb{R}^d - \{0\}$, $K(\xi)A_0$ *is a skew-symmetric matrix and*

$$\frac{1}{2}(K(\xi)A(\xi)A_0 + A(\xi)A_0K^T(\xi)) - \frac{1}{2}(BA_0 + A_0B^T)$$

  is strictly positive definite;

- *if* $\lambda(z)$ *is an eigenvalue of* $E(z) = B - iA(z)$, *then* $Re(\lambda(i\xi)) < 0$ *for every* $\xi \in \mathbb{R}^d - \{0\}$*;*

- *there exists* $c > 0$ *such that*

$$Re(\lambda(i\xi)) \leq -c\frac{|\xi|^2}{1+|\xi|^2}$$

  for every $\xi \in \mathbb{R}^d - \{0\}$.

Consider a *constant right symmetrizer* for system (3.0.3).

Definition 3.0.4. *A* right symmetrizer *for system (3.0.3) is a positive definite symmetric matrix* A_0 *such that, if* $\boldsymbol{u} = A_0\boldsymbol{w}$ *in (3.0.3), it holds*

$$A_0\partial_t \boldsymbol{w} + \sum_{j=1}^{d} \tilde{A}_j \partial_{x_j} \boldsymbol{w} = \tilde{B}\boldsymbol{w},$$

where $\tilde{A}_j = A_jA_0$ *for* $j = 1, \cdots, d$ *are symmetric, and* $\tilde{B} = BA_0$.

It is possible to define a standard change of variables to put a *right symmetrizable system* as in (3.0.3) in (C-D) form. This method is presented in [15]. To this end, a preliminary step is to find a *right symmetrizer* A_0, which highlights the dissipative properties of a right symmetrizable system as

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} A_j \partial_{x_j} \mathbf{u} = \begin{pmatrix} 0 & 0 \\ D_1 & D_2 \end{pmatrix} \mathbf{u}.$$

Precisely, we look for a right symmetrizer A_0 such that, by changing variable $\mathbf{u} = A_0\mathbf{w}$, the previous system reads

$$A_0\partial_t\mathbf{w} + \sum_{j=1}^{d} \tilde{A}_j = \tilde{B}\mathbf{w} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \mathbf{w},$$

with $\tilde{A}_j = \tilde{A}_j^T$, where $\tilde{A}_j = A_j A_0$, $\tilde{B} = BA_0$, and D is strictly negative definite. This property will be essential in Chapter 10. Notice that, in Definition 3.0.3, A_0 is replaced by the identity matrix. This means that the symmetrized system before is not exactly in (C-D) form. However, there exists a change of variables, described in [15], starting from a system as the one before, to put it in the standard (C-D) form. This procedure will be discussed in Chapter 9, where the (C-D) form will wear a crucial role. Let us point out the role of *right symmetrizers* in the context of partially dissipative mechanisms of systems of high dimensions. This is clear once we applied the right product BA_0, where

$$B = \begin{pmatrix} 0 & 0 \\ D_1 & D_2 \end{pmatrix}, \quad \text{and} \quad A_0 = \begin{pmatrix} a_1 & a_2 \\ a_2^T & b_2 \end{pmatrix}$$

is a general symmetrizer, with $a_1 \in \mathcal{M}^{N_1 \times N_1}, a_2 \in \mathcal{M}^{N_1 \times N_2}, b_1 \in \mathcal{M}^{N_2 \times N_2}$. Thanks to the right product, the first N_1 lines of the resulting matrix are vanishing too, i.e.

$$BA_0 = \begin{pmatrix} 0 & 0 \\ \tilde{D}_1 & \tilde{D}_2 \end{pmatrix},$$

where $\tilde{D}_1, \tilde{D}_2$ depend on a_1, a_2, b_1, b_2, and so we can find a_1, a_2, b_1, b_2 such that

$$\begin{pmatrix} 0 & 0 \\ \tilde{D}_1 & \tilde{D}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix},$$

with D negative definite. Notice that the vanishing N_1 lines cannot be obtained by a left product, namely a classical left symmetrizer. At least for systems of a huge amount of equations, *right* rather than *left* symmetrizers provide a simple way to highlight the conservative-dissipative properties of systems.

Part II

Models in mixture theory

Chapter 4

A multiphase model for the growth of biofilms

Models arising from *mixture theory*, see [65], have been introduced in their general formulation in Section 1, and they are used in many fields, as tumor growth and vasculogenesis in [3], biological tissues and porous media in [32]. In particular, among several applications, we refer to the model proposed in [27], which describes biological structures called *biofilms*, namely complex gel-like aggregations of microorganisms like bacteria, algae, protozoa and fungi, embedded in a self-produced polymeric matrix called EPS. This model has been derived by applying the theory of mixture models in Section 1 to four different phases: bacteria, dead bacteria, extracellular polymeric matrix, and the liquid phase, with respective volume fractions B, D, E, L. Here, all the phases were assumed to have the same constant density $\rho_B = \rho_D = \rho_E = \rho_L = 1$. Recalling that, from mixture theory, there is no stress in the momentum equation for the liquid phase besides the hydrostatic pressure, an additional physical assumption has been performed on the excess stress tensor for the solid phases in (1.0.8), i.e.

$$\sum_{\phi \neq L} \phi T_\phi = \Sigma I,$$

where Σ is a monotone decreasing scalar function depending on the solid phases $B + D + E = 1 - L$, i.e.

$$\Sigma = -\gamma(1 - L), \tag{4.0.1}$$

where γ is an experimental constant value. This model satisfies the following equations:

$$\begin{cases} \partial_t B + \nabla \cdot (B v_S) = \Gamma_B, \\ \partial_t D + \nabla \cdot (D v_S) = \Gamma_D, \\ \partial_t E + \nabla \cdot (E v_S) = \Gamma_E, \\ \partial_t L + \nabla \cdot (L v_L) = \Gamma_L, \\ \partial_t ((1-L) v_S) + \nabla \cdot ((1-L) v_S \otimes v_S) + \gamma \nabla (1-L) + (1-L) \nabla P \\ = (M - \Gamma_L) v_L - M v_S, \\ \partial_t (L v_L) + \nabla \cdot (L v_L \otimes v_L) + L \nabla P = -(M - \Gamma_L) v_L + M v_S, \end{cases} \tag{4.0.2}$$

together with the saturation condition

$$B + D + E + L = 1, \tag{4.0.3}$$

and the conservation of the total mass

$$\Gamma_B + \Gamma_E + \Gamma_D + \Gamma_L = 0, \tag{4.0.4}$$

where v_ϕ, Γ_ϕ are respectively the velocities and the source terms for $\phi = B, D, E, L$, and ∇P is the incompressible pressure. The mass constraint in (4.0.4) states that the mixture is closed, namely there is no net production of mass for the mixture. According to [27], the reaction terms are given by:

$$\begin{aligned}
\Gamma_B &= k_B BL - k_D B, \\
\Gamma_D &= \alpha k_D B - k_N D, \\
\Gamma_E &= k_E BL - \epsilon E.
\end{aligned}$$

The birth of new cells depends on the quantity of liquid available in the neighborhood of the point, that is why the birth term in Γ_B is a product between the volume fraction B of active cells and the volume fraction L of liquid. This way, the mass production term Γ_B is the difference between a birth term and a death term, where the second is proportional to the fraction B of bacteria, with rate k_D. The death term in the expression of Γ_B gives rise to a creation term for the mass exchange rate of the dead cells Γ_D, with a proportional coefficient α, since a part of the active cells goes into liquid when the cell dies. In Γ_D, we also find a natural decay of dead cells with a constant decay rate k_N. The EPS is produced by active cells in presence of liquid, and then the production term will be proportional to BL, where k_E is the growth rate of EPS. There is also a natural decay of EPS with rate ϵ. Finally, we choose Γ_L in order to enforce condition (4.0.4). See again [27] for more details. Actually, system (4.0.2) is part of a general class of problems arising from mixture theory, see [65, 19, 20, 31], which have been introduced in Section 1 and present the coehexistence of the hydrostatic pressure and a compressible pressure term. For instance, consider a simplified version of the model in [27], composed of just two constituents, a solid phase B, which stands for "bacteria", but it can represents a general solid component, and a liquid phase L:

$$\begin{cases}
\partial_t B + \nabla \cdot (B v_S) = \Gamma_B, \\
\partial_t L + \nabla \cdot (L v_L) = \Gamma_L, \\
\partial_t (B v_S) + \nabla \cdot (B v_S \otimes v_S) + \gamma \nabla B + B \nabla P = (M - \Gamma_L) v_L - M v_S, \\
\partial_t (L v_L) + \nabla \cdot (L v_L \otimes v_L) + L \nabla P = -(M - \Gamma_L) v_L + M v_S, \\
B + L = 1, \\
\Gamma_B + \Gamma_L = 0,
\end{cases} \tag{4.0.5}$$

where v_S, v_L are the velocities of the solid and the liquid phase respectively, and γ, M are experimental constants. In (4.0.5), the momentum equations for the solid and the liquid phases are different: in the first one, $\gamma \nabla B$ is the excess stress tensor, while, as explained in Section 1 and in [32, 3], there is no excess stress tensor for the liquid part. Summing the first and the second equation, using the two last conditions in (4.0.5), and setting $L = 1 - B$, yields:

$$\begin{cases} \partial_t B + \nabla \cdot (B v_S) = \Gamma_B, \\ \partial_t v_S + v_S \cdot \nabla v_S + \dfrac{\gamma \nabla B}{B} + \nabla P = \dfrac{(M + \Gamma_B)(v_L - v_S)}{B}, \\ \partial_t v_L + v_L \cdot \nabla v_L + \nabla P = \dfrac{M(v_S - v_L)}{(1-B)}, \\ \nabla \cdot (B v_S + (1-B) v_L) = 0, \end{cases} \tag{4.0.6}$$

where the last equation,

$$\nabla \cdot (B v_S + (1-B) v_L) = 0, \tag{4.0.7}$$

namely the divergence free condition of the averaged velocity of the mixture, represents incompressibility of the mixture as a whole. Let us point out that the equation for the solid phase velocity v_S presents a pressure term composed of two parts, an incompressible pressure, ∇P, and the compressible one, $\frac{\gamma \nabla B}{B} = \gamma \nabla log(B)$. Actually, system (4.0.6) is just the two-phase case of the multiphase model in (4.0.2), where the three "solid" species are lumped together.

In the following, first we provide a complete analytical study of the model in (4.0.2) in one space dimension, with a proof of global existence and uniqueness of smooth solutions for initial data that are small perturbation of the unique non-trivial equilibrium point, and an investigation on the long time behavior of these solutions. As a matter of facts, the analytical study of model (4.0.2) in more than one space dimension is much more difficult, also at the level of existence and uniqueness of local smooth solutions. A detailed analysis on these difficulties will be presented in Section 6, where we also consider a simplified version of the two-phase model (4.0.5) in the general d-dimensional case in space. Indeed, in Section 6 we aim to understand the right method to study the original model (4.0.2) or, without loss of generality, the two-phase system (4.0.5), by studying the well-posedness of a density dependent fluid of the incompressible Euler equations type, which also presents an additional compressible pressure, in the spirit of $\frac{\gamma \nabla B}{B}$ in (4.0.6). Finally, in Section 7, we apply, with some technical modifications, the first method of Section 6, which makes use of paradifferential techniques, and so we are able to get the well-posedness of system (4.0.6) in two dimensions in space. In the end, the arguments will be briefly generalized to the case of the multiphase model in (4.0.2).

Chapter 5

A multiphase model in one space dimension

Here we provide a first analytical study in one space dimension of the multiphase model (4.0.2), originally introduced in the general d-dimensional case in Chapter 4. This chapter is based on [12]. The one dimensional system reads:

$$\begin{cases} \partial_t B + \partial_x(Bv_S) = \Gamma_B, \\ \partial_t E + \partial_x(Ev_S) = \Gamma_E, \\ \partial_t D + \partial_x(Dv_S) = \Gamma_D, \\ \partial_t L + \partial_x(Lv_L) = \Gamma_L, \\ \partial_t((1-L)v_S) + \partial_x((1-L)v_S^2) = -(1-L)\partial_x P - \gamma\partial_x(1-L) \\ \quad + (M-\Gamma_L)v_L - Mv_S; \\ \partial_t(Lv_L) + \partial_x(Lv_L^2) = -L\partial_x P - (M-\Gamma_L)v_L + Mv_S, \end{cases} \tag{5.0.1}$$

where the reaction terms are

$$\begin{array}{l} \Gamma_B = k_B BL - k_D B, \\ \Gamma_E = k_E BL - \varepsilon E, \\ \Gamma_D = \alpha k_D B - k_N D, \\ \Gamma_L = -(\Gamma_B + \Gamma_E + \Gamma_D), \end{array} \tag{5.0.2}$$

and Γ_L follows from the total mass constraint (4.0.4).
The equations of system (5.0.1) can be written in a more simplified form. First, thanks to the saturation condition (4.0.3),

$$L = 1 - (B + D + E),$$

and so the equation for the liquid volume fraction L is no more necessary. Furthermore, summing the equations for B, E, D, L in (5.0.1) and using again the volume constraint (4.0.3), we get an incompressibility condition on the averaged velocity, which is the following:

$$\partial_x((1-L)v_S + Lv_L) = 0. \tag{5.0.3}$$

Moreover, the one dimensional space setting, together with equation (5.0.3), allows us to solve for v_L, namely

$$v_L = \frac{L-1}{L} v_S. \tag{5.0.4}$$

Now, summing the fifth and the sixth equation of system (5.0.1) and using equality (5.0.4), we can finally solve for $\partial_x P$, i.e.

$$\partial_x P = -\gamma \partial_x (1-L) - \partial_x \left(\frac{1-L}{L} v_S^2 \right). \tag{5.0.5}$$

Setting $v := v_S$, further simplifications and the previous results lead to write system (5.0.1) in the following form:

$$\begin{cases} \partial_t B + \partial_x (Bv) = \Gamma_B, \\ \partial_t E + \partial_x (Ev) = \Gamma_E, \\ \partial_t D + \partial_x (Dv) = \Gamma_D, \\ \partial_t v + \partial_x \left[\dfrac{(3L-2)v^2}{2L} + \gamma(L + \log(1-L)) \right] = \dfrac{\Gamma_L - M}{L(1-L)} v = \Gamma_v, \end{cases} \tag{5.0.6}$$

while the velocity for the liquid phase v_L and the pressure term P are given by (5.0.4) and (5.0.5) respectively. As we will see in details in Section 5.1, system (5.0.6) is hyperbolic symmetrizable, and so, at least for results concerning the local existence and uniqueness of smooth solutions, the standard theory, see Section 2, applies. On the other hand, here we aim to provide a complete analytical study of this model in one space dimension, which also encloses an investigation on the long time behavior of the solution to the Cauchy problem. To this end, according to their physical meanings, in Section 5.2 we establish some reasonable conditions on the parameters of the model, which provide a dissipation property for the system. In Section 5.3, the dissipative property of the source allows us to prove that the solutions to the Cauchy problem for initial data that are small perturbation of the equilibrium point are actually global in time, and they decay exponentially to the equilibrium itself. The main tool here was the use of the *Nishida functional*, see [61]. Let us point out that the reformulation (5.0.6) of the original system in (5.0.1), where the incompressible pressure no more appears, is strictly related to the one dimensional nature of the problem. Actually, in more space dimensions that simplification no more occurs, and, as we will see in the next two chapters, the analytical study of the multiphase model results to be much more difficult, also at the level of the local in time existence of smooth solutions.

5.1 Hyperbolicity and symmetrizability

Let us set

$$\mathbf{u} = (B, E, D, v),$$

and let us rewrite system (5.0.6) in compact form

$$\partial_t \mathbf{u} + A(\mathbf{u}) \partial_x \mathbf{u} = G(\mathbf{u}), \tag{5.1.7}$$

with

$$A(\mathbf{u}) = \begin{pmatrix} v & 0 & 0 & B \\ 0 & v & 0 & E \\ 0 & 0 & v & D \\ \eta & \eta & \eta & \frac{(3L-2)}{L} v \end{pmatrix}, \tag{5.1.8}$$

where

$$\eta := \frac{\gamma L}{(1-L)} - \frac{v^2}{L^2}, \tag{5.1.9}$$

and

$$G(\mathbf{u}) = (\Gamma_B, \Gamma_E, \Gamma_D, \Gamma_v). \tag{5.1.10}$$

It easy to see that a Friedrich symmetrizer for system (5.1.7) is given by

$$A_0(\mathbf{u}) = \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \frac{B\eta}{E} & 0 & 0 \\ 0 & 0 & \frac{B\eta}{D} & 0 \\ 0 & 0 & 0 & B \end{pmatrix}, \tag{5.1.11}$$

namely system (5.1.7) is symmetrizable hyperbolic in the following domain

$$\begin{aligned} W &= \left\{ \mathbf{u} = (B,E,D,v) \in [0,1]^3 \times \mathbb{R} : \eta > 0 \right\} \\ &= \left\{ \mathbf{u} = (B,E,D,v) \in [0,1]^3 \times \mathbb{R} : -\frac{\gamma^{1/2} L^{3/2}}{(1-L)^{1/2}} < v < \frac{\gamma^{1/2} L^{3/2}}{(1-L)^{1/2}} \right\}, \end{aligned} \tag{5.1.12}$$

where $L = 1 - (B + E + D)$. This yields the following symmetrized compact form of system (5.1.7):

$$A_0(\mathbf{u})\partial_t \mathbf{u} + A_1(\mathbf{u})\partial_x \mathbf{u} = A_0 G(\mathbf{u}), \tag{5.1.13}$$

where $A_1(\mathbf{u}) = A_0 A(\mathbf{u})$, and $A_0(\mathbf{u}), G(\mathbf{u})$ are given by (5.1.11) and (5.1.10) respectively. Now, the standard theory on symmetrizable hyperbolic systems, see Section 2, guarantees the existence and uniqueness of local smooth solutions for the Cauchy problem associated to problem (5.1.7)-(5.1.8)-(5.1.10).

5.2 Dissipation property

Here we want to show that, under some reasonable assumptions on the physical parameters, system (5.0.6) is *totally dissipative*, according to Definition 3.0.1. In the following, we adapt the definition of *totally dissipative hyperbolic system* to the case of symmetrizable systems.

Definition 5.2.1 ((D)-Condition). *Consider a general one-dimensional $n \times n$ hyperbolic symmetrizable system in the compact formulation (5.1.7), where $\boldsymbol{u} \in \Omega \subseteq \mathbb{R}^n$, Ω is a convex open subset of the domain of symmetrizability W, and $A(\boldsymbol{u}), G(\boldsymbol{u})$ are smooth enough. Assume that Ω contains a unique equilibrium point $\bar{\boldsymbol{u}}$ for (5.1.7), such that $G(\bar{\boldsymbol{u}}) = 0$. Then, system (5.1.7) is totally dissipative in Ω if there exists a matrix $D = D(\boldsymbol{u}, \bar{\boldsymbol{u}}) \in M^{n\times n}$ such that, for every $\boldsymbol{u} \in \Omega$:*

- *$G(\boldsymbol{u}) = D(\boldsymbol{u}, \bar{\boldsymbol{u}})(\boldsymbol{u} - \bar{\boldsymbol{u}})$;*
- *$A_0 D(\boldsymbol{u}, \bar{\boldsymbol{u}})$ is strictly negative definite.*

Let us check the dissipation property of system (5.0.6). First of all, we determine the expression of the point where the source term vanishes, i.e. the equilibrium point $\bar{\mathbf{u}}$. Setting

$$\bar{B} = \left(1 - \frac{k_D}{k_B}\right) \Big/ \left(1 + \frac{\alpha k_D}{k_N} + \frac{k_D k_E}{\varepsilon k_B}\right),$$

then

$$\bar{\mathbf{u}} = \begin{pmatrix} \bar{B} \\ \bar{E} \\ \bar{D} \\ \bar{v} \end{pmatrix} = \bar{B} \begin{pmatrix} 1 \\ \dfrac{k_E k_D}{\varepsilon k_B} \\ \dfrac{\alpha k_D}{k_N} \\ 0 \end{pmatrix} = \frac{k_B - k_D}{k_B(1 + \frac{\alpha k_D}{k_N} + \frac{k_D k_E}{\varepsilon k_B})} \begin{pmatrix} 1 \\ \dfrac{k_E k_D}{\varepsilon k_B} \\ \dfrac{\alpha k_D}{k_N} \\ 0 \end{pmatrix}. \tag{5.2.14}$$

Since the volume fractions $\bar{B}, \bar{E}, \bar{D}, \bar{L}$ take positive values, from (5.2.14) we have to assume

$$k_B > k_D. \tag{5.2.15}$$

Let us point out that inequality (5.1.12), which describes the region of hyperbolic symmetrizability of system (5.0.6), is satisfied if $\mathbf{u}$ is close enough to the equilibrium point $\bar{\mathbf{u}}$. For this reason, in the following we will take the initial datum $\mathbf{u}_0$ in a convex and compact neighborhood of the equilibrium point $\bar{\mathbf{u}}$. To this end, we set

$$\Omega = \mathrm{B}_r(\bar{\mathbf{u}}),$$

for a fixed $r > 0$. According to **(D)-Condition**, we consider $G(\mathbf{u})$ in (5.1.10) and we write

$$G(\mathbf{u}) = D(\mathbf{u}, \bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}),$$

with

$$D(\mathbf{u}, \bar{\mathbf{u}}) = \begin{pmatrix} -B k_B & -B k_B & -B k_B & 0 \\ k_E(L - \bar{B}) & \frac{-\varepsilon(\bar{E}+\bar{L})}{L} & -\bar{B} k_E & 0 \\ \alpha k_D & 0 & -k_N & 0 \\ 0 & 0 & 0 & \frac{\Gamma_L - M}{L(1-L)} \end{pmatrix}.$$

In order to prove that system in (5.1.13)- (5.1.11)-(5.1.8)- (5.1.10) satisfies **(D)-Condition** in $\Omega = \mathrm{B}_r(\bar{\mathbf{u}})$, we show the strict negativity of matrix $A_0 D(\bar{\mathbf{u}}, \bar{\mathbf{u}})$ by using the *Routh-Hurwitz conditions* on $\frac{(A_0 D)+(A_0 D)^T}{2}$, see[58]. From (5.2.14),

$$A_0(\bar{\mathbf{u}}) = \begin{pmatrix} \frac{k_D \gamma}{(k_B - k_D)} & 0 & 0 & 0 \\ 0 & \frac{\varepsilon k_B \gamma}{k_E(k_B - k_D)} & 0 & 0 \\ 0 & 0 & \frac{k_N \gamma}{\alpha(k_B - k_D)} & 0 \\ 0 & 0 & 0 & \bar{B} \end{pmatrix},$$

and

$$D(\bar{\mathbf{u}}, \bar{\mathbf{u}}) = \begin{pmatrix} -\bar{B} k_B & -\bar{B} k_B & -\bar{B} k_B & 0 \\ k_E(\bar{L} - \bar{B}) & -\varepsilon - \bar{B} k_E & -\bar{B} k_E & 0 \\ \alpha k_D & 0 & -k_N & 0 \\ 0 & 0 & 0 & -\frac{k_B^2 M}{k_D(k_B - k_D)} \end{pmatrix}.$$

Thus,

$$\left(\frac{(A_0D)+(A_0D)^T}{2}\right)(\bar{\mathbf{u}},\bar{\mathbf{u}}) = \frac{\gamma}{k_B-k_D}$$

$$\cdot\begin{pmatrix} -\bar{B}k_Bk_D & \frac{k_B}{2}[\varepsilon(\bar{L}-\bar{B})-k_D\bar{B}] & \frac{k_D}{2}[k_N-\bar{B}k_B] & 0 \\ \frac{k_B}{2}[\varepsilon(\bar{L}-\bar{B})-k_D\bar{B}] & -\frac{\varepsilon k_B(\varepsilon+k_E\bar{B})}{k_E} & -\frac{\varepsilon k_B\bar{B}}{2} & 0 \\ \frac{k_D}{2}[k_N-\bar{B}k_B] & -\frac{\varepsilon k_B\bar{B}}{2} & -\frac{k_N^2}{\alpha} & 0 \\ 0 & 0 & 0 & -\frac{k_B^2M\bar{B}}{\gamma k_D(k_B-k_D)} \end{pmatrix}.$$

Consider the 3×3 matrix which contains the main diagonal of the original 4×4 matrix. Its characteristic polynomial is

$$P(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_3, \tag{5.2.16}$$

where the coefficients a_i $(i=1,2,3)$ are real. We establish the conditions on the a_i, such that the zeros of $P(\lambda)$ have $Re\lambda<0$. The necessary and sufficient conditions for this are the *Routh-Hurwitz conditions* [58]. For the cubic equation in (5.2.16), they are given by the following inequalities:

$$[RH]\begin{cases} a_1>0; \\ a_3>0; \\ a_1a_2-a_3>0. \end{cases} \tag{5.2.17}$$

In our case, the coefficients of (5.2.16) are the following:

$$\begin{cases} a_1=\bar{B}k_Bk_D+\dfrac{\varepsilon^2k_B}{k_E}+\varepsilon k_B\bar{B}+\dfrac{k_N^2}{\alpha}; \\[2ex] a_2=\dfrac{\varepsilon^2\bar{B}k_B^2k_D}{k_E}+\dfrac{\varepsilon\bar{B}k_Bk_N^2}{\alpha}+\dfrac{\bar{B}k_Bk_Dk_N^2}{\alpha}+\dfrac{\bar{B}k_Bk_D(\varepsilon\bar{B}k_B+\varepsilon k_D+k_Dk_N+\varepsilon^2)}{2} \\ \quad+\dfrac{\varepsilon^2k_Bk_N^2}{\alpha k_E}-\dfrac{\bar{B}^2k_B^2(k_D^2+\varepsilon^2)}{2}-\dfrac{k_D^2(k_N^2+\varepsilon^2)}{4}; \\[2ex] a_3=\dfrac{\varepsilon^2\bar{B}k_B^2k_D^2k_N}{2k_E}+\dfrac{\varepsilon^2\bar{B}k_B^2k_Dk_N^2}{\alpha k_E}+\dfrac{\varepsilon\bar{B}k_Bk_Dk_N^2(\bar{B}k_B+\varepsilon+k_D)}{2\alpha} \\ \quad+\dfrac{\varepsilon\bar{B}k_Bk_D^2k_N(\varepsilon+\bar{B}k_B)}{4}-\dfrac{\varepsilon\bar{B}k_Bk_D(\varepsilon\bar{B}k_Bk_N+\varepsilon\bar{B}k_Bk_D+k_Dk_N^2)}{4} \\ \quad-\dfrac{\varepsilon^2k_Bk_D^2(k_N^2+\bar{B}^2k_B^2)}{4k_E}-\dfrac{k_N^2(\varepsilon^2k_D^2+\varepsilon^2\bar{B}^2k_B^2+\bar{B}^2k_B^2k_D^2)}{4\alpha}. \end{cases} \tag{5.2.18}$$

Now, we can state our result.

Proposition 5.2.1. *Assume that (5.2.15) holds. Then, if $[RH]$ condition is verified for a_1,a_2,a_3 in (5.2.18), system (5.0.6) satisfies **(D)-Condition**, and so it is totally dissipative in a neighborhood of its equilibrium point.*

Now, the first condition of (5.2.17) is always satisfied, since a_1 in (5.2.17) is positive. In particular, the last two conditions of (5.2.17) hold for $\varepsilon,\alpha,k_B,k_D$, k_E,k_N in the following table:

Table 5.1: A list of (dimensional) parameters.

Param.	Value	Unit of meas.	Indications
k_B	$8 \cdot 10^{-6}$	$1/\text{sec}$	Bact. growth rate
k_E	$3 \cdot 10^{-6}$	$1/\text{sec}$	EPS growth rate
k_D	$2 \cdot 10^{-7}$	$1/\text{sec}$	Bact. death rate
k_N	$1 \cdot 10^{-6}$	$1/\text{sec}$	Dead cells consumption
ε	$1.25 \cdot 10^{-7}$	$1/\text{sec}$	EPS death rate
α	0.25	dimensionless	coeff. liquid dead-cells

This is a list of parameters in [27], with $k_E = 3 \cdot 10^{-6}$.

More generally, if we take a real parameter a, we can restrict our attention to the class of coefficients such that

$$\varepsilon = 1.25 \cdot 10^{-7} \simeq 10^{-7}, \quad k_N = 10a\varepsilon, \quad , k_E = 100a\varepsilon, \quad k_D = 2a\varepsilon, \quad k_B = 80a\varepsilon.$$

By this reduction, the third condition of (5.2.17) gives a second degree polynomial inequality that can be easily solved, and it holds true for a in a precise interval of the real line. Moreover, it can be seen that the second condition in (5.2.17) is also verified in the same interval.

5.3 Global existence of smooth solutions for small initial data and asymptotic behavior

Now, we prove that the solution to the Cauchy problem associated to (5.0.6) with initial datum in a small neighborhood of the equilibrium point is actually global in time. To this end, we take inspiration from the proof of global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy in [39]. However, we point out that in [39] the authors make use of a convex dissipative mathematical entropy and the so called Shizuta-Kawashima condition, see [46]. On the contrary, in our proof we do not use any of these last two properties, since the totally dissipative structure of system (5.0.1), **(D)-Condition**, allows us to get the energy estimates which provide the global existence result by means of the Nishida functional, see [61]. Besides, we are able to prove that the global solution decays exponentially in time to the unique equilibrium point of system (5.0.6).
Let us state now our main result.

Theorem 5.3.1. *Consider system (5.0.6) and its unique equilibrium point $\bar{\boldsymbol{u}}$ in (5.2.14), and assume that (5.2.15) holds. If this system satisfies **(D)-Condition**, there exists a positive constant $\delta < r$ such that, if $||\boldsymbol{u}_0 - \bar{\boldsymbol{u}}||_2 \leq \delta$, then there is a unique global solution $\boldsymbol{u}$ with initial datum $\boldsymbol{u}_0$, which verifies*

$$\boldsymbol{u} - \bar{\boldsymbol{u}} \in C([0, +\infty), H^2(\mathbb{R})) \cap C^1([0, +\infty), H^1(\mathbb{R})),$$

and

$$sup_{\ 0 \leq t < +\infty} ||\boldsymbol{u}(t) - \bar{\boldsymbol{u}}||_2^2 + \int_0^{+\infty} ||\boldsymbol{u}(\tau) - \bar{\boldsymbol{u}}||_2^2 \, d\tau \leq C(\delta) ||\boldsymbol{u}_0 - \bar{\boldsymbol{u}}||_2,$$

where $C(\delta)$ is a positive constant. Moreover, the global solution $\boldsymbol{u}$ decays exponentially in time to the equilibrium point $\bar{\mathbf{u}}$, i.e.

$$||\boldsymbol{u}(t)-\bar{\boldsymbol{u}}||_{H^2(\mathbb{R})} \leq C_1 e^{-\beta t}||\boldsymbol{u}_0-\bar{\boldsymbol{u}}||_{H^2(\mathbb{R})}, \quad t>0, \tag{5.3.19}$$

where C_1, β are positive constants.

According to **(D)-Condition**, we consider a neighborhood $\Omega = B_r(\bar{\mathbf{u}})$ of the equilibrium point $\bar{\mathbf{u}}$. Let us introduce the following translation:

$$\mathbf{w} := \mathbf{u} - \bar{\mathbf{u}}.$$

Then, system (5.1.13) reads

$$A_0(\mathbf{w}+\bar{\mathbf{u}})\partial_t \mathbf{w} + A_1(\mathbf{w}+\bar{\mathbf{u}})\partial_x \mathbf{w} = A_0 G(\mathbf{w}+\bar{\mathbf{u}}). \tag{5.3.20}$$

In order to prove that the solution to the Cauchy problem associated to (5.0.6) is global in time, we follow the approach proposed in [61], see also [46] and [39], and we introduce the functional

$$N_l^2(t) := sup_{\ 0\leq\tau\leq t}\ ||\mathbf{w}(\tau)||_l^2 + \int_0^t ||\mathbf{w}(\tau)||_l^2\ d\tau, \tag{5.3.21}$$

for $l = 0, 1, 2$.

Proposition 5.3.1. *Let $T > 0$, and assume that there exists a local smooth solution $\boldsymbol{w}$ to the Cauchy problem associated to system (5.0.6) in $[0, T]$. Then, there exists $\varepsilon > 0$ and $C > 0$ such that, if $N_2(T) \leq \varepsilon$,*

$$N_2^2(T) \leq C(N_2^2(0) + N_2^3(T)). \tag{5.3.22}$$

The existence and uniqueness of a local smooth solution to system (5.0.6) with initial datum $\mathbf{u}_0 \in H^2(\mathbb{R})$ is guaranteed by the theory on quasilinear symmetrizable hyperbolic systems, see Section 2. Besides, the first part of Theorem 5.3.1 follows directly from Proposition 5.3.1, see [39], [61], so providing, for a constant value c, the uniform estimate

$$N_2(T) \leq cN_2(0). \tag{5.3.23}$$

In order to prove Proposition 5.3.1 above, we need the following two lemmas.

Lemma 5.3.1. *If $N_2(T) \leq \varepsilon \leq \frac{\delta}{\alpha}$, where α is the Sobolev embedding constant, then*

$$N_0^2(T) \leq C_1(N_2^2(0) + N_2^3(T)). \tag{5.3.24}$$

Usually, to state an estimate as (5.3.24), a function of convex entropy is used, but here, in our proof of Lemma 5.3.1, we do not use anything but the dissipative property of system (5.0.6).

Proof. Using **(D)-Condition** and (5.3.20), let us consider system (5.0.6) in the following symmetric form:

$$A_0(\mathbf{w}+\bar{\mathbf{u}})\partial_t \mathbf{w} + A_1(\mathbf{w}+\bar{\mathbf{u}})\partial_x \mathbf{w} = A_0(\mathbf{w}+\bar{\mathbf{u}})D(\mathbf{w}+\bar{\mathbf{u}},\bar{\mathbf{u}})\mathbf{w}.$$

In the previous equation, the new reaction term is

$$D_1(\mathbf{w}, \bar{\mathbf{u}}) := A_0(\mathbf{w} + \bar{\mathbf{u}})D(\mathbf{w} + \bar{\mathbf{u}}, \bar{\mathbf{u}}).$$

Therefore, we have

$$A_0(\mathbf{w} + \bar{\mathbf{u}})\partial_t\mathbf{w} + A_1(\mathbf{w} + \bar{\mathbf{u}})\partial_x\mathbf{w} = D_1(\mathbf{w}, \bar{\mathbf{u}})\mathbf{w}. \tag{5.3.25}$$

We have the following identities:

$$(A_0(\mathbf{w} + \bar{\mathbf{u}})\partial_t\mathbf{w}, \mathbf{w}) = \frac{1}{2}\partial_t(A_0(\mathbf{w} + \bar{\mathbf{u}})\mathbf{w}, \mathbf{w}) - \frac{1}{2}(\partial_t A_0(\mathbf{w} + \bar{\mathbf{u}})\mathbf{w}, \mathbf{w}); \tag{5.3.26}$$

$$(A_1(\mathbf{w} + \bar{\mathbf{u}})\partial_x\mathbf{w}, \mathbf{w}) = \frac{1}{2}\partial_x(A_1(\mathbf{w} + \bar{\mathbf{u}})\mathbf{w}, \mathbf{w}) - \frac{1}{2}(\partial_x A_1(\mathbf{w} + \bar{\mathbf{u}})\bar{\mathbf{w}}, \mathbf{w}). \tag{5.3.27}$$

We consider (5.3.25) and take the inner product with $\mathbf{w}$, which yields

$$(A_0(\mathbf{w} + \bar{\mathbf{u}})\partial_t\mathbf{w}, \mathbf{w}) + (A_1(\mathbf{w} + \bar{\mathbf{u}})\partial_x\mathbf{w}, \mathbf{w}) = (D_1(\mathbf{w}, \bar{\mathbf{u}})\mathbf{w}, \mathbf{w}). \tag{5.3.28}$$

Using the identities (5.3.26) and (5.3.27) in (5.3.28), we obtain

$$\frac{1}{2}\partial_t(A_0(\mathbf{w} + \bar{\mathbf{u}})\mathbf{w}, \mathbf{w}) + \frac{1}{2}\partial_x(A_1(\mathbf{w} + \bar{\mathbf{u}}))\mathbf{w}, \mathbf{w})$$

$$= \frac{1}{2}(\partial_t A_0(\mathbf{w} + \bar{\mathbf{u}})\mathbf{w}, \mathbf{w}) + \frac{1}{2}(\partial_x A_1(\mathbf{w} + \bar{\mathbf{u}})\mathbf{w}, \mathbf{w}) + (D_1(\mathbf{w}, \bar{\mathbf{u}})\mathbf{w}, \mathbf{w}).$$

Therefore, if we integrate equality (5.3.28) over $\mathbb{R} \times [0, T]$, we have

$$\frac{1}{2}\int_{\mathbb{R}}(A_0(\mathbf{w}(T) + \bar{\mathbf{u}})\mathbf{w}(T), \mathbf{w}(T))\ dx\ - \frac{1}{2}\int_{\mathbb{R}}(A_0(\mathbf{w}(0) + \bar{\mathbf{u}})\mathbf{w}(0), \mathbf{w}(0))\ dx$$

$$= \int_0^T dt\ \int_{\mathbb{R}}\left(\left[\frac{1}{2}\partial_t A_0(\mathbf{w} + \bar{\mathbf{u}}) + \frac{1}{2}\partial_x A_1(\mathbf{w} + \bar{\mathbf{u}}) + D_1(\mathbf{w}, \bar{\mathbf{u}})\right]\mathbf{w}, \mathbf{w}\right)\ dx =: \mathrm{I}.$$

To estimate I, we use (5.3.25) in the following form:

$$\partial_t\mathbf{w} = -A(\mathbf{w} + \bar{\mathbf{u}})\partial_x\mathbf{w} + D(\mathbf{w}, \bar{\mathbf{u}})\mathbf{w}.$$

Then

$$\partial_t A_0(\mathbf{w} + \bar{\mathbf{u}}) = A_0'(\mathbf{w} + \bar{\mathbf{u}})\partial_t\mathbf{w},$$

and

$$\partial_t A_0(\mathbf{w} + \bar{\mathbf{u}}) = A_0'(\mathbf{w} + \bar{\mathbf{u}})(-A(\mathbf{w} + \bar{\mathbf{u}})\partial_x\mathbf{w} + D(\mathbf{w}, \bar{\mathbf{u}})\mathbf{w}).$$

Then, we have

$$\mathrm{I} = -\int_0^T dt\ \int_{\mathbb{R}}\frac{1}{2}((A_0'A(\mathbf{w} + \bar{\mathbf{u}})\partial_x\mathbf{w}, \mathbf{w}), \mathbf{w})\ dx$$

$$+\int_0^T dt\ \int_{\mathbb{R}]}\frac{1}{2}((A_1'(\mathbf{w} + \bar{\mathbf{u}})\partial_x\mathbf{w}, \mathbf{w}), \mathbf{w})\ dx$$

$$+\int_0^T dt\ \int_{\mathbb{R}} D_1(\mathbf{w}, \bar{\mathbf{u}})\ dx + \int_0^T dt\ \int_{\mathbb{R}}\frac{1}{2}(A_0'D(\mathbf{w}, \bar{\mathbf{u}})\bar{\mathbf{w}} \cdot \mathbf{w}, \mathbf{w})\ dx.$$

Then

$$\frac{1}{2}\int_{\mathbb{R}}(A_0(\mathbf{w}(T)+\bar{\mathbf{u}})\mathbf{w}(T),\mathbf{w}(T))\ dx\ -\int_0^T dt\ \int_{\mathbb{R}}(D_1(\mathbf{w},\bar{\mathbf{u}})\ dx$$

$$=\frac{1}{2}\int_{\mathbb{R}}(A_0(\mathbf{w}(0)+\bar{\mathbf{u}})\mathbf{w}(0),\mathbf{w}(0))\ dx\ +\int_0^T dt\ \int_{\mathbb{R}}\frac{1}{2}((A_1'(\mathbf{w}+\bar{\mathbf{u}})\partial_x\mathbf{w},\mathbf{w}),\mathbf{w})\ dx$$

$$+\int_0^T dt\ \int_{\mathbb{R}}\frac{1}{2}((A_0'D(\mathbf{w},\bar{\mathbf{u}})\mathbf{w},\mathbf{w}),\mathbf{w})-\int_0^T dt\ \int_{\mathbb{R}}\frac{1}{2}((A_0'A(\mathbf{w}+\bar{\mathbf{u}})\partial_x\mathbf{w}),\mathbf{w},\mathbf{w})\ dx. \tag{5.3.29}$$

From **(D)-Condition**, D_1 is negative definite and since $A_0, A_1', A_0'D, A_0'A$ are bounded in a neighborhood of the equilibrium point $\bar{\mathbf{u}}$, we have

$$\frac{c}{2}||\mathbf{w}(T)||_0^2+c_1\int_0^T\ ||\mathbf{w}(t)||_0^2\ dt$$

$$\leq\frac{c_2}{2}||\mathbf{w}(0)||_0^2+\frac{c_3}{2}\int_0^T dt\ \int_{\mathbb{R}}|\partial_x\mathbf{w}||\mathbf{w}|^2\ dx+\frac{c_4}{2}\int_0^T dt\ \int_{\mathbb{R}}|\partial_x\mathbf{w}||\mathbf{w}|^2\ dx$$

$$+\frac{c_5}{2}\int_0^T dt\ \int_{\mathbb{R}}|\mathbf{w}||\mathbf{w}|^2\ dx$$

$$\leq\frac{c_2}{2}||\mathbf{w}(0)||_0^2+\frac{c_6}{2}\sup_{t\in[0,T]}|\partial_x\mathbf{w}(t)|_\infty\int_0^T dt\ ||\mathbf{w}||_0^2+\frac{c_5}{2}\sup_{t\in[0,T]}|\mathbf{w}(t)|_\infty\int_0^T dt\ ||\mathbf{w}||_0^2. \tag{5.3.30}$$

The embedding of H^1 in L^∞, where α is the Sobolev embedding constant, yields

$$|\mathbf{w}|_\infty\leq\alpha||\mathbf{w}||_{H^1}=\alpha(||\mathbf{w}||_0+||\partial_x\mathbf{w}||_0).$$

Thus, from the definition of the functional $N_2(t)$ in (5.3.21), we have

$$|\partial_x\mathbf{w}|_\infty\leq\alpha(||\partial_x\mathbf{w}||_0+||\partial_{xx}\mathbf{w}||_0)\leq N_2(T),$$

and

$$|\mathbf{w}|_\infty\leq\alpha(||\mathbf{w}||_0+||\partial_x\mathbf{w}||_0)\leq N_2(T).$$

The last term in (5.3.30) is estimated by

$$\frac{c_2}{2}N_2^2(0)+\frac{c_6}{2}N_2^3(T)+\frac{c_5}{2}N_2^3(T).$$

So, using (5.3.30), we have

$$||\mathbf{w}(T)||_0^2+\int_0^T\ ||\mathbf{w}(t)||_0^2\ dt\leq C_1(N_2^2(0)+N_2^3(T)).$$

□

Let us now estimate the first and second order derivatives.

Lemma 5.3.2. *If $N_2(T) \leq \varepsilon \leq \frac{\delta}{\alpha}$, then, for $l = 1, 2$,*

$$sup_{\ 0\leq t\leq T}\ ||\boldsymbol{w}(t)||_l^2 + \int_0^T ||\boldsymbol{w}(t)||_l^2\ dt \leq C(N_2^2(0) + N_2^3(T)).$$

Proof. Apply the first space derivative to system (5.3.20) and take the inner product with $\partial_x \mathbf{w}$, which provides

$$\partial_x(A_0(\mathbf{w}+\bar{\mathbf{u}})\partial_t\mathbf{w} + A_1(\mathbf{w}+\bar{\mathbf{u}})\partial_x\mathbf{w})\cdot\partial_x\mathbf{w} = [\partial_x(A_0D)\mathbf{w} + (A_0D)\partial_x\mathbf{w}]\cdot\partial_x\mathbf{w}. \quad (5.3.31)$$

We have the following identities:

$$\partial_x(A_0(\mathbf{w}+\bar{\mathbf{u}})\partial_t\mathbf{w})\cdot\partial_x\mathbf{w} = \frac{1}{2}\partial_t((A_0(\mathbf{w}+\bar{\mathbf{u}})\partial_x\mathbf{w})\cdot\partial_x\mathbf{w}) - \frac{1}{2}(\partial_t A_0\partial_x\mathbf{w})\cdot\partial_x\mathbf{w}$$

$$+(\partial_x A_0\partial_t\mathbf{w})\cdot\partial_x\mathbf{w};$$

$$\partial_x(A_1(\mathbf{w}+\bar{\mathbf{u}})\partial_x\mathbf{w})\cdot\partial_x\mathbf{w} = \frac{1}{2}\partial_x((A_1(\mathbf{w}+\bar{\mathbf{u}})\partial_x\mathbf{w})\cdot\partial_x\mathbf{w}) + \frac{1}{2}(\partial_x A_1\partial_x\mathbf{w})\cdot\partial_x\mathbf{w}.$$

If we integrate equality (5.3.31) over $\mathbb{R}$ and use the previous identities, the term

$$\partial_x((A_1(\mathbf{w}+\bar{\mathbf{u}})\partial_x\mathbf{w})\cdot\partial_x\mathbf{w})$$

vanishes, and then we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(A_0(\mathbf{w}+\bar{\mathbf{u}})\partial_x\mathbf{w})\cdot\partial_x\mathbf{w}\ dx - \int_{\mathbb{R}}((A_0D)\partial_x\mathbf{w})\cdot\partial_x\mathbf{w}\ dx$$

$$= \int_{\mathbb{R}}\left\{\frac{1}{2}\partial_t A_0\partial_x\mathbf{w} - \partial_x A_0\partial_t\mathbf{w} - \frac{1}{2}\partial_x A_1\partial_x\mathbf{w}\right\}\cdot\partial_x\mathbf{w} + (\partial_x(A_0D)\mathbf{w})\cdot\partial_x\mathbf{w}\ dx. \quad (5.3.32)$$

To estimate the right-end side of (5.3.32), we use (5.3.20) in the following form:

$$\partial_t\mathbf{w} = -A\partial_x\mathbf{w} + D\mathbf{w}.$$

Then

$$\partial_t A_0 = A_0'\partial_t\mathbf{w} = A_0'(-A\partial_x\mathbf{w} + D\mathbf{w}).$$

Thus, equality (5.3.32) is

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(A_0(\mathbf{w}+\bar{\mathbf{u}})\partial_x\mathbf{w})\cdot\partial_x\mathbf{w}\ dx - \int_{\mathbb{R}}((A_0D)\partial_x\mathbf{w})\cdot\partial_x\mathbf{w}\ dx$$

$$= \int_{\mathbb{R}} -\frac{1}{2}((A_0'A\partial_x\mathbf{w},\partial_x\mathbf{w}),\partial_x\mathbf{w}) + \frac{1}{2}(A_0'\partial_x\mathbf{w}A\partial_x\mathbf{w},\partial_x\mathbf{w}) + \frac{1}{2}(A_0'D\mathbf{w}\partial_x\mathbf{w},\partial_x\mathbf{w})\ dx$$

$$-\frac{1}{2}((A_0A'\partial_x\mathbf{w},\partial_x\mathbf{w}),\partial_x\mathbf{w}) + \int_{\mathbb{R}}(A_0D'(\partial_x\mathbf{w})\mathbf{w},\partial_x\mathbf{w})\ dx. \quad (5.3.33)$$

Using (5.3.33), we have

$$||\partial_x \mathbf{w}(T)||_0^2 + \int_0^T ||\partial_x \mathbf{w}(t)||_0^2 \, dt$$

$$\leq C_1(\varepsilon)\left(||\partial_x \mathbf{w}(0)||_0^2 + \int_0^T \int_{\mathbb{R}} |\partial_x \mathbf{w}|^2 |\partial_x \mathbf{w}| \, dxdt \right)$$

$$\leq C_1(\varepsilon)\left(||\partial_x \mathbf{w}(0)||_0^2 + (\sup_{t\in[0,T]} |\mathbf{w}(t)|_\infty + \sup_{t\in[0,T]} |\partial_x \mathbf{w}(t)|_\infty) \int_0^T ||\partial_x \mathbf{w}(t)||_0^2 \, dt \right). \quad (5.3.34)$$

In the same way, we perform the second space derivative of (5.3.25) and take the inner product with $\partial_{xx}\mathbf{w}$, which provides

$$\partial_{xx}(A_0 \partial_t \mathbf{w} + A_1 \partial_x \mathbf{w}) \cdot \partial_{xx}\mathbf{w} = \partial_{xx}(A_0 D\mathbf{w}) \cdot \partial_{xx}\mathbf{w}. \quad (5.3.35)$$

We have the following identities:

$$\partial_{xx}(A_0 \partial_t \mathbf{w}) \cdot \partial_{xx}\mathbf{w}$$

$$= \frac{1}{2}\partial_t((A_0 \partial_{xx}\mathbf{w}) \cdot \partial_{xx}\mathbf{w}) - \frac{1}{2}(\partial_t A_0 \partial_{xx}\mathbf{w}) \cdot \partial_{xx}\mathbf{w}$$

$$+2(\partial_x A_0 \partial_{xt}\mathbf{w}) \cdot \partial_{xx}\mathbf{w} + (\partial_{xx} A_0 \partial_t \mathbf{w}) \cdot \partial_{xx}\mathbf{w}; \quad (5.3.36)$$

$$\partial_{xx}(A_1(\mathbf{W})\partial_x \mathbf{w}) \cdot \partial_{xx}\mathbf{w}$$

$$= \frac{1}{2}\partial_x((A_1 \partial_{xx}\mathbf{w}) \cdot \partial_{xx}\mathbf{w}) + \frac{3}{2}(\partial_x A_1 \partial_{xx}\mathbf{w}) \cdot \partial_{xx}\mathbf{w} + (\partial_{xx} A_1 \partial_x \mathbf{w}) \cdot \partial_{xx}\mathbf{w}. \quad (5.3.37)$$

If we integrate (5.3.35) over $\mathbb{R}$ and we use the previous identities (5.3.36)-(5.3.37), the term

$$\partial_x((A_1 \partial_{xx}\mathbf{w}) \cdot \partial_{xx}\mathbf{w})$$

vanishes.

This way, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (A_0 \partial_{xx}\mathbf{w}) \cdot \partial_{xx}\mathbf{w} \, dx - \int_{\mathbb{R}} (A_0 D \partial_{xx}\mathbf{w}, \partial_{xx}\mathbf{w}) \, dx$$

$$= \int_{\mathbb{R}} \left\{ \frac{1}{2}\partial_t A_0 \partial_{xx}\mathbf{w} - 2\partial_x A_0 \partial_{xt}\mathbf{w} - \partial_{xx} A_0 \partial_t \mathbf{w} \right\} \cdot \partial_{xx}\mathbf{w} \, dx$$

$$- \int_{\mathbb{R}} \left\{ \frac{3}{2}\partial_x A_1 \partial_{xx}\mathbf{w} + \partial_{xx} A_1 \partial_x \mathbf{w} \right\} \cdot \partial_{xx}\mathbf{w} + (\partial_{xx}(A_0 D)\mathbf{w}, \partial_{xx}\mathbf{w}) + ((A_0 D)\partial_x \mathbf{w}, \partial_{xx}\mathbf{w}) \, dx$$

$$= \int_{\mathbb{R}} \Bigg\{ -\frac{1}{2}(A_0' A \partial_x \mathbf{w}, \partial_{xx}\mathbf{w}) + \frac{1}{2}(A_0' D\mathbf{w}, \partial_{xx}\mathbf{w}) + (A_0'\partial_x \mathbf{w} A' \partial_x \mathbf{w}, \partial_x \mathbf{w})$$

$$+\frac{1}{2}A_0'\partial_x\mathbf{w}A\partial_{xx}\mathbf{w} - 2(A_0'\partial_x\mathbf{w}D'\partial_x\mathbf{w},\mathbf{w}) - 2A_0'\partial_x\mathbf{w}D\partial_x\mathbf{w} - A_0'\partial_{xx}\mathbf{w}D\mathbf{w}$$

$$-(A_0A''\partial_x\mathbf{w}\cdot\partial_x\mathbf{w},\partial_x\mathbf{w}) - (A_0A'\partial_{xx}\mathbf{w},\partial_x\mathbf{w}) + (A_0'D'\partial_{xx}\mathbf{w},\mathbf{w})\Bigg\}\cdot\partial_{xx}\mathbf{w}\ dx.$$

Then,

$$||\partial_{xx}\mathbf{w}(T)||_0^2 + \int_0^T ||\partial_{xx}\mathbf{w}(t)||_0^2\ dt$$

$$\leq C_2(\varepsilon)\Bigg\{||\partial_{xx}\mathbf{w}(0)||_0^2 + \int_0^T\int_{[0,1]}(|\partial_x\mathbf{w}|^2 + |\partial_{xx}\mathbf{w}|^2)(|\partial_x\mathbf{w}| + |\mathbf{w}| + |\partial_x\mathbf{w}|)\ dxdt\Bigg\}$$

$$\leq C_2(\varepsilon)\Bigg\{||\partial_{xx}\mathbf{w}(0)||_0^2 + (\sup_{t\in[0,T]}|\mathbf{w}|_\infty + \sup_{t\in[0,T]}|\partial_x\mathbf{w}|_\infty)\int_0^T(||\partial_x\mathbf{w}(t)||_0^2 + ||\partial_{xx}\mathbf{w}(t)||_0^2)\ dt\Bigg\}. \tag{5.3.38}$$

□

Now, Lemma 5.3.1 and Lemma 5.3.2 prove inequality (5.3.22). Then, Proposition 5.3.1 follows immediately.
It remains to study the asymptotic behavior of the solution. Let us set

$$\mathrm{E}(t) = \frac{1}{2}||\mathbf{w}(t)||_2^2. \tag{5.3.39}$$

Taking the time derivative of (5.3.29) and using the embedding of $H^2(\mathbb{R})$ in $L^\infty(\mathbb{R})$, we have

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}(t)||_0^2 + c_1||\mathbf{w}(t)||_0^2 \leq c_2||\mathbf{w}(t)||_2||\mathbf{w}(t)||_0^2. \tag{5.3.40}$$

Taking the time derivative of (5.3.34), we obtain, in the same way, the estimate on the first derivative of $\mathbf{w}$,

$$\frac{1}{2}\frac{d}{dt}||\partial_x\mathbf{w}(t)||_0^2 + c_3||\partial_x\mathbf{w}(t)||_0^2 \leq c_4||\mathbf{w}(t)||_2||\partial_x\mathbf{w}(t)||_0^2. \tag{5.3.41}$$

Finally, from the time derivative of (5.3.38) and using Morrey's Theorem, we have the second order estimate

$$\frac{1}{2}\frac{d}{dt}||\partial_{xx}\mathbf{w}(t)||_0^2 + c_5||\partial_{xx}\mathbf{w}(t)||_0^2 \leq c_6||\mathbf{w}(t)||_2||\partial_{xx}\mathbf{w}(t)||_0^2. \tag{5.3.42}$$

Summing (5.3.40), (5.3.41) and (5.3.42), from (5.3.39) we have

$$\partial_t\mathrm{E} + \mu\mathrm{E} \leq \nu\mathrm{E}^{3/2}, \tag{5.3.43}$$

where $\mu = c_1 + c_3 + c_5$ and $\nu = c_2 + c_4 + c_6$.

