PhD admissions exam

41^{th} cycle – 30^{th} June 2025

The exam consists of 14 exercises divided into seven different topics. The exercises are independent and can be solved in any order.

- Each exercise is evaluated with **9 points** with a full correct answer. Partial answers receive a correspondingly partial score. The minimum score for the admission to the oral exam is **18 points**. The maximum total score for the exam is capped at **32 points**.
- Solutions should be written in **English**, except for candidates applying for positions as Italian public employees, who may write their solutions in **Italian**.
- During the exam, it is **forbidden** to consult notes, communicate with others, or use any external resources, in particular sources on the web.

Candidates are encouraged to select a manageable number of exercises that can be completed within the given time while maintaining high-quality solutions. Solutions should be clearly written, mathematically rigorous and appropriately motivated.

Contents

1 Algebra

Exercise 1.1. Consider the element $x^2 - 2 \in \mathbb{Z}[x]$ and the ring

$$B = \mathbb{Z}[x]/(x^2 - 2)$$

i) Determine whether the ideal generated by $x^2 - 2$ in $\mathbb{Z}[x]$ is prime and whether it is maximal; if it is not maximal, find an ideal strictly between $(x^2 - 2)$ and (1).

ii) Determine whether the ring B is an integral domain, whether it is a unique factorization domain, a principal ideal domain, a Euclidean domain, and whether it is a field.

iii) Determine whether the elements $3, x, x + 1, 3x$ in B can be factored into irreducibles; if so, provide a factorization into irreducibles.

Exercise 1.2. Let $V = \mathbb{R}^4$ with standard basis $\mathcal{B} = \{e_1, e_2, e_3, e_4\}$. Let $G = S_3$ be the symmetric group on three elements. Define an action of G on V as follows: for each $\sigma \in S_3$ and for each $v = (v_1, v_2, v_3, v_4) \in V$ define $\sigma \cdot v \in V$ as

$$\sigma \cdot v = (v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, v_{\sigma^{-1}(3)}, v_4).$$

1. Show that this defines a linear action of G on V, i.e., a group homomorphism $\phi : G \to \mathrm{GL}(V)$.

2. Let $\tau = (1\,2) \in S_3$ and $\rho = (1\,2\,3) \in S_3$. Decide whether $\phi(\tau)$ and/or $\phi(\rho)$ is diagonalisable.

3. Determine the subspace of G-invariant vectors:

   $$V^G = \{v \in V \mid \sigma \cdot v = v \text{ for all } \sigma \in G\}.$$

   Find a basis and its dimension.

4. Compute the dimension of the space of coinvariants:

   $$V_G = V/\langle \sigma \cdot v - v \mid \sigma \in G, v \in V\rangle.$$

5. Let $W = \mathrm{span}\{e_1, e_2, e_3\} \subseteq V$. Prove that W is G-stable (as a set) and decide whether it is a reducible representation or not, that is whether it admits a proper non-zero G-stable subspace or not.

2 Analysis

Exercise 2.1. Consider the Cauchy problem for the first order differential equation

$$\begin{cases} y' = \dfrac{xe^y(y^3 - y^4)}{2 + \cos(x + y^2)} \\ y(0) = y_0. \end{cases}$$

Show that, for every $y_0 \in \mathbb{R}$, the above problem has a global solution. Depending on the value of $y_0 \in \mathbb{R}$, study the limits of $y(x)$ as $x \to -\infty$ and $x \to +\infty$.

Exercise 2.2. Let us define, for $n \in \mathbb{N}$,

$$f_n(x) = \frac{1}{\pi} \frac{n^3}{n^4(nx - 1)^2 + 1}, \qquad x \in \mathbb{R}.$$

(i) Compute the antiderivative (indefinite integral) $\displaystyle\int f_n(x)\, dx$.

(ii) Describe the intervals $[a, b] \subset \mathbb{R}$ such that f_n converges uniformly on $[a, b]$.

