Some results of geometric representation theory

Lecturers: Luca Casarin, Luca Francone.

Course length: 20 hours.

Contents: The aim of this course is to introduce some fundamental results of geometric representation theory. More concretely, we will focus on geometric methods for constructing representations of reductive algebraic groups and Lie algebras over an algebraically closed field of characteristic zero. The lectures will be divided into two main parts.

In the first half of the lectures we will introduce and prove the Borel-Weil-Bott theorem, which allows us to realize all the irreducible representations of a reductive group G as cohomology spaces of line bundles on the flag variety of G. Time permitting, we will then see how the BWB theorem can be used to study the decomposition of tensor product of irreducible representations of G [PRRV67, Theorem 2.1], [FR23], and to motivate interest in good, perfect, and biperfect bases and the theory of crystals [Kam22], [BFZ05].

In the second part of the lectures we will introduce $\mathscr{D}$-modules [HT07], which are an algebro-geometric version of the theory of PDE's and discuss the Beilinson-Bernstein localization theorem [BB93][Theorem 3.3.1]. The latter establishes an equivalence between the category of representations of a semisimple Lie algebra with a fixed central character and the category of twisted $\mathscr{D}$-modules on the flag variety of G, it is a cornerstone of geometric representation theory and has been used by the authors to prove the Kazhdan-Lusztig conjectures. We will discuss the theory of $\mathscr{D}$-modules in general, but with a focus on the concrete case of $\mathfrak{sl}_2$ acting via vector fields on $\mathbb{P}^1$.

Exams: Exercises to be carried out during the lecture period and oral exam at the end of the lectures. The oral exam will consist in a talk on a subject related to the ones discussed during the course, to be decided with the lecturers.

Prerequisites: The course will be as self contained as possible. As representation theory is concerned we require some knowledge of (reductive, simple) algebraic groups, Lie algebras (over $\mathbb{C}$) and their representations. For these topics, we refer to the classical books of Humphreys [Hum75], [Hum12]. For the algebraic geometry aspects we will assume some familiarity with algebraic varieties and quasi-coherent sheaves on them. The knowledge of basic results from [Har13] will be largely sufficient. Some homological algebra (derived categories) will be used, but we will recall the tools that we need.

Program: The course will take place during the months of January/February 2025, with the precise schedule to be decided with the participants. We will conduct a preliminary meeting with those interested in the course, with the aim to calibrate the content of the first lessons and make the course more accessible. Each part of the course will last 10 hours. It will be possible to attend the two parts independently, but we strongly discourage that since some notions presented in the first part will be used in the second one.

Here is a more detailed program:

- **Part I: The Borel-Weil-Bott Theorem**

  In the first lecture we will recall some fundamental structural results of reductive groups and their representations. Depending on the needs of the audience, we will discuss the classification of irreducible representations of reductive groups, the structure of the flag variety, and the properties of the Bruhat decomposition.

  We will then introduce G-linearisations of line bundles on algebraic varieties carrying a group action and study them, following [Bri18], and see how these objects provide a geometric framework for constructing representations.

  We will then prove the Borel-Weil theorem (Theorem 2 of [Lur07]).

  In the last lectures of Part I we will discuss some of the previously mentioned applications of the Borel-Weil Theorem. We will finally prove the Borel-Weil-Bott Theorem, relying on the proof of Demazure [Dem76].

- **Part II: The Beilinson-Bernstein localization Theorem**

  Introductory talk: we will give an historical overview (and prove) of the Riemann-Hilbert correspondence as a motivation to the theory of $\mathscr{D}$-modules;

  The bulk of the talks will be concerned with developing the theory of $\mathscr{D}$-modules. We define them on affine algebraic varieties and then in their sheaf theoretic version, we explain the difference between left and right $\mathscr{D}$-modules. We study pushforward and pullback between such objects, prove Kashiwara's Theorem and analyze the structure of coherent $\mathscr{D}$-modules. We will pick the material we need from standard references such as Bernstein notes `https://gauss.math.yale.edu/~il282/Bernstein_D_mod.pdf` and [HT07];

  The last talks will be devoted to state and prove (at least the $\mathfrak{sl}_2$ version) of the Beilinson Bernstein localization Theorem. We will rely on the original paper [BB93] for the proof and on [Rom21] for examples.

Technical aspects: The first half of the lectures will be in the university of Sapienza, while the second half in Tor Vergata. The schedule will be decided with the participants. It will be possible to stream the course under request.

References

[BB93] Alexander Beilinson and John Andrew Bernstein. A proof of jantzen conjectures. 1993.

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras iii: Upper bounds and double bruhat cells. *Duke Mathematical Journal*, 126(1):1–52, 2005.

[Bri18] Michel Brion. Linearization of algebraic group actions. *Handbook of group actions*, 4:291–340, 2018.

[Dem76] Michel Demazure. A very simple proof of bott's theorem. *Inventiones mathematicae*, 33:271–272, 1976.

[FR23] Luca Francone and Nicolas Ressayre. On the multiplicity spaces of spherical subgroups of minimal rank. *Bulletin of the London Mathematical Society*, 2023.

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[HT07] Ryoshi Hotta and Toshiyuki Tanisaki. *D-modules, perverse sheaves, and representation theory*, volume 236. Springer Science & Business Media, 2007.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[Hum12] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.

[Kam22] Joel Kamnitzer. Perfect bases in representation theory: three mountains and their springs. *arXiv preprint arXiv:2201.02289*, 2022.

[Lur07] Jacob Lurie. A proof of the borel-weil-bott theorem. *unpublished, available at the author's webpage: http://www. math. harvard. edu/ lurie/bwb. pdf*, 2007.

[PRRV67] K. R. Parthasarathy, R. Ranga Rao, and V. S. Varadarajan. Representations of complex semi-simple Lie groups and Lie algebras. *Ann. of Math. (2)*, 85:383–429, 1967.

[Rom21] Anna Romanov. Four examples of beilinson–bernstein localization. *Lie Groups, Number Theory, and Vertex Algebras*, 768:65–85, 2021.