Now, we take the initial datum small enough such that

$$\mathrm{E}(0) < \left(\frac{\gamma}{\nu}\right)^2,$$

for some $0 < \gamma < min\{\mu, \nu\}$. Thus,

$$\nu \mathrm{E}^{3/2} < \gamma \mathrm{E},$$

at least for a small interval of time and, by using (5.3.43),

$$\partial_t \mathrm{E} \leq (\gamma - \mu)\mathrm{E},$$

while (5.3.19) follows directly from the Gronwall inequality, taking $\beta = \mu - \gamma$.

Chapter 6

A density dependent compressible-incompressible Euler model in several space dimensions

Here, we consider a fluid described by the following equations in $\mathbb{R}^d$:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0, \\ \partial_t v + v \cdot \nabla v + f(\rho, v) \nabla \rho + \nabla P = 0, \\ \nabla \cdot v = 0, \end{cases} \tag{6.0.1}$$

with initial data

$$\rho(0, x) = \rho_0(x), \qquad v(0, x) = v_0(x) \text{ such that } \nabla \cdot v_0(x) = 0, \tag{6.0.2}$$

where $f(\rho, v)$ is a scalar function of $(\rho, v) \in \mathbb{R}^{d+1}$. This section is based on [14]. System (6.0.1) describes the motion of a nonhomogeneous, also called density-dependent, fluid. The nonnegative scalar function ρ is the density of the fluid, $v \in \mathbb{R}^d$ its velocity, and P is the incompressible hydrostatic pressure generated by the divergence free constraint. The term $f(\rho, v)\nabla\rho$ is a slight generalization of a compressible pressure. This system is intended as a toy model for a general class of problems presented in (4.0.5). More generally, many problems characterized by the interaction between compressible and incompressible pressure terms, where the compressible part can be generalized replacing $\gamma log(B)$ in (4.0.5) with a function $\phi(B)$ only depending on the solid phase B, arise from mixture theory and are similar to system (4.0.5), as, for instance, models of biofilms [27] in (4.0.2), tumor growth [3] and organic tissues and vasculogenesis [32]. A complete analytical study of the one dimensional model in (4.0.5) is given in [12]. As a matter of facts, in more space dimensions, model (4.0.5) presents several analytical difficulties, which we are trying to understand by studying a simplified version. In order to do this, the first idea would be to consider a model where the solid phase B and the liquid L have the same transport velocity $v = v_S = v_L$, whose equation contains a compressible pressure term. These assumptions give the following model:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0, \\ \partial_t v + v \cdot \nabla v + \nabla \phi(\rho) + \nabla P = 0, \\ \nabla \cdot v = 0, \end{cases} \tag{6.0.3}$$

where ρ is the density of the fluid and $\phi(\rho)$ a general compressible pressure. It can be checked that the techniques developed in this paper in the following continue to work on it, but there is also a more trivial way to proceed. Namely, by defining a new pressure $Q := P + \phi(\rho)$, model (6.0.3) can be reduced to

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0, \\ \partial_t v + v \cdot \nabla v + \nabla Q = 0, \\ \nabla \cdot v = 0, \\ P = Q - \phi(\rho), \end{cases} \tag{6.0.4}$$

which is just the homogeneous incompressible Euler equations plus a transport equation for the density variable and can be solved separately, see [49]. So, since there is a too simple way to solve (6.0.4), we are going to study (6.0.1), which is a mathematical generalization of model (6.0.3), endowed with most of the analytical difficulties of (4.0.5). Let us compare system (6.0.1) with the density-dependent incompressible Euler equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho v) = 0, \\ \partial_t v + v \cdot \nabla v + \frac{\nabla P}{\rho} = 0, \\ \nabla \cdot v = 0, \end{cases} \tag{6.0.5}$$

which have been studied by many authors, see for instance *J. E. Marsden* [53], *H. Beirão da Veiga* [7], *A. Valli* [76] & *R. Danchin* [29]. Let us remark that, in [76], *Valli* and *Zajaczkowski* have studied model (6.0.5) by using an approximating system where the divergence of the velocity field vanishes in a similar way with respect to to the Chorin-Temam projection method in [75]. Although model (6.0.1) looks quite similar to (6.0.5), most of the ideas used to solve it do not apply to our system. In fact, consider the elliptic equation for the pressure term

$$\Delta P + \nabla \cdot (f(\rho, v)\nabla \rho) = -\sum_{i,j=1}^{d} \partial_{x_i} v_j \partial_{x_j} v_i,$$

which is obtained by applying the divergence free operator to the velocity equation in (6.0.1). By contrast with what happens for system (6.0.5), the pressure term P in (6.0.1) does not gain one more space derivative of regularity with respect to the other unknowns ρ, v, and we are not able to get energy estimates. Besides, even the vorticity method from [7] does not seem to work for (6.0.1) and (4.0.5), so we have to proceed in a different way. Here, we establish the well-posedness of system (6.0.1), using an approximation based on the projection onto the space of the divergence free velocity field and paradifferential calculus. Besides, we show the convergence of a new singular perturbation approximation that can be considered as a continuous version of the projection method in [75], which turns out to work also on the homogeneous incompressible Euler equations. We point out that our proof of convergence of the second approximation can be adapted with slight modifications to prove the convergence of the classical fractional step of Chorin-Temam projection method [75], so providing a constructive approximation to system (6.0.1). To have a complete view, we briefly show that also the more classical artificial compressibility method in [75] works on system (6.0.1). In Remarks 6.1.2, 6.2.3 and 6.3.3,

we see that these three kinds of approximations do not work on the density dependent Euler equations (6.0.5), showing, this way, the deep analytical dissimilarity among these two models.
First of all, we notice that, by using the divergence free condition $\nabla \cdot v = 0$, the mass balance equation of (6.0.1) yields

$$\partial_t \rho + v \cdot \nabla \rho = 0,$$

and system (6.0.1) reads:

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0, \\ \partial_t v + v \cdot \nabla v + f(\rho, v) \nabla \rho + \nabla P = 0, \\ \nabla \cdot v = 0. \end{cases} \tag{6.0.6}$$

Now, let $\mathbf{u} = (\rho, v)$ and $F_P = (0, \nabla P)$. System (6.0.6) can be written in the following compact formulation:

$$\begin{cases} \partial_t \mathbf{u} + \sum_{j=1}^d A_j(\mathbf{u}) \partial_{x_j} \mathbf{u} + F_P = 0, \\ \nabla \cdot v = 0, \end{cases} \tag{6.0.7}$$

with initial data (6.0.2)

$$\mathbf{u}(0, x) = \mathbf{u}_0(x) = (\rho_0(x), v_0(x)), \tag{6.0.8}$$

where, in the two dimensional case

$$A_1(\mathbf{u}) = \begin{pmatrix} v_1 & 0 & 0 \\ f(\mathbf{u}) & v_1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}, \; A_2(\mathbf{u}) = \begin{pmatrix} v_2 & 0 & 0 \\ 0 & v_2 & 0 \\ f(\mathbf{u}) & 0 & v_2 \end{pmatrix}, \tag{6.0.9}$$

and, in the general d-dimensional case, for $j = 1, \cdots, d$,

$$A_j(\mathbf{u}) = \begin{pmatrix} v_j & 0 & 0 & \cdots & 0 \\ \delta_{1j} f(\mathbf{u}) & v_j & 0 & \cdots & 0 \\ \delta_{2j} f(\mathbf{u}) & 0 & v_j & \cdots & 0 \\ \cdots & \cdots & \cdots & v_j & \cdots \\ \delta_{dj} f(\mathbf{u}) & 0 & 0 & \cdots & v_j \end{pmatrix}. \tag{6.0.10}$$

Remark 6.0.1. *As we will see later, in order to apply the techniques in the last two sections, we need to stay far away from the vacuum, i.e. $\rho(t, x) \neq 0$ for every time t and $x \in \mathbb{R}^d$. This way, ρ cannot belong to $L^2(\mathbb{R}^d)$, while the translated variable $\rho - \bar{\rho}$, with $\bar{\rho}$ an arbitrary positive constant, does. For this reason, choosing a constant value $\bar{\rho}$, in the following we are going to use the variable $\rho - \bar{\rho}$ to get a translated version of system (6.0.7). The analytical motivations of the translation are discussed at the beginning of Section 6.2, anyway in Section 6.1 we can also admit $\bar{\rho} = 0$. We notice that the non-vanishing density variable is a condition also required by the general system (4.0.5), where the density ρ is replaced with the volume fraction of the solid phase S and the compressible pressure is $\frac{\gamma \nabla S}{S}$.*

Taking into account Remark 6.0.1, first of all we make a slight modification of system (6.0.7), defining

$$\tilde{\rho} := \rho - \bar{\rho}, \qquad \bar{\mathbf{u}} := (\bar{\rho}, 0), \qquad \text{and} \qquad \tilde{\mathbf{u}} = (\tilde{\rho}, \tilde{v}) := \mathbf{u} - \bar{\mathbf{u}} = (\rho - \bar{\rho}, v),$$

with $\bar{\rho} > 0$. By this change of variable, the Cauchy problem (6.0.7)-(6.0.8) reads:

$$\begin{cases} \partial_t \tilde{\mathbf{u}} + \sum_{j=1}^d A_j(\tilde{\mathbf{u}} + \bar{\mathbf{u}})\partial_{x_j}\tilde{\mathbf{u}} + F_P = 0, \\ \nabla \cdot v = 0, \end{cases} \tag{6.0.11}$$

with initial data

$$\tilde{\mathbf{u}}(0, x) = \tilde{\mathbf{u}}_0(x) = (\rho_0 - \bar{\rho}, v_0), \tag{6.0.12}$$

where ρ_0, v_0 are defined in (6.0.2). Now, we provide the definition of classical local solutions to system (6.0.1), with initial data (6.0.2).

Definition 6.0.1. *Let $m > [d/2] + 1$ be fixed, $m \in \mathbb{N}$. The term (ρ, v, P), with $\rho > 0$, is a classical solution to the Cauchy problem (6.0.6) - (6.0.2) if, fixed a constant value $\bar{\rho} > 0$, there exists $T > 0$ such that $\tilde{\boldsymbol{u}} = (\rho - \bar{\rho}, v)$ belongs to $C([0,T], H^m(\mathbb{R}^d)) \cap C^1([0,T], H^{m-1}(\mathbb{R}^d))$, ∇P belongs to $C([0,T], H^{m-1}(\mathbb{R}^d))$, and $(\tilde{\boldsymbol{u}}, P)$ solve*

$$\begin{cases} \partial_t \tilde{\boldsymbol{u}} + \sum_{j=1}^d A_j(\tilde{\boldsymbol{u}} + \bar{\boldsymbol{u}})\partial_{x_j}\tilde{\boldsymbol{u}} + (0, \nabla P) = 0, \\ \nabla \cdot v = 0, \end{cases}$$

with initial data in (6.0.12) belonging to $H^m(\mathbb{R}^d)$.

In the next three sections, we will prove the existence and uniqueness of the solution to (6.0.6)-(6.0.2), according to Definition 6.0.1, by using three different techniques.

6.1 Well-posedness via the Leray projector

This section is devoted to the proof of existence and uniqueness of the local solution to (6.0.1)-(6.0.2), by using the so called *Leray projector* and related paradifferential calculus. Following [9], first of all we approximate the translated version (6.0.11) of system (6.0.7) by a standard regularization, using mollifiers J_ε, which we define here.

Definition 6.1.1. *Let $\Phi(x) \in C_0^\infty(\mathbb{R}^d)$ be any positive, radial function such that $\int_{\mathbb{R}^d} \Phi \; dx = 1$. Fix $\varepsilon > 0$, and let $j_\varepsilon = \frac{1}{\varepsilon^d}\Phi(x/\varepsilon)$. The mollification $J_\varepsilon \boldsymbol{w}$ of functions $\boldsymbol{w} \in L^2(\mathbb{R}^d)$ is defined by*

$$J_\varepsilon \boldsymbol{w}(x) = (j_\varepsilon * \boldsymbol{w})(x) = \frac{1}{\varepsilon^d}\int_{\mathbb{R}^d} \Phi\left(\frac{x-y}{\varepsilon}\right)\boldsymbol{w}(y) \; dy. \tag{6.1.13}$$

Now, we regularize (6.0.11) using mollifiers, and we get the following approximating system:

$$\begin{cases} \partial_t \tilde{\mathbf{u}}^\varepsilon + \sum_{j=1}^d J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon + (0, \nabla P^\varepsilon) = 0, \\ \nabla \cdot v^\varepsilon = 0, \end{cases} \tag{6.1.14}$$

with initial data

$$\tilde{\mathbf{u}}_0^\varepsilon(x) = \tilde{\mathbf{u}}_0(x) = (\rho_0(x) - \bar{\rho}, v_0(x)) \tag{6.1.15}$$

in (6.0.12). Here, the idea is to eliminate the pressure term ∇P^ε in (6.1.14), by applying the projector operator onto the space of the divergence free vectors, which is known as the *Leray projector*, to the equation for the velocity v^ε in (6.1.14). Precisely, since system (6.1.14) is written in terms of the unknown $\tilde{\mathbf{u}}^\varepsilon = (\rho^\varepsilon - \bar{\rho}, v^\varepsilon)$, and we will work in the framework of the Sobolev spaces, we are looking for an operator $\tilde{\mathbf{P}}$ that projects any vector $\tilde{\mathbf{u}}^\varepsilon = (\rho^\varepsilon - \bar{\rho}, v^\varepsilon) \in H^s(\mathbb{R}^d)$ onto the space

$$V^s := \{(\rho - \bar{\rho}, v) \in H^s(\mathbb{R}^d) : \nabla \cdot v = 0\}. \tag{6.1.16}$$

It results that $\tilde{\mathbf{P}}$ is a generalization of the Leray projector, i.e.

$$\tilde{\mathbf{P}} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbb{P} \end{pmatrix}, \tag{6.1.17}$$

where $\mathbf{0}$ is the d-dimensional null vector and $\mathbb{P}$ is the standard Leray projector, whose symbol is given by

$$\mathbb{P}(\xi) = (\mathbb{P}_{ij}(\xi))_{i,j=1,\cdots,d}, \quad \text{where} \quad \mathbb{P}_{ij}(\xi) = \delta_{ij}(\xi) - \frac{\xi_i \xi_j}{|\xi|^2}, \tag{6.1.18}$$

defined in [4]. Now, we want to apply the operator $\tilde{\mathbf{P}}$ in (6.1.17) to the approximating system (6.1.14)-(6.1.15). Notice that, since $\nabla \cdot v^\varepsilon = 0$ in (6.1.14), then $\tilde{\mathbf{u}}^\varepsilon \in V^s$ by definition (6.1.16), i.e.

$$\tilde{\mathbf{P}}\tilde{\mathbf{u}}^\varepsilon = (\rho^\varepsilon - \bar{\rho}, v^\varepsilon) = \tilde{\mathbf{u}}^\varepsilon.$$

Then, applying $\tilde{\mathbf{P}}$ to (6.1.14)-(6.1.15), we get

$$\partial_t \tilde{\mathbf{u}}^\varepsilon + \sum_{j=1}^d \tilde{\mathbf{P}}(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon) = 0, \tag{6.1.19}$$

with the same initial data in (6.1.15). Notice also that now the divergence free condition $\nabla \cdot v^\varepsilon = 0$ in (6.1.14) is implicitly contained in (6.1.19), which can be treated as an hyperbolic system of the first order. We prove the following theorem.

Theorem 6.1.1. *(Local existence of the approximating solution to the first type approximation) Let $\tilde{\mathbf{u}}_0^\varepsilon$ as in (6.1.15) be belonging to V^s defined in (6.1.16), with $s > d/2 + 1$. Then, for every $\varepsilon > 0$, there exists a time T, independent of ε, such that system (6.1.19) has a unique solution $\tilde{\mathbf{u}}^\varepsilon = (\rho^\varepsilon - \bar{\rho}, v^\varepsilon) \in C^1([0,T], V^s)$.*

Proof. First of all, we show that existence and uniqueness follow from the Picard theorem (see [9]). Then, we prove that the time of local existence T_ε can be bounded from below by a time $T > 0$, which is independent of ε. System (6.1.19) reduces to an ordinary differential equation:

$$\partial_t \tilde{\mathbf{u}}^\varepsilon = F^\varepsilon(\tilde{\mathbf{u}}^\varepsilon), \qquad \tilde{\mathbf{u}}^\varepsilon|_{t=0} = \tilde{\mathbf{u}}_0^\varepsilon(x),$$

where

$$F^\varepsilon(\tilde{\mathbf{u}}^\varepsilon) = -\sum_{j=1}^d \tilde{\mathbf{P}}(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon). \tag{6.1.20}$$

Notice that $J_\varepsilon \tilde{\mathbf{u}}^\varepsilon$ and $J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}})$ are C^∞ functions and, from [8, 55], $\mathbb{P}$ is associated to an analytic pseudodifferential operator of order 0, modulo an infinitely smooth remainder, then

$$F^\varepsilon : V^s \to V^s.$$

In order to apply the Picard theorem, we have to prove that $F^\varepsilon(\tilde{\mathbf{u}}^\varepsilon)$ in (6.1.20) is Lipschitz continuous. To do this, we take two vectors $\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2$. In the following, we omit the index ε in the unknown functions, where there is no ambiguity. Let c_S be the Sobolev embedding constant. Then

$$||F^\varepsilon(\tilde{\mathbf{u}}_1) - F^\varepsilon(\tilde{\mathbf{u}}_2)||_0 \leq \sum_{j=1}^{d} ||\tilde{\mathbf{P}} J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_1 - \tilde{\mathbf{P}} J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_2||_0$$

$$\leq \sum_{j=1}^{d} \{||[\tilde{\mathbf{P}} J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}})) - \tilde{\mathbf{P}} J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}}))]\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_1||_0$$
$$+ ||\tilde{\mathbf{P}} J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)||_0\}$$

$$= \sum_{j=1}^{d} \left\{ \left|\left| \tilde{\mathbf{P}} \left[\int_0^1 \frac{d}{dr}(J_\varepsilon A_j(rJ_\varepsilon(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}}) + (1-r)J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}})))\ dr \right] \partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_1 \right|\right|_0 \right.$$
$$\left. + c(|J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}})|_\infty)||\partial_{x_j} J_\varepsilon(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)||_0 \right\}$$

$$= \sum_{j=1}^{d} \left\{ \left|\left| \tilde{\mathbf{P}} \left[\int_0^1 (J_\varepsilon A_j(rJ_\varepsilon(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}}) + (1-r)J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}})))' dr \right] J_\varepsilon(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)\ \partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_1 \right|\right|_0 \right.$$
$$\left. + c(|J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}})|_\infty)||\partial_{x_j} J_\varepsilon(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)||_0 \right\}$$

$$\leq \sum_{j=1}^{d} \{c(|J_\varepsilon(\tilde{\mathbf{u}}_{1,2} + \bar{\mathbf{u}})|_\infty, |\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_1|_\infty)(||J_\varepsilon(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)||_0 + ||\partial_{x_j} J_\varepsilon(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)||_0)\}$$
$$\leq c(|J_\varepsilon(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}})|_\infty, |J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}})|_\infty, |\nabla J_\varepsilon \tilde{\mathbf{u}}_1|_\infty)||J_\varepsilon(\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)||_1$$

$$\leq c(c_S, ||\tilde{\mathbf{u}}_1||_s, ||\tilde{\mathbf{u}}_2||_s, \bar{\rho})||\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2||_1,$$

where the last inequality follows from Moser estimates and properties of mollifiers. Taking the α ($|\alpha| \leq s$) derivative, we have

$$||D^\alpha(F^\varepsilon(\tilde{\mathbf{u}}_1) - F^\varepsilon(\tilde{\mathbf{u}}_2))||_0$$

$$\leq \sum_{j=1}^{d} ||D^\alpha(\tilde{\mathbf{P}} J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_1 - \tilde{\mathbf{P}} J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_2)||_0$$

$$\leq \sum_{j=1}^{d} \{||\tilde{\mathbf{P}} D^\alpha[(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}})) - J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}})))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_1]||_0$$

$$+||\tilde{\mathbf{P}}D^\alpha(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}}))\partial_{x_j}J_\varepsilon(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2))||_0\}$$

$$\le c_S\sum_{j=1}^d\{||\tilde{\mathbf{P}}D^s(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_1+\bar{\mathbf{u}}))-J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})))||_0|\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}_1|_\infty$$

$$+|\tilde{\mathbf{P}}[J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_1+\bar{\mathbf{u}}))-J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}}))]|_\infty||D^s\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}_1||_0$$

$$+||\tilde{\mathbf{P}}D^s(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})))||_0|\partial_{x_j}J_\varepsilon(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)|_\infty$$

$$+|\tilde{\mathbf{P}}J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}}))|_\infty||D^s\partial_{x_j}J_\varepsilon(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)||_0\}$$

$$=c_S\sum_{j=1}^d\left\{\left|\left|\tilde{\mathbf{P}}D^s\left[\int_0^1\frac{d}{dr}(J_\varepsilon A_j(rJ_\varepsilon(\tilde{\mathbf{u}}_1+\bar{\mathbf{u}})+(1-r)J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})))\;dr\right]\right|\right|_0|\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}_1|_\infty\right.$$

$$+\left|\tilde{\mathbf{P}}\int_0^1\frac{d}{dr}(J_\varepsilon A_j(rJ_\varepsilon(\tilde{\mathbf{u}}_1+\bar{\mathbf{u}})+(1-r)J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})))\;dr\right|_\infty||D^s\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}_1||_0$$

$$+||\tilde{\mathbf{P}}D^s(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})))||_0|\partial_{x_j}J_\varepsilon(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)|_\infty$$

$$\left.+|\tilde{\mathbf{P}}J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}}))|_\infty||D^s\partial_{x_j}J_\varepsilon(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)||_0\right\}$$

$$=c_S\sum_{j=1}^d\left\{\left|\left|\tilde{\mathbf{P}}D^s\left[\int_0^1dr(J_\varepsilon A_j(rJ_\varepsilon(\tilde{\mathbf{u}}_1+\bar{\mathbf{u}})+(1-r)J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})))'J_\varepsilon(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)\right]\right|\right|_0\right.$$

$$|\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}_1|_\infty$$

$$+\left|\tilde{\mathbf{P}}\int_0^1dr(J_\varepsilon A_j(rJ_\varepsilon(\tilde{\mathbf{u}}_1+\bar{\mathbf{u}})+(1-r)J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})))'J_\varepsilon(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)\right|_\infty$$

$$||D^s\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}_1||_0$$

$$+||\tilde{\mathbf{P}}D^s(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})))||_0|\partial_{x_j}J_\varepsilon(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)|_\infty$$

$$\left.+|\tilde{\mathbf{P}}J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}}))|_\infty||D^s\partial_{x_j}J_\varepsilon(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)||_0\right\}$$

$$\le c(c_S,||\tilde{\mathbf{u}}_1||_m,||\tilde{\mathbf{u}}_2||_m,\bar{\rho},\varepsilon^{-1})||D^s(\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2)||_0$$

$$=c(c_S,||\tilde{\mathbf{u}}_1||_m,||\tilde{\mathbf{u}}_2||_m,\bar{\rho},\varepsilon^{-1})||\tilde{\mathbf{u}}_1-\tilde{\mathbf{u}}_2||_s, \tag{6.1.21}$$

where, once again, the last inequality follows from Moser estimates and properties of mollifiers, as we can see in the following remark.

Remark 6.1.1. *Define* $G(\tilde{\boldsymbol{u}}_2) := \tilde{\boldsymbol{P}} J_\varepsilon[A_j(J_\varepsilon(\tilde{\boldsymbol{u}}_2 + \bar{\boldsymbol{u}})) - A_j(\bar{\boldsymbol{u}})]$*. Then,* $G(\boldsymbol{0}) = 0$*. Applying Theorem C. 12 in [8] to* $G(\tilde{\boldsymbol{u}}_2)$*, we have*

$$||\tilde{\boldsymbol{P}} D^s(J_\varepsilon A_j(J_\varepsilon(\tilde{\boldsymbol{u}}_2 + \bar{\boldsymbol{u}})))||_0 = ||\tilde{\boldsymbol{P}} D^s(J_\varepsilon A_j(J_\varepsilon(\tilde{\boldsymbol{u}}_2 + \bar{\boldsymbol{u}})) - \ J_\varepsilon A_j(\bar{\boldsymbol{u}})) + \tilde{\boldsymbol{P}} D^s(J_\varepsilon A_j(\bar{\boldsymbol{u}}))||_0$$

$$= ||\tilde{\boldsymbol{P}}[D^s(J_\varepsilon A_j(J_\varepsilon(\tilde{\boldsymbol{u}}_2 + \bar{\boldsymbol{u}})) - J_\varepsilon A_j(\bar{\boldsymbol{u}}))]||_0 \leq c(|\tilde{\boldsymbol{u}}_2|_\infty)||\tilde{\boldsymbol{u}}_2||_s \leq c(c_S||\tilde{\boldsymbol{u}}_2||_s)||\tilde{\boldsymbol{u}}_2||_s.$$

The last inequality (6.1.21) implies that, for fixed ε, F^ε is locally Lipschitz continuous on any open set

$$\mathcal{U}^M = \{\tilde{\mathbf{u}}^\varepsilon \in V^s : ||\tilde{\mathbf{u}}^\varepsilon||_s < M\}.$$

By using the Picard theorem, there exists the unique solution $\tilde{\mathbf{u}}^\varepsilon \in C^1([0, T_\varepsilon), \mathcal{U}^M)$ for any $T_\varepsilon > 0$.

Now, we show that the time of existence T_ε is bounded from below by a strictly positive time T that is independent of ε. Let $\tilde{\mathbf{P}}$ be the analytic - modulo an infinitely smooth remainder, see [8, 55, 74] - pseudodifferential operator defined in (6.1.17), and, according to the notations in [55], let T_{iA} be the paradifferential operator so defined

$$T_{iA} := \sum_{j=1}^{d} T_{A_j} \partial_{x_j}, \tag{6.1.22}$$

where $T_{A_j} = T_{A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))}$ is the paradifferential operator related to the symbol $A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))$, for $j = 1, \cdots, d$, according to Section 2. There is also another way to define T_{iA}, i.e. we consider the symbolic matrix $A(\xi, \mathbf{u}) := \sum_{j=1}^d A_j(\mathbf{u})\xi_j$

$$= \begin{pmatrix} \sum_{j=1}^d v_j\xi_j & 0 & 0 & \cdots & 0 \\ f(\mathbf{u})\xi_1 & \sum_{j=1}^d v_j\xi_j & 0 & \cdots & 0 \\ f(\mathbf{u})\xi_2 & 0 & \sum_{j=1}^d v_j\xi_j & \cdots & 0 \\ \cdots & \cdots & \cdots & \sum_{j=1}^d v_j\xi_j & \cdots \\ f(\mathbf{u})\xi_d & 0 & 0 & \cdots & \sum_{j=1}^d v_j\xi_j \end{pmatrix}, \tag{6.1.23}$$

and we indicate with T_{iA} the paradifferential operator associated to the regularized version of the symbol $iA(\xi, \mathbf{u})$, evaluated in $J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}})$, with $A(\xi, \mathbf{u})$ in (6.1.23). Now, we write (6.1.19) as

$$\partial_t \tilde{\mathbf{u}}^\varepsilon + \tilde{\mathbf{P}} J_\varepsilon T_{iA} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon = -\left[\sum_{j=1}^{d} \tilde{\mathbf{P}}(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon) - \tilde{\mathbf{P}} J_\varepsilon T_{iA} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon\right]. \tag{6.1.24}$$

Let $\Lambda = (1 - \Delta)^{\frac{1}{2}}$, where Δ is the Laplace operator. From (6.1.24), we have

$$\frac{1}{2}\frac{d}{dt}||\tilde{\mathbf{u}}^\varepsilon||_s^2 = (\Lambda^s \partial_t \tilde{\mathbf{u}}^\varepsilon, \Lambda^s \tilde{\mathbf{u}}^\varepsilon)_0 = -Re(\Lambda^s \tilde{\mathbf{P}} J_\varepsilon T_{iA} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \Lambda^s \tilde{\mathbf{u}}^\varepsilon)_0 + Q, \tag{6.1.25}$$

with

$$Q := \sum_{j=1}^{d} (\Lambda^s \tilde{\mathbf{P}}(J_\varepsilon[T_{A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))} - A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))]\partial_{x_j} J_\varepsilon \mathbf{u}^\varepsilon), \Lambda^s \tilde{\mathbf{u}}^\varepsilon)_0.$$

Lemma 7.2.3 in [55] states that

$$||\sum_{j=1}^{d}[T_{A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon+\bar{\mathbf{u}}))} - A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))]\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon||_s \le c(||\tilde{\mathbf{u}}^\varepsilon||_s, \bar{\rho})||J_\varepsilon \tilde{\mathbf{u}}^\varepsilon||_s,$$

then, we have the following estimate:

$$|Q| \le c(||\tilde{\mathbf{u}}^\varepsilon||_s)||\tilde{\mathbf{u}}^\varepsilon||_s^2.$$

It remains to discuss the first term of the right hand side of (6.1.25), which is

$$Re(\Lambda^s \tilde{\mathbf{P}} J_\varepsilon T_{iA} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \Lambda^s \tilde{\mathbf{u}}^\varepsilon)_0.$$

We need the following Lemma.

Lemma 6.1.1. *The operator $\Lambda^s \tilde{\boldsymbol{P}}$, with $\tilde{\boldsymbol{P}}$ in (6.1.17), commutes with the diagonal matrix with mollifiers J_ε entries.*

Proof. The symbols associated to $\Lambda^s \tilde{\mathbf{P}}$ and J_ε are both Fourier multipliers, which commute (see [55]). □

Now, by using Lemma 6.1.1, we have

$$(\Lambda^s \tilde{\mathbf{P}} J_\varepsilon T_{iA} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \Lambda^s \tilde{\mathbf{u}}^\varepsilon)_0 = (\Lambda^s \tilde{\mathbf{P}} T_{iA} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \Lambda^s J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0$$

$$= (\tilde{\mathbf{P}} T_{iA} \Lambda^s J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \Lambda^s J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0 + ([\Lambda^s, \tilde{\mathbf{P}} T_{iA}] J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \Lambda^s J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0.$$

Since the symbol of Λ^s is $\Delta^s(\xi)Id$, where Id is the $d+1$ dimensional identity matrix, the commutation rule (see [55], [34], [8], [74]) implies that $[\Lambda^s, \mathbf{P}T_{iA}]$ is an operator of order less than or equal to s, i.e.

$$([\Lambda^s, \tilde{\mathbf{P}} T_{iA}] J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \Lambda^s J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0 \le ||[\Lambda^s, \tilde{\mathbf{P}} T_{iA}] J_\varepsilon \tilde{\mathbf{u}}^\varepsilon||_0 ||\Lambda^s J_\varepsilon \tilde{\mathbf{u}}^\varepsilon||_0 \le c(||\tilde{\mathbf{u}}^\varepsilon||_s)||\tilde{\mathbf{u}}^\varepsilon||_s^2.$$

It remains to deal with $Re(\tilde{\mathbf{P}} T_{iA} \Lambda^s J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \Lambda^s J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0$. From Proposition 1.10 in [34], the symbol associated to the composition $\tilde{\mathbf{P}} T_{iA}$ is made by a sum, in the α multi-index, of terms

$$\partial_\xi^\alpha \tilde{\mathbf{P}}^\phi(\xi) D_x^\alpha (iA^\phi(\xi, J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))),$$

where $\tilde{\mathbf{P}}^\phi(\xi)$ and $A^\phi(\xi, J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))$ are the regularized versions of the symbols $\tilde{\mathbf{P}}(\xi)$, $A(\xi, J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))$, through the standard Littlewood-Paley decomposition, see Section 2, and $D_x = \frac{1}{i}\partial_x$. Apart from $|\alpha| = 0$, the others are terms of order less than or equal to 0, namely the symbol related to the operator $\tilde{\mathbf{P}} T_{iA}$ can be written as

$$\tilde{\mathbf{P}}^\phi(\xi) iA^\phi(\xi, J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}})) + R_\alpha, \tag{6.1.26}$$

where R_α is a remainder of order less than or equal to 0. Now, taking a generic vector $\mathbf{u} = (\rho, v)$, from (6.1.17), (6.0.9), and (6.0.10) we have

$$\tilde{\mathbf{P}} A(\xi, \mathbf{u}) = \begin{pmatrix} v_1\xi_1 + v_2\xi_2 & 0 & 0 \\ 0 & \frac{\xi_2^2}{|\xi|^2}(v_1\xi_1 + v_2\xi_2) & -\frac{\xi_1\xi_2}{|\xi|^2}(v_1\xi_1 + v_2\xi_2) \\ 0 & -\frac{\xi_1\xi_2}{|\xi|^2}(v_1\xi_1 + v_2\xi_2) & \frac{\xi_1^2}{|\xi|^2}(v_1\xi_1 + v_2\xi_2) \end{pmatrix},$$

in the two dimensional case, while, in the general d-dimensional case, we have

$$\tilde{\mathbf{P}}A(\xi,\mathbf{u}) = \sum_{j=1}^{d} v_j\xi_j \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1-\frac{\xi_1^2}{|\xi|^2} & -\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{\xi_1\xi_3}{|\xi|^2} & \cdots & -\frac{\xi_1\xi_d}{|\xi|^2} \\ 0 & -\frac{\xi_1\xi_2}{|\xi|^2} & 1-\frac{\xi_2^2}{|\xi|^2} & -\frac{\xi_2\xi_3}{|\xi|^2} & \cdots & -\frac{\xi_2\xi_d}{|\xi|^2} \\ 0 & -\frac{\xi_1\xi_3}{|\xi|^2} & -\frac{\xi_2\xi_3}{|\xi|^2} & 1-\frac{\xi_3^2}{|\xi|^2} & \cdots & -\frac{\xi_3\xi_d}{|\xi|^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & -\frac{\xi_1\xi_d}{|\xi|^2} & -\frac{\xi_2\xi_d}{|\xi|^2} & -\frac{\xi_3\xi_d}{|\xi|^2} & \cdots & 1-\frac{\xi_d^2}{|\xi|^2} \end{pmatrix}. \tag{6.1.27}$$

Since $\tilde{\mathbf{P}}A(\xi,\mathbf{u})$ in (6.1.27) is a symmetric symbolic matrix, it follows that

$$Re(i\mathbf{P}A(\xi,\mathbf{u})) = 0.$$

Then, by using (6.1.26), we have

$$Re(\mathbf{P}T_{iA}\Lambda^s J_\varepsilon\tilde{\mathbf{u}}^\varepsilon, \Lambda^s J_\varepsilon\tilde{\mathbf{u}}^\varepsilon)_0 \leq c(||\tilde{\mathbf{u}}^\varepsilon||_s,\bar{\rho})||\tilde{\mathbf{u}}^\varepsilon||_s^2. \tag{6.1.28}$$

Putting it all together in (6.1.25), we get

$$\frac{d}{dt}||\tilde{\mathbf{u}}^\varepsilon||_s^2 \leq c(||\tilde{\mathbf{u}}^\varepsilon||_s,\bar{\rho})||\tilde{\mathbf{u}}^\varepsilon||_s^2. \tag{6.1.29}$$

Let T_ε be the maximum time of existence of the solution to system (6.1.19)-(6.1.15). We want to show that there exists a time $T>0$, which is independent of ε, such that $T \leq T_\varepsilon$ for every $\varepsilon > 0$. From the statement of Theorem 6.1.1, there exists a constant M such that $||\tilde{\mathbf{u}}_0^\varepsilon||_s \leq M$. Fixed a constant value $\tilde{M} > M$, let $T_0^\varepsilon \leq T_\varepsilon$ be a positive time such that the smooth solution $\tilde{\mathbf{u}}^\varepsilon$ verifies

$$sup_{\ 0\leq\tau\leq T_0^\varepsilon}\ ||\tilde{\mathbf{u}}^\varepsilon(\tau)||_s \leq \tilde{M}. \tag{6.1.30}$$

By (6.1.29), we get

$$||\tilde{\mathbf{u}}^\varepsilon(t)||_s \leq ||\tilde{\mathbf{u}}_0^\varepsilon||_s e^{c(\tilde{M},\bar{\rho})t} \tag{6.1.31}$$

for $t\in[0,T_0^\varepsilon]$. Let T, with $0 < T \leq T_0^\varepsilon$, be such that

$$Me^{c(\tilde{M},\bar{\rho})T} \leq \tilde{M}.$$

This yields

$$T \leq \frac{log(\frac{\tilde{M}}{M})}{c(\tilde{M},\bar{\rho})}. \tag{6.1.32}$$

Since $M,\tilde{M},\bar{\rho}$ are independent of the parameter ε, estimate (6.1.32) implies that T is independent of ε and $\tilde{\mathbf{u}}^\varepsilon$ is uniformly bounded provided that $T \leq \dfrac{log(\frac{\tilde{M}}{M})}{c(\tilde{M},\bar{\rho})}$. □

We also need a uniform bound for the time derivatives $\partial_t\tilde{\mathbf{u}}^\varepsilon(t)$, which is easily obtained from (6.1.19) and (6.1.30). Thus, we have

$$||\partial_t\tilde{\mathbf{u}}^\varepsilon(t)||_{s-1} \leq c(M,\tilde{M},\bar{\rho}) \qquad \text{for} \quad t\in[0,T]. \tag{6.1.33}$$

6.1.1 Convergence to the compressible-incompressible model - I method

This section is devoted to the proof of the following theorem:

Theorem 6.1.2. *Let $\tilde{\mathbf{u}}_0 = (\rho_0 - \bar{\rho}, v_0)$ be the translated initial data in (6.0.12), $\tilde{\mathbf{u}}_0$ belonging to $H^m(\mathbb{R}^d)$, with $m > [d/2]+1$ integer. There is a positive time T, such that there exists the unique $\tilde{\mathbf{u}} \in C([0,T], H^m(\mathbb{R}^d)) \cap C^1([0,T], H^{m-1}(\mathbb{R}^d))$ and a function P such that $\nabla P \in C([0,T], H^{m-1}(\mathbb{R}^d))$ which solve (6.0.11). The solution $(\tilde{\mathbf{u}}, P)$ to (6.0.11) is the limit of the sequence of the solutions to the approximating system (6.1.14) with initial data (6.1.15).*

Proof. Let us consider the uniform bounds that we have just proved in (6.1.30) and (6.1.33):

$$sup_{\ 0\le t\le T}||\tilde{\mathbf{u}}^\varepsilon||_m \le M_1, \tag{6.1.34}$$

and

$$sup_{\ 0\le t\le T}||\partial_t\tilde{\mathbf{u}}^\varepsilon||_{m-1} \le M_2, \tag{6.1.35}$$

for fixed constant values M_1, M_2. Now, we need the following Lemma.

Lemma 6.1.2. *The sequence of the solutions to the approximating system (6.1.19)-(6.1.15) is a Cauchy sequence in $C([0,T], L^2(\mathbb{R}^d))$.*

Proof. For $\varepsilon, \varepsilon' > 0$, let $\tilde{\mathbf{u}}^\varepsilon, \tilde{\mathbf{u}}^{\varepsilon'}$ two solutions to (6.1.19)-(6.1.15). We have

$$\frac{1}{2}\frac{d}{dt}||\tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'}||_0^2$$

$$+\sum_{j=1}^d(\tilde{\mathbf{P}}(J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon+\bar{\mathbf{u}})\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}^\varepsilon - J_{\varepsilon'}A_j(J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}})\partial_{x_j}J_{\varepsilon'}\tilde{\mathbf{u}}^{\varepsilon'}), \tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'})_0$$

$$\begin{aligned}
&= \frac{1}{2}\frac{d}{dt}||\tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'}||_0^2 + \sum_{j=1}^d\{(\tilde{\mathbf{P}}[(J_\varepsilon - J_{\varepsilon'})A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon+\bar{\mathbf{u}}))\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}^\varepsilon], \tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'})_0\\
&+(\tilde{\mathbf{P}}[J_{\varepsilon'}(A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon+\bar{\mathbf{u}})) - A_j(J_\varepsilon(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}})))\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}^\varepsilon], \tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'})_0\\
&+(\tilde{\mathbf{P}}[J_{\varepsilon'}(A_j(J_\varepsilon(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}})) - A_j(J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}})))\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}^\varepsilon], \tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'})_0\\
&+(\tilde{\mathbf{P}}[J_{\varepsilon'}A_j(J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}}))\partial_{x_j}(J_\varepsilon - J_{\varepsilon'})\tilde{\mathbf{u}}^\varepsilon], \tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'})_0\\
&+(\tilde{\mathbf{P}}[J_{\varepsilon'}A_j(J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}}))\partial_{x_j}J_{\varepsilon'}(\tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'})], \tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'})_0\\
&= \frac{1}{2}\frac{d}{dt}||\tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'}||_0^2 + I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Here,

$$|I_1| \le c(M_1)\max\{\varepsilon, \varepsilon'\}||\tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'}||_0,$$

where the last inequality follows from the Sobolev embedding theorem and from (6.1.34). For I_3, I_4 we get a similar estimate, while

$$|I_2| \le c(M_1)||\tilde{\mathbf{u}}^\varepsilon - \tilde{\mathbf{u}}^{\varepsilon'}||_0^2.$$

In order to estimate I_5, we use the paradifferential techniques, as done before to get the energy estimates. Then, by using (6.1.22), we write

$$I_5 = Re(\tilde{\mathbf{P}} J_{\varepsilon'} T_{iA(J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}}))} J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'}), \tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'})_0$$

$$+\sum_{j=1}^{d}(\tilde{\mathbf{P}}[J_{\varepsilon'}(A_j(J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}})) - T_{A_j(J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}}))})\partial_{x_j} J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'})], \tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'})_0 = I_5' + I_5''. \tag{6.1.36}$$

The first term of (6.1.36) can be estimated through the same argument that leads to (6.1.28). This way

$$|I_5'| \leq c(M_1)||\tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'}||_0^2.$$

Now, applying Lemma 7.1.5 in [55] and using again (6.1.34), we get

$$||\sum_{j=1}^{d}(A_j(J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}})) - T_{A_j(J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon'}+\bar{\mathbf{u}}))})\partial_{x_j} J_{\varepsilon'}(\tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'})||_0 \leq c(M_1)||\tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'}||_0.$$

Thus, the symmetric property of $\tilde{\mathbf{P}}$, the divergence free condition of $\tilde{\mathbf{u}}^{\varepsilon}, \tilde{\mathbf{u}}^{\varepsilon'}$, and the Cauchy-Schwarz inequality imply that

$$|I_5''| \leq c(M_1)||\tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'}||_0^2.$$

Putting it all together, we have

$$\frac{d}{dt}||\tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'}||_0 \leq c(M_1)(\max\{\varepsilon, \varepsilon'\} + ||\tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'}||_0),$$

and so, since $\tilde{\mathbf{u}}^{\varepsilon}(0,x) = \tilde{\mathbf{u}}^{\varepsilon'}(0,x)$ in (6.1.15), by the Gronwall inequality

$$\sup_{t\in[0,T]} ||\tilde{\mathbf{u}}^{\varepsilon} - \tilde{\mathbf{u}}^{\varepsilon'}||_0 \leq c(M_1, T)\max\{\varepsilon, \varepsilon'\}.$$

□

Lemma 6.1.2 implies that there exists $\tilde{\mathbf{u}}^{\star} \in C([0,T], L^2(\mathbb{R}^d))$ such that

$$\tilde{\mathbf{u}}^{\varepsilon} \to \tilde{\mathbf{u}}^{\star} \quad \text{in} \;\; C([0,T], L^2(\mathbb{R}^d)) \;\; \text{as } \varepsilon \to 0.$$

Furthermore, by using (6.1.34) together with the interpolation lemma in Sobolev spaces, see [9], for $m' < m$ we get

$$\tilde{\mathbf{u}}^{\varepsilon} \to \tilde{\mathbf{u}}^{\star} \quad \text{in} \;\; C([0,T], H^{m'}(\mathbb{R}^d)) \;\; \text{as } \varepsilon \to 0. \tag{6.1.37}$$

Next, from (6.1.34), $\tilde{\mathbf{u}}^{\varepsilon}$ is uniformly bounded in $L^2([0,T], H^m(\mathbb{R}^d))$, so there exists a subsequence such that

$$\tilde{\mathbf{u}}^{\varepsilon} \rightharpoonup \tilde{\mathbf{u}}^{\star} \;\; \text{in} \;\; L^2([0,T], H^m(\mathbb{R}^d)).$$

Besides, for fixed $t \in [0,T]$, $\tilde{\mathbf{u}}^{\varepsilon}(t)$ is uniformly bounded in $H^m(\mathbb{R}^d)$, so $\tilde{\mathbf{u}}^{\star}(t)$ is bounded in $H^m(\mathbb{R}^d)$ and this fact, together with $\tilde{\mathbf{u}}^{\star} \in L^2([0,T], H^m(\mathbb{R}^d))$, implies that $\tilde{\mathbf{u}}^{\star} \in L^{\infty}([0,T], H^m(\mathbb{R}^d))$.

Now, let $\psi \in C_c^\infty((0,T))$ and $\phi = (\rho, v)$ so that $v \in V^0 = \{v \in L^2(\mathbb{R}^d) \mid \nabla \cdot v = 0\}$ with compact support. The weak formulation of system (6.1.19) is

$$\int_0 -\psi'(t)(\tilde{\mathbf{u}}^\varepsilon, \phi)_0 \, dt + \sum_{j=1}^d \int_0 \psi(t)(\mathbf{P} J_\varepsilon A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \phi)_0 \, dt = 0. \tag{6.1.38}$$

Passing to the limit in (6.1.38) and using (6.1.37), we get

$$\int_0 -\psi'(t)(\tilde{\mathbf{u}}^\star, \phi)_0 \, dt + \sum_{j=1}^d \int_0 \psi(t)(\mathbf{P} A_j(\tilde{\mathbf{u}}^\star + \bar{\mathbf{u}})\partial_{x_j} \tilde{\mathbf{u}}^\star, \phi)_0 \, dt = 0,$$

i.e.

$$\partial_t \tilde{\mathbf{u}}^\star + \sum_{j=1}^d \mathbf{P}(A_j(\tilde{\mathbf{u}}^\star + \bar{\mathbf{u}})\partial_{x_j} \tilde{\mathbf{u}}^\star) = 0 \tag{6.1.39}$$

in the sense of distributions, and so $\tilde{\mathbf{u}}^\star \in Lip([0,T], H^{m-1}(\mathbb{R}^d))$. Moreover, from (6.1.39) and the Helmholtz-Hodge decomposition theorem, there exists

$$\nabla P^\star \in L^\infty([0,T], H^{m-1}(\mathbb{R}^d)),$$

such that

$$\partial_t \tilde{\mathbf{u}}^\star + \sum_{j=1}^d A_j(\tilde{\mathbf{u}}^\star + \bar{\mathbf{u}})\partial_{x_j} \tilde{\mathbf{u}}^\star = (0, -\nabla P^\star). \tag{6.1.40}$$

Next, by using (6.1.35) and passing to a subsequence, we have $\partial_t \tilde{\mathbf{u}}^\varepsilon \rightharpoonup^* \partial_t \tilde{\mathbf{u}}^\star$ in $L^\infty([0,T], H^{m-1}(\mathbb{R}^d))$. Besides, (6.1.34) and (6.1.35) yield

$$sup_{\ 0 \le t \le T} \ ||\nabla P^\varepsilon||_{m-1} \le c(M_1, M_2),$$

and then

$$\nabla P^\varepsilon \rightharpoonup^* \nabla P^\star \quad in \quad L^\infty([0,T], H^{m-1}(\mathbb{R}^d)).$$

Now, we want to show that $\tilde{\mathbf{u}}^\star \in C([0,T], H^m_w(\mathbb{R}^d))$ Since $\tilde{\mathbf{u}}^\star \in C([0,T], H^{m'}(\mathbb{R}^d))$, then $\tilde{\mathbf{u}}^\star \in C([0,T], H^{m'}_w(\mathbb{R}^d))$, i.e., for all $\varepsilon > 0$, for all $\phi' \in H^{-m'}(\mathbb{R}^d)$, there exists $\delta > 0$ such that, for $|h| < \delta$,

$$|(\tilde{\mathbf{u}}^\star(t+h) - \tilde{\mathbf{u}}^\star(t), \phi')_{-m',m'}| \le \frac{\varepsilon}{2}.$$

Moreover, the density of $H^{-m'} \subset H^{-m}$ $(m' < m)$ implies that, for all $\varepsilon > 0$ and for all $\phi \in H^{-m}(\mathbb{R}^d)$, there exists $\phi' \in H^{-m'}(\mathbb{R}^d)$ such that

$$||\phi - \phi'||_{-m} \le \frac{\varepsilon}{4M_1},$$

where M_1 is the uniform bound in (6.1.34). Putting it all together, we get

$$|(\tilde{\mathbf{u}}^\star(t+h) - \tilde{\mathbf{u}}^\star(t), \phi)_{-m,m}|$$

$$\le |(\tilde{\mathbf{u}}^\star(t+h) - \tilde{\mathbf{u}}^\star(t), \phi - \phi')_{-m,m}| + |(\tilde{\mathbf{u}}^\star(t+h) - \tilde{\mathbf{u}}^\star(t), \phi')_{-m',m'}|$$

$$\leq 2M_1||\phi - \phi'||_{-m} + |(\tilde{\mathbf{u}}^\star(t+h) - \tilde{\mathbf{u}}^\star(t), \phi')_{-m',m'}| \leq \varepsilon,$$

namely $\tilde{\mathbf{u}}^\star$ belongs to $C([0,T], H^m_w(\mathbb{R}^d))$. Putting it all together, we have

$$\tilde{\mathbf{u}}^\star \in L^\infty([0,T], H^m(\mathbb{R}^d)) \cap Lip([0,T], H^{m-1}(\mathbb{R}^d)) \cap C_w([0,T], H^m(\mathbb{R}^d)),$$

and

$$\nabla P^\star \in L^\infty([0,T], H^{m-1}(\mathbb{R}^d)) \cap C_w([0,T], H^{m-1}(\mathbb{R}^d)).$$

The additional regularity $\tilde{\mathbf{u}} \in C([0,T], H^m(\mathbb{R}^d)) \cap C^1([0,T], H^{m-1}(\mathbb{R}^d))$ can be achieved in a standard way, following [9]. We sketch the proof. First, it is sufficient to prove that $\tilde{\mathbf{u}}^\star \in C([0,T], H^m(\mathbb{R}^d))$, since the regularity $C^1([0,T], H^{m-1}(\mathbb{R}^d))$ follows directly from (6.1.40). Moreover, we only need to prove the continuity of $\tilde{\mathbf{u}}^\star$ in the strong norm $||\cdot||_m$ at the time $t=0$, in fact the same argument can be adapted to any other time $\tilde{T}$, $0 \leq \tilde{T} \leq T$. Furthermore, since system (6.1.40) is time reversible, it is sufficient to prove just the right continuity at time $t=0$ in the strong norm $||\cdot||_m$. From (6.1.34), passing to a subsequence, we have

$$\limsup_{\varepsilon\to 0} ||\tilde{\mathbf{u}}^\varepsilon||_m \geq ||\tilde{\mathbf{u}}^\star||_m.$$

Moreover, from (6.1.31),

$$||\tilde{\mathbf{u}}^\varepsilon||_m \leq e^{c(\tilde{M},\bar{\rho})t} ||\tilde{\mathbf{u}}^\varepsilon_0||_m.$$

This implies

$$\sup_{0\leq t\leq T} ||\tilde{\mathbf{u}}^\varepsilon||_m - ||\tilde{\mathbf{u}}^\varepsilon_0||_m \leq e^{c(\tilde{M},\bar{\rho})T} ||\tilde{\mathbf{u}}^\varepsilon_0||_m - ||\tilde{\mathbf{u}}^\varepsilon_0||_m.$$

Last estimates give

$$\limsup_{t\to 0^+} ||\tilde{\mathbf{u}}^\star||_m \leq ||\tilde{\mathbf{u}}_0||_m.$$

Now, since $\tilde{\mathbf{u}}^\star \in C_w([0,T], H^m(\mathbb{R}^d))$,

$$\liminf_{t\to 0^+} ||\tilde{\mathbf{u}}^\star||_m \geq ||\tilde{\mathbf{u}}_0||_m.$$

In particular,

$$\lim_{t\to 0^+} ||\tilde{\mathbf{u}}^\star(t)||_m = ||\tilde{\mathbf{u}}_0||_m.$$

Then, the strong right continuity at $t=0$ is proved. □

Remark 6.1.2. *This kind of approximation does not work on system (6.0.5), since* $\frac{\nabla P}{\rho}$ *is not a gradient and it cannot be eliminated by applying the projector operator to the system.*

6.1.2 Uniqueness

We end this section with the proof of uniqueness of the solution to the Cauchy problem (6.0.11)-(6.0.12).

