PhD admissions exam

39th cycle – 6th July 2023

This exam contains many exercises. They are organised into different topics, each topic in a different section of this document. Applicants are required to choose a few of the questions and to submit written solutions in a clear and readable manner. Solutions should be submitted for at most four exercises and the exercises should be chosen to show a breadth of knowledge as well as depth. Solutions should be written in English apart from those applying to the position for Italian public servants who may write their solutions in Italian. During the exam it is forbidden to consult notes, communicate with others, or in any other way use external resources.

Some questions are longer than others, every candidate comes from a slightly different background and will have studied different material, different people have different mathematical interests. In this sense this exam is very different to a standard undergraduate exam. In particular there is no benefit to answering the maximum number of questions, instead it is best to answer a limited number of exercises, the amount of material that can be done comfortably in the allotted time, and produce solutions of a high standard.

This is an opportunity for candidates to exhibit their mathematical skills. Among other qualities, we look for solutions to be well communicated, rigorous and fully reasoned.

The 2023 PhD admission committee wish every applicant "in bocca al lupo", all the best for the exam!

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1 Algebra

Exercise 1.1. Consider the following linear operator Φ on $\mathbb{C}^4$

$$\Phi \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -6z - 3w \\ x - 7z - 2w \\ -2z - w \\ 4z + 2w \end{pmatrix}.$$

(A) Compute the characteristic polynomial and the minimal polynomial of Φ.

(B) Determine the eigenvalues of Φ and, for each of them, compute the corresponding generalized eigenspaces.

(C) Determine the Jordan canonical form of Φ and compute a Jordan basis for Φ.

Exercise 1.2. Consider the group

$$G = \langle a, x | a^8 = 1, x^2 = 1, xa = a^3 x \rangle,$$

that is, the group with generators a and x and whose relations are generated by $a^8 = 1$, $x^2 = 1$ and $xa = a^3 x$.

(A) Prove that G is a finite group of order 16 and write down the multiplication table of G.

(B) Compute the conjugacy classes of G.

(C) Compute the center $Z(G)$ of G and prove that $G/Z(G)$ is isomorphic to the dihedral group of order 8.

(D) Compute the commutator subgroup $[G, G]$ of G and prove that

$$G/[G, G] \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

(E) Compute all the subgroups of G of order 8 and, for each such subgroup H, say if G is isomorphic to a semidirect product $H \rtimes S$, for some subgroup S.

2 Analysis

Esercise 2.1. Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval and $f : \mathcal{I} \to \mathbb{R}$ a continuos function. Consider the subset of $[0, +\infty)$ given by:

$$\mathcal{C}(f) := \{|x - y| : \ x, y \in \mathcal{I} \text{ and } f(x) = f(y)\}$$

(i) Prove that if $h \in \mathcal{C}(f)$, then $\frac{h}{n} \in \mathcal{C}(f)$ for every $n \geq 1$.

(ii) Prove that if $\mathcal{I} = \mathbb{R}$ and f is a continuos periodic fucntion of period $T > 0$, then $\mathcal{C}(f) = [0, +\infty)$.

Esercise 2.2. Let U be a connected open non-empty subset of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(A) Prove that if U is simply connected, then the function $1/z$ admits a primitive on U (that is, a function $F : U \to \mathbb{C}$ such that $F'(z) = 1/z$).

(B) Prove that if there exists a primitive of $1/z$ on U then there exists a logarithm on U (that is, a holomorphic function $\ln : U \to C$ such that $e^{\ln(z)} = z$ for every $z \in U$).

(C) Is it true that if there exists a logarithm on U, then U is simply connected?

Esercise 2.3. Let $AC_0[0, 1]$ be the vector space of real functions, absolutely continuous on the interval $[0, 1]$, vanishing in zero.

1. Define a norm on $AC_0[0, 1]$ which makes it a Banach space.
2. Describe the dual of the Banach space $AC_0[0, 1]$.
3. Explain how the concepts of “absolutely continuous function” and “absolutely continuous measure” are connected to each other.

3 Didactics and the history of mathematics

Exercise 3.1.

- Il candidato discuta le nozioni di numeri interi, razionali e reali, e proponga delle esperienze didattiche a riguardo.
- The applicant should discuss the notions of integer, rational and real numbers, and they should propose teaching activities on this subject.

Exercise 3.2.