(iii) For $u \in L^\infty(\mathbb{R})$ and $n \in \mathbb{N}$, define

$$T_n u(x) = \int_{-\infty}^{\infty} u(x - y) f_n(y)\, dy, \qquad x \in \mathbb{R}.$$

Show that T_n is well defined and is a bounded linear operator from $L^\infty(\mathbb{R})$ to itself. Compute its operator norm $||T_n||$.

(iv) Show that, if $u \in L^\infty(\mathbb{R})$ is continuous at a point x, then we have

$$\lim_{n\to\infty} T_n u(x) = u(x).$$

What if u has a jump discontinuity at x?

3 Didactics and history of mathematics

Exercise 3.1. Presentare una esperienza didattica ispirata ai risultati della matematica ellenistica, indicando i punti di forza di una didattica basata sulla storia del pensiero matematico.

Present a teaching experience inspired by the results of Hellenistic mathematics, indicating the strengths of a teaching approach based on the history of mathematical thought.

Exercise 3.2. Discutere vantaggi e svantaggi della presentazione della geometria analitica attraverso software dedicati.

Discuss advantages and disadvantages of a [presentation of analytical geometry through dedicated software.

4 Geometry

Exercise 4.1. Let $\mathbb{E}^2(\mathbb{R})$ be the Euclidean plane endowed with origin O and Cartesian coordinates (x, y), and let C be the conic with cartesian equation

$$3x^2 + 2xy + 3y^2 + 2x - 10y + 7 = 0.$$

(i) Determine the standard metric form of C in a suitable coordinate system and draw a picture of C in such coordinate system

Let C be a real ellipse in $\mathbb{E}^2(\mathbb{R})$ and let $X \subseteq \mathbb{E}^2(\mathbb{R})$ be the subset of points $P \in \mathbb{E}^2(\mathbb{R})$ such that there exists at least a tangent line to C through P.

(ii) Discuss whether X is closed and whether X is compact in the standard Euclidean topology of $\mathbb{E}^2(\mathbb{R})$.

(iii) Prove that X is connected.

(iv) Compute the fundamental group $\pi_1(X, x)$

Exercise 4.2. Let $\mathbb{P}^3(\mathbb{R})_{[x_0:x_1:x_2:x_3]}$ be real projective space with homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$. Consider the hyperplane $\{x_2 = 0\} = H \cong \mathbb{P}^2(\mathbb{R})$ and the point $P = [0 : 0 : 1 : 0] \in \mathbb{P}^3(\mathbb{R})$. We define:

$$\begin{aligned} \pi \colon \mathbb{P}^3(\mathbb{R}) \setminus \{P\} &\to H \cong \mathbb{P}^2(\mathbb{R}) \\ Q &\to \pi(Q) = L_{P,Q} \cap H \end{aligned}$$

where $L_{P,Q}$ is the projective line through P and Q.

(i) Show that π is well defined and smooth by finding an explicit expression for π in homogeneous coordinates

(*Hint: to understand π it might be convenient to think of points, lines, planes in $\mathbb{P}^3(\mathbb{R})$ as, respectively, lines, planes, three-dimensional subspaces through the origin in $\mathbb{R}^4$*)

(ii) Let $F \colon \mathbb{P}^1(\mathbb{R})_{[t:s]} \to \mathbb{P}^3(\mathbb{R})_{[x_0:x_1:x_2:x_3]}$ be the morphism defined in homogeneous coordinates by

$$[t : s] \to [t^3 : t^2s : ts^2 : s^3].$$

Show, by looking at the various charts, that F is a regular and injective parametrization.