Theorem 6.1.3. *There is a unique solution* $\tilde{\mathbf{u}}$ *to problem (6.0.11)-(6.0.12) in the space* $L^\infty([0,T], Lip(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))$.

Proof. According to Definition 6.0.1, let $\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2$ and P_1, P_2 be two solutions to system (6.0.11), with initial data (6.0.12). We have

$$\partial_t(\tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1) + \sum_{j=1}^{d}\{A_j(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}})\partial_{x_j}(\tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1) + (A_j(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}}) - A_j(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}}))\partial_{x_j}\tilde{\mathbf{u}}_1\}$$

$$+(0, \nabla P_2 - \nabla P_1) = 0. \tag{6.1.41}$$

Applying the operator $\tilde{\mathbf{P}}$ to (6.1.41), we get

$$\partial_t(\tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1) + \sum_{j=1}^{d}\tilde{\mathbf{P}}\{A_j(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}})\partial_{x_j}(\tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1) + (A_j(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}}) - A_j(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}}))\partial_{x_j}\tilde{\mathbf{u}}_1\} = 0.$$

As done before, we write the paradifferential version of the previous equation

$$\partial_t(\tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1) + \tilde{\mathbf{P}}T_{iA(\xi,\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})}(\tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1) + \tilde{\mathbf{P}}(T_{iA(\xi,\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})} - T_{iA(\xi,\tilde{\mathbf{u}}_1+\bar{\mathbf{u}})})\tilde{\mathbf{u}}_1$$

$$= \tilde{\mathbf{P}}[T_{iA(\xi,\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})} - \sum_{j=1}^{2} A_j(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}})\partial_{x_j}](\tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1)$$

$$+\tilde{\mathbf{P}}[T_{iA(\xi,\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})} - T_{iA(\xi,\tilde{\mathbf{u}}_1+\bar{\mathbf{u}})} - \sum_{j=1}^{d}(A_j(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}}) - A_j(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}}))\partial_{x_j}]\tilde{\mathbf{u}}_1.$$

We set $\mathbf{w} := \tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1$ and take the scalar product with $\mathbf{w}$. From [55] and (6.1.22) we have

$$|(\tilde{\mathbf{P}}(T_{iA(\xi,\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})} - T_{iA(\xi,\tilde{\mathbf{u}}_1+\bar{\mathbf{u}})})\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1)_0|$$

$$= |(\tilde{\mathbf{P}}\sum_{j=1}^{d}(T_{A_j(\tilde{\mathbf{u}}_2+\bar{\mathbf{u}})} - T_{A_j(\tilde{\mathbf{u}}_1+\bar{\mathbf{u}})})\partial_{x_j}\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1)_0|$$

$$= |(\mathcal{F}^{-1}\{[\sum_{\alpha}\partial_\xi^\alpha\tilde{\mathbf{P}}^\chi(\xi)\sum_{j=1}^{d}(D_x^\alpha A_j^\chi(\tilde{\mathbf{u}}_2 + \bar{\mathbf{u}}) - D_x^\alpha A_j^\chi(\tilde{\mathbf{u}}_1 + \bar{\mathbf{u}}))]\mathcal{F}(\partial_{x_j}\tilde{\mathbf{u}}_1)\}, \tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1)_0|$$

$$\le c(|\nabla\tilde{\mathbf{u}}_1|_\infty)||\tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1||_0^2,$$

where $\mathcal{F}$ is the Fourier transform, and $\tilde{\mathbf{P}}^\chi A_j^\chi$, $j = 1, \cdots, d$ are the regularized versions of $\tilde{\mathbf{P}}, A_j$, through the standard Littlewood-Paley decomposition, see Section 2. Thus, we obtain

$$\frac{d}{dt}||\mathbf{w}||_0^2 \le c(|\tilde{\mathbf{u}}_1|_\infty, |\tilde{\mathbf{u}}_2|_\infty, |\nabla\tilde{\mathbf{u}}_1|_\infty, |\nabla\tilde{\mathbf{u}}_2|_\infty)||\mathbf{w}||_0^2,$$

i.e. $\mathbf{w} = 0$, since $\tilde{\mathbf{u}}_1(0, x) = \tilde{\mathbf{u}}_2(0, x) = \tilde{\mathbf{u}}_0 = (\rho_0(x) - \bar{\rho}, v_0(x))$ in (6.0.12). □

6.2 The continuous projection approximation

First of all, we want to point out that, although it involves again the use of the Leray projector, the idea inside this other kind of approximation is quite different from that discussed before. Roughly speaking, the main feature is that we will use the projector operator as a singular source term. The divergence of the velocity field vanishes as long as the parameter ε goes to zero and this is the reason why this approximation can be viewed as a continuous version of the Chorin-Temam projection method. Moreover, the proof below can be adapted with slight modifications to the classical discrete version of the projection method, providing a constructive solution to system (6.0.1). Now, let us go back to the original system (6.0.1). Unlike (6.0.9), (6.0.10), we consider the compact formulation of system (6.0.1), where the mass balance equation is

$$\partial_t \rho + \nabla \cdot (\rho v) = 0.$$

This way, we need to redefine the matrices $A_j, j = 1, \cdots, d$ in (6.0.7). Namely, now we have

$$A_1(\mathbf{u}) = \begin{pmatrix} v_1 & \rho & 0 \\ f(\mathbf{u}) & v_1 & 0 \\ 0 & 0 & v_1 \end{pmatrix}, \; A_2(\mathbf{u}) = \begin{pmatrix} v_2 & 0 & \rho \\ 0 & v_2 & 0 \\ f(\mathbf{u}) & 0 & v_2 \end{pmatrix} \tag{6.2.42}$$

in the two dimensional case, and, in the general d-dimensional case

$$A_j(\mathbf{u}) = \begin{pmatrix} v_j & \delta_{1j}\rho & \delta_{2j}\rho & \cdots & \delta_{dj}\rho \\ \delta_{1j}f(\mathbf{u}) & v_j & 0 & \cdots & 0 \\ \delta_{2j}f(\mathbf{u}) & 0 & v_j & \cdots & 0 \\ \cdots & \cdots & \cdots & v_j & \cdots \\ \delta_{dj}f(\mathbf{u}) & 0 & 0 & \cdots & v_j \end{pmatrix} \tag{6.2.43}$$

for $j = 1, \cdots, d$. First, let us neglect for a while the incompressible pressure term F_P. It is easy to check that (6.0.7) with A_j in (6.2.43) is a Friedrichs symmetrizable hyperbolic system, whose classical symmetrizer is the diagonal $(d+1) \times (d+1)$ matrix

$$A_0(\mathbf{u}) = diag\left(\frac{f(\mathbf{u})}{\rho}, 1, 1, \cdots, 1\right), \tag{6.2.44}$$

and

$$A_0 A_j = \begin{pmatrix} \frac{f(\mathbf{u})v_j}{\rho} & \delta_{1j}f(\mathbf{u}) & \delta_{2j}f(\mathbf{u}) & \cdots & \delta_{dj}f(\mathbf{u}) \\ \delta_{j1}f(\mathbf{u}) & v_j & 0 & \cdots & 0 \\ \delta_{j2}f(\mathbf{u}) & 0 & v_j & \cdots & 0 \\ \cdots & \cdots & \cdots & v_j & \cdots \\ \delta_{jd}f(\mathbf{u}) & 0 & 0 & \cdots & v_j \end{pmatrix},$$

for $j = 1, \cdots d$.

Remark 6.2.1. *We point out that, for $A_0(\boldsymbol{u})$ to be a classical symmetrizer, we need $\rho, f(\boldsymbol{u}) > 0$. The second one will be an assumption, as we are going to precise in the following, while here we discuss the first one. Taking into account the density equation in (6.0.6), it can be seen that, if the initial datum ρ_0 in (6.0.2) does not vanish for all $x \in \mathbb{R}^d$, then, under some standard assumptions of regularity, $\rho(t, x)$ cannot vanish too, as we will see* a posteriori. *However, we are going to translate again the density variable, as done in the previous section, but here, unlike Section 6.1, see Remark 6.0.1, we assume $\bar{\rho} > 0$.*

Applying the symmetrizer $A_0(\mathbf{u})$ to the compact system (6.0.7), with $A_j, j = 1, \cdots, d$ in (6.2.42), (6.2.43), we get the symmetric formulation

$$A_0(\mathbf{u})\partial_t \mathbf{u} + \sum_{j=1}^{d} A_0 A_j(\mathbf{u})\partial_{x_j}\mathbf{u} + A_0(\mathbf{u})F_P = 0.$$

We want to focus on the fact that

$$A_0(\mathbf{u})F_P = diag\left(\frac{f(\mathbf{u})}{\rho}, 1, 1, \cdots, 1\right) \cdot (0, \nabla P) = (0, \nabla P) = F_P, \tag{6.2.45}$$

namely the A_0-scalar product preserves the gradient function ∇P and this is the reason why, in the following, we will be able perform classical energy estimates, despite the presence of the gradient of the incompressible pressure or, equivalently, the Leray projector. Thus, (6.2.45) yields

$$A_0(\mathbf{u})\partial_t \mathbf{u} + \sum_{j=1}^{d} A_0 A_j(\mathbf{u})\partial_{x_j}\mathbf{u} + F_P = 0.$$

To get uniform energy estimates, $f(\mathbf{u})$ has to satisfy some properties, then here we provide the definition of *admissible* scalar functions $f(\mathbf{u})$.

Definition 6.2.1. *The scalar function $f(\boldsymbol{u})$ in (6.0.7) is an* admissible *function if*

- $f(\boldsymbol{u})$ *is strictly positive;*
- $\nabla_v f(\boldsymbol{u}) = \alpha(\rho, |v|)v$, *where α is a positive and continuous scalar function, only depending on the density ρ and the norm $|v|$ of the velocity field.*

An example of an admissible function is given by

$$f(\boldsymbol{u}) = \bar{f} + \beta(\rho, v^2), \qquad \bar{f} + \beta(\rho_0, v_0^2) \geq 0,$$

where $\bar{f}$ is a constant value and $\nabla_v \beta(\rho, v^2) = 2\partial_{v^2}\beta(\rho, v^2)v$, with $\partial_{v^2}\beta(\rho, v^2) \geq 0$.

Following [9] and [75], we look for a suitable approximation to system (6.0.1), which is:

$$A_0(J_\varepsilon \mathbf{u}^\varepsilon)\partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^{d} J_\varepsilon A_0 A_j(J_\varepsilon \mathbf{u}^\varepsilon)\partial_{x_j} J_\varepsilon \mathbf{u}^\varepsilon + (0, \nabla P^\varepsilon) = 0,$$

where $\mathbf{u}^\varepsilon = (\rho^\varepsilon, v^\varepsilon)$, $A_j, j = 1, \cdots, d$ in (6.2.42), (6.2.43), A_0 in (6.2.44), and v^ε is no more divergence free. We choose the approximating sequence ∇P^ε so that, for each fixed ε, ∇P^ε is proportional to the gradient part of v^ε. Namely, using the Helmholtz-Hodge decomposition theorem, we can set

$$v^\varepsilon = \mathbb{P}v^\varepsilon + \varepsilon \nabla P^\varepsilon.$$

This way,

$$\nabla P^\varepsilon = \frac{(\mathbb{I} - \mathbb{P})v^\varepsilon}{\varepsilon}.$$

Then, the approximating system becomes

$$A_0(J_\varepsilon \mathbf{u}^\varepsilon)\partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^{d} J_\varepsilon A_0 A_j(J_\varepsilon \mathbf{u}^\varepsilon)\partial_{x_j} J_\varepsilon \mathbf{u}^\varepsilon = -(0, \frac{(\mathbb{I}-\mathbb{P})v^\varepsilon}{\varepsilon}), \tag{6.2.46}$$

with initial data

$$\rho_0^\varepsilon(x) = \rho_0(0,x), \qquad v_0^\varepsilon(0,x) = v_0(x) + \varepsilon v_0^1(x), \tag{6.2.47}$$

with ρ_0, v_0 in (6.0.2) and $v_0^1(x) \in H^m(\mathbb{R}^d)$.

Remark 6.2.2. *Similarly to the incompressible limit of the Euler equations in [52], [47], the "slightly compressible" form of the initial data in (6.1.15) guarantees the uniform bound of the time derivative of v^ε in the L^2-norm, as we will see later.*

According to Remark 6.2.1, as done in (6.0.11), we translate the approximating system (6.2.46) and the related initial data (6.2.47). Setting $\tilde{\mathbf{u}}^\varepsilon = (\tilde{\rho}^\varepsilon, \tilde{v}^\varepsilon) = (\rho^\varepsilon - \bar{\rho}, v^\varepsilon)$, we get

$$A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\partial_t \tilde{\mathbf{u}}^\varepsilon + \sum_{j=1}^{d} J_\varepsilon A_0 A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon = -(0, \frac{(\mathbb{I}-\mathbb{P})v^\varepsilon}{\varepsilon}), \tag{6.2.48}$$

with

$$\tilde{\mathbf{u}}_0^\varepsilon = (\tilde{\rho}_0^\varepsilon, \tilde{v}_0^\varepsilon) = (\rho_0^\varepsilon - \bar{\rho}, v_0^\varepsilon), \tag{6.2.49}$$

and $\rho_0^\varepsilon, v_0^\varepsilon$ in (6.2.47). We prove the following theorem.

Theorem 6.2.1. *(Local existence of approximating solutions for the second approximation) Let $\tilde{\mathbf{u}}_0^\varepsilon = (\tilde{\rho}_0^\varepsilon, \tilde{v}_0^\varepsilon) \in H^m(\mathbb{R}^d)$ as in (6.2.49) and $m \in \mathbb{N}$, with $m > [d/2]+1$. Then, for any $\varepsilon > 0$, there exists a time T, independent of ε, such that system (6.2.48) has a unique solution $\tilde{\mathbf{u}}^\varepsilon = (\tilde{\rho}^\varepsilon, \tilde{v}^\varepsilon) \in C^1([0,T], H^m(\mathbb{R}^d))$.*

Proof. Once applied the Picard theorem in [9], we get uniform energy estimates to start the compactness tools. Comparing to Section 6.1 , here we just have to consider $F_2^\varepsilon(\tilde{\mathbf{u}}^\varepsilon) = (0, \frac{(\mathbb{I}-\mathbb{P})v^\varepsilon}{\varepsilon})$. Then

$$||F_2^\varepsilon(\tilde{\mathbf{u}}_1) - F_2^\varepsilon(\tilde{\mathbf{u}}_2)||_m = (0, \frac{||(\mathbb{I}-\mathbb{P})(\tilde{v}_1 - \tilde{v}_2)||_m}{\varepsilon}) \leq (0, \frac{1}{\varepsilon}||\tilde{v}_1 - \tilde{v}_2||_m) \leq \frac{1}{\varepsilon}||\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2||_m.$$

Putting it all together, we get

$$||F^\varepsilon(\tilde{\mathbf{u}}_1) - F^\varepsilon(\tilde{\mathbf{u}}_2)||_m \leq c(||\tilde{\mathbf{u}}_1||_m, ||\tilde{\mathbf{u}}_2||_m, \bar{\rho}, \varepsilon^{-1})||\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2||_m.$$

Thus, for fixed ε, F^ε is locally Lipschitz continuous on any open set

$$\mathcal{U}^M = \{\tilde{\mathbf{u}}^\varepsilon \in H^m(\mathbb{R}^d) : ||\tilde{\mathbf{u}}^\varepsilon||_m < M\}.$$

From the Picard theorem, there exists the unique solution $\tilde{\mathbf{u}}^\varepsilon \in C^1([0,T_\varepsilon), \mathcal{U}^M)$, for any $T_\varepsilon > 0$. Now, we need a uniform bound on $\tilde{\mathbf{u}}^\varepsilon$ in the higher H^m-norm. From (6.2.48), we have

$$\partial_t \tilde{\mathbf{u}}^\varepsilon = -\sum_{j=1}^{d} A_0^{-1} J_\varepsilon A_0 A_j(J_\varepsilon(\tilde{\mathbf{u}^\varepsilon} + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon - \left(0, \frac{(\mathbb{I}-\mathbb{P})v^\varepsilon}{\varepsilon}\right).$$

Taking the α-derivative for $|\alpha| \leq m$, we get

$$\partial_t D^\alpha \tilde{\mathbf{u}}^\varepsilon + \sum_{j=1}^d A_0^{-1} J_\varepsilon A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}^\varepsilon} + \bar{\mathbf{u}}))\partial_{x_j} D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon + \left(0, \frac{(\mathbb{I}-\mathbb{P})D^\alpha v^\varepsilon}{\varepsilon}\right) = F_\alpha, \quad (6.2.50)$$

where

$$F_\alpha = -\sum_{j=1}^d [D^\alpha (A_0^{-1} J_\varepsilon A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}^\varepsilon} + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon) - A_0^{-1} J_\varepsilon A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}^\varepsilon} + \bar{\mathbf{u}}))\partial_{x_j} D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon].$$

Multiplying (6.2.50) by $D^\alpha \tilde{\mathbf{u}}^\varepsilon$ through the A_0 inner product $(A_0 \cdot, \cdot)_0$, where A_0 is the symmetrizer in (6.2.44), we obtain

$$\frac{1}{2}\frac{d}{dt}(A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0 + \frac{1}{\varepsilon}((\mathbb{I}-\mathbb{P})D^\alpha v^\varepsilon, D^\alpha v^\varepsilon)_0$$

$$= \frac{1}{2}(\partial_t A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0 + \sum_{j=1}^d (\partial_{x_j}(A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0$$

$$+(A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))F_\alpha, D^\alpha \tilde{\mathbf{u}}^\varepsilon)_0.$$

This implies that

$$\frac{1}{2}\frac{d}{dt}(A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0 + \frac{1}{\varepsilon}((\mathbb{I}-\mathbb{P})D^\alpha v^\varepsilon, D^\alpha v^\varepsilon)_0$$

$$\leq c(|\tilde{\mathbf{u}}^\varepsilon|_\infty, |\nabla \tilde{\mathbf{u}}^\varepsilon|_\infty, \bar{\rho})||D^\alpha \tilde{\mathbf{u}}^\varepsilon||_0^2 + c(|\tilde{\mathbf{u}}^\varepsilon|_\infty)||F_\alpha||_0||D^\alpha \tilde{\mathbf{u}}^\varepsilon||_0,$$

where we are able to control $\frac{1}{2}(\partial_t A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0$ thanks to the properties of $f(\mathbf{u})$ in Definition 6.2.1. Now,

$$||F_\alpha||_0$$

$$= \left|\left| \sum_{j=1}^d [D^\alpha (A_0^{-1} J_\varepsilon A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}^\varepsilon} + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon) - A_0^{-1} J_\varepsilon A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}^\varepsilon} + \bar{\mathbf{u}}))\partial_{x_j} D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon] \right|\right|_0$$

$$\leq \sum_{j=1}^d \{|D(A_0^{-1} J_\varepsilon A_0 A_j (\tilde{\mathbf{u}^\varepsilon} + \bar{\mathbf{u}}))|_\infty ||D^{m-1}\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon||_0$$

$$+||D^m (A_0^{-1} J_\varepsilon A_0 A_j (\tilde{\mathbf{u}^\varepsilon} + \bar{\mathbf{u}}))||_0 |\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon|_\infty\}$$

and then, using Remark 6.1.1,

$$\leq c(|\tilde{\mathbf{u}}^\varepsilon|_\infty, |\nabla \tilde{\mathbf{u}}^\varepsilon|_\infty, \bar{\rho})||D^m \tilde{\mathbf{u}}^\varepsilon||_0^2.$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}(A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0 + \frac{1}{\varepsilon}((\mathbb{I}-\mathbb{P})D^\alpha v^\varepsilon, D^\alpha v^\varepsilon)_0 \quad (6.2.51)$$

$$\leq c(|\tilde{\mathbf{u}}^\varepsilon|_\infty, |\nabla \tilde{\mathbf{u}}^\varepsilon|_\infty, \bar{\rho})||D^m \tilde{\mathbf{u}}^\varepsilon||_0^2.$$

Notice that the Helmholtz-Hodge decomposition theorem provides a positive sign for the source term in the left hand side of (6.2.51), i.e.

$$\frac{1}{\varepsilon}((\mathbb{I}-\mathbb{P})D^\alpha v^\varepsilon, D^\alpha v^\varepsilon)_0 = \frac{1}{\varepsilon}((\mathbb{I}-\mathbb{P})D^\alpha v^\varepsilon, D^\alpha((\mathbb{I}-\mathbb{P})v^\varepsilon + \mathbb{P}v^\varepsilon))_0 = \frac{1}{\varepsilon}||(\mathbb{I}-\mathbb{P})v^\varepsilon||_0^2.$$

Summing up to $|\alpha| \leq m$, we have

$$\frac{d}{dt}\sum_{|\alpha|\leq m}(A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, D^\alpha J_\varepsilon \tilde{\mathbf{u}}^\varepsilon)_0 \leq c(|\tilde{\mathbf{u}}^\varepsilon|_\infty, |\nabla \tilde{\mathbf{u}}^\varepsilon|_\infty, \bar{\rho})||\tilde{\mathbf{u}}^\varepsilon||_m^2.$$

Since A_0 is positive definite and using the properties of mollifiers, last estimate yields

$$\frac{d}{dt}||\tilde{\mathbf{u}}^\varepsilon||_m^2 \leq c(\tilde{\mathbf{u}}^\varepsilon|_\infty, |\nabla \tilde{\mathbf{u}}^\varepsilon|_\infty, \bar{\rho})||\tilde{\mathbf{u}}^\varepsilon||_m^2. \tag{6.2.52}$$

As seen in the previous section, estimate (6.2.52) gives

$$||\tilde{\mathbf{u}}^\varepsilon(t)||_m \leq M \quad \text{ for } t \in [0,T]. \tag{6.2.53}$$

□

To obtain a uniform bound for the time derivatives $\partial_t \tilde{\mathbf{u}}^\varepsilon$ in the low norm L^2, we take the time derivative of equation (6.2.48) and let

$$\mathbf{w}^\varepsilon := \partial_t \tilde{\mathbf{u}}^\varepsilon = (\partial_t \rho^\varepsilon, \partial_t v^\varepsilon).$$

Then, we have

$$\partial_t \mathbf{w}^\varepsilon + \sum_{j=1}^d A_0^{-1} J_\varepsilon A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \mathbf{w}^\varepsilon + \sum_{j=1}^d (A_0^{-1} J_\varepsilon A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}})))' J_\varepsilon \mathbf{w}^\varepsilon \partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon$$

$$= -\left(0, \frac{(\mathbb{I}-\mathbb{P})\partial_t v^\varepsilon}{\varepsilon}\right).$$

Taking the $(A_0\cdot,\cdot)_0$ inner product with $\mathbf{w}^\varepsilon$, we get

$$\frac{1}{2}\frac{d}{dt}(A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\mathbf{w}^\varepsilon, \mathbf{w}^\varepsilon)_0 + \frac{||(\mathbb{I}-\mathbb{P})\partial_t v^\varepsilon||_0^2}{\varepsilon}$$

$$= \frac{1}{2}((A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}})))' J_\varepsilon \mathbf{w}^\varepsilon \cdot \mathbf{w}^\varepsilon, \mathbf{w}^\varepsilon)_0 + \frac{1}{2}\sum_{j=1}^d ((A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}})))' \partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon \cdot J_\varepsilon \mathbf{w}^\varepsilon, J_\varepsilon \mathbf{w}^\varepsilon)_0$$

$$+ \sum_{j=1}^d (A_0(A_0^{-1} J_\varepsilon A_j (J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}})))' J_\varepsilon \mathbf{w}^\varepsilon \partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}^\varepsilon, \mathbf{w}^\varepsilon)_0.$$

We obtain

$$\frac{d}{dt}(A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\mathbf{w}^\varepsilon, \mathbf{w}^\varepsilon)_0 \leq c(|\tilde{\mathbf{u}}^\varepsilon|_\infty, |\nabla \tilde{\mathbf{u}}^\varepsilon|_\infty, \bar{\rho})||\mathbf{w}^\varepsilon||_0^2.$$

From (6.2.53), we have

$$\frac{d}{dt}(A_0(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon + \bar{\mathbf{u}}))\mathbf{w}^\varepsilon, \mathbf{w}^\varepsilon)_0 \le M||\mathbf{w}^\varepsilon||_0^2,$$

i.e.

$$||\mathbf{w}^\varepsilon(t)||_0^2 \le ||\mathbf{w}^\varepsilon(0)||_0^2 e^{Mt},$$

and

$$||\partial_t \tilde{\mathbf{u}}^\varepsilon(t)||_0^2 \le ||\partial_t \tilde{\mathbf{u}}^\varepsilon(0)||_0^2 e^{Mt}.$$

Then, $\partial_t \tilde{\mathbf{u}}^\varepsilon$ is uniformly bounded in $L^2(\mathbb{R}^d)$ for each $t \in [0,T]$, provided that $||\mathbf{w}^\varepsilon(0)||_0^2 = ||\partial_t \tilde{\mathbf{u}}^\varepsilon(0)||_0^2$ is uniformly bounded in ε. This is guaranteed by the structural conditions on the initial data in (6.2.47). In fact, from (6.2.48), we have

$$\partial_t^\varepsilon \tilde{\mathbf{u}}^\varepsilon(0) = -\sum_{j=1}^d A_0^{-1} J_\varepsilon A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}}_0^\varepsilon + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_0^\varepsilon - \left(0, \frac{(\mathbb{I}-\mathbb{P})v_0^\varepsilon}{\varepsilon}\right).$$

By using (6.2.47),

$$v_0^\varepsilon(x) = v_0(x) + \varepsilon v_0^1(x),$$

with $\nabla \cdot v_0(x) = 0$, namely $\mathbb{P}v_0 = v_0$ and $\frac{1}{\varepsilon}(\mathbb{I}-\mathbb{P})v_0 = 0$, and so

$$\partial_t \tilde{\mathbf{u}}^\varepsilon(0) = -\sum_{j=1}^d A_0^{-1} J_\varepsilon A_0 A_j (J_\varepsilon(\tilde{\mathbf{u}}_0^\varepsilon + \bar{\mathbf{u}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{u}}_0^\varepsilon - (\mathbb{I}-\mathbb{P})v_0^1(x).$$

Thus, since $\partial_t \tilde{\mathbf{u}}^\varepsilon(0)$ in uniformly bounded in $H^m(\mathbb{R}^d)$, we have

$$||\partial_t \tilde{\mathbf{u}}^\varepsilon(t)||_0 \le M.$$

Similarly, we get

$$||\partial_t \tilde{\mathbf{u}}^\varepsilon(t)||_{m-1} \le M. \tag{6.2.54}$$

6.2.1 Convergence to the compressible-incompressible system - II method

We prove the following theorem.

Theorem 6.2.2. *Let $\tilde{\mathbf{u}}_0 = (\tilde{\rho}_0, \tilde{v}_0)$ be the translated initial data in (6.0.12), $\tilde{\mathbf{u}}_0 \in H^m(\mathbb{R}^d)$ with $m > [d/2]+1$. There is a positive time T, such that there exists the unique $\tilde{\mathbf{u}} \in C([0,T], H^m(\mathbb{R}^d)) \cap C^1([0,T], H^{m-1}(\mathbb{R}^d))$ and an incompressible pressure P such that $\nabla P \in C([0,T], H^{m-1}(\mathbb{R}^d))$ which solve (6.0.11). The solution $(\tilde{\mathbf{u}}, P)$ to (6.0.11) is the limit of the sequence of the solutions to the approximating system (6.2.48) with initial data (6.2.49).*

Proof. The first part is completely analogous to Section 3. We start from some facts:

$$\begin{aligned}
&\tilde{\mathbf{u}}^\varepsilon \to \tilde{\mathbf{u}}^\star \ \text{ as } \ \varepsilon \to 0 \ \text{ in } \ C([0,T], H^{m'}(\mathbb{R}^d)) \ \text{ with } \ m' < m,\\
&\tilde{\mathbf{u}}^\varepsilon \rightharpoonup \tilde{\mathbf{u}}^\star \ \text{ as } \ \varepsilon \to 0 \ \text{ in } \ L^2([0,T], H^m(\mathbb{R}^d)),\\
&\tilde{\mathbf{u}}^\star \in L^\infty([0,T], H^m(\mathbb{R}^d)) \cap C([0,T], H^m_w(\mathbb{R}^d)).
\end{aligned}$$

From (6.2.48), (6.2.53) and (6.2.54), we have

$$sup_{\ 0\le t\le T}\ \frac{1}{\varepsilon}||(\mathbb{I}-\mathbb{P})v^\varepsilon||_{m-1} \le M,$$

i.e. $||(\mathbb{I}-\mathbb{P})v^\varepsilon||_{m-1} \to 0$ as $\varepsilon \to 0$ and, since $v^\varepsilon \to \tilde{v}^\star$ in $C([0,T],H^{m'}(\mathbb{R}^d))$, then $\mathbb{P}\tilde{v}^\star = \tilde{v}^\star$, namely

$$\nabla \cdot \tilde{v}^\star = 0.$$

Next, let $\psi \in C_c^\infty((0,T))$ and $\phi = (\rho, v)$ so that $v \in V^0 = \{v \in L^2(\mathbb{R}^d) \mid \nabla \cdot v = 0\}$ with compact support. Writing a weak formulation of system (6.2.48), we have

$$\int_0^T -\psi'(t)(\tilde{\mathbf{u}}^\varepsilon, \phi)_0\ dt + \sum_{j=1}^d \int_0^T \psi(t)(A_0^{-1}J_\varepsilon A_0 A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon+\bar{\mathbf{u}}))\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}^\varepsilon, \phi)_0\ dt$$
$$= -\int_0^T \psi(t)\left(\frac{(\mathbb{I}-\mathbb{P})v^\varepsilon}{\varepsilon}, v\right)_0\ dt.$$

Since $(\mathbb{I}-\mathbb{P})v^\varepsilon$ is a gradient for every ε, the right hand side of last equality vanishes, then

$$\int_0^T -\psi'(t)(\tilde{\mathbf{u}}^\varepsilon, \phi)_0\ dt + \sum_{j=1}^d \int_0^T \psi(t)(A_0^{-1}J_\varepsilon A_0 A_j(J_\varepsilon(\tilde{\mathbf{u}}^\varepsilon+\bar{\mathbf{u}}))\partial_{x_j}J_\varepsilon\tilde{\mathbf{u}}^\varepsilon, \phi)_0\ dt = 0. \tag{6.2.55}$$

As done before, passing to the limit in (6.2.55), we obtain

$$\int_0^T -\psi'(t)(\tilde{\mathbf{u}}^\star, \phi)_0\ dt + \sum_{j=1}^d \int_0^T \psi(t)(A_j(\tilde{\mathbf{u}}^\star+\bar{\mathbf{u}})\partial_{x_j}\tilde{\mathbf{u}}^\star, \phi)_0\ dt = 0.$$

This yields $\partial_t\tilde{\mathbf{u}}^\varepsilon \rightharpoonup^* \partial_t\tilde{\mathbf{u}}^\star$ in $L^\infty([0,T],H^{m-1}(\mathbb{R}^d))$ and equation (6.1.39), i.e.

$$\partial_t\tilde{\mathbf{u}}^\star + \sum_{j=1}^d \mathbf{P}(A_j(\tilde{\mathbf{u}}^\star+\bar{\mathbf{u}})\partial_{x_j}\tilde{\mathbf{u}}^\star) = 0.$$

This way, we get the additional regularity $\tilde{\mathbf{u}}^\star \in Lip([0,T],H^{m-1}(\mathbb{R}^d))$ and the existence of $\nabla P^\star \in L^\infty([0,T],H^{m-1}(\mathbb{R}^d))$ such that

$$\partial_t\tilde{\mathbf{u}}^\star + \sum_{j=1}^d A_j(\tilde{\mathbf{u}}^\star+\bar{\mathbf{u}})\partial_{x_j}\tilde{\mathbf{u}}^\star = (0, -\nabla P^\star).$$

Thus, $\tilde{\mathbf{u}}^\star \in L^\infty([0,T],H^m(\mathbb{R}^d)) \cap Lip([0,T],H^{m-1}(\mathbb{R}^d)) \cap C_w([0,T],H^m(\mathbb{R}^d))$ is a weak solution to (6.0.11)-(6.0.12). The last part of the proof of Section 6.1 yields $\tilde{\mathbf{u}}^\star$ belonging to $C([0,T],H^m(\mathbb{R}^d)) \cap C^1([0,T],H^{m-1}(\mathbb{R}^d))$ and $P^\star \in C([0,T],H^m(\mathbb{R}^d))$. □

Remark 6.2.3. *This approximation for the density-dependent incompressible Euler equations (6.0.5) in the two-dimensional case is:*

$$\partial_t \boldsymbol{u}^\varepsilon + J_\varepsilon \begin{pmatrix} v_1^\varepsilon & 0 & 0 \\ 0 & v_1^\varepsilon & 0 \\ 0 & 0 & v_1^\varepsilon \end{pmatrix} \partial_x J_\varepsilon \boldsymbol{u}^\varepsilon + J_\varepsilon \begin{pmatrix} v_2^\varepsilon & 0 & 0 \\ 0 & v_2^\varepsilon & 0 \\ 0 & 0 & v_2^\varepsilon \end{pmatrix} \partial_y J_\varepsilon \boldsymbol{u}^\varepsilon + \left(0, \frac{(\mathbb{I}-\mathbb{P})v^\varepsilon}{\varepsilon\rho^\varepsilon}\right) = 0,$$

where $\boldsymbol{u}^\varepsilon = (\rho^\varepsilon, v^\varepsilon)$. The scalar product of the α-derivative of the singular term against the α-derivative of the velocity field has no more a positive definite sign, then this method does not work in a simple way on (6.0.5).

6.3 The artificial compressibility method

Following [75], we consider another kind of approximation of system (6.0.1), based on a family of perturbed system, which, in order to approximate the divergence constraint $\nabla \cdot v = 0$, contains the following artificial equation for the pressure term P^ε:

$$\varepsilon^2 \partial_t P^\varepsilon + \nabla \cdot v^\varepsilon = 0.$$

We consider the artificial state equation

$$P^\varepsilon = P_0 + \varepsilon \tilde{P}^\varepsilon,$$

where P_0 is constant. Without loss of generality, we take $P_0 = 1$. Setting $\mathbf{u}^\varepsilon := (\rho^\varepsilon, \tilde{P}^\varepsilon, v^\varepsilon)$, the approximating system reads:

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon v^\varepsilon) = 0, \\ \partial_t \tilde{P}^\varepsilon + \frac{\nabla \cdot v^\varepsilon}{\varepsilon} = 0, \\ \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + f(\rho^\varepsilon, v^\varepsilon) \nabla \rho^\varepsilon + \frac{\nabla \tilde{P}^\varepsilon}{\varepsilon} = 0, \end{cases} \tag{6.3.56}$$

with the following initial data as in (6.2.47):

$$\rho_0^\varepsilon(x) = \rho_0(0, x), \qquad v_0^\varepsilon(0, x) = v_0(x) + \varepsilon v_0^1(x), \tag{6.3.57}$$

where ρ_0, v_0 are the initial data (6.0.2) of the original problem (6.0.1).

Remark 6.3.1. *Although it is needed, we do not make explicitly the translation of (6.3.56) to simplify the notation.*

Again, we can write system (6.3.56) in the compact form:

$$\partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^{d} A_j(\mathbf{u}^\varepsilon) \partial_{x_j} \mathbf{u}^\varepsilon = 0, \tag{6.3.58}$$

with initial data

$$\mathbf{u}_0^\varepsilon = (\rho_0^\varepsilon, \tilde{P}_0^\varepsilon, v_0^\varepsilon), \tag{6.3.59}$$

where the function $\tilde{P}_0^\varepsilon$ is arbitrarily chosen, provided that $\tilde{P}_0^\varepsilon \in H^m(\mathbb{R}^d)$, and $\rho_0^\varepsilon, v_0^\varepsilon$ in (6.3.57). The matrices $A_j(\mathbf{u}^\varepsilon)$ have the following structural form:

$$A_j(\mathbf{u}^\varepsilon) = \tilde{A}_j(\mathbf{u}^\varepsilon) + \frac{A_j^0}{\varepsilon}$$

for $j = 1, \cdots, d$. In the two dimensional case, we have

$$A_1(\mathbf{u}^\varepsilon) = \tilde{A}_1(\mathbf{u}^\varepsilon) + \frac{A_1^0}{\varepsilon} = \begin{pmatrix} v_1^\varepsilon & 0 & \rho^\varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ f(\mathbf{u}^\varepsilon) & 0 & v_1^\varepsilon & 0 \\ 0 & 0 & 0 & v_1^\varepsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\varepsilon} & 0 \\ 0 & \frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2(\mathbf{u}^\varepsilon) = \tilde{A}_2(\mathbf{u}^\varepsilon) + \frac{A_2^0}{\varepsilon} = \begin{pmatrix} v_2^\varepsilon & 0 & 0 & \rho^\varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & v_2^\varepsilon & 0 \\ f(\mathbf{u}^\varepsilon) & 0 & 0 & v_2^\varepsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\varepsilon} & 0 & 0 \end{pmatrix}$$

and, in the general d-case, for $j = 1, \cdots, d$,

$$A_j(\mathbf{u}^\varepsilon) = \tilde{A}_j(\mathbf{u}^\varepsilon) + \frac{A_j^0}{\varepsilon}$$

$$= \begin{pmatrix} v_j^\varepsilon & 0 & \delta_{1j}\rho^\varepsilon & \delta_{2j}\rho^\varepsilon & \cdots & \cdots & \delta_{dj}\rho^\varepsilon \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \delta_{1j} f(\mathbf{u}^\varepsilon) & 0 & v_j^\varepsilon & 0 & \cdots & \cdots & 0 \\ \delta_{2j} f(\mathbf{u}^\varepsilon) & 0 & 0 & v_j^\varepsilon & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & v_j^\varepsilon & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & v_j^\varepsilon & \cdots \\ \delta_{dj} f(\mathbf{u}^\varepsilon) & 0 & 0 & \cdots & \cdots & \cdots & v_j^\varepsilon \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \frac{\delta_{1j}}{\varepsilon} & \frac{\delta_{2j}}{\varepsilon} & \cdots & \frac{\delta_{dj}}{\varepsilon} \\ 0 & \frac{\delta_{1j}}{\varepsilon} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\delta_{2j}}{\varepsilon} & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{\delta_{dj}}{\varepsilon} & 0 & \cdots & \cdots & \cdots \end{pmatrix}.$$

System (6.3.58) is Friedrichs-symmetrizable by the $(d+2) \times (d+2)$ - symmetrizer

$$A_0(\mathbf{u}^\varepsilon) = diag\left(\frac{f(\mathbf{u}^\varepsilon)}{\rho^\varepsilon}, 1, 1, \cdots, 1\right).$$

Remark 6.3.2. *We point out that here we need just the first assumption of Definition 6.2.1 on $f(\boldsymbol{u})$.*

Now, looking at the matrices A_j for $j = 1, \cdots, d$, we notice that they satisfy the structural conditions required by *Majda* and *Klainerman* in [52] and [47] to prove the convergence of the compressible Euler equations to the incompressible ones. Moreover, the initial data (6.3.59) associated to system (6.3.58) are consistent with respect to the hypothesis of 'slightly compressible initial data' in [52]. Then, the proof in [52] can be adapted to this context, providing us a result of existence and uniqueness of the solution to (6.0.1)-(6.0.2) in the Sobolev spaces, as in the previous sections.

Remark 6.3.3. *Applying the artificial compressibility method to system (6.0.5), we obtain an approximation system whose matrices and the related Friedrichs symmetrizer do not satisfy the assumptions stated in [52]. For instance, in the two-dimensional case, setting $\boldsymbol{u}^\varepsilon := (\rho^\varepsilon, \tilde{P}^\varepsilon, v^\varepsilon)$, we have the system*

$$\partial_t \boldsymbol{u}^\varepsilon + \begin{pmatrix} v_1^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\varepsilon} & 0 \\ 0 & \frac{1}{\varepsilon\rho^\varepsilon} & v_1^\varepsilon & 0 \\ 0 & 0 & 0 & v_1^\varepsilon \end{pmatrix} \partial_x \boldsymbol{u}^\varepsilon + \begin{pmatrix} v_2^\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & v_2^\varepsilon & 0 \\ 0 & \frac{1}{\varepsilon\rho^\varepsilon} & 0 & v_2^\varepsilon \end{pmatrix} \partial_x \boldsymbol{u}^\varepsilon = 0,$$

where the singular parts of the matrices above are not constant.

Chapter 7

A multiphase model in two space dimensions

Although mixture models are largely diffused, up to now the analytical theory has been mainly developed in one space dimension, see for instance [38], [72], [79], and [41], while some results about linear stability and numerical approximations were considered in [35]. Moreover, a complete analytical study of the one dimensional biofilms model (4.0.2) and the related two phases system (4.0.6), with the proof of the global existence and uniqueness of the smooth solution and the analysis of its asymptotic behavior for initial data, that are small perturbations of the equilibrium point, were given in Section 5, which is based on [12]. Let us recall that, in the one dimensional case, the incompressibility condition (1.0.5) allows us to solve for the incompressible pressure ∇P in (5.0.1), obtaining (5.0.5) and (5.0.4). Besides, the remaining system (5.0.6) is symmetrizable hyperbolic (see [52], [8] [74]), and so the standard theory applies. On the other hand, in several space dimensions there is not a simple way to deal with the term ∇P, since the incompressibility inequality is given by (4.0.7). In order to work using a divergence free formulation, we define

$$w := Bv_S + (1 - B)v_L$$

and we could try to apply classical methods used for incompressible fluids, see [75], [52], [76], and [9], which are essentially based on the projection of the velocity field onto the space of the divergence free vectors.

However, in our case, even in the divergence free variables, there are some difficulties. The first one is given by the interaction between the *Friedrichs* symmetrizer of the hyperbolic part of system (4.0.6) and the gradient of the incompressible pressure term. Actually, the scalar product induced by the classical symmetrizer does not preserve the orthogonality of the gradient of the incompressible pressure with respect to the divergence free average velocity. This happens since the symmetrizer and the pressure part of the system do not commute and, moreover, their commutator is still a first order operator, see Section 7.2 below. Therefore, we cannot get rid of the incompressible pressure, unlike in the case of the incompressible Euler equations, see for instance [49]. Furthermore, it is not obvious how to get useful energy estimates in Sobolev spaces for system (4.0.6), since our hydrostatic pressure does not possess enough regularity in space. In fact, looking at the elliptic equation for the pressure P, which is obtained applying the divergence operator to the momentum equations in system (4.0.6), we have

$$\Delta P = -\sum_{j=1}^{d}\sum_{i=1}^{d}\partial_{x_j} w_i \partial_{x_i} w_j - \nabla\cdot\nabla\cdot(B(1-B)z\otimes z) - \gamma\Delta B, \tag{7.0.1}$$

where $z := v_S - v_L$. Let us compare (7.0.1) with the elliptic equation for the pressure P^E of the incompressible Euler equations with velocity v^E, see [49], namely

$$\Delta P^E = -\sum_{j=1}^{d}\sum_{i=1}^{d}\partial_{x_j} {v^E}_i \partial_{x_i} {v^E}_j. \tag{7.0.2}$$

Starting from velocity fields w, z in (7.0.1) and v^E in (7.0.2) with the same H^s regularity for some $s > [d/2] + 1$, our pressure P in (7.0.1) is only in H^s, while P^E in (7.0.2) is in H^{s+1}. So, because of this lack of regularity, which is due not only to the inertial term $\nabla\cdot(B(1-B)z\otimes z)$, but also to the compressible pressure term $\gamma\nabla B$, we are unable to close the energy estimates for system (4.0.6).
For all these reasons, the different approaches used for incompressible fluids do not work for (4.0.6). For instance, even if the numerical simulations in [27], which use the *Chorin-Temam* projection method [75], seem to yield some reliable results, we do not know how to prove any rigorous convergence result for this approximation scheme in this case. In fact, while the L^2-norm of the projected solution is estimated step by step by the L^2-norm of the non-projected vector, thanks to the *Hodge* decomposition theorem [75], this property no more holds for the scalar product induced by the symmetrizer and so we are unable to control the energy estimates. This structural difficulty is also the reason why the singular perturbation approximation in [15], which can be viewed as a continuous version of the projection method, does not work for system (4.0.6). Also, we are not able to prove the convergence of the approximation used by *Valli* & *Zajazckowski* in [76] to solve the incompressible Euler equations, since, again, we cannot get the necessary energy estimates from the elliptic equation (7.0.1). For completeness, we notice that the same holds for the artificial compressibility method of *Temam* in [75], since there is no classical symmetrizer for the related approximating compressible system and the *Lax* symmetrizer that we have found does not satisfy the assumptions required in studying singular perturbations approximations, as in [34].
In spite of these negative remarks, in the following we prove the convergence of one approximation to system (4.0.6), made by the composition of some smoothing operators and the *Leray* projector, see [9] and [15] for different applications of this technique. This section is based on [13]. Here, the main idea is as follows. First, we apply the projector onto the space of the vectors such that the averaged velocity w is divergence free. Then, we consider the paradifferential operator associated to the projected system (4.0.6), we notice that its highest order part is a strongly hyperbolic operator of the first order, and therefore it is possible to construct a *Lax* symmetrizer for it (see Section 2). The construction of this symmetrizer is essentially based on the techniques explained in Section 2, developed in [55], which are combined to some ideas in [34]. We point out that, the main point here is to symmetrize the whole projected operator, rather than just use the symmetrizer of the hyperbolic part of (4.0.6). Using paradifferential calculus, we are able to establish some uniform energy estimates and the convergence of this method to (4.0.6), as well as in the case of the more general model (4.0.2), both in two space dimensions.

In the following, we discuss the general setting and the main properties of the two phases system (4.0.6) in two space dimensions. Section 7.2 is devoted to the well-posedness of our approximation, using an approach based on paradifferential calculus, and a proof of its convergence. Section 7.3 is dedicated to the explanation of the failure of some more classical approaches to first order models of incompressible fluids with respect to the paradifferential one. In Section 7.4, we show how to apply the arguments of Section 7.2 to the more general system (4.0.2), always in two space dimensions. Finally, in Section 7.5 we discuss the difficulties we have found to extend these results to the three dimensional case.

7.1 Basic formulation

Let $\mathbf{u} = (B, v_S, v_L)$. System (4.0.6) can be written in the following compact form:

$$\begin{cases} \partial_t \mathbf{u} + \sum_{j=1}^{d} A_j(\mathbf{u})\partial_{x_j}\mathbf{u} + F_P = G(\mathbf{u}), \\ \nabla \cdot (Bv_S + (1-B)v_L) = 0, \end{cases} \tag{7.1.3}$$

where the term F_P is given by the gradient of the hydrostatic incompressible pressure

$$F_P = (0, \nabla P, \nabla P), \tag{7.1.4}$$

and the source term has the following expression

$$G(\mathbf{u}) = (\Gamma_B, \Gamma_{v_S}, \Gamma_{v_L}), \tag{7.1.5}$$

where

$$\Gamma_B = B(k_B(1-B) - k_D), \quad \Gamma_{v_S} = \frac{(M+\Gamma_B)(v_L - v_S)}{B}, \quad \Gamma_{v_L} = \frac{M(v_S - v_L)}{(1-B)}, \tag{7.1.6}$$

and k_B, k_D, M are experimental constants. The initial data related to (7.1.3) are the following:

$$\mathbf{u}(0,x) = \mathbf{u}_0(x) = (B_0(x), v_{S_0}(x), v_{L_0}(x)) \quad \text{such that} \quad \nabla \cdot (B_0 v_{S_0} + (1-B_0)v_{L_0}) = 0. \tag{7.1.7}$$

Although most of the calculations in this first section hold in the general d-dimensional case, we limit our consideration only to the two dimensional case. In one space dimension, in fact, system (7.1.3) is a particular version of that already discussed in [12], while in three space dimensions there are some structural problems that lead to technical difficulties, as we will see in Section 6. Setting $d = 2$, system (7.1.3) reads

$$\begin{cases} \partial_t \mathbf{u} + A_1(\mathbf{u})\partial_x \mathbf{u} + A_2(\mathbf{u})\partial_y \mathbf{u} + F_P = G(\mathbf{u}), \\ \nabla \cdot (Bv_S + (1-B)v_L) = 0, \end{cases} \tag{7.1.8}$$

with F_p in (7.1.4), $G(\mathbf{u})$ in (7.1.5) and the initial data $\mathbf{u}_0$ in (7.1.7). The flux matrices are:

$$A_1(\mathbf{u}) = \begin{pmatrix} v_{S_1} & B & 0 & 0 & 0 \\ \frac{\gamma}{B} & v_{S_1} & 0 & 0 & 0 \\ 0 & 0 & v_{S_1} & 0 & 0 \\ 0 & 0 & 0 & v_{L_1} & 0 \\ 0 & 0 & 0 & 0 & v_{L_1} \end{pmatrix}, \qquad A_2(\mathbf{u}) = \begin{pmatrix} v_{S_2} & 0 & B & 0 & 0 \\ 0 & v_{S_2} & 0 & 0 & 0 \\ \frac{\gamma}{B} & 0 & v_{S_2} & 0 & 0 \\ 0 & 0 & 0 & v_{L_2} & 0 \\ 0 & 0 & 0 & 0 & v_{L_2} \end{pmatrix}. \tag{7.1.9}$$

Assumption. From (7.1.9), (7.1.5) and (7.1.6) the terms B and $(1-B)$ cannot vanish. Then, we take $0 < B < 1$.

Remark 7.1.1. *The assumption above is quite natural, in fact, from the mass balance equation for B in (4.0.6), if the initial data B_0 (and $1-B_0$) in (7.1.7) does not vanish for all $x \in \mathbb{R}^d$, then, under some standard assumptions of regularity, $B(t,x)$ (and $1-B(t,x)$) cannot vanish too. However, this will be proved* a posteriori.
In the following, we prove that, fixing a constant value $\bar{B}$ and taking B_0 such that $B_0 - \bar{B} \in H^s(\mathbb{R}^2)$, with $s > [d/2] + 1 = 2$, then $(B - \bar{B}, v_S) \in C([0,T], H^s(\mathbb{R}^2)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^2))$.

As discussed in Remark 7.1.1, system (7.1.8) is singular in $B = 0$, then the unknown B cannot belong to $L^2(\mathbb{R}^2)$. In order to work in the natural setting of the Sobolev spaces, we make a slight modification. From the form of the source term G in (7.1.5)–(7.1.6), the admissible equilibrium point of system (4.0.6) is the following:

$$\bar{\mathbf{u}} = (\bar{B}, \bar{v}_S, \bar{v}_L) = (1 - \frac{k_D}{k_B}, \bar{v}, \bar{v}), \tag{7.1.10}$$

where $\bar{v}$ is a two dimensional constant vector arbitrarily chosen. Taking $\bar{v} = 0$, we have

$$\bar{\mathbf{u}} = (\bar{B}, \mathbf{0}, \mathbf{0}). \tag{7.1.11}$$

In this section, to simplify the presentation, we define the translated system, which will be considered in Section 3. Let

$$\tilde{\mathbf{u}} = (\tilde{B}, \tilde{v}_S, \tilde{v}_L) := \mathbf{u} - \bar{\mathbf{u}},$$

with $\bar{\mathbf{u}}$ in (7.1.11). Then, we will study the following system:

$$\begin{cases} \partial_t \tilde{\mathbf{u}} + \sum_{j=1}^2 A_j(\tilde{\mathbf{u}} + \bar{\mathbf{u}}) \partial_{x_j} \tilde{\mathbf{u}} + F_P = G(\tilde{\mathbf{u}} + \bar{\mathbf{u}}), \\ \nabla \cdot ((\tilde{B} + \bar{B}) \tilde{v}_S + (1 - (\tilde{B} + \bar{B})) \tilde{v}_L) = 0, \end{cases} \tag{7.1.12}$$

with initial data

$$\tilde{\mathbf{u}}(0,x) = \tilde{\mathbf{u}}_0 = \mathbf{u}_0 - \bar{\mathbf{u}}, \tag{7.1.13}$$

and $\mathbf{u}_0$ in (7.1.7). We provide now the definition of classical local solutions to (4.0.6).