- Il candidato presenti un risultato di scienza antica e una esperienza didattica ispirata al risultato scelto.
- The applicant should present a result of ancient science and a teaching experience inspired by the chosen result.

4 Geometry

Exercise 4.1. Consider the vector space $\mathbb{R}^3$ endowed with the canonical basis $\mathcal{E}$ and the standard scalar product $\langle -, - \rangle$. Consider the following matrix

$$A = \frac{1}{2}\begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(A) Let B be the symmetric bilinear form B on $\mathbb{R}^3$ whose matrix with respect to the canonical basis $\mathcal{E}$ is equal to A. Determine a basis of $\mathbb{R}^3$ with respect to which B is in its Sylvester canonical form.

(B) Let S be the linear operator on $(\mathbb{R}^3, \langle -, - \rangle)$ whose matrix with respect to the canonical basis $\mathcal{E}$ is equal to A. Determine if S is orthonormally diagonalizable and, in case it is, determine an orthonormal basis of $\mathbb{R}^3$ with respect to which S is diagonalizable.

Exercise 4.2. Consider the real projective space $\mathbb{P} := \mathbb{P}^3(\mathbb{R})$, with homogenous coordinates $[x_0, x_1, x_2, x_3]$. Let $\mathbb{P}^*$ the dual projective space, that is the space that parametrizes the projective planes of $\mathbb{P}$, whose homogenous coordinates will be denoted by $[a_0, a_1, a_2, a_3]$.

(A) Let $P := [0, 1, 1, 1] \in \mathbb{P}$ and consider the set $\Lambda(P)$ of projective planes passing through P. Denote by

$$\delta : \{\text{hyperplanes of } \mathbb{P}\} \to \mathbb{P}^*$$

the duality map associating to a plane $\pi \subset \mathbb{P}$ the corresponding point $\delta(\pi) \in \mathbb{P}^*$.

Show that $\delta(\Lambda(P))$ is a projective subspace of $\mathbb{P}^*$, compute its dimension and determine its cartesian equations in the coordinates $[a_0, a_1, a_2, a_3]$.

(B) Consider the projective quadric $\mathcal{Q}$ of $\mathbb{P}$ of equation

$$3x_0^2 + 5x_0x_1 + 5x_0x_2 + x_0x_3 + 3x_1^2 + 6X_1X_2 + 3X_2^2 = 0.$$

- Classify the quadric $\mathcal{Q}$ and reduce it in standard canonical form in suitable homogenous coordinates.
- Determine its singular locus and establish if $\mathcal{Q}$ can be ruled by lines, and if yes, then describe the family of lines contained in $\mathcal{Q}$.

(C) Consider the affine chart $\mathbb{A}_0$ of $\mathbb{P}$ given by $\mathbb{A}_0 := \{[x_0, x_1, x_2, x_3] \in \mathbb{P} \mid x_0 \neq 0\}$. Consider the affine quadric in $\mathbb{A}_0$

$$\mathcal{Q}_0 := \mathcal{Q} \cap \mathbb{A}_0 \subset \mathbb{A}_0 \cong \mathbb{A}^3(\mathbb{R}).$$

- Classify the quadric $\mathcal{Q}_0$ and reduce it in standard canonical form in suitable affine coordinates.
- Classify the conic sections obtained by intersecting $\mathcal{Q}_0$ with the tangent plane at $\mathcal{Q}_0$ at any non-singular point of $\mathcal{Q}_0$.

5 Mathematical physics

Exercise 5.1. Let us consider the one-dimensional mechanical system governed by the differential equation

$$\ddot{x} = -x - ax^2 ,$$

where $x \in \mathbb{R}$, $\ddot{x}$ denotes the second derivative of x with respect to time and a is a real *positive* parameter. In particular, let us study the motion law $t \mapsto x(t)$ that is solution of the equation above when the initial conditions are $x(0) = 0$ and $\dot{x}(0) = v_0 > 0$.

For each value of the initial velocity v_0, determine the corresponding value of a such that the motion law $t \mapsto x(t)$ is a *bounded* and *non-periodic* function.

