“Criticality theory and nonlinear elliptic equations with local and nonlocal interactions”

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Abstract: This course is an introduction to Agmon's Criticality Theory of Schrödinger operators and its applications in the elliptic nonlinear problems. It will focus on Allegretto-Piepenbrink positivity principle and Phragmen-Lindelöf comparison principle. We will show how these fundamental principles enable to prove a range of Hardy type inequalities and provide a powerful tool in the analysis of the structure of positive solutions for large classes of nonlinear elliptic equations, including NLS and nonlinear equations with nonlocal interactions, such as Choquard equations. The course splits into three parts:

I. Linear theory:
Allegretto-Piepenbrink positivity principle for linear Schrödinger operators and some corollaries
Connection with Hardy inequalities
Maximum principle on bounded and ubounded domains
Phragmen-Lindelöf comparison principle: large and small positive solutions
Weak, strong and critical potentials

II. Nonlinear applications – local equations:
exponential vs polynomial decay
nonlinear Liouville theorems, Serrin's critical exponent, fast and slow decay solutions
singular solutions of semilinear elliptic equations, local and global Keller-Osserman bound, removable singularities

III. Nonlinear applications – nonlocal problems:
the origin and motivation for nonlocal interactions in elliptic PDEs
Riesz potentials; fundamental L^1-decay estimate
Nonlocal AAP positivity principle and nonlocal nonlinear Liouville theorems
Nonuniversality of decay in nonlocal problems

Prerequisites are limited to basic concepts of elliptic PDEs: weak solutions, classical maximum principle, basic understanding of Sobolev spaces.

The course will be an extended version of the online lectures given at *Mini-courses in Mathematical Analysis* at University of Padova (2020) and at the *TCC (The Taught Course Centre)* at University of Oxford (2022). For an outline of the first two parts of the course see DOI: 10.13140/RG.2.2.10445.05601