Definition 7.1.1. *Let $s > 2$ be fixed. The function $\tilde{\mathbf{u}} = (\tilde{B}, \tilde{v}_S, \tilde{v}_L)$ is a classical solution to system (7.1.3), if $\tilde{\mathbf{u}} \in C([0,T], H^s(\mathbb{R}^2)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^2))$ for any time $T > 0$, and $\tilde{\mathbf{u}}$ solves system (7.1.12) in the classical sense, with initial data $\tilde{\mathbf{u}}_0 \in H^s(\mathbb{R}^2)$ in (7.1.13), where P is a function such that $\nabla P \in C([0,T], H^{s-1}(\mathbb{R}^2))$.*

In the remainder of this section, we take into account the translation, but we just omit the tilde to simplify the notations. Now, in order to deal with the divergence free vector field, we change variables. Define

$$w := B v_S + (1-B) v_L, \quad z := v_S - v_L, \tag{7.1.14}$$

and let $\phi(\mathbf{u})$ be the diffeomorphism so defined

$$\mathbf{v} = (B, w, z) = \phi(\mathbf{u}) = (B, B v_S + (1-B) v_L, v_S - v_L). \tag{7.1.15}$$

System (7.1.8), can be written in the following compact form:

$$\begin{cases} \partial_t \mathbf{v} + \sum_{j=1}^2 \tilde{A}_j(\mathbf{v})\partial_{x_j}\mathbf{v} + \tilde{F}_P = \tilde{G}(\mathbf{v}); \\ \nabla \cdot w = 0, \end{cases} \tag{7.1.16}$$

with initial data

$$\mathbf{v}(0,x) = \mathbf{v}_0(x) = (B_0(x), w_0(x), z_0(x)) \quad \text{such that} \quad \nabla \cdot w_0 = 0, \tag{7.1.17}$$

where

$$\tilde{A}_j(\mathbf{v}) = (\phi' A_j \phi'^{-1})(\phi^{-1}(\mathbf{v})), \quad \text{for} \quad j = 1,2, \qquad \tilde{G}(\mathbf{v}) = (\phi' G \phi'^{-1})(\phi^{-1}(\mathbf{v})),$$

and

$$\tilde{F}_P = (0, \nabla P, \mathbf{0}).$$

Explicitly,

$$\tilde{A}_1(\mathbf{v}) = \begin{pmatrix} w_1 + z_1(1-2B) & B & 0 & B(1-B) & 0 \\ \gamma + z_1^2(1-2B) & w_1 + Bz_1 & 0 & 2Bz_1(1-B) & 0 \\ z_1 z_2(1-2B) & Bz_2 & w_1 & Bz_2(1-B) & Bz_1(1-B) \\ \frac{\gamma}{B} - z_1^2 & z_1 & 0 & w_1 + z_1(1-2B) & 0 \\ -z_1 z_2 & 0 & z_1 & 0 & w_1 + z_1(1-2B) \end{pmatrix},$$

$$\tilde{A}_2(\mathbf{v}) = \begin{pmatrix} w_2 + z_2(1-2B) & 0 & B & 0 & B(1-B) \\ z_1 z_2(1-2B) & w_2 & Bz_1 & Bz_2(1-B) & Bz_1(1-B) \\ \gamma + z_2^2(1-2B) & 0 & w_2 + Bz_2 & 0 & 2Bz_2(1-B) \\ -z_1 z_2 & z_2 & 0 & w_2 + z_2(1-2B) & 0 \\ \frac{\gamma}{B} - z_2^2 & 0 & z_2 & 0 & w_2 + z_2(1-2B) \end{pmatrix},$$

while

$$\tilde{G}(\mathbf{v}) = (\Gamma_B, \mathbf{0}, \frac{-z(M + \Gamma_B(1-B))}{B(1-B)}),$$

with Γ_B in (7.1.6). Let us define the generalized projector operator:

$$\mathbf{P}(\xi) := \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbb{P} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}, \tag{7.1.18}$$

where $\mathbb{P}$ is the standard Leray projector, namely the projector onto the divergence free vector valued functions. If we apply the operator $\mathbf{P}$ to system (7.1.16), with the aim of eliminating ∇P, we get

$$\partial_t \mathbf{v} + \sum_{j=1}^{2} \mathbf{P}\tilde{A}_j(\mathbf{v})\partial_{x_j}\mathbf{v} = \mathbf{P}\tilde{G}(\mathbf{v}), \tag{7.1.19}$$

since $\mathbf{P}\tilde{F}_P = (0, \mathbb{P}\nabla P, \mathbf{0}) = 0$ by definition, and

$$\mathbf{P}\mathbf{v} = (B, \mathbb{P}w, z) = (B, w, z),$$

by the divergence free condition $\nabla \cdot w = 0$.

7.1.1 A new formulation

Taking inspiration from our preliminary work [15], we aim to propose a different symmetrization strategy for our problem, to be able to estimate correctly the pressure term. We apply first the operator $\mathbf{P}$ to system (7.1.16). Notice that the initial datum does not change by projection, since the initial average velocity w_0 is already a divergence free vector and then, applying $\mathbf{P}$ to (7.1.17), we have

$$\mathbf{P}\mathbf{v}_0(x) = (B_0(x), \mathbb{P}w_0(x), z_0(x)) = (B_0(x), w_0(x), z_0(x)).$$

Moreover, $\mathbf{P}\tilde{F}_P = (0, \mathbb{P}\nabla P, \mathbf{0}) = \mathbf{0}$, and the divergence free constraint $\nabla \cdot w = 0$ in (7.1.16) is implicitly contained in system (7.1.19). We consider the paradifferential version of system (7.1.19):

$$\partial_t \mathbf{v} + \mathbf{P}T_{i\tilde{A}(\xi,\mathbf{v})}\mathbf{v} = \mathbf{P}T_{\tilde{G}(\mathbf{v})} + \sum_{j=1}^{d}[\mathbf{P}T_{\tilde{A}_j(\mathbf{v})} - \mathbf{P}\tilde{A}_j(\mathbf{v})]\partial_{x_j}\mathbf{v} - [\mathbf{P}T_{\tilde{G}(\mathbf{v})} - \mathbf{P}\tilde{G}(\mathbf{v})], \quad (7.1.20)$$

where, from [55],

$$T_{i\tilde{A}(\xi,\mathbf{v})} = \sum_{j=1}^{2} T_{\tilde{A}_j(\mathbf{v})}\partial_{x_j}\mathbf{v} \quad (7.1.21)$$

is the paradifferential operator associated to the x-dependent matrix symbol

$$i\tilde{A}(\xi, \mathbf{v}) = \sum_{j=1}^{2} i\xi_j \tilde{A}_j(\mathbf{v}) = \sum_{j=1}^{2} i\xi_j \tilde{A}_j(\mathbf{v}(t,x)), \quad (7.1.22)$$

and similarly for $\tilde{G}(\mathbf{v})$ and $T_{\tilde{G}(\mathbf{v})}$. As we will see in details in the next section, in (7.1.20) there is only one operator of order 1, which is $\mathbf{P}T_{i\tilde{A}(\xi,\mathbf{v})}$. We want now to show that we can apply Theorem 2.4.2 and the corollary above to this operator. From Section 2, the symbol associated to the composition is made by the sum over the multi-index α of terms of type

$$\partial_\xi^\alpha \mathbf{P} D_x^\alpha \tilde{A}(\xi, \mathbf{v}),$$

where $D_x = \frac{1}{i}\partial_x$. The expansion above implies that there is only one term of degree 1 in ξ, which is given for $|\alpha| = 0$, namely $\mathbf{P}(\xi)\tilde{A}(\xi, \mathbf{v})$. Thus, the symbol of $\mathbf{P}T_{i\tilde{A}(\xi,\mathbf{v})}$ can be written as

$$\mathbf{P}(\xi) i\tilde{A}(\xi, \mathbf{v}) + R(\xi, \mathbf{v})$$

$$= i \begin{pmatrix} (w + (1-2B)z)\cdot\xi & B\xi_1 & B\xi_2 & B(1-B)\xi_1 & B(1-B)\xi_2 \\ \frac{\xi_2(1-2B)\mu_1}{|\xi|^2} & \frac{\xi_2\mu_2}{|\xi|^2} & \frac{-\xi_2\mu_3}{|\xi|^2} & \frac{B(1-B)\xi_2\mu_4}{|\xi|^2} & \frac{-B(1-B)\xi_2\mu_5}{|\xi|^2} \\ \frac{-\xi_1(1-2B)\mu_1}{|\xi|^2} & \frac{-\xi_1\mu_2}{|\xi|^2} & \frac{\xi_1\mu_3}{|\xi|^2} & \frac{-B(1-B)\xi_1\mu_4}{|\xi|^2} & \frac{B(1-B)\xi_1\mu_5}{|\xi|^2} \\ \frac{\gamma\xi_1}{B} - z_1(z\cdot\xi) & z\cdot\xi & 0 & (w + (1-2B))\cdot\xi & 0 \\ \frac{\gamma\xi_2}{B} - z_2(z\cdot\xi) & 0 & z\cdot\xi & 0 & (w + (1-2B)z)\cdot\xi \end{pmatrix}$$

$$+R(\xi, \mathbf{v}), \quad (7.1.23)$$

where $R(\xi, \mathbf{v})$ is a remainder of order less than or equal to 0, and

$$\begin{cases} \mu_1 := (z\cdot\xi)(\xi_2 z_1 - \xi_1 z_2), \\ \mu_2 := (w\cdot\xi)\xi_2 + B\xi_1(\xi_2 z_1 - \xi_1 z_2), \\ \mu_3 := (w\cdot\xi)\xi_1 + B\xi_2(\xi_2 z_1 - \xi_1 z_2), \\ \mu_4 := z_2(\xi_2^2 - \xi_1^2) + 2z_1\xi_1\xi_2, \\ \mu_5 := z_1(\xi_1^2 - \xi_2^2) + 2z_2\xi_1\xi_2. \end{cases}$$

The eigenvalues of $\mathbf{P}\tilde{A}(\xi, \mathbf{v})$ are the following:

$$\lambda_1 = 0, \quad \lambda_2 = (w - Bz)\cdot\xi, \quad \lambda_3 = (w + (1-B)z)\cdot\xi,$$

$$\lambda_{4/5} = (w + (1-2B)z)\cdot\xi \pm \sqrt{(1-B)\Delta_2}. \tag{7.1.24}$$

Its eigenvectors are the columns of $V(\xi, \mathbf{v})$

$$= \begin{pmatrix} \frac{B|\xi|(w-Bz)\cdot\xi}{\Delta_1} & 0 & 0 & \frac{-B|\xi|\sqrt{1-B}}{\sqrt{\Delta_2}} & \frac{B|\xi|\sqrt{1-B}}{\sqrt{\Delta_2}} \\ \frac{p_1}{|\xi|\Delta_1} & \frac{(1-B)\xi_2}{|\xi|} & \frac{-B\xi_2}{|\xi|} & \frac{B\xi_2(\xi_1 z_2-\xi_2 z_1)\sqrt{1-B}}{|\xi|\sqrt{\Delta_2}} & \frac{-B\xi_2(\xi_1 z_2-\xi_2 z_1)\sqrt{1-B}}{|\xi|\sqrt{\Delta_2}} \\ \frac{p_2}{|\xi|\Delta_1} & \frac{-(1-B)\xi_1}{|\xi|} & \frac{B\xi_1}{|\xi|} & \frac{-B\xi_1(\xi_1 z_2-\xi_2 z_1)\sqrt{1-B}}{|\xi|\sqrt{\Delta_2}} & \frac{B\xi_1(\xi_1 z_2-\xi_2 z_1)\sqrt{1-B}}{|\xi|\sqrt{\Delta_2}} \\ \frac{\xi_1}{|\xi|} & \frac{-\xi_2}{|\xi|} & \frac{-\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{\xi_2}{|\xi|} & \frac{\xi_2}{|\xi|} \end{pmatrix}, \tag{7.1.25}$$

where $p_1 = p_1(\xi, \mathbf{v}), p_2 = p_2(\xi, \mathbf{v})$ are polynomial functions of degree 3 in ξ depending on $\mathbf{v}$, and

$$\Delta_1 := (1-B)(z\cdot\xi)^2 - \gamma|\xi|^2 + (w\cdot\xi)(z\cdot\xi), \quad \Delta_2 := \gamma|\xi|^2 - B(z\cdot\xi)^2. \tag{7.1.26}$$

Its inverse matrix $V^{-1}(\xi, \mathbf{v})$

$$= \begin{pmatrix} 0 & \frac{-\xi_1\Delta_1}{|\xi|\Delta_3} & \frac{-\xi_2\Delta_1}{|\xi|\Delta_3} & 0 & 0 \\ \frac{\xi_1 z_2-\xi_2 z_1}{|\xi|} & \frac{\xi_2}{|\xi|} & \frac{-\xi_1}{|\xi|} & \frac{-B\xi_2}{|\xi|} & \frac{B\xi_1}{|\xi|} \\ \frac{-\xi_1 z_2+\xi_2 z_1}{|\xi|} & \frac{-\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{-(1-B)\xi_2}{|\xi|} & \frac{(1-B)\xi_1}{|\xi|} \\ \frac{-\sqrt{\Delta_2}}{2B|\xi|\sqrt{1-B}} & \frac{\xi_1 q_1}{2|\xi|\Delta_3\sqrt{(1-B)\Delta_2}} & \frac{\xi_2 q_1}{2|\xi|\Delta_3\sqrt{(1-B)\Delta_2}} & \frac{\xi_1}{2|\xi|} & \frac{\xi_2}{2|\xi|} \\ \frac{\sqrt{\Delta_2}}{2B|\xi|\sqrt{(1-B)}} & \frac{\xi_1 q_2}{2|\xi|\Delta_3\sqrt{(1-B)\Delta_2}} & \frac{\xi_2 q_2}{2|\xi|\Delta_3\sqrt{(1-B)\Delta_2}} & \frac{\xi_1}{2|\xi|} & \frac{\xi_2}{2|\xi|} \end{pmatrix}, \tag{7.1.27}$$

where $q_1 = q_1(\xi, \mathbf{v}), q_2 = q_2(\xi, \mathbf{v})$ are polynomial functions of degree 3 in ξ and

$$\Delta_3 := (1-3B(1-B))(z\cdot\xi)^2 + (w\cdot\xi)^2 - \gamma(1-B)|\xi|^2 + 2(1-2B)(w\cdot\xi)(z\cdot\xi). \tag{7.1.28}$$

Proposition 7.1.1. *Under the following assumptions*

$$\Delta_1 \neq 0, \quad \Delta_2 > 0 \quad \textit{and} \quad \Delta_3 \neq 0 \quad \textit{for} \quad \xi \neq (0,0),$$

the first order operator of system (7.1.20) is strongly hyperbolic.

Proof. Considering the symbolic matrix (7.1.23) and the related eigenvalues in (7.1.24) and eigenvectors in (7.1.25), it follows by the definition of strong hyperbolicity, see [55]. □

Proposition 7.1.2. *Under the following conditions*

$$\begin{cases} 2\gamma > (1-B)|z|^2 + (w\cdot z), \\ \gamma^2 > \gamma(1-B)|z|^2 + \gamma(w\cdot z) + \frac{w_1 z_2^2}{4} + \frac{w_2 z_1^2}{4}, \\ \gamma > B|z|^2, \\ 2\gamma > B|z|^2, \\ 2\gamma(1-B) > (1-3B(1-B))|z|^2 + 2(1-2B)(w\cdot z) + |w|^2, \\ \gamma^2(1-B)^2 > \gamma(1-B)((1-3B(1-B))|z|^2 + |w|^2 + 2(1-2B)(w\cdot z)) \\ ((3B(1-B)-1)z_1^2 + 2w_1z_1(1-2B) + w_1^2)((1-3B(1-B))z_2^2 \\ +2w_2z_2(1-2B) + w_2^2) + ((1-3B(1-B))z_1z_2 \\ +(1-2B)(w_1z_2+w_2z_1) + w_1w_2)^2, \end{cases}$$

the value $\xi = (0,0)$ is a strict maximum, minimum and maximum point for Δ_1, Δ_2 and Δ_3 respectively and $\Delta_1|_{\xi_1=\xi_2=0} = \Delta_2|_{\xi_1=\xi_2=0} = \Delta_3|_{\xi_1=\xi_2=0} = 0$. Therefore, Proposition 7.1.1 is verified.

Proposition 7.1.3. *For any $\boldsymbol{v} = (B, w, z)$ in a small neighborhood of the equilibrium point $\bar{\boldsymbol{v}} = \phi(\bar{\mathbf{u}})$, with ϕ in (7.1.15) and $\bar{\mathbf{u}}$ in (7.1.11), the first order operator of system (7.1.20) is strongly hyperbolic.*

Proof. It follows directly from Proposition 7.1.2 and Proposition 7.1.1. □

As we pointed out in the Introduction, the main problem with our original system (7.1.16) is that it is difficult to give for it a direct energy estimate, since, as we will see in Section 4 in details, the pressure term is not well behaved against both the symmetrizers of the first order hyperbolic part, the classical one and the *Lax* one, that work only on the hyperbolic part of system (7.1.16), disregarding the pressure. However, we just proved that system (7.1.19) is strongly hyperbolic near the equilibrium point $\bar{\mathbf{v}} = \phi(\bar{\mathbf{u}})$, in (7.1.11), and so we can construct an appropriate symmetrizer for this system, which in this case is forced to be a paradifferential operator. Our construction in the following is essentially based on the techniques developed in [55], which are combined to the ideas in [34] and adapted to our specific operator.

7.2 Main result

In this section we prove a local existence result in the Sobolev spaces for the Cauchy problem associated to (7.1.20). To the best of our knowledge, such a result is not explicitly stated in all the relevant works about paradifferential calculus. For instance, in the lecture notes [55], only linear and quasi-linear equations of differential operators are considered, while in [34] the discussion is extended to evolution equations of pseudodifferential operators, but the proof makes use of some particular structural characteristics that our system does not satisfy. Therefore, we give our proof of existence and uniqueness of the solution to the Cauchy problem associated to the translated version of (7.1.20). We state here our main result. Since we will work in Sobolev spaces, we define

$$V^s := \{\mathbf{v} = (B, w, z) \in H^s(\mathbb{R}^2) | \nabla \cdot w = 0\}.$$

Theorem 7.2.1. *Let* $\tilde{\boldsymbol{v}}_0 := \boldsymbol{v}_0 - \bar{\boldsymbol{v}}$*, with* $\boldsymbol{v}_0$ *in (7.1.17),* $\bar{\boldsymbol{v}} = \phi(\bar{\mathbf{u}})$ *in (7.1.11), and* $\tilde{\boldsymbol{v}}_0 \in V^s$ *with* $s > 2$*. There is a positive time* T*, such that there exists the unique* $\tilde{\boldsymbol{v}} \in C([0,T],V^s) \cap C^1([0,T],V^{s-1})$ *and a function* P *such that* $\nabla P \in C([0,T],H^{s-1}(\mathbb{R}^2))$ *which solve*

$$\begin{cases} \partial_t \tilde{\boldsymbol{v}} + \sum_{j=1}^{2} \tilde{A}_j((\tilde{\boldsymbol{v}}+\bar{\boldsymbol{v}}))\partial_{x_j}\tilde{\boldsymbol{v}} + \tilde{F}_P = \tilde{G}((\tilde{\boldsymbol{v}}+\bar{\boldsymbol{v}})), \\ \nabla \cdot \tilde{w} = 0. \end{cases} \tag{7.2.29}$$

The solution $(\tilde{\boldsymbol{v}}, P)$ *to (7.2.29) is the limit of the sequence of the solutions to the approximating system (7.2.30) below, with initial data (7.2.31).*

The proof follows by combining in a classical ways, see for instance [9], Theorem 7.2.2 and 7.2.4 below. First, following [9], we write our approximation to system (7.2.29) via a regularization of the operator and the Picard iterations. Namely, let J_ε be a standard mollifier, then solve

$$\begin{cases} \partial_t \tilde{\mathbf{v}}^\varepsilon + \sum_{j=1}^{2} J_\varepsilon \tilde{A}_j(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon + \tilde{F}_P^\varepsilon = J_\varepsilon \tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}})), \\ \nabla \cdot \tilde{w}^\varepsilon = 0, \end{cases} \tag{7.2.30}$$

where $\tilde{F}_P^\varepsilon = (0, \nabla P^\varepsilon, \mathbf{0})$, the initial data are

$$\tilde{\mathbf{v}}^\varepsilon(0,x) = \tilde{\mathbf{v}}_0^\varepsilon(x) = (\tilde{B}_0^\varepsilon, \tilde{w}_0^\varepsilon, \tilde{z}_0^\varepsilon) = \tilde{\mathbf{v}}_0 := \mathbf{v}_0 - \bar{\mathbf{v}}, \tag{7.2.31}$$

and $\tilde{\mathbf{v}}_0$ as in Theorem 7.2.1. We apply now $\mathbf{P}$ to (7.2.30) to get the projected version

$$\partial_t \tilde{\mathbf{v}}^\varepsilon + \sum_{j=1}^{2} \mathbf{P} J_\varepsilon \tilde{A}_j(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon = \mathbf{P} J_\varepsilon \tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}})), \tag{7.2.32}$$

with initial data in (7.2.31).

Theorem 7.2.2. *(Local existence of the approximating solution) Let* $\tilde{\boldsymbol{v}}_0^\varepsilon = (\tilde{B}_0^\varepsilon, \tilde{w}_0^\varepsilon, \tilde{z}_0^\varepsilon) \in V^s$ *in (7.2.31), with* $s > 2$*. Then, for every* $\varepsilon > 0$*, there exists a time* T*, independent of* ε*, such that system (7.2.32) has a unique solution* $\tilde{\boldsymbol{v}}^\varepsilon = (\tilde{B}^\varepsilon, \tilde{w}^\varepsilon, \tilde{z}^\varepsilon) \in C^1([0,T],V^s)$*.*

Proof. First, we show that existence and uniqueness follow from the Picard theorem (see [9]). System (7.2.32) can be reduced to an ordinary differential equation

$$\partial_t \tilde{\mathbf{v}}^\varepsilon = F^\varepsilon(\tilde{\mathbf{v}}^\varepsilon), \qquad \tilde{\mathbf{v}}^\varepsilon(0,x) = \tilde{\mathbf{v}}_0^\varepsilon(x),$$

where

$$F^\varepsilon(\tilde{\mathbf{v}}^\varepsilon) = -\sum_{j=1}^{2} \mathbf{P} J_\varepsilon \tilde{A}_j(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))\partial_{x_j} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon + \mathbf{P} J_\varepsilon \tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}})) =: F_1^\varepsilon(\tilde{\mathbf{v}}^\varepsilon) + F_2^\varepsilon(\tilde{\mathbf{v}}^\varepsilon). \tag{7.2.33}$$

Notice that $J_\varepsilon \tilde{\mathbf{v}}^\varepsilon$ and $J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}})$ are C^∞ functions and, from [55], $\mathbb{P}$ is associated to an analytic pseudo-differential operator of order 0, modulo an infinitely smooth remainder, so that

$$F^\varepsilon : V^s \to V^s.$$

In order to apply the Picard theorem, we have to prove that $F^\varepsilon(\tilde{\mathbf{v}}^\varepsilon)$ in (7.2.33) is Lipschitz continuous. To do this, we take two vectors $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2$ in V^s. In the following, we omit the index ε in the unknown functions, where there is no ambiguity. We state the following theorem proved in [8].

Theorem 7.2.3. *If $F \in C^\infty(\mathbb{R})$, $F(0) = 0$, and $s > \frac{d}{2}$, then there exists a continuous function $C : [0, \infty) \to [0, \infty)$ such that, for all $u \in H^s(\mathbb{R}^d)$,*

$$\|F(u)\|_s \le C(\|u\|_\infty)\|u\|_s.$$

It is straightforward here to prove that

$$||F_1^\varepsilon(\tilde{\mathbf{v}}_1) - F_1^\varepsilon(\tilde{\mathbf{v}}_2)||_s \le c(c_S, ||\tilde{\mathbf{v}}_1||_s, ||\tilde{\mathbf{v}}_2||_s, \bar{B}, \varepsilon^{-1})||\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2||_s, \tag{7.2.34}$$

where c_S is the Sobolev embedding constant and the last inequality follows from Moser estimates and properties of mollifiers. Similarly, we have

$$||F_2^\varepsilon(\tilde{\mathbf{v}}_1) - F_2^\varepsilon(\tilde{\mathbf{v}}_2)||_s \le c(c_S, ||\tilde{\mathbf{v}}_1||_s, ||\tilde{\mathbf{v}}_2||_s, \bar{B})||\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2||_s. \tag{7.2.35}$$

From (7.2.34) and (7.2.35) we have that, for fixed ε, F^ε is locally Lipschitz continuous on any open set

$$\mathcal{U}^M = \{\tilde{\mathbf{v}}^\varepsilon \in V^s : ||\tilde{\mathbf{v}}^\varepsilon||_s < M\}.$$

Then, the Picard theorem provides a unique solution $\tilde{\mathbf{v}}^\varepsilon \in C^1([0, T_\varepsilon), \mathcal{U}^M)$ for any $T_\varepsilon > 0$. Now, we want to show that the time of existence T_ε is bounded from below by any strictly positive time T that is independent of ε.
According to (7.1.20), from (7.2.32) we have

$$\partial_t \tilde{\mathbf{v}}^\varepsilon + \mathbf{P} J_\varepsilon T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon = \sum_{j=1}^{2} \mathbf{P} J_\varepsilon [T_{\tilde{A}_j(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))} - \tilde{A}_j(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))] \partial_{x_j} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon$$

$$+ \mathbf{P} J_\varepsilon T_{\tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))} - \mathbf{P} J_\varepsilon [T_{\tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))} - \tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))]. \tag{7.2.36}$$

From Lemma 2.4.1 in Section 2, properties of mollifiers and the Leray projector, we get

$$\begin{aligned} &||\mathbf{P} J_\varepsilon \{[T_{\tilde{A}_j(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))} - \tilde{A}_j(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))] \partial_{x_j} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon\}||_s \le c(||\tilde{\mathbf{v}}^\varepsilon||_s, \bar{B})||\tilde{\mathbf{v}}^\varepsilon||_s, \text{ and} \\ &||\mathbf{P} J_\varepsilon \{T_{\tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))} - \tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))\}||_s \le c(||\tilde{\mathbf{v}}^\varepsilon||_s, \bar{B})||\tilde{\mathbf{v}}^\varepsilon||_s. \end{aligned} \tag{7.2.37}$$

Then, we can focus on the paradifferential part of (7.2.36), which is

$$\partial_t \tilde{\mathbf{v}}^\varepsilon + J_\varepsilon \mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon - J_\varepsilon \mathbf{P} T_{\tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))}.$$

From (7.1.23), we know that the symbolic matrix associated to the composition

$$\mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\mathbf{v}^\varepsilon + \bar{\mathbf{v}}))}$$

can be written as

$$\mathbf{P}(\xi) i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}})) + R,$$

where $\mathbf{P}(\xi)\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))$ is the symbolic part of degree 1, while R is a remainder of order less than or equal to 0. Now, by construction

$$\mathbf{P}(\xi)\tilde{A} = VDV^{-1}, \tag{7.2.38}$$

with D the diagonal matrix of the eigenvalues of $\mathbf{P}\tilde{A}$ in (7.1.24), namely

$$(V^{-1})^* V^{-1}\mathbf{P}\tilde{A} = (V^{-1})^* D V^{-1}$$

is symmetric. This way, we set

$$W(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}})) := (1 - \theta_\lambda(\xi))V^{-1}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}})), \tag{7.2.39}$$

with $V^{-1}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))$ in (7.1.27). Now, following *Métivier* in [55], we define

$$\theta_\lambda(\xi)Id = \theta(\lambda^{-1}\xi)Id, \tag{7.2.40}$$

for any fixed parameter λ and for any $\theta(\xi) \in C_c^\infty(\mathbb{R}^2)$ such that $0 \leq \theta \leq 1$ for $1 < |\xi| < 2$, $\theta = 1$ for $|\xi| \leq 1$ and $\theta = 0$ for $|\xi| \geq 2$. We define the regularized symmetrizer

$$\Sigma := (T_W)^* T_W + \theta_\lambda^2(D_x)Id, \tag{7.2.41}$$

where $(T_W)^*$ is the adjoint of the paradifferential operator T_W associated to (7.2.39). Thus, by construction, Σ in (7.2.41) is symmetric. Moreover,

$$(\Sigma\mathbf{u}, \mathbf{u})_0 = ||T_W\mathbf{u}||_0^2 + ||\theta_\lambda(D_x)\mathbf{u}||_0^2, \tag{7.2.42}$$

for every $\mathbf{u} \in L^2(\mathbb{R}^2)$. In order to get energy estimates, an important element is the equivalence of (7.2.42) with respect to the L^2-norm, i.e.

$$\underline{c}||\mathbf{u}||_0^2 \leq (\Sigma\mathbf{u}, \mathbf{u})_0 \leq \bar{c}||\mathbf{u}||_0^2,$$

for some $c, \underline{c}$. This is proved in Lemma 2.4.2, which is an adapted version of Lemma 7.1.6 in [55], where we replace the square root of a more general *Lax*-symmetrizer with V^{-1} in (7.1.27).

Now, we are ready to get energy estimates. Applying Λ^s and the symmetrizer (7.2.41) to (7.2.36), we have

$$\frac{d}{dt}(\Sigma\Lambda^s\tilde{\mathbf{v}}^\varepsilon, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0 = (\partial_t\Sigma\Lambda^s\tilde{\mathbf{v}}^\varepsilon, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0 + 2(\Sigma\Lambda^s\partial_t\tilde{\mathbf{v}}^\varepsilon, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0. \tag{7.2.43}$$

The operator of the first term of the right-hand side,

$$\partial_t\Sigma = (T_{\partial_t W})^* T_W + (T_W)^* T_{\partial_t W},$$

has order 0 and depends on $\partial_t\tilde{\mathbf{v}}^\varepsilon$, i.e.

$$|(\partial_t\Sigma\Lambda^s\tilde{\mathbf{v}}^\varepsilon, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0| \leq c(|\partial_t\tilde{\mathbf{v}}^\varepsilon|_\infty)||\tilde{\mathbf{v}}^\varepsilon||_s^2 \leq c(|\tilde{\mathbf{v}}^\varepsilon|_\infty, |\partial_{x_j}\tilde{\mathbf{v}}^\varepsilon|_\infty)||\tilde{\mathbf{v}}^\varepsilon||_s^2 \leq c(||\tilde{\mathbf{v}}^\varepsilon||_s)||\tilde{\mathbf{v}}^\varepsilon||_s^2,$$

where the inequalities follow from (7.2.36) and the Sobolev embedding theorem. The last term of (7.2.43) yields

$$\begin{aligned}(\Sigma\Lambda^s\partial_t\tilde{\mathbf{v}}^\varepsilon, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0 = &-Re(\Sigma\Lambda^s\mathbf{P}J_\varepsilon T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))}J_\varepsilon\tilde{\mathbf{v}}^\varepsilon, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0 \\ &+(\Sigma\Lambda^s\mathbf{P}J_\varepsilon T_{\tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))}, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0 + Q^\varepsilon,\end{aligned}$$

where

$$Q^{\varepsilon} = \sum_{j=1}^{2} (\Lambda^s \mathbf{P} J_\varepsilon [T_{\tilde{A}_j(J_\varepsilon(\mathbf{v}^\varepsilon+\bar{\mathbf{v}}))} - \tilde{A}_j(J_\varepsilon(\mathbf{v}^\varepsilon + \bar{\mathbf{v}}))]\partial_{x_j} J_\varepsilon \mathbf{v}^\varepsilon, \Lambda^s \tilde{\mathbf{v}}^\varepsilon)_0$$

$$-(\Lambda^s \mathbf{P} J_\varepsilon [T_{\tilde{G}(J_\varepsilon(\mathbf{v}^\varepsilon+\bar{\mathbf{v}}))} - \tilde{G}(J_\varepsilon(\mathbf{v}^\varepsilon + \bar{\mathbf{v}}))], \Lambda^s \tilde{\mathbf{v}}^\varepsilon)_0.$$

From (7.2.37),

$$|Q^\varepsilon| \leq c(||\tilde{\mathbf{v}}^\varepsilon||_s)||\tilde{\mathbf{v}}^\varepsilon||_s$$

and, from the composition theorem in Section 2,

$$|(\Sigma \Lambda^s \mathbf{P} J_\varepsilon T_{\tilde{G}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}, \Lambda^s \tilde{\mathbf{v}}^\varepsilon)_0| \leq c(||\tilde{\mathbf{v}}^\varepsilon||_s)||\tilde{\mathbf{v}}^\varepsilon||_s.$$

It remains to deal with

$$Re(\Sigma \Lambda^s \mathbf{P} J_\varepsilon T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon, \Lambda^s \tilde{\mathbf{v}}^\varepsilon)_0 = Re(\Sigma J_\varepsilon \Lambda^s \mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon, \Lambda^s \tilde{\mathbf{v}}^\varepsilon)_0$$

$$= Re(\Sigma \Lambda^s \mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))} J_\varepsilon \tilde{\mathbf{v}}^\varepsilon, \Lambda^s J_\varepsilon \tilde{\mathbf{v}}^\varepsilon)_0 + Re([\Sigma, J_\varepsilon] \Lambda^s \mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}] J_\varepsilon \tilde{\mathbf{v}}^\varepsilon, \Lambda^s \tilde{\mathbf{v}}^\varepsilon)_0$$

$$= Re(\Sigma \mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))} \Lambda^s J_\varepsilon \tilde{\mathbf{v}}^\varepsilon, \Lambda^s J_\varepsilon \tilde{\mathbf{v}}^\varepsilon)_0 + Re([\Sigma, J_\varepsilon] \mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))} \Lambda^s J_\varepsilon \tilde{\mathbf{v}}^\varepsilon, \Lambda^s \tilde{\mathbf{v}}^\varepsilon)_0$$

$$+Re(\Sigma[\Lambda^s, \mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}] J_\varepsilon \tilde{\mathbf{v}}^\varepsilon, \Lambda^s J_\varepsilon \tilde{\mathbf{v}}^\varepsilon)_0. \tag{7.2.44}$$

In the last term of the expression above, from Section 2, the symbol of the commutator $[\Lambda^s, \mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}]$ is given by

$$\sum_{|\alpha| \geq 0} \left[\partial_\xi^\alpha \Lambda^s D_x^\alpha (\sum_{|\beta| \geq 0} \partial_\xi^\beta \mathbf{P} i D_x^\beta \tilde{A}) - \partial_\xi^\alpha (\sum_{|\beta| \geq 0} \partial_\xi^\beta \mathbf{P} i D_x^\beta \tilde{A}) D_x^\alpha \Lambda^s \right],$$

where $\sum_{|\beta| \geq 0} \partial_\xi^\beta \mathbf{P} i D_x^\beta \tilde{A}$ is the symbol of the composition $\mathbf{P} T_{i\tilde{A}}$. Since $\Lambda^s(\xi)$ only depends on the parameter ξ, the sum can be written as

$$\Lambda^s (\sum_{|\beta| \geq 0} \partial_\xi^\beta \mathbf{P} i D_x^\beta \tilde{A}) - (\sum_{|\beta| \geq 0} \partial_\xi^\beta \mathbf{P} i D_x^\beta \tilde{A}) \Lambda^s + \sum_{|\alpha| > 0} \partial_\xi^\alpha \Lambda^s D_x^\alpha (\sum_{|\beta| \geq 0} \partial_\xi^\beta \mathbf{P} i D_x^\beta \tilde{A}).$$

Now, since $\Lambda^s = (1 + |\xi|^2)^{\frac{s}{2}} Id$, then

$$\Lambda^s (\sum_{|\beta| \geq 0} \partial_\xi^\beta \mathbf{P} i D_x^\beta \tilde{A}) - (\sum_{|\beta| \geq 0} \partial_\xi^\beta \mathbf{P} i D_x^\beta \tilde{A}) \Lambda^s = 0,$$

namely the commutator $[\Lambda^s, \mathbf{P} T_{i\tilde{A}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}]$ has order less than or equal to s, and

$$|(\Sigma[\Lambda^s, \mathbf{P} T_{i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}] \tilde{\mathbf{v}}^\varepsilon, \Lambda^s \tilde{\mathbf{v}}^\varepsilon)_0| \leq c(||\tilde{\mathbf{v}}^\varepsilon||_s)||\tilde{\mathbf{v}}^\varepsilon||_s^2.$$

In a similar way, the commutator of the middle term of (7.2.44) has the following expansion:

$$[\Sigma, J_\varepsilon] = \Sigma J_\varepsilon - J_\varepsilon \Sigma + \sum_{|\alpha| > 0} D_\xi^\alpha J_\varepsilon D_x^\alpha \Sigma,$$

and, since $J_\varepsilon = j_\varepsilon(\xi)Id$, it results that $\Sigma J_\varepsilon - J_\varepsilon\Sigma = 0$. Then, the commutator $[\Sigma, J_\varepsilon]$ has order less than or equal to -1 and, since $\mathbf{P}T_{i\tilde{A}(\xi,J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}$ has order 1, from the composition theorem in Section 2 we have

$$([\Sigma, J_\varepsilon]\mathbf{P}T_{i\tilde{A}(\xi,J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}\Lambda^s\tilde{\mathbf{v}}^\varepsilon, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0 \leq c(||\tilde{\mathbf{v}}^\varepsilon||_s)||\tilde{\mathbf{v}}^\varepsilon||_s^2.$$

It remains to consider the last first of (7.2.44). From (7.2.41), $\Sigma = (T_W)^*T_W + \theta_\lambda^2(D_x)$, and, from Section 2 and the definition of the symbolic matrix W in (7.2.39), the symbol of degree 1 in the expansion of $\Sigma\mathbf{P}T_{i\tilde{A}(\xi,J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}$ is given by

$$(V^{-1})^*V^{-1}\mathbf{P}i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))(1-\theta_\lambda(\xi))^2 + \theta_\lambda^2(\xi)\mathbf{P}i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}})).$$

By construction, from (7.1.25) and (7.1.27), we have

$$\mathbf{P}i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}})) = ViDV^{-1}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}})),$$

where $D(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))$ is the diagonal matrix (7.2.38) of the real terms (7.1.24), and

$$(V^{-1})^*V^{-1}\mathbf{P}i\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))(1-\theta_\lambda(\xi))^2 = (V^{-1})^*iDV^{-1}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))(1-\theta_\lambda(\xi))^2.$$

We define

$$N := (V^{-1})^*iDV^{-1}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))(1-\theta_\lambda(\xi))^2.$$

Then

$$Re(i(V^{-1})^*V^{-1}\mathbf{P}\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))(1-\theta_\lambda(\xi))^2) = N + N^* = 0.$$

The second addend of the symbolic symmetrizer (7.2.41) gives

$$|Re(i\theta_\lambda^2(D_x)\mathbf{P}\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))\Lambda^s\tilde{\mathbf{v}}^\varepsilon, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0| \leq ||\theta_\lambda(D_x)\mathbf{P}\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))\Lambda^s\tilde{\mathbf{v}}^\varepsilon||_0||\Lambda^s\tilde{\mathbf{v}}^\varepsilon||_0.$$

From (7.2.40), we get

$$||\theta_\lambda(D_x)\mathbf{P}\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))\Lambda^s\tilde{\mathbf{v}}^\varepsilon||_0 \leq \sqrt{1+4\lambda^2}||\theta_\lambda(D_x)\mathbf{P}\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))\Lambda^s\tilde{\mathbf{v}}^\varepsilon||_{H^{-1}}$$

$$\leq 3\lambda||\mathbf{P}\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))\Lambda^s\tilde{\mathbf{v}}^\varepsilon||_{H^{-1}} \leq c(||\tilde{\mathbf{v}}^\varepsilon||_s)||\tilde{\mathbf{v}}^\varepsilon||_s.$$

This way,

$$|Re(\Sigma\mathbf{P}T_{i\tilde{A}(J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon+\bar{\mathbf{v}}))}\Lambda^s J_\varepsilon\tilde{\mathbf{v}}^\varepsilon, \Lambda^s J_\varepsilon\tilde{\mathbf{v}}^\varepsilon)_0| \leq c(||\tilde{\mathbf{v}}^\varepsilon||_s)||\tilde{\mathbf{v}}^\varepsilon||_s^2,$$

and, putting it all together, we have

$$\frac{d}{dt}(\Sigma\Lambda^s\tilde{\mathbf{v}}^\varepsilon, \Lambda^s\tilde{\mathbf{v}}^\varepsilon)_0 \leq c(||\tilde{\mathbf{v}}^\varepsilon||_s)||\tilde{\mathbf{v}}^\varepsilon||_s^2. \tag{7.2.45}$$

Let T_ε be the maximum time of existence of the solution to system (7.2.30). We want to show that there exists a time $T > 0$, which is independent of ε, such that $T \leq T_\varepsilon$ for every $\varepsilon > 0$. From Theorem 7.2.2, there exists a constant M such that $||\tilde{\mathbf{u}}_0^\varepsilon||_s \leq M$. Fixed a constant value $\tilde{M} > M$, let $T_0^\varepsilon \leq T_\varepsilon$ be a positive time such that the smooth solution $\tilde{\mathbf{v}}^\varepsilon$ verifies

$$sup_{\ 0\leq\tau\leq T_0^\varepsilon}\ ||\tilde{\mathbf{v}}^\varepsilon(\tau)||_s \leq \tilde{M}.$$

From (7.2.45), we get

$$||\tilde{\mathbf{v}}^{\varepsilon}(t)||_s \leq ||\tilde{\mathbf{v}}_0^{\varepsilon}||_s e^{c(\tilde{M})t}$$

for $t \in [0, T_0^{\varepsilon}]$. Let T, with $0 < T \leq T_0^{\varepsilon}$, be such that

$$Me^{c(\tilde{M})T} \leq \tilde{M}.$$

This holds if

$$T \leq \frac{log(\frac{\tilde{M}}{M})}{c(\tilde{M})}. \tag{7.2.46}$$

Since $M, \tilde{M}$ are independent of the parameter ε, estimate (7.2.46) implies that the time T is independent of ε and $(\tilde{\mathbf{v}}^{\varepsilon})_{\varepsilon \geq 0}$ is uniformly bounded provided that inequality (7.2.46) holds. □

7.2.1 Uniqueness

We can establish a uniqueness result in space larger than one where we are going to prove the existence of the solutions.

Theorem 7.2.4. *There is a unique solution* $\tilde{\mathbf{v}}$ *to problem (7.2.29) in the space*

$$Lip([0,T], Lip(\mathbb{R}^2)|\nabla \cdot w = 0) \cap L^{\infty}([0,T], V^0).$$

Proof. According to Definition 7.1.1, let $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2$ be two solutions to system (7.1.1), with the respective pressure terms P_1, P_2 and the same initial data $\tilde{\mathbf{v}}_1(0,x) = \tilde{\mathbf{v}}_2(0,x) = \tilde{\mathbf{v}}_0$. From (7.1.19), we have

$$\Sigma(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})\partial_t(\tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_1) + \sum_{j=1}^{2} \Sigma(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})\mathbf{P}T_{\tilde{A}_j(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})}\partial_{x_j}(\tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_1)$$

$$+\Sigma(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})\sum_{j=1}^{2}[\mathbf{P}\tilde{A}_j(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}}) - \mathbf{P}T_{\tilde{A}_j(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})}]\partial_{x_j}(\tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_1)$$

$$= \Sigma(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})\tilde{G}(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}}) - \Sigma(\tilde{\mathbf{v}}_1 + \bar{\mathbf{v}})\tilde{G}(\tilde{\mathbf{v}}_1 + \bar{\mathbf{v}})$$

$$+[\Sigma(\tilde{\mathbf{v}}_1 + \bar{\mathbf{v}}) - \Sigma(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})]\partial_t\tilde{\mathbf{v}}_1 + \sum_{j=1}^{2}[\Sigma(\tilde{\mathbf{v}}_1 + \bar{\mathbf{v}})\mathbf{P}T_{\tilde{A}_j(\tilde{\mathbf{v}}_1 + \bar{\mathbf{v}})} - \Sigma(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})\mathbf{P}T_{\tilde{A}_j(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})}]\partial_{x_j}\tilde{\mathbf{v}}_1$$

$$+\sum_{j=1}^{2}[\Sigma(\tilde{\mathbf{v}}_1 + \bar{\mathbf{v}})\mathbf{P}\tilde{A}_j(\tilde{\mathbf{v}}_1 + \bar{\mathbf{v}}) - \Sigma(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})\mathbf{P}\tilde{A}_j(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})]\partial_{x_j}\tilde{\mathbf{v}}_1$$

$$+\sum_{j=1}^{2}[\Sigma(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})\mathbf{P}T_{\tilde{A}_j(\tilde{\mathbf{v}}_2 + \bar{\mathbf{v}})} - \Sigma(\tilde{\mathbf{v}}_1 + \bar{\mathbf{v}})\mathbf{P}T_{\tilde{A}_j(\tilde{\mathbf{v}}_1 + \bar{\mathbf{v}})}]\partial_{x_j}\tilde{\mathbf{v}}_1.$$

As done before, this provides the following estimate:

$$\frac{d}{dt}(\Sigma\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2)_0 \leq c||\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2||_0^2, \tag{7.2.47}$$

namely $\tilde{\mathbf{v}}_1 = \tilde{\mathbf{v}}_2 = 0$, since $(\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2)(0,x) = \tilde{\mathbf{v}}_1(0,x) - \tilde{\mathbf{v}}_2(0,x) = 0$, where the constant value c in (7.2.47) only depends on $|\tilde{\mathbf{v}}|_{\infty}$, $|\partial_t\tilde{\mathbf{v}}|_{\infty}$ and $|\nabla\tilde{\mathbf{v}}|_{\infty}$. □

7.3 Some failed attempts

Let us consider system (7.1.8) in the old variables $\mathbf{u}$. Looking at (7.1.9), it is easy to find a diagonal matrix that symmetrizes the first order part $A_1(\mathbf{u})\partial_x\mathbf{u}, A_2(\mathbf{u})\partial_y\mathbf{u}$ of system (7.1.8). The *Friedrichs* (or classical) symmetrizer is

$$S_0(\mathbf{u}) = diag(\gamma/B, B, B, (1-B), (1-B)). \tag{7.3.48}$$

The existence of this symmetrizer for $A_1(\mathbf{u}), A_2(\mathbf{u})$ implies that, disregarding the pressure term, system (7.1.8) is hyperbolic. Nevertheless, that classical symmetrizer is not useful to close some energy estimates, since we have to deal also with the incompressible pressure term $F_P = (0, \nabla P, \nabla P)$. In fact, in the Sobolev spaces $H^s(\mathbb{R}^2)$ with $s > 2$, when we take the s-derivative of system (7.1.8) and multiply by $\nabla^s\mathbf{u}$ in order to get energy estimates, the right-hand side of the equation contains the following scalar product in $L^2(\mathbb{R}^2)$:

$$\begin{aligned}(S_0(\mathbf{u})\nabla^s F_P, \nabla^s\mathbf{u})_0 &= ((0, B\nabla^{s+1}P, (1-B)\nabla^{s+1}P), (\partial_x^s B, \nabla^s v_S, \nabla^s v_L))_0 \\ &= (B\nabla^{s+1}P, \nabla^s v_S)_0 + ((1-B)\nabla^{s+1}P, \nabla^s v_L)_0.\end{aligned}$$

Unfortunately, taking $\mathbf{u} \in H^s(\mathbb{R}^2)$, the pressure term P has not enough regularity, as shown by the elliptic equation (7.0.1), and then we are unable to close our estimates. Besides, the symmetrizer (7.3.48) depends on the variable $\mathbf{u}$, whose components do not depend explicitly on the average velocity $Bv_S + (1-B)v_L$, which is, instead, the divergence free vector field associated to (7.1.8). Then, we can try to use the new variables in (7.1.14), i.e.

$$w := Bv_S + (1-B)v_L, \quad z := v_S - v_L,$$

so setting $\mathbf{v} = (B, w, z)$. As we noticed in Section 2, passing to the new variable $\mathbf{v}$, the fourth equation of (4.0.6) yields $\nabla \cdot w = 0$, which is exactly the incompressibility condition for the mixture as a whole. Moreover, the equation for the average velocity w, which is

$$\partial_t w + w\cdot\nabla w + \nabla\cdot(B(1-B)z\otimes z) + \gamma\nabla B + \nabla P = 0,$$

contains the gradient of the incompressible pressure ∇P alone, without multiplication by any phase volume fraction, while the equation for the relative velocity z,

$$\partial_t z + w\cdot\nabla z + z\cdot\nabla w + z\cdot\nabla((1-B)z) - Bz\cdot\nabla z + \frac{\gamma\nabla B}{B} = -\frac{z(M+\Gamma_B(1-B))}{B(1-B)},$$

is free from the incompressible pressure. In the new variables $\mathbf{v}$, we get the compact system (7.1.16), that can be written as

$$\partial_t\mathbf{v} + T_{i\tilde{A}(\xi,\mathbf{v})}\mathbf{v} = T_{\tilde{G}(\xi,\mathbf{v})} + [T_{i\tilde{A}(\xi,\mathbf{v})} - \sum_{j=1}^{2}\tilde{A}_j(\mathbf{v})\partial_{x_j}]\mathbf{v} + [\tilde{G}(\mathbf{v}) - T_{\tilde{G}(\mathbf{v})}],$$

with $T_{i\tilde{A}(\xi,\mathbf{v})}$ in (7.1.21). The symbolic matrix $\tilde{A}(\xi,\mathbf{v})$ in (7.1.22) has the following eigenvalues:

$$\begin{cases}\lambda_1 = \lambda_2 = (w - Bz)\cdot\xi, \\ \lambda_3 = (w + (1-B)z)\cdot\xi, \\ \lambda_4 = (w + (1-B)z)\cdot\xi - \sqrt{\gamma}|\xi|, \\ \lambda_5 = (w + (1-B)z)\cdot\xi + \sqrt{\gamma}|\xi|.\end{cases} \tag{7.3.49}$$

They are real, then, as long as we neglect the incompressible pressure $\tilde{F}_P$, the remaining system in (7.1.16) is hyperbolic, as the symmetrizable system (7.1.8) in the old variable $\mathbf{u}$. Moreover, the eigenvectors of (7.1.22) are the columns of

$$U(\xi,\mathbf{u}) = \begin{pmatrix} 0 & 0 & 0 & -\frac{B}{\sqrt{\gamma}} & \frac{B}{\sqrt{\gamma}} \\ -(1-B) & 0 & \frac{-B\xi_2}{|\xi|} & \frac{B(\gamma\xi_1 - z_1\sqrt{\gamma}|\xi|)}{\gamma|\xi|} & \frac{B(\gamma\xi_1 + z_1\sqrt{\gamma}|\xi|)}{\gamma|\xi|} \\ 0 & -(1-B) & \frac{B\xi_1}{|\xi|} & \frac{B(\gamma\xi_2 - z_2\sqrt{\gamma}|\xi|)}{\gamma|\xi|} & \frac{B(\gamma\xi_2 + z_2\sqrt{\gamma}|\xi|)}{\gamma|\xi|} \\ 1 & 0 & \frac{-\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{\xi_1}{|\xi|} \\ 0 & 1 & \frac{\xi_1}{|\xi|} & \frac{\xi_2}{|\xi|} & \frac{\xi_2}{|\xi|} \end{pmatrix}, \quad (7.3.50)$$

while its inverse matrix is

$$U^{-1}(\xi,\mathbf{u}) = \begin{pmatrix} z_1 & -1 & 0 & B & 0 \\ z_2 & 0 & -1 & 0 & B \\ \frac{\xi_2 z_1 - \xi_1 z_2}{|\xi|} & \frac{-\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{-(1-B)\xi_2}{|\xi|} & \frac{(1-B)\xi_1}{|\xi|} \\ -\frac{\sqrt{\gamma}|\xi| + B(z\cdot\xi)}{2B|\xi|} & \frac{\xi_1}{2|\xi|} & \frac{\xi_2}{2|\xi|} & \frac{(1-B)\xi_1}{2|\xi|} & \frac{(1-B)\xi_2}{2|\xi|} \\ \frac{\sqrt{\gamma}|\xi| - B(z\cdot\xi)}{2B|\xi|} & \frac{\xi_1}{2|\xi|} & \frac{\xi_2}{2|\xi|} & \frac{(1-B)\xi_1}{2|\xi|} & \frac{(1-B)\xi_2}{2|\xi|} \end{pmatrix}. \quad (7.3.51)$$

Since (7.3.50) and (7.3.51) are bounded for each $\xi \in \mathbb{R}^2 - \{\mathbf{0}\}$, the regularized the symbolic matrix

$$\tilde{S}(\xi,\mathbf{v}) := (U^{-1}(1-\theta_\lambda(\xi)))^* U^{-1}(\xi,\mathbf{v})(1-\theta_\lambda(\xi)) + \theta_\lambda^2(\xi) Id \quad (7.3.52)$$

can be associated to a *Lax* symbolic symmetrizer $T_{\tilde{S}}$, as done before in Section 3. Therefore, the hyperbolic part of (7.1.16) is symmetrizable. Unfortunately, again, the mere existence of a symmetrizer is not enough to get energy estimates for system (7.1.16), since we have to deal with the incompressible pressure term and then, by definition, with the projector operator in (7.1.18). As a matter of fact, the interaction between the symbolic symmetrizer $T_{\tilde{S}}$ and the gradient of the pressure term ∇P gives structural problems. The operators $T_{\tilde{S}}$ and the projector $\mathbf{P}$ in (7.1.18) do not commute and their commutator does not improve on the order of the original symbols. By construction, S symmetrizes $A(\xi,\mathbf{v})$ in (7.1.22), then we write (7.1.19) as

$$\partial_t \mathbf{v} + \sum_{j=1}^{2} \tilde{A}_j(\mathbf{v})\partial_{x_j}\mathbf{v} + \sum_{j=1}^{2} [\tilde{A}_j, \mathbf{P}]\partial_{x_j}\mathbf{v} = \mathbf{P}\tilde{G}(\mathbf{v})$$

and we apply $T_{\tilde{S}}$ to its paradifferential formulation. Unfortunately, from (7.1.18), (7.1.22) and (7.1.21), the first term of the symbolic commutator in $\sum_{j=1}^{2}[\tilde{A}_j, \mathbf{P}]\partial_{x_j}\mathbf{v}$ contains the following term of degree 1 in ξ:

$$\tilde{A}(\xi,\mathbf{v})\mathbf{P}(\xi) - \mathbf{P}(\xi)\tilde{A}(\xi,\mathbf{v}),$$

then the commutator between $\tilde{A}(\xi,\mathbf{v})$ and $\mathbf{P}$ is still a symbol of degree 1, and it is not symmetrized by S. On the other hand, if at first we symmetrize the system by using the paradifferential operator $T_{\tilde{S}}$, the pressure gives the term $T_{\tilde{S}}\tilde{F}_P$, which is still an operator of the first order. After that, when we project the equation by applying $\mathbf{P}$ to it, the latter term reads

$$\mathbf{P}T_{\tilde{S}}\tilde{F}_P = [T_{\tilde{S}}, \mathbf{P}]\tilde{F}_P,$$

whose symbol contains the smoothed version of the following term of degree 1 in ξ:

$$S\mathbf{P}(0, i\xi_1 P, i\xi_2 P, 0, 0)$$
$$= \frac{i}{2|\xi|^2}(P(z\cdot\xi)(2\xi_1^2+3\xi_2^2), 0, 0, P\xi_1(2B\xi_1^2+3B\xi_2^2-\xi_2^2), P\xi_2(2B\xi_1^2+3B\xi_2^2-\xi_2^2)),$$

which is still a first order operator, then neither we are able to get energy estimates because of the lack of regularity of P, as discussed before and shown in (7.0.1), nor to get rid of the pressure term P by using the projector operator $\mathbf{P}$.
To be complete, we point out that system (7.1.8) in the new variable $\mathbf{v} = (B, w, z)$ has also a classical symmetrizer (see Section 2), given by

$$A_0(\mathbf{v}) = \begin{pmatrix} \frac{\gamma}{B}+|z|^2 & -z_1 & -z_2 & 0 & 0 \\ -z_1 & 1 & 0 & 0 & 0 \\ -z_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & B(1-B) & 0 \\ 0 & 0 & 0 & 0 & B(1-B) \end{pmatrix}, \tag{7.3.53}$$

which is strictly positive for z small enough and under some assumptions on B, discussed in Remark 7.1.1, then its positivity is verified in a small neighbourhood of the admissible equilibrium point (7.1.11) in the variables $\mathbf{v}$ in (7.1.15). Anyway, again, the classical symmetrizer (7.3.53) is not compatible with the projector operator (7.1.18), in the sense that the scalar product induced by the classical symmetrizer does not preserve the orthogonality between the gradient of the incompressible pressure ∇P and the divergence free average velocity w, i.e.

$$(A_0(\mathbf{v})\tilde{F}_P, \mathbf{v})_0 \neq 0, \quad \text{while} \quad (F_P, \mathbf{v})_0 = (\nabla P, w)_0 = 0,$$

and the resulting incompatibility can be seen by arguing as for the *Lax* symmetrizer $T_{\tilde{S}}$.