Exercise 5.2. According to observations made in the last decade, the star TRAPPIST–1 is orbited by seven exoplanets, namely TRAPPIST–1b, 1c, 1d, 1e, 1f, 1g, and 1h in alphabetic order going out from the star. Let us denote the mass of TRAPPIST–1 as M; all these exoplanets have very small masses with respect to M and their orbits are quasi-circular and nearly coplanar. In the following, we use the symbols T_1 , T_2 , T_3 , T_4 , T_5 , T_6 and T_7 to refer to the revolution periods of TRAPPIST–1b, 1c, 1d, 1e, 1f, 1g, and 1h, respectively, while we denote their (average) values of the distance from the star as r_1 , r_2 , r_3 , r_4 , r_5 , r_6 and r_7 . This extrasolar planetary system is characterized by an impressive chain of resonances; indeed, the ratios of the periods between neighbouring planet pairs are such that $T_2/T_1 \simeq 8/5$, $T_3/T_2 \simeq 5/3$, $T_4/T_3 \simeq 3/2$, $T_5/T_4 \simeq 3/2$, $T_6/T_5 \simeq 4/3$ and $T_7/T_6 \simeq 3/2$.

After having justified the general relation existing between the radius of a circular planetary orbit and its period of revolution (by neglecting the mutual gravitational interactions between planets), calculate the ratios r_{j+1}/r_j $\forall\ j = 1, \ldots , 6$.

6 Numerical analysis

Exercise 6.1. Let us consider the functions

$$B_i^{(p)}(x) := \binom{p}{i} x^i (1-x)^{p-i}, \quad i = 0, \cdots, p, \tag{6.1}$$

and let $x_0, \ldots, x_p$ be real numbers belonging to $[0,1]$. Let us consider the matrix

$$B := \begin{pmatrix} B_0^p(x_0) & B_1^p(x_0) & \cdots & B_p^p(x_0) \\ B_0^p(x_1) & B_1^p(x_1) & \cdots & B_p^p(x_1) \\ \vdots & \vdots & & \vdots \\ B_0^p(x_p) & B_1^p(x_p) & \cdots & B_p^p(x_p) \end{pmatrix}.$$

1. Prove that the vector $\mathbf{v} := (1, 1, \cdots, 1)^T$ is an eigenvector of B and compute the corresponding eigenvalue, $\lambda_{\mathbf{v}}$;

2. prove that for any eigenvalue λ_i of B we have $|\lambda_i| \leq \lambda_{\mathbf{v}}$;

3. analyze existence and uniqueness of the solution of the following interpolation problem
   Find $g(x) := \sum_{j=0}^{p} c_j B_j^{(p)}(x)$, $c_0, \ldots, c_p \in \mathbb{R}$, such that

   $$g(x_i) = f_i, \ i = 0, \ldots, p$$

   where $f_0, \ldots, f_p$ are given real numbers.

Exercise 6.2. Let us consider the matrix

$$A = \alpha I - \mathbf{u}\mathbf{u}^T, \ \mathbf{u} \in \mathbb{R}^n, \ \alpha \in \mathbb{R}, \ \|\mathbf{u}\|_2 = 1.$$

1. Compute the eigenvalues of A as a function of α and the corresponding eigenvectors;

2. determine α such that the iterative method

   $$\mathbf{x}^{(i+1)} = A\mathbf{x}^{(i)} + \mathbf{q}, \ \mathbf{q} \in \mathbb{R}^n$$

   is convergent;

3. compute A^{-1} if it exists;

4. compute

   $$\min_{0<\alpha<1} \mu_2(A),$$

   where $\mu_2(A) := \|A\|_2 \|A^{-1}\|_2$, and $\|A\|_2 := \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$.

Exercise 6.3. Let $a =: z_0 < z_1 < \cdots < z_{n+1} := b$ be given points belonging to the interval $[a, b]$ and let $\mathbb{P}_3$ be the space of algebraic polynomials of degree less than or equal to 3.

Let us consider the space of functions

$$\mathbb{S} := \{s \in C^2([a, b]) : s|_{[z_i, z_{i+1}]} \in \mathbb{P}_3,\ i = 0, \ldots, n,\ s''(a) = s''(b) = 0\}. \quad (6.2)$$

1. Determine the dimension of $\mathbb{S}$ and discuss the existence and uniqueness of the solution of the following problem

   Find $s \in \mathbb{S}$, such that

   $$s(z_i) = f_i,\ i = 0, \ldots, n + 1$$

   where $f_0, \ldots, f_{n+1}$ are given real numbers;

2. prove that the solution of the previous problem, whenever it exists, minimize the following functional

   $$\int_a^b (g''(x))^2 \mathrm{d}x,$$

   in the class of functions

   $$\{g \in C^2([a, b]) :\ g(z_i) = f_i,\ i = 0, \ldots, n + 1\}.$$

7 Probability

Exercise 7.1. Let X be a random variable taking values in $\mathbb{R}$ and let $\varphi_X : \mathbb{R} \to \mathbb{C}$ denote its characteristic function: $\varphi_X(t) = \mathbb{E}(e^{itX})$, $\mathbb{E}$ denoting the expectation.