(iii) After checking that $P \notin F(\mathbb{P}^1(\mathbb{R}))$, show that $\pi \circ F \colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^2(\mathbb{R})$ is an injective, continuous but *not* regular parametrization

(iv) Describe the singularities of $\mathrm{Im}(\pi \circ F)$ (i.e. node, cusps, triple points etc.) in a suitable chart

5 Mathematical physics

Exercise 5.1. Consider a mechanical system composed by a pair of point of mass $m/2$ constrained to move with fixed distance $2l$ (dumbbell). The centre of mass of such dumbbell is constrained to move on a circumference of radius $R >> l$, and centre O in the origin of a reference frame in the vertical plane containing the circumference. A mass M is placed at the point O. The points of the dumbbell interact gravitationally with the point in O, and are subjected to a constant gravitational field oriented as the axis y of the reference frame, with gravity acceleration g.

a) Write the Lagrangian of the system up to the second order in the parameter $\frac{l}{R}$

b) Discuss the equilibrium configurations of the system, and their stability

c) Write the Hamiltonian of the system

d) Posing $g = 0$ solve the motion of the system in terms of indefinite integrals by means of the Hamilton-Jacobi method

Exercise 5.2. In statistical mechanics consider a spin system containing 4 spins $\sigma_i = \pm 1$, $i = 1, ..., 4$. Let the Hamiltonian of the system be

$$H(\sigma) = - \sum_{1 \leq i < j \leq 4} \sigma_i \sigma_j$$

and consider the Gibbs measure

$$\pi(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z}$$

with Z being the partition function of the system.

a) Compute explicitly Z

b) Calling $\sigma = +$ the configuration in which $\sigma_i = +1 \quad \forall\, i = 1, ..., 4$, find a value of the inverse temperature β such that $\pi(+) > 9/20$ and a value of the inverse temperature β such that $\pi(+) < 1/10$

6 Numerical analysis

Exercise 6.1. Let $\alpha \in \mathbb{R}$, $n \geq 4$ an integer and $A = (a_{i,j})$ a $n \times n$ matrix defined as $a_{i,i} = i\alpha$ for $i = 1, \ldots, n$, $a_{i+1,i} = -1$, $a_{i,i+1} = 1$ for $i = 1, \ldots, n-1$ and $a_{i,j} = 0$ otherwise.

1. Determine the values of α such that the Gerschgorin circles of A are pairwise disjoint.

2. Prove that for such values of α the matrix A has real eigenvalues.

3. Given $\alpha = 4$, find, if possible, a diagonal similarity transformation that yields the tightest possible inclusion disks for each eigenvalue of A.

Exercise 6.2. Let $a \in \mathbb{R} \backslash \{0\}$ and $f_a(x) = x \log |x + a|$.

1. Determine the values of α, if any, such that the Newton method can by applied to provide an approximation of the zeros of $f_a(x)$.

2. Discuss the local convergence of the Newton method at each zero varying a and, if possible, establish the convergence order.

3. What can be said if we now assume that for $a = 0$ $f_a(0) := 0$? Is it possible to apply the Newton method to approximate the zero at the origin?

7 Probability

Exercise 7.1. Let X, Y be jointly Gaussian random variables.

(a) Let $Z = X - \alpha Y$. Find $\alpha \in \mathbb{R}$ such that Z is independent of Y.

(b) Using (a), compute $\mathbb{E}[e^{itX} \mid Y]$ for $t \in \mathbb{R}$.

(c) Deduce from (b) that $X \mid Y$ is Gaussian, and determine $\mathbb{E}[X \mid Y]$ and $\mathrm{Var}(X \mid Y)$.

[Recall that the characteristic function of the Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$ is $e^{it\mu - \frac{1}{2}t^2\sigma^2}$]

Exercise 7.2. Let (X_n) be a sequence of random variables such that $X_n \sim \mathrm{Unif}(0, 1/n)$, and let $Y_n = n^\gamma X_n$ with $\gamma \in \mathbb{R}$.

(a) Study the convergence in distribution of (Y_n) as γ varies.

(b) Prove that a sequence of independent, identically distributed, not almost surely constant random variables can not converge in probability.

(c) Assuming that (X_n) is independent, study the convergence in probability of (Y_n) as γ varies, using (a) and (b).