7.4 The original biofilms system: a multi-solid-phases model

We consider system (4.0.2), which can be written as

$$\begin{cases} \partial_t B + \nabla\cdot(Bv_S) = \Gamma_B := k_B BL - k_D B, \\ \partial_t D + \nabla\cdot(Dv_S) = \Gamma_D := \alpha B k_D - k_N D, \\ \partial_t E + \nabla\cdot(Ev_S) = \Gamma_E := BLk_E - \varepsilon E, \\ \partial_t v_S + v_S\cdot\nabla v_S + \frac{\gamma\nabla(B+D+E)}{(B+D+E)} + \nabla P = \Gamma_{v_S} := \frac{(M+\Gamma_B+\Gamma_D+\Gamma_E)(v_L-v_S)}{B+D+E}, \\ \partial_t v_L + v_L\cdot\nabla v_L + \nabla P = \Gamma_{v_L} := \frac{M(v_S-v_L)}{1-(B+D+E)}, \\ \nabla\cdot((B+D+E)v_S + (1-(B+D+E))v_L) = 0, \\ B+D+E+L = 1, \end{cases} \tag{7.4.54}$$

where $k_B, k_D, k_E, k_N, \alpha, \varepsilon$ are experimental constants. Now, $\mathbf{P}\tilde{A}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))$ has the following eigenvalues:

$$\begin{aligned} &\lambda_1 = 0, \\ &\lambda_2 = (w - Bz)\cdot\xi, \\ &\lambda_3 = \lambda_4 = \lambda_5 = (w + (1-B)z)\cdot\xi, \\ &\lambda_{6/7} = (w + (1-B-\nu))\cdot\xi \pm \sqrt{(1-\nu)(\gamma|\xi|^2 - \nu(z\cdot\xi)^2)}. \end{aligned} \tag{7.4.55}$$

Besides, the eigenvectors are the columns of $V(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))$

$$\begin{pmatrix} \frac{q_{11}}{|\xi|\Delta_1} & 0 & 1 & -1 & 0 & \frac{B|\xi|\Delta_4}{\Delta_2} & \frac{-B|\xi|\Delta_4}{\Delta_2} \\ \frac{q_{21}}{|\xi|\Delta_1} & 0 & -1 & 0 & 0 & \frac{D|\xi|\Delta_4}{\Delta_2} & \frac{-D|\xi|\Delta_4}{\Delta_2} \\ \frac{q_{31}}{|\xi|\Delta_1} & 0 & 0 & 1 & 0 & \frac{E|\xi|\Delta_4}{\Delta_2} & \frac{-E|\xi|\Delta_4}{\Delta_2} \\ \frac{q_{41}}{|\xi|\Delta_1} & \frac{(1-\nu)\xi_2}{|\xi|} & 0 & 0 & \frac{-\nu\xi_2}{|\xi|} & \frac{-\xi_2\Delta_5}{|\xi|\Delta_2} & \frac{\xi_2\Delta_5}{|\xi|\Delta_2} \\ \frac{q_{51}}{|\xi|\Delta_1} & \frac{-(1-\nu)\xi_1}{|\xi|} & 0 & 0 & \frac{\nu\xi_1}{|\xi|} & \frac{\xi_1\Delta_5}{|\xi|\Delta_2} & \frac{-\xi_1\Delta_5}{|\xi|\Delta_2} \\ \frac{\xi_1}{|\xi|} & \frac{-\xi_2}{|\xi|} & 0 & 0 & \frac{-\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{\xi_1}{|\xi|} \\ \frac{\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & 0 & 0 & \frac{\xi_1}{|\xi|} & \frac{\xi_2}{|\xi|} & \frac{\xi_2}{|\xi|} \end{pmatrix}, \tag{7.4.56}$$

where

$$q_{11} = q_{11}(\xi, J_\varepsilon\tilde{\mathbf{v}}^\varepsilon, \bar{\mathbf{v}}), q_{21} = q_{21}(\xi, J_\varepsilon\tilde{\mathbf{v}}^\varepsilon, \bar{\mathbf{v}}), q_{31} = q_{31}(\xi, J_\varepsilon\tilde{\mathbf{v}}^\varepsilon, \bar{\mathbf{v}}), q_{41} = q_{41}(\xi, J_\varepsilon\tilde{\mathbf{v}}^\varepsilon, \bar{\mathbf{v}}),$$

$$q_{51} = q_{51}(\xi, J_\varepsilon\tilde{\mathbf{v}}^\varepsilon, \bar{\mathbf{v}})$$

are polynomial functions of degree 3 in the ξ variable, Δ_1 in (7.1.26), and

$$\begin{cases} \nu := B + D + E, \\ \Delta_2 := \gamma|\xi|^2 - (B + D + E)(z \cdot \xi)^2, \\ \Delta_4 := \sqrt{(1-\nu)\Delta_2}, \\ \Delta_5 := (\xi_1 z_2 - \xi_2 z_1)\nu\Delta_4, \end{cases}$$

while $V^{-1}(\xi, J_\varepsilon(\tilde{\mathbf{v}}^\varepsilon + \bar{\mathbf{v}}))$

$$= \begin{pmatrix} 0 & 0 & 0 & \frac{-\xi_1}{|\xi|\sqrt{1-\nu}} & \frac{-\xi_2}{|\xi|\sqrt{1-\nu}} & 0 & 0 \\ 0 & 0 & 0 & \frac{\xi_2}{|\xi|} & \frac{-\xi_1}{|\xi|} & \frac{-\xi_2\nu}{|\xi|} & \frac{\xi_1\nu}{|\xi|} \\ \frac{-D}{\nu} & \frac{B+E}{\nu} & \frac{-D}{\nu} & 0 & 0 & 0 & 0 \\ \frac{-E}{\nu} & \frac{-E}{\nu} & \frac{B+D}{\nu} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{-(1-\nu)\xi_2}{|\xi|} & \frac{(1-\nu)\xi_1}{|\xi|} \\ \frac{\sqrt{\gamma}}{2\nu\sqrt{1-\nu}} & \frac{\sqrt{\gamma}}{2\nu\sqrt{1-\nu}} & \frac{\sqrt{\gamma}}{2\nu\sqrt{1-\nu}} & \frac{\xi_1}{2|\xi|(1-\nu)} & \frac{\xi_2}{2|\xi|(1-\nu)} & \frac{\xi_1}{2|\xi|} & \frac{\xi_2}{2|\xi|} \\ \frac{-\sqrt{\gamma}}{2\nu\sqrt{1-\nu}} & \frac{-\sqrt{\gamma}}{2\nu\sqrt{1-\nu}} & \frac{-\sqrt{\gamma}}{2\nu\sqrt{1-\nu}} & \frac{\xi_1}{2|\xi|(1-\nu)} & \frac{\xi_2}{2|\xi|(1-\nu)} & \frac{\xi_1}{2|\xi|} & \frac{\xi_2}{2|\xi|} \end{pmatrix}.$$

Since V and V^{-1} are bounded for each $\xi \in \mathbb{R}^2 - \{\mathbf{0}\}$, we can apply the arguments developed for system (4.0.6) to the complete case (7.4.54).

7.5 An open problem: the three dimensional two phases model

The three dimensional case contains structural difficulties that we are not able to solve. In three space dimensions, if we project the main operator we obtain

$$\mathbf{P}\tilde{A}(\xi, \mathbf{v}) = \mathbf{P}(\xi)(\tilde{A}_1(\mathbf{v})\xi_1 + \tilde{A}_2(\mathbf{v})\xi_2 + \tilde{A}_3(\mathbf{v})\xi_3), \tag{7.5.57}$$

with the following eigenvalues:

$$\begin{aligned}
&\lambda_1 = 0,\\
&\lambda_2 = \lambda_3 = (w - Bz)\cdot\xi,\\
&\lambda_4 = \lambda_5 = (w + (1-B)z)\cdot\xi,\\
&\lambda_{6/7} = (w + (1-2B)z)\cdot\xi \pm \sqrt{(1-B)\Delta_2},
\end{aligned} \tag{7.5.58}$$

where $\Delta_2 = \gamma|\xi|^2 - B(z\cdot\xi)^2$. To simplify the discussion, the matrix with the eigenvectors on the columns, $V(\xi, \mathbf{v})$, has been calculated in the equilibrium point (7.1.10) $\bar{\mathbf{v}} = (\bar{B}, \bar{w}, \bar{z}) = (\bar{B}, \mathbf{0}, \mathbf{0})$, with $\bar{B} = 1 - \frac{k_D}{k_B}$. We have

$$V(\xi, \mathbf{v}) = \frac{1}{|\xi|}$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \bar{B}\sqrt{\frac{1-\bar{B}}{\gamma}}|\xi| & -\bar{B}\sqrt{\frac{1-\bar{B}}{\gamma}}|\xi| \\
-(1-\bar{B})\xi_1 & \xi_2(1-\bar{B}) & \xi_3(1-\bar{B}) & -\bar{B}\xi_2 & -\bar{B}\xi_3 & 0 & 0 \\
-(1-\bar{B})\xi_2 & -(1-\bar{B})\xi_1 & 0 & \bar{B}\xi_1 & 0 & 0 & 0 \\
-(1-\bar{B})\xi_3 & 0 & -(1-\bar{B})\xi_1 & 0 & \bar{B}\xi_1 & 0 & 0 \\
\xi_1 & -\xi_2 & -\xi_3 & -\xi_2 & -\xi_3 & \xi_1 & \xi_1 \\
\xi_2 & \xi_1 & 0 & \xi_1 & 0 & \xi_2 & \xi_2 \\
\xi_3 & 0 & \xi_1 & 0 & \xi_1 & \xi_3 & \xi_3
\end{pmatrix}. \tag{7.5.59}$$

For $\xi_1 = 0$, we get

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \bar{B}\sqrt{\frac{1-\bar{B}}{\gamma}}|\xi| & -\bar{B}\sqrt{\frac{1-\bar{B}}{\gamma}}|\xi| \\
0 & (1-\bar{B})\xi_2 & (1-\bar{B})\xi_3 & -\bar{B}\xi_2 & -\bar{B}\xi_3 & 0 & 0 \\
-(1-\bar{B})\xi_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-(1-\bar{B})\xi_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\xi_2 & -\xi_3 & -\xi_2 & -\xi_3 & 0 & 0 \\
\xi_2 & 0 & 0 & 0 & 0 & \xi_2 & \xi_2 \\
\xi_3 & 0 & 0 & 0 & 0 & \xi_3 & \xi_3
\end{pmatrix}. \tag{7.5.60}$$

The second and the third columns of (7.5.60), namely the eigenvectors of the second and the third coincident eigenvalues in (7.5.58), degenerate in the same vector when $\xi_1 = 0$. This happens also to the fourth and the fifth columns of (7.5.60), i.e. the fourth and the fifth coincident eigenvalues in (7.5.58). For this reason, in the three dimensional case the symbol $\mathbf{P}\tilde{A}(\xi, \mathbf{v})$ in (7.5.57) loses the property of *strong symmetrizability* and related *microlocal symmetrizability*, according to the definitions given in [55].

Part III

BGK approximations to hydrodynamics equations

Chapter 8

Introduction to the BGK models

This chapter is devoted to the presentation of the BGK models, which are a class of kinetic approximations to hyperbolic and parabolic systems. The present introduction is based on [16, 59]. These BGK models were introduced by Bhatnagar, Gross and Krook as a modified version of the Boltzmann equation, characterized by a simplification of the collision operator. Later, these models have been generalized in order to approximate different systems. Originally, they presented continuous velocities, see [64], but there exists also a subclass of discrete velocities BGK models. Here we will focus on discrete velocities BGK approximations. The main advantage of this approach is to deal with semilinear systems, in the spirit of the relaxation approximation, see [59, 22, 36]. Let us define the general framework of these systems.
Consider a general system of conservation laws,

$$\partial_t \mathbf{u} + \sum_{j=1}^{d} \partial_{x_j} F_j(\mathbf{u}) = 0, \tag{8.0.1}$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, $\mathbf{u}(t, x)$ belongs to a convex subset $\mathcal{U} \in \mathbb{R}^N$, and F_j, $j = 1, \cdots d$ are given smooth functions. We assume that there exists an *entropy* for system (8.0.1), i.e. a function $\eta : \mathcal{U} \to \mathbb{R}$ such that there exist function $g_j : \mathcal{U} \to \mathbb{R}$ satisfying

$$\nabla_{\mathbf{u}} g_j = (F_j')^T \nabla_{\mathbf{u}} \eta, \qquad j = 1, \cdots, d.$$

Notice that to guarantee the existence of an entropy, we need to prove the previous condition, namely the differential form $(\nabla_{\mathbf{u}} \eta) F_j'$ has to be exact. This is true if $\eta'' F_j'$ is symmetric. Moreover, if the entropy η is strictly convex, η'' defines a scalar product on the Sobolev spaces which symmetrizes system (8.0.1) in the classical sense, see Chapter 2. Then, let $\mathcal{E}$ be a non-empty set of convex entropies for (8.0.1). Assume also that $\mathcal{E}$ is separable. A BGK model reads as follows

$$\partial_t f_i + \lambda_i \cdot \nabla_x f_i = \frac{1}{\varepsilon}(M_i(\mathbf{u}) - f_i), \qquad i = 1, \cdots, L, \tag{8.0.2}$$

where $\varepsilon > 0$ is the parameter of the singular approximation, $L \geq d$ and, for $i = 1, \cdots, L$,

$$\begin{aligned} &f_i(t, x) = (f_i^1, \cdots, f_i^N) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^N, \\ &\lambda_i = (\lambda_i^1, \cdots, \lambda_i^d), \\ &Mi(\mathbf{u}) = (M_i^1, \cdots, M_i^N) : \mathbb{R}^N \to \mathbb{R}^N. \end{aligned}$$

Here we set

$$\mathbf{u} = \sum_{i=1}^{L} f_i,$$

while $M_i(\mathbf{u})$ are the so-called *Mawxellian functions.* In order to have the consistency of the BGK approximation with respect to the limit system (8.0.1), we have to assume some compatibility conditions,

$$\sum_{i=1}^{L} M_i(\mathbf{u}) = \mathbf{u}, \tag{8.0.3}$$

$$\sum_{i=1}^{L} \lambda_{ij} M_i(\mathbf{u}) = F_j(\mathbf{u}), \qquad j = 1, \cdots, d. \tag{8.0.4}$$

Remark 8.0.1. *We point out that BGK models have been used also to approximate parabolic systems. In this case, in order to get the second order derivative in the limit, additional compatibility conditions on the Maxwellian functions are needed. We remind to [2, 17] for a detailed discussion.*

An important feature of these approximation is the existence, under some reasonable conditions, of a kinetic entropy. Set $\mathcal{D}_i := \{M_i(\mathbf{u}) : \mathbf{u} \in \mathcal{U}\}$.

Definition 8.0.1. *A kinetic entropy for system (8.0.2) is a convex function $H(f) = \sum_{i=1}^{L} H_i(f_i)$, with $H_i : \mathcal{D}_i \to \mathbb{R}$, such that*

- $H(M(\boldsymbol{u})) = \eta(\boldsymbol{u})$ *for every* $\boldsymbol{u} \in \mathcal{U}$*;*

- $H(M(\boldsymbol{u}_f)) \leq H(f)$*, where* $\boldsymbol{u}_f := \sum_{i=1}^{L} f_i \in \mathcal{U}$*,* $f_i \in \mathcal{D}_i$*.*

This property provides an energy inequality which gives robustness for the scheme. Indeed, it is easy to see that, multiplying the BGK system (8.0.2) by $\nabla_{f_i} H_i(f_i)$, we obtain

$$\partial_t H(f) + \sum_{j=1}^{d} \partial_{x_j} \left(\sum_{i=1}^{L} \lambda_{ij} H_i(f_i) \right) \leq 0.$$

In order to state the existence result for kinetic entropies due to *Bouchut*, see [16], we need to introduce some preliminary notions. Let us define the space of Maxwellians

$$\mathcal{M}^{\varepsilon} = \{M : \mathcal{U} \to \mathbb{R}^{LN} \,|\, \forall \eta \in \mathcal{E},\, \forall i : \, (M_i')^T \eta'' \quad \text{is symmetric everywhere in } \mathcal{U}\},$$

and the convex cone of nondecreasing Maxwellians

$$\mathcal{M}^{\varepsilon}_{+} = \{M : \mathcal{U} \to \mathbb{R}^{LN} \,|\, \forall \eta \in \mathcal{E},\, \forall i : \, (M_i')^T \eta'' \geq 0 \quad \text{everywhere in } \mathcal{U}\}.$$

We also introduce the notion of *microscopic entropy*, i.e.

$$\tilde{H}(\mathbf{u}) = \sum_{i=1}^{L} \tilde{H}_i(\mathbf{u}) = \sum_{i=1}^{L} H_i(M_i(\mathbf{u})).$$

Thus, the first condition in Definition 8.0.1 can be written as

$$\tilde{H}(\mathbf{u}) = \eta(\mathbf{u}).$$

As mentioned before, here we present a characterization of kinetic entropies proved in [16].

Theorem 8.0.1. *Let $\mathcal{U}$ be an open set of $\mathbb{R}^N$, and $M \in C^1(\mathcal{U})$. We assume that:*

- *for η in a dense subset of $\mathcal{E}$, $\eta'' > 0$ and $\nabla_{\boldsymbol{u}}\eta(\mathcal{U})$ is convex;*
- *for $i = 1, \cdots, L$, M_i is a C^1 diffeomorphism from $\mathcal{U}$ onto the convex open subset $\mathcal{D}_i$ defined above.*

Then, the existence of convex functions $(H)_{\eta\in\mathcal{E}}$ satisfying Definition 8.0.1 and such that $\tilde{H}(\boldsymbol{u})$ defined above is $C^1(\mathcal{U})$ is equivalent to

$$M \in \mathcal{M}_+^{\varepsilon}.$$

Moreover, if this holds, then

$$\forall \boldsymbol{u} \in \mathcal{U}, \forall i: \; \nabla_{\boldsymbol{u}}\tilde{H}_i = (M_i')^T \nabla_{\boldsymbol{u}}\eta.$$

In order to use in practice the previous result, we present an important characterization of the space of the positive Maxwellians $\mathcal{M}_+^{\varepsilon}$, see [16].

Proposition 8.0.1. *Consider and open subset $\mathcal{U} \in \mathbb{R}^N$. Assume $\mathcal{E}$ contains at least a strictly convex entropy η_0, and $M \in \mathcal{M}_+^{\varepsilon}$ belongs to $C^1(\mathcal{U})$. Then, for all $\boldsymbol{u} \in \mathcal{U}$, and $i = 1, \cdots, L$, the Jacobian matrix M_i' is diagonalizable (and thus has only real eigenvalues). Moreover, $M \in \mathcal{M}_+^{\varepsilon}$ if and only if*

$$\forall \boldsymbol{u} \in \mathcal{U}, \forall i = 1, \cdots, L: \quad \sigma(M_i') \subset [0, \infty),$$

where σ is the spectrum.

The BGK models come from the ideas of kinetic approximations for compressible flows. They are inspired by the hydrodynamic limits of the Boltzmann equation: see [5, 6, 24] for the limit to the compressible Euler equations, and see [30, 33] for the incompressible Navier-Stokes equations. In this regard, one of the main directions has been the approximation of hyperbolic systems with continuous or discrete velocities BGK models, as in [21, 43, 59, 16, 64]. Similar results have been obtained for convection-diffusion systems under the diffusive scaling [50, 17, 48, 2]. In the framework of the BGK approximations, one of the first important contributions was given in computational physics by the so- called *Lattice-Boltzmann methods*, see for instance [73, 77]. Under some assumptions on the physical parameters, LBMs approximate the incompressible Navier-Stokes equations by scalar velocities models of kinetic equations, and a rigorous mathematical result on the validity of these kinds of approximations was proved in [44]. Other partially hyperbolic approximations of the Navier-Stokes equations were developed in [22, 63, 37, 36]. The vector BGK systems studied here are a combination of the ideas of discrete velocities BGK approximations and LBMs. They are called *vector BGK models* since, unlike the LBMs [73, 77], they associate every scalar velocity with one vector of unknowns. As we metioned before, another fruitful property of vector BGK models is their natural compatibility with a mathematical entropy, [16], which provides a nice analytical structure and stability properties.
In the context of semilinear relaxation approximations, in Chapter 9 we consider a singular parabolic scaling to the *Jin-Xin* approximation for conservation laws, see [43]. In the one dimensional case, this system can be written as a very simple BGK model.

Thus, we study its smooth solutions, and, by using an approach based on the Green function associated with the system, in the spirit of the work in [15], we also investigate their asymptotic behavior in the singular perturbation limit. This work can be seen as an introduction to a more complex problem, which is studied in Chapter 10. The work of the Chapter 10 takes its roots in [23, 18], where vector BGK approximations for the incompressible Navier-Stokes equations were introduced. We prove a rigorous local in time convergence result for the smooth solutions to the vector BGK system to the smooth solutions to the Navier-Stokes equations. A further investigation would be to try to apply the study of the Green function of the scaled Jin-Xin system in Chapter 9 to the vector BGK model in Chapter 10, in order to extend the local in time result for smooth solutions.

Chapter 9

The Jin-Xin model under the diffusion scaling: uniform asymptotic and convergence estimates

We consider the following scaled version of the Jin-Xin approximation for systems of conservation laws in [43]:

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \varepsilon^2 \partial_t v + \lambda^2 \partial_x u = f(u) - v, \end{cases} \tag{9.0.1}$$

where $\lambda > 0$ is a positive constant, u, v depend on $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and take values in $\mathbb{R}$, while $f(u) : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that $f(0) = 0$, and $f'(0) = a$, with a a constant value. The diffusion limit of this system for $\varepsilon \to 0$ has been studied in [42, 17], where the convergence to the following equations is proved:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ v = f(u) - \lambda^2 \partial_x u. \end{cases} \tag{9.0.2}$$

From [59, 17], it is well-known that system (9.0.1) can be written in BGK formulation, [16], by means of the linear change of variables:

$$u = f_1^\varepsilon + f_2^\varepsilon, \qquad v = \frac{\lambda}{\varepsilon}(f_1^\varepsilon - f_2^\varepsilon). \tag{9.0.3}$$

Precisely, the BGK form of (9.0.1) reads:

$$\begin{cases} \partial_t f_1^\varepsilon + \dfrac{\lambda}{\varepsilon} \partial_x f_1^\varepsilon = \dfrac{1}{\varepsilon^2}(M_1(u) - f_1^\varepsilon), \\ \partial_t f_2^\varepsilon - \dfrac{\lambda}{\varepsilon} \partial_x f_2^\varepsilon = \dfrac{1}{\varepsilon^2}(M_2(u) - f_2^\varepsilon), \end{cases} \tag{9.0.4}$$

where the so-called Maxwellians are:

$$M_1(u) = \frac{u}{2} + \frac{\varepsilon f(u)}{2\lambda}, \qquad M_2(u) = \frac{u}{2} - \frac{\varepsilon f(u)}{2\lambda}. \tag{9.0.5}$$

According to the theory on diffusive limits of the Boltzmann equation and related BGK models, see [33, 67], we take some fluctuations of the Maxwellian functions as initial data for the Cauchy problem associated with system (9.0.1). Namely, given a function $\bar{u}_0(x)$, depending on the spatial variable, we assume

$$(u(0,x), v(0,x)) = (u_0, v_0) = (\bar{u}_0,\ f(\bar{u}_0) - \lambda^2 \partial_x \bar{u}_0), \tag{9.0.6}$$

indeed perturbations of the Maxwellians, as it is clear by expressing the initial data (9.0.6) through the change of variables (9.0.3), i.e.

$$(f_1^\varepsilon(0,x), f_2^\varepsilon(0,x)) = \left(M_1(\bar{u}_0) - \frac{\varepsilon\lambda}{2}\partial_x \bar{u}_0,\ M_2(\bar{u}_0) + \frac{\varepsilon\lambda}{2}\partial_x \bar{u}_0 \right), \tag{9.0.7}$$

where the fluctuations are given by $\pm \frac{\varepsilon\lambda}{2}\partial_x \bar{u}_0$.
System (9.0.1) is the parabolic scaled version of the hyperbolic relaxation approximation for systems of conservation laws, the Jin-Xin system, introduced in [43] in 1995. This model has been studied in [60, 26, 43], and the hyperbolic relaxation limit has been investigated. A complete review on hyperbolic conservation laws with relaxation, and a focus on the Jin-Xin system is presented in [54]. By means of the Chapman-Enskog expansion, local attractivity of diffusion waves for the Jin-Xin model was established in [26]. In [51], the authors showed that, under some assumptions on the initial data and the function $f(u)$, the first component of system (9.0.1) with $\varepsilon = 1$ decays asymptotically towards the fundamental solution to the Burgers equation, for the case of $f(u) = \alpha u^2/2$. Besides, [62] is a complete study of the long time behavior of this model for a more general class of functions $f(u) = |u|^{q-1}u$, with $q \geq 2$. The method developed in [62] can be also extended to the multidimensional case in space, and provides sharp decay rates. Here we study the parabolic scaled version of the system studied in [62], i.e. (9.0.1), and we consider a more general function $f(u) = au + h(u)$, where a is a constant, and $h(u)$ is a quadratic function. We point out that only the case $a = 0$ has been handled in [62], and in many previous works as well. In accordance with the theory presented in [15] on partially dissipative hyperbolic systems, we are able to cover also the case $a \neq 0$. Furthermore, besides the aymptotic behavior of the solutions, here we are interested in studying the diffusion limit, for vanishing ε, of the Jin-Xin system, which is the main improvement of the present chapter with respect to the results achieved in [15]. Indeed, because of the presence of the singular parameter, we cannot approximate the analysis of the Green function of the linearized problems, as the authors did in [15], and explicit calculations in that context are needed.
The diffusive Jin-Xin system has been already investigated in the following works below. In [42], initial data around a traveling wave were considered, while in [17] the authors write system (9.0.1) in terms of a BGK model, and the diffusion limit is studied by using monotonicity properties of the solution. In all these cases, u, v are scalar functions. For simplicity, here we also take scalar unknowns u, v. However, our approach, which takes its roots in [15], can be generalized to the case of vectorial functions $u, v \in \mathbb{R}^N$. As mentioned before, the novelty of the present chapter consists in dealing with the singular approximation and, in the meanwhile, with the large time asymptotic of system (9.0.1), which behaves like the limit parabolic equation (9.0.2), without using monotonicity arguments. We obtain, indeed, sharp decay estimates in time to the solution to system (9.0.1) in the Sobolev spaces, which are uniform with respect to the singular parameter.

This provides the convergence to the limit nonlinear parabolic equation (9.0.2) both asymptotically in time, and in the vanishing ε-limit. To this end, we perform an crucial change of variables that highlights the dissipative property of the Jin-Xin system, and provides a faster decay of the dissipative variable with respect to the conservative one, which allows to close the estimates. Next, a deep investigation on the Green function of the linearized system (9.0.1) and the related spectral analysis is provided, since explicit expressions are needed in order to deal with the singular parameter ε. The dissipative property of the diffusive Jin-Xin system, together with the uniform decay estimates discussed above, and the Green function analysis combined with the Duhamel formula provide our main result. Consider the following equation

$$\partial_t w_p + a\partial_x w_p + \partial_x h(w_p) - \lambda^2 \partial_{xx} w_p = 0, \tag{9.0.8}$$

and the definition below, where $\|\cdot\|_m$ stands for the $H^m(\mathbb{R})$ Sobolev norm and $H^0(\mathbb{R}) = L^2(\mathbb{R})$,

$$E_m = \max\{\|u_0\|_{L^1} + \varepsilon\|v_0 - au_0\|_{L^1}, \|u_0\|_m + \varepsilon\|v_0 - au_0\|_m\}.$$

Our main result is stated here.

Theorem 9.0.1. *Let w_p be the solution to the nonlinear equation (9.0.8) with sufficiently smooth initial data*

$$w_p(0) = u(0) = u_0,$$

where u_0 in (9.0.6) is the initial datum for the Jin-Xin system (9.0.1). For any $\mu \in [0, 1/2)$, if E_1 is sufficiently small with respect to $(1/2 - \mu)$, then we have the following decay estimate:

$$\|D^\beta(u(t) - w_p(t))\|_0 \leq C\varepsilon \min\{1, t^{-1/4-\mu-\beta/2}\} E_{|\beta|+4}, \tag{9.0.9}$$

with $C = C(E_{|\beta|+\sigma})$ for σ large enough.

Once we identified the right scaled variable to study system (9.0.1), $(u, \varepsilon^2 v)$, which are expressed at the beginning of Section 9.1, and we found the strategy, discussed in Section 9.1 and 9.2, to achieve the so-called *conservative-dissipative* (C-D) form in [15] for our model, our approach essentially relies on the method developed in [15], with substantial differences listed here.

- We need an explicit Green function analysis of the linearized system rather than expansions and approximations, in order to deal with the singular parameter ε. The analysis performed in first part of Section 9.3 is as precise as it is possible.

- Some estimates in [15] rely on the use of the Shizuta-Kawashima (SK) condition, introduced in Chapter 3, and recalled here. Consider a linear first order system in compact form: $\partial_t \mathbf{u} + A\partial_x \mathbf{u} = G\mathbf{u}$. Passing to the Fourier transform, define $E(i\xi) = G - iA\xi$. The (SK) condition states that, if $\lambda(z)$ is an eigenvalue of $E(z)$, then $Re(\lambda(i\xi)) \leq -c\frac{|\xi|^2}{1+|\xi|^2}$, for some constant $c > 0$ and for every $\xi \in \mathbb{R} - \{0\}$. As it can be seen in (9.3.57), these eigenvalues for the compact linearized system in (C-D) form (9.2.22) of system (9.0.1) have different weights in ε. Thus, we cannot simply apply the (SK) condition to estimate the remainders in paragraph *Remainders in between* as the authors did in [15], since the weights in ε are essential to deal with the singular nonlinear term in the Duhamel formula (9.4.66). Again, a further analysis is needed.

- Differently from [15], we are not assuming to have a global in time solution, uniformly bounded in ε, for our singular system. The uniform global existence follows implicitly from the uniform asymptotic estimates in our calculations.
- The coupling between the convergence to the limit equation (9.0.2) for vanishing ε and for large time in the last section is the main novelty of the present chapter, and new ideas are needed to get this result.

This chapter is based on [10].

9.1 General setting

First of all, we write system (9.0.1) in the following form:

$$\begin{cases} \partial_t u + \dfrac{\partial_x(\varepsilon^2 v)}{\varepsilon^2} = 0, \\ \partial_t(\varepsilon^2 v) + \lambda^2 \partial_x u = f(u) - \dfrac{\varepsilon^2 v}{\varepsilon^2}. \end{cases} \tag{9.1.10}$$

The unknown variable is $\mathbf{u} = (u, \varepsilon^2 v)$, in the spirit of the scaled variables introduced in [11], which are the right scaling to get the conservative-dissipative form discussed below. Here we write $f(u) = au + h(u)$, where $a = f'(0)$, and system (9.1.10) reads

$$\begin{cases} \partial_t u + \dfrac{\partial_x(\varepsilon^2 v)}{\varepsilon^2} = 0, \\ \partial_t(\varepsilon^2 v) + \lambda^2 \partial_x u = au + h(u) - \dfrac{\varepsilon^2 v}{\varepsilon^2}. \end{cases} \tag{9.1.11}$$

Equations (9.1.11) can be written in compact form:

$$\partial_t \mathbf{u} + A \partial_x \mathbf{u} = -B\mathbf{u} + N(u), \tag{9.1.12}$$

where

$$A = \begin{pmatrix} 0 & \dfrac{1}{\varepsilon^2} \\ \lambda^2 & 0 \end{pmatrix}, \qquad -B = \begin{pmatrix} 0 & 0 \\ a & -\dfrac{1}{\varepsilon^2} \end{pmatrix}, \qquad N(u) = \begin{pmatrix} 0 \\ h(u) \end{pmatrix}. \tag{9.1.13}$$

In particular, $-B\mathbf{u}$ is the linear part of the source term, while $N(u)$ is the remaining nonlinear one, which only depends on the first component of $\mathbf{u} = (u, \varepsilon^2 v)$. Now, we look for a right constant symmetrizer Σ for system (9.1.12), which also highlights the dissipative properties of the linear source term. Thus, we find

$$\Sigma = \begin{pmatrix} 1 & a\varepsilon^2 \\ a\varepsilon^2 & \lambda^2\varepsilon^2 \end{pmatrix}. \tag{9.1.14}$$

Taking $\mathbf{w}$ such that

$$\mathbf{u} = \begin{pmatrix} u \\ \varepsilon^2 v \end{pmatrix} = \Sigma \mathbf{w} = \begin{pmatrix} (\Sigma\mathbf{w})_1 \\ (\Sigma\mathbf{w})_2 \end{pmatrix}, \text{ where } \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \dfrac{u\lambda^2 - a\varepsilon^2 v}{\lambda^2 - a^2\varepsilon^2} \\ \\ \dfrac{v - au}{\lambda^2 - a^2\varepsilon^2} \end{pmatrix}, \tag{9.1.15}$$

system (9.1.12) reads

$$\Sigma\partial_t \mathbf{w} + A_1\partial_x \mathbf{w} = -B_1\mathbf{w} + N((\Sigma\mathbf{w})_1), \tag{9.1.16}$$

where

$$A_1 = A_1^T = A\Sigma = \begin{pmatrix} a & \lambda^2 \\ \lambda^2 & a\lambda^2\varepsilon^2 \end{pmatrix}, \qquad -B_1 = -B\Sigma = \begin{pmatrix} 0 & 0 \\ 0 & a^2\varepsilon^2 - \lambda^2 \end{pmatrix},$$

$$N((\Sigma\mathbf{w})_1) = \begin{pmatrix} 0 \\ h(w_1 + a\varepsilon^2 w_2) \end{pmatrix}. \tag{9.1.17}$$

By using the Cauchy inequality we get the following lemma.

Lemma 9.1.1. *The symmetrizer Σ is definite positive. Precisely*

$$\frac{1}{2}\|w_1\|_0^2 + \varepsilon^2\|w_2\|_0^2(\lambda^2 - 2a^2\varepsilon^2) \leq (\Sigma\boldsymbol{w}, \boldsymbol{w})_0 \leq \|w_1\|_0^2(1 + a\varepsilon^2) + \|w_2\|_0^2(a + \lambda^2)\varepsilon^2. \tag{9.1.18}$$

Notice that from the theory on hyperbolic systems, [52], the Cauchy problem for (9.1.16) with initial data $\mathbf{w}_0$ in $H^m(\mathbb{R})$, $m \geq 2$, has a unique local smooth solution $\mathbf{w}^\varepsilon$ for each fixed $\varepsilon > 0$. We denote by T^ε the maximum time of existence of this local solution and, hereafter, we consider the time interval $[0, T^*]$, with $T^* \in [0, T^\varepsilon)$ for every ε. In the following, we study the Green function of system (9.1.16), and we establish some uniform energy estimates and decay rates of the smooth solution to system (9.1.16).

9.1.1 The conservative-dissipative form

In this section, we introduce a linear change of variable, so providing a particular structure for our system, the so-called *conservative-dissipative form* (C-D) defined in [15]. The (C-D) form allows to identify a conservative variable and a dissipative one for system (9.0.1), such that in the following a crucial faster decay of the dissipative variable is observed. Thanks to this change of variables, we are able indeed to handle the case $a \neq 0$ in (9.1.11). Hereafter, $(\cdot,\cdot)$ denotes the standard scalar product in $L^2(\mathbb{R})$, and $\|\cdot\|_m$ is the $H^m(\mathbb{R})$-norm, for $m \in \mathbb{N}$, where $H^0(\mathbb{R}) = L^2(\mathbb{R})$.

Proposition 9.1.1. *Given the right symmetrizer Σ in (9.1.14) for system (9.1.12), denoting by*

$$\tilde{\boldsymbol{w}} = M\boldsymbol{u} = \begin{pmatrix} 1 & 0 \\ \dfrac{-a\varepsilon}{\sqrt{\lambda^2 - a^2\varepsilon^2}} & \dfrac{1}{\varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2}} \end{pmatrix}\boldsymbol{u} = \begin{pmatrix} u \\ \dfrac{\varepsilon(v - au)}{\sqrt{\lambda^2 - a^2\varepsilon^2}} \end{pmatrix}, \tag{9.1.19}$$

system (9.1.12) can be written in (C-D) form defined in [15], i.e.

$$\partial_t\tilde{\boldsymbol{w}} + \tilde{A}\partial_x\tilde{\boldsymbol{w}} = -\tilde{B}\tilde{\boldsymbol{w}} + \tilde{N}(\tilde{w}_1), \tag{9.1.20}$$

where

$$\tilde{A} = \begin{pmatrix} a & \dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} \\ \dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} & -a \end{pmatrix}, \; \tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{\varepsilon^2} \end{pmatrix}, \; \tilde{N}(\tilde{w}_1) = \begin{pmatrix} 0 \\ \dfrac{h(\tilde{w}_1)}{\varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2}} \end{pmatrix}. \tag{9.1.21}$$

9.2 The Green function of the linear partially dissipative system

We consider the linear part of the (C-D) system (9.1.20)-(9.1.21) without the *tilde* for simplicity,

$$\partial_t \mathbf{w} + A\partial_x \mathbf{w} = -B\mathbf{w}. \tag{9.2.22}$$

We want to apply the approach developed in [15], to study the singular approximation system above. The main difficulty here is to deal with the singular perturbation parameter ε. We consider the Green kernel $\Gamma(t,x)$ of (9.2.22), which satisfies

$$\begin{cases} \partial_t \Gamma + A\partial_x \Gamma = -B\Gamma, \\ \Gamma(0,x) = \delta(x)I. \end{cases} \tag{9.2.23}$$

Taking the Fourier transform $\hat{\Gamma}$, we get

$$\begin{cases} \frac{d}{dt}\hat{\Gamma} = (-B - i\xi A)\hat{\Gamma}, \\ \hat{\Gamma}(0,\xi) = I. \end{cases} \tag{9.2.24}$$

Consider the entire function

$$E(z) = -B - zA = \begin{pmatrix} -az & -\dfrac{z\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} \\ -\dfrac{z\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} & az - \dfrac{1}{\varepsilon^2} \end{pmatrix}. \tag{9.2.25}$$

Formally, the solution to (9.2.24) is given by

$$\hat{\Gamma}(t,\xi) = e^{E(i\xi)t} = \sum_{n=0}^{\infty} (-B - i\xi A)^n. \tag{9.2.26}$$

Since $E(z)$ in (9.2.25) is symmetric, if z is not exceptional we can write

$$E(z) = \lambda_1(z)P_1(z) + \lambda_2(z)P_2(z),$$

where $\lambda_1(z), \lambda_2(z)$ are the eigenvalues of $E(z)$, and $P_1(z), P_2(z)$ the related eigenprojections, given by

$$P_j(z) = -\frac{1}{2\pi i}\oint_{|\xi-\lambda_j(z)|<<1} (E(z) - \xi I)^{-1}\, d\xi, \qquad j = 1,2.$$

Following [15], we study the low frequencies (case $z = 0$) and the high frequencies (case $z = \infty$) separately.

Case $z = 0$ The total projector for the eigenvalues near to 0 is

$$P(z) = -\frac{1}{2\pi i}\oint_{|\xi|<<1} (E(z) - \xi I)^{-1}\, d\xi. \tag{9.2.27}$$

Besides, it has the following expansion, see [45],

$$P(z) = P_0 + \sum_{n\geq 1} z^n P^n(z), \tag{9.2.28}$$

where P_0 is the eigenprojection for $E(0) - \xi I = -B - \xi I$, i.e.

$$P_0 = -\frac{1}{2\pi i}\oint_{|\xi|<<1} (-B-\xi I)^{-1}\, d\xi =: Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; P^n(z) = -\frac{1}{2\pi i}\oint R^{(n)}(\xi)\, d\xi, \tag{9.2.29}$$

with $R^{(n)}$ the n-th term in the expansion of the resolvent (9.2.30). Here Q_0 is the projection onto the null space of the source term, while we denote by $Q_- = I - Q_0$ the complementing projection, and by L_-, L_0 and R_-, R_0 the related left and right eigenprojectors, see [45, 15], i.e.

$$L_- = R_-^T = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad L_0 = R_0^T = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$Q_- = R_- L_-, \qquad Q_0 = R_0 L_0.$$

On the other hand, from [45],

$$\begin{aligned} R(\xi, z) = (E(z) - \xi I)^{-1} = (-B - zA - \xi I)^{-1} &= (-B-\xi I)^{-1}\sum_{n=0}^{\infty}(Az(-B-\xi I)^{-1})^n \\ &= (-B-\xi I)^{-1} + \sum_{n\geq 1}(-B-\xi I)^{-1} z^n (A(-B-\xi I)^{-1})^n \\ &= R_0(\xi) + \sum_{n\geq 1} R^{(n)}(\xi), \end{aligned}$$

i.e.

$$R^{(n)} = z^n(-B-\xi I)^{-1}(A(-B-\xi I)^{-1})^n. \tag{9.2.30}$$

Since a neighborhood of $z = 0$ is considered, at this point the authors in [15] take the first two terms of the asymptotic expansion of the total projector (9.2.28), so obtaining an expression with a remainder $O(z^2)$. We cannot approximate the projector in the same way, since we need to check the singular terms in ε. Thus, we perform an explicit spectral analysis for the Green function of our problem. First of all,

$$A(-B-\xi I)^{-1} = \begin{pmatrix} -\dfrac{a}{\xi} & -\dfrac{\varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2}}{1+\varepsilon^2\xi} \\ -\dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon\xi} & \dfrac{a\varepsilon^2}{1+\varepsilon^2\xi} \end{pmatrix}, \tag{9.2.31}$$

which is diagonalizable, i.e.

$$A(-B-\xi I)^{-1} = VDV^{-1},$$

where D is the diagonal matrix with entries given by the eigenvalues, and V is the matrix with the eigenvectors on the columns. Explicitly, setting

$$\Box := a^2 + 4\varepsilon^2\lambda^2\xi^2 + 4\lambda^2\xi, \tag{9.2.32}$$

we have

$$D = diag\left\{\frac{-a \pm \sqrt{\Box}}{2\xi(1+\varepsilon^2\xi)}\right\},\ V = \begin{pmatrix} \dfrac{\varepsilon(a+\sqrt{\Box}+2a\varepsilon^2\xi)}{2(1+\varepsilon^2\xi)\sqrt{\lambda^2-a^2\varepsilon^2}} & \dfrac{\varepsilon(a-\sqrt{\Box}+2a\varepsilon^2\xi)}{2(1+\varepsilon^2\xi)\sqrt{\lambda^2-a^2\varepsilon^2}} \\ 1 & 1 \end{pmatrix}, \tag{9.2.33}$$

$$V^{-1} = \begin{pmatrix} \dfrac{(1+\varepsilon^2\xi)\sqrt{\lambda^2-a^2\varepsilon^2}}{\varepsilon\sqrt{\Box}} & \dfrac{-a+\sqrt{\Box}-2a\varepsilon^2\xi}{2\sqrt{\Box}} \\ -\dfrac{(1+\varepsilon^2\xi)\sqrt{\lambda^2-a^2\varepsilon^2}}{\varepsilon\sqrt{\Box}} & \dfrac{a+\sqrt{\Box}+2a\varepsilon^2\xi}{2\sqrt{\Box}} \end{pmatrix}. \tag{9.2.34}$$

This way, denoting by

$$\Diamond_1 = a-\sqrt{\Box}+2a\varepsilon^2\xi,\ \Diamond_2 = a+\sqrt{\Box}+2a\varepsilon^2\xi,\ \triangle_1 = -\frac{a+\sqrt{\Box}}{2\xi(1+\varepsilon^2\xi)},\ \triangle_2 = -\frac{a-\sqrt{\Box}}{2\xi(1+\varepsilon^2\xi)},$$

with $\Box$ in (9.2.32), from (9.2.30) we have

$$R^{(n)} = z^n(-B-\xi I)^{-1}(A(-B-\xi I)^{-1})^n = z^n(-B-\xi I)^{-1}(VD^nV^{-1})$$
$$= z^n \begin{pmatrix} \dfrac{\Diamond_1\triangle_2^n - \Diamond_2\triangle_1^n}{2\sqrt{\Box}\xi} & -\varepsilon\sqrt{\dfrac{\lambda^2-a^2\varepsilon^2}{\Box}}(\triangle_1^n-\triangle_2^n) \\ -\varepsilon\sqrt{\dfrac{\lambda^2-a^2\varepsilon^2}{\Box}}(\triangle_1^n-\triangle_2^n) & -\dfrac{\varepsilon^2}{2(1+\varepsilon^2\xi)\sqrt{\Box}}(\Diamond_2\triangle_2^n-\Diamond_1\triangle_1^n) \end{pmatrix}.$$

The matrix above is completely bounded in ε, and so we can approximate the expression of the total projector (9.2.28) up to the second order. To this end, we consider the previous expression of $R^{(n)}$ for $n = 0, 1, 2$, we apply the integral formula (9.2.29) and we obtain

$$P(z) = \begin{pmatrix} 1+O(z^2) & -\varepsilon z\sqrt{\lambda^2-a^2\varepsilon^2}+\varepsilon O(z^2) \\ -\varepsilon z\sqrt{\lambda^2-a^2\varepsilon^2}+\varepsilon O(z^2) & \varepsilon^2 z^2(\lambda^2-a^2\varepsilon^2)+\varepsilon^2 O(z^3) \end{pmatrix}. \tag{9.2.35}$$

Now, we consider the left $L(z)$ and the right $R(z)$ eigenprojectors of $P(z)$, i.e.

$$P(z) = R(z)L(z), \quad L(z)R(z) = I$$
$$L(z)P(z) = L(z), \quad P(z)R(z) = R(z).$$

We can limit ourselves to the second order approximation, according to (9.2.35). Then, we consider

$$\tilde{P}(z) = \begin{pmatrix} 1 & -\varepsilon z\sqrt{\lambda^2-a^2\varepsilon^2} \\ -\varepsilon z\sqrt{\lambda^2-a^2\varepsilon^2} & \varepsilon^2 z^2(\lambda^2-a^2\varepsilon^2) \end{pmatrix}, \tag{9.2.36}$$

and, by applying the conditions above, we obtain

$$\tilde{L}(z) = \begin{pmatrix} 1 & -\varepsilon z\sqrt{\lambda^2-a^2\varepsilon^2} \end{pmatrix}, \qquad \tilde{R}(z) = \begin{pmatrix} 1 \\ -\varepsilon z\sqrt{\lambda^2-a^2\varepsilon^2} \end{pmatrix},$$

where

$$\begin{aligned} \tilde{P}(z) &= \tilde{R}(z)\tilde{L}(z), & \tilde{L}(z)\tilde{R}(z) &= 1+\varepsilon^2 O(z^2), \\ \tilde{P}(z)\tilde{R}(z) &= \tilde{R}(z)+\varepsilon^2 O(z^2), & \tilde{L}(z)\tilde{P}(z) &= \tilde{L}(z)+\varepsilon^2 O(z^2), \end{aligned} \tag{9.2.37}$$

and so

$$P(z) = \tilde{P}(z) + O(z^2), \quad R(z) = \tilde{R}(z) + O(z^2), \qquad L(z) = \tilde{L}(z) + O(z^2).$$

Let us point out that further expansions of $L(z), R(z)$ are not singular in ε too, since the weights in ε of these vectors come from (9.2.35). Precisely, one can see that $L^\varepsilon(z)$ depends on ε as follows:

$$L^\varepsilon(\cdot) = \begin{pmatrix} 1 & O(\varepsilon) \end{pmatrix} = [R(\cdot)^\varepsilon]^T.$$

Now, by using the left and the right operators, we decompose $E(z)$ in the following way, see [15],

$$E(z) = R(z)F(z)L(z) + R_-(z)F_-(z)L_-(z), \tag{9.2.38}$$

where $L_-(z), R_-(z)$ are left and right eigenprojectors of $P_-(z) = I - P(z)$, while

$$F(z) = L(z)E(z)R(z), \quad F_-(z) = L_-(z)E(z)R_-(z).$$

We use the approximations of $L(z), R(z)$ above, and so

$$F(z) = (\tilde{L}(z)+O(z^2))(-B-Az)(\tilde{R}(z)+O(z^2)) = -az+(\lambda^2-a^2\varepsilon^2)z^2+O(z^3). \tag{9.2.39}$$

We study $F_-(z)$. Matrix (9.2.35) and the definition above imply that

$$P_-(z) = \begin{pmatrix} O(z^2) & z\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}+\varepsilon O(z^2) \\ z\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}+\varepsilon O(z^2) & 1+\varepsilon^2 O(z^2) \end{pmatrix}, \tag{9.2.40}$$

and, approximating again,

$$L_-(z) = \tilde{L}_-(z) + O(z^2) = \begin{pmatrix} z\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2} & 1 \end{pmatrix} + O(z^2),$$

$$R_-(z) = \tilde{R}_-(z) + O(z^2) = \begin{pmatrix} z\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2} \\ 1 \end{pmatrix} + O(z^2).$$

Thus,

$$F_-(z) = \tilde{L}_-(z)(-B-Az)\tilde{R}_-(z) + O(z^2) = -\frac{1}{\varepsilon^2} + az + O(z^2). \tag{9.2.41}$$

This yields the proposition below.

Proposition 9.2.1. *We have the following decomposition near $z = 0$:*

$$E(z) = F(z)P(z) + E_-(z), \tag{9.2.42}$$

with $F(z)$ in (9.2.39), $P(z)$ in (9.2.35), $E_-(z) = R_-(z)F_-(z)L_-(z)$, and $F_-(z)$ in (9.2.41).