(a) Prove that $\overline{\varphi_X}(t)$, $\varphi_X(t)^2$ and $|\varphi_X(t)|^2$ are still characteristic functions and find the random variables underlying them.

(b) Suppose that X is square integrable. Prove that $\varphi_X''(0) \leq (\varphi_X'(0))^2$.

(c) Suppose that φ_X is a polynomial. Prove that $\varphi_X(t) \equiv 1$ and deduce what X is.

Exercise 7.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathbb{E}$ denote the expectation under $\mathbb{P}$. Assume that $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$ and

$$\mathbb{P}(A) = \frac{1}{\sqrt{4\pi}} \int_A e^{-\frac{1}{4}x^2} dx, \quad A \in \mathcal{B}(\mathbb{R}).$$

Take $X(\omega) = \omega$ and let $\mathbb{Q}$ denote the measure on $(\Omega, \mathcal{F})$ defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{X-1}$, that is, $\mathbb{Q}(A) = \mathbb{E}(e^{X-1} 1\!\!1_A)$ for every $A \in \mathcal{F}$. In the following, $\mathrm{N}(\mu, \sigma^2)$ denotes the Gaussian law with mean μ and variance σ^2 (recall that if $Z \sim \mathrm{N}(0,1)$ then $\mathbb{E}(e^{\lambda Z}) = e^{\lambda^2/2}$.)

(a) Prove that $X \sim \mathrm{N}(0,2)$ and that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$.

(b) Let Y_1 and Y_2 be random variables such that X, Y_1 and Y_2 are independent in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that Y_1 and Y_2 are independent also in $(\Omega, \mathcal{F}, \mathbb{Q})$. Is there some link between their joint law in $(\Omega, \mathcal{F}, \mathbb{P})$ and in $(\Omega, \mathcal{F}, \mathbb{Q})$?

(c) On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{Y_n\}_n$ denote a sequence of i.i.d. random variables which are independent of X. Assume that Y_1 is square integrable and $\mathbb{E}(Y_1) = 0$, $\mathrm{Var}(Y_1) = \sigma^2$. Study the weak convergence, for $n \to \infty$, of $\{\frac{1}{n^\alpha} \sum_{k=1}^n Y_k\}_n$ in the space $(\Omega, \mathcal{F}, \mathbb{Q})$, for every $\alpha \in \mathbb{R}$.

Exercise 7.3. Let $\{X_n\}_{n=1,2,\ldots}$ be a sequence of random variables defined on a probability space $\{\Omega, \Im, \mathbb{P}\}$. We say that $\{X_n\}_{n=1,2,\ldots}$ converges almost certainly to zero ($X_n \to_{qc} 0$) if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}(\cup_{m\geq n} \{\omega \in \Omega : |X_m(\omega)| > \varepsilon\}) = 0 \ ,$$

or equivalently

$$\mathbb{P}(\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = 0) = 1 \ .$$

X_n is said to converge completely to zero ($X_n \to_{cc} 0$) if for each $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\omega \in \Omega : |X_m(\omega)| > \varepsilon\right\} < \infty \ .$$

1) Prove that $\{X_n \to_{cc} 0\} \Rightarrow \{X_n \to_{qc} 0\}$
2) Prove with a counterexample that the implication

$$\{X_n \to_{qc} 0\} \Rightarrow \{X_n \to_{cc} 0\}$$

is false.

3) Let $X_n(\alpha) = n^{-\alpha} \sum_{i=1}^{n} Y_i Z_i$, with $Z_i \sim NID(0,1)$ (standard independent Gaussian) and $Y_i \sim i.i.d.\ Exp(1)$ (exponentials of 1 independent parameter); we also assume that the Y are independent of the Z for any index value. Calculate $\lim_{n\to\infty} \mathbb{E}[X_n^4(\frac{1}{2})]$.

4) Prove that $X_n(\frac{4}{5}) = n^{-4/5} \sum_{i=1}^{n} Y_i Z_i \to_{cc} 0$.