Case $z = \infty$ We consider $E(z) = -B - Az = z(-B/z - A) = zE_1(1/z)$ and, setting $z = i\xi$ and $\zeta = 1/z = -i\eta$, with $\xi, \eta \in \mathbb{R}$, we have $E_1(\zeta) = -A - \zeta B$

$$= \begin{pmatrix} -a & -\dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} \\ -\dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} & a - \dfrac{\zeta}{\varepsilon^2} \end{pmatrix} = \begin{pmatrix} -a & -\dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} \\ -\dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} & a + \dfrac{i\eta}{\varepsilon^2} \end{pmatrix}.$$

Since $E_1(\zeta)$ is symmetric, we determine the eigenvalues and the right eigenprojectors,

$$-A - \zeta B = \lambda_1^{E_1}(\zeta) R_1(\zeta) R_1^T(\zeta) + \lambda_2^{E_1}(\zeta) R_2(\zeta) R_2^T(\zeta),$$

such that, for $j = 1, 2$, $R_j^T(\zeta) R_j(\zeta) = I$. The following expression for the eigenvalues of $E_1(i\eta)$ is provided

$$\lambda_{1,2}^{E_1}(z) = \frac{i\eta}{2\varepsilon^2} \pm \frac{\sqrt{4\varepsilon^2\lambda^2 + 4a\eta\varepsilon^2 i - \eta^2}}{2\varepsilon^2},$$

and it is simple to prove that both the corresponding eigenvalues of $E(z)$, which can be obtained multiplying $\lambda_1^{E_1}(z)$ and $\lambda_2^{E_1}(z)$ above by $z = i\xi = i/\eta$, have a strictly negative real part in the high frequencies regime ($|\zeta| = |\eta| << 1$) and in the vanishing ε limit. Moreover, setting $\delta_{1,2} = \sqrt{8\varepsilon^2\lambda^2 + 2\zeta^2 - 8a\varepsilon^2\zeta \pm (-2\zeta\sqrt{\mu} + 4a\varepsilon^2\sqrt{\mu})}$, where $\mu = 4\varepsilon^2\lambda^2 + \zeta^2 - 4a\varepsilon^2\zeta$, the normalized right eigenprojectors are given by:

$$R_1(\zeta) = \frac{1}{\delta_1} \begin{pmatrix} (2a\varepsilon^2 - \zeta) + \sqrt{\mu} \\ 2\varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2} \end{pmatrix}, \quad R_2(\zeta) = \frac{1}{\delta_2} \begin{pmatrix} (2a\varepsilon^2 - \zeta) - \sqrt{\mu} \\ 2\varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2} \end{pmatrix}.$$

The eigenprojectors are bounded in ε, even for ζ near zero. Thus, we can approximate the total projector of $E_1(\zeta) = -A - \zeta B$ in a more convenient way, i.e. we decompose

$$A = \lambda_1 R_1 R_1^T + \lambda_2 R_2 R_2^T,$$

where $\lambda_1 = \lambda/\varepsilon$, $\lambda_2 = -\lambda/\varepsilon$, and the corresponding eigenprojectors

$$R_1 = \frac{1}{\sqrt{2\lambda}} \begin{pmatrix} \sqrt{\dfrac{\lambda^2 - a^2\varepsilon^2}{(\lambda - a\varepsilon)}} \\ \sqrt{\lambda - a\varepsilon} \end{pmatrix}, \ R_2 = \frac{1}{\sqrt{2\lambda}} \begin{pmatrix} -\sqrt{\dfrac{\lambda^2 - a^2\varepsilon^2}{(\lambda + a\varepsilon)}} \\ \sqrt{\lambda + a\varepsilon} \end{pmatrix}.$$

Now, by considering the total projector for the family of eigenvalues going to $\lambda_j = \pm\lambda/\varepsilon$ as $\zeta \approx 0$, we obtain the following approximations:

$$F_{1j}(\zeta) = -\lambda_j I + \zeta R_j^T(-B) R_j + O(\zeta^2). \tag{9.2.43}$$

Explicitly,

$$F_{11}(\zeta) = -\frac{\lambda}{\varepsilon} - \frac{(\lambda - a\varepsilon)\zeta}{2\lambda\varepsilon^2} + O(\zeta^2), \quad F_{12}(\zeta) = \frac{\lambda}{\varepsilon} - \frac{(\lambda + a\varepsilon)\zeta}{2\lambda\varepsilon^2} + O(\zeta^2). \tag{9.2.44}$$

Since $E(z) = zE_1(1/z)$, we multiply $F_1(\zeta) = F_1(1/z)$ by z and, for $|z| \to +\infty$,

$$\lambda_1(z) = -\frac{\lambda}{\varepsilon} z - \frac{\lambda - a\varepsilon}{2\lambda\varepsilon^2} + O(1/z), \quad \lambda_2(z) = \frac{\lambda}{\varepsilon} z - \frac{\lambda + a\varepsilon}{2\lambda\varepsilon^2} + O(1/z), \tag{9.2.45}$$

while the projectors are

$$\mathcal{P}_j(z) = R_j R_j^T + O(1/z), \quad j = 1, 2. \tag{9.2.46}$$

Remark 9.2.1. *Notice that the term $O(1/z)$ in (9.2.45) could be singular in ε. However, from the previous discussion, the eigenvalues of $E(z)$ have a strictly negative real part. This implies that the coefficients of the even powers of z in (9.2.45) have a negative sign, while the others are imaginary terms. Thus, $e^{\lambda_{1,2}(z)}$ are bounded in ε.*

Proposition 9.2.2. *We have the following decomposition near $z = \infty$:*

$$E(z) = \lambda_1(z)\mathcal{P}_1(z) + \lambda_2(z)\mathcal{P}_2(z), \tag{9.2.47}$$

with $\lambda_1(z), \lambda_2(z)$ in (9.2.45), and $\mathcal{P}_1(z), \mathcal{P}_2(z)$ in (9.2.46).

9.3 Green function estimates

Green function estimates near $z = 0$ We associate to (9.2.39) the parabolic equation

$$\partial_t w + a\partial_x w = (\lambda^2 - a^2\varepsilon^2)\partial_{xx} w.$$

We can write the explicit solution

$$g(t,x) = \frac{1}{2\sqrt{(\lambda^2 - a^2\varepsilon^2)\pi t}} \exp\left\{ -\frac{(x-at)^2}{4(\lambda^2 - a^2\varepsilon^2)t} \right\}. \tag{9.3.48}$$

This means that, for some $c_1, c_2 > 0$,

$$|g(t,x)| \leq \frac{c_1}{\sqrt{t}} e^{-(x-at)^2/ct}, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \qquad \forall \varepsilon > 0. \tag{9.3.49}$$

Now, recalling Proposition 9.2.1 and considering the approximation $\tilde{P}(z)$ in (9.2.36) of the total projector $P(z)$ in (9.2.35),

$$e^{E(z)t} = \hat{g}(z)\tilde{P}(z) + R_-(z)e^{F_-(z)t}L_-(z) + \hat{R}_1(t,z),$$

where $\hat{g}(z) = -az - (\lambda^2 - \varepsilon^2 a^2)z^2$, and $R_1(t,x)$ is a remainder term, we take the inverse of the Fourier transform of

$$\hat{K}(z) = \hat{g}(z)\tilde{P}(z), \tag{9.3.50}$$

which yields the expression of the first part of the Green function near $z = 0$, i.e.

$$K(t,x) = \begin{pmatrix} g(t,x) & \varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2}\left(\dfrac{dg(t,x)}{dx}\right) \\ \varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2}\left(\dfrac{dg(t,x)}{dx}\right) & \varepsilon^2(\lambda^2 - a^2\varepsilon^2)\left(\dfrac{d^2 g(t,x)}{d^2 x}\right) \end{pmatrix}. \tag{9.3.51}$$

Here, $\hat{K}(t,\xi)$ is the approximation of $\hat{\Gamma}(t,\xi)$ in (9.2.26) for $|\xi| \approx 0$. Thus, for $\xi \in [-\delta, \delta]$ with $\delta > 0$ sufficiently small, we consider the following remainder term

$$\begin{aligned} R_1(t,x) &= \frac{1}{2\pi}\int_{-\delta}^{\delta} (e^{E(i\xi)t} - e^{\hat{K}(t,\xi)t})e^{i\xi x}\, d\xi \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{i\xi(x-at) - \xi^2(\lambda^2 - a^2\varepsilon^2)t}(e^{O(\xi^3 t)}P(i\xi) - \tilde{P}(i\xi))\, d\xi \\ &+ \frac{1}{2\pi}\int_{-\delta}^{\delta} R_-(i\xi)e^{F_-(i\xi)t}L_-(i\xi)e^{i\xi x}\, d\xi. \end{aligned} \tag{9.3.52}$$

We need an estimate for the remainder above. First of all, from (9.2.41) and (9.2.40),

$$\left|\frac{1}{2\pi}\int_{-\delta}^{\delta} R_-(i\xi)e^{F_-(i\xi)t}L_-(i\xi)e^{i\xi x}\,d\xi\right| \leq \left|\frac{1}{2\pi}\int_{-\delta}^{\delta} P_-(i\xi)e^{(-1/\varepsilon^2+ai\xi+O(\xi^2))t}e^{i\xi x}\,d\xi\right|$$

$$\leq Ce^{-t/\varepsilon^2}$$

for some constant C. Following [15],

$$|e^{O(\xi^3 t)}P(i\xi)-\tilde{P}(i\xi)| = |z^3|te^{2\mu|z|^2 t}\begin{pmatrix} O(1) & O(\varepsilon)|z| \\ O(\varepsilon)|z| & O(\varepsilon^2)|z|^2 \end{pmatrix},$$

for a constant $\mu > 0$. This way,

$$R_1(t,x) = e^{-(x-at)^2/(ct)}\begin{pmatrix} O(1)(1+t)^{-1} & O(\varepsilon)(1+t)^{-3/2} \\ O(\varepsilon)(1+t)^{-3/2} & O(\varepsilon^2)(1+t)^{-2} \end{pmatrix}.$$

Green function estimates near $z=\infty$ We associate to (9.2.45) the following equations:

$$\partial_t w + \frac{\lambda}{\varepsilon}\partial_x w = -\frac{\lambda - a\varepsilon}{2\lambda\varepsilon^2}w, \quad \partial_t w - \frac{\lambda}{\varepsilon}\partial_x w = -\frac{\lambda + a\varepsilon}{2\lambda\varepsilon^2}w.$$

We can write explicitly the solutions

$$g_1(t,x) = \delta(x-\lambda t/\varepsilon)e^{-(\lambda-a\varepsilon)t/(2\lambda\varepsilon^2)}, \quad g_2(t,x) = \delta(x+\lambda t/\varepsilon)e^{-(\lambda+a\varepsilon)t/(2\lambda\varepsilon^2)}.$$

Thus,

$$|g_j(t,x)| \leq C\delta(x\pm\lambda t/\varepsilon)e^{-ct/\varepsilon^2}, \quad j=1,2.$$

We determine the Fourier transform of the Green function for $|z|$ going to infinity,

$$\hat{\mathcal{K}}(t,\xi) = \exp\left\{-i\frac{\lambda t\xi}{\varepsilon} - \frac{(\lambda-a\varepsilon)t}{2\lambda\varepsilon^2}\right\}\mathcal{P}_1(\infty) + \exp\left\{i\frac{\lambda t\xi}{\varepsilon} - \frac{(\lambda+a\varepsilon)t}{2\lambda\varepsilon^2}\right\}\mathcal{P}_2(\infty). \quad (9.3.53)$$

This way, from Proposition 9.2.2, the remainder term here is

$$R_2(t,x) = \frac{1}{2\pi}\int_{|\xi|\geq N}(e^{E(i\xi)t} - \hat{\mathcal{K}}(t,\xi))e^{i\xi x}\,d\xi, \qquad \text{and} \qquad (9.3.54)$$

$$\begin{aligned}|R_2| \leq{}& \frac{1}{2\pi}\left|\int_{|\xi|\geq N} e^{i\xi(x-\lambda t/\varepsilon)-(\lambda-a\varepsilon)t/(2\lambda\varepsilon^2)}\cdot(e^{O(1)t/(i\xi)+O(1)t/\xi^2}\mathcal{P}_1(i\xi)-\mathcal{P}_1(\infty))\,d\xi\right| \\ &+\frac{1}{2\pi}\left|\int_{|\xi|\geq N} e^{i\xi(x+\lambda t/\varepsilon)-(\lambda+a\varepsilon)t/(2\lambda\varepsilon^2)}\cdot(e^{O(1)t/(i\xi)+O(1)t/\xi^2}\mathcal{P}_2(i\xi)-\mathcal{P}_2(\infty))\,d\xi\right|.\end{aligned}$$

Following [15] and thanks to Remark 9.2.1,

$$|R_2(t,x)| \leq Ce^{-ct/\varepsilon^2}\left[\left|\int_{|\xi|\geq N}\frac{e^{i\xi(x\pm\lambda t/\varepsilon)}}{\xi}\,d\xi\right| + \int_{|\xi|\geq N}\frac{1}{\xi^2}\,d\xi\right] \leq Ce^{-ct/\varepsilon^2}.$$

Remainders in between Until now, we studied the Green function of the linearized diffusive Jin-Xin system for $z \approx 0$, which yields the parabolic kernel $\hat{K}$ in (9.3.50), and for $z \approx \infty$, so obtaining $\hat{\mathcal{K}}$ in (9.3.53). In these two cases, we also provided estimates for the remainder terms:

- R_1 in (9.3.52) for the parabolic kernel K for $|\xi| \leq \delta$, with δ sufficiently small;
- R_2 in (9.3.54) for the transport kernel $\mathcal{K}$ for $|\xi| \geq N$, with N big enough.

It remains to estimate the last remainder terms, namely the parabolic kernel K for $|\xi| \geq \delta$, $t \geq 1$, the transport kernel $\mathcal{K}$ for $|\xi| \leq N$, and the kernel $E(z)$ for $\delta \leq |\xi| \leq N$.

Parabolic kernel $K(t,x)$ for $|\xi| \geq \delta$, $\delta << 1$ Let us define

$$R_3(t,x) = \frac{1}{2\pi} \int_{|\xi| \geq \delta} \hat{K}(t,\xi) e^{i\xi x}\, d\xi. \tag{9.3.55}$$

Thus, from (9.3.50), for $t \geq 1$,

$$\begin{aligned} |R_3(t,x)| &\leq C \left| \int_{|\xi|\geq\delta} e^{i\xi(x-at)} e^{-(\lambda^2 - a^2\varepsilon^2)\xi^2 t} \tilde{P}(i\xi)\, d\xi \right| \\ &\leq \frac{Ce^{-t/C}}{\sqrt{t}} \begin{pmatrix} O(1) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon^2) \end{pmatrix}. \end{aligned}$$

Transport kernel for $|\xi| \leq N$ Set

$$R_4(t,x) = \frac{1}{2\pi} \int_{|\xi| \leq N} \hat{\mathcal{K}}(t,\xi) e^{i\xi x}\, d\xi, \tag{9.3.56}$$

and, from (9.3.53),

$$\begin{aligned} |R_4(t,x)| &\leq Ce^{-(\lambda + |a|\varepsilon)t/(2\lambda\varepsilon^2)} \sum \left| \int_{-N}^{N} e^{i\xi(x \pm \lambda t/\varepsilon)} d\xi \right| \\ &\leq Ce^{-ct/\varepsilon^2} \min\left\{ N, \frac{1}{|x \pm \lambda t/\varepsilon|} \right\}. \end{aligned}$$

Kernel $E(z)$ for $\delta \leq |\xi| \leq N$ Finally, we set

$$R_5(t,x) = \frac{1}{2\pi} \int_{\delta \leq |\xi| \leq N} e^{E(i\xi)t} e^{i\xi t} d\xi.$$

The eigenvalues of $E(i\xi) = -i\xi A - B$ are expressed here:

$$\lambda_{1/2} = \frac{1}{2\varepsilon^2} \left(-1 \pm \sqrt{1 - 4\varepsilon^2(ia\xi + \lambda^2\xi^2)} \right) = \frac{-2(ai\xi + \lambda^2\xi^2)}{1 \pm \sqrt{1 - 4\varepsilon^2(ai\xi + \lambda^2\xi^2)}}. \tag{9.3.57}$$

By using the Taylor expansion for $\varepsilon \approx 0$,

$$\lambda_1 = -\frac{ai\xi + \lambda^2\xi^2}{1 - \varepsilon^2(ai\xi + \lambda^2\xi^2)}, \quad \lambda_2 = -\frac{1}{\varepsilon^2}.$$

Explicitly, denoting by

$$\triangle = \sqrt{1-4\varepsilon^2\xi(\lambda^2\xi+ia)}, \quad \square_1 = -1+\triangle+2ia\xi\varepsilon^2, \quad \square_2 = 1+\triangle-2ia\xi\varepsilon^2,$$

one can find that $e^{E(i\xi)t}$

$$= \begin{pmatrix} \frac{e^{\lambda_2 t}}{2\triangle}\square_1 + \frac{e^{\lambda_1 t}}{2\triangle}\square_2 & -\frac{i\xi\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}(e^{\lambda_1 t}-e^{\lambda_2 t})}{\triangle} \\ -\frac{i\xi\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}(e^{\lambda_1 t}-e^{\lambda_2 t})}{\triangle} & \frac{e^{\lambda_1 t}}{2\triangle}\square_1 + \frac{e^{\lambda_2 t}}{2\triangle}\square_2 \end{pmatrix},$$

where $\square_1 = -1+\triangle = -2\varepsilon^2\xi(\lambda^2\xi+ia)+O(\varepsilon^2) = O(\varepsilon^2), \quad \square_2 = 1+\triangle = O(1),$

and, in terms of the singular parameter ε, this yields

$$e^{E(i\xi)t} = \begin{pmatrix} O(1)(e^{\lambda_1 t}+e^{\lambda_2 t}) & O(\varepsilon)(e^{\lambda_1 t}-e^{\lambda_2 t}) \\ O(\varepsilon)(e^{\lambda_1 t}-e^{\lambda_2 t}) & e^{\lambda_1 t}O(\varepsilon^2)+O(1)e^{\lambda_2 t} \end{pmatrix}.$$

Putting the calculations above all together and integrating in space with respect to the Fourier variable for $\delta \leq |\xi| \leq N$, we get

$$|R_5(t,x)| \leq C \begin{pmatrix} O(1)e^{-t/C} & O(\varepsilon)e^{-t/C} \\ O(\varepsilon)e^{-t/C} & O(\varepsilon^2)e^{-t/C}+O(1)e^{-t/\varepsilon^2} \end{pmatrix}. \tag{9.3.58}$$

From (9.3.52), (9.3.54), (9.3.55), (9.3.56), (9.3.58), we denote the remainder by

$$R(t) = R_1(t)+R_2(t)+R_3(t)+R_4(t)+R_5(t). \tag{9.3.59}$$

The estimates above provide the following lemma.

Lemma 9.3.1. *Let $\Gamma(t,x)$ be the Green function of the linear system (9.2.22). We have the following decomposition:*

$$\Gamma(t,x) = K(t,x)+\mathcal{K}(t,x)+R(t,x),$$

with $K(t,x), \mathcal{K}(t,x), R(t,x)$ in (9.3.51), (9.3.53) and (9.3.59) respectively. Moreover, for some constant c, C,

- $|K(t,x)| \leq e^{-(x-at)^2/(ct)} \begin{pmatrix} O(1)(1+t)^{-1} & O(\varepsilon)(1+t)^{-3/2} \\ O(\varepsilon)(1+t)^{-3/2} & O(\varepsilon^2)(1+t)^{-2} \end{pmatrix};$
- $|\mathcal{K}(t,x)| \leq Ce^{-ct/\varepsilon^2};$
- $$\begin{aligned} |R(t)| \leq e^{-(x-at)^2/(ct)} & \begin{pmatrix} O(1)(1+t)^{-1} & O(\varepsilon)(1+t)^{-3/2} \\ O(\varepsilon)(1+t)^{-3/2} & O(\varepsilon^2)(1+t)^{-2} \end{pmatrix} \\ & + \begin{pmatrix} O(1) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon^2) \end{pmatrix} e^{-ct} + Id\, e^{-ct/\varepsilon^2}. \end{aligned}$$

Decay estimates Let us consider the solution to the Cauchy problem associated with the linear system (9.2.22) and initial data $\mathbf{w}_0$,

$$\hat{\mathbf{w}}(t,\xi) = \hat{\Gamma}(t,\xi)\hat{\mathbf{w}}_0(\xi) = e^{E(i\xi)t}\hat{\mathbf{w}}_0(\xi).$$

By using the decomposition provided by Lemma 9.3.1, we get the following theorem.

Theorem 9.3.1. *Consider the linear system in (9.2.22), i.e.*

$$\partial_t \boldsymbol{w} + A\partial_x \boldsymbol{w} = -B\boldsymbol{w},$$

and let $Q_0 = R_0 L_0$ and $Q_- = R_- L_-$ as before, i.e. the eigenprojectors onto the null space and the negative definite part of $-B$ respectively. Then, for any function $\boldsymbol{w}_0 \in L^1 \cap L^2(\mathbb{R},\mathbb{R})$, the solution $\boldsymbol{w}(t) = \Gamma(t)\boldsymbol{w}_0$ to the related Cauchy problem can be decomposed as

$$\boldsymbol{w}(t) = \Gamma(t)\boldsymbol{w}_0 = K(t)\boldsymbol{w}_0 + \mathcal{K}(t)\boldsymbol{w}_0 + R(t)\boldsymbol{w}_0.$$

Moreover, for any index β, the following estimates hold:

$$\begin{aligned}\|L_0 D^\beta K(t)\boldsymbol{w}_0\|_0 \le{}& C\min\{1, t^{-1/4-|\beta|/2}\}\|L_0\boldsymbol{w}_0\|_{L^1}\\ &+ C\varepsilon \min\{1, t^{-3/4-|\beta|/2}\}\|L_-\boldsymbol{w}_0\|_{L^1},\end{aligned} \tag{9.3.60}$$

$$\begin{aligned}\|L_- D^\beta K(t)\boldsymbol{w}_0\|_0 \le{}& C\varepsilon\min\{1, t^{-3/4-|\beta|/2}\}\|L_0\boldsymbol{w}_0\|_{L^1}\\ &+ C\varepsilon^2 \min\{1, t^{-5/4-|\beta|/2}\}\|L_-\boldsymbol{w}_0\|_{L^1},\end{aligned} \tag{9.3.61}$$

$$\|D^\beta \mathcal{K}(t)\boldsymbol{w}_0\|_0 \le Ce^{-ct/\varepsilon^2}\|D^\beta \boldsymbol{w}_0\|_0, \tag{9.3.62}$$

$$\begin{aligned}\|L_0 D^\beta R(t)\boldsymbol{w}_0\|_0 \le{}& C\min\{1, t^{-1/4-|\beta|/2}\}\|L_0\boldsymbol{w}_0\|_{L^1}\\ &+ C\varepsilon \min\{1, t^{-3/4-|\beta|/2}\}\|L_-\boldsymbol{w}_0\|_{L^1}\\ &+ Ce^{-ct}\|L_0\boldsymbol{w}_0\|_{L^1} + C\varepsilon e^{-ct}\|L_-\boldsymbol{w}_0\|_{L^1} + Ce^{-ct/\varepsilon^2}\|\boldsymbol{w}_0\|_{L^1},\end{aligned} \tag{9.3.63}$$

$$\begin{aligned}\|L_- D^\beta R(t)\boldsymbol{w}_0\|_0 \le{}& C\varepsilon\min\{1, t^{-3/4-|\beta|/2}\}\|L_0\boldsymbol{w}_0\|_{L^1}\\ &+ C\varepsilon^2 \min\{1, t^{-5/4-|\beta|/2}\}\|L_-\boldsymbol{w}_0\|_{L^1}\\ &+ C\varepsilon e^{-ct}\|L_0\boldsymbol{w}_0\|_{L^1} + C\varepsilon^2 e^{-ct}\|L_-\boldsymbol{w}_0\|_{L^1} + Ce^{-ct/\varepsilon^2}\|\boldsymbol{w}_0\|_{L^1}.\end{aligned} \tag{9.3.64}$$

Proof. From Lemma 9.3.1, for some constants $c, C > 0$, and for an index β, it holds

$$\|D^\beta \mathcal{K}(t)\mathbf{w}_0\|_0 \le Ce^{-ct/\varepsilon^2}\|D^\beta \mathbf{w}_0\|_0. \tag{9.3.65}$$

On the other hand, the hyperbolic kernel (9.3.50) can be estimated as

$$\begin{aligned}|L_0\widehat{K(t)\mathbf{w}_0}| &\le Ce^{-c|\xi|^2 t}(|L_0\hat{\mathbf{w}}_0| + \varepsilon|\xi||L_-\hat{\mathbf{w}}_0|),\\ |L_-\widehat{K(t)\mathbf{w}_0}| &\le Ce^{-c|\xi|^2 t}(\varepsilon|\xi||L_0\hat{\mathbf{w}}_0| + \varepsilon^2|\xi|^2|L_-\hat{\mathbf{w}}_0|).\end{aligned}$$

This yields

$$\begin{aligned}\|L_0K(t)\mathbf{w}_0\|_0^2 &\le C\int_0^\infty\int_{S^0} e^{-2c|\xi|^2t}(|L_0\hat{\mathbf{w}}_0(\xi)|^2+\varepsilon^2|\xi|^2|L_-\hat{\mathbf{w}}_0(\xi)|^2)\,d\zeta d\xi\\ &\le C\min\{1,t^{-1/2}\}\|L_0\hat{\mathbf{w}}_0\|_\infty^2+C\varepsilon^2\min\{1,t^{-3/2}\}\|L_-\hat{\mathbf{w}}_0\|_\infty^2\\ &\le C\min\{1,t^{-1/2}\}\|L_0\mathbf{w}_0\|_{L^1}^2+C\varepsilon^2\min\{1,t^{-3/2}\}\|L_-\mathbf{w}_0\|_{L^1}^2,\end{aligned}$$

and

$$\begin{aligned}\|L_-K(t)\mathbf{w}_0\|_0^2 &\le C\int_0^\infty\int_{S^0} e^{-2c|\xi|t}(\varepsilon^2|\xi|^2|L_0\hat{\mathbf{w}}_0(\xi)|^2+\varepsilon^4|\xi|^2|L_-\hat{\mathbf{w}}_0(\xi)|^2)\,d\zeta d\xi\\ &\le C\varepsilon^2\min\{1,t^{-3/2}\}\|L_0\mathbf{w}_0\|_{L^1}^2+C\varepsilon^4\min\{1,t^{-5/2}\}\|L_-\mathbf{w}_0\|_{L^1}^2.\end{aligned}$$

Besides, for every β we multiply by $\xi^{2\beta}$ the integrand and we get

$$\begin{aligned}\|L_0D^\beta K(t)\mathbf{w}_0\|_0 &\le C\min\{1,t^{-1/4-|\beta|/2}\}\|L_0\mathbf{w}_0\|_{L^1}\\ &\quad+C\varepsilon\min\{1,t^{-3/4-|\beta|/2}\}\|L_-\mathbf{w}_0\|_{L^1},\end{aligned}$$

$$\begin{aligned}\|L_-D^\beta K(t)\mathbf{w}_0\|_0 &\le C\varepsilon\min\{1,t^{-3/4-|\beta|/2}\}\|L_0\mathbf{w}_0\|_{L^1}\\ &\quad+C\varepsilon^2\min\{1,t^{-5/4-|\beta|/2}\}\|L_-\mathbf{w}_0\|_{L^1}.\end{aligned}$$

The estimates for $R(t)$ are obtained in a similar way. □

9.4 Decay estimates and convergence

Consider the local solution $\mathbf{w}$ to the Cauchy problem associated with (9.1.20), where we drop the *tilde*, and initial data $\mathbf{w}_0$. The solution to the nonlinear problem (9.1.20) can be expressed by using the Duhamel formula

$$\begin{aligned}\mathbf{w}(t) &= \Gamma(t)\mathbf{w}_0+\int_0^t\Gamma(t-s)(N(w_1(s))-DN(0)w_1(s))\,ds\\ &= \Gamma(t)\mathbf{w}_0+\int_0^t\Gamma(t-s)\begin{pmatrix}0\\ \dfrac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}}\end{pmatrix}ds\quad t\in[0,T^*].\end{aligned}\tag{9.4.66}$$

From (9.2.29) and the formulas below, we recall that $w_1=L_0\mathbf{w}=(1-L_-)\mathbf{w}$ is the conservative variable, while $w_2=L_-\mathbf{w}$ is the dissipative one. We remind the Green function decomposition given by Lemma 9.3.1. For the β-derivative,

$$\begin{aligned}
D^\beta \mathbf{w}(t) &= D^\beta K(t)\mathbf{w}(0) + \mathcal{K}(t)D^\beta \mathbf{w}(0) + R(t)D^\beta \mathbf{w}(0) \\
&\quad + \int_0^{t/2} D^\beta K(t-s)R_-L_- \begin{pmatrix} 0 \\ \dfrac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}} \end{pmatrix} ds \\
&\quad + \int_{t/2}^{t} K(t-s)R_-D^\beta L_- \begin{pmatrix} 0 \\ \dfrac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}} \end{pmatrix} ds \\
&\quad + \int_0^{t} \mathcal{K}(t-s)D^\beta \begin{pmatrix} 0 \\ \dfrac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}} \end{pmatrix} ds \\
&\quad + \int_0^{t/2} D^\beta R(t-s)R_-L_- \begin{pmatrix} 0 \\ \dfrac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}} \end{pmatrix} ds \\
&\quad + \int_{t/2}^{t} R(t-s)R_-D^\beta L_- \begin{pmatrix} 0 \\ \dfrac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}} \end{pmatrix} ds
\end{aligned}$$

$$\begin{aligned}
&= D^\beta K(t)\mathbf{w}(0) + \mathcal{K}(t)D^\beta \mathbf{w}(0) + R(t)D^\beta \mathbf{w}(0) \\
&\quad + \int_0^{t/2} \begin{pmatrix} D^\beta K_{12}(t-s) \\ D^\beta K_{22}(t-s) \end{pmatrix} \frac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}}\, ds \\
&\quad + \int_{t/2}^{t} \begin{pmatrix} K_{12}(t-s) \\ K_{22}(t-s) \end{pmatrix} D^\beta \frac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}}\, ds \\
&\quad + \int_0^{t} \mathcal{K}(t-s)D^\beta \begin{pmatrix} 0 \\ \dfrac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}} \end{pmatrix} ds \\
&\quad + \int_0^{t/2} \begin{pmatrix} D^\beta R_{12}(t-s) \\ D^\beta R_{22}(t-s) \end{pmatrix} \frac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}}\, ds \\
&\quad + \int_{t/2}^{t} \begin{pmatrix} R_{12}(t-s) \\ R_{22}(t-s) \end{pmatrix} D^\beta \frac{h(w_1(s))}{\varepsilon\sqrt{\lambda^2-a^2\varepsilon^2}}\, ds.
\end{aligned}$$

Notice that, from (9.3.51), K_{12}, K_{22} are of order ε and ε^2 respectively, and the same holds for

$$\begin{aligned}
R_{12} &= O(\varepsilon)(1+t)^{-3/2}e^{-(x-at)^2/ct} + O(\varepsilon)e^{-ct} + O(1)e^{-ct/\varepsilon^2}, \\
R_{22} &= O(\varepsilon^2)(1+t)^{-2}e^{-(x-at)^2/ct} + O(\varepsilon^2)e^{-ct} + O(1)e^{-ct/\varepsilon^2}.
\end{aligned}$$

From the assumptions above, $f(u) = f(w_1) = aw_1 + h(w_1)$, where $h(w_1) = w_1^2\tilde{h}(w_1)$ for some function $\tilde{h}(w_1)$. Thus, by using the estimates of Theorem 9.3.1, and recalling that

$\|\cdot\|_m = \|\cdot\|_{H^m(\mathbb{R})}$, for $m = 0, 1, 2$, ($H^0 = L^2$), we have, for $j = 1, 2$,

$$\begin{aligned}\|\mathbf{w}(t)\|_m \leq{}& C\min\{1, t^{-1/4}\}\|\mathbf{w}_0\|_{L^1} + Ce^{-ct/\varepsilon^2}\|\mathbf{w}_0\|_m\\ &+ C\int_0^t \min\{1, (t-s)^{-3/4}\}(\|w_1^2\tilde{h}(w_1)\|_{L^1} + \|w_1^2\tilde{h}(w_1)\|_m)\, ds\\ &+ C\int_0^t e^{-c(t-s)}\|w_1^2\tilde{h}(w_1)\|_m\, ds\\ &+ C\int_0^t \frac{1}{\varepsilon}e^{-c(t-s)/\varepsilon^2}\|w_1^2\tilde{h}(w_1)\|_m\, ds.\end{aligned}$$

For m big enough,

$$\begin{aligned}\|\mathbf{w}(t)\|_m \leq{}& C\min\{1, t^{-1/4}\}\|\mathbf{w}_0\|_{L^1} + Ce^{-ct/\varepsilon^2}\|\mathbf{w}_0\|_m\\ &+ \int_0^t \min\{1, (t-s)^{-3/4}\}C(|w_1|_\infty)\|w_1\|_m^2\, ds\\ &+ \int_0^t e^{-c(t-s)}C(|w_1|_\infty)\|w_1\|_m^2\, ds\\ &+ \int_0^t \frac{1}{\varepsilon}e^{-c(t-s)/\varepsilon^2}C(|w_1|_\infty)\|w_1\|_j^2\, ds.\end{aligned}$$

From (9.1.19), we recall that $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} u \\ \dfrac{\varepsilon(v - au)}{\sqrt{\lambda^2 - a^2\varepsilon^2}} \end{pmatrix}$, and so, for $m = 2$,

$$\begin{aligned}\|u(t)\|_2 + c\varepsilon\|v(t) - au(t)\|_2 \leq{}& C\min\{1, t^{-1/4}\}(\|u_0\|_{L^1} + c\varepsilon\|v_0 - au_0\|_{L^1})\\ &+ e^{-ct/\varepsilon^2}(\|u_0\|_2 + c\varepsilon\|v_0 - au_0\|_2)\\ &+ \int_0^t \min\{1, (t-s)^{-3/4}\}C(|u|_\infty)\|u\|_2^2\, ds\\ &+ \int_0^t e^{-c(t-s)}C(|u|_\infty)\|u\|_2^2\, ds\\ &+ \int_0^t \frac{1}{\varepsilon}e^{-c(t-s)/\varepsilon^2}C(|u|_\infty)\|u\|_2^2\, ds.\end{aligned}$$

Let us denote by

$$E_m = \max\{\|u_0\|_{L^1} + \varepsilon\|v_0 - au_0\|_{L^1}, \|u_0\|_m + \varepsilon\|v_0 - au_0\|_m\}, \tag{9.4.67}$$

where, according to (9.0.6), $v_0 = f(u_0) - \lambda^2\partial_x u_0$, and

$$M_0(t) = \sup_{0\leq\tau\leq t}\{\max\{1, \tau^{1/4}\}(\|u(\tau)\|_2 + \varepsilon\|v(\tau) - au(\tau)\|_2)\}. \tag{9.4.68}$$

The first term of the right hand side of the estimate above gives

$$\begin{aligned}C\min\{1, t^{-1/4}\}(\|u_0\|_{L^1} + c\varepsilon\|v_0 - au_0\|_{L^1}) + Ce^{-ct/\varepsilon^2}(\|u_0\|_2 + c\varepsilon\|v_0 - au_0\|_2)&\\ \leq C\min\{1, t^{-1/4}\}E_2.&\end{aligned}$$

Besides,

$$C(|u|_\infty)\|u\|_2^2 \leq C(|u|_\infty)\min\{1, s^{-1/2}\}(M_0(s))^2.$$

Thus,

$$\begin{aligned}\|u(t)\|_2 + \varepsilon\|v(t) - au(t)\|_2 \leq C\min\{1, t^{-1/4}\}E_2 \\ + (M_0(t))^2\int_0^t e^{-c(t-s)}c(|u|_\infty)\min\{1, s^{-1/2}\}\,ds \\ + (M_0(t))^2\int_0^t \frac{1}{\varepsilon}e^{-c(t-s)/\varepsilon^2}c(|u|_\infty)\min\{1, s^{-1/2}\}\,ds \\ + (M_0(t))^2\int_0^t c(|u|_\infty)\min\{1, (t-s)^{-3/4}\}\min\{1, s^{-1/2}\}\,ds.\end{aligned}$$

From the Sobolev embedding theorem,

$$c(|u(s)|_\infty) \leq c(\|u(s)\|_2) \leq C\min\{1, s^{-1/4}\}M_0(s) \leq CM_0(s).$$

This way,

$$\begin{aligned}\|u(t)\|_2 + \varepsilon\|v(t) - au(t)\|_2 \leq C\min\{1, t^{-1/4}\}E_2 \\ + C(M_0(t))^3\int_0^t e^{-c(t-s)}\min\{1, s^{-1/2}\}\,ds \\ + C(M_0(t))^3\int_0^t \frac{1}{\varepsilon}e^{-c(t-s)/\varepsilon^2}\min\{1, s^{-1/2}\}\,ds \\ + C(M_0(t))^3\int_0^t \min\{1, (t-s)^{-3/4}\}\min\{1, s^{-1/2}\}\,ds.\end{aligned}$$

Notice that

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t e^{-c(t-s)/\varepsilon^2}\min\{1, s^{-1/2}\}\,ds &= \varepsilon e^{-ct/\varepsilon^2}\int_0^{t/\varepsilon^2} e^{c\tau}\min\{1, \varepsilon\sqrt{\tau}\}\,d\tau \\ &\leq \varepsilon e^{-ct/\varepsilon^2}\int_0^{t/\varepsilon^2} e^{c\tau}\,d\tau \\ &= \frac{\varepsilon}{c}[1 - e^{-ct/\varepsilon^2}] \\ &\leq C\varepsilon.\end{aligned}$$

By using this inequality in the estimate above,

$$\begin{aligned}\|u(t)\|_2 + \varepsilon\|v(t) - au(t)\|_2 \leq C\min\{1, t^{-1/4}\}E_2 \\ + C(M_0(t))^3\int_0^t e^{-c(t-s)}\min\{1, s^{-1/2}\}\,ds \\ + \varepsilon C(M_0(t))^3 \\ + C(M_0(t))^3\int_0^t \min\{1, (t-s)^{-3/4}\}\min\{1, s^{-1/2}\}\,ds.\end{aligned}$$

By applying usual lemmas on integration, as Lemma 5.2 in [15], we get the following inequality

$$M_0(t) \leq C(E_2 + (M_0(t))^3).$$

Then, if E_2 is small enough,

$$M_0(t) \leq CE_2,$$

i.e.

$$\|u(t)\|_2 + \varepsilon\|v(t) - au(t)\|_2 \leq C\min\{1, t^{-1/4}\}E_2. \tag{9.4.69}$$

By arguing as before and following [15], we have the proposition below.

Proposition 9.4.1. *The following estimates hold, with C a constant independent of ε,*

$$\|D^\beta \boldsymbol{w}(t)\|_0 \leq C\min\{1, t^{-1/4-|\beta|/2}\}E_{|\beta|+3/2}, \tag{9.4.70}$$

$$\|D^\beta w_2(t)\|_0 \leq C\min\{1, t^{-3/4-|\beta|/2}\}E_{|\beta|+3/2}, \tag{9.4.71}$$

$$\|D^\beta \partial_t \boldsymbol{w}(t)\|_0 \leq C\min\{1, t^{-3/4-|\beta|/2}\}E_{|\beta|+5/2}, \tag{9.4.72}$$

$$\|D^\beta \partial_t w_2(t)\|_0 \leq C\min\{1, t^{-5/4-|\beta|/2}\}E_{|\beta|+7/2}. \tag{9.4.73}$$

Remark 9.4.1. *Notice that the estimates for the partial derivative in time of the solution (9.4.72), (9.4.73) are uniform in ε thanks to the particular form of the initial data (9.0.7). In fact, these estimates can be obtained by applying again the Duhamel formula as before and, similarly to (9.4.69), we get a bound for $\|D^\beta\partial_t \boldsymbol{w}(t)\|_0$ which depends on $\|D^\beta\partial_t \boldsymbol{w}|_{t=0}\|$. This norm is not singular in ε thanks to the particular form of the initial data, as it is shown below. The initial data satisfy (9.0.6), i.e.*

$$v_0 = f(u_0) - \lambda^2\partial_x u_0 = au_0 + h(u_0) - \lambda^2\partial_x u_0.$$

In terms of the (C-D)-variable $\boldsymbol{w}$,

$$\begin{cases} u = w_1, \\ v = aw_1 + \dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} w_2, \end{cases}$$

this gives the following relation:

$$\frac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon} w_2^0 = h(w_1^0) - \lambda^2\partial_x w_1^0 \tag{9.4.74}$$

Using (9.4.74) in system (9.1.20), this yields

$$\begin{cases} \partial_t w_1|_{t=0} = -a\partial_x w_1^0 - \dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon}\partial_x w_2^0, \\ \partial_t w_2|_{t=0} = -\dfrac{\sqrt{\lambda^2 - a^2\varepsilon^2}}{\varepsilon}\partial_x w_1^0 + a\partial_x w_2^0 + \dfrac{\lambda^2}{\varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2}}\partial_x w_1^0 \\ = \dfrac{a^2\varepsilon}{\sqrt{\lambda^2 - a^2\varepsilon^2}}\partial_x w_1^0 + a\partial_x w_2^0. \end{cases}$$

In terms of the original variable,

$$\begin{cases} \partial_t w_1|_{t=0} = -\partial_x f(\bar{u}_0) + \lambda^2\partial_{xx}\bar{u}_0, \\ \partial_t w_2|_{t=0} = \dfrac{a\varepsilon}{\sqrt{\lambda^2 - a^2\varepsilon^2}}(\partial_x f(\bar{u}_0) - \lambda^2\partial_{xx}\bar{u}_0). \end{cases}$$

Convergence in the diffusion limit and asymptotic behavior We perform the one dimensional Chapman-Enskog expansion. Recalling that

$$w_1 = u, \qquad w_2 = u_d,$$

where u is the conservative variable and u_d is the dissipative one, system (9.1.20) is

$$\partial_t \begin{pmatrix} u \\ u_d \end{pmatrix} + A\partial_x \begin{pmatrix} u \\ u_d \end{pmatrix} = \begin{pmatrix} 0 \\ q(u) \end{pmatrix},$$

with A in (9.1.21) and $q(u) = -\dfrac{w_2}{\varepsilon^2} + \dfrac{h(w_1)}{\varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2}}$. We consider the following nonlinear parabolic equation

$$\partial_t u + a\partial_x u + \partial_x h(u) - (\lambda^2 - a^2\varepsilon^2)\partial_{xx} u = \partial_x S,$$

where

$$S = \varepsilon\sqrt{\lambda^2 - a^2\varepsilon^2}\{\partial_t u_d - a\partial_x u_d\}. \tag{9.4.75}$$

The homogeneous equation is

$$\partial_t w_p + a\partial_x w_p + \partial_x h(w_p) - (\lambda^2 - a^2\varepsilon^2)\partial_{xx} w_p = 0, \tag{9.4.76}$$

and associated Green function is provided here

$$\Gamma_p(t) = K_{11}(t) + \tilde{\mathcal{K}}(t) + \tilde{R}(t),$$

with K_{11} in (9.3.51). We take the difference between the conservative variable $u = w_1$ and w_p,

$$\begin{aligned} D^\beta(u(t) - w_p(t)) = & \int_0^{t/2} D^\beta D(K_{11}(t-s) + \tilde{R}(t-s))(h(w_p(s)) - h(u(s)))\,ds \\ & + \int_0^{t/2} D^\beta D(K_{11}(t-s) + \tilde{R}(t-s))S(s)\,ds \\ & + \int_{t/2}^{t} D(K_{11}(t-s) + \tilde{R}(t-s))D^\beta(h(w_p(s)) - h(u(s)) + S(s))\,ds \\ & + \int_0^{t} \tilde{\mathcal{K}}(t-s)D^\beta D(h(w_p(s)) - h(u(s)) + S(s))\,ds. \end{aligned} \tag{9.4.77}$$

By using (9.4.70), (9.4.71), (9.4.73), we have

$$\|D^\beta S\|_0 \le C\varepsilon \min\{1, t^{-5/4-\beta/2}\}E_{\beta+1}.$$

Let us define, for $\mu \in [0, 1/2)$,

$$m_0(t) = \sup_{\tau\in[0,t]} \{\max\{1, \tau^{1/4+\mu}\}\|u(\tau) - w_p(\tau)\|_0\}. \tag{9.4.78}$$

For $\beta = 0$,

$$
\begin{aligned}
\|u(t) - w_p(t)\|_0 \le{}& CE_1 m_0(t) \int_0^t \min\{1, (t-s)^{-3/4}\} \min\{1, s^{-1/2-\mu}\}\, ds \\
&+ C\varepsilon E_3 \int_0^t \min\{1, (t-s)^{-3/4}\} \min\{1, s^{-1}\}\, ds \\
&+ C(E_1 m_0(t) + \varepsilon E_4) \int_0^t e^{-c(t-s)} \min\{1, s^{-5/4}\}\, ds \\
\le{}& C \min\{1, s^{-1/4-\mu}\}(E_1 m_0(t) + \varepsilon E_1 + \varepsilon E_4 + (1/2-\mu)^{-1} E_1 m_0(t)),
\end{aligned}
$$

i.e., if E_1 is small enough,

$$m_0(t) \le C\varepsilon E_4. \tag{9.4.79}$$

Similarly, it can be proved by induction that if, for $\gamma < \beta$, defining

$$m_\beta(t) = \sup_{\tau \in [0,t]} \{\max\{1, \tau^{1/4+\mu+\beta/2}\} \|D^\beta(u(\tau) - w_p(\tau))\|_0\}, \tag{9.4.80}$$

and assuming $m_\gamma(t) \le C(\mu)\varepsilon E_{\gamma+4}$, then

$$\|D^\beta(h(u(s)) - h(w_p(s)))\|_0 \le C \min\{1, t^{-1/2-\mu-\beta/2}\}(C(\mu)E_{\beta+1}E_{\beta+3} + E_1 m_\beta(t)).$$

Using this inequality, (9.4.75) and (9.4.79) in (9.4.77), finally we get

$$m_\beta(t) \le C(\mu)\varepsilon E_{\beta+4},$$

which ends the proof of Theorem 9.0.1.

Chapter 10

A BGK model for hydrodynamic equations

We want to study the convergence of a singular perturbation approximation to the Cauchy problem for the incompressible Navier-Stokes equations on the d dimensional torus $\mathbb{T}^d$:

$$\begin{cases} \partial_t \mathbf{u}^{NS} + \nabla \cdot (\mathbf{u}^{NS} \otimes \mathbf{u}^{NS}) + \nabla P^{NS} = \nu \Delta \mathbf{u}^{NS}, \\ \nabla \cdot \mathbf{u}^{NS} = 0, \end{cases} \tag{10.0.1}$$

with $(t, x) \in [0, +\infty) \times \mathbb{T}^d$, and initial data

$$\mathbf{u}^{NS}(0, x) = \mathbf{u}_0(x), \qquad \nabla \cdot \mathbf{u}_0 = 0. \tag{10.0.2}$$

Here $\mathbf{u}^{NS}$ and ∇P^{NS} are respectively the velocity field and the gradient of the pressure term, and $\nu > 0$ is the viscosity coefficient. This chapter is based on [11].
We consider a semilinear hyperbolic approximation, called *vector BGK model*, [23, 18], to the incompressible Navier-Stokes equations (10.0.1). The general form of this approximation is as follows:

$$\partial_t f_l^\varepsilon + \frac{\lambda_l}{\varepsilon} \cdot \nabla_x f_l^\varepsilon = \frac{1}{\tau \varepsilon^2}(M_l(\rho^\varepsilon, \varepsilon \rho^\varepsilon \mathbf{u}^\varepsilon) - f_l^\varepsilon), \tag{10.0.3}$$

with initial data

$$f_l^\varepsilon(0, x) = \bar{M}_l^\varepsilon(\bar{\rho}, \varepsilon \bar{\rho} \mathbf{u}_0) = M_l^\varepsilon(\bar{\rho}, \varepsilon \bar{\rho} \mathbf{u}_0) + \varepsilon g(\nabla \mathbf{u}_0), \quad \mathbf{u}_0 \text{ in } (10.0.2), \quad l = 1, \cdots, L, \tag{10.0.4}$$

where f_l^ε and M_l^ε take values in $\mathbb{R}^{d+1}$, with the Maxwellian functions M_l^ε Lipschitz continuous, $\lambda_l = (\lambda_{l1}, \cdots, \lambda_{ld})$ are constant velocities, and $L \geq d + 1$. The $\bar{M}_l^\varepsilon$ are the perturbed Maxwellian functions, which will be expressed later, where g is the first order correction of the Maxwellians in the Chapman-Enskog expansion. Moreover, $\bar{\rho} > 0$ is a given constant value, and ε and τ are positive parameters. Denoting by $f_{lj}^\varepsilon, M_{lj}^\varepsilon$, for $j = 0, \cdots, d$, the $d + 1$ components of $f_l^\varepsilon, M_l^\varepsilon$ for each $l = 1, \cdots, L$, let us set

$$\rho^\varepsilon = \sum_{l=1}^L f_{l0}^\varepsilon(t, x) \quad \text{and} \quad q_j^\varepsilon = \varepsilon \rho^\varepsilon u_j^\varepsilon = \sum_{l=1}^L f_{lj}^\varepsilon(t, x). \tag{10.0.5}$$

In [23, 18], the convergence of the solutions to the vector BGK model introduced above to the solutions to the incompressible Navier-Stokes equations is studied numerically. More precisely, assuming that, in a suitable functional space,

$$\rho^\varepsilon \to \hat{\rho}, \qquad \mathbf{u}^\varepsilon \to \hat{\mathbf{u}}, \qquad \text{and} \qquad \frac{\rho^\varepsilon - \bar{\rho}}{\varepsilon^2} \to \hat{P},$$

under some consistency conditions of the BGK approximation with respect to the Navier-Stokes equations, see [23], it can be shown that the couple $(\hat{\mathbf{u}}, \hat{P})$ is a solution to the incompressible Navier-Stokes equations. The aim of this chapter is to provide a rigorous proof of this convergence in the Sobolev spaces.
We focus on the two dimensional case in space. Following [23], let us set $d = 2$, $L = 5$, and

$$w^\varepsilon = (\rho^\varepsilon, \mathbf{q}^\varepsilon) = (\rho^\varepsilon, q_1^\varepsilon, q_2^\varepsilon) = (\rho^\varepsilon, \varepsilon\rho^\varepsilon u_1^\varepsilon, \varepsilon\rho^\varepsilon u_2^\varepsilon) = \sum_{l=1}^{5} f_l^\varepsilon \in \mathbb{R}^3. \tag{10.0.6}$$

Fix $\lambda, \tau > 0$ and let $\varepsilon > 0$ be a small parameter, which is going to zero in the singular perturbation limit. Thus, we get a five velocities model (15 scalar equations):

$$\begin{cases} \partial_t f_1^\varepsilon + \frac{\lambda}{\varepsilon}\partial_x f_1^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_1(w^\varepsilon) - f_1^\varepsilon), \\ \partial_t f_2^\varepsilon + \frac{\lambda}{\varepsilon}\partial_y f_2^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_2(w^\varepsilon) - f_2^\varepsilon), \\ \partial_t f_3^\varepsilon - \frac{\lambda}{\varepsilon}\partial_x f_3^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_3(w^\varepsilon) - f_3^\varepsilon), \\ \partial_t f_4^\varepsilon - \frac{\lambda}{\varepsilon}\partial_y f_4^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_4(w^\varepsilon) - f_4^\varepsilon), \\ \partial_t f_5^\varepsilon = \frac{1}{\tau\varepsilon^2}(M_5(w^\varepsilon) - f_5^\varepsilon). \end{cases} \tag{10.0.7}$$

Here the Maxwellian functions $M_j \in \mathbb{R}^3$ have the following expressions:

$$M_{1,3}(w^\varepsilon) = aw^\varepsilon \pm \frac{A_1(w^\varepsilon)}{2\lambda}, \qquad M_{2,4}(w^\varepsilon) = aw^\varepsilon \pm \frac{A_2(w^\varepsilon)}{2\lambda}, \qquad M_5(w^\varepsilon) = (1-4a)w^\varepsilon, \tag{10.0.8}$$

where

$$A_1(w^\varepsilon) = \begin{pmatrix} q_1^\varepsilon \\ \frac{(q_1^\varepsilon)^2}{\rho^\varepsilon} + P(\rho^\varepsilon) \\ \frac{q_1^\varepsilon q_2^\varepsilon}{\rho^\varepsilon} \end{pmatrix}, \qquad A_2(w^\varepsilon) = \begin{pmatrix} q_2^\varepsilon \\ \frac{q_1^\varepsilon q_2^\varepsilon}{\rho^\varepsilon} \\ \frac{(q_2^\varepsilon)^2}{\rho^\varepsilon} + P(\rho^\varepsilon) \end{pmatrix}, \tag{10.0.9}$$

$$P(\rho^\varepsilon) = \rho^\varepsilon - \bar{\rho}, \tag{10.0.10}$$

and

$$a = \frac{\nu}{2\lambda^2\tau}, \tag{10.0.11}$$

where ν is the viscosity coefficient in (10.0.1). In the following, our main goal is to obtain uniform energy estimates for the solutions to the vector BGK model (10.0.7) in the Sobolev spaces and to get the convergence by compactness. In [23, 18], an L^2 estimate was obtained by using the entropy function associated with the vector BGK model, whose existence is proved in [16]. However, there is no explicit expression for the kinetic entropy, so we do not know the weights, with respect to the singular parameter, of the terms of the classical symmetrizer derived by the entropy, see [39] for the one dimensional case and [15, 44] for the general case. For this reason, the existence of an entropy is not enough to control the higher order estimates. Moreover, our pressure term is given by (10.0.10) and it is linear with respect to ρ^ε, so the estimates in [23, 18] no more hold. To solve this problem, we use a constant right symmetrizer, whose entries are weighted in terms of the singular parameter in a suitable way. Besides, the symmetrization obtained by the right multiplication provides the conservative-dissipative form introduced in Chapter 3. The dissipative property of the symmetrized system holds under the following hypothesis.

Assumption 10.0.1 (Dissipation condition)**.** *We assume the following structural condition:*

$$0 < a < \frac{1}{4}.$$

Finally, we point out that Assumption 10.0.1 is a necessary condition, also in the case of nonlinear pressure terms, for the existence of a kinetic entropy for the approximating system, see [16].

10.1 General framework

Let us set

$$U^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon, f_4^\varepsilon, f_5^\varepsilon) \in \mathbb{R}^{3\times 5},$$

and let us write the compact formulation of equations (10.0.7)-(10.0.4), which reads

$$\partial_t U^\varepsilon + \Lambda_1 \partial_x U^\varepsilon + \Lambda_2 \partial_y U^\varepsilon = \frac{1}{\tau\varepsilon^2}(M(U^\varepsilon) - U^\varepsilon), \tag{10.1.12}$$

with initial data

$$U_0^\varepsilon = f_l^\varepsilon(0,x) = \bar{M}_l^\varepsilon(\bar\rho, \varepsilon\bar\rho\mathbf{u}_0) = M_l^\varepsilon(\bar\rho, \varepsilon\bar\rho\mathbf{u}_0) + \varepsilon g(\nabla\mathbf{u}_0), \quad l = 1, \cdots, 5, \tag{10.1.13}$$

where $\bar{M}_l^\varepsilon$ are the perturbed Maxwellian functions, with M_l^ε the Maxwellians in (10.0.8), and

$$g(\nabla\mathbf{u}_0) = \begin{pmatrix} -a\lambda\tau\partial_x w_0 \\ -a\lambda\tau\partial_y w_0 \\ a\lambda\tau\partial_x w_0 \\ a\lambda\tau\partial_y w_0 \\ 0 \end{pmatrix}, \quad w_0 = (\bar\rho, \varepsilon\bar\rho\mathbf{u}_0), \tag{10.1.14}$$

$$\Lambda_1 = \begin{pmatrix} \frac{\lambda}{\varepsilon}Id & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\lambda}{\varepsilon}Id & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda}{\varepsilon}Id & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\lambda}{\varepsilon}Id & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Id is the 3×3 identity matrix, and

$$M(U^\varepsilon) = (M_1^\varepsilon(w^\varepsilon), M_2^\varepsilon(w^\varepsilon), M_3^\varepsilon(w^\varepsilon), M_4^\varepsilon(w^\varepsilon), M_5^\varepsilon(w^\varepsilon)). \tag{10.1.15}$$

10.1.1 Conservative variables

We define the following change of variables:

$$w^\varepsilon = \sum_{l=1}^5 f_l^\varepsilon, \quad m^\varepsilon = \tfrac{\lambda}{\varepsilon}(f_1^\varepsilon - f_3^\varepsilon), \quad \xi^\varepsilon = \tfrac{\lambda}{\varepsilon}(f_2^\varepsilon - f_4^\varepsilon), \quad k^\varepsilon = f_1^\varepsilon + f_3^\varepsilon, \quad h^\varepsilon = f_2^\varepsilon + f_4^\varepsilon. \tag{10.1.16}$$

This way, the vector BGK model (10.0.7) reads:

$$\begin{cases} \partial_t w^\varepsilon + \partial_x m^\varepsilon + \partial_y \xi^\varepsilon = 0; \\ \partial_t m^\varepsilon + \frac{\lambda^2}{\varepsilon^2}\partial_x k^\varepsilon = \frac{1}{\tau\varepsilon^2}\left(\frac{A_1(w^\varepsilon)}{\varepsilon} - m^\varepsilon\right), \\ \partial_t \xi^\varepsilon + \frac{\lambda^2}{\varepsilon^2}\partial_y h^\varepsilon = \frac{1}{\tau\varepsilon^2}\left(\frac{A_2(w^\varepsilon)}{\varepsilon} - \xi^\varepsilon\right), \\ \partial_t k^\varepsilon + \partial_x m^\varepsilon = \frac{1}{\tau\varepsilon^2}(2aw^\varepsilon - k^\varepsilon), \\ \partial_t h^\varepsilon + \partial_y \xi^\varepsilon = \frac{1}{\tau\varepsilon^2}(2aw^\varepsilon - h^\varepsilon). \end{cases} \tag{10.1.17}$$

We make a slight modification of system (10.1.17). Set $\bar{w} = (\bar{\rho}, 0, 0)$ and

$$w^{\varepsilon\star} := w^\varepsilon - \bar{w} = (w_1^\varepsilon - \bar{\rho}, w_2^\varepsilon, w_3^\varepsilon), \quad k^{\varepsilon\star} = k^\varepsilon - 2a\bar{w}, \quad h^{\varepsilon\star} = h^\varepsilon - 2a\bar{w}. \tag{10.1.18}$$

In the following, we are going to work with the modified variables. System (10.1.17) reads:

$$\begin{cases} \partial_t w^{\varepsilon\star} + \partial_x m^\varepsilon + \partial_y \xi^\varepsilon = 0; \\ \partial_t m^\varepsilon + \frac{\lambda^2}{\varepsilon^2}\partial_x k^{\varepsilon\star} = \frac{1}{\tau\varepsilon^2}\left(\frac{A_1(w^{\varepsilon\star}+\bar{w})}{\varepsilon} - m^\varepsilon\right), \\ \partial_t \xi^\varepsilon + \frac{\lambda^2}{\varepsilon^2}\partial_y h^{\varepsilon\star} = \frac{1}{\tau\varepsilon^2}\left(\frac{A_2(w^{\varepsilon\star}+\bar{w})}{\varepsilon} - \xi^\varepsilon\right), \\ \partial_t k^{\varepsilon\star} + \partial_x m^\varepsilon = \frac{1}{\tau\varepsilon^2}(2aw^{\varepsilon\star} - k^{\varepsilon\star}), \\ \partial_t h^{\varepsilon\star} + \partial_y \xi^\varepsilon = \frac{1}{\tau\varepsilon^2}(2aw^{\varepsilon\star} - h^{\varepsilon\star}). \end{cases} \tag{10.1.19}$$

Notice from (10.0.9) that

$$A_1(w^\varepsilon) = \begin{pmatrix} q_1^\varepsilon \\ \frac{(q_1^\varepsilon)^2}{\rho^\varepsilon} + \rho^\varepsilon - \bar{\rho} \\ \frac{q_1^\varepsilon q_2^\varepsilon}{\rho^\varepsilon} \end{pmatrix} = \begin{pmatrix} w_2^{\varepsilon\star} \\ \frac{(w_2^{\varepsilon\star})^2}{w_1^{\varepsilon\star}+\bar{\rho}} + w_1^{\varepsilon\star} \\ \frac{w_2^{\varepsilon\star}w_3^{\varepsilon\star}}{w_1^{\varepsilon\star}+\bar{\rho}} \end{pmatrix} = A_1(w^{\varepsilon\star} + \bar{w}),$$

and, similarly,

$$A_2(w^\star) = \begin{pmatrix} q_2^\varepsilon \\ \frac{q_1^\varepsilon q_2^\varepsilon}{\rho^\varepsilon} \\ \frac{(q_2^\varepsilon)^2}{\rho^\varepsilon} + \rho^\varepsilon - \bar{\rho} \end{pmatrix} = \begin{pmatrix} w_3^{\varepsilon\star} \\ \frac{w_2^{\varepsilon\star}w_3^{\varepsilon\star}}{w_1^{\varepsilon\star}+\bar{\rho}} \\ \frac{(w_3^{\varepsilon\star})^2}{w_1^{\varepsilon\star}+\bar{\rho}} + w_1^{\varepsilon\star} \end{pmatrix} = A_2(w^{\varepsilon\star} + \bar{w}).$$

Hereafter, we will omit the apexes $\varepsilon\star$ for $w^{\varepsilon\star}, k^{\varepsilon\star}, h^{\varepsilon\star}$, and the apex ε for $m^\varepsilon, \xi^\varepsilon$, when there is no ambiguity.
Let us define the 15×15 matrix

$$C = \begin{pmatrix} Id & Id & Id & Id & Id \\ \varepsilon\lambda Id & 0 & -\varepsilon\lambda Id & 0 & 0 \\ 0 & \varepsilon\lambda Id & 0 & -\varepsilon\lambda Id & 0 \\ \varepsilon^2 Id & 0 & \varepsilon^2 Id & 0 & 0 \\ 0 & \varepsilon^2 Id & 0 & \varepsilon^2 Id & 0 \end{pmatrix}, \tag{10.1.20}$$

and set

$$W = (w, \varepsilon^2 m, \varepsilon^2 \xi, \varepsilon^2 k, \varepsilon^2 h) := CU - (\bar{w}, 0, 0, 0, 0). \tag{10.1.21}$$

Thus, we can write the translated system (10.1.19) in the compact form

$$\partial_t W + B_1 \partial_x W + B_2 \partial_y W = \frac{1}{\tau\varepsilon^2}(\tilde{M}(W) - W), \tag{10.1.22}$$

with initial conditions

$$W_0 = CU_0 - (\bar{w}, 0, 0, 0, 0), \tag{10.1.23}$$

where U_0 is given by (10.1.13),

$$B_1 = C\Lambda_1 C^{-1}, \qquad B_2 = C\Lambda_2 C^{-1},$$

$$B_1 = \begin{pmatrix} 0 & \frac{1}{\varepsilon^2} Id & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda^2}{\varepsilon^2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & Id & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & \frac{1}{\varepsilon^2} Id & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\lambda^2}{\varepsilon^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Id & 0 & 0 \end{pmatrix}, \tag{10.1.24}$$

and

$$\tilde{M}(W) = CM(C^{-1}W) = CM(U).$$

Here,

$$\frac{1}{\tau\varepsilon^2}(\tilde{M}(W) - W) = \frac{1}{\tau} \begin{pmatrix} 0 \\ \frac{A_1(w+\bar{w})}{\varepsilon} - \frac{\varepsilon^2 m}{\varepsilon^2} \\ \frac{A_2(w+\bar{w})}{\varepsilon} - \frac{\varepsilon^2 \xi}{\varepsilon^2} \\ 2aw - \frac{\varepsilon^2 k}{\varepsilon^2} \\ 2aw - \frac{\varepsilon^2 h}{\varepsilon^2} \end{pmatrix} = \frac{1}{\tau} \begin{pmatrix} 0 \\ \frac{1}{\varepsilon}\begin{pmatrix} w_2 \\ \frac{w_2^2}{w_1+\bar{\rho}} + w_1 \\ \frac{w_2 w_3}{w_1+\bar{\rho}} \end{pmatrix} - \frac{\varepsilon^2 m}{\varepsilon^2} \\ \frac{1}{\varepsilon}\begin{pmatrix} w_3 \\ \frac{w_2 w_3}{w_1+\bar{\rho}} \\ \frac{w_3^2}{w_1+\bar{\rho}} + w_1 \end{pmatrix} - \frac{\varepsilon^2 \xi}{\varepsilon^2} \\ 2aw - \frac{\varepsilon^2 k}{\varepsilon^2} \\ 2aw - \frac{\varepsilon^2 h}{\varepsilon^2} \end{pmatrix}$$

$$= \frac{1}{\tau} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\varepsilon}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & -\frac{1}{\varepsilon^2} Id & 0 & 0 & 0 \\ \frac{1}{\varepsilon}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & 0 & -\frac{1}{\varepsilon^2} Id & 0 & 0 \\ 2aId & 0 & 0 & -\frac{1}{\varepsilon^2} Id & 0 \\ 2aId & 0 & 0 & 0 & -\frac{1}{\varepsilon^2} Id \end{pmatrix} W + \frac{1}{\tau} \begin{pmatrix} 0 \\ \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ \frac{w_2^2}{w_1+\bar{\rho}} \\ \frac{w_2 w_3}{w_1+\bar{\rho}} \end{pmatrix} \\ \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ \frac{w_2 w_3}{w_1+\bar{\rho}} \\ \frac{w_3^2}{w_1+\bar{\rho}} \end{pmatrix} \\ 0 \\ 0 \end{pmatrix}$$

$$=: -LW + N(w + \bar{w}), \tag{10.1.25}$$

where $-L$ is the linear part of the source term of (10.1.22), while N is the remaining nonlinear one. Thus, we can rewrite system (10.1.22) as follows:

$$\partial_t W + B_1 \partial_x W + B_2 \partial_y W = -LW + N(w + \bar{w}). \tag{10.1.26}$$

10.2 The weighted constant right symmetrizer and the conservative-dissipative form

According to the theory of semilinear hyperbolic systems, see for instance [52, 8], we need a symmetric formulation of system (10.1.26) in order to get energy estimates. However, we are dealing with a singular perturbation system, so any symmetrizer for system (10.1.26) is not enough. In other words, we look for a symmetrizer which provides a suitable dissipative structure for system (10.1.26). In this context, notice that the first equation of system (10.1.26) reads

$$\partial_t w + \partial_x m + \partial_y \xi = 0,$$

i.e. the first term of the source vanishes, and w is a conservative variable. We want to take advantage of this conservative property, in order to simplify the algebraic structure of the linear part of the source term. To this end, rather than a classical Friedrichs left symmetrizer, see again [52, 8], we look for a right symmetrizer for (10.1.26), which allows to get the conservative-dissipative form introduced in Chapter 3. More precisely, the right multiplication easily provides the conservative structure in [15], while the dissipation is proved a posteriori. Besides, the symmetrizer Σ presents constant ε-weighted entries and this allows us to control the nonlinear part N of the source term (10.1.25) of system (10.1.26). To be complete, we point out that the inverse matrix Σ^{-1} is a left symmetrizer for system (10.1.26), according to Chapter 2. However, the product $-\Sigma^{-1}L$ is a full matrix, so the symmetrized version of system (10.1.26), obtained by the left multiplication by Σ^{-1}, does not provide the conservative-dissipative form in Chapter 3.
Let us explicitly write the symmetrizer

$$\Sigma = \begin{pmatrix} Id & \varepsilon\sigma_1 & \varepsilon\sigma_2 & 2a\varepsilon^2 Id & 2a\varepsilon^2 Id \\ \varepsilon\sigma_1 & 2\lambda^2 a\varepsilon^2 Id & 0 & \varepsilon^3\sigma_1 & 0 \\ \varepsilon\sigma_2 & 0 & 2\lambda^2 a\varepsilon^2 Id & 0 & \varepsilon^3\sigma_2 \\ 2a\varepsilon^2 Id & \varepsilon^3\sigma_1 & 0 & 2a\varepsilon^4 Id & 0 \\ 2a\varepsilon^2 Id & 0 & \varepsilon^3\sigma_2 & 0 & 2a\varepsilon^4 Id \end{pmatrix}, \tag{10.2.27}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{10.2.28}$$

It is easy to check that Σ is a constant right symmetrizer for system (10.1.26) since, taking B_1, B_2 and L in (10.1.24) and (10.1.25) respectively,

$$B_1\Sigma = \Sigma B_1^T, \qquad B_2\Sigma = \Sigma B_2^T,$$

$$-L\Sigma = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0}^T & -\tilde{L} \end{pmatrix}$$

$$= \frac{1}{\tau}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2\lambda^2 aId + \sigma_1^2 & \sigma_1\sigma_2 & (2a-1)\varepsilon\sigma_1 & 2a\varepsilon\sigma_1 \\ 0 & \sigma_1\sigma_2 & -2\lambda^2 aId + \sigma_2^2 & 2a\varepsilon\sigma_2 & (2a-1)\varepsilon\sigma_2 \\ 0 & (2a-1)\varepsilon\sigma_1 & 2a\varepsilon\sigma_2 & 2a(2a-1)\varepsilon^2 Id & 4a^2\varepsilon^2 Id \\ 0 & 2a\varepsilon\sigma_1 & (2a-1)\varepsilon\sigma_2 & 4a^2\varepsilon^2 Id & 2a(2a-1)\varepsilon^2 Id \end{pmatrix}. \tag{10.2.29}$$

Now, we define the following change of variables:

$$W = \Sigma\tilde{W} = \Sigma(\tilde{w}, \varepsilon^2\tilde{m}, \varepsilon^2\tilde{\xi}, \varepsilon^2\tilde{k}, \varepsilon^2\tilde{h}), \tag{10.2.30}$$

with W in (10.1.21). System (10.1.26) reads:

$$\Sigma\partial_t\tilde{W} + B_1\Sigma\partial_x\tilde{W} + B_2\Sigma\partial_y\tilde{W} = -L\Sigma\tilde{W} + N((\Sigma\tilde{W})_1 + \bar{w}), \tag{10.2.31}$$

where $(\Sigma\tilde{W})_1$ is the first component of the unknown vector $\Sigma\tilde{W}$. Now, we want to show that Σ in (10.2.27) is strictly positive definite. Thus,

$$\begin{aligned}(\Sigma\tilde{W}, \tilde{W})_0 &= ||\tilde{w}||_0^2 + 2\lambda^2 a\varepsilon^6(||\tilde{m}||_0^2 + ||\tilde{\xi}||_0^2) + 2a\varepsilon^8(||\tilde{k}||_0^2 + ||\tilde{h}||_0^2) + 2(\varepsilon^3\sigma_1\tilde{m}, \tilde{w})_0 \\ &\quad + 2(\varepsilon^3\sigma_2\tilde{\xi}, \tilde{w})_0 + 4a\varepsilon^4(\tilde{k} + \tilde{h}, \tilde{w})_0 + 2\varepsilon^7(\sigma_1\tilde{k}, \tilde{m})_0 + 2\varepsilon^7(\sigma_2\tilde{h}, \tilde{\xi})_0 \\ &= ||\tilde{w}||_0^2 + 2\lambda^2 a\varepsilon^6(||\tilde{m}||_0^2 + ||\tilde{\xi}||_0^2) + 2a\varepsilon^8(||\tilde{k}||_0^2 + ||\tilde{h}||_0^2) + I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

Taking two positive constants δ, μ and by using the Cauchy inequality, we have:

$$I_1 = 2\varepsilon^3[(\tilde{m}_2, \tilde{w}_1)_0 + (\tilde{m}_1, \tilde{w}_2)_0] \geq -\delta\varepsilon^6||\tilde{m}_2||_0^2 - \tfrac{||\tilde{w}_1||_0^2}{\delta} - \delta\varepsilon^6||\tilde{m}_1||_0^2 - \tfrac{||\tilde{w}_2||_0^2}{\delta};$$

$$I_2 = 2\varepsilon^3[(\tilde{\xi}_3, \tilde{w}_1)_0 + (\tilde{\xi}_1, \tilde{w}_3)_0] \geq -\delta\varepsilon^6||\tilde{\xi}_3||_0^2 - \tfrac{||\tilde{w}_1||_0^2}{\delta} - \delta\varepsilon^6||\tilde{\xi}_1||_0^2 - \tfrac{||\tilde{w}_3||_0^2}{\delta};$$

$$I_3 = 4a\varepsilon^4[(\tilde{k}, \tilde{w})_0 + (\tilde{h}, \tilde{w})_0] \geq -2a\mu||\tilde{w}||_0^2 - \tfrac{2a\varepsilon^8}{\mu}||\tilde{k}||_0^2 - 2a\mu||\tilde{w}||_0^2 - \tfrac{2a\varepsilon^8}{\mu}||\tilde{h}||_0^2;$$

$$I_4 = 2\varepsilon^7[(\tilde{k}_2, \tilde{m}_1)_0 + (\tilde{k}_1, \tilde{m}_2)_0] \geq -\tfrac{\varepsilon^8}{\delta}||\tilde{k}_2||_0^2 - \delta\varepsilon^6||\tilde{m}_1||_0^2 - \tfrac{\varepsilon^8}{\delta}||\tilde{k}_1||_0^2 - \delta\varepsilon^6||\tilde{m}_2||_0^2;$$

$$I_5 = 2\varepsilon^7[(\tilde{h}_3, \tilde{\xi}_1)_0 + (\tilde{h}_1, \tilde{\xi}_3)_0] \geq -\tfrac{\varepsilon^8}{\delta}||\tilde{h}_3||_0^2 - \delta\varepsilon^6||\tilde{\xi}_1||_0^2 - \tfrac{\varepsilon^8}{\delta}||\tilde{h}_1||_0^2 - \delta\varepsilon^6||\tilde{\xi}_3||_0^2.$$

Thus, putting them all together,

$$
\begin{aligned}
(\Sigma\tilde{W},\tilde{W})_0 \geq{}& ||\tilde{w}_1||_0^2\left[1-\frac{2}{\delta}-4a\mu\right]+||\tilde{w}_2||_0^2\left[1-\frac{1}{\delta}-4a\mu\right]+||\tilde{w}_3||_0^2\left[1-\frac{1}{\delta}-4a\mu\right]\\
&+\varepsilon^6||\tilde{m}_1^\varepsilon||_0^2[2\lambda^2a-2\delta]+\varepsilon^6||\tilde{m}_2^\varepsilon||_0^2[2\lambda^2a-2\delta]+\varepsilon^6||\tilde{m}_3^\varepsilon||_0^2[2\lambda^2a]\\
&+\varepsilon^6||\tilde{\xi}_1^\varepsilon||_0^2[2\lambda^2a-2\delta]+\varepsilon^6||\tilde{\xi}_2^\varepsilon||_0^2[2\lambda^2a]+\varepsilon^6||\tilde{\xi}_3^\varepsilon||_0^2[2\lambda^2a-2\delta]\\
&+\varepsilon^8||\tilde{k}_1||_0^2\left[2a-\frac{2a}{\mu}-\frac{1}{\delta}\right]+\varepsilon^8||\tilde{k}_2||_0^2\left[2a-\frac{2a}{\mu}-\frac{1}{\delta}\right]+\varepsilon^8||\tilde{k}_3||_0^2\left[2a-\frac{2a}{\mu}\right]\\
&+\varepsilon^8||\tilde{h}_1||_0^2\left[2a-\frac{2a}{\mu}-\frac{1}{\delta}\right]+\varepsilon^8||\tilde{h}_2||_0^2\left[2a-\frac{2a}{\mu}\right]+\varepsilon^8||\tilde{h}_3||_0^2\left[2a-\frac{2a}{\mu}-\frac{1}{\delta}\right].
\end{aligned}
\tag{10.2.32}
$$

Now, we can prove the following lemma.

Lemma 10.2.1. *If Assumption 10.0.1 is satisfied and λ is big enough, then Σ is strictly positive definite.*

Proof. From (10.2.32), we take

$$
\begin{cases}
1<\mu<\frac{1}{4a};\\
\delta>\max\{\frac{2}{1-4a\mu},\frac{1}{2a(1-\frac{1}{\mu})}\};\\
\lambda>\sqrt{\frac{\delta}{a}}.
\end{cases}
\tag{10.2.33}
$$

Notice that we can choose the constant velocity λ as big as we need, therefore the third inequality is automatically verified. □

Now, we consider the linear part $-L\Sigma$ of the source term of (10.2.31). Thus,

$$\tau(-L\Sigma\tilde{W},\tilde{W})_0 = -2\lambda^2 a\varepsilon^4(||\tilde{m}||_0^2+||\tilde{\xi}||_0^2)+2a(2a-1)\varepsilon^6(||\tilde{k}||_0^2+||\tilde{h}||_0^2)$$

$$+\varepsilon^4||\tilde{m}_1||_0^2+\varepsilon^4||\tilde{m}_2||_0^2+\varepsilon^4||\tilde{\xi}_1||_0^2+\varepsilon^4||\tilde{\xi}_3||_0^2+2\varepsilon^4(\sigma_1\sigma_2\tilde{\xi},\tilde{m})_0$$

$$+2(2a-1)\varepsilon^5(\sigma_1\tilde{k},\tilde{m})_0+4a\varepsilon^5(\sigma_1\tilde{h},\tilde{m})_0+4a\varepsilon^5(\sigma_2\tilde{k},\tilde{\xi})_0$$

$$+2(2a-1)\varepsilon^5(\sigma_2\tilde{h},\tilde{\xi})_0+8a^2\varepsilon^6(\tilde{h},\tilde{k})_0$$

$$=(-2\lambda^2a+1)\varepsilon^4(||\tilde{m}_1||_0^2+||\tilde{m}_2||_0^2+||\tilde{\xi}_1||_0^2+||\tilde{\xi}_3||_0^2)$$

$$-2\lambda^2a\varepsilon^4(||\tilde{m}_3||_0^2+||\tilde{\xi}_2||_0^2)$$

$$+2a(2a-1)\varepsilon^6(||\tilde{k}||_0^2+||\tilde{h}||_0^2)+J_1+J_2+J_3+J_4+J_5+J_6.$$

Now, taking a positive constant ω and by using the Cauchy inequality, we have

$$J_1=2\varepsilon^4(\tilde{\xi}_3,\tilde{m}_2)_0\le\varepsilon^4(||\tilde{\xi}_3||_0^2+||\tilde{m}_2||_0^2);$$

$$J_2=(4a-2)\varepsilon^5[(\tilde{k}_2,\tilde{m}_1)_0+(\tilde{k}_1,\tilde{m}_2)_0]\le(1-2a)\Bigg\{\frac{\varepsilon^6}{\omega}(||\tilde{k}_2||_0^2+||\tilde{k}_1||_0^2)$$

$$+\varepsilon^4\omega(||\tilde{m}_1||_0^2+||\tilde{m}_2||_0^2)\Bigg\};$$

$$J_3=4a\varepsilon^5[(\tilde{h}_2,\tilde{m}_1)_0+(\tilde{h}_1,\tilde{m}_2)_0]\le 2a\Bigg\{\frac{\varepsilon^6}{\omega}||\tilde{h}_2||_0^2+\varepsilon^4\omega||\tilde{m}_1||_0^2+\frac{\varepsilon^6}{\omega}||\tilde{h}_1||_0^2+\varepsilon^4\omega||\tilde{m}_2||_0^2\Bigg\};$$

$$J_4=4a\varepsilon^5[(\tilde{k}_3,\tilde{\xi}_1)_0+(\tilde{k}_1,\tilde{\xi}_3)_0]\le 2a\Bigg\{\frac{\varepsilon^6}{\omega}||\tilde{k}_3||_0^2+\varepsilon^4\omega||\tilde{\xi}_1||_0^2+\frac{\varepsilon^6}{\omega}||\tilde{k}_1||_0^2+\varepsilon^4\omega||\tilde{\xi}_3||_0^2\Bigg\};$$

$$J_5=2(2a-1)\varepsilon^5[(\tilde{h}_3,\tilde{\xi}_1)_0+(\tilde{h}_1,\tilde{\xi}_3)_0]\le(1-2a)\Bigg\{\frac{\varepsilon^6}{\omega}||\tilde{h}_3||_0^2+\varepsilon^4\omega||\tilde{\xi}_1||_0^2$$

$$+\frac{\varepsilon^6}{\omega}||\tilde{h}_1||_0^2+\varepsilon^4\omega||\tilde{\xi}_3||_0^2\Bigg\};$$

$$J_6=8a^2\varepsilon^6(\tilde{h},\tilde{k})_0\le 4a^2\varepsilon^6\{||\tilde{h}||_0^2+||\tilde{k}||_0^2\}.$$

Putting them all together, we have

$$\tau(-L\Sigma\tilde{W},\tilde{W})_0\le\varepsilon^4||\tilde{m}_1||_0^2[-2\lambda^2a+1+\omega]+\varepsilon^4||\tilde{m}_2||_0^2[-2\lambda^2a+2+\omega]-2\lambda^2a\varepsilon^4||\tilde{m}_3||_0^2$$

$$+\varepsilon^4||\tilde{\xi}_1||_0^2[-2\lambda^2a+1+\omega]-2\lambda^2a\varepsilon^4||\tilde{\xi}_2||_0^2+\varepsilon^4||\tilde{\xi}_3||_0^2[-2\lambda^2a+2+\omega]+\varepsilon^6||\tilde{k}_1||_0^2\left[2a(4a-1)+\frac{1}{\omega}\right]$$

$$+\varepsilon^6||\tilde{k}_2||_0^2\left[2a(4a-1)+\frac{(1-2a)}{\omega}\right]+\varepsilon^6||\tilde{k}_3||_0^2\left[2a(4a-1)+\frac{2a}{\omega}\right]+\varepsilon^6||\tilde{h}_1||_0^2\left[2a(4a-1)+\frac{1}{\omega}\right]$$

$$+\varepsilon^6||\tilde{h}_2||_0^2\left[2a(4a-1)+\frac{2a}{\omega}\right]+\varepsilon^6||\tilde{h}_3||_0^2\left[2a(4a-1)+\frac{(1-2a)}{\omega}\right]. \tag{10.2.34}$$

This way, we obtain the following Lemma.

Lemma 10.2.2. *If Assumption 10.0.1 is satisfied and λ is big enough, then the symmetrized linear part of the source term $-L\Sigma$ given by (10.2.29) is negative definite.*

Proof. We need ω and λ satisfying:

$$\begin{cases} \omega > \frac{1}{2a(1-4a)}; \\ \lambda > \sqrt{\frac{2+\omega}{2a}}. \end{cases} \tag{10.2.35}$$

Recalling (10.2.33), we assume

$$\lambda > \max\left\{\sqrt{\frac{\delta}{a}}, \sqrt{\frac{4a(1-4a)+1}{4a^2(1-4a)}}\right\}. \tag{10.2.36}$$

Then, we take $\omega > \frac{1}{2a(1-4a)}$, which ends the proof. □

10.3 Energy estimates

Here we provide ε-weighted energy estimates for the solution W^ε to (10.1.26). Let us introduce T^ε the maximal time of existence of the unique solution $\tilde{W}^\varepsilon$ for fixed ε to system (10.2.31), see [52]. In the following, we consider the time interval $[0, T]$, with $T \in [0, T^\varepsilon)$. Our setting is represented by the Sobolev spaces $H^s(\mathbb{T}^2)$, with $s > 3$.

10.3.1 Zero order estimate

We assume the following condition.

Assumption 10.3.1. *Let λ satisfies (10.2.36) and*

$$\lambda > \sqrt{\frac{5+\frac{1}{a(1-4a)}}{4a}}. \tag{10.3.37}$$

Lemma 10.3.1. *If Assumptions 10.0.1 and 10.3.1 are satisfied, then the following zero order energy estimate holds:*

$$||\tilde{w}(T)||_0^2+\varepsilon^6(||\tilde{m}(T)||_0^2+||\tilde{\xi}(T)||_0^2)+\varepsilon^8(||\tilde{k}(T)||_0^2+||\tilde{h}(T)||_0^2)$$

$$+\int_0^T \varepsilon^4(||\tilde{m}(t)||_0^2+||\tilde{\xi}(t)||_0^2)+\varepsilon^6(||\tilde{k}(t)||_0^2+||\tilde{h}(t)||_0^2)\,dt \le c\varepsilon^2(||\boldsymbol{u}_0||_0^2+||\nabla \boldsymbol{u}_0||_0^2)$$

$$+c(||\boldsymbol{u}||_{L^\infty([0,T]\times\mathbb{T}^2)})\int_0^T ||\tilde{w}(t)||_0^2+\varepsilon^6(||\tilde{m}(t)||_0^2+||\tilde{\xi}(t)||_0^2)+\varepsilon^8(||\tilde{k}(t)||_0^2+||\tilde{h}(t)||_0^2)\ dt. \tag{10.3.38}$$

Proof. We consider the symmetrized compact system (10.2.31) and we multiply $\tilde{W}$ through the L^2-scalar product. Thus, we have:

$$\frac{1}{2}\frac{d}{dt}(\Sigma\tilde{W},\tilde{W})_0+(L\Sigma\tilde{W},\tilde{W})_0=(N((\Sigma\tilde{W})_1+\bar{w}),\tilde{W})_0.$$

Integrating in time, we get:

$$\frac{1}{2}(\Sigma\tilde{W}(T),\tilde{W}(T))_0+\int_0^T(L\Sigma\tilde{W}(t),\tilde{W}(t))_0\ dt\leq\frac{1}{2}(\Sigma\tilde{W}(0),\tilde{W}(0))_0$$

$$+\int_0^T|(N((\Sigma\tilde{W}(t))_1+\bar{w}),\tilde{W}(t))_0|\ dt. \tag{10.3.39}$$

Consider (10.2.32) and let us introduce the following positive constants:

$$\Gamma_\Sigma:=1-4a\mu-\tfrac{2}{\delta},\quad \Delta_\Sigma:=2(\lambda^2a-\delta)\quad \Theta_\Sigma:=2a(1-\tfrac{1}{\mu})-\tfrac{1}{\delta}. \tag{10.3.40}$$

Similarly, from (10.2.34), we define:

$$\Delta_{L\Sigma}:=2(\lambda^2a-1)-\omega,\quad \Theta_{L\Sigma}:=2a(1-4a)-\tfrac{1}{\omega}. \tag{10.3.41}$$

Thus, from (10.3.39), we get:

$$\Gamma_\Sigma||\tilde{w}(T)||_0^2+\varepsilon^6\Delta_\Sigma(||\tilde{m}(T)||_0^2+||\tilde{\xi}(T)||_0^2)+\varepsilon^8\Theta_\Sigma(||\tilde{k}(T)||_0^2+||\tilde{h}(T)||_0^2)$$

$$+\frac{2}{\tau}\int_0^T \varepsilon^4\Delta_{L\Sigma}(||\tilde{m}(t)||_0^2+||\tilde{\xi}(t)||_0^2)+\varepsilon^6\Theta_{L\Sigma}(||\tilde{k}(t)||_0^2+||\tilde{h}(t)||_0^2)\ dt$$

$$\leq(\Sigma\tilde{W}_0,\tilde{W}_0)_0+2\int_0^T|(N((\Sigma\tilde{W}(t))_1+\bar{w}),\tilde{W}(t))_0|\ dt. \tag{10.3.42}$$

Notice that, from (10.2.30),

$$(\Sigma\tilde{W}_0,\tilde{W}_0)_0=(\Sigma\Sigma^{-1}W_0,\Sigma^{-1}W_0)_0=(\Sigma^{-1}W_0,W_0)_0,$$

where $W_0=W(0,x)=(w(0,x),\varepsilon^2m(0,x),\varepsilon^2\xi(0,x),\varepsilon^2k(0,x),\varepsilon^2h(0,x))$, and, from (10.1.16), (10.1.18) and the initial conditions (10.0.4),

$$\begin{aligned}
w(0,x)&=w_0-\bar{w}=(0,\varepsilon\bar{\rho}u_{01},\varepsilon\bar{\rho}u_{02});\\
m(0,x)&=\tfrac{\lambda}{\varepsilon}(f_{10}-f_{30})=\tfrac{A_1(w_0)}{\varepsilon}-2a\lambda^2\tau\partial_xw_0\\
&=(\bar{\rho}u_{01},\varepsilon\bar{\rho}{u_{01}}^2-2a\varepsilon\bar{\rho}\partial_xu_{01},\varepsilon\bar{\rho}u_{01}u_{02}-2a\varepsilon\bar{\rho}\partial_xu_{02});\\
\xi(0,x)&=\tfrac{\lambda}{\varepsilon}(f_{20}-f_{40})=\tfrac{A_2(w_0)}{\varepsilon}-2a\lambda^2\tau\partial_yw_0\\
&=(\bar{\rho}u_{02},\varepsilon\bar{\rho}u_{01}u_{02}-2a\varepsilon\bar{\rho}\partial_yu_{01},\varepsilon\bar{\rho}{u_{02}}^2-2a\varepsilon\bar{\rho}\partial_yu_{02});\\
k(0,x)&=f_{10}+f_{30}-2a\bar{w}=2aw_0-2a\bar{w}=2a(0,\varepsilon\bar{\rho}u_{01},\varepsilon\bar{\rho}u_{02});\\
h(0,x)&=f_{20}+f_{40}-2a\bar{w}=2aw_0-2a\bar{w}=2a(0,\varepsilon\bar{\rho}u_{01},\varepsilon\bar{\rho}u_{02}).
\end{aligned}$$

Besides, the explicit expression of the constant symmetric matrix Σ^{-1} is given by

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{1-4a}Id & 0 & 0 & \frac{-1}{\varepsilon^2(1-4a)}Id & \frac{-1}{\varepsilon^2(1-4a)}Id \\ 0 & H_1 & 0 & \frac{1}{\varepsilon^3(1-4\lambda^2a^2)}\sigma_1 & 0 \\ 0 & 0 & H_2 & 0 & \frac{1}{\varepsilon^3(1-4\lambda^2a^2)}\sigma_2 \\ \frac{-1}{\varepsilon^2(1-4a)}Id & \frac{1}{\varepsilon^3(1-4\lambda^2a^2)}\sigma_1 & 0 & H_3 & \frac{1}{\varepsilon^4(1-4a)}Id \\ \frac{-1}{\varepsilon^2(1-4a)}Id & 0 & \frac{1}{\varepsilon^3(1-4\lambda^2a^2)}\sigma_2 & \frac{1}{\varepsilon^4(1-4a)}Id & H_4 \end{pmatrix}, \tag{10.3.43}$$

where

$$H_1 = \begin{pmatrix} \frac{2a}{\varepsilon^2(4\lambda^2a^2-1)} & 0 & 0 \\ 0 & \frac{2a}{\varepsilon^2(4\lambda^2a^2-1)} & 0 \\ 0 & 0 & \frac{1}{2\lambda^2a\varepsilon^2} \end{pmatrix}; \quad H_2 = \begin{pmatrix} \frac{2a}{\varepsilon^2(4\lambda^2a^2-1)} & 0 & 0 \\ 0 & \frac{1}{2\lambda^2a\varepsilon^2} & 0 \\ 0 & 0 & \frac{2a}{\varepsilon^2(4\lambda^2a^2-1)} \end{pmatrix};$$

$$H_3 = \begin{pmatrix} \frac{4\lambda^2a^2-2\lambda^2a+1}{\varepsilon^4(4a-1)(4\lambda^2a^2-1)} & 0 & 0 \\ 0 & \frac{4\lambda^2a^2-2\lambda^2a+1}{\varepsilon^4(4a-1)(4\lambda^2a^2-1)} & 0 \\ 0 & 0 & \frac{2a-1}{2a\varepsilon^4(4a-1)} \end{pmatrix};$$

$$H_4 = \begin{pmatrix} \frac{4\lambda^2a^2-2\lambda^2a+1}{\varepsilon^4(4a-1)(4\lambda^2a^2-1)} & 0 & 0 \\ 0 & \frac{2a-1}{2a\varepsilon^4(4a-1)} & 0 \\ 0 & 0 & \frac{4\lambda^2a^2-2\lambda^2a+1}{\varepsilon^4(4a-1)(4\lambda^2a^2-1)} \end{pmatrix}.$$

It is easy to check that

$$\begin{aligned}(\Sigma^{-1}W_0, W_0)_0 &= \bar\rho^2\varepsilon^2\|\mathbf{u}_0\|_0^2 + \frac{2a\bar\rho^2\varepsilon^4}{4\lambda^2a^2-1}(\|u_{01}^2 - 2a\lambda^2\partial_x u_{01}\|_0^2 + \|u_{02}^2 - 2a\lambda^2\partial_y u_{02}\|_0^2) \\ &\quad + \frac{\bar\rho^2\varepsilon^4}{2\lambda^2 a}(\|u_{01}u_{02} - 2a\lambda^2\partial_x u_{02}\|_0^2 + \|u_{01}u_{02} - 2a\lambda^2\partial_y u_{01}\|_0^2) \\ &\quad \le c\varepsilon^2(\|\mathbf{u}_0\|_0^2 + \|\nabla\mathbf{u}_0\|_0^2),\end{aligned}$$

and so, from (10.3.42) we get the following inequality:

$$\begin{aligned}&\Gamma_\Sigma||\tilde w(T)||_0^2 + \varepsilon^6\Delta_\Sigma(||\tilde m(T)||_0^2 + ||\tilde\xi(T)||_0^2) + \varepsilon^8\Theta_\Sigma(||\tilde k(T)||_0^2 + ||\tilde h(T)||_0^2) \\ &+\frac{2}{\tau}\int_0^T \varepsilon^4\Delta_{L\Sigma}(||\tilde m(t)||_0^2 + ||\tilde\xi(t)||_0^2) + \varepsilon^6\Theta_{L\Sigma}(||\tilde k(t)||_0^2 + ||\tilde h(t)||_0^2)\ dt \\ &\le c\varepsilon^2(\|\mathbf{u}_0\|_0^2 + \|\nabla\mathbf{u}_0\|_0^2) + 2\int_0^T |(N((\Sigma\tilde W(t))_1 + \bar w), \tilde W(t))_0|\ dt.\end{aligned} \tag{10.3.44}$$

It remains to deal with the last term of (10.3.44). Recall that $w = (\rho - \bar\rho, \varepsilon\rho u_1, \varepsilon\rho u_2)$.

From (10.1.25),

$$N((\Sigma\tilde{W})_1+\bar{w})=N(w+\bar{w})=\frac{1}{\tau}\begin{pmatrix}0\\ \begin{pmatrix}0\\u_1w_2\\u_1w_3\end{pmatrix}\\ \begin{pmatrix}0\\u_2w_2\\u_2w_3\end{pmatrix}\\0\\0\end{pmatrix}. \tag{10.3.45}$$

Thus,

$$(N(w+\bar{w}),\tilde{W})_0=\frac{1}{\tau}\{(u_1w_2,\varepsilon^2\tilde{m}_2)_0+(u_1w_3,\varepsilon^2\tilde{m}_3)_0+(u_2w_2,\varepsilon^2\tilde{\xi}_2)_0+(u_2w_3,\varepsilon^2\tilde{\xi}_3)_0\}$$

$$\leq\frac{1}{2\tau}\{||u_1w_2||_0^2+\varepsilon^4||\tilde{m}_2||_0^2+||u_1w_3||_0^2+\varepsilon^4||\tilde{m}_3||_0^2+||u_2w_2||_0^2+\varepsilon^4||\tilde{\xi}_2||_0^2+||u_2w_3||_0^2+\varepsilon^4||\tilde{\xi}_3||_0^2\}$$

$$\leq c(||\mathbf{u}||_\infty)||w||_0^2+\frac{\varepsilon^4}{2\tau}(||\tilde{m}||_0^2+||\tilde{\xi}||_0^2).$$

By definition (10.2.30), explicitly we have:

$$w=(\Sigma\tilde{W}^\varepsilon)_1=\tilde{w}+\varepsilon^3\sigma_1\tilde{m}+\varepsilon^3\sigma_2\tilde{\xi}+2a\varepsilon^4(\tilde{k}+\tilde{h}), \tag{10.3.46}$$

and so,

$$|(N(w+\bar{w}),\tilde{W})_0|\leq c(||\mathbf{u}||_\infty)\{||\tilde{w}||_0^2+\varepsilon^6(||\tilde{m}||_0^2+||\tilde{\xi}||_0^2)+\varepsilon^8(||\tilde{k}||_0^2+||\tilde{h}||_0^2)\}+\frac{\varepsilon^4}{2\tau}(||\tilde{m}||_0^2+||\tilde{\xi}||_0^2).$$

Putting them all together, (10.3.44) yields:

$$\Gamma_\Sigma||\tilde{w}(T)||_0^2+\varepsilon^6\Delta_\Sigma(||\tilde{m}(T)||_0^2+||\tilde{\xi}(T)||_0^2)+\varepsilon^8\Theta_\Sigma(||\tilde{k}(T)||_0^2+||\tilde{h}(T)||_0^2)$$

$$+\frac{2}{\tau}\int_0^T\varepsilon^4\Delta_{L\Sigma}(||\tilde{m}(t)||_0^2+||\tilde{\xi}(t)||_0^2)+\varepsilon^6\Theta_{L\Sigma}(||\tilde{k}(t)||_0^2+||\tilde{h}(t)||_0^2)\ dt$$

$$\leq c\varepsilon^2(\|\mathbf{u}_0\|_0^2+\|\nabla\mathbf{u}_0\|_0^2)+\int_0^T\frac{\varepsilon^4}{\tau}(||\tilde{m}(t)||_0^2+||\tilde{\xi}(t)||_0^2)\ dt$$

$$+c(||\mathbf{u}||_{L^\infty([0,T]\times\mathbb{T}^2)})\int_0^T||\tilde{w}(t)||_0^2+\varepsilon^6(||\tilde{m}(t)||_0^2+||\tilde{\xi}(t)||_0^2)+\varepsilon^8(||\tilde{k}(t)||_0^2+||\tilde{h}(t)||_0^2)\ dt. \tag{10.3.47}$$

This gives:

$$\Gamma_\Sigma||\tilde{w}(T)||_0^2 + \varepsilon^6\Delta_\Sigma(||\tilde{m}(T)||_0^2 + ||\tilde{\xi}(T)||_0^2) + \varepsilon^8\Theta_\Sigma(||\tilde{k}(T)||_0^2 + ||\tilde{h}(T)||_0^2)$$

$$+\frac{1}{\tau}\int_0^T \varepsilon^4(2\Delta_{L\Sigma} - 1)(||\tilde{m}(t)||_0^2 + ||\tilde{\xi}(t)||_0^2) + 2\varepsilon^6\Theta_{L\Sigma}(||\tilde{k}(t)||_0^2 + ||\tilde{h}(t)||_0^2)\ dt$$

$$\leq c\varepsilon^2(\|\mathbf{u}_0\|_0^2 + \|\nabla\mathbf{u}_0\|_0^2)$$

$$+c(||\mathbf{u}||_{L^\infty([0,T]\times\mathbb{T}^2)})\int_0^T ||\tilde{w}(t)||_0^2 + \varepsilon^6(||\tilde{m}(t)||_0^2 + ||\tilde{\xi}(t)||_0^2) + \varepsilon^8(||\tilde{k}(t)||_0^2 + ||\tilde{h}(t)||_0^2)\ dt, \tag{10.3.48}$$

where, by definition (10.3.41), $2\Delta_{L\Sigma} - 1 = 4\lambda^2 a - 4 - 2\omega$ is positive thanks to condition (10.3.37). This gives estimate (10.3.38). □

10.3.2 Higher order estimates

Lemma 10.3.2. *If Assumptions 10.0.1 and 10.3.1 are satisfied, then the following H^s energy estimate holds:*

$$||\tilde{w}(T)||_s^2 + \varepsilon^6(||\tilde{m}(T)||_s^2 + ||\tilde{\xi}(T)||_s^2) + \varepsilon^8(||\tilde{k}(T)||_s^2 + ||\tilde{h}(T)||_s^2)$$

$$+\int_0^T \varepsilon^4(||\tilde{m}(t)||_s^2 + ||\tilde{\xi}(t)||_s^2) + \varepsilon^6(||\tilde{k}(t)||_s^2 + ||\tilde{h}(t)||_s^2)\ dt \leq c\varepsilon^2(||\boldsymbol{u}_0||_s^2 + ||\nabla\boldsymbol{u}_0||_s^2)$$

$$+c(||\boldsymbol{u}||_{L_t^\infty H_x^s})\int_0^T ||\tilde{w}(t)||_s^2 + \varepsilon^6(||\tilde{m}(t)||_s^2 + ||\tilde{\xi}(t)||_s^2) + \varepsilon^8(||\tilde{k}(t)||_s^2 + ||\tilde{h}(t)||_s^2)\ dt,$$

where, hereafter,

$$L_t^\infty H_x^s := L^\infty([0,T], H^s(\mathbb{T}^2)), \quad \textit{for } s \in \mathbb{R}.$$

Proof. We take the $|\alpha|$-derivative, $0 < |\alpha| \leq s$, of the semilinear system given by (10.1.26). As done previously, we get:

$$\Gamma_\Sigma||D^\alpha\tilde{w}(T)||_0^2 + \varepsilon^6\Delta_\Sigma(||D^\alpha\tilde{m}(T)||_0^2 + ||D^\alpha\tilde{\xi}(T)||_0^2) + \varepsilon^8\Theta_\Sigma(||D^\alpha\tilde{k}(T)||_0^2 + ||D^\alpha\tilde{h}(T)||_0^2)$$

$$+\frac{2}{\tau}\int_0^T \varepsilon^4\Delta_{L\Sigma}(||D^\alpha\tilde{m}(t)||_0^2 + ||D^\alpha\tilde{\xi}(t)||_0^2) + \varepsilon^6\Theta_{L\Sigma}(||D^\alpha\tilde{k}(t)||_0^2 + ||D^\alpha\tilde{h}(t)||_0^2)\ dt$$

$$\leq c\varepsilon^2(||D^\alpha\mathbf{u}_0||_0^2 + ||D^{\alpha+1}\mathbf{u}_0||_0^2) + 2\int_0^T |D^\alpha(N((\Sigma\tilde{W}(t))_1 + \bar{w}), D^\alpha\tilde{W}(t))_0|\ dt. \tag{10.3.49}$$

Now, from (10.3.45),

$$\begin{aligned}|(D^\alpha N(w+\bar{w}), D^\alpha \tilde{W})_0| &\le \frac{1}{\tau}\{|(D^\alpha(u_1w_2), D^\alpha\varepsilon^2\tilde{m}_2)_0| + |D^\alpha(u_1w_3), D^\alpha\varepsilon^2\tilde{m}_3)_0| \\ &+ |(D^\alpha(u_2w_2), D^\alpha\varepsilon^2\tilde{\xi}_2)_0| + |D^\alpha(u_2w_3), D^\alpha\varepsilon^2\tilde{\xi}_3)_0|\} \\ &\le \frac{1}{2\tau}\{||(D^\alpha(u_1w_2)||_0^2 + ||D^\alpha(u_1w_3)||_0^2 \\ &+ ||D^\alpha(u_2w_2)||_0^2 + ||D^\alpha(u_2w_3)||_0^2 \\ &+ \varepsilon^4(||D^\alpha\tilde{m}||_0^2 + ||D^\alpha\tilde{\xi}||_0^2)\} \\ &\le c(||\mathbf{u}||_s)||w||_s^2 + \frac{\varepsilon^4}{2\tau}(||\tilde{m}||_s^2 + ||\tilde{\xi}||_s^2).\end{aligned}$$

By using (10.3.46) we have:

$$\begin{aligned}|(D^\alpha N(w+\bar{w}), D^\alpha \tilde{W})_0| \quad &\le c(||\mathbf{u}||_s)(||\tilde{w}||_s^2 + \varepsilon^6(||\tilde{m}||_s^2 + ||\tilde{\xi}||_s^2) + \varepsilon^8(||\tilde{k}||_s^2 + ||\tilde{h}||_s^2)) \\ &+\frac{\varepsilon^4}{2\tau}(||\tilde{m}||_s^2 + ||\tilde{\xi}||_s^2).\end{aligned}$$

Thus, from (10.3.49),

$$\begin{aligned}&\Gamma_\Sigma||\tilde{w}(T)||_s^2 + \varepsilon^6\Delta_\Sigma(||\tilde{m}(T)||_s^2 + ||\tilde{\xi}(T)||_s^2) + \varepsilon^8\Theta_\Sigma(||\tilde{k}(T)||_s^2 + ||\tilde{h}(T)||_s^2) \\ &\quad + \frac{2}{\tau}\int_0^T \varepsilon^4(2\Delta_{L\Sigma}-1)(||\tilde{m}(t)||_s^2 + ||\tilde{\xi}(t)||_s^2) + 2\varepsilon^6\Theta_{L\Sigma}(||\tilde{k}(t)||_s^2 + ||\tilde{h}(t)||_s^2)\, dt \\ &\quad \le c\varepsilon^2(||\mathbf{u}_0||_s^2 + ||\nabla\mathbf{u}_0||_s^2) \\ &\quad + c(||\mathbf{u}||_{L_t^\infty H_x^s})\int_0^T ||\tilde{w}(t)||_s^2 + \varepsilon^6(||\tilde{m}(t)||_s^2 + ||\tilde{\xi}(t)||_s^2) + \varepsilon^8(||\tilde{k}(t)||_s^2 + ||\tilde{h}(t)||_s^2)\, dt.\end{aligned}$$

□

Remark 10.3.1. *In the case $s > 3$ is not an integer, by using the pseudodifferential operator $\lambda^s(\xi) = (1+|\xi|^2)^{s/2}$ in the Fourier space, we get the same estimates in a standard way.*

Now, we need a bound in the H^s-norm for the original variable $w = (\rho-\bar{\rho}, \varepsilon\rho\mathbf{u})$, which is the first component of the unknown vector W in (10.1.21). By using estimate (10.3.49) and definition (10.2.30), we can prove the following proposition.

Proposition 10.3.1. *If Assumptions 10.0.1 and 10.3.1 are satisfied, then the following estimate holds:*

$$||w(t)||_s^2+\varepsilon^6(||\tilde{m}(t)||_s^2+||\tilde{\xi}(t)||_s^2)+\varepsilon^8(||\tilde{k}(t)||_s^2+||\tilde{h}(t)||_s^2) \le c\varepsilon^2(||\boldsymbol{u}_0||_s^2+||\nabla\boldsymbol{u}_0||_s^2)e^{c(||\boldsymbol{u}||_{L_t^\infty H_x^s})t},$$

and

$$\frac{||\rho(t)-\bar{\rho}||_s^2}{\varepsilon^2} + ||\rho\boldsymbol{u}(t)||_s^2 \le c(||\boldsymbol{u}_0||_s^2 + ||\nabla\boldsymbol{u}_0||_s^2)e^{c(||\boldsymbol{u}||_{L_t^\infty H_x^s})t}, \tag{10.3.50}$$

for $t \in [0, T^\varepsilon)$.

Proof. The Gronwall inequality applied to (10.3.49) yields:

$$\Gamma_\Sigma||\tilde{w}(t)||_s^2 + \varepsilon^6\Delta_\Sigma(||\tilde{m}(t)||_s^2 + ||\tilde{\xi}(t)||_s^2) + \varepsilon^8\Theta_\Sigma(||\tilde{k}(t)||_s^2 + ||\tilde{h}(t)||_s^2)$$
$$\leq c\varepsilon^2(||\mathbf{u}_0||_s^2 + ||\nabla\mathbf{u}_0||_s^2)e^{c(||\mathbf{u}||_{L_t^\infty H_x^s})t}. \tag{10.3.51}$$

Recalling (10.3.46),

$$\tilde{w} = w - \varepsilon^3\sigma_1\tilde{m} - \varepsilon^3\sigma_2\tilde{\xi} - 2a\varepsilon^4(\tilde{k}+\tilde{h}).$$

Thus,

$$\begin{aligned}
||\tilde{w}||_s^2 &= ||w||_s^2 + \varepsilon^6(||\tilde{m}_1||_s^2 + ||\tilde{m}_2||_s^2 + ||\tilde{\xi}_1||_s^2 + ||\tilde{\xi}_3||_s^2) + 4a^2\varepsilon^8||\tilde{k}+\tilde{h}||_s^2 \\
&\quad - 2\varepsilon^3(w,\sigma_1\tilde{m})_s - 2\varepsilon^3(w,\sigma_2\tilde{\xi})_s - 4a\varepsilon^4(w,\tilde{k}+\tilde{h})_s + 2\varepsilon^6(\sigma_1\tilde{m},\sigma_2\tilde{\xi})_s \\
&\quad + 4a\varepsilon^7(\sigma_1\tilde{m},\tilde{k}+\tilde{h})_s + 4a\varepsilon^7(\sigma_2\tilde{\xi},\tilde{k}+\tilde{h})_s \qquad (10.3.52) \\
&= ||w||_s^2 + \varepsilon^6(||\tilde{m}_1||_s^2 + ||\tilde{m}_2||_s^2 + ||\tilde{\xi}_1||_s^2 + ||\tilde{\xi}_3||_s^2) + 4a^2\varepsilon^8||\tilde{k}+\tilde{h}||_s^2 \\
&\quad + Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6.
\end{aligned}$$

Now, taking two positive constants η, ζ and using the Cauchy inequality, from (10.3.52) we have:

$$\begin{aligned}
&Y_1 = -2\varepsilon^3(w,\sigma_1\tilde{m})_s \geq -\frac{||w_1||_s^2}{\eta} - \varepsilon^6\eta||\tilde{m}_2||_s^2 - \frac{||w_2||_s^2}{\eta} - \varepsilon^6\eta||\tilde{m}_1||_s^2; \\
&Y_2 = -2\varepsilon^3(w,\sigma_2\tilde{\xi})_s \geq -\frac{||w_1||_s^2}{\eta} - \varepsilon^6\eta||\tilde{\xi}_3||_s^2 - \frac{||w_3||_s^2}{\eta} - \varepsilon^6\eta||\tilde{\xi}_1||_s^2; \\
&Y_3 = -4a\varepsilon^4(w,\tilde{k}+\tilde{h})_s \geq \frac{-2a}{\zeta}||w||_s^2 - 2a\zeta\varepsilon^8||\tilde{k}+\tilde{h}||_s^2; \\
&Y_4 = 2\varepsilon^6(\tilde{m}_2,\tilde{\xi}_3)_s \geq -\varepsilon^6(||\tilde{m}_2^\varepsilon||_s^2 + ||\tilde{\xi}_3^\varepsilon||_s^2); \\
&Y_5 = 4a\varepsilon^7[(\tilde{m}_2,\tilde{k}_1+\tilde{h}_1)_s + (\tilde{m}_1,\tilde{k}_2+\tilde{h}_2)_s] \geq -2a\varepsilon^6\eta||\tilde{m}_2||_s^2 - \frac{2a\varepsilon^8}{\eta}||\tilde{k}_1+\tilde{h}_1||_s^2 \\
&- 2a\varepsilon^6\eta||\tilde{m}_1||_s^2 - \frac{2a\varepsilon^8}{\eta}||\tilde{k}_2+\tilde{h}_2||_s^2; \\
&Y_6 = 4a\varepsilon^7[(\tilde{\xi}_3,\tilde{k}_1+\tilde{h}_1)_s + (\tilde{\xi}_1,\tilde{k}_3+\tilde{h}_3)_s] \geq -2a\varepsilon^6\eta||\tilde{\xi}_3||_s^2 - \frac{2a\varepsilon^8}{\eta}||\tilde{k}_1+\tilde{h}_1||_s^2 \\
&- 2a\varepsilon^6\eta||\tilde{\xi}_1||_s^2 - \frac{2a\varepsilon^8}{\eta}||\tilde{k}_3+\tilde{h}_3||_s^2.
\end{aligned}$$

The left hand side of (10.3.51) and the previous calculations yield the following inequality:

$$\Gamma_\Sigma||\tilde{w}||_s^2+\varepsilon^6\Delta_\Sigma(||\tilde{m}||_s^2+||\tilde{\xi}||_s^2)+\varepsilon^8\Theta_\Sigma(||\tilde{k}||_s^2+||\tilde{h}||_s^2)\geq\Gamma_\Sigma\left[1-\frac{2}{\eta}-\frac{2a}{\zeta}\right]||w_1||_s^2$$

$$+\Gamma_\Sigma\left[1-\frac{1}{\eta}-\frac{2a}{\zeta}\right](||w_2||_s^2+||w_3||_s^2)+\varepsilon^8\theta_\Sigma(||\tilde{k}||_s^2+||\tilde{h}||_s^2)$$

$$+\varepsilon^6(||\tilde{m}_1||_s^2+||\tilde{\xi}_1||_s^2)[\Delta_\Sigma+\Gamma_\Sigma(1-\eta-2a\eta)]+\varepsilon^6(||\tilde{m}_2||_s^2+||\tilde{\xi}_3||_s^2)[\Delta_\Sigma+\Gamma_\Sigma(-\eta-2a\eta)]$$

$$+\varepsilon^6(||\tilde{m}_3||_s^2+||\tilde{\xi}_2||_s^2)\Delta_\Sigma+\varepsilon^8||\tilde{k}_1+\tilde{h}_1||_s^2\Gamma_\Sigma\left[4a^2-2a\zeta-\frac{4a}{\eta}\right]$$

$$+\varepsilon^8\Gamma_\Sigma(||\tilde{k}_2+\tilde{h}_2||_s^2+||\tilde{k}_3+\tilde{h}_3||_s^2)\left[4a^2-2a\zeta-\frac{2a}{\eta}\right]. \tag{10.3.53}$$

Fixed $\beta>1$, the Cauchy inequality yields $||\tilde{k}+\tilde{h}||_s^2\geq(1-\frac{1}{\beta})||\tilde{k}||_s^2+(1-\beta)||\tilde{h}||_s^2$, then the last term of (10.3.53) is bounded from below by the following expression:

$$\varepsilon^8(||\tilde{k}_1||_s^2+||\tilde{h}_1||_s^2)\left[\Theta_\Sigma+(1-1/\beta)\Gamma_\Sigma\left[4a^2-2a\zeta-\frac{4a}{\eta}\right]\right]$$

$$+\varepsilon^8(||\tilde{k}_2||_s^2+||\tilde{k}_3||_s^2+||\tilde{h}_2||_s^2+||\tilde{h}_3||_s^2)\left[\Theta_\Sigma+(1-\beta)\Gamma_\Sigma\left[4a^2-2a\zeta-\frac{2a}{\eta}\right]\right]. \tag{10.3.54}$$

Thus, in order to get estimate (10.3.50), we require:

$$\begin{cases}1-\frac{2}{\eta}-\frac{4a}{\zeta}>0;\\ \Delta_\Sigma-\eta\Gamma_\Sigma(1+2a)>0;\\ \Theta_\Sigma+(1-1/\beta)\Gamma_\Sigma\left[4a^2-2a\zeta-\frac{4a}{\eta}\right]>0;\\ \Theta_\Sigma+(1-\beta)\Gamma_\Sigma\left[4a^2-2a\zeta-\frac{4a}{\eta}\right]>0.\end{cases} \tag{10.3.55}$$

Recalling definition (10.3.40), $\Delta_\Sigma=2(\lambda^2a-\delta)$, and so the second inequality is satisfied for λ big enough. Precisely, we take λ as in Assumption 10.3.1 and

$$\lambda>\sqrt{\frac{\delta}{a}+\frac{\eta\Gamma_\Sigma(1+2a)}{2a}}.$$

Moreover, the first condition of (10.3.55) is verified if

$$\boxed{\eta>\frac{2\zeta}{\zeta-4a},\quad\zeta>4a}. \tag{10.3.56}$$

Since Θ_Σ and Γ_Σ are positive, taking $1-\beta<0$, i.e. $\beta>1$, the last inequality is verified if

$$2a\zeta + \frac{4a}{\eta} - 4a^2 > 0.$$

From (10.3.56),

$$2a\zeta + \frac{4a}{\eta} - 4a^2 > 8a^2 + \frac{4a}{\eta} - 4a^2 = 4a^2 + \frac{4a}{\eta} > 0,$$

then the last inequality in (10.3.55) holds under (10.3.56). Now, the third condition in (10.3.55) is satisfied if

$$\zeta < \frac{\Theta_\Sigma}{2a\Gamma_\Sigma(1-1/\beta)} + 2(a-1/\eta).$$

Thus, if $\boxed{\eta > \frac{1}{a}}$, we can take

$$4a < \zeta < \frac{\Theta_\Sigma}{2a\Gamma_\Sigma(1-1/\beta)}, \tag{10.3.57}$$

with η and ζ satisfying (10.3.56). In particular, we show that there exists $\beta>1$ such that:

$$4a < \frac{\Theta_\Sigma}{2a\Gamma_\Sigma(1-1/\beta)}, \quad \text{i.e.} \quad 8a^2\Gamma_\Sigma(1-1/\beta) < \Theta_\Sigma. \tag{10.3.58}$$

From (10.3.40), $\Gamma_\Sigma = 1-4a\mu-\frac{2}{\delta}$ and, from Lemma 10.2.1, $0<\Gamma_\Sigma<1$. Thus, in order to verify (10.3.58), we require:

$$8a^2(1-1/\beta) < \Theta_\Sigma,$$

which is automatically verified if $8a^2 \leq \Theta_\Sigma$. Otherwise, it yields $\beta < \dfrac{8a^2}{8a^2-\Theta_\Sigma}$.
Finally, since $\beta>1$, we need

$$1 < \frac{8a^2}{8a^2-\Theta_\Sigma}, \quad \text{i.e.} \quad \Theta_\Sigma > 0,$$

which is already satisfied thanks to Lemma 10.2.1.
This way, from (10.3.54), (10.3.51) and (10.3.55), we get some positive constants $\Gamma^1_\Sigma, \Delta^1_\Sigma, \Theta^1_\Sigma$ such that

$$\Gamma^1_\Sigma||w(t)||_s^2 + \varepsilon^6\Delta^1_\Sigma(||\tilde{m}(t)||_s^2 + ||\tilde{\xi}(t)||_s^2) + \varepsilon^8\Theta^1_\Sigma(||\tilde{k}(t)||_s^2 + ||\tilde{h}(t)||_s^2)$$

$$\leq c\varepsilon^2(||\mathbf{u}_0||_s^2 + ||\nabla\mathbf{u}_0||_s^2)e^{c(||\mathbf{u}||_{L^\infty_t H^s_x})t}, \tag{10.3.59}$$

and, in particular,

$$||w(t)||_s^2 \leq c\varepsilon^2(||\mathbf{u}_0||_s^2 + ||\nabla\mathbf{u}_0||_s^2)e^{c(||\mathbf{u}||_{L^\infty_t H^s_x})t},$$

i.e.

$$\frac{||\rho(t)-\bar{\rho}||_s^2}{\varepsilon^2} + ||\rho\mathbf{u}(t)||_s^2 \leq c(||\mathbf{u}_0||_s^2 + ||\nabla\mathbf{u}_0||_s^2)e^{c(||\mathbf{u}||_{L^\infty_t H^s_x})t}. \tag{10.3.60}$$

□

Thus, we are able to prove that the time T^ε of existence of the solutions to the vector BGK scheme is bounded form below by a positive time $T^\star$, which is independent of ε.

Proposition 10.3.2. *There exist ε_0 and $T^\star$ fixed such that $T^\star < T^\varepsilon$ for all $\varepsilon \leq \varepsilon_0$. This also yields, for $\varepsilon \leq \varepsilon_0$, the uniform bounds:*

$$||\boldsymbol{u}(t)||_s \leq M, \quad t \in [0, T^\star], \tag{10.3.61}$$

$$||\rho(t) - \bar{\rho}||_s \leq \varepsilon M, \quad i.e. \quad ||\rho(t)||_s \leq \bar{\rho}|\mathbb{T}^2| + \varepsilon M, \quad t \in [0, T^\star], \tag{10.3.62}$$

and

$$||\rho \boldsymbol{u}(t)||_s \leq M(\bar{\rho}|\mathbb{T}^2| + \varepsilon M), \quad t \in [0, T^\star]. \tag{10.3.63}$$

Proof. Let $\mathbf{u}_0 \in H^{s+1}(\mathbb{T}^2)$ and, from (10.0.4), recall that $\rho_0 = \bar{\rho}$. Then, there exists a positive constant M_0 such that $||\mathbf{u}_0||_{s+1} \leq M_0$, and

$$||\rho_0 \mathbf{u}_0||_{s+1} = \bar{\rho}||\mathbf{u}_0||_{s+1} \leq \bar{\rho} M_0 =: \tilde{M}_0. \tag{10.3.64}$$

Let $M > \tilde{M}_0$ be any fixed constant, and

$$T_0^\varepsilon := \sup \left\{ t \in [0, T^\varepsilon] \;\middle|\; \frac{||\rho(t) - \bar{\rho}||_s^2}{\varepsilon^2} + ||\rho \mathbf{u}(t)||_s^2 \leq M^2, \quad \forall \varepsilon \leq \varepsilon_0 \right\}. \tag{10.3.65}$$

Notice that, from (10.3.65),

$$||\rho - \bar{\rho}||_\infty \leq c_S ||\rho - \bar{\rho}||_s \leq c_S M \varepsilon, \quad t \in [0, T_0^\varepsilon],$$

where c_S is the Sobolev embedding constant, i.e.

$$\bar{\rho} - c_S M \varepsilon \leq \rho \leq \bar{\rho} + c_S M \varepsilon, \quad t \in [0, T_0^\varepsilon].$$

Taking ε_0 such that $\bar{\rho} - c_S M \varepsilon_0 > \frac{\bar{\rho}}{2}$, i.e. $\bar{\rho} > 2 c_S M \varepsilon_0$, we have

$$\rho > \frac{\bar{\rho}}{2}, \quad t \in [0, T_0^\varepsilon]. \tag{10.3.66}$$

Now, since $s > 3 = \frac{d}{2} + 2$,

$$||\mathbf{u}||_s \leq ||\rho \mathbf{u}||_s ||1/\rho||_s.$$

Moreover,

$$||1/\rho||_s \leq c \left(\frac{|\mathbb{T}^2|}{\bar{\rho}} + \frac{||\rho||_s}{c(\bar{\rho})} \right) \leq c_1 + c_2 ||\rho||_s.$$

From (10.3.65),

$$||\rho||_s \leq c(|\mathbb{T}^2|\bar{\rho} + M\varepsilon),$$

so

$$||1/\rho||_s \leq c_1 + c_2 M \varepsilon,$$

and

$$||\mathbf{u}||_s \leq c M (c_1 + c_2 M \varepsilon).$$

From (10.3.60),

$$\frac{||\rho(t)-\bar{\rho}||_s^2}{\varepsilon^2} + ||\rho\mathbf{u}(t)||_s^2 \leq cM_0^2 e^{c(M(c_1+c_2M\varepsilon))t}, \quad t\in[0,T_0^\varepsilon].$$

We take $T^\star \leq T_0^\varepsilon$ such that

$$cM_0^2 e^{c(M(c_1+c_2M\varepsilon_0))T^\star} \leq M^2,$$

i.e.

$$T^\star \leq \frac{1}{c(M(c_1+c_2M\varepsilon_0))} log(M^2/(cM_0^2)) \quad \forall\varepsilon\leq\varepsilon_0. \tag{10.3.67}$$

This way,

$$||\mathbf{u}(t)||_s \leq cM(c_1+c_2M\varepsilon), \quad t\in[0,T^\star] \quad \text{and} \quad ||\rho\mathbf{u}||_s\leq M \quad \forall\varepsilon\leq\varepsilon_0. \tag{10.3.68}$$

□

10.3.3 Time derivative estimate

In order to use the compactness tools, we need a uniform bound for the time derivative of the unknown vector field.

Proposition 10.3.3. *If Assumptions 10.0.1 and 10.3.1 hold, for M_0 in (10.3.64) and M in (10.3.61), we have:*

$$\begin{aligned} &||\partial_t w||_{s-1}^2 + \varepsilon^6(||\partial_t\tilde{m}||_{s-1}^2 + ||\partial_t\tilde{\xi}||_{s-1}^2) + \varepsilon^8(||\partial_t\tilde{k}||_{s-1}^2 + ||\partial_t\tilde{h}||_{s-1}^2) \\ &\qquad \leq \varepsilon^2 c(||\boldsymbol{u}_0||_{s+1})e^{c(M)t} \leq \varepsilon^2 c(M_0,M) \quad in \quad [0,T^\star], \end{aligned} \tag{10.3.69}$$

with $T^\star$ in (10.3.67). This also yields the uniform bound:

$$\frac{||\partial_t(\rho-\bar{\rho})||_{s-1}^2}{\varepsilon^2} + ||\partial_t(\rho\boldsymbol{u})||_{s-1}^2 \leq c(||\boldsymbol{u}_0||_{s+1}) \leq M^2 \quad in \quad [0,T^\star]. \tag{10.3.70}$$

Proof. Let us take the time derivative of system (10.2.31). Defining $\tilde{V}=\partial_t\tilde{W}^\varepsilon$, from (10.1.25) we get:

$$\partial_t\Sigma\tilde{V} + \tilde{\Lambda}_1\Sigma\partial_x\tilde{V} + \tilde{\Lambda}_2\Sigma\partial_y\tilde{V} = -L\Sigma\tilde{V} + \partial_t N((\Sigma\tilde{W})_1+\bar{w}) = -L\Sigma\tilde{V} + \partial_t N(w+\bar{w}), \tag{10.3.71}$$

where

$$\partial_t N(w+\bar{w}) = \frac{1}{\tau}\left(\begin{array}{c} 0 \\ \left(\begin{array}{c} 0 \\ 2u_1\partial_t w_2 - \varepsilon u_1^2\partial_t w_1 \\ u_2\partial_t w_2 + u_1\partial_t w_3 - \varepsilon u_1u_2\partial_t w_1 \end{array}\right) \\ \left(\begin{array}{c} 0 \\ u_2\partial_t w_2 + u_1\partial_t w_3 - \varepsilon u_1u_2\partial_t w_1 \\ 2u_2\partial_t w_3 - \varepsilon u_2^2\partial_t w_1 \end{array}\right) \\ 0 \\ 0 \end{array}\right). \tag{10.3.72}$$

Taking the scalar product with $\tilde{V}$, we have:

$$\frac{1}{2}\frac{d}{dt}(\Sigma\tilde{V},\tilde{V})_0 + (L\Sigma\tilde{V},\tilde{V})_0 \leq |(\partial_t N(w+\bar{w}), V)_0|. \tag{10.3.73}$$

Here,

$$|(\partial_t N(w+\bar{w}),\tilde{V})_0| = \frac{1}{\tau}|(2u_1\partial_t w_2 - \varepsilon u_1^2\partial_t w_1, \varepsilon^2\partial_t\tilde{m}_2)_0$$

$$+(u_2\partial_t w_2 + u_1\partial_t w_3 - \varepsilon u_1 u_2\partial_t w_1, \varepsilon^2\partial_t\tilde{m}_3 + \varepsilon^2\partial_t\tilde{\xi}_2)_0$$

$$+(2u_2\partial_t w_3 - \varepsilon u_2^2\partial_t w_1, \varepsilon^2\partial_t\tilde{\xi}_3)_0| \leq c(||\mathbf{u}||_\infty)||\partial_t w||_0^2 + \frac{\varepsilon^4}{2\tau}(||\partial_t\tilde{m}||_0^2 + ||\partial_t\tilde{\xi}||_0^2).$$

Similarly to (10.3.48), we get:

$$\Gamma_\Sigma||\partial_t\tilde{w}||_0^2 + \varepsilon^6\Delta_\Sigma(||\partial_t\tilde{m}||_0^2 + |||\partial_t\tilde{\xi}||_0^2) + \varepsilon^8\Theta_\Sigma(||\partial_t\tilde{k}||_0^2 + ||\partial_t\tilde{h}||_0^2)$$

$$+\frac{1}{\tau}\int_0^T (2\Delta_{L\Sigma} - 1)\varepsilon^4(||\partial_t\tilde{m}||_0^2 + ||\partial_t\tilde{\xi}||_0^2) + 2\varepsilon^6\Theta_{L\Sigma}(||\partial_t\tilde{k}||_0^2 + ||\partial_t\tilde{h}||_0^2)\, dt$$

$$\leq c\varepsilon^2||\partial_t w|_{t=0}||_0^2$$

$$+c(||\mathbf{u}||_{L^\infty([0,T]\times\mathbb{T}^2)})\int_0^T ||\partial_t\tilde{w}||_0^2 + \varepsilon^6(||\partial_t\tilde{m}||_0^2 + ||\partial_t\tilde{\xi}||_0^2) + \varepsilon^8(||\partial_t\tilde{k}||_0^2 + ||\partial_t\tilde{h}||_0^2)\, dt.$$

Now, from the first equation given by (10.1.19),

$$\partial_t w|_{t=0} = -\partial_x m|_{t=0} - \partial_y\xi|_{t=0},$$

where, from (10.1.16), (10.0.4), and (10.0.9),

$$m|_{t=0} = \frac{A_1(w_0)}{\varepsilon} - 2a\lambda^2\tau\partial_x w_0 = \bar{\rho}\begin{pmatrix} u_{01} \\ \varepsilon u_{01}^2 - 2a\lambda^2\varepsilon\partial_x u_{01} \\ \varepsilon u_{01}u_{02} - 2a\lambda^2\varepsilon\partial_x u_{02}\end{pmatrix},$$

$$\xi|_{t=0} = \frac{A_2(w_0)}{\varepsilon} - 2a\lambda^2\tau\partial_y w_0 = \bar{\rho}\begin{pmatrix} u_{02} \\ \varepsilon u_{01}u_{02} - 2a\lambda^2\tau\varepsilon\partial_y u_{01} \\ \varepsilon u_{02}^2 - 2a\lambda^2\tau\varepsilon\partial_y u_{02}\end{pmatrix}.$$

By definition of w in (10.0.6), $\partial_t w|_{t=0} = (\partial_t\rho|_{t=0}, \varepsilon\partial_t(\rho\mathbf{u})|_{t=0})$. This implies that

$$\partial_t\rho|_{t=0} = -\bar{\rho}(\nabla\cdot\mathbf{u}_0) = 0,$$

since $\mathbf{u}_0$ is divergence free. This way,

$$\partial_t\mathbf{u}|_{t=0} = -\partial_x\begin{pmatrix} u_{01}^2 - 2a\lambda^2\tau\partial_x u_{01} \\ u_{01}u_{02} - 2a\lambda^2\tau\partial_x u_{02}\end{pmatrix} - \partial_y\begin{pmatrix} u_{01}u_{02} - 2a\lambda^2\tau\partial_y u_{01} \\ u_{02}^2 - 2a\lambda^2\tau\partial_y u_{02}\end{pmatrix}.$$

Thus,

$$\Gamma_\Sigma||\partial_t\tilde{w}||_0^2+\varepsilon^6\Delta_\Sigma(||\partial_t\tilde{m}||_0^2+|||\partial_t\tilde{\xi}||_0^2)+\varepsilon^8\Theta_\Sigma(||\partial_t\tilde{k}||_0^2+||\partial_t\tilde{h}||_0^2)$$

$$+\frac{1}{\tau}\int_0^T(2\Delta_{L\Sigma}-1)\varepsilon^4(||\partial_t\tilde{m}||_0^2+||\partial_t\tilde{\xi}||_0^2)+2\varepsilon^6\Theta_{L\Sigma}(||\partial_t\tilde{k}||_0^2+||\partial_t h||_0^2)\,dt$$

$$\leq c\varepsilon^2(||\mathbf{u}_0||_0^2+||\nabla\mathbf{u}_0||_0^2+||\nabla^2\mathbf{u}_0||_0^2)$$

$$+c(M)\int_0^T||\partial_t\tilde{w}||_0^2+\varepsilon^6(||\partial_t\tilde{m}||_0^2+||\partial_t\tilde{\xi}||_0^2)+\varepsilon^8(||\partial_t\tilde{k}||_0^2+||\partial_t\tilde{h}||_0^2)\,dt,$$

where the last inequality follows form the Sobolev embedding theorem and from (10.3.61).
Similarly, taking the $|\alpha|$-derivative, for $|\alpha|\leq s-1$, of (10.3.71) and multiplying by $D^\alpha\tilde{V}$ through the scalar product, we get:

$$\frac{1}{2}\frac{d}{dt}(\Sigma D^\alpha\tilde{V},D^\alpha\tilde{V})_0+(L\Sigma D^\alpha\tilde{V},D^\alpha\tilde{V})_0\leq|(D^\alpha\partial_tN(w+\bar{w}),D^\alpha V)_0|,$$

where

$$|(D^\alpha\partial_tN(w+\bar{w}),D^\alpha\tilde{V})_0|=\frac{1}{\tau}|(D^\alpha(2u_1\partial_tw_2-\varepsilon u_1^2\partial_tw_1),\varepsilon^2\partial_tD^\alpha\tilde{m}_2)_0$$

$$+(D^\alpha(u_2\partial_tw_2+u_1\partial_tw_3-\varepsilon u_1u_2\partial_tw_1),\varepsilon^2\partial_tD^\alpha\tilde{m}_3+\varepsilon^2\partial_tD^\alpha\tilde{\xi}_2)_0$$

$$+(D^\alpha(2u_2\partial_tw_3-\varepsilon u_2^2\partial_tw_1),\varepsilon^2\partial_tD^\alpha\tilde{\xi}_3)_0|$$

$$\leq c(||\mathbf{u}||_{s-1})||\partial_tw||_{s-1}^2+\frac{\varepsilon^4}{2\tau}(||\partial_t\tilde{m}||_{s-1}^2+||\partial_t\tilde{\xi}||_{s-1}^2)$$

$$\leq c(M)||\partial_tw||_{s-1}^2+\frac{\varepsilon^4}{2\tau}(||\partial_t\tilde{m}||_{s-1}^2+||\partial_t\tilde{\xi}||_{s-1}^2),$$

where the last inequality follows from (10.3.61). Finally, we obtain:

$$\Gamma_\Sigma||\partial_t\tilde{w}||_{s-1}^2+\varepsilon^6\Delta_\Sigma(||\partial_t\tilde{m}||_{s-1}^2+||\partial_t\tilde{\xi}||_{s-1}^2)+\varepsilon^8\Theta_\Sigma(||\partial_t\tilde{k}||_{s-1}^2+||\partial_t\tilde{h}||_{s-1}^2)$$

$$+\frac{1}{\tau}\int_0^T(2\Delta_{L\Sigma}-1)\varepsilon^4(||\partial_t\tilde{m}||_{s-1}^2+||\partial_t\tilde{\xi}||_{s-1}^2)+\varepsilon^6\Theta_{L\Sigma}(||\partial_t\tilde{k}||_{s-1}^2+||\partial_t\tilde{h}||_{s-1}^2)\,dt$$

$$\leq c\varepsilon^2(||\mathbf{u}_0||_{s-1}^2+||\nabla\mathbf{u}_0||_{s-1}^2+||\nabla^2\mathbf{u}_0||_{s-1}^2)$$

$$+c(M)\int_0^T||\partial_t\tilde{w}||_{s-1}^2+\varepsilon^6(||\partial_t\tilde{m}||_{s-1}^2+||\partial_t\tilde{\xi}||_{s-1}^2)+\varepsilon^8(||\partial_t\tilde{k}||_{s-1}^2+||\partial_t\tilde{h}||_{s-1}^2)\,dt. \tag{10.3.74}$$

Lemma 10.3.3. *If Assumption 10.0.1 and 10.3.1 hold, then there exists a positive constant c such that:*

$$||\partial_tw||_{s-1}^2\leq c(||\partial_t\tilde{w}||_{s-1}^2+\varepsilon^6(||\partial_t\tilde{m}||_{s-1}^2+||\partial_t\tilde{\xi}||_{s-1}^2)+\varepsilon^8(||\partial_t\tilde{k}||_{s-1}^2+||\partial_t\tilde{h}||_{s-1}^2)).$$

Proof. The proof of Proposition 10.3.1 can be adapted here with slight modifications. □

We end the proof by applying the Gronwall inequality to (10.3.74) and using Lemma 10.3.3. □

10.4 Convergence to the Navier-Stokes equations

Now we state our main result.

Theorem 10.4.1. *Let $s > 3$. If Assumptions 10.0.1 and 10.3.1 hold, there exists a subsequence $W^\varepsilon = (w^{\varepsilon\star}, \varepsilon^2 m^\varepsilon, \varepsilon^2\xi^\varepsilon, \varepsilon^2 k^{\varepsilon\star}, \varepsilon^2 h^{\varepsilon\star})$, with $w^{\varepsilon\star} = (\rho^\varepsilon - \bar\rho, \varepsilon\rho^\varepsilon \boldsymbol{u}^\varepsilon)$ and $\bar\rho > 0$, of the solutions to the vector BGK model (10.1.22) with initial data (10.1.23) and $\boldsymbol{u}_0 \in H^{s+1}(\mathbb{T}^2)$ in (10.0.2), such that*

$$(\rho^\varepsilon, \boldsymbol{u}^\varepsilon) \to (\bar\rho, \boldsymbol{u}^{NS}) \quad in \quad C([0,T^\star], H^{s'}(\mathbb{T}^2)),$$

with $T^\star$ in (10.3.67), $s-1 < s' < s$, and where $\boldsymbol{u}^{NS}$ is the unique solution to the Navier-Stokes equations in (10.0.1), with initial data $\boldsymbol{u}_0$ above and P^{NS} the incompressible pressure. Moreover,

$$\frac{\nabla(\rho^\varepsilon - \bar\rho)}{\varepsilon^2} \rightharpoonup^\star \nabla P^{NS} \quad in \quad L^\infty_t H^{s-3}_x.$$

Proof. First of all, consider the previous bounds in (10.3.61), (10.3.62), (10.3.63) and (10.3.70):

$$\sup_{t\in[0,T^\star]} \frac{||\rho^\varepsilon - \bar\rho||_s}{\varepsilon} \le M, \quad \sup_{t\in[0,T^\star]} \frac{||\partial_t(\rho^\varepsilon - \bar\rho)||_{s-1}}{\varepsilon} \le M_1, \tag{10.4.75}$$

$$\sup_{t\in[0,T^\star]} ||\rho^\varepsilon \mathbf{u}^\varepsilon||_s \le N, \quad \sup_{t\in[0,T^\star]} ||\partial_t(\rho^\varepsilon \mathbf{u}^\varepsilon)||_{s-1} \le N_1, \tag{10.4.76}$$

where M, M_1, N, N_1 are positive constants. The Lions-Aubin Lemma in [9] implies that, for $s-1 < s' < s$,

$$\rho^\varepsilon \to \bar\rho \quad \text{strongly} \quad \text{in} \quad C([0,T^\star], H^{s'}(\mathbb{T}^2)),$$

and there exists $\mathbf{m}^\star$ such that

$$\mathbf{m}^\varepsilon = \rho^\varepsilon \mathbf{u}^\varepsilon \to \mathbf{m}^\star \quad \text{strongly} \quad \text{in} \quad C([0,T^\star], H^{s'}(\mathbb{T}^2)).$$

Notice also that $\mathbf{u}^\varepsilon = \dfrac{\mathbf{m}^\varepsilon}{\rho^\varepsilon}$, where

$$1/\rho^\varepsilon \to 1/\bar\rho \quad \text{strongly} \quad \text{in} \quad C([0,T^\star], H^{s'}(\mathbb{T}^2)),$$

since we can take $\bar\rho$ such that $\rho^\varepsilon > \frac{\bar\rho}{2}$ as in (10.3.66). Then

$$\mathbf{u}^\varepsilon = \frac{\mathbf{m}^\varepsilon}{\rho^\varepsilon} \to \frac{\mathbf{m}^\star}{\bar\rho} =: \mathbf{u}^\star \quad \text{strongly} \quad \text{in} \quad C([0,T^\star], H^{s'}(\mathbb{T}^2)).$$

Now, consider system (10.1.19) in the following formulation:

$$\begin{cases} \partial_t w^\varepsilon + \partial_x m^\varepsilon + \partial_y \xi^\varepsilon = 0; \\ \varepsilon\partial_t m^\varepsilon + \frac{\lambda^2}{\varepsilon}\partial_x k^\varepsilon = \frac{1}{\tau}(\frac{A_1(w^\varepsilon+\bar w)}{\varepsilon^2} - \frac{m^\varepsilon}{\varepsilon}), \\ \varepsilon\partial_t \xi^\varepsilon + \frac{\lambda^2}{\varepsilon}\partial_y h^\varepsilon = \frac{1}{\tau}(\frac{A_2(w^\varepsilon+\bar w)}{\varepsilon^2} - \frac{\xi^\varepsilon}{\varepsilon}), \\ \varepsilon\partial_t k^\varepsilon + \varepsilon\partial_x m^\varepsilon = \frac{(2aw^\varepsilon - k^\varepsilon)}{\tau\varepsilon}, \\ \varepsilon\partial_t h^\varepsilon + \varepsilon\partial_y \xi^\varepsilon = \frac{(2aw^\varepsilon - h^\varepsilon)}{\tau\varepsilon}, \end{cases} . \tag{10.4.77}$$

From (10.4.77) and $2a\lambda^2\tau = \nu$ as in (10.0.11), it follows that

$$\begin{cases} m^\varepsilon = \frac{A_1(w^\varepsilon+\bar{w})}{\varepsilon} - \nu\partial_x w^\varepsilon + \varepsilon^2\lambda^2\tau^2(\partial_{tx}k^\varepsilon + \partial_{xx}m^\varepsilon) - \varepsilon^2\tau\partial_t m^\varepsilon; \\ \xi^\varepsilon = \frac{A_2(w^\varepsilon+\bar{w})}{\varepsilon} - \nu\partial_y w^\varepsilon + \varepsilon^2\lambda^2\tau^2(\partial_{ty}h^\varepsilon + \partial_{yy}\xi^\varepsilon) - \varepsilon^2\tau\partial_t \xi^\varepsilon. \end{cases}$$

Substituting the expansions above in the first equation of (10.4.77), we get the following equation:

$$\partial_t w^\varepsilon + \frac{\partial_x A_1(w^\varepsilon+\bar{w})}{\varepsilon} + \frac{\partial_y A_2(w^\varepsilon+\bar{w})}{\varepsilon} - \nu\Delta w^\varepsilon$$
$$= \varepsilon^2\tau\partial_{tx}m^\varepsilon + \varepsilon^2\tau\partial_{ty}\xi^\varepsilon - \varepsilon^2\lambda^2\tau^2(\partial_{txx}k^\varepsilon + \partial_{xxx}m^\varepsilon + \partial_{tyy}h^\varepsilon + \partial_{yyy}\xi^\varepsilon).$$

We recall that $W^\varepsilon = \Sigma\tilde{W}^\varepsilon$ by definition (10.2.30), with $W^\varepsilon, \tilde{W}^\varepsilon$ in (10.1.21) and (10.2.30) respectively. This yields:

$$\begin{cases} w^\varepsilon = \tilde{w}^\varepsilon + \varepsilon^3\sigma_1\tilde{m}^\varepsilon + \varepsilon^3\sigma_2\tilde{\xi}^\varepsilon + 2a\varepsilon^4\tilde{k}^\varepsilon + 2a\varepsilon^4\tilde{h}^\varepsilon; \\ \varepsilon^2 m^\varepsilon = \varepsilon\sigma_1\tilde{w}^\varepsilon + 2a\lambda^2\varepsilon^4\tilde{m}^\varepsilon + \varepsilon^5\sigma_1\tilde{k}^\varepsilon; \\ \varepsilon^2\xi^\varepsilon = \varepsilon\sigma_2\tilde{w}^\varepsilon + 2a\lambda^2\varepsilon^4\tilde{\xi}^\varepsilon + \varepsilon^5\sigma_2\tilde{h}^\varepsilon; \\ \varepsilon^2 k^\varepsilon = 2a\varepsilon^2\tilde{w}^\varepsilon + \varepsilon^5\sigma_1\tilde{m}^\varepsilon + 2a\varepsilon^6\tilde{k}^\varepsilon; \\ \varepsilon^2 h^\varepsilon = 2a\varepsilon^2\tilde{w}^\varepsilon + \varepsilon^5\sigma_2\tilde{\xi}^\varepsilon + 2a\varepsilon^6\tilde{h}^\varepsilon. \end{cases} \tag{10.4.78}$$

From (10.3.69), (10.3.50)-(10.3.68) and (10.4.78) it follows that, for a fixed constant value $c > 0$,

$$\tau\varepsilon^2||\partial_{tx}m^\varepsilon + \partial_{ty}\xi^\varepsilon - \lambda^2\tau(\partial_{txx}k^\varepsilon + \partial_{xxx}m^\varepsilon + \partial_{tyy}h^\varepsilon + \partial_{yyy}\xi^\varepsilon)||_{s-3} = O(\varepsilon^2),$$

then

$$\left\|\partial_t w^\varepsilon + \frac{\partial_x A_1(w^\varepsilon+\bar{w})}{\varepsilon} + \frac{\partial_y A_2(w^\varepsilon+\bar{w})}{\varepsilon} - \nu\Delta w^\varepsilon\right\|_{s-3} = O(\varepsilon^2). \tag{10.4.79}$$

The last two equations and the previous bounds (10.4.75) and (10.4.76) yield:

$$\left\|\partial_t(\rho^\varepsilon\mathbf{u}^\varepsilon) + \nabla\cdot(\rho^\varepsilon\mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon) + \frac{\nabla(\rho^\varepsilon-\bar{\rho})}{\varepsilon^2} - \nu\Delta(\rho\mathbf{u}^\varepsilon)\right\|_{s-3} = O(\varepsilon), \tag{10.4.80}$$

and, in particular,

$$\frac{||\nabla(\rho^\varepsilon-\bar{\rho})||_{s-3}}{\varepsilon^2} \le c,$$

i.e. there exists $\nabla P^\star \in L^\infty_t H^{s-3}_x$ such that

$$\frac{\nabla(\rho^\varepsilon-\bar{\rho})}{\varepsilon^2} \rightharpoonup^\star \nabla P^\star \quad \text{in} \quad L^\infty_t H^{s-3}_x.$$

Moreover, since $\rho^\varepsilon \to \bar{\rho}$ and $\mathbf{u}^\varepsilon \to \mathbf{u}^\star$ in $C([0,T^\star], H^{s'}(\mathbb{T}^2))$, from $||\partial_t(\rho^\varepsilon\mathbf{u}^\varepsilon)||_{s-1} \le N_1$ as in (10.4.76), it follows also that

$$\partial_t(\rho^\varepsilon\mathbf{u}^\varepsilon) \rightharpoonup^\star \bar{\rho}\partial_t\mathbf{u}^\star \quad \text{in} \quad L^\infty_t H^{s-3}_x,$$

while
$$\nabla\cdot(\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) \rightharpoonup^\star \bar{\rho}\nabla\cdot(\mathbf{u}^\star\otimes\mathbf{u}^\star) \ \text{ in } \ L^\infty_t H^{s-3}_x.$$

Thus, from (10.4.80) we have the weak* convergence in $L^\infty_t H^{s-3}_x$, i.e.

$$\partial_t(\rho^\varepsilon\mathbf{u}^\varepsilon)+\nabla\cdot(\rho^\varepsilon\mathbf{u}^\varepsilon\otimes\mathbf{u}^\varepsilon)+\frac{\nabla(\rho^\varepsilon-\bar{\rho})}{\varepsilon^2}-\nu\Delta(\rho^\varepsilon\mathbf{u}^\varepsilon) \rightharpoonup^\star \bar{\rho}\left(\partial_t\mathbf{u}^\star+\nabla\cdot(\mathbf{u}^\star\otimes\mathbf{u}^\star)+\frac{\nabla P^\star}{\bar{\rho}}-\nu\Delta\mathbf{u}^\star\right).$$

On the other hand, the first equation of (10.4.79) yields

$$\partial_t(\rho^\varepsilon-\bar{\rho})+\nabla\cdot(\rho^\varepsilon\mathbf{u}^\varepsilon)-\nu\Delta(\rho^\varepsilon-\bar{\rho}) = O(\varepsilon^2). \tag{10.4.81}$$

Notice that $||\partial_t(\rho^\varepsilon-\bar{\rho})||_{s-1} = O(\varepsilon)$ and $||\Delta(\rho^\varepsilon-\bar{\rho})||_{s-2} = O(\varepsilon)$ thanks to (10.4.75), while $\rho^\varepsilon\to\bar{\rho}$ and $\mathbf{u}^\varepsilon\to\mathbf{u}^\star$ in $C([0,T^\star],H^{s'}(\mathbb{T}^2))$. This way, from (10.4.81) we finally recover the divergence free condition

$$\nabla\cdot\mathbf{u}^\star = 0.$$

□

Bibliography

[1] S. Alinhac, P. Gérard, Pseudo-differential operators and the Nash-Moser Theorem, *Graduate Studies in Mathematics* Vol. 82, *American Mathematical Society* (2007).

[2] D. Aregba-Driollet, R. Natalini, S.Q. Tang, Diffusive kinetic explicit schemes for nonlinear degenerate parabolic systems, *Math. Comp.* **73** (2004), 63-94.

[3] S. Astanin, L. Preziosi, Multiphase models of tumor growth., Selected topics in cancer modeling, *Model. Simul. Sci. Eng. Technol.,* Birkhäuser Boston, Boston, MA (2008), 223-253.

[4] H. Bahouri, J.-Y. Chemin and R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, *Grundlehren der Mathematischen Wissenschaften* Vol. 343, *Springer Heidelberg* (2011).

[5] C. Bardos, F. Golse, C. D. Levermore, Fluid dynamic limits of hyperbolic equations. I. Formal derivations, *J. Stat. Phys.* **63** (1991), 323-344.

[6] C. Bardos, F. Golse, C. D. Levermore, Fluid dynamic limits of kinetic equations - II Convergence proofs for the Boltzmann-equation, *Comm. Pure Appl. Math.* **46** (1993), 667-753.

[7] H. Beirão da Veiga, A. Valli, Existence of C^∞ solutions of the Euler Equations for non-homogeneous fluids, *Comm. Part. Diff. Eq.* **5** (1980), 95-107.

[8] S. Benzoni-Gavage, D. Serre, Multidimensional Hyperbolic Partial Differential Equations, *Oxford University Press* (2007).

[9] A. Bertozzi, A. Majda, Vorticity and Incompressible Flow, *Cambridge University Press* (2002).

[10] R. Bianchini, Uniform asymptotic and convergence estimates for the Jin-Xin model under the diffusion scaling, *submitted* (2017).

[11] R. Bianchini, R. Natalini, Convergence of a vector BGK approximation for the incompressible Navier-Stokes equations, *submitted* (2017).

[12] R. Bianchini, R. Natalini, Global existence and asymptotic stability of smooth solutions to a fluid dynamics model of biofilms in one space dimension, *J. Math. Anal. Appl.,* **434** (2016), 1909-1923.

[13] R. Bianchini, R. Natalini, The paradifferential approach to the local well-posedness of some problems in mixture theory in two space dimensions, *submitted* (2016).

[14] R. Bianchini, R. Natalini, Well-posedness of a model of nonhomogeneous compressible-incompressible fluids, *J. Hyp. Diff. Eq.* (2017). **14** (03) (2017), 487-516.

[15] S. Bianchini, B. Hanouzet, R. Natalini, Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy, *Comm. Pure Appl. Math.* **60** (2007), 1559-1622.

[16] F. Bouchut, Construction of BGK Models with a Family of Kinetic Entropies for a Given System of Conservation Laws, *J. Stat. Phys.* **95** (2003).

[17] F. Bouchut, F. Guarguaglini, R. Natalini, Diffusive BGK Approximations for Nonlinear Multidimensional Parabolic Equations, *Indiana Univ. Math. J.* **49** (2000), 723-749.

[18] F. Bouchut, Y. Jobic, R. Natalini, R. Occelli, V. Pavan, Second-order entropy satisfying BGK-FVS schemes for incompressible Navier-Stokes equations, preprint June 2016, submitted.

[19] R. M. Bowen, Theory of mixtures, *Continuum physics, Vol.3, Eringen A.C. ed., Academic Press* (1976).

[20] R. M. Bowen, Incompressible porous media model by use of theory of mixtures, *Int. J. Eng. Sci.* **18** (1980), 1129-1148.

[21] Y. Brenier, Averaged multivalued solutions for scalar conservation laws, *SIAM J. Numer. Anal.* **21** (1984), 1013-1037.

[22] Y. Brenier, R. Natalini, M. Puel, On a relaxation approximation of the incompressible Navier-Stokes equations, *Proc. Amer. Math. Soc.* **132** (4) (2003), 1021-1028.

[23] M. Carfora, R. Natalini, A discrete kinetic approximation for the incompressible Navier-Stokes equations, *ESAIM: Math. Modelling Numer. Anal.* **42** (2008), 93-112.

[24] C. Cercignani, R. Illner, M. Pulvirenti, The Mathematical Theory of Dilute Gases, *Springer-Verlag, New York* (1994).

[25] J.-Y. Chemin, Perfect incompressible fluids, *Oxford, Clarendon Press, New York, Oxford University Press* (1998).

[26] I. L. Chern, Long-time effect of relaxation for hyperbolic conservation laws, *Comm. Math. Phys.* **172** (1995), 39-55.

[27] F. Clarelli, C. Di Russo, R. Natalini, M. Ribot, A fluid dynamics model of the growth of phototrophic biofilms, *J. Math. Biol.* **66** (7) (2013), 1387-1408.

[28] C. M: Dafermos, Hyperbolic conservation laws in continuum physics, *Springer-Verlag, Berlin* (2000).

[29] R. Danchin, On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework, *J. Diff. Eq.* **24** (8) (2010), 2130-2170.

[30] A. DeMasi, R. Esposito, J. Lebowitz, Incompressible Navier-Stokes and Euler Limits of the Boltzmann equation, *Comm. Pure Appl. Math.* **42** (1990), 1189-1214.

[31] W. Ehlers, Constitutive equations for granular materials in geomechanical context, *Environmental Sciences and Geophysics, CISM Courses and Lectures N.337, Hutter K. ed., Springer-Verlag*, (1993).

[32] A. Farina, L. Preziosi, On Darcy's law for growing porous media, *In. J. Non-Lin. Mech.* **37** (3) (2002), 485-491.

[33] F. Golse, L. Saint-Raymond, The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels, *Invent. math.* **155** (81) (2004).

[34] E. Grenier, Pseudo-Differential Energy Estimates of Singular Perturbations, *Comm. Pure Appl. Math.* **50** (9) (1997), 821-865.

[35] R. L. Gudmundsson, On the well-posedness of the two-fluid model for dispersed two-phase flow in 2D, *Technical Report TRITA-NA-0223, Royal Institute of Technology* (2002).

[36] I. Hachicha, Approximations hyperboliques des eq́uations de Navier-Stokes, Ph. D. Thesis, Université d'Évry-Val d'Essone (2013).

[37] I. Hachicha, Global existence for a damped wave equation and convergence towards a solution of the Navier-Stokes problem. *Nonlinear Anal.* **96** (2014), 68-86.

[38] L. Hanich, M. Louaked, C.P. Thompson, Well-posedness of incompressible models of two - and three - phase flow, *IMA Journal of Applied Mathematics* **68** (2003), 595-620.

[39] B. Hanouzet, R. Natalini, Global Existence of Smooth Solutions for Partially Dissipative Hyperbolic Systems with a Convex Entropy, *Arch. Rational Mech. Anal.* **169** (2003), 89-117.

[40] L. Hsiao, T. P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, *Comm. Math. Phys.* **143** (1993) n. 3, 599-605.

[41] M. Ishii, J. H. Song, The well-posedness of incompressible one dimensional two-fluid model, *International Journal of Heat and Mass Transfer* **43** (2000), 2221-2231.

[42] S. Jin, H. L. Liu, Diffusion limit of a hyperbolic system with relaxation, *Meth. and Appl. Anal.* **5** (1998), 317-334.

[43] S. Jin, Z. Xin, The relaxation schemes for system of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.* **48** (1995), 235-277.

[44] M. Junk, W. A. Yong, Rigorous Navier-Stokes Limit of the Lattice Boltzmann Equation, *Asymptotic Anal.* **35** (165) (2003).

[45] T. Kato, Perturbation theory for linear operator, 2nd ed. Grundlehren der Mathematischen Wissenschaften, 132. *Springer, New York* (1976).

[46] S. Kawashima, Large-time behavior of solutions to hyperbolic-parabolic systems of conservation laws and applications, *Proc. Roy. Soc. Edinburgh Sect. A* **106** (1987), 169-194.

[47] S. Klainerman, A. Majda, Singular Limits of Quasilinear Hyperbolic Systems with Large Parameters and the Incompressible Limit of Compressible Fluids, *Comm. Pure Appl. Math.* **34** (1981), 481-524.

[48] C. Lattanzio, R. Natalini, Convergence of diffusive BGK approximations for nonlinear strongly parabolic systems, *Proc. Roy. Soc. Edinburgh Sect. A* **132** (2002) n. 2, 341-358.

[49] P. L. Lions, Mathematical Topics in Fluid Mechanics, Vol. 1, *Oxford Univ. Press, Oxford* (1996).

[50] P. L. Lions, G. Toscani, Diffusive limits for finite velocity Boltzmann kinetic models, *Revista Mat. Iberoamer.* **13** (1997), 473-513.

[51] H. Liu, R. Natalini, Long-time diffusive behavior of solutions to a hyperbolic relaxation system, *Asymptot. Anal.* **25** (1) (2001), 21-38.

[52] A. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, *Springer-Verlag, New York* (1984).

[53] J. E. Marsden, Well-posedness of the equations of a non-homogeneous perfect fluid, *Comm. Part. Diff. Eq.* **1** (1976), 215-230.

[54] C. Mascia, Twenty-eight years with “Hyperbolic Conservation Laws with Relaxation”, *Acta Math. Scientia* **35** (4) (2015), 807-831.

[55] G. Métivier, Para-differential Calculus and Application to the Cauchy Problem for Nonlinear Systems, *CRM Series, Edizioni della Scuola Normale Superiore* (2008).

[56] I. Muller, Thermodynamics of mixture of fluids, *J. Mec.,* **14** (1975), 267-303.

[57] I. Muller, Rational Thermodynamics of mixtures of fluids, *Thermodynamics and Constitutive Equations, Lecture Notes in Physics 228, Grioli G. ed., Springer-Verlag.* (1985).

[58] J. D. Murray, Mathematical Biology: An Introduction (Third Edition), Interdisciplinary Applied Mathematics, Vol. 17 *Springer-Verlag, New York* (2002).

[59] R. Natalini, A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws, *J. Diff. Eq.* **148** (1998), 292-317.

[60] R. Natalini, Convergence to equilibrium for the relaxation approximations of conservation laws, *Comm. Pure Appl. Math.* **49** (8) (1996), 795-823.

[61] T. Nishida, Nonlinear hyperbolic equations and related topics in fluid dynamics, *Départment de Mathématique, Université de Paris-Sud, Orsay, Publications Mathématiques d'Orsay* (1978), 78-02.

[62] R. Orive, E. Zuazua, Long-time behavior of solutions to a non-linear hyperbolic relaxation system, *J. Diff. Eq.* **228** (2006), 17-38.

[63] M. Paicu and G. Raugel, A hyperbolic perturbation of the Navier-Stokes equations, (Une perturbation hyperbolique des équations de Navier-Stokes.). *ESAIM, Proc.* **21** (2007), 65-87.

[64] B. Perthame, Kinetic formulation of conservation laws, *Oxford Lecture Series in Mathematics and its Applications* 21, *Oxford University Press* (2000).

[65] K. R. Rajagopal, L. Tao, Mechanics of Mixtures, Series on Advances in Mathematics for Applied Sciences, 35. World Scientific Publishing Co., River Edge, NJ (1995).

[66] K. R. Rajagopal, On a hierarchy of approximate models for flows of incompressible fluids through porous solids, Math. Models Methods Appl. Sci., **17**, 215 (2007).

[67] L. Saint-Raymond, From the BGK model to the Navier–Stokes equations, *Annales scientifiques de l'École Normale Supérieure* **36**.2 (2003), 271-317.

[68] A. Sepe, Convergence of a BGK Approximation of the Isentropic Euler Equations, Report n. 30/2009, Math Department, Università degli Studi di Bari.

[69] D. Serre, Systems of conservation laws Vol. I, Hyperbolicity, entropies, shock waves, *Cambridge University Press* (2000).

[70] Y. Shizuta, S. Kawashima, Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation, *Hokkaido Math. J.* **14** (1985) n.2, 249-275.

[71] T. C. Sideris, B. Thomases, D. Wang, Long time behavior of solutions to the 3D compressible Euler equations with damping, *Comm. Partial Differential Equations* **28** (2003) n. 3-4, 795-816.

[72] H. B. Stewart, B. Wendroff, Review article two-phase flow: Models and methods, *Journal of Computational Physics* **56** (1984), 363-409.

[73] S. Succi, The lattice Boltzmann equation for fluid dynamics and beyond, *Numerical Mathematics and Scientific Computation, Oxford Science Publications, the Clarendon Press, Oxford University Press, New York* (2001).

[74] M. Taylor, Partial differential equations III, *Applied Mathematical Sciences 117, Springer* (1996).

[75] R. Temam, Navier-Stokes Equations -Theory and Numerical Analysis, *North-Holland Publishing Company* (1977).

[76] A. Valli, W. M. Zajazckowski, About the motion of nonhomogeneous ideal incompressible fluids, *Nonlinear Analysis, Theory, Methods and Applications* **12** (1) (1988), 43-50.

[77] D. A. Wolf-Gladrow, Lattice-gas cellular automata and Lattice Boltzmann models. An introduction, Lecture Notes in Mathematics, *Springer-Verlag, Berlin* (2000).

[78] W.-A. Yong, Entropy and global existence for hyperbolic balance laws, *Arch. Ration. Mech. Anal.* **172** (2004) n. 2, 247-266.

[79] J. Yström, On two-fluid equations for dispersed incompressible two-phase flow, *Comput. Visual. Sci.* **4** (2001), 